Contents

1 Euclidean Geometry

1.1 Jane Street's 3b1b Puzzler

Let C be a closed bounded convex set in $\mathbb{R}^3$, and let $B = \partial C$. Now imagine creating a new set by taking any two points from B and adding them together, doing this for every possible pair of points. Denote this new set of vector sums as $D = \{\vec{p} + \vec{q} \,|\, \vec{p}, \vec{q} \in B\}$.
Prove that D is also a convex set.

Proof. Let $\vec{x}, \vec{y} \in D$. This implies that $\vec{x} = \vec{p} + \vec{q}$ and $\vec{y} = \vec{r} + \vec{s}$, where $\vec{p}, \vec{q}, \vec{r}, \vec{s} \in B$. It is first important to note that since C is closed, it contains all its boundary points, and since C is bounded, the distance between any two boundary points is within the bound defined by the distance function d_C, say $M \in \mathbb{R}$. Then, since D is just the vector sums of all possible pairs of boundary points in C, we can find the bound for d_D with our given points in D as follows:

$$d_D(\vec{x}, \vec{y}) = |\vec{x} - \vec{y}| = |(\vec{p} + \vec{q}) - (\vec{r} + \vec{s})| \leq |\vec{p} - \vec{r}| + |\vec{q} - \vec{s}| \leq M + M = 2M.$$

With this bound, let $\lambda \in \mathbb{R}$, where $0 < \lambda < 1$, and see that

$$\begin{aligned}
|(1-\lambda)\vec{x} + \lambda\vec{y}| &= |\vec{x} - \lambda\vec{x} + \lambda\vec{y}| \\
&= |(\vec{p} + \vec{q}) - \lambda(\vec{p} + \vec{q}) + \lambda(\vec{r} + \vec{s})| \\
&= |\vec{p} - \lambda\vec{p} + \lambda\vec{r} + \vec{q} - \lambda\vec{q} + \lambda\vec{s}| \\
&= |((1-\lambda)\vec{p} + \lambda\vec{r}) + ((1-\lambda)\vec{q} + \lambda\vec{s})| \\
&\leq |(1-\lambda)\vec{p} + \lambda\vec{r}| + |(1-\lambda)\vec{q} + \lambda\vec{s})| \\
&\leq M + M = 2M.
\end{aligned}$$

Hence $(1-\lambda)\vec{x} + \lambda\vec{y} \in D$, implying that D is indeed also a convex set. $\square$