Contents

1 Prior Concepts to Know

1.1 Important inequalities

Theorem (Young's inequality)**.** Let $p > 1$, $q < \infty$, and $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

Proof. Suppose we $t = \frac{1}{q}$ and $(1-t) = \frac{1}{p}$, implying $t \in (0,1)$. Then since we know ln is a concave function, we have

$$\begin{aligned} \ln\left((1-t)a^p + tb^q\right) &\geq (t-1)\ln\left(a^p\right) + t\ln\left(b^q\right) \\ &= p(t-1)\ln(a) + qt\ln(b) \\ &= \ln(a) + \ln(b) \\ &= \ln(ab). \end{aligned}$$

□

Theorem (Hölder's inequality)**.** Assume $1 \leq p, q \leq \infty$, and $\frac{1}{p} + \frac{1}{p} = 1$. If $u \in L^p(\mathbb{R}^n)$ and $v \in L^q(\mathbb{R}^n)$, then

$$\int_{\mathbb{R}^n} |uv| dx \leq \|u\|_{L^p(\mathbb{R}^n)} \|v\|_{L^q(\mathbb{R}^n)}.$$

Proof. Letting $0 < t \in \mathbb{R}$, we can say

$$|uv| = |ut||v/t|,$$

and by Young's inequality

$$|uv| = |ut||v/t| \leq \frac{|ut|^p}{p} + \frac{|v/t|^q}{q} = \frac{|u|^p t^p}{p} + \frac{|u|^q}{qt^q}.$$

Now integrating with respect to $x \in \mathbb{R}^n$, results in

$$\int_{\mathbb{R}^n} |uv| dx \leq \int_{\mathbb{R}^n} \left(\frac{|u|^p t^p}{p} + \frac{|u|^q}{qt^q} \right) dx = \frac{t^p}{p} \int_{\mathbb{R}^n} |u|^p dx + \frac{1}{qt^q} \int_{\mathbb{R}^n} |v|^q dx = g(t). \tag{1}$$

For the sake of simplicity let

$$a = \int_{\mathbb{R}^n} |u|^p dx, \text{ and } b = \int_{\mathbb{R}^n} |v|^q dx.$$

Now we wish to find the smallest $t > 0$ that satisfies . To do this we can employ Calculus, which says the minimum t we are looking for, which we denote as t_0, satisfies $g'(t_0) = 0$. First, we need to compute $g'(t)$ as follows

$$g'(t) = \frac{d}{dt}\left(\frac{t^p}{p} a + \frac{1}{qt^q} b \right) = t^{p-1} a - \frac{b}{t^{q+1}}.$$

Then, let $g'(t) = 0$ and solve for t as follows

$$\begin{aligned}
&g'(t) = t^{p-1}a - \frac{b}{t^{q+1}} = 0\\
&\Rightarrow t^{p-1}a = \frac{b}{t^{q+1}}\\
&\Rightarrow t^{p-1}t^{q+1} = \frac{b}{a}\\
&\Rightarrow t^{p+q} = \frac{b}{a}\\
&\Rightarrow t = \left(\frac{b}{a}\right)^{\frac{1}{p+q}} = t_0.
\end{aligned}$$

Finally, compute $g(t_0)$ as follows

$$\begin{aligned}
g(t_0) &= \frac{\left(\left(\frac{b}{a}\right)^{\frac{1}{p+q}}\right)^p}{p}a + \frac{1}{q\left(\left(\frac{b}{a}\right)^{\frac{1}{p+q}}\right)^q}b\\
&= \frac{\left(\frac{b}{a}\right)^{\frac{p}{p+q}}a}{p} + \frac{b}{q\left(\frac{b}{a}\right)^{\frac{q}{p+q}}}\\
&= \frac{\left(\frac{b}{a}\right)^{\frac{p-1}{p}}a}{p} + \frac{b}{q\left(\frac{b}{a}\right)^{\frac{1}{p}}}\\
&= \frac{b^{\frac{1}{q}}a^{\frac{1}{p}}}{p} + \frac{b^{\frac{1}{q}}a^{\frac{1}{p}}}{q}\\
&= b^{\frac{1}{q}}a^{\frac{1}{p}}\left(\frac{1}{p} + \frac{1}{q}\right)\\
&= b^{\frac{1}{q}}a^{\frac{1}{p}}\\
&= \left(\int_{\mathbb{R}^n} |v|^q dx\right)^{\frac{1}{q}}\left(\int_{\mathbb{R}^n} |u|^p dx\right)^{\frac{1}{p}}\\
&= \|v\|_{L^q(\mathbb{R}^n)}\|u\|_{L^p(\mathbb{R}^n)}.
\end{aligned}$$

$\square$

2 Sobolev Spaces

2.1 Hölder spaces

Motivation. Hölder spaces are function spaces which contain well-behaved functions, and where the function space itself is a Banach space. Functions being well-behaved in this case just means they are differentiable and continuous to certain degrees. Hölder spaces become relevant when discussing Sobolev inequalities.

Definition (Hölder continuous functions)**.** Assume $U \subset \mathbb{R}^n$ is open and $0 < \gamma \leq 1$. *Lipschitz continuous functions* $u : U \to \mathbb{R}$ satisfy the estimate

$$|u(x) - u(y)| \leq C|x - y|$$

where $x, y \in U$ and C is some constant. Now with the same assumptions, consider the variant

$$|u(x) - u(y)| \leq C|x - y|^\gamma,$$

where u is now said to be *Hölder continuous with exponent* γ.

Definition (Supremum norm and γ^{th}-Hölder seminorm)**.** (i) If $u : U \to \mathbb{R}$ is bounded and continuous, the supremum norm is denoted

$$\|u\|_{C(\overline{U})} = \sup_{x \in U} |u(x)|.$$

(ii) The γ^{th}-Hölder seminorm of $u : U \to \mathbb{R}$ is

$$[u]_{C^{0,\gamma}(\overline{U})} = \sup_{\substack{x,y \in U \\ x \neq y}} \left\{ \frac{|u(x) - u(y)|}{|x - y|^\gamma} \right\},$$

and the γ^{th}-Hölder norm is

$$\|u\|_{C^{0,\gamma}(\overline{U})} = \|u\|_{C(\overline{U})} + [u]_{C^{0,\gamma}(\overline{U})}.$$

Definition (Hölder spaces)**.** The Hölder space $C^{k,\gamma}(\overline{U})$ consists of all functions $u \in C^k(\overline{U})$, where the norm

$$\|u\|_{C^{k,\gamma}(\overline{U})} = \sum_{|\alpha| \leq k} \|D^\alpha u\|_{C(\overline{U})} + \sum_{|\alpha| = k} [D^\alpha u]_{C^{0,\gamma}(\overline{U})} < \infty.$$

Theorem (Hölder spaces as function spaces)**.** The space of functions $C^{k,\gamma}(\overline{U})$ is a Banach space.

Proof. Problem 5.1. □

2.2 Approximation

Motivation. Sobolev spaces are nice function spaces as we find that elements in Sobolev spaces can be approximated by sequences of smooth functions. These approximations can be done locally, globally up to and excluding the boundary, and globally up to and including the boundary. As expected these require three different Theorems (taken from Evans) which increase in difficulty to prove.

Theorem (Local approximation by smooth functions)**.** Let $U \subset \mathbb{R}^n$ be a bounded open set. Suppose $u \in W^{k,p}(U)$ for some $1 \leq p < \infty$, and set

$$u^\epsilon = \eta_\epsilon * u \text{ in } U_\epsilon.$$

Then

(i) $u^\epsilon \in C^\infty(U_\epsilon)\ \forall \epsilon > 0$, and

(ii) $u^\epsilon \to u$ in $W^{k,p}_{\mathrm{loc}}(U)$, as $\epsilon \to 0$.

Proof. In progress... □

Theorem (Global approximation by smooth functions)**.** Let $U \subset \mathbb{R}^n$ be a bounded open set. Suppose $u \in W^{k,p}(U)$ for some $1 \leq p < \infty$. Then there exists functions $u_m \in C^\infty(U) \cap W^{k,p}(U)$ such that

$$u_m \to u \text{ in } W^{k,p}(U).$$

Proof. In progress... □

Since $U \subset \mathbb{R}^n$ is a bounded open set, this Theorem says nothing about the smoothness of the boundary of U, which is the goal of the next Theorem.

Theorem (Global approximation by functions smooth up to the boundary)**.** Let $U \subset \mathbb{R}^n$ be a bounded open set. Suppose ∂U is C^1 and suppose $u \in W^{k,p}(U)$ for some $1 \leq p < \infty$. Then there exists functions $u_m \in C^\infty(\overline{U})$ such that

$$u_m \to u \text{ in } W^{k,p}(U).$$

Proof. In progress... □

2.3 Extensions

Motivation. Naturally we should desire to extend functions in $W^{k,p}(U)$ to functions in $W^{k,p}(\mathbb{R}^n)$. Evans discusses and proves this extension which becomes useful for some of the Sobolev inequalities proofs.

Theorem. Suppose $1 \leq p \leq \infty$. Let $U \subset \mathbb{R}^n$ be a bounded open set, where ∂U is C^1. Let $V \subset \mathbb{R}^n$ be a bounded open set such that $U \subset\subset V$. Then there exists a bounded linear operator

$$E : W^{1,p}(U) \to W^{1,p}(\mathbb{R}^n)$$

such that $\forall u \in W^{1,p}(U)$:

(i) $Eu = u$ a.e. in U,

(ii) Eu has support within V, and

(iii) $\|Eu\|_{W^{1,p}(\mathbb{R}^n)} \leq C(p, U, V)\|u\|_{W^{1,p}(U)}$.

Proof. In progress... $\square$

2.4 Sobolev inequalities

Motivation. Sobolev inequalities are useful for finding embeddings of Sobolev spaces in other function spaces that may be easier to work with, more desirable, or simply imply nice properties. Many of the following theorems are taken from Evans.

Theorem (Gagliardo-Nirenberg-Sobolev inequality)**.** Suppose $1 \leq p < n$. There exists a constant $C(p,n) > 0$ such that

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C(p,n)\|Du\|_{L^p(\mathbb{R}^n)},$$

for all $u \in C_c^1(\mathbb{R}^n)$.

Proof. In progress... □

Theorem (Estimates for $W^{1,p}$, where $1 \leq p < n$)**.** Let $U \subset \mathbb{R}^n$ be a bounded open set, and suppose ∂U is C^1. Suppose $u \in W^{1,p}(U)$ for some $1 \leq p < n$. It then follows that $u \in L^{p^*}$, with the estimate that

$$\|u\|_{L^{p^*}(U)} \leq C(p,n,U)\|u\|_{W^{1,p}(U)},$$

where $C(p,n,U) > 0$ is a constant.

Proof. In progress... □

Theorem (Estimates for $W_0^{1,p}$, where $1 \leq p < n$)**.** Let $U \subset \mathbb{R}^n$ be a bounded open set. Suppose $u \in W_0^{1,p}(U)$ for some $1 \leq p < n$. Then we get the estimate

$$\|u\|_{L^q(U)} \leq C(p,q,n,U)\|Du\|_{L^p(U)},$$

for all $1 \leq q \leq p^*$, where $C(p,q,n,U) > 0$ is a constant.

Proof. In progress... □

Remark. If U is bounded as we supposed, then on $W_0^{1,p}$, the norm $\|Du\|_{L^p} \equiv \|u\|_{W^{1,p}(U)}$, meaning this is still an estimate for functions in Sobolev spaces.

Theorem (Morrey's inequality)**.** Suppose $n < p \leq \infty$. Then there exists a constant $C(p,n) > 0$ such that

$$\|u\|_{C^{0,\gamma}(\mathbb{R}^n)} \leq C(p,n)\|u\|_{W^{1,p}(\mathbb{R}^n)},$$

for all $u \in W^{1,p}(U)$ with $u \in C^1(\mathbb{R}^n)$, where $\gamma = 1 - \frac{n}{p}$.

Proof. In progress... □

Theorem (Estimates for $W^{1,p}$, where $n < p \leq \infty$)**.** Let $U \subset \mathbb{R}^n$ be a bounded open set, and suppose ∂U is C^1. Suppose $u \in W^{1,p}(U)$ for some $n < p \leq \infty$. It then follows that a continuous version $u^* \in C^{0,\gamma}(\overline{U})$, where $\gamma = 1 - \frac{n}{p}$, with the estimate that

$$\|u^*\|_{C^{0,\gamma}(\overline{U})} \leq C(p,n,U)\|u\|_{W^{1,p}(U)},$$

where $C(p,n,U) > 0$ is a constant.

Proof. In progress... □

3 Notation

(i) A *multiindex* is a vector $\alpha = (\alpha_1, \ldots, \alpha_n)$ where each component $\alpha_i \in \mathbb{N}_0$. A multiindex has an order defined by

$$|\alpha| = \alpha_1 + \cdots + \alpha_n.$$

(ii) Using our definition of a multiindex and letting $u(x)$ be some function, we define

$$D^\alpha u(x) = \frac{\partial^{|\alpha|} u(x)}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}} = \partial_{x_1}^{\alpha_1} \ldots \partial_{x_n}^{\alpha_n} u(x).$$

(iii) Let $U, V \subset \mathbb{R}^n$. Then define

$$V \subset\subset U$$

to be when $V \subset \overline{V} \subset U$ and $\overline{V}$ is compact. In plain english this means V is *compactly contained* in U.

(iv) Let f and g be functions. Then define $*$ to be the *convolution operator* where

$$(f * g)(x) = \int_{-\infty}^{\infty} f(\tau) g(x - \tau) d\tau = \int_{-\infty}^{\infty} f(x - \tau) g(\tau) d\tau$$

is the *convolution* of the functions f and g which results in a third function that expresses how one of the functions modifies the other. Note that I am assuming f and g are both supported on an infinite interval, which may not always be the case.

(v) Let $u(x)$ be a function where $x \in \mathbb{R}^n$. Then the *gradient vector* of u is

$$Du(x_1, \ldots, x_n) = (u_{x_1}, ..., u_{x_n}).$$