Number Theory Exercises

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1 Induction Exercises

1.1

Prove that

$$1^2 + 2^2 + 3^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6} (*)$$

Proof. If $n = 1$, then

$$1(1+1)(2(1)+1) = (2)(3) = 6$$

which is divisible by 6, making the statement true for $n = 1$.

Assuming the statement is true for $n \leq k$, where n exists in the natural numbers. We then know that

$$k(k+1)(2k+1) = 6m, \text{ where } m \in \mathbf{N},$$

and so

$$\begin{aligned} (k+1)(k+2)(2(k+1)+1) &= (k+2)(k+1)((2k+1)+2) \\ &= (k+1)(k(2k+1) + 2k + 2(2k+1) + 4) \\ &= k(k+1)(2k+1) + 2k(k+1)(k + (2k+1) + 2) \\ &= 6m + 2(k+1)(3k+3) \\ &= 6m + 6(k+1)^2 \\ &= 6(m + (k+1)^2) \end{aligned}$$

where the last equation is divisible by 6. Thus, by the principle of mathematical induction, $(*)$ is true for all $n \in \mathbf{N}$. □