Recalling from Michael's notes, equations (24-25), using Forward Euler for the advection correction, we have $c_A^{(k)} = 0$. Additionally, we notice that if the residual $\tilde{\boldsymbol{R}}(\phi^{(k)}, t_{n+1}) = 0$, then we have that the term $c_{AD}^{(k)} = 0$ satisfies the Backward Euler equation

$$c_{AD}^{(k)} = \Delta t \left(d(u^{(k),n+1} + c_{AD}^{(k)}) - d(u^{(k),n+1}) \right) + \tilde{\boldsymbol{R}}(\phi^{(k)}, t_{n+1}). \tag{1}$$

Therefore, since the correction is given by the ODE

$$\begin{aligned} (c^{(k)})'(t) &= ac_A(t) + dc_{AD}(t) + rc^{(k)}(t) + \tilde{\boldsymbol{R}}' &(2)\\ &= rc^{(k)} &(3) \end{aligned}$$

with zero initial conditions, we have that $c^{(k)}(t) = 0$. Therefore, $\phi^{(k+1)} = \phi^{(k)}$.

Additionally, the update ODE for the new solution is given by

$$\begin{aligned} (c^{(k)} + \phi^{(k)})'(t) &= ac_A^{(k)}(t) + dc_{AD}^{(k)}(t) + r(c^{(k)} + \phi^{(k)})(t) + \ell_k(t) &(4)\\ &= r(c^{(k)} + \phi^{(k)})(t) + \ell_k(t) &(5) \end{aligned}$$

We write $\phi^{(k+1)} = c^{(k)} + \phi^{(k)}$. We will additionally assume $t_n = 0$. This is not an essential assumption, it entirely for the sake of simplicity. Since $\ell_k(t)$ is a linear approximation, we can write

$$\ell_k(t) = mt + b. \tag{6}$$

Our update ODE is therefore

$$\phi_t^{(k+1)}(t) = r\phi^{(k+1)}(t) + mt + b, \qquad \phi^{(k+1)}(0) = \phi_n \tag{7}$$

the solution to which is given by

$$\phi^{(k+1)}(t) = \frac{1}{r^2} \left(e^{rt} r^2 \phi_n + be^{rt} r + e^{rt} m - mrt - m \right) \tag{8}$$

Recall that ℓ_k is given by

$$\ell_k(t) = (a + d) \left(\phi_n \frac{t_{n+1} - t}{\Delta t} + \frac{t - t_n}{\Delta t} \phi_{n+1}^{(k)} \right). \tag{9}$$

Since $\phi^{(k+1)} = \phi^{(k)}$, we must have

$$\ell_k(t) = (a + d) \left(\phi_n \frac{t_{n+1} - t}{\Delta t} + \frac{t}{\Delta t} \phi_{n+1}^{(k+1)} \right). \tag{10}$$

In other words,

$$b = (a + d)\phi_n, \qquad m = \frac{(a + d) \left(\phi_{n+1}^{(k+1)} - \phi_n \right)}{\Delta t}. \tag{11}$$

We can then write

$$\phi_{n+1}^{(k+1)} = \frac{m\Delta t}{a + d} + \phi_n. \tag{12}$$

Solving simultaneously equations (8) and (12), we see that

$$m = \frac{\phi_n}{\frac{\Delta t}{(a-d)(e^{r\Delta t}-1)} - \frac{1}{r(a+d+r)}}. \tag{13}$$

Using this expression to substitute for m and b in equation (8), we find that

$$\phi^{(k+1)}(t) = \frac{\phi_n(a+d)\left(e^{\Delta tr}(t(a+d+r)+1) - (t-\Delta t)(a+d+r) - 1\right) - \Delta t \phi_n e^{rt}(a+d+r)2}{a\left(\Delta t(-r) + e^{\Delta tr} - 1\right) + de^{\Delta tr} - \Delta tr(d+r) - d}. \quad (14)$$

Expanding the Taylor series for $\phi^{(k+1)}(t_{n+1})$ around 0, and comparing with the exact solution $\phi(t_{n+1}) = \phi_n e^{(a+d+r)t_{n+1}}$, we see that

$$\begin{aligned} \phi_{n+1}^{(k+1)} - \phi(t_{n+1}) &= \frac{1}{12}\phi_n(a+d)(a+d+r)^2\Delta t^3 + \frac{1}{12}\phi_n(a+d)(a+d+r)^3\Delta t^4 \\ &\quad + \frac{1}{720}\phi_n(a+d)(a+d+r)^2\left(72r(a+d) + 39(a+d)^2 + 32r^2\right)\Delta t^5 + \mathcal{O}(\Delta t^6). \quad (15) \end{aligned}$$

In other words, the solution to which the MISDC iterations "should" converge (given that the residual goes to zero) is locally third-order accurate, in the case of a piecewise linear approximation for the advection and diffusion terms. I have performed a similar analysis for the case of piecewise cubic, and found (as expected) that the error is $\mathcal{O}(\Delta t^5)$.