1 Induction Exercises

1.1

Prove that

$$\sum_{i=0}^{n} i^2 = 1^2 + 2^2 + 3^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6} (*)$$

Proof. If $n = 1$, then

$$\sum_{i=0}^{1} i^2 = 1 = \frac{1(1+1)(2+1)}{6}$$

thus $(*)$ is true for $n = 1$.

Now assume $(*)$ is true for $n = k$, such that

$$\sum_{i=0}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$$

We then need to prove that $(*)$ is true for $n = k + 1$. Adding $(k + 1)^2$ to both sides of our assumption gets

$$\begin{aligned}
\sum_{i=0}^{k+1} i^2 &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\
&= (k+1)\left[\frac{k(2k+1)}{6} + (k+1)\right] \\
&= (k+1)\left[\frac{k(2k+1) + 6(k+1)}{6}\right] \\
&= (k+1)\left[\frac{2k^2 + k + 6k + 6}{6}\right] \\
&= (k+1)\left[\frac{2k^2 + 4k + 3k + 6}{6}\right] \\
&= (k+1)\left[\frac{2k(k+2) + 3(k+2)}{6}\right] \\
&= (k+1)\left[\frac{(k+2)(2k+3)}{6}\right] \\
&= \frac{(k+1)(k+2)(2k+3)}{6} \\
&= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}
\end{aligned}$$

where the last equation is the same as $(*)$ with $n = k+1$. Hence by the priniciple of mathematical induction $(*)$ is true for all $n \in \mathbf{N}$. $\square$