Equation (5) from my notes is

$$\phi_t^{(k+1)} = R(\phi^{(k+1)}, t) + D(\phi_{AD}^{(k+1),n+1}) + \frac{1}{2}\left(A(\phi^n) + D(\phi^n) + A(\phi^{(k+1),n+1}) - D(\phi^{(k+1),n+1})\right).$$

Integrating from t^n to t^{n+1}, we have

$$\begin{aligned}\int_{t^n}^{t^{n+1}} \phi_t^{(k+1)}(\tau)d\tau = &\int_{t^n}^{t^{n+1}} R(\phi^{(k+1)}, \tau)d\tau \\ &+ \int_{t^n}^{t^{n+1}} \left\{D(\phi_{AD}^{(k+1),n+1}) + \frac{1}{2}\left(A(\phi^n) + D(\phi^n) + A(\phi^{(k+1),n+1}) - D(\phi^{(k+1),n+1})\right)\right\}d\tau\end{aligned}$$

Here we are considering the case where the advection and diffusion terms are approximated as piecewise constants, so all the terms in the second integral on the right-hand side are actually constants, and can therefore be integrated exactly. The case for piecewise polynomials of any degree is the same. Therefore,

$$\begin{aligned}\int_{t^n}^{t^{n+1}} R(\phi^{(k+1)}, \tau)d\tau = &\phi^{(k+1),n+1} - \phi^n \\ &- \Delta t\left\{D(\phi_{AD}^{(k+1),n+1}) + \frac{1}{2}\left(A(\phi^n) + D(\phi^n) + A(\phi^{(k+1),n+1}) - D(\phi^{(k+1),n+1})\right)\right\},\end{aligned}$$

where the right-hand side conists entirely of known quantities.