Deadline: 22nd November 2023

1. Calculate the limits
   (a)
$$\lim_{x\to 0}\frac{(1+x)^{\frac{1}{n}}-x}{x},$$
   where $n\in\mathbb{N}$ is a given natural number,
   (b)
$$\lim_{x\to\frac{\pi}{4}}(\tan x)^{\tan 2x},$$
   (c)
$$\lim_{x\to+\infty}\left(1+\ln\left(1+\frac{1}{x}\right)\right)^{\sqrt{x^2+1}},$$
   (d)
$$\lim_{x\to 0}(\cos x)^{\frac{1}{x}},$$
   (e)
$$\lim_{x\to 0}\frac{\ln(1+\sin x)}{\sqrt{1+2x}-\sqrt{1+x}},$$
   (f)
$$\lim_{x\to 0}\frac{1-\cos(\arctan x)}{x^2},$$
   (g)
$$\lim_{x\to+\infty}\left(\sqrt{x^2+2x}-x\right)^{x}.$$
   For $a,b,c\in\mathbb{R}$, with $a>0$, find α,β so that
   (h)
$$\lim_{x\to+\infty}\sqrt{ax^2+bx+c}-\alpha x-\beta=0.$$

Deadline: 15th November 2023

1. Calculate the limits
   (a)
$$\lim_{x\to\frac{\pi}{4}}\frac{\sin x-\cos x}{x-\frac{\pi}{4}},$$
   (b)
$$\lim_{x\to x_0}\frac{\cos x-\cos x_0}{x-x_0},$$
   (c)
$$\lim_{x\to\infty}x[\ln x-\ln(x+2)]$$
   (d)
$$\lim_{x\to 5}\frac{-5+x-5x^2-4x^3+x^4}{10-7x+x^2},$$
   (e)
$$\lim_{x\to 5}\frac{25-10x+26x^2+15x^3-9x^4+x^5}{-50+45x-12x^2+x^3},$$
   (f)
$$\lim_{x\to 0}\frac{\sin\left(x+x^3+x^5+\cdots+x^{2n+1}\right)}{x},$$
   where $n\in\mathbb{N}$ is a given natural number,
   (g)
$$\lim_{x\to\infty}\sqrt{x+\sqrt{x}}-\sqrt{x}$$

(h)
$$\lim_{x\to 0+} \frac{\frac{1}{x} - \sin\frac{1}{x}}{\frac{1}{x}},$$

(i)
$$\lim_{x\to 0+} \frac{b\mathrm{e}^{-\frac{a}{x^2}} - a\mathrm{e}^{-\frac{b}{x^2}}}{x},$$

where $a, b \in \mathbb{R}^+$ are given positive real numbers.

2. Optional: Find the limit
$$\lim_{x\to 0} \frac{\sin\tan x - \tan\sin x}{\arcsin\arctan x - \arctan\arcsin x}$$

Deadline: 25th October 2023

1. Hyperbolic functions
   1. Draw a unit hyperbola up to some hyperbolic angle ψ. (To do this, you can use a computer program and, e.g. hyperbolic functions). Next to the unit hyperbola ($x^2 - y^2 = 1$) draw a unit circle ($x^2 + y^2 = 1$). Where do the plots touch ?
   2. Now, insert the point $\tanh\psi = \frac{\sinh\psi}{\cosh\psi}$. From this point draw a line parallel to the x axis ($x > 0$). Mark a new point A where it intersects the unit circle. Using A and the origin, mark an angle ϕ in the unit circle for which $\tanh\psi = \sin\phi$ holds.
   3. Show that for this ϕ also $\tan\phi = \sinh\psi$, and mark the corresponding point B.
   4. Connect the points A and B, see that the line that goes through them intersects with the x-axis, the unit circle and the unit hyperbola at $x = -1$. As an **optional** task, show this using trigonometric identities.
2. Find the supremum of the the set $Y = \{a - b : a, b \in \mathbb{R}, 1 < a < 2, 3 < b < 4\}$. Find a proof to convince yourself that the supremum for a set is unique if it exists.
3. [**optional**] We have seen that if we take $\phi \in [0, 2\pi)$, then the equations
$$x = r\cos\varphi, \tag{1a}$$
$$y = r\sin\varphi, \tag{1b}$$
allow us to draw a circle or radius r in the plane. (The meaning of (1) is the following—take a φ from the interval $[0, 2\pi]$ and draw a point at the position $\begin{bmatrix} x & y \end{bmatrix} \in \mathbb{R}^2$ in the plane. Repeat for all φ in $[0, 2\pi]$.) Use your favourite plotting tool and plot the following variant of (1),
$$x = r\cos\varphi + \frac{r}{a}\cos(b\varphi), \tag{2a}$$
$$y = r\sin\varphi + \frac{r}{a}\sin(b\varphi), \tag{2b}$$
where $a \in \mathbb{R}$ and $b \in \mathbb{R}$ are some real numbers; you might try $a = 5$ and $b = 10$. (In your favourite plotting tool you will typically need something that is referred to as the "parametric plot".) Can you guess—before using your favourite plotting tool—how the plot will look like? You can also try to find out what *epicycles* are.

Deadline: 18th October 2023

1. For x $\in \mathbb{R}$ with $x > -1, x \neq 0$ and $n = 2, 3, 4, \ldots$ prove by induction that the inequality
$$(1 + x)^n > 1 + nx$$
holds. Furthermore, prove that for $0 < x < 1$ and $n \in \mathbb{N}$
$$(1 - x)^n < \frac{1}{1 + nx}.$$
2. Last time, we used the trigonometric identity
$$\sin(x + y) - \sin(x - y) = 2\cos(x)\sin(y).$$
Show why this is trivially true using the representations given for the sin and cos function in terms of the complex exponential function. Select two suitable, different examples of your choice from a list of trigonometric identities involving sin and cos or tan (e.g. on wikipedia) and prove them using the complex exponential representation.
We discussed the roots of an equation, show that for $z \in \mathbb{C}$ the solutions of $z^6 = 1$ form a regular hexagon. **optional**

3. Any mapping $f : X \to Y$ between two sets allows for a mapping $f^I : P(Y) \to P(X)$ between the respective power sets defined by
$$f^I(E) := \{x \in X; f(x) \in E\}.$$
Show that f^I preserves unions and intersections. Use as much of the notation we learned, as possible. For now it is enough to consider finite sets.
def: A power set P(A) of a set A is the set of all subsets of A, including the empty set and A itself.

4. **[optional]** So far we have been dealing with very nice iterated maps in the sense that the sequence $\{a_n\}_{n=0}^{+\infty}$ has always approached a limit. (Later on we shall write $\lim_{n\to+\infty} a_n = L$.) Use your favourite computer language and write a code that computes the elements of the iterated map defined as
$$a_{n+1} =_{\text{def}} \mathrm{e}^{-\alpha a_n^2} + \beta, \tag{3}$$
with $a_0 =_{\text{def}} 0$. Check what happens for different values of parameter β. (Recommended values are $\alpha = \frac{49}{10}$ and $\beta \in [-1, 1]$. You should definitely try $\beta = -1$, $\beta = 0$ and $\beta = 1$.) This map is referred to as the Gauss map or *mouse map*. Why mouse?

Deadline: 11th October 2023

1. Show that the number x defined as
$$x =_{\text{def}} 1.107\,107\,107\,107\ldots \tag{4}$$
is a rational number, that is it can be written in the form $x = \frac{p}{q}$, where p and q are coprime integers. Find the corresponding values of p and q.

2. We have seen that that the iteratively defined sequence $\{a_n\}_{n=0}^{+\infty}$,
$$a_{n+1} =_{\text{def}} \frac{1}{2}\left(a_n + \frac{2}{a_n}\right), \tag{5}$$
converges, for a suitably chosen initial value a_0, to $\sqrt{2}$. (Later on we shall write $\lim_{n\to+\infty} a_n = \sqrt{2}$.)
   (a) Find a modification of the definition (5) such that the newly defined sequence converges, for a suitably chosen initial value, to $\sqrt{3}$. Provide an explanation for your construction!
   (b) **[optional]** Use your favourite computer language and write a code that computes a decent approximation of $\sqrt{3}$.

3. Assume that $x \neq k\pi$, $k \in \mathbb{Z}$. Show that
$$\sum_{n=0}^{N-1} \cos(a + nx) = \frac{\sin\left(\frac{N}{2}x\right)\cos\left(a + \frac{N-1}{2}x\right)}{\sin\left(\frac{x}{2}\right)}. \tag{6}$$
Hint: There are several tricks how to do this, in particular the identity $\sin(x+y) - \sin(x-y) = 2\cos(x)\sin(y)$, and the concept of a *telescoping* sum. Alternatively, complex exponentials can be used, e.g. the fact that $\sin(nx) = \Im\left(\mathrm{e}^{\mathrm{i}x}\right)$ and $\cos(nx) = \Re\left(\mathrm{e}^{\mathrm{i}x}\right)$, and where $\Im$ denotes the imaginary part of the corresponding complex number and $\Re$ the real part, and the formula for the sum of the geometric sequence
$$\sum_{n=0}^{N} q^n = \frac{1 - q^{N+1}}{1 - q}.$$
Other trigonometric identities might be useful as well.

4. **[optional]** Consider the object defined as the limit of the sequence shown in 1 below. As indicated in the figure, we

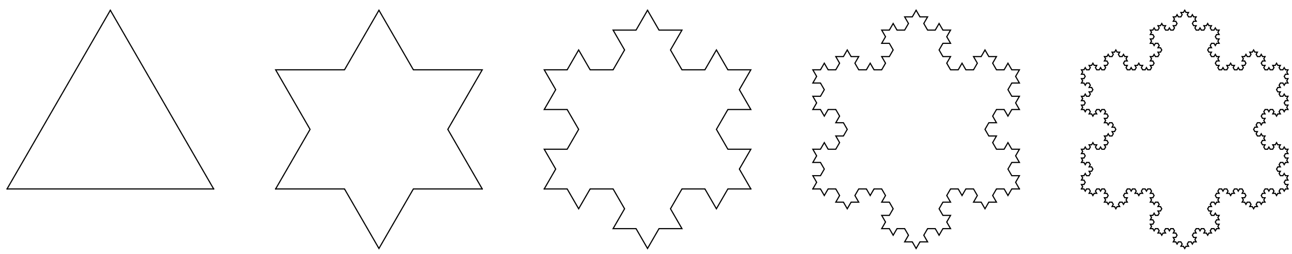

Figure 1: Iterative construction of an planar object.

start with an equilateral triangle with the surface area A_{init}, and we are adding small equilateral triangles. Show that the surface area of A_∞ is $A_\infty = \frac{8}{5}A_{\text{init}}$ and that, *surprise*, the *length of the boundary* of A_∞ is *infinite*. Thus the march to the infinity allowed us to design a *finite area* object which is however enclosed by an *infinite boundary*.

Hint: The best strategy is to first find the formula for the *number of line segments* in each step and then the formula for the *length of the line segment* and proceed similarly for the area. The object you are investigating is referred to as *Koch snowflake* or *Koch island*. You can have a conversation about it with ChatGPT or with your another favourite LLM. Check the answers!