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3.1 Approximation

Motivation. The main motivation of studying Sobolev spaces in general is that functions in these spaces can be approximated by smooth functions. These approximations can be done locally, globally up until the boundary, and globally up to and including the boundary. As expected these require three different Theorems (taken from Evans) which increase in difficulty to prove.

Theorem (Local approximation by smooth functions)**.** Suppose $u \in W^{k,p}(U)$ for some $1 \leq p < \infty$, and set

$$u^\epsilon = \eta_\epsilon * u \text{ in } U_\epsilon.$$

Then

(i) $u^\epsilon \in C^\infty(U_\epsilon)\ \forall\, \epsilon > 0$, and

(ii) $u^\epsilon \to u$ in $W^{k,p}_{\text{loc}}(U)$, as $\epsilon \to 0$.

3.2 Sobolev Inequalities

Motivation. The main goal of Sobolev inequalities is to find embeddings of Sobolev spaces in other spaces that may be easier to work with, more desirable, or simply imply nice properties. These inequalities essentially provide estimates for functions in Sobolev spaces using other well know function spaces. Many of the following proofs are taken from Evans.

Theorem (Gagliardo-Nirenberg-Sobolev Inequality)**.** Suppose $1 \leq p < n$. There exists a constant $C > 0$, depending only on p and n, such that

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|Du\|_{L^p(\mathbb{R}^n)},$$

for all $u \in C^1_c(\mathbb{R}^n)$.

Theorem (Estimates for $W^{1,p}$, where $1 \leq p < n$)**.** Let $U \subset \mathbb{R}^n$ be a bounded open set, and suppose ∂U is C^1. Suppose $u \in W^{1,p}(U)$ for some $1 \leq p < n$. It then follows that $u \in L^{p^*}$, with the estimate that

$$\|u\|_{L^{p^*}(U)} \leq C\|u\|_{W^{1,p}(U)},$$

where $C > 0$ is constant and only depending on p, n, and U.

Theorem (Estimates for $W^{1,p}_0$, where $1 \leq p < n$)**.** Let $U \subset \mathbb{R}^n$ be a bounded open set. Suppose $u \in W^{1,p}_0(U)$ for some $1 \leq p < n$. Then we get the estimate

$$\|u\|_{L^q(U)} \leq C\|Du\|_{L^p(U)},$$

for all $1 \leq q \leq p^*$, where $C > 0$ is constant and only depending on p, q, n, and U.

Remark. If U is bounded as we supposed, then on $W^{1,p}_0$, the norm $\|Du\|_{L^p} \equiv \|u\|_{W^{1,p}(U)}$, meaning this is still an estimate for functions in Sobolev spaces.

Theorem (Morrey's Inequality)**.** Suppose $n < p \leq \infty$. Then there exists a constant $C > 0$, depending only on p and n, such that

$$\|u\|_{C^{0,\gamma}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(\mathbb{R}^n)},$$

for all $u \in W^{1,p}(U)$ with $u \in C^1(\mathbb{R}^n)$, where $\gamma = 1 - \frac{n}{p}$.

Theorem (Estimates for $W^{1,p}$, where $n < p \leq \infty$)**.** Let $U \subset \mathbb{R}^n$ be a bounded open set, and suppose ∂U is C^1. Suppose $u \in W^{1,p}(U)$ for some $n < p \leq \infty$. It then follows that a continuous version $u^* \in C^{0,\gamma}(\overline{U})$, where $\gamma = 1 - \frac{n}{p}$, with the estimate that

$$\|u^*\|_{C^{0,\gamma}(\overline{U})} \leq C\|u\|_{W^{1,p}(U)},$$

where $C > 0$ is constant and only depending on p, n, and U.

4 Notation

(i) A *multiindex* is a vector $\alpha = (\alpha_1, \dots, \alpha_n)$ where each component $\alpha_i \in \mathbb{N}_0$. A multiindex has an order defined by

$$|\alpha| = \alpha_1 + \cdots + \alpha_n.$$

(ii) Using our definition of a multiindex and letting $u(x)$ be some function, we define

$$D^\alpha u(x) = \frac{\partial^{|\alpha|} u(x)}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}} = \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n} u(x).$$

(iii) Let $U, V \subset \mathbb{R}^n$. Then define

$$V \subset\subset U$$

to be when $V \subset \overline{V} \subset U$ and $\overline{V}$ is compact. In plain english this means V is *compactly conatined* in U.

(iv) Let f and g be functions. Then define $*$ to be the *convolution operator* where

$$(f * g)(x) = \int_{-\infty}^{\infty} f(\tau) g(x - \tau) d\tau = \int_{-\infty}^{\infty} f(x - \tau) g(\tau) d\tau$$

is the *convolution* of the functions f and g which results in a third function that expresses how one of the functions modifies the other. Note that I am assuming f and g are both supported on an infinite interval, which may not always be the case.

(v) Let $u(x)$ be a function where $x \in \mathbb{R}^n$. Then the *gradient vector* of u is

$$Du(x_1, \dots, x_n) = (u_{x_1}, ..., u_{x_n}).$$