Contents

5 Sobolev Spaces

5.1

Suppose $k \in \{0, 1, \dots\}$, $0 < \gamma < 1$. Prove $C^{k,\gamma}(\overline{U})$ is a Banach space.

Proof. A Banach space is a *complete normed linear space*, so we need to show that $C^{k,\gamma}(\overline{U})$ is each of the following:

1. A Linear space.

2. A Normed linear space.

3. A Complete space.

Much of the proof will rely on the fact that we know $C(\overline{U})$ is a Banach space itself.

1. Suppose $u, v, w \in C^{k,\gamma}(\overline{U})$ and $a, b \in \mathbb{R}$. It follows that $u, v, w \in C^k(\overline{U})$ and that the Hölder norm of each u, v, and w is finite. With this knowledge we will loosely prove the linear space axioms as follows:

   (i) $u + v \in C^{k,\gamma}(\overline{U})$ because $\|u + v\|_{C^{k,\gamma}(\overline{U})} < \infty$, which follows from the triangle inequality of norms.
   (ii) $u + v = v + u$ because $u, v \in C^k(\overline{U})$, which we know is itself a linear space.
   (iii) $u + (v + w) = (u + v) + w$ following the same argument as (ii).
   (iv) $C^k(\overline{U}) \subset C^{k,\gamma}(\overline{U})$ implies $\exists\, \vec{0} \in C^{k,\gamma}(\overline{U})$ such that $u + \vec{0} = u$.
   (v) Following the same argument as (iv) we get that $\forall\, u \in C^{k,\gamma}(\overline{U})\, \exists\, (-u)$ such that $u + (-u) = \vec{0}$.
   (vi) $a(bu) = (ab)u$ following the same argument as (ii).
   (vii) $a(u + v) = au + av$ following the same argument as (ii).
   (viii) $(a + b)u = au + bu$ following the same argument as (ii).
   (ix) Following the same argument as (iv) we get that $\exists\, \vec{1} \in C^{k,\gamma}(\overline{U})$ such that $\vec{1} \cdot u = u$.
   (x) Following the same argument as (iv) we get that $\exists\, \vec{0} \in C^{k,\gamma}(\overline{U})$ such that $\vec{0} \cdot u = \vec{0}$.

   These axioms prove that $C^{k,\gamma}(\overline{U})$ is indeed a linear space.

2. Using the fact that $C(\overline{U})$ is itself a normed linear space, with the norm

   $$\|u(x)\|_{C(\overline{U})} := \sup_{x \in U} |u(x)|,$$

   we can prove the normed linear space axioms as follows:

   (i) Suppose $u, v \in C^{k,\gamma}(\overline{U})$ and that α is a fixed multi-index such that $|\alpha| \leq k$. It then follows that

   $$\begin{aligned}
   \|u + v\|_{C^{k,\gamma}(\overline{U})} &= \sum_{|\alpha| \leq k} \|D^\alpha (u + v)\|_{C(\overline{U})} + \sum_{|\alpha| = k} [D^\alpha (u + v)]_{C^{0,\gamma}(\overline{U})} \\
   &\leq \sum_{|\alpha| \leq k} (\|D^\alpha u\|_{C(\overline{U})} + \|D^\alpha v\|_{C(\overline{U})}) + \sum_{|\alpha| = k} [D^\alpha (u + v)]_{C^{0,\gamma}(\overline{U})} \\
   &\leq \sum_{|\alpha| \leq k} (\|D^\alpha u\|_{C(\overline{U})} + \|D^\alpha v\|_{C(\overline{U})}) + \sum_{|\alpha| = k} ([D^\alpha u]_{C^{0,\gamma}(\overline{U})} + [D^\alpha v]_{C^{0,\gamma}(\overline{U})}) \\
   &= \|u\|_{C^{k,\gamma}(\overline{U})} + \|v\|_{C^{k,\gamma}(\overline{U})}.
   \end{aligned}$$

(ii) Suppose $m \in \mathbb{R}$, $u \in C^{k,\gamma}(\overline{U})$, and that α is a multi-index such that $|\alpha| \leq k$. It then follows that

$$\begin{aligned}
\|mu\|_{C^{k,\gamma}(\overline{U})} &= \sum_{|\alpha|\leq k} \|mD^\alpha u\|_{C^k(\overline{U})} + \sum_{|\alpha|=k} [mD^\alpha u]_{C^{0,\gamma}(\overline{U})} \\
&= |m| \sum_{|\alpha|\leq k} \|D^\alpha u\|_{C^k(\overline{U})} + |m| \sum_{|\alpha|=k} [D^\alpha u]_{C^{0,\gamma}(\overline{U})} \\
&= |m| (\sum_{|\alpha|\leq k} \|D^\alpha u\|_{C^k(\overline{U})} + \sum_{|\alpha|=k} [D^\alpha u]_{C^{0,\gamma}(\overline{U})}) \\
&= |m| \|u\|_{C^{k,\gamma}(\overline{U})}.
\end{aligned}$$

(iii) We now want to show that $\|u\|_{C^{k,\gamma}(\overline{U})} \geq 0, \forall\, u \in C^{k,\gamma}(\overline{U})$. So suppose $u \in C^{k,\gamma}(\overline{U})$, and from (ii) we know that

$$\| - u\|_{C^{k,\gamma}(\overline{U})} = \|(-1)u\|_{C^{k,\gamma}(\overline{U})} = \|u\|_{C^{k,\gamma}(\overline{U})},$$

which implies that

$$\begin{aligned}
\|0\|_{C^{k,\gamma}(\overline{U})} &= \|u + (-u)\|_{C^{k,\gamma}(\overline{U})} \\
&\leq \|u\|_{C^{k,\gamma}(\overline{U})} + \| - u\|_{C^{k,\gamma}(\overline{U})} \\
&= \|u\|_{C^{k,\gamma}(\overline{U})} + \|u\|_{C^{k,\gamma}(\overline{U})} \\
&= 2\|u\|_{C^{k,\gamma}(\overline{U})}.
\end{aligned}$$

Since u is arbitrary we can say

$$\|0\|_{C^{k,\gamma}(\overline{U})} \leq 2\|0\|_{C^{k,\gamma}(\overline{U})},$$

which further implies

$$0 \leq \|0\|_{C^{k,\gamma}(\overline{U})}$$

when we substract through by $\|0\|_{C^{k,\gamma}(\overline{U})}$. Combining inequalities from above gives us

$$0 \leq \|0\|_{C^{k,\gamma}(\overline{U})} \leq 2\|u\|_{C^{k,\gamma}(\overline{U})},$$

where dividing through by $1/2$ results in

$$0 \leq \|u\|_{C^{k,\gamma}(\overline{U})}.$$

(iv) We finally want to show that $\|u\|_{C^{k,\gamma}(\overline{U})} = 0$ if and only if $u = 0$. Firstly, suppose $u \in C^{k,\gamma}(\overline{U})$ and that $\|u\|_{C^{k,\gamma}(\overline{U})} = 0$. Letting α be some fixed multi-index such that $|\alpha| \leq k$, it then follows that

$$\|u\|_{C^{k,\gamma}(\overline{U})} = \sum_{|\alpha|\leq k} \|D^\alpha u\|_{C(\overline{U})} + \sum_{|\alpha|=k} [D^\alpha u]_{C^{0,\gamma}(\overline{U})} = 0.$$

By definition we know that each $\|D^\alpha u\|_{C(\overline{U})} \geq 0$, and also by definition we know that

$$[D^\alpha u]_{C^{0,\gamma}(\overline{U})} = \sup_{x,y\in U, x\neq y} \left\{ \frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x-y|^\gamma} \right\} \geq 0.$$

Therefore, if $\|D^\alpha u\|_{C^{k,\gamma}(\overline{U})} = 0$, then it must be that each $\|D^\alpha u\|_{C(\overline{U})} = 0$ and each $[D^\alpha u]_{C^{0,\gamma}(\overline{U})} = 0$. This further implies from the definitions that $D^\alpha u = 0$ implying $u = 0$, $\forall\, x \in U$. Conversely, suppose $u = 0$. Then by (ii) we can see that

$$\begin{aligned}\|u\|_{C^{k,\gamma}(\overline{U})} &= \|0\|_{C^{k,\gamma}(\overline{U})}\\ &= \|0 \cdot 0\|_{C^{k,\gamma}(\overline{U})}\\ &= |0|\|0\|_{C^{k,\gamma}(\overline{U})}\\ &= 0.\end{aligned}$$

These axioms prove that $C^{k,\gamma}(\overline{U})$ is indeed a normed linear space.

3. Suppose $\{u_m\}_{m=1}^\infty$ is a Cauchy sequence in $C^{k,\gamma}(\overline{U})$. This implies each $u_m \in C^k(\overline{U})$, and since $C^k(\overline{U})$ is a complete space, then there exists $u \in C^k(\overline{U})$ such that $u_m \to u$ in $C^k(\overline{U})$, as $m \to \infty$. Now if we wish to show $u \in C^{k,\gamma}(\overline{U})$, then we need to show that

$$\|u\|_{C^{k,\gamma}(\overline{U})} = \sum_{|\alpha| \le k} \|D^\alpha u\|_{C(\overline{U})} + \sum_{|\alpha| = k} [D^\alpha u]_{C^{0,\gamma}(\overline{U})} < \infty,$$

where α is some fixed multi-index such that $|\alpha| \le k$. $u \in C^k(\overline{U})$ implies that $D^\alpha u$ is uniformly continuous on bounded subsets of U, for all $|\alpha| \le k$, and since U is a bounded open set then it follows that each $D^\alpha u$ is bounded, futher implying that

$$\sum_{|\alpha| \le k} \|D^\alpha u\|_{C(\overline{U})} = \sum_{|\alpha| \le k} \sup_{x \in U} |D^\alpha u(x)| < \infty$$

since there is only a finite number of k^{th} or less partial derivatives of u. Now to show that each $[D^\alpha u]_{C^{0,\gamma}(\overline{U})}$ is finite observe that for some fixed $x, y \in U$, where $x \neq y$, we get

$$\begin{aligned}\frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x-y|^\gamma} &= \frac{|D^\alpha u(x) - D^\alpha u_m(x) + D^\alpha u_m(x) - D^\alpha u_m(y) + D^\alpha u_m(y) - D^\alpha u(y)|}{|x-y|^\gamma}\\ &\le \frac{|D^\alpha u(x) - D^\alpha u_m(x)|}{|x-y|^\gamma} + \frac{|D^\alpha u_m(x) - D^\alpha u_m(y)|}{|x-y|^\gamma} + \frac{|D^\alpha u_m(y) - D^\alpha u(y)|}{|x-y|^\gamma},\end{aligned}$$

where taking $m \to \infty$ makes the first and third summands disappear, and since u_m is a Cauchy sequence then we can say the middle summand is bounded. This means

$$\frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x-y|^\gamma} < \infty,$$

which implies

$$\sum_{|\alpha| = k} [D^\alpha u]_{C^{0,\gamma}(\overline{U})} = \sum_{|\alpha| = k} \sup_{x,y \in U, x \neq y} \frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x-y|^\gamma} < \infty$$

as again, there is only a finite number of k^{th} partial derivatives of u. Then since both summations are finite, it follows that $\|u\|_{C^{k,\gamma}(\overline{U})} < \infty$, proving that $u \in C^{k,\gamma}(\overline{U})$. Now we need to show that $u_m \to u$ in $C^{k,\gamma}(\overline{U})$. This requires that we show

$$\|u_m - u\|_{C^{k,\gamma}(\overline{U})} = \sum_{|\alpha| \le k} \|D^\alpha (u_m - u)\|_{C(\overline{U})} + \sum_{|\alpha| = k} [D^\alpha (u_m - u)]_{C^{0,\gamma}(\overline{U})} < \infty.$$

Now since we know $u_m \to u$ in $C^k(\overline{U})$, then we could follow a similar argument as before to conclude

$$\sum_{|\alpha|\leq k} \|D^\alpha(u_m - u)\|_{C(\overline{U})} < \infty.$$

To show each $[D^\alpha(u_m - u)]_{C^{0,\gamma}(\overline{U})}$ is finite observe again that

$$\begin{aligned}\frac{|D^\alpha(u_m-u)(x) - D^\alpha(u_m-u)(y)|}{|x-y|^\gamma} &= \frac{|D^\alpha u_m(x) - D^\alpha u(x) - D^\alpha u_m(y) + D^\alpha u(y)|}{|x-y|^\gamma} \\ &= \lim_{n\to\infty} \frac{|(D^\alpha u_m(x) - D^\alpha u_n(x)) + (D^\alpha u_m(y) - D^\alpha u_n(y))|}{|x-y|^\gamma},\end{aligned}$$

which can be made arbitrarily small choosing m large enough, since both u_m and u_n are Cauchy sequences. Then since Cauchy implies boundedness, we get that

$$\sum_{|\alpha|=k} [D^\alpha(u_m - u)]_{C^{0,\gamma}(\overline{U})} =< \infty.$$

This proves that $\|u_m - u\|_{C^{k,\gamma}(\overline{U})} < \infty$ for any $m \in \mathbb{N}$, and thus $u_m \to u$ in $C^{k,\gamma}(\overline{U})$.

With this last proof we have shown that $C^{k,\gamma}(\overline{U})$ is indeed a complete space, which completes the proof for $C^{k,\gamma}(\overline{U})$ being a Banach space. $\square$

5.3

Denote by U the open square $\{x \in \mathbb{R}^2 \,|\, |x_1| < 1, |x_2| < 1\}$. Define

$$u(x) = \begin{cases} 1 - x_1, & \text{if } x_1 > 0, |x_2| < x_1 \\ 1 + x_1, & \text{if } x_1 < 0, |x_2| < -x_1 \\ 1 - x_2, & \text{if } x_2 > 0, |x_1| < x_2 \\ 1 + x_2, & \text{if } x_2 < 0, |x_1| < -x_2. \end{cases}$$

For which $1 \leq p \leq \infty$ does u belong to $W^{1,p}(U)$.

Proof. In progress... □

5.5

Let U, V be open sets, with $V \subset\subset U$. Show there exists a smooth function ζ such that $\zeta \equiv 1$ on V, $\zeta = 0$ near ∂U. (Hint: Take $V \subset\subset W \subset\subset U$ and mollify χ_W.)

Proof. In progress... □

5.6

Assume U is bounded and $U \subset\subset \bigcup_{i=1}^N V_i$. Show there exist C^∞ functions ζ_i $(i = 1, \ldots, N)$ such that

$$\begin{cases} 0 \leq \zeta_i \leq 0, \text{ supp } \zeta_i \subset V_i \,(i = 1, \ldots, N) \\ \sum_{i=1}^N \zeta_i \text{ on } U. \end{cases}$$

Proof. In progress... □