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# 1 Basis Representation

## 1.1 Principle of Mathematical Induction

### 1.1.1

Prove that

$$\sum_{i=0}^n i^2 = 1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

*Proof.* If  $n = 1$ , then

$$\sum_{i=0}^1 i^2 = 1 = \frac{1(1+1)(2+1)}{6}.$$

Thus, the identity holds for  $n = 1$ . Now assume that

$$\sum_{i=0}^n i^2 = \frac{n(n+1)(2n+1)}{6}$$

for some  $n \geq 1$ , and compute

$$\begin{aligned} \sum_{i=0}^{n+1} i^2 &= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \\ &= (n+1) \left[ \frac{n(2n+1)}{6} + (n+1) \right] \\ &= (n+1) \left[ \frac{n(2n+1) + 6(n+1)}{6} \right] \\ &= (n+1) \left[ \frac{2n^2 + n + 6n + 6}{6} \right] \\ &= (n+1) \left[ \frac{2n^2 + 4n + 3n + 6}{6} \right] \\ &= (n+1) \left[ \frac{2n(n+2) + 3(n+2)}{6} \right] \\ &= (n+1) \left[ \frac{(n+2)(2n+3)}{6} \right] \\ &= \frac{(n+1)(n+2)(2n+3)}{6} \\ &= \frac{(n+1)((n+1)+1)(2(n+1)+1)}{6}, \end{aligned}$$

so the identity also holds for  $n+1$ . Thus, by the principle of mathematical induction, the identity holds for all  $n \geq 1$ , where  $n \in \mathbb{N}$ .  $\square$

### 1.1.2

Prove that

$$\sum_{i=0}^n i^3 = 1^3 + 2^3 + 3^3 + \dots + n^3 = (1 + 2 + 3 + \dots + n)^3.$$

*Proof.* If  $n = 1$ , then

$$\sum_{i=0}^1 i^3 = 1 = (1)^3.$$

Thus, the identity holds for  $n = 1$ . Now assume that

$$\sum_{i=0}^n i^3 = (1 + 2 + 3 + \dots + n)^3$$

for some  $n \geq 1$ , and compute

$$\begin{aligned} \sum_{i=0}^{n+1} i^3 &= \sum_{i=0}^n i^3 + (n+1)^3 \\ &= (1 + 2 + 3 + \dots + n)^3 + (n+1)^3 \\ &= \left( \frac{n(n+1)}{2} \right)^3 + (n+1)^3 \\ &= \left( \left( \frac{n(n+1)}{2} \right) + (n+1) \right)^3 \\ &= \left( \frac{n(n+1) + 2(n+1)}{2} \right)^3 \\ &= \left( \frac{(n+1)(n+2)}{2} \right)^3 \\ &= \left( \frac{(n+1)((n+1)+1)}{2} \right)^3 \\ &= (1 + 2 + 3 + \dots + (n+1))^3, \end{aligned}$$

so the identity also holds for  $n + 1$ . Thus, by the principle of mathematical induction, the identity holds for all  $n \geq 1$ , where  $n \in \mathbb{N}$ .  $\square$

**1.1.4**

Prove that

$$\sum_{i=0}^n i(i+1) = 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \dots + n(n+1) = \frac{n(n+1)(n+2)}{3}.$$

*Proof.* If  $n = 1$ , then

$$\sum_{i=0}^1 i(i+1) = 2 = \frac{1(2)(3)}{3}.$$

Thus, the identity holds for  $n = 1$ . Now assuming that

$$\sum_{i=0}^n i(i+1) = \frac{n(n+1)(n+2)}{3}$$

for some  $n \geq 1$ , and compute

$$\begin{aligned} \sum_{i=0}^{n+1} i(i+1) &= \left( \sum_{i=0}^n i(i+1) \right) + (n+1)((n+1)+1) \\ &= \frac{n(n+1)(n+2)}{3} + (n+1)(n+2) \\ &= \frac{n(n+1)(n+2) + 3(n+1)(n+2)}{3} \\ &= \frac{(n+1)(n+2)(n+3)}{3} \\ &= \frac{(n+1)((n+1)+1)((n+1)+2)}{3}, \end{aligned}$$

so the identity also holds for  $n+1$ . Thus, by the principle of mathematical induction, the identity holds for all  $n \geq 1$ , where  $n \in \mathbb{N}$ .  $\square$

**1.1.5**

Prove that

$$\sum_{i=1}^n (2i - 1) = 1 + 3 + 5 + \dots + (2n - 1) = n^2.$$

*Proof.* If  $n = 1$ , then

$$\sum_{i=1}^1 (2i - 1) = 1 = 1^2.$$

Thus, the identity holds for  $n = 1$ . Now assume that

$$\sum_{i=1}^n (2i - 1) = n^2$$

for some  $n \geq 1$ , and compute

$$\begin{aligned} \sum_{i=1}^{n+1} (2i - 1) &= \left( \sum_{i=1}^n (2i - 1) \right) + (2(n + 1) - 1) \\ &= n^2 + 2(n + 1) - 1 \\ &= n^2 + 2n + 1 \\ &= (n + 1)(n + 1) \\ &= (n + 1)^2, \end{aligned}$$

so the identity also holds for  $n + 1$ . Thus, by the principle of mathematical induction, the identity holds for all  $n \geq 1$ , where  $n \in \mathbb{N}$ .  $\square$