Exercise 5.3

Given any circle graph C_n we know that the adjacency matrix A has entries (1) for (i,j) adjacent and 0 otherwise, and looks like this:

$$A: \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 1 \\ 1 & 0 & 1 & 0 & \dots & 0 \\ 0 & 1 & 0 & 1 & 0 \,.. & 0 \\ & \ddots & & \ddots & & \end{bmatrix}$$, which is a cyclical shift of the first row.

$D_{ii} = 2$ for any i, naturally.

For i vertices, then, we have that $A_{i,i+1} = 1$ and $A_{i,i-1} = 0$.

Therefore the Laplacian is defined for any C_n as

$$L f(i) = 2f(i) - f(i+1) - f(i-1).$$

If we apply $u_k(i)$ to $L f(i)$, we get $L u_k(i) = 2\sin\left(\frac{2\pi k i}{n}\right) - \sin\left(\frac{2\pi k(i+1)}{n}\right) - \sin\left(\frac{2\pi k(i-1)}{n}\right)$, which simplifies to

$$L u_k(i) = 2\left(1 - \cos\left(\frac{2\pi k}{n}\right)\right) \underbrace{\sin\left(\frac{2\pi k i}{n}\right)}_{\substack{\downarrow \\ \text{function proportional} \\ \text{to } u_k(i).}} ,$$

$\Rightarrow$ eigenvector ✓

and $\lambda_k = 2 - 2\cos\left(\frac{2\pi k}{n}\right)$.

The same applies to $L v_k(i)$. *

*

call $\frac{2\pi k}{n} := p$

$u_k(i)$:

$$\mathcal{L} = 2\sin(pi) - (\sin(pi)\cos(p) + \cos(pi)\sin(p))$$
$$- (\sin(pi)\cos(p) - \cos(pi)\sin(p))$$

$$= 2\sin(pi) - 2\sin(pi)\cos(p)$$

$$= 2(1 - \cos(p))\sin(pi)$$

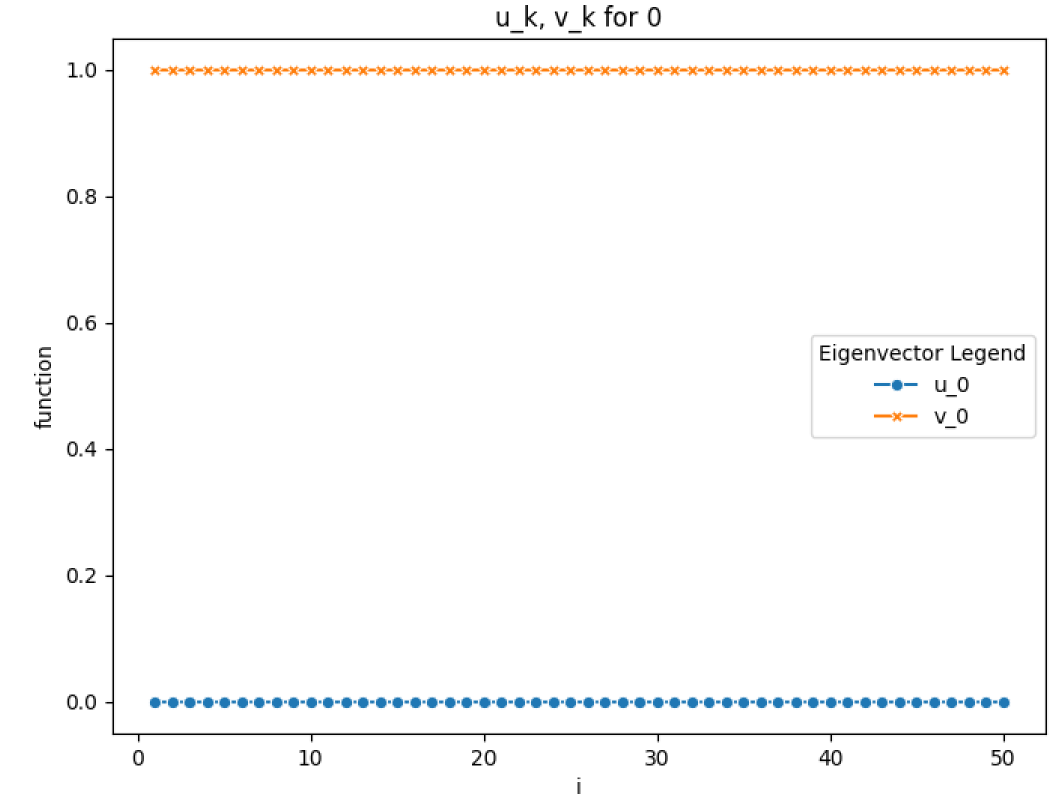
$v_k(i)$:

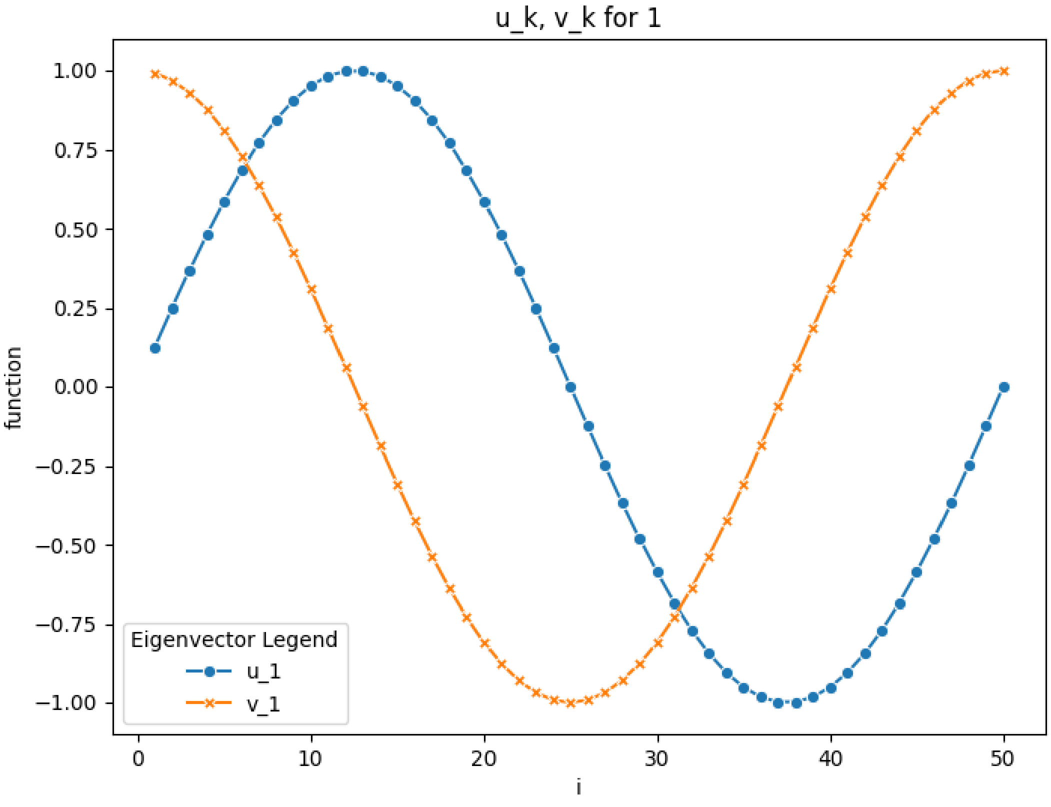
$$\mathcal{L} = 2\cos(pi) - \left[\cos(pi\cos(p) - \sin(pi \cdot \sin(p)))\right]$$
$$- \left[\cos(pi\cos(p) - \sin(p\sin(p)))\right]$$

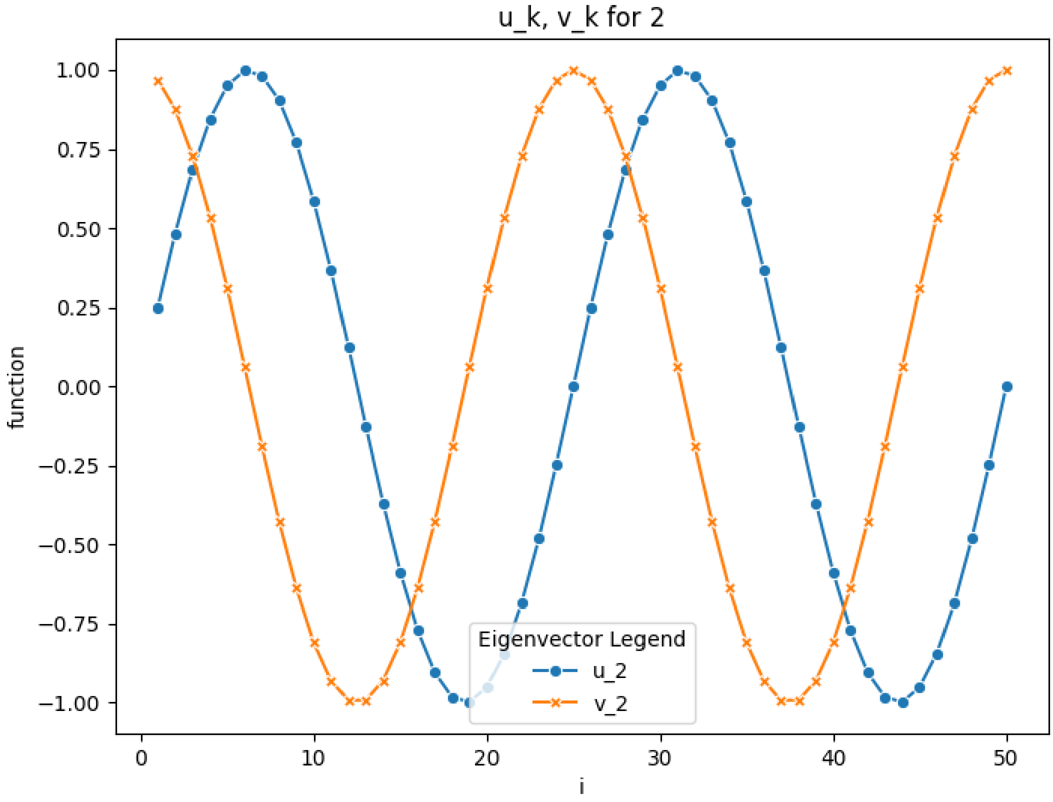
$$= 2\cos(pi) - 2\cos(pi)\cos(p)$$

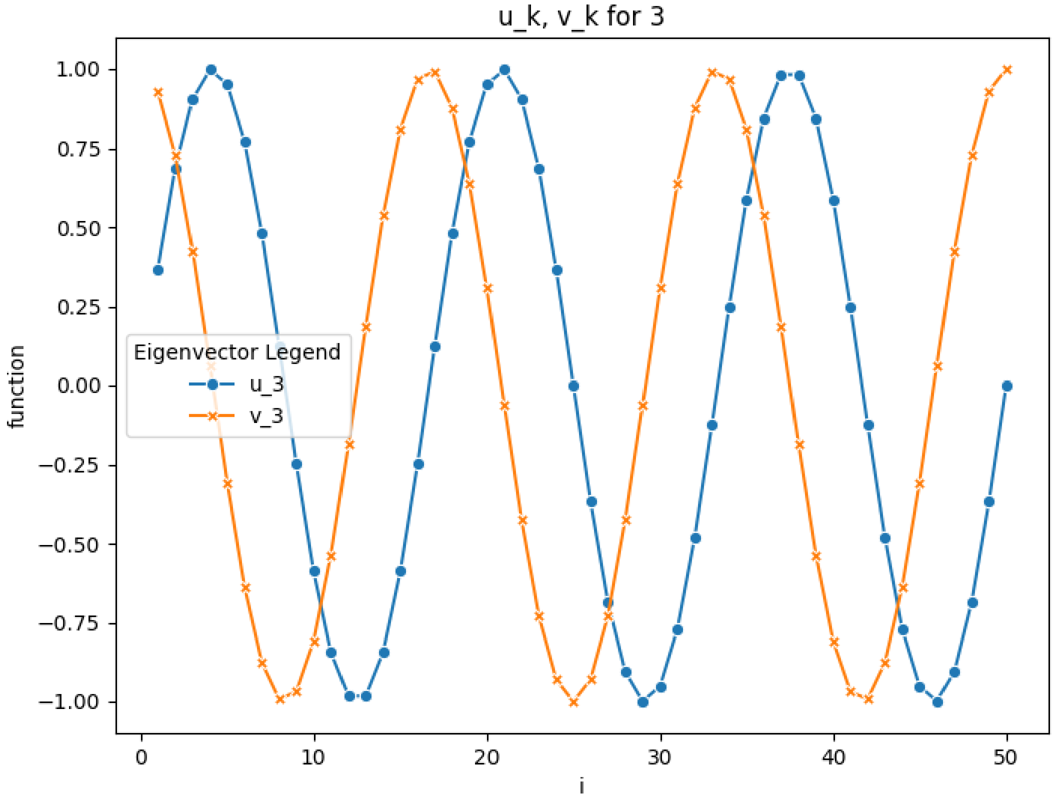
$$= 2\cos(pi)(1 - \cos(p))$$

$$= 2(1 - \cos(p))\cos(pi)$$

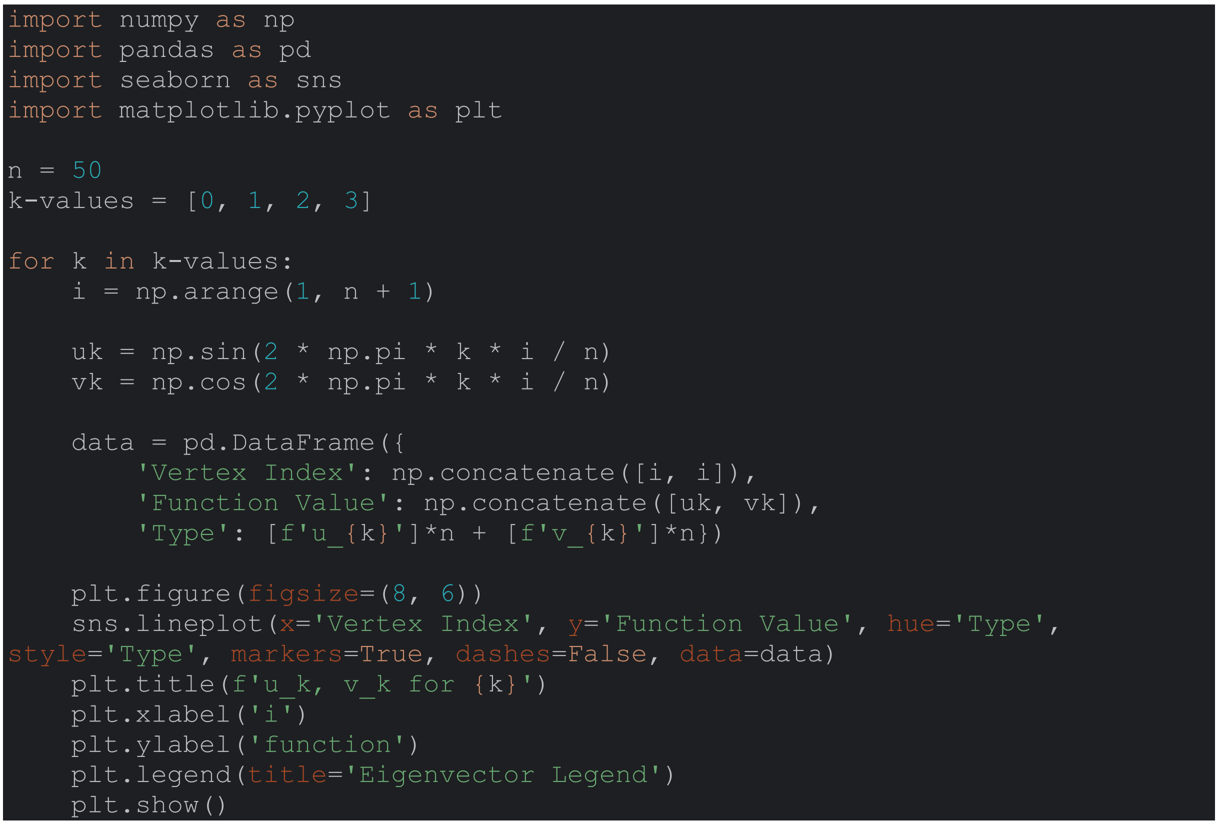

u_k, v_k for 0
function
1.0
0.8
0.6
0.4
0.2
0.0
0
10
20
30
40
50
i
Eigenvector Legend
u_0
v_0


u_k, v_k for 1
function
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
0
10
20
30
40
50
i
Eigenvector Legend
u_1
v_1


u_k, v_k for 2
function
i
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
0
10
20
30
40
50
Eigenvector Legend
u_2
v_2


u_k, v_k for 3
function
i
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
0
10
20
30
40
50
Eigenvector Legend
u_3
v_3

Code snippet for plotting:



```
import numpy as np
import pandas as pd
import seaborn as sns
import matplotlib.pyplot as plt

n = 50
k-values = [0, 1, 2, 3]

for k in k-values:
    i = np.arange(1, n + 1)

    uk = np.sin(2 * np.pi * k * i / n)
    vk = np.cos(2 * np.pi * k * i / n)

    data = pd.DataFrame({
        'Vertex Index': np.concatenate([i, i]),
        'Function Value': np.concatenate([uk, vk]),
        'Type': [f'u_{k}']*n + [f'v_{k}']*n})

    plt.figure(figsize=(8, 6))
    sns.lineplot(x='Vertex Index', y='Function Value', hue='Type',
style='Type', markers=True, dashes=False, data=data)
    plt.title(f'u_k, v_k for {k}')
    plt.xlabel('i')
    plt.ylabel('function')
    plt.legend(title='Eigenvector Legend')
    plt.show()
```