Induction Exercise 1-2

Prove that

$$\sum_{i=0}^{n} i^3 = 1^3 + 2^3 + 3^3 + ... + n^3 = (1 + 2 + 3 + ... + n)^3$$

Proof. If $n = 1$, then

$$\sum_{i=0}^{1} i^3 = 1 = (1)^3.$$

Thus the statement is true for $n = 1$.

Now assuming that

$$\sum_{i=0}^{k} i^3 = (1 + 2 + 3 + ... + k)^3$$

we find that

$$\begin{aligned}
\sum_{i=0}^{k+1} i^3 &= \sum_{i=0}^{k} i^3 + (k+1)^3 \\
&= (1 + 2 + 3 + ... + k)^3 + (k+1)^3 \\
&= \left(\frac{k(k+1)}{2}\right)^3 + (k+1)^3 \\
&= \left(\left(\frac{k(k+1)}{2}\right) + (k+1)\right)^3 \\
&= \left(\frac{k(k+1) + 2(k+1)}{2}\right)^3 \\
&= \left(\frac{(k+1)(k+2)}{2}\right)^3 \\
&= \left(\frac{(k+1)((k+1)+1)}{2}\right)^3 \\
&= (1 + 2 + 3 + ... + (k+1))^3.
\end{aligned}$$

Hence by the priniciple of mathematical induction we have established that the statement is true for all $n \in \mathbb{N}$. $\square$