Contents

1 Basis Representation

1.1 Principle of Mathematical Induction

1.1.1

Prove that
$$\sum_{i=0}^{n} i^2 = 1^2 + 2^2 + 3^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Proof. If $n = 1$, then
$$\sum_{i=0}^{1} i^2 = 1 = \frac{1(1+1)(2+1)}{6}.$$

Thus, the identity holds for $n = 1$. Now assume that
$$\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

for some $n \geq 1$, and compute
$$\begin{aligned}
\sum_{i=0}^{n+1} i^2 &= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \\
&= (n+1)\left[\frac{n(2n+1)}{6} + (n+1)\right] \\
&= (n+1)\left[\frac{n(2n+1)+6(n+1)}{6}\right] \\
&= (n+1)\left[\frac{2n^2+n+6n+6}{6}\right] \\
&= (n+1)\left[\frac{2n^2+4n+3n+6}{6}\right] \\
&= (n+1)\left[\frac{2n(n+2)+3(n+2)}{6}\right] \\
&= (n+1)\left[\frac{(n+2)(2n+3)}{6}\right] \\
&= \frac{(n+1)(n+2)(2n+3)}{6} \\
&= \frac{(n+1)((n+1)+1)(2(n+1)+1)}{6},
\end{aligned}$$

so the identity also holds for $n + 1$. Thus, by the principle of mathematical induction, the identity holds for all $n \geq 1$, where $n \in \mathbb{N}$. $\square$

1.1.2

Prove that
$$\sum_{i=0}^{n} i^3 = 1^3 + 2^3 + 3^3 + ... + n^3 = (1 + 2 + 3 + ... + n)^3.$$

Proof. If $n = 1$, then
$$\sum_{i=0}^{1} i^3 = 1 = (1)^3.$$

Thus, the identity holds for $n = 1$. Now assume that
$$\sum_{i=0}^{n} i^3 = (1 + 2 + 3 + ... + n)^3$$

for some $n \geq 1$, and compute
$$\begin{aligned}\sum_{i=0}^{n+1} i^3 &= \sum_{i=0}^{n} i^3 + (n+1)^3 \\ &= (1 + 2 + 3 + ... + n)^3 + (n+1)^3 \\ &= \left(\frac{n(n+1)}{2}\right)^3 + (n+1)^3 \\ &= \left(\left(\frac{n(n+1)}{2}\right) + (n+1)\right)^3 \\ &= \left(\frac{n(n+1) + 2(n+1)}{2}\right)^3 \\ &= \left(\frac{(n+1)(n+2)}{2}\right)^3 \\ &= \left(\frac{(n+1)((n+1)+1)}{2}\right)^3 \\ &= (1 + 2 + 3 + ... + (n+1))^3,\end{aligned}$$

so the identity also holds for $n + 1$. Thus, by the principle of mathematical induction, the identity holds for all $n \geq 1$, where $n \in \mathbb{N}$. $\square$

1.1.4

Prove that

$$\sum_{i=0}^{n} i(i+1) = 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + ... + n(n+1) = \frac{n(n+1)(n+2)}{3}.$$

Proof. If $n = 1$, then

$$\sum_{i=0}^{1} i(i+1) = 2 = \frac{1(2)(3)}{3}.$$

Thus, the identity holds for $n = 1$. Now assuming that

$$\sum_{i=0}^{n} i(i+1) = \frac{n(n+1)(n+2)}{3}$$

for some $n \geq 1$, and compute

$$\begin{aligned}
\sum_{i=0}^{n+1} i(i+1) &= \left(\sum_{i=0}^{n} i(i+1)\right) + (n+1)((n+1)+1) \\
&= \frac{n(n+1)(n+2)}{3} + (n+1)(n+2) \\
&= \frac{n(n+1)(n+2) + 3(n+1)(n+2)}{3} \\
&= \frac{(n+1)(n+2)(n+3)}{3} \\
&= \frac{(n+1)((n+1)+1)((n+1)+2)}{3},
\end{aligned}$$

so the identity also holds for $n + 1$. Thus, by the principle of mathematical induction, the identity holds for all $n \geq 1$, where $n \in \mathbb{N}$. $\square$

1.1.5

Prove that

$$\sum_{i=1}^{n}(2i-1) = 1+3+5+...+(2n-1) = n^2.$$

Proof. If $n=1$, then

$$\sum_{i=1}^{1}(2i-1) = 1 = 1^2.$$

Thus, the identity holds for $n=1$. Now assume that

$$\sum_{i=1}^{n}(2i-1) = n^2$$

for some $n \geq 1$, and compute

$$\begin{aligned}\sum_{i=1}^{n+1}(2i-1) &= \left(\sum_{i=1}^{n}(2i-1)\right) + (2(n+1)-1)\\ &= n^2 + 2(n+1) - 1\\ &= n^2 + 2n + 1\\ &= (n+1)(n+1)\\ &= (n+1)^2,\end{aligned}$$

so the identity also holds for $n+1$. Thus, by the principle of mathematical induction, the identity holds for all $n \geq 1$, where $n \in \mathbb{N}$. $\square$