# Contents

## Preface

The following are notes for my summer research in Adaptive Finite Element Methods (AFEM). The books and resources I follow closely will be documented in the references section, and will be cited when I closely follow them. By no means should any of MY proofs be taken as actual proofs, but instead they should be taken as my best attempts. My proof attempts will be labeled, and proofs that I followed very closely will just be labeled as proofs. Also, proofs may not be provided for most propositions as these are just notes to keep track of important concepts that I need to know.

# 1 Prior Concepts to Know

**Motivation.** The following are just some concepts, ideas, and interesting topics (mainly in analysis) that help in understanding these notes. The main focus here is on background material required for studying Sobolev spaces and AFEM.

## 1.1 Banach spaces

**Motivation.** Banach spaces are very important in the study of Sobolev spaces as Sobelev spaces themselves are Banach spaces, which will be proven later on. Being able to prove a space is a Banach space is very helpful because Banach spaces are very nice mathematical structures that imply a lot of nice properties, as we will see.

**Definition 1.** *A* `linear space` *over a field $F$ (usually a subfield of the complex numbers) is a non-empty set $X$ equipped with a special element 0, three operations, and a set of axioms that the three operations must satisfy. These operations and axioms are laid out nicely for the real numbers by Professor Tao at https://www.math.ucla.edu/~tao/resource/general/121.1.00s/vector_axioms.html.*

**Definition 2.** *A* `normed linear space` *is a linear space $X$ over a subfield $F$ of the complex numbers equipped with a norm, where the norm is a real-valued function $\|\cdot\| : X \to \mathbb{R}$ satisfying the following four axioms:*

*(i)* $\|u+v\| \leq \|u\| + \|v\|$ *for all* $u, v \in X$*.*

*(ii)* $\|\lambda u\| = |\lambda|\|u\|$ *for all* $u \in X$ *and* $\lambda \in F$*.*

*(iii)* $\|u\| \geq 0$ *for all* $u \in V$*.*

*(iv)* $\|u\| = 0$ *if and only if* $u = 0$*.*

*A normed linear space $X$ is also a* `metric space` *under the metric $\rho$ defined by*

$$\rho(u, v) = \|u - v\|$$

*for all $u, v \in X$.*

**Definition 3.** *A* `complete metric space` *is a metric space $X$ where every Cauchy sequence of points in $X$ converge to a point also in $X$. i.e. a space that has no missing points.*

**Definition 4.** *A* `Banach space` *is a complete normed linear space.*

**Definition 5** (Continuous embedding)**.** *Let $X$ and $Y$ be two normed vector spaces, with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively, such that $X \subseteq Y$. If the identity function*

$$i : X \hookrightarrow Y : x \mapsto x$$

*is continuous, i.e. if there exists a constant $C > 0$ such that*

$$\|x\|_Y \leq C\|x\|_X$$

*for every $x \in X$, then $X$ is said to be* `continuously embedded` *in $Y$.*

**Definition 6** (Compact embedding)**.** *Let $X$ and $Y$ be as previously defined. We say that $X$ is* `compactly embedded` *in $Y$, written $X \subset\subset Y$, if*

*(i) $X$ is continuously embedded in $Y$, and*

*(ii) each bounded sequence in $X$ is precompact in $Y$.*

### 1.2 $L^p$ spaces

**Motivation.** $L^p$ spaces are spaces of functions whose $p^{th}$ powers are integrable. These spaces are very nice and incredibly important, however, they take a considerable amount of work to develop, so we will just go over their definitions here.

**Definition 1.** *Let $(X, S, \mu)$ be a measure space and fix $1 \leq p < \infty$. The space $L^p(X)$ consists of equivalence classes of measurable functions $f : X \to \mathbb{R}$ such that*

$$\int |f|^p d\mu < \infty.$$

*The $L^p$-norm of $f \in L^p(U)$ is defined by*

$$\|f\|_{L^p(X)} = \left( \int |f|^p d\mu \right)^{1/p}.$$

**Remark.** $L^\infty$ spaces require a notion of the essential supremum and essential boundedness, which we will need to define first.

**Definition 2.** *Let $f : X \to \mathbb{R}$ be a measurable function on a measure space $(X, S, \mu)$. The **essential supremum** of $f$ on $X$ is*

$$\operatorname*{ess\,sup}_X f = \inf \{a \in \mathbb{R} \,|\, \mu\{x \in X \mid f(x) > a\} > 0\}.$$

*We also say that $f$ is **essentially bounded** on $X$ if*

$$\operatorname*{ess\,sup}_X |f| < \infty.$$

**Remark.** The essential supremum gives a loose sense of weight, where instead of just being the largest point, it looks at the largest point with other points around it. This is sort of the intuition of why the definition uses the infimum.

**Definition 3.** *Let $(X, S, \mu)$ be a measure space. The space $L^\infty(X)$ consists of pointwise a.e.-equivalence classes of essentially bounded function $f : X \to \mathbb{R}$ with norm*

$$\|f\|_{L^\infty(X)} = \operatorname*{ess\,sup}_X |f|.$$

**Remark.** The spaces $L^p$ for $1 \leq p < \infty$, and $L^\infty$ are both Banach spaces.

## 1.3 Hilbert spaces

**Motivation.** Hilbert spaces are important in the study of Sobolev spaces as some Sobolev spaces are themselves Hilbert spaces. Hilbert spaces are also important as there are some very important representation theorems for boundary value problems that involve Hilbert spaces.

**Definition 1.** *Let $H$ be a real linear space. A mapping $B(\cdot,\cdot) : H \times H \to \mathbb{R}$ such that each of the maps $u \mapsto B(u, v)$ and $v \mapsto B(u, v)$ is a linear form on $H$ is called a* `bilinear form` *on $H$. A bilinear form is* `symmetric` *if $B(u, v) = B(v, u)$ for all $u, v \in H$. A bilinear form is* `continuous or bounded` *if there exists a constant $C < \infty$ such that*

$$|B(u, v)| \leq C\|u\|\|v\|$$

*for any $u, v \in H$. A bilinear form is* `coercive` *on $H$ if there exists a number $\nu > 0$ such that*

$$B(v, v) \geq \nu\|v\|^2$$

*for any $v \in H$.*

**Definition 2.** *Let $H$ be a real linear space. The symmetric bilinear form $(\cdot,\cdot) : H \times H \to \mathbb{R}$ is called the* `inner product`*, and satifies*

*(i)* $(u, v) = (v, u)$ *for all* $u, v \in H$,

*(ii)* $u \mapsto (u, v)$ *is linear for each* $v \in H$,

*(iii)* $(u, v) \geq 0$ *for all* $v \in H$,

*(iv)* $(u, u) = 0$ *if and only if* $u = 0$.

**Definition 3.** *A linear space $H$ together with and inner product defined on it is called an* `inner-product space`*.*

**Definition 4.** *The inner product is also related to the norm of $H$ as*

$$\|u\| = (u, u)^{1/2}, \tag{1}$$

*and by the Cauchy-Schwarz inequality, we have*

$$|(u, v)| \leq \|u\|\|v\| \tag{2}$$

*for all $u, v \in H$. Equation (2) verifies that equation (1) defines a norm on $H$.*

**Definition 5.** *A* `Hilbert space` *$H$ is a Banach space equipped with an inner product which defines the norm on $H$.*

**Definition 6.** *Let $H$ be a Hilbert space and $S \subset H$ be a linear subset that is closed in $H$. Then $S$ is called a* `subspace` *of $H$.*

**Theorem 1.** *If $S$ is a subspace of $H$, then $S$ is also a Hilbert space.*

**Theorem 2.** *Let $H$ be a Hilbert space, and suppose $B(\cdot,\cdot)$ is a symmetric bilinear form that is continuous on $H$ and coercive on a subspace $V$ of $H$. Then $V$, equipped with the same bilinear form, is a Hilbert space.*

## 1.4 Hölder spaces

**Motivation.** Hölder spaces are function spaces which contain well-behaved functions, and where the function space itself is a Banach space. Functions being well-behaved in this case just means they are differentiable and continuous to certain degrees. Hölder spaces become relevant when discussing Sobolev inequalities.

Let $U \subset \mathbb{R}^n$ be a bounded open set and $0 < \gamma \leq 1$.

**Definition 1** (Hölder continuous functions)**.** ***Lipschitz continuous functions*** *$u : U \to \mathbb{R}$ satisfy the estimate*

$$|u(x) - u(y)| \leq C|x - y|$$

*where $x, y \in U$ and $C$ is some constant. Now with the same assumptions, consider the variant*

$$|u(x) - u(y)| \leq C|x - y|^\gamma,$$

*where $u$ is now said to be* ***Hölder continuous with exponent*** *$\gamma$.*

**Definition 2** (Supremum norm and $\gamma^{th}$-Hölder seminorm)**.**

*(i) If $u : U \to \mathbb{R}$ is bounded and continuous, the supremum norm is denoted*

$$\|u\|_{C(\overline{U})} = \sup_{x \in U} |u(x)|.$$

*(ii) The $\gamma^{th}$-Hölder seminorm of $u : U \to \mathbb{R}$ is*

$$[u]_{C^{0,\gamma}(\overline{U})} = \sup_{\substack{x,y \in U \\ x \neq y}} \left\{ \frac{|u(x) - u(y)|}{|x - y|^\gamma} \right\},$$

*and the $\gamma^{th}$-Hölder norm is*

$$\|u\|_{C^{0,\gamma}(\overline{U})} = \|u\|_{C(\overline{U})} + [u]_{C^{0,\gamma}(\overline{U})}.$$

**Definition 3.** *The Hölder space $C^{k,\gamma}(\overline{U})$ consists of all functions $u \in C^k(\overline{U})$, where the norm*

$$\|u\|_{C^{k,\gamma}(\overline{U})} = \sum_{|\alpha| \leq k} \|D^\alpha u\|_{C(\overline{U})} + \sum_{|\alpha| = k} [D^\alpha u]_{C^{0,\gamma}(\overline{U})} < \infty.$$

**Theorem 1** (Hölder spaces as function spaces)**.** *The space of functions $C^{k,\gamma}(\overline{U})$ is a Banach space.*

*My proof attempt.* `https://github.com/Aidenwjt/some-math-notes/blob/master/exercises/evans/exercises.pdf`. □

### 1.5 Mollifiers

**Motivation.** Mollifiers are smooth functions which allow us to "smooth over" other functions via convolution. Since mollifiers are smooth functions, we find that they become very important in the study of Sobolev spaces, especially for approximations, Sobolev inequalities, and the Sobolev embedding theorem.

Let $U \subset \mathbb{R}^n$ be a bounded open set.

**Definition 1** (Standard mollifier)**.**

*(i) Define* $\eta \in C^\infty(\mathbb{R}^n)$ *by*

$$\eta(x) = \begin{cases} Ce^{\frac{1}{|x|^2-1}}, & \text{if } |x| < 1 \\ 0, & \text{if } |x| \geq 1, \end{cases}$$

*where* $C$ *is chosen to satisfy*

$$\int_{\mathbb{R}^n} \eta(x)dx = 1.$$

$\eta$ *is then called the* ***standard mollifier****.*

*(ii) Now for each* $\epsilon > 0$*, define*

$$\eta_\epsilon(x) = \frac{1}{\epsilon^n}\eta\left(\frac{x}{\epsilon}\right).$$

*The functions* $\eta_\epsilon \in C^\infty(\mathbb{R}^n)$ *also satisfy*

$$\int_{\mathbb{R}^n} \eta_\epsilon(x)dx = 1,$$

*where support*$(\eta_\epsilon) \in B(0,\epsilon)$*.*

**Definition 2.** *If* $f : U \to \mathbb{R}$ *is locally integrable, then define its* ***mollification*** *as the convolution of* $f$ *and* $\eta_\epsilon$ *in* $U_\epsilon = \{x \in U \,|\, dist(x, \partial U) > \epsilon\}$*, written as*

$$f^\epsilon = f * \eta_\epsilon \text{ in } U_\epsilon.$$

*In terms of the convolution, we can write this as*

$$f^\epsilon(x) = \int_U \eta_\epsilon(x-y)f(y)dy = \int_{B(0,\epsilon)} \eta_\epsilon(y)f(x-y)dy$$

*for* $x \in U_\epsilon$*.*

**Theorem 1** (Properties of mollifiers)**.**

*(i)* $f^\epsilon \in C^\infty(U_\epsilon)$*.*

*(ii)* $f^\epsilon \to f$ *a.e., as* $\epsilon \to 0$*.*

*(iii) If* $f \in C(U)$*, then* $f^\epsilon \to f$ *uniformly on compact subsets of* $U$*.*

*(iv) If* $1 \leq p < \infty$ *and* $f \in L^p_{loc}(U)$*, then* $f^\epsilon \to f$ *in* $L^p_{loc}(U)$*.*

## 1.6 Weak derivatives

**Motivation.** The motivation for weak derivatives in our case comes from the desire to satisfy certain integral relations given by the integration by parts formula. Let $U \subset \mathbb{R}^n$ be a bounded open set and let $u \in C^1(U)$. Then if $\phi \in C_c^\infty(U)$, we can see that

$$\int_U u\phi_{x_i} dx = [u\phi]_U - \int_U \phi u_{x_i} dx = -\int_U \phi u_{x_i} dx$$

when we use the integration by parts formula. Now letting $k \in \mathbb{N}$, $u \in C^k(U)$, and $\alpha$ be a multi-index, we find that

$$\begin{aligned}
\int_U u D^\alpha \phi dx &= \int_U u \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} \phi dx \\
&= -\int_U \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} u \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_{n-1}}}{\partial x_{n-1}^{\alpha_{n-1}}} \phi dx \\
&= \qquad \vdots \\
&= (-1)^{|\alpha|} \int_U \phi \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} u dx \\
&= (-1)^{|\alpha|} \int_U \phi D^\alpha u dx.
\end{aligned}$$

This gives us the relation

$$\int_U u D^\alpha \phi dx = (-1)^{|\alpha|} \int_U D^\alpha u \phi dx \tag{3}$$

where the LHS of equation (3) also holds if $u \in L^1_{\text{loc}}(U)$. The question now is "*does $u$ need to be k-times partially differentiable, or can we replace $D^\alpha u$ with something that also satisfies equation (3), but is just locally summable instead?*"

**Definition 1.** *Let $u, v \in L^1_{loc}(U)$ and let $\alpha$ be a multi-index. Then we say that $v$ is the $\alpha^{th}$* `weak partial derivative of` *$u$ provided*

$$\int_U u D^\alpha \phi dx = (-1)^{|\alpha|} \int_U v \phi dx.$$

*In this case, we also say $D^\alpha u = v$ "in the weak sense."*

**Definition 2.** *The space of k-times weakly differentiable functions described above is denoted as $W^k(U)$, where clearly $C^k(U) \subset W^k(U)$.*

## 1.7 Fourier transform

In progress...

## 1.8 Important inequalities

**Motivation.** A lot of analysis proofs in general can be boiled down to figuring out if some element is smaller than, bigger than, or equal to some other element. For this reason, it is very helpful to know some of these important inequalities.

**Theorem 1** (Young's inequality)**.** *Let $p > 1$, $q < \infty$, and $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

*My proof attempt.* Suppose we $t = \frac{1}{q}$ and $(1-t) = \frac{1}{p}$, implying $t \in (0,1)$. Then since we know ln is a concave function, we have

$$\begin{aligned}\ln\left((1-t)a^p + tb^q\right) &\geq (t-1)\ln\left(a^p\right) + t\ln\left(b^q\right)\\ &= p(t-1)\ln(a) + qt\ln(b)\\ &= \ln(a) + \ln(b)\\ &= \ln(ab).\end{aligned}$$

□

**Theorem 2** (Hölder's inequality)**.** *Assume $1 \leq p,\ q \leq \infty$, and $\frac{1}{p} + \frac{1}{p} = 1$. If $u \in L^p(\mathbb{R}^n)$ and $v \in L^q(\mathbb{R}^n)$, then*

$$\int_{\mathbb{R}^n} |uv| dx \leq \|u\|_{L^p(\mathbb{R}^n)} \|v\|_{L^q(\mathbb{R}^n)}.$$

*My proof attempt.* Letting $0 < t \in \mathbb{R}$, we can say

$$|uv| = |ut||v/t|,$$

and by Young's inequality

$$|uv| = |ut||v/t| \leq \frac{|ut|^p}{p} + \frac{|v/t|^q}{q} = \frac{|u|^p t^p}{p} + \frac{|u|^q}{qt^q}.$$

Now integrating with respect to $x \in \mathbb{R}^n$, results in

$$\int_{\mathbb{R}^n} |uv| dx \leq \int_{\mathbb{R}^n} \left(\frac{|u|^p t^p}{p} + \frac{|u|^q}{qt^q}\right) dx = \frac{t^p}{p}\int_{\mathbb{R}^n} |u|^p dx + \frac{1}{qt^q}\int_{\mathbb{R}^n} |v|^q dx = g(t). \tag{4}$$

For the sake of simplicity let

$$a = \int_{\mathbb{R}^n} |u|^p dx, \text{ and } b = \int_{\mathbb{R}^n} |v|^q dx.$$

Now we wish to find the smallest $t > 0$ that satisfies equation (4). To do this we can employ Calculus, which says the minimum $t$ we are looking for, which we denote as $t_0$, satisfies $g'(t_0) = 0$. First, we need to compute $g'(t)$ as follows

$$g'(t) = \frac{d}{dt}\left(\frac{t^p}{p}a + \frac{1}{qt^q}b\right) = t^{p-1}a - \frac{b}{t^{q+1}}.$$

Then, let $g'(t) = 0$ and solve for $t$ as follows

$$\begin{aligned}
&g'(t) = t^{p-1}a - \frac{b}{t^{q+1}} = 0 \\
&\Rightarrow t^{p-1}a = \frac{b}{t^{q+1}} \\
&\Rightarrow t^{p-1}t^{q+1} = \frac{b}{a} \\
&\Rightarrow t^{p+q} = \frac{b}{a} \\
&\Rightarrow t = \left(\frac{b}{a}\right)^{\frac{1}{p+q}} = t_0.
\end{aligned}$$

Finally, compute $g(t_0)$ as follows

$$\begin{aligned}
g(t_0) &= \frac{\left(\left(\frac{b}{a}\right)^{\frac{1}{p+q}}\right)^p}{p}a + \frac{1}{q\left(\left(\frac{b}{a}\right)^{\frac{1}{p+q}}\right)^q}b \\
&= \frac{\left(\frac{b}{a}\right)^{\frac{p}{p+q}} a}{p} + \frac{b}{q\left(\frac{b}{a}\right)^{\frac{q}{p+q}}} \\
&= \frac{\left(\frac{b}{a}\right)^{\frac{p-1}{p}} a}{p} + \frac{b}{q\left(\frac{b}{a}\right)^{\frac{1}{p}}} \\
&= \frac{b^{\frac{1}{q}}a^{\frac{1}{p}}}{p} + \frac{b^{\frac{1}{q}}a^{\frac{1}{p}}}{q} \\
&= b^{\frac{1}{q}}a^{\frac{1}{p}}\left(\frac{1}{p} + \frac{1}{q}\right) \\
&= b^{\frac{1}{q}}a^{\frac{1}{p}} \\
&= \left(\int_{\mathbb{R}^n} |v|^q dx\right)^{\frac{1}{q}} \left(\int_{\mathbb{R}^n} |u|^p dx\right)^{\frac{1}{p}} \\
&= \|v\|_{L^q(\mathbb{R}^n)}\|u\|_{L^p(\mathbb{R}^n)}.
\end{aligned}$$

□

**Theorem 3** (Cauchy-Schwarz inequality)**.** *Let $u$ and $v$ be arbitrary vectors in an inner product space over the scalar field $\mathbb{F}$, where $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$. Then*

$$|(u, v)| \leq \|u\|\|v\|$$

*with equality holding ("=") if and only if $u$ and $v$ are linearly dependent. Moreover, if $|(u, v)| = \|u\|\|v\|$ and $v \neq 0$, then*

$$u = \frac{(u, v)}{\|v\|^2} v.$$

*Proof.* When $u = 0$ or $v = 0$ then the inequality is trivial, so assume WLOG that $v \neq 0$. Let

$$z = u - \frac{(u, v)}{(v, v)} v,$$

then it follows from the linearity of the inner product that

$$(z, v) = \left( u - \frac{(u, v)}{(v, v)} v \right) = (u, v) - \left( \frac{(u, v)}{(v, v)} v, v \right) = (u, v) - \frac{(u, v)}{(v, v)} (v, v) = 0.$$

This implies $z$ is orthogonal to $v$. Now applying Pythagorean's Theorem to

$$u = \frac{(u, v)}{(v, v)} v + z$$

results in

$$\begin{aligned}
\|u\|^2 &= \left\| \frac{(u, v)}{(v, v)} v \right\|^2 + \|z\|^2 \\
&= \left| \frac{(u, v)}{(v, v)} \right|^2 \|v\|^2 + \|z\|^2 \\
&= \frac{|(u, v)|^2}{|(v, v)^{1/2}(v, v)^{1/2}|^2} \|v\|^2 + \|z\|^2 \\
&= \frac{|(u, v)|^2}{|\|v\|^2|^2} \|v\|^2 + \|z\|^2 \\
&= \frac{|(u, v)|^2}{\|v\|^2} + \|z\|^2 \\
&\geq \frac{|(u, v)|^2}{\|v\|^2}.
\end{aligned}$$

Multiplying through by $\|v\|^2$ results in our desired inequality. We can also observe if $z = 0$, we get

$$u = \frac{(u, v)}{(v, v)} v = \frac{(u, v)}{(v, v)^{1/2}(v, v)^{1/2}} v = \frac{(u, v)}{\|v\|^2} v,$$

as desired. □

**Theorem 4** (Minkowski's inequality)**.** *If $u, v \in L^p(U)$ with $1 \leq p \leq \infty$, then we have*

$$\|u + v\|_{L^p(U)} \leq \|u\|_{L^p(U)} + \|v\|_{L^p(U)}.$$

*My proof attempt.* In progress... □

# 2 Sobolev Spaces

**Motivation.** Sobolev spaces are motivated by the study of elliptic PDEs, for example, the Poisson equation

$$\Delta u = f.$$

Multiplying this equation on both sides by $\phi \in C_c^\infty$, then integrating both sides, then integrating the left side by parts, results in

$$\begin{aligned} \int \phi \Delta u &= \int \phi f \\ \Rightarrow \int \phi \nabla^2 u &= \int \phi f \\ \Rightarrow \phi \nabla u - \int \phi \nabla u &= \int \phi f \\ \Rightarrow - \int \phi \nabla u &= \int \phi f. \end{aligned}$$

It can then be shown that for any $u \in C^2$, if the above integral relation holds, then $u$ is a classical solution of the original equation. However, one may notice that the integral relation only needs $\nabla u \in L^p$, for some fixed $p$, to be satisfied. Also, one may notice the similarity of the integral relation to the definition of weak derivatives. Both these observations lead to a more generalized case where $u \in W^k$ and $u \notin C^1$, for some fixed $p$. A space of functions like this is very useful as the functions in these spaces don't necessarily need to be continuous or totally differentiable, yet they still satisfy the above integral relation.

## 2.1 Definition of the $W^{k,p}$ spaces

**Motivation.** We now desire to explicitly state the generalized formal definition for the space of functions described above, as well as layout some useful properties that come along with this space. Most of this section is taken from Evans [1], and Gilbarg and Trudinger [2].

Fix $1 \leq p \leq \infty$, let $k \in \mathbb{N}$, and let $U \subset \mathbb{R}^n$ be a bounded open set.

**Definition 1.** *The Sobolev space*

$$W^{k,p}(U) = \{u \in W^k(U) \mid D^\alpha u \in L^p(U) \ \forall |\alpha| \leq k\},$$

*consists of all locally summable functions $u : U \to \mathbb{R}$ such that for each $|\alpha| \leq k$, $D^\alpha u$ exists in the weak sense and belongs to $L^p(U)$.*

**Remark.** When $p = 2$ we will instead use the notation

$$W^{k,2}(U) = H^k(U),$$

because these Sobolev spaces are also Hilbert spaces.

**Definition 2.** *If $u \in W^{k,p}(U)$, we define its norm to be*

$$\|u\|_{W^{k,p}(U)} = \sum_{|\alpha| \leq k} \|D^\alpha u\|_{L^p(U)}.$$

**Definition 3.**

*(i)* *Let* $\{u_m\}_{m=1}^{\infty}$ *be a sequence of functions and let* $u \in W^{k,p}(U)$*. Then* $u_m$ *converges to* $u$ *in* $W^{k,p}(U)$*, written*

$$u_m \to u \text{ in } W^{k,p}(U),$$

*provided that*

$$\lim_{m\to\infty} \|u_m - u\|_{W^{k,p}(U)} = 0.$$

*(ii)* *We write*

$$u_m \to u \text{ in } W^{k,p}_{loc}(U)$$

*to mean*

$$u_m \to u \text{ in } W^{k,p}(V)$$

*for each* $V \subset\subset U$.

**Definition 4.** *Denote by*

$$W_0^{k,p}(U)$$

*the closure of* $C_c^{\infty}(U)$ *in* $W^{k,p}(U)$*. Thus* $u \in W_0^{k,p}(U)$ *if and only if there exists functions* $u_m \in C_c^{\infty}(U)$ *such that* $u_m \to u$ *in* $W^{k,p}(U)$.

**Theorem 1** (Properties of weak derivatives)**.** *Suppose* $u \in W^{k,p}(U)$.

*(i)* *If* $D^{\alpha}u \in W^{k-|\alpha|,p}(U)$*, then*

$$D^{\beta}u(D^{\alpha}u) = D^{\alpha}u(D^{\beta}u) = D^{\alpha+\beta}u$$

*for all multi-indices* $\alpha, \beta$ *such that* $|\alpha| + |\beta| \leq k$.

*(ii)* $D^{\alpha}$ *is a linear operator on* $W^{k,p}(U)$ *for all multi-indices* $\alpha$*, such that* $|\alpha| \leq k$.

*(iii)* *If* $V \subset U$ *is open, then* $u \in W^{k,p}(V)$.

*(iv)* *If* $\zeta \in C_c^{\infty}(U)$*, then* $\zeta u \in W^{k,p}(U)$.

**Theorem 2** (Sobolev spaces as function spaces)**.** *For each* $k \in \mathbb{N}$ *and* $1 \leq p \leq \infty$*, the Sobolev space* $W^{k,p}(U)$ *is a Banach space.*

*Proof.* In progress... □

## 2.2 Approximations and Density

**Motivation.** Sobolev spaces are nice function spaces as we find that elements in Sobolev spaces can be approximated by sequences of smooth functions. These approximations can be done locally, globally up to but excluding the boundary, and globally up to and including the boundary. As expected, these require three different theorems which increase in difficulty to prove, where these theorems are taken from Evans [1]. Also, Gilbarg and Trudinger [2] phrase the global approximation result as a density result in a nice concise theorem that we will mention as well.

Let $U \subset \mathbb{R}^n$ be a bounded open set and suppose $u \in W^{k,p}(U)$ for some $1 \leq p < \infty$.

**Theorem 1** (Local approximation by smooth functions)**.** *Set*

$$u^\epsilon = \eta_\epsilon * u \text{ in } U_\epsilon.$$

*Then*

*(i)* $u^\epsilon \in C^\infty(U_\epsilon)$ $\forall\, \epsilon > 0$, *and*

*(ii)* $u^\epsilon \to u$ *in* $W^{k,p}_{loc}(U)$, *as* $\epsilon \to 0$.

**Theorem 2** (Global approximation by smooth functions)**.** *There exists functions* $u_m \in C^\infty(U) \cap W^{k,p}(U)$ *such that*

$$u_m \to u \text{ in } W^{k,p}(U).$$

**Remark.** Since $U \subset \mathbb{R}^n$ is a bounded open set, this Theorem says nothing about the smoothness of the boundary of $U$, which is the goal of the next Theorem.

**Theorem 3** (Global approximation by functions smooth up to the boundary)**.** *Suppose* $\partial U$ *is* $C^1$. *Then there exists functions* $u_m \in C^\infty(\overline{U})$ *such that*

$$u_m \to u \text{ in } W^{k,p}(U).$$

**Theorem 4.** *The subspace* $C^\infty(U) \cap W^{k,p}(U)$ *is dense in* $W^{k,p}(U)$.

## 2.3 Extensions

**Motivation.** Naturally, we should desire to extend functions in $W^{k,p}(U)$ to functions in $W^{k,p}(\mathbb{R}^n)$. Evans [1] discusses and proves this extension which becomes useful for proving Sobolev inequalities.

**Theorem 1.** *Suppose $1 \leq p \leq \infty$. Let $U \subset \mathbb{R}^n$ be a bounded open set, where $\partial U$ is $C^1$. Let $V \subset \mathbb{R}^n$ be a bounded open set such that $U \subset\subset V$. Then there exists a bounded linear operator*

$$E : W^{1,p}(U) \to W^{1,p}(\mathbb{R}^n)$$

*such that $\forall u \in W^{1,p}(U)$:*

*(i) $Eu = u$ a.e. in $U$,*

*(ii) $Eu$ has support within $V$, and*

*(iii) $\|Eu\|_{W^{1,p}(\mathbb{R}^n)} \leq C(p, U, V)\|u\|_{W^{1,p}(U)}$.*

### 2.4 Traces

**Motivation.** In order to deal with Dirichlet boundary conditions in boundary value problems, it becomes necessary to assign meaning to boundary values of functions in Sobolev spaces. This is because Sobolev spaces contain unbounded functions, meaning we can no longer interpret the boundary conditions in a pointwise sense for $n \geq 2$. Instead, we should interpret the boundary of a $n$-dimensional domain as an $n-1$-dimensional object, i.e. a manifold. We now seek an interpretation of restrictions of Sobolev-class functions to manifolds of the dimension $n-1$. The following is a combination of definitions and propositions taken from Bartels [3] and from Brenner and Scott [9].

Firstly, we need to know what a Lipshitz domain is as this type of domain is necessary for the following definitions and propositions. Also, we will instead use $d$ to denote the dimension and $\Omega$ to denote the domain.

**Definition 1.** *A set $\Omega \subset \mathbb{R}^d$ is called* ***`Lipschitz domain`****, if it is open and connected, and if for each $x \in \partial\Omega$ there exists a transformation*

$$\Phi(y) = My + r$$

*with an orthogonal matrix $M \in \mathbb{R}^{d\times d}$ and a vector $r \in \mathbb{R}^d$, a parameter $\delta > 0$, an open set $Q' \subset \mathbb{R}^{d-1}$, and a Lipschitz continuous function $h : Q' \to \mathbb{R}$ such that*

*(i)* $\Omega \cap B_\delta(x) = \Phi(\{(y', y_d) \in Q' \times \mathbb{R} \,|\, h(y') < y_d\}) \cap B_\delta(x)$,

*(ii)* $\partial\Omega \cap B_\delta(x) = \Phi(\{(y', y_d) \in Q' \times \mathbb{R} \,|\, h(y') = y_d\}) \cap B_\delta(x)$,

*(iii)* $\overline{\Omega}^c \cap B_\delta(x) = \Phi(\{(y', y_d) \in Q' \times \mathbb{R} \,|\, h(y') > y_d\}) \cap B_\delta(x)$.

We now find that smooth bounded functions are dense in Sobolev spaces on Lipschitz domains if $p < \infty$, which gives us an "*improved*" density result from the Approximations and Density section.

**Theorem 1** (Density of smooth functions)**.** *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Then the set $C^\infty(\overline{\Omega})$ is dense in $W^{k,p}(\Omega)$ for $k \in \mathbb{N}_0$ and $1 \leq p < \infty$.*

Now since Lebesgue functions can be modified on sets of measure zero, it is generally not meaningful to specify boundary values for these functions. However, because of Morrey (`https://math.stackexchange.com/a/2361510/1122760`) this is possible for Sobolev functions, which brings us to the next proposition.

**Theorem 2** (Trace operator)**.** *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain and fix $1 \leq p \leq \infty$. There exists a uniquely defined bounded linear operator*

$$T : W^{1,p}(\Omega) \to L^p(\partial\Omega)$$

*such that for all $u \in C^\infty(\overline{\Omega}) \cap W^{1,p}(\Omega)$ we have $T(u) = u|_{\partial\Omega}$.*

Intuitively, the trace operator literally "*traces*" the boundary of a function $u \in W^{1,p}(\Omega)$. This has obvious importance for boundary value problems where we can't work with the boundary in a pointwise sense. As alluded to before, we will now explicitly define the subset of $W^{1,p}(\Omega)$ that contains these elements.

**Definition 2.** *Let* $\Omega \subset \mathbb{R}^d$ *be a Lipschitz domain and fix* $1 \leq p \leq \infty$. *We will use the notation* $W_0^{1,p}(\Omega)$ *to denote the subset of* $W^{1,p}(\Omega)$ *consisting of functions whose trace on* $\partial\Omega$ *is zero, that is*

$$W_0^{1,p}(\Omega) = \{v \in W^{1,p}(\Omega) \,|\, v|_{\partial\Omega} = 0 \text{ in } L^2(\partial\Omega)\}.$$

*More generally, we let* $W_0^{k,p}(\Omega)$ *denote the subset of* $W^{k,p}(\Omega)$ *consisting of functions whose derivatives of order* $k-1$ *are in* $W^{1,p}(\Omega)$, *that is*

$$W_0^{k,p}(\Omega) = \{v \in W^{k,p}(\Omega) \,|\, D^\alpha v|_{\partial\Omega} = 0 \text{ in } L^2(\partial\Omega)\ \forall\, |\alpha| < k\}.$$

## 2.5 Sobolev inequalities, estimates, and embeddings

**Motivation.** Sobolev inequalities are useful for finding embeddings of Sobolev spaces in other function spaces which may be easier to work with, be more desirable, or simply imply nice properties. Many of the following theorems are taken from Evans [1], and Gilbarg and Trundinger [2].

Let $U \subset \mathbb{R}^n$ be a bounded open set.

**Definition 1.** *If $1 \le p < n$, then the* `Sobolev conjugate` *of $p$ is*

$$p^* = \frac{np}{n-p}.$$

**Theorem 1** (Gagliardo-Nirenberg-Sobolev inequality)**.** *Suppose $1 \le p < n$. There exists a constant $C(p,n)$ such that*

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \le C(p,n)\|Du\|_{L^p(\mathbb{R}^n)},$$

*for all $u \in C_c^1(\mathbb{R}^n)$.*

*Proof.* In progress... □

**Theorem 2** (Estimates for $W^{1,p}$, where $1 \le p < n$)**.** *Suppose $\partial U$ is $C^1$, and suppose $u \in W^{1,p}(U)$ for some $1 \le p < n$. It then follows that $u \in L^{p^*}$, with the estimate that*

$$\|u\|_{L^{p^*}(U)} \le C(p,n,U)\|u\|_{W^{1,p}(U)}.$$

*Proof.* In progress... □

**Theorem 3** (Estimates for $W_0^{1,p}$, where $1 \le p < n$)**.** *Suppose $u \in W_0^{1,p}(U)$ for some $1 \le p < n$. Then we have the estimate*

$$\|u\|_{L^q(U)} \le C(p,q,n,U)\|Du\|_{L^p(U)},$$

*for all $1 \le q \le p^*$.*

*Proof.* In progress... □

**Remark.** If $U$ is bounded as we supposed, then on $W_0^{1,p}$, the norm $\|Du\|_{L^p(U)} \equiv \|u\|_{W^{1,p}(U)}$, meaning this is still an estimate for functions in Sobolev spaces.

**Theorem 4** (Morrey's inequality)**.** *Suppose $n < p \le \infty$. Then there exists a constant $C(p,n)$ such that*

$$\|u\|_{C^{0,\gamma}(\mathbb{R}^n)} \le C(p,n)\|u\|_{W^{1,p}(\mathbb{R}^n)},$$

*for all $u \in W^{1,p}(U)$ with $u \in C^1(\mathbb{R}^n)$, where $\gamma = 1 - \frac{n}{p}$.*

*Proof.* In progress... □

**Theorem 5** (Estimates for $W^{1,p}$, where $n < p \le \infty$)**.** *Suppose $\partial U$ is $C^1$, and suppose $u \in W^{1,p}(U)$ for some $n < p \le \infty$. It then follows that a continuous version $u^* \in C^{0,\gamma}(\overline{U})$, where $\gamma = 1 - \frac{n}{p}$, with the estimate that*

$$\|u^*\|_{C^{0,\gamma}(\overline{U})} \le C(p,n,U)\|u\|_{W^{1,p}(U)}.$$

*Proof.* In progress... $\square$

**Theorem 6** (General Sobolev inequalities). *Suppose $\partial U$ is $C^1$, and suppose $u \in W^{k,p}(U)$.*

*(i) If $k < \frac{n}{p}$, then $u \in L^q(U)$, where*

$$\frac{1}{q} = \frac{1}{p} - \frac{k}{n},$$

*with the estimate*

$$\|u\|_{L^q(U)} \leq C(k,p,n,U)\|u\|_{W^{k,p}(U)}.$$

*(ii) If $k > \frac{n}{p}$, then $u \in C^{k-\left\lfloor \frac{n}{p} \right\rfloor -1,\gamma}(\overline{U})$, where*

$$\gamma = \begin{cases} \left\lfloor \frac{n}{p} \right\rfloor + 1 - \frac{n}{p}, & \textit{if } \frac{n}{p} \notin \mathbb{Z} \\ \textit{any } x \in (0,1), & \textit{if } \frac{n}{p} \in \mathbb{Z}, \end{cases}$$

*with the estimate*

$$\|u\|_{C^{k-\left\lfloor \frac{n}{p} \right\rfloor -1,\gamma}(\overline{U})} \leq C(k,p,n,\gamma,U)\|u\|_{W^{k,p}(U)}.$$

*Proof.* In progress... $\square$

**Remark.** With our notion of continuous embeddings between Banach spaces, we can see that the General Sobolev inequalities imply that

$$W^{k,p}(U) \hookrightarrow L^{\frac{np}{n-kp}}(U) \text{ for } k < \frac{n}{p},$$

and

$$W^{k,p}(U) \hookrightarrow C^{k-\left\lfloor \frac{n}{p} \right\rfloor -1,\gamma}(\overline{U}) \text{ for } k > \frac{n}{p}.$$

## 2.6 Compactness results

**Motivation.** Some of the previous Sobolev embeddings are actually compact embeddings as well. These results, layed out by Rellich and Kondrachov, can be found in Evans [1], and Gilbarg and Trudinger [2]. I will use Gilbarg and Trudinger's version here as Evans doesn't mention the second result.

**Theorem 1** (Rellich-Kondrachov compactness theorem)**.** *Let $U \subset \mathbb{R}^n$ be a bounded open set. The spaces $W_0^{1,p}(U)$ are compactly embedded in*

*(i)* $L^q(U)$ *for any* $1 \leq q < p^*$, *if* $p < n$, *and*

*(ii)* $C^0(\overline{U})$, *if* $p > n$.

## 2.7 Difference quotients and $W^{1,p}$

**Motivation.** "In PDE, the weak or classical differentiability of functions may often be deduced through a consideration of their difference quotients." [2] However, for functions in Sobolev spaces, we will need to consider difference quotient approximations to weak derivatives as these functions may not be continuous or totally differentiable. Since Evans [1], and Gilbarg and Trudinger's [2] sections on difference quotients are so similar, I will use a combination of both their definitions and propositions here.

Let $U \subset \mathbb{R}^n$ be a bounded open set.

**Definition 1.** *Suppose $u : U \to \mathbb{R}$ is a locally integrable function, where $V \subset\subset U$. Also, denote $e_i$ as the $i^{th}$ canonical vector in the $x_i$ direction. The $i^{th}$* ***`difference quotient`*** *of size $h$ is*

$$D^h u(x) = D_i^h u(x) = \frac{u(x + he_i) - u(x)}{h} \qquad (i = 1, \ldots, n)$$

*for $x \in V$ and $h \in \mathbb{R}$, such that $0 < |h| < dist(V, \partial U)$.*

**Theorem 1.** *Let $u \in W^{1,p}(U)$ for some fixed $1 \leq p < \infty$. Then $D^h u \in L^p(V)$ for each $V \subset\subset U$, where we have*

$$\|D^h u\|_{L^p(V)} \leq \|Du\|_{L^p(U)}$$

*for all $0 < |h| < dist(V, \partial U)$.*

*Proof.* In progress... □

**Theorem 2.** *Let $u \in L^p(U)$ for some fixed $1 < p < \infty$, and suppose that there exists a constant $C$ such that $D^h u \in L^p(V)$ and*

$$\|D^h u\|_{L^p(V)} \leq C$$

*for all $0 < |h| < dist(V, \partial U)$ and $V \subset\subset U$. Then $u \in W^{1,p}(U)$, with*

$$\|Du\|_{L^p(U)} \leq C.$$

*Proof.* In progress... □

## 2.8 Fourier transform methods involving $H^k$

**Motivation.** Since Sobolev spaces are also Hilbert spaces when $p = 2$ it may be useful to have an alternate characterization of the $H^k$ spaces. Specifically, the $H^k$ spaces can be characterized by the Fourier transform, which is a very useful transformation method. This section follows Evans [1] very closely.

Note that all functions here are complex-valued like in the Fourier transform section.

**Theorem 1** (Characterization of $H^k$ by Fourier transform)**.** *Let $k \in \mathbb{N}_0$.*

*(i) A function $u \in L^2(\mathbb{R}^n)$ belongs to $H^k(\mathbb{R}^n)$ if and only if*

$$(1 + |y|^k)\hat{u} \in L^2(\mathbb{R}^n) \tag{5}$$

*for $y \in \mathbb{R}^n$.*

*(ii) In addition, there exists a constant $C$ such that*

$$\frac{1}{C}\|u\|_{H^k(\mathbb{R}^n)} \leq \|(1 + |y|^k)\hat{u}\|_{L^2(\mathbb{R}^n)} \leq C\|u\|_{H^k(\mathbb{R}^n)} \tag{6}$$

*for each $u \in H^k(\mathbb{R}^n)$.*

*Proof.* In progress... □

## 2.9 Characterization of the $H_0^1$ dual space

**Motivation.** Remember that $H_0^1 = W_0^{1,2}$. In this section we desire to have an explicit characterization of the dual of this space, which we will find useful in some propositions for spaces involving time. Again, this section closely follows Evans [1].

Let $U \subset \mathbb{R}^n$ be a bounded open set.

**Definition 1.** *We denote by $H^{-1}(U)$ the* `dual space` *to $H_0^1(U)$.*

**Remark.** Although $H^{-1}$ is the dual space to $H_0^1$, it is not true they are reflexive, in fact we find that

$$H_0^1(U) \subsetneq L^2(U) \subset H^{-1}(U).$$

**Definition 2.** *If $f \in H^{-1}(U)$, we define the norm*

$$\|f\|_{H^{-1}(U)} = \sup \{ < f, v > \mid v \in H_0^1(U), \|v\|_{H_0^1(U)} \leq 1.$$

**Theorem 1** (Characterization of $H^{-1}$)**.**

*(i) Suppose $f \in H^{-1}(U)$. Then there exists functions $f^0, f^1, \ldots, f^n \in L^2(U)$ such that*

$$< f, v >= \int_U \left( f^0 v + \sum_{i=1}^{n} f^i v_{x_i} \right) dx \tag{7}$$

*for any $u \in H_0^1(U)$.*

*(ii) Further,*

$$\|f\|_{H^{-1}(U)} = \inf \left\{ \left( \int_U \sum_{i=1}^{n} |f_i|^2 dx \right)^{1/2} \mid f \text{ satifies } (7) \text{ for } f^0, \ldots, f^n \in L^2(U) \right\}.$$

*(iii) In particular, we have*

$$(u, v)_{L^2(U)} =< u, v > \tag{8}$$

*for all $u \in H_0^1(U)$ and $v \in L^2(U) \subset H^{-1}(U)$.*

*Note that if identity (7) holds, we say $f = f^0 - \sum_{i=1}^{n} f^i_{x_i}$.*

*Proof.* In progress... □

## 2.10 Sobolev spaces involving time

In progress...

# 3 Adaptive Finite Element Methods

In Progress...

### 3.1 Formulation of variational problems

**Motivation.** Taken from Brenner and Scott [9], consider the two-point boundary value problem

$$\begin{cases} -\frac{d^2u}{dx^2} = f \text{ in } (0,1) \\ u(0) = 0, u'(1) = 0. \end{cases} \tag{9}$$

If $u$ is the solution and $v$ is any function such that $v(0) = 0$, then integration by parts yields

$$\begin{aligned}(f, v) &:= \int_0^1 f(x)v(x)dx \\ &= \int_0^1 -u''(x)v(x)dx \\ &= \int_0^1 u'(x)v'(x)dx \\ &=: B(u, v).\end{aligned}$$

Let us then define

$$V = \{v \in L^2(0,1) \,|\, B(v,v) < \infty \text{ and } v(0) = 0\}.$$

Then we can say that the solution $u$ to (9) is characterized by

$$u \in V \text{ such that } B(u,v) = (f,v)$$

for any $v \in V$, which is called the ***variational*** or ***weak*** formulation of (9).

We now wish to generalize this example by applying abstract Hilbert space theory to get existence and uniqueness results for variational formulations of boundary value problems, both symmetric and nonsymmetric. To do this, we will first need to introduce one of the important representation theorems alluded to before in the Hilbert spaces section.

**Theorem 1** (Riesz representation theorem)**.** *For every bounded linear functional $F$ on a Hilbert space $H$, there exists a uniquely determined element $f \in H$ such that $F(x) = (x, f)$ for all $x \in H$. Furthermore, $\|F\|_{op} = \|f\|_H$.*

*My proof attempt.* For the case when $H = N(F) = \{x \in H \,|\, F(x) = 0\}$ we find that our uniquely determined element is $f = 0$ as the zero vector is orthogonal to everything including itself, implying $F(x) = 0 = (x, 0)$ for all $x \in H$. However, assume $H \neq N(F)$. $N(F)$ is now a closed subspace of $H$, which by virtue of the Projection Theorem implies there exists non-zero elements in $N(F)^\perp$. Take $z \in N(F)^\perp$ such that $z \neq 0$, where it follows that $(u, z) = 0$ for all $u \in N(F)$. Furthermore, $F(z) \neq 0$ and for any $x \in H$

$$F\left(x - \frac{F(x)}{F(z)}z\right) = F(x) - \frac{F(x)}{F(z)}F(z) = 0,$$

implying that $x - \frac{F(x)}{F(z)}z \in N(F)$, which further implies that $\left(x - \frac{F(x)}{F(z)}z, z\right) = 0$. Now by the properties of inner products, we have

$$
\begin{aligned}
(x, z) &= \left(x - \frac{F(x)}{F(z)}z, z\right) + \left(\frac{F(x)}{F(z)}z, z\right) \\
&= \left(\frac{F(x)}{F(z)}z, z\right) \\
&= \frac{F(x)}{F(z)}(z, z) \\
&= \frac{F(x)}{F(z)}(z, z)^{1/2}(z, z)^{1/2} \\
&= \frac{F(x)}{F(z)}\|z\|^2
\end{aligned}
$$

for any $x \in H$. Rearranging the above equation results in

$$F(x) = (x, z)\frac{F(z)}{\|z\|^2} = \left(x, \frac{F(z)}{\|z\|^2}z\right),$$

where we now just need to show $f = \frac{F(z)}{\|z\|^2}z$ is unique. To prove the uniqueness of $f$ assume there exists $f_1, f_2 \in H$ such that $F(x) = (x, f_1) = (x, f_2)$ for all $x \in H$, and assume we have $x \in H$ such that $\|x\| > 0$. We can see that this implies

$$0 = |F(x) - F(x)| = |(x, f_1) - (x, f_2)| = |(x, f_1 - f_2)| \leq \|x\|\|f_1 - f_2\|.$$

This implies $\|f_1 - f_2\| = 0$, which futher implies $f_1 - f_2 = 0$, proving the uniqueness of $f$. Finally, to show $\|F\|_{\text{op}} = \|f\|_H$ we can use the fact that

$$\|F\|_{\text{op}} = \sup\left\{\frac{\|F(x)\|_{\mathbb{R}}}{\|x\|_H} : x \neq 0 \text{ and } x \in H\right\}$$

to say

$$\|F\|_{\text{op}} = \sup_{x\neq 0}\frac{\|F(x)\|_{\mathbb{R}}}{\|x\|_H} = \sup_{x\neq 0}\frac{|F(x)|}{\|x\|_H} = \sup_{x\neq 0}\frac{|(x, f)|}{\|x\|_H} \leq \sup_{x\neq 0}\frac{\|x\|_H\|f\|_H}{\|x\|_H} = \|f\|_H.$$

We can also say

$$\|f\|_H^2 = \|f\|_H\|f\|_H = (f, f)^{1/2}(f, f)^{1/2} = (f, f) = F(f) \leq \|F(f)\|_{\text{op}}\|f\|_H.$$

These inequalities imply that $\|F\|_{\text{op}} \leq \|f\|_H$ and $\|f\|_H \leq \|F\|_{\text{op}}$, respectively, which can only be true if $\|F\|_{\text{op}} = \|f\|_H$. □

In general, a symmetric variational problem is posed as follows. Suppose the following three conditions are valid:

(i) $H$ is a Hilbert space.

(ii) $V$ is a subspace of $H$.

(iii) $B(\cdot, \cdot)$ is a bounded symmetric bilinear form coercive on $V$.

Then the ***symmetric variational problem*** is the following.

*Given $F \in V^*$, find $u \in V$ such that $B(u, v) = F(v)$ for any $v \in V$.*

**Theorem 2.** *Suppose conditions (i)-(iii) of the symmetric variational problem hold. Then there exists a unique $u \in V$ that solves the symmetric variational problem.*

The ***Ritz-Galerkin approximation problem*** is the following.

> *Given a finite-dimensional subspace $V_h \subset V$ and $F \in V^*$, find $u_h \in V_h$ such that*
> $$B(u_h, v) = F(v)$$
> *for any $v \in V_h$.*

**Theorem 3.** *Suppose conditions (i)-(iii) of the symmetric variation problem hold. Then there exists a unique $u_h$ that solves the Ritz-Galerkin approximation problem.*

**Theorem 4.** *Let $u$ and $u_h$ be solutions to the symmetric variational problem and the Ritz-Galerkin approximation problem, respectively. Then*

$$B(u - u_h, v) = 0$$

*for any $v \in V_h$.*

**Corollary 1.**
$$\|u - u_h\|_E = \min_{v \in V_h} \|u - v\|_E.$$

A nonsymmetric variational problem is posed as follows. Suppose that the following five conditions are valid:

(i) $H$ is a Hilbert space.

(ii) $V$ is a subspace of $H$.

(iii) $B(\cdot, \cdot)$ is a bilinear form on $V$, not necessarily symmetric.

(iv) $B(\cdot, \cdot)$ is continuous on $V$.

(v) $B(\cdot, \cdot)$ is coercive on $V$.

The variational and approximation problems are posed in the exact same way as in the symmetric case, but since the bilinear form is not guaranteed to be symmetric this means we can no longer rely on the Riesz representation theorem to guarantee existence and uniqueness. This gives rise to the Lax-Milgram theorem.

**Theorem 5** (Lax-Milgram Theorem)**.** *Let $B$ be a bounded, coercive bilinear form on a Hilbert space $H$. Then for every bounded linear functional $F$ on $H$ there exists a uniquely determined element $f \in H$ such that*

$$B(x, f) = F(x)$$

*for all $x \in H$.*

*My proof attempt.* For any $f \in H$, define the mapping

$$f \mapsto (F = B(\cdot, f)). \tag{10}$$

Then since $F = B(\cdot, f) \in H^*$, we can use the Riesz Representation Theorem (RRT) to define another mapping

$$(F = B(\cdot, f)) \overset{RRT}{\mapsto} \tilde{f}, \tag{11}$$

where $\tilde{f}$ is defined to be a uniquely determined element in $H$. Combining mappings (10) and (11) allows us to define one more mapping

$$f \mapsto \tilde{f}, \tag{12}$$

where we denote this mapping as $T$ such that $Tf = \tilde{f}$. With these mapping we now have

$$F(z) = B(z, f) = (z, Tf)$$

for any $z \in H$. This may look like we are done, but notice that we have no idea if $f$ is unique or not. So our goal now is to show that $T$ is a bijective bounded linear operator from $H$ to $H$, which will allow us to use the Bounded Inverse Theorem to say $T^{-1}$ exists and is also bounded. Firstly, by the linearity and bilinearity of the inner product and the bilinear form $B$, respectively, we can say that for all $x, y, z \in H$ and $\lambda, \mu \in \mathbb{C}$, we have

$$\begin{aligned}
(z, T(\lambda x + \mu y)) &= B(z, \lambda x + \mu y) \\
&= B(z, \lambda x) + B(z, \mu y) \\
&= \lambda B(z, x) + \mu B(z, y) \\
&= \lambda(z, Tx) + \mu(z, Ty) \\
&= (z, \lambda Tx) + (z, \mu Ty) \\
&= (z, \lambda Tx + \mu Ty),
\end{aligned}$$

implying that $T(\lambda x + \mu y) = \lambda Tx + \mu Ty$, which is the definition of a linear operator. Next, for any $z \in H$, we have

$$|(z, Tf)| = |B(z, f)| \leq k\|z\|\|f\|$$

for some $k > 0$, where letting $z = Tf$ results in

$$\|Tf\|^2 = |(Tf, Tf)| = |B(Tf, f)| \leq k\|Tf\|\|f\|,$$

implying that

$$\|Tf\| \leq k\|f\|. \tag{13}$$

Thus, $T$ is bounded. Now, also note that since

$$(z, Tf) = B(z, f)$$

it follows that we can use the Cauchy-Schwarz inequality and the coercivity of $B$ to say that if $z = f$, then

$$\|f\|\|Tf\| \geq |(f, Tf)| = |B(f, f)| \geq |\nu\|f\|^2| = \nu\|f\|^2 \tag{14}$$

for some $\nu > 0$. Inequalities (13) and (14) imply

$$\nu\|f\| \leq \|Tf\| \leq k\|f\| \tag{15}$$

for some $\nu > 0$ and $k > 0$. By definition $T$ is injective provided that for any $x, y \in H$, if $Tx = Ty$, then $x = y$. But note that the linearity of $T$ implies that we can instead say "$T$ is injective provided that for all $w \in H$, if $Tw = 0$, the $w = 0$ ." It is then easy to see that inequality (15) satisfies this definition, and thus $T$ is injective. Now to prove that $T$ is also surjective, we first need to prove that $T$ has a closed range so that we can invoke the Hilbert Projection Theorem. To prove this, let $T(H)$ denote the range of $T$. Then since $T(H)$ is a subspace of a Hilbert space we know that there exists a Cauchy sequence $\{\tilde{f}_k\}_{k=1}^{\infty} \subset H$ such that $\tilde{f}_k \to \tilde{f}$ in $T(H)$, as $k \to \infty$. Then, since $T$ is injective, there exists a unique sequence $\{f_k\}_{k=1}^{\infty} \subset H$ such that $\tilde{f}_k = Tf_k$, for all $k \in \mathbb{N}$. Now $\{\tilde{f}_k\}_{k=1}^{\infty}$ being Cauchy implies that for any $\epsilon > 0$ there exists $N > 0$ such that

$$\epsilon > \|\tilde{f}_k - \tilde{f}_\ell\| = \|Tf_k - Tf_\ell\| = \|T(f_k - f_\ell)\| \geq \nu\|f_k - f_\ell\|$$

for any $k, \ell \in \mathbb{N}$ and some $\nu > 0$. This implies $\{f_k\}_{k=1}^{\infty}$ is also Cauchy, further implying there exists a unqiue $f \in H$ such that $f_k \to f$ in $H$, as $k \to \infty$. Finally, to show $Tf = \tilde{f}$, observe that

$$Tf = \lim_{k\to\infty} Tf_k = \lim_{k\to\infty} \tilde{f}_k = \tilde{f}.$$

Thus, $T$ has a closed range. Now to prove $T$ is surjective, suppose by contradiction that $T$ is not surjective. Then this implies $T(H)$ is a closed proper subspace of $H$, where we know by the Hilbert Projection Theorem that there exists non-zero elements $z \in T(H)^{\perp}$ such that $(z, Tf) = 0$. Letting $f = z$ results in

$$0 = |(z, Tz)| = |B(z, z)| \geq \nu\|z\|^2 \geq 0$$

which can only be true if $z = 0$. Thus, $T$ must be surjective. Collecting ourselves, we have now proven that $T$ is a bijective bounded linear operator with a closed range. This means by virtue of the Bounded Inverse Theorem, $T^{-1}$ exists and is bounded. Then invoking the RRT again, we can say for any $F \in H^*$, there exists a uniquely determined element $\hat{f} \in H$ such that

$$F(z) = (z, \hat{f})$$

for any $z \in H$. Now since we know

$$F = B(\cdot, f)$$

for any $f \in H$, we can say for any $z \in H$

$$F(z) = (z, TT^{-1}\hat{f}) = B(z, T^{-1}\hat{f})$$

because we have proven that $T$ is a bijection. Letting $T^{-1}\hat{f} = g$ results in

$$F(z) = B(z, g)$$

for any $z \in H$. Now all that is left is to prove the uniqueness of $g$. Suppose we have $g_1, g_2 \in H$ such that $Tg_1$ and $Tg_2$ are unique elements satisfying the RRT. This means

$$\begin{aligned}
&B(z, g_1) = F(z) = B(z, g_2) \\
&\Rightarrow B(z, g_1) = B(z, g_2) \\
&\Rightarrow B(z, g_1) - B(z, g_2) = 0 \\
&\Rightarrow B(z, g_1 - g_2) = 0.
\end{aligned}$$

Letting $z = g_1 - g_2$ results in

$$0 = B(g_1 - g_2, g_1 - g_2) \geq \nu\|g_1 - g_2\|^2 \geq 0$$

for some $\nu > 0$. This can only be true if $g_1 - g_2 = 0$, proving the uniqueness of $g$. □

**Corollary 2.** *Suppose conditions (i)-(v) of the nonsymmetric variational problem hold. Then there exists a unique $u$ that solves the nonsymmetric variatonal problem.*

**Corollary 3.** *Suppose conditions (i)-(v) of the nonsymmetric variational problem hold. Then there exists a unique* $u_h$ *that solves the nonsymmetric approximation problem.*

Now let $u$ be a solution to the nonsymmetric variational problem, and let $u_h$ be a solution to the nonsymmetric approximation problem. We desire to estimate the error $\|u - u_h\|_V$ like we did in the symmetric case.

**Theorem 6** (Céa's theorem)**.** *Suppose condtitions (i)-(v) of the nonsymmetric variational problem hold. For the* `finite element variational problem`*, i.e. the approximation problem, we have*

$$\|u - u_h\|_V \leq \frac{C}{\alpha} \min_{v \in V_h} \|u - v\|_V,$$

*where* $C$ *is the* `continuity constant` *and* $\alpha$ *is the* `coercivity constant` *of* $B(\cdot, \cdot)$ *on* $V$*.*

# Notation

(i) A `multi-index` is a vector $\alpha = (\alpha_1, \ldots, \alpha_n)$ where each component $\alpha_i \in \mathbb{N}_0$. A multi-index has an order defined by

$$|\alpha| = \alpha_1 + \cdots + \alpha_n.$$

(ii) Using our definition of a multi-index and letting $u(x)$ be some function, we define

$$D^\alpha u(x) = \frac{\partial^{|\alpha|} u(x)}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}} = \partial_{x_1}^{\alpha_1} \ldots \partial_{x_n}^{\alpha_n} u(x).$$

(iii) Let $U, V \subset \mathbb{R}^n$. Then define

$$V \subset\subset U$$

to be when $V \subset \overline{V} \subset U$ and $\overline{V}$ is compact. In plain english this means $V$ is `compactly contained` in $U$.

(iv) Let $f$ and $g$ be functions. Then define $*$ to be the `convolution operator` where

$$(f * g)(x) = \int_{-\infty}^{\infty} f(\tau) g(x - \tau) d\tau = \int_{-\infty}^{\infty} f(x - \tau) g(\tau) d\tau$$

is the `convolution` of the functions $f$ and $g$ which results in a third function that expresses how one of the functions modifies the other. Note that I am assuming $f$ and $g$ are both supported on an infinite interval, which may not always be the case.

(v) Let $u(x)$ be a function where $x \in \mathbb{R}^n$. Then the `gradient vector` of $u$ is

$$Du(x_1, \ldots, x_n) = (u_{x_1}, ..., u_{x_n}).$$

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