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5 Sobolev Spaces

5.1

Suppose $k \in \{0, 1, \dots\}$, $0 < \gamma < 1$. Prove $C^{k,\gamma}(\overline{U})$ is a Banach space.

Proof. A Banach space is a *complete normed linear space*, so we need to show that $C^{k,\gamma}(\overline{U})$ is each of the following:

1. A Linear space.
2. A Normed linear space.
3. A Complete space.

Much of the proof will rely on the fact that we know $C(\overline{U})$ is a Banach space itself.

1. Suppose $u, v, w \in C^{k,\gamma}(\overline{U})$ and $a, b \in \mathbb{R}$. It follows that $u, v, w \in C^k(\overline{U})$ and that the Hölder norm of each u , v , and w is finite. With this knowledge we will loosely prove the linear space axioms as follows:

- (i) $u + v \in C^{k,\gamma}(\overline{U})$ because $\|u + v\|_{C^{k,\gamma}(\overline{U})} < \infty$, which follows from the triangle inequality of norms.
- (ii) $u + v = v + u$ because $u, v \in C^k(\overline{U})$, which we know is itself a linear space.
- (iii) $u + (v + w) = (u + v) + w$ following the same argument as (ii).
- (iv) $C^k(\overline{U}) \subset C^{k,\gamma}(\overline{U})$ implies $\exists \vec{0} \in C^{k,\gamma}(\overline{U})$ such that $u + \vec{0} = u$.
- (v) Following the same argument as (iv) we get that $\forall u \in C^{k,\gamma}(\overline{U}) \exists (-u)$ such that $u + (-u) = \vec{0}$.
- (vi) $a(bu) = (ab)u$ following the same argument as (ii).
- (vii) $a(u + v) = au + av$ following the same argument as (ii).
- (viii) $(a + b)u = au + bu$ following the same argument as (ii).
- (ix) Following the same argument as (iv) we get that $\exists \vec{1} \in C^{k,\gamma}(\overline{U})$ such that $\vec{1} \cdot u = u$.
- (x) Following the same argument as (iv) we get that $\exists \vec{0} \in C^{k,\gamma}(\overline{U})$ such that $\vec{0} \cdot u = \vec{0}$.

These axioms prove that $C^{k,\gamma}(\overline{U})$ is indeed a linear space.

2.

3.

□

5.3

Denote by U the open square $\{x \in \mathbb{R}^2 \mid |x_1| < 1, |x_2| < 1\}$. Define

$$u(x) = \begin{cases} 1 - x_1, & \text{if } x_1 > 0, |x_2| < x_1 \\ 1 + x_1, & \text{if } x_1 < 0, |x_2| < -x_1 \\ 1 - x_2, & \text{if } x_2 > 0, |x_1| < x_2 \\ 1 + x_2, & \text{if } x_2 < 0, |x_1| < -x_2. \end{cases}$$

For which $1 \leq p \leq \infty$ does u belong to $W^{1,p}(U)$.

Proof. In progress...

□

5.5

Let U, V be open sets, with $V \subset\subset U$. Show there exists a smooth function ζ such that $\zeta \equiv 1$ on V , $\zeta = 0$ near ∂U . (Hint: Take $V \subset\subset W \subset\subset U$ and mollify χ_W .)

Proof. In progress...

□

5.6

Assume U is bounded and $U \subset\subset \bigcup_{i=1}^N V_i$. Show there exist C^∞ functions ζ_i ($i = 1, \dots, N$) such that

$$\begin{cases} 0 \leq \zeta_i \leq 1, \text{ supp } \zeta_i \subset V_i \text{ } (i = 1, \dots, N) \\ \sum_{i=1}^N \zeta_i = 1 \text{ on } U. \end{cases}$$

Proof. In progress...

□