Functional Programming and Verification
revision course

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Outline

1 Functional Programming and Haskell
- Basic Haskell
- Recursion, guards, pattern matching
- List comprehensions
- QuickCheck
- Polymorphism
- Currying, partial application, higher-order functions

2. Types
- Type aliases
- Type Classes
- Algebraic Data Types
- Modules, Abstract Data Types
- Type inference

1.1 Basic Haskell

function types	`f :: a -> b -> c`
function definitions	`f a b = a + b`
function application	`f 1 2`
function composition	`f . g` means $f(g(x))$
conditional	`if True then a else b`
prefix/infix precedence	``f a `g` b`` means ``(f a) `g` b``
$ sign	``f $ a `g` b`` means ``f (a `g` b)``

Types

`Bool`	`True` or `False`
`Int`	fixed-width integers
`Integer`	unbounded integers
`Char`	`'a'`
`String`	`"hello"` (type `[Char]`)
`(a,b)` (Tuple)	`("hello",1) :: (String,Int)`

Tuples

```
(1,"hello") :: (Int,String)
(x,y,z) :: (a,b,c)
-- ...
```

Prelude functions: `fst`, `snd`

Lists

Two ways of constructing a list:

```
a = [1,2,3]
b = 1 : 2 : 3 : []
```

Cons `(:)` and `[]` are constructors of lists, that is a function that uniquely constructs a value of the list type.

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a = [1,2,3]
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```

Cons `(:)` and `[]` are constructors of lists, that is a function that uniquely constructs a value of the list type.

Intuitively: `(:) :: a -> [a] -> [a]`.

Prelude functions

`head :: [a] -> a`	first element
`last :: [a] -> a`	last element
`init :: [a] -> [a]`	every element but last element
`tail :: [a] -> [a]`	every element but first element
`elem :: a -> [a] -> Bool`	element in list?
`(++) :: [a] -> [a] -> [a]`	append lists
`reverse :: [a] -> [a]`	reverse list
`length :: [a] -> Int`	length of list
`null :: [a] -> Bool`	empty?
`concat :: [[a]] -> [a]`	flatten list
`zip :: [a] -> [b] -> [(a,b)]`	combine lists element-wise
`unzip :: [(a,b)] -> ([a],[b])`	separate list of tuples into list of components

Prelude functions (2)

`replicate :: Int -> a -> [a]`	build list from repeated element
`take :: Int -> [a] -> [a]`	prefix of list with given length
`drop :: Int -> [a] -> [a]`	suffix of list with given length
`and ::[Bool] -> Bool`	conjunction over all elements
`or ::[Bool] -> Bool`	disjunction over all elements
`sum ::[Int] -> Int`	sum over all elements
`product ::[Int] -> Int`	product over all elements

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search for functions by type signature on `https://hoogle.haskell.org/`.

Ranges

```
[1..5]
```

Ranges

```
[1..5]
= [1,2,3,4,5]

[1,3..10]
```

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```

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[1..5]
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[1..]
= [1,2,3...]

[1,3..]
```

Ranges

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[1..]
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```

Local definitions

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defines `x` locally in e_2.

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`let x =` e_1 `in` e_2

defines `x` locally in e_2.

e_2 `where x =` e_1

also defines `x` locally in e_2 where e_2 has to be a function definition.

1.2 Recursion, guards, pattern matching

Guards

Example: maximum of two integers.

```
max2 :: Integer -> Integer -> Integer
max2 x y
  | x >= y    = x
  | otherwise = y
```

Recursion

Reduce problem into a solving a series of smaller problems of a similar kind.

Example

```
factorial :: Integer -> Integer
factorial n
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```
factorial :: Integer -> Integer
factorial n
  | n == 0 = 1                              -- base case
  | n > 0  = n * factorial (n - 1)    -- recursive case
```

Accumulating parameter

Alternatively, `factorial` could be defined as

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The resulting function is tail recursive, that is the recursive call is located at the very end of its body.
Therefore, no computation is done after the recursive function call returns.

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Therefore, no computation is done after the recursive function call returns.

In general, recursion using accumulating parameters is less readable.

Pattern matching

A more compact syntax for recursion:

```
factorial 0 = 1
factorial n | n > 0 = n * factorial (n - 1)
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Pattern matching

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factorial 0 = 1
factorial n | n > 0 = n * factorial (n - 1)
```

Patterns are expressions consisting only of constructors, variables, and literals.

Pattern matching

Examples

```
head :: [a] -> a
```

Pattern matching

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```
head :: [a] -> a
head (x : _) = x

tail :: [a] -> [a]
```

Pattern matching

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head :: [a] -> a
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tail :: [a] -> [a]
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null :: [a] -> Bool
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```
head :: [a] -> a
head (x : _) = x

tail :: [a] -> [a]
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null :: [a] -> Bool
null []      = True
null (_ : _) = False
```

Constructors vs Types

What is the difference between `True` and `Bool`?

- `True` is a constructor, `Bool` is a type.
- `True` can be used in expressions to build values of a type.
- `Bool` can be used in type signatures to hint at the type of bindings.

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Constructor or type?

`False`

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```
False    yes
(:)
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False     yes
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Maybe
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Constructor or type?

```
False    yes
(:)      yes
Maybe    no
Just     yes
Nothing  yes
```

Case

Pattern matching in nested expressions

```
singleOrEmpty :: [a] -> Bool
singleOrEmpty xs = case xs of []  -> True
                              [_] -> True
                              _   -> False
```

1.3 List comprehensions

$$[\ expr\ \mid\ E_1,\ \ldots,\ E_n\ ]$$

where *expr* is an expression and each E_i is a generator or a test.

- a generator is of the form *pattern* `<-` *listexpression*
- a test is a Boolean expression

List comprehensions

Examples

```
[x ^ 2 | x <- [1..5]]
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[(x, even x) | x <- [1..3]]
```

List comprehensions

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= [1, 4, 9, 16, 25]

[toLower c | c <- “Hello World!”]
= “hello world!”

[(x, even x) | x <- [1..3]]
= [(1, False), (2, True), (3, False)]
```

Multiple generators

Generators are reduced from left to right.
A generator or test can depend on any generator to its left.

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```
[(i,j) | i <- [1 .. 3], j <- [i .. 3]]
= [(1,j) | j <- [1..3]] ++
  [(2,j) | j <- [2..3]] ++
  [(3,j) | j <- [3..3]]
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Example

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[(i,j) | i <- [1 .. 3], j <- [i .. 3]]
= [(1,j) | j <- [1..3]] ++
  [(2,j) | j <- [2..3]] ++
  [(3,j) | j <- [3..3]]
= [(1,1), (1,2), (1,3), (2,2), (2,3), (3,3)]
```

The meaning of list comprehensions

```
[e | x <- [a1,...,an]]
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```
[e | x <- [a1,...,an]]
= (let x = a1 in [e]) ++ · · · ++ (let x = an in [e])

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= (let x = a1 in [e | E]) ++ · · · ++
  (let x = an in [e | E])

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QuickCheck tests check if a proposition holds true for a large number of random arguments. It can be used to *test* the equivalence of two functions.

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import Test.QuickCheck

prop_max2 x y =
  max2 x y = max x y
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Run `quickCheck prop_max2` from GHCI to check the property.

1.5 Polymorphism

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`length :: [a] -> Int` is defined for all types `a` where `a` is a type variable.

Subtype vs parametric polymorphism

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- subtype polymorphism - any object of type `T'` where `T'` is a subtype of `T` can be used in place of objects of type `T`.

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Haskell uses parametric polymorphism.

Type constraints

Type variables can be constrained by type constraints.

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(+) :: Num a => a -> a -> a
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Function `(+)` has type `a -> a -> a` for any type `a` of the type class `Num`.

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Some type classes:

1. `Num`
2. `Integral`
3. `Fractional`
4. `Ord`
5. `Eq`
6. `Show`

Quiz

```
f x y z = if x then y else z
```

Quiz

```
f x y z = if x then y else z
f :: Bool -> a -> a -> a

f x y = [(x,y), (y,x)]
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f :: a -> a -> [(a,a)]

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```

Quiz

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f :: a -> a -> [(a,a)]

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f :: [([a],Int)] -> [Int]

f x y = [ u ++ x | u <- y, length u < x ]
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f x y = [[ (u,v) | u <- w, u, v <- x] | w <- y]
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f x y = [[ (u,v) | u <- w, u, v <- x] | w <- y]
f :: [a] -> [[Bool]] -> [[(Bool, a)]]
```

1.6 Currying, partial application, higher-order functions

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Example

```
f :: Int -> Int -> Int
f x y = x + y

f a b
= a + b
```

```
f :: Int -> (Int -> Int)
f x = \y -> x + y

(f a) b
= (\y -> a + y) b
= a + b
```

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```

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f :: Int -> (Int -> Int)
f x = \y -> x + y

(f a) b
= (\y -> a + y) b
= a + b
```

Any function of two arguments can be viewed as a function of the first argument that returns a function of the second argument.

Anonymous functions (lambdas)

An anonymous function (or lambda abstractionanonymous function) is a function without a name.

Examples

```
(\x -> x + 1) 4
```

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= 5

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(\x y -> x + y) 3 5
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What is the type of `\n -> iter n succ` where

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succ :: Integer -> Integer
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succ :: Integer -> Integer
```

```
Integer -> (Integer -> Integer)
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Partial application

Every function of n parameters can be applied to less than n arguments.

A function is partially applied when some arguments have already been applied to a function (that is some parameters are already *fixed*), but some parameters are missing.

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(`elem` [1..5]) 0   no
```

Expressions of the form (*infixop expr*) or (*expr infixop*) are called sections.

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all, any :: (a -> Bool) -> [a] -> Bool

takeWhile, dropWhile :: (a -> Bool) -> [a] -> [a]

(.) :: (b -> c) -> (a -> b) -> (a -> c)
```

Fold

Folding is the most elementary way of combining elements of a list.

Right-associative (`foldr`):

```
foldr :: (b -> a -> a) -> a -> [b] -> a
foldr f a []     = a
foldr f a (x:xs) = f x (foldr f a xs)
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= 1 + (2 + (3 + foldr (+) 0 []))
```

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Why is this right-associative?

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foldr (+) 0 [1,2,3]
= 1 + foldr (+) 0 [2,3]
= 1 + (2 + foldr (+) 0 [3])
= 1 + (2 + (3 + foldr (+) 0 []))
= 1 + (2 + (3 + 0))
```

Fold

Folding is the most elementary way of combining elements of a list.

Right-associative (`foldr`):

```
foldr :: (b -> a -> a) -> a -> [b] -> a
foldr f a []     = a
foldr f a (x:xs) = f x (foldr f a xs)
```

Why is this right-associative?

```
foldr (+) 0 [1,2,3]
= 1 + foldr (+) 0 [2,3]
= 1 + (2 + foldr (+) 0 [3])
= 1 + (2 + (3 + foldr (+) 0 []))
= 1 + (2 + (3 + 0))
= 1 + (2 + 3)
```

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= 1 + (2 + foldr (+) 0 [3])
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= 1 + (2 + (3 + 0))
= 1 + (2 + 3)
= 1 + 5 = 6
```

2.1 Type aliases

Allows the renaming of a more complex type expression.

Examples

```
type String = [Char]
type List a = [a]
```

2.2 Type Classes

Type classes are collections of types that implement some fixed set of functions.

Similar concepts are commonly called *interfaces*.

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Similar concepts are commonly called *interfaces*.

Creating and using a type class:

1. creating a type class $\sim$ creating an interface (define set of functions)
2. instantiating a type class $\sim$ implementing an interface (implement a set of functions for a member of a type class)

2.2 Type Classes

Examples

```
class Eq a where
  (==) :: a -> a -> Bool
```

2.2 Type Classes

Examples

```
class Eq a where
  (==) :: a -> a -> Bool

instance Eq Bool where
  True  == True  = True
  False == False = True
  _     == _     = False
```

Constrained instances

Instances of type classes can be constrained.

Example

```
instance (Eq a) => Eq [a] where
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  (x:xs) == (y:ys) = x == y && xs == ys
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Constrained instances

Instances of type classes can be constrained.

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instance (Eq a) => Eq [a] where
  [] == [] = True
  (x:xs) == (y:ys) = x == y && xs == ys
  _ == _ = False
```

Subclasses

Example

```
class (Eq a) => Ord a where
  (<=), (<) :: a -> a -> Bool
```

Class `Ord` inherits all functions of class `Eq`.

Subclasses

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Before instantiating a subclass with a type, the type must be an instance of all *"superclasses"*.

Subclasses

Example

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instance Ord Bool where
```

Subclasses

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Class `Ord` inherits all functions of class `Eq`.

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```
instance Ord Bool where
  b1 <= b2 = not b1 || b2
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Subclasses

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class (Eq a) => Ord a where
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Class `Ord` inherits all functions of class `Eq`.

Before instantiating a subclass with a type, the type must be an instance of all *"superclasses"*.

```
instance Ord Bool where
  b1 <= b2 = not b1 || b2
  b1 < b2 = b1 <= b2 && not(b1 == b2)
```

2.3 Algebraic Data Types

- data types with a custom shape
- defines a type along with constructors to build values of that type

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- defines a type along with constructors to build values of that type

$$\texttt{data}\ \mathit{type}\ a_1 \ldots a_n\ \texttt{=}\ \mathit{constructor}\ a_1 \ldots a_n\ \texttt{|}\ \texttt{...}$$

2.3 Algebraic Data Types

Examples

```
data Bool = False | True
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2.3 Algebraic Data Types

Examples

```
data Bool = False | True

data Maybe a = Nothing | Just a
  deriving (Eq, Show)

data Nat = Zero | Suc Nat
  deriving (Eq, Show)

data [a] = [] | (:) a [a]
  deriving Eq

data Tree a = Empty | Node a (Tree a) (Tree a)
  deriving (Eq, Show)
```

Constructors vs Types (again)

Repetition:

- constructors can be used in expressions to build values of a type.
- types can be used in type signatures to hint at the type of bindings.

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- constructors can be used in expressions to build values of a type.
- types can be used in type signatures to hint at the type of bindings.

An algebraic data types is a custom type with one or more constructors.

Constructors vs Types (again)

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Constructors can have varying arity:

- a constructor with arity 0 acts like a value of the algebraic data type
- a constructor with arity k combines k values of different types into a single value of the algebraic data type

Constructors are functions that unambiguously construct the value of a type.

Pattern matching

Pattern matching works just the same for custom constructors as for predefined constructors.

Examples

```
find :: Ord a => a -> Tree a -> Bool
```

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find :: Ord a => a -> Tree a -> Bool
find _ Empty = False
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Examples

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find :: Ord a => a -> Tree a -> Bool
find _ Empty = False
find x (Node a l r)
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Pattern matching

Pattern matching works just the same for custom constructors as for predefined constructors.

Examples

```
find :: Ord a => a -> Tree a -> Bool
find _ Empty = False
find x (Node a l r)
  | x < a     = find x l
```

Pattern matching

Pattern matching works just the same for custom constructors as for predefined constructors.

Examples

```
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find _ Empty = False
find x (Node a l r)
  | x < a     = find x l
  | a < x     = find x r
```

Pattern matching

Pattern matching works just the same for custom constructors as for predefined constructors.

Examples

```
find :: Ord a => a -> Tree a -> Bool
find _ Empty = False
find x (Node a l r)
  | x < a     = find x l
  | a < x     = find x r
  | otherwise = True

insert :: Ord => a -> Tree a -> Tree a
```

Pattern matching

Pattern matching works just the same for custom constructors as for predefined constructors.

Examples

```
find :: Ord a => a -> Tree a -> Bool
find _ Empty = False
find x (Node a l r)
  | x < a     = find x l
  | a < x     = find x r
  | otherwise = True

insert :: Ord => a -> Tree a -> Tree a
insert x Empty = Node x Empty Empty
```

Pattern matching

Pattern matching works just the same for custom constructors as for predefined constructors.

Examples

```
find :: Ord a => a -> Tree a -> Bool
find _ Empty = False
find x (Node a l r)
  | x < a     = find x l
  | a < x     = find x r
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insert x Empty = Node x Empty Empty
insert x (Node a l r)
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Pattern matching

Pattern matching works just the same for custom constructors as for predefined constructors.

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find :: Ord a => a -> Tree a -> Bool
find _ Empty = False
find x (Node a l r)
  | x < a     = find x l
  | a < x     = find x r
  | otherwise = True

insert :: Ord => a -> Tree a -> Tree a
insert x Empty = Node x Empty Empty
insert x (Node a l r)
  | x < a     = Node a (insert x l) r
```

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insert x Empty = Node x Empty Empty
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Pattern matching works just the same for custom constructors as for predefined constructors.

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find :: Ord a => a -> Tree a -> Bool
find _ Empty = False
find x (Node a l r)
  | x < a     = find x l
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  | otherwise = True

insert :: Ord => a -> Tree a -> Tree a
insert x Empty = Node x Empty Empty
insert x (Node a l r)
  | x < a     = Node a (insert x l) r
  | a < x     = Node a l (insert x r)
  | otherwise = Node a l r
```

2.4 Modules, Abstract Data Types

Modules

Collection of type, function, class and other definitions.

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Examples

```
module M where
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exports everything defined in `M`

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Modules

Collection of type, function, class and other definitions.

Examples

`module M where`
exports everything defined in `M`

`module M (T, f, ...) where`
exports everything defined in `T, f, ...`

Exporting data types

```
module M (T) where
data T = ...
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exports only `T` but not its constructors

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```
module M (T(..)) where
data T = ...
```

exports `T` and all its constructors

Exporting data types

```
module M (T) where
data T = ...
```

exports only T but not its constructors

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module M (T(C,D,...)) where
data T = ...
```

exports T and its constructors C, D, ...

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data T = ...
```

exports T and all its constructors

Not allowed (why?):

```
module M (T,C,D) where
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Exporting data types

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```
module M (T(..)) where
data T = ...
```

exports `T` and all its constructors

Not allowed (why?):

```
module M (T,C,D) where
```

Constructors could have the same name as a type.

Abstract Data Types

Hides data representation by wrapping data in a constructor that is not exported.

Abstract Data Types

Example

```
module Set (Set, empty, insert, isin, size) where
-- Interface
empty :: Set a
insert :: Eq a => a -> Set a -> Set a
isin :: Eq a => a -> Set a -> Bool
size :: Set a -> Int
```

Abstract Data Types

Example

```
module Set (Set, empty, insert, isin, size) where
-- Interface
empty :: Set a
insert :: Eq a => a -> Set a -> Set a
isin :: Eq a => a -> Set a -> Bool
size :: Set a -> Int
-- Implementation
newtype Set a = Set [a]
```

Abstract Data Types

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empty = Set []
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size :: Set a -> Int
-- Implementation
newtype Set a = Set [a]

empty = Set []
insert x (Set xs) = Set (if elem x xs then xs else x:xs)
```

Abstract Data Types

Example

```
module Set (Set, empty, insert, isin, size) where
-- Interface
empty :: Set a
insert :: Eq a => a -> Set a -> Set a
isin :: Eq a => a -> Set a -> Bool
size :: Set a -> Int
-- Implementation
newtype Set a = Set [a]

empty = Set []
insert x (Set xs) = Set (if elem x xs then xs else x:xs)
isin x (Set xs) = elem x xs
```

Abstract Data Types

Example

```
module Set (Set, empty, insert, isin, size) where
-- Interface
empty :: Set a
insert :: Eq a => a -> Set a -> Set a
isin :: Eq a => a -> Set a -> Bool
size :: Set a -> Int
-- Implementation
newtype Set a = Set [a]

empty = Set []
insert x (Set xs) = Set (if elem x xs then xs else x:xs)
isin x (Set xs) = elem x xs
size (Set xs) = length xs
```

`type` vs `data` vs `newtype`

- `type` is used to create type aliases

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- `data` is used to create algebraic data types (types witha custom shape)

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`type` vs `data` vs `newtype`

- `type` is used to create type aliases
- `data` is used to create algebraic data types (types witha custom shape)
- `newtypre` is used to create a custom constructor for a single type
  - syntax "subset" of the syntax for `data`
  - may only have a *single* constructor taking a *single* argument
  - introduces no runtime overhead
  - creates *strict* types while `data` creates *lazy* types

2.5 Type inference

How to infer/reconstruct the type of an expression.

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Given an expression e.

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3. for each sub-expression in e set up an equation linking the type of parameters and arguments

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Given an expression e.

1. give all variables in e distinct type variables
2. give each function $f :: T$ in e a new general type with fresh type variables
3. for each sub-expression in e set up an equation linking the type of parameters and arguments
4. simplify the set of equations by replacing equivalences

2.5 Type inference

Example

Given `f u v = min (head u) (last (concat v))`

2.5 Type inference

Example

Given `f u v = min (head u) (last (concat v))`

Step 1

2.5 Type inference

Example

Given `f u v = min (head u) (last (concat v))`

Step 1

1. `u :: a`
2. `v :: b`

Step 2

2.5 Type inference

Example

Given `f u v = min (head u) (last (concat v))`

Step 1

1. `u :: a`
2. `v :: b`

Step 2

1. `head :: [c] -> c`

2.5 Type inference

Example

Given `f u v = min (head u) (last (concat v))`

Step 1

1. `u :: a`
2. `v :: b`

Step 2

1. `head :: [c] -> c`
2. `concat :: [[d]] -> [d]`

2.5 Type inference

Example

Given `f u v = min (head u) (last (concat v))`

Step 1

1. `u :: a`
2. `v :: b`

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1. `head :: [c] -> c`
2. `concat :: [[d]] -> [d]`
3. `last :: [e] -> e`

2.5 Type inference

Example

Given `f u v = min (head u) (last (concat v))`

Step 1

1. `u :: a`
2. `v :: b`

Step 2

1. `head :: [c] -> c`
2. `concat :: [[d]] -> [d]`
3. `last :: [e] -> e`
4. `min :: Ord f => f -> f -> f`

2.5 Type inference

Example

Given `f u v = min (head u) (last (concat v))`

Step 3

2.5 Type inference

Example

Given `f u v = min (head u) (last (concat v))`

Step 3

1. from `head u` derive `[c] = a`

2.5 Type inference

Example

Given `f u v = min (head u) (last (concat v))`

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1. from `head u` derive `[c] = a`
2. from `concat v` derive `[[d]] = b`

2.5 Type inference

Example

Given `f u v = min (head u) (last (concat v))`

Step 3

1. from `head u` derive `[c] = a`
2. from `concat v` derive `[[d]] = b`
3. from `last (concat v)` derive `[e] = [d]`

2.5 Type inference

Example

Given `f u v = min (head u) (last (concat v))`

Step 3

1. from `head u` derive `[c] = a`
2. from `concat v` derive `[[d]] = b`
3. from `last (concat v)` derive `[e] = [d]`
4. from `min (head u) (last (concat v))` derive `f = c` and `f = e`

2.5 Type inference

Example

Given `f u v = min (head u) (last (concat v))`
Goal `f :: a -> b -> f`

Step 4

2.5 Type inference

Example

Given `f u v = min (head u) (last (concat v))`
Goal `f :: a -> b -> f`

Step 4

1. apply `[c] = a` and update

2.5 Type inference

Example

Given `f u v = min (head u) (last (concat v))`
Goal `f :: a -> b -> f`

Step 4

1. apply `[c] = a` and update
   - `u :: [c]`

2.5 Type inference

Example

Given `f u v = min (head u) (last (concat v))`
Goal `f :: a -> b -> f`

Step 4

1. apply `[c] = a` and update
   - `u :: [c]`
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Example

Given `f u v = min (head u) (last (concat v))`
Goal `f :: a -> b -> f`

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Given `f u v = min (head u) (last (concat v))`
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1. apply `[c] = a` and update
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   - `v :: [[d]]`
3. apply `[e] = [d]` to get `e = d` and update

2.5 Type inference

Example

Given `f u v = min (head u) (last (concat v))`
Goal `f :: a -> b -> f`

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1. apply `[c] = a` and update
   - `u :: [c]`
2. apply `[[d]] = b` and update
   - `v :: [[d]]`
3. apply `[e] = [d]` to get `e = d` and update
   - `v :: [[e]]`

2.5 Type inference

Example

Given `f u v = min (head u) (last (concat v))`

Goal `f :: a -> b -> f`

Step 4

1. apply `[c] = a` and update
   - `u :: [c]`
2. apply `[[d]] = b` and update
   - `v :: [[d]]`
3. apply `[e] = [d]` to get `e = d` and update
   - `v :: [[e]]`
   - `concat :: [[e]] -> [e]`

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Example

Given `f u v = min (head u) (last (concat v))`
Goal `f :: a -> b -> f`

Step 4

1. apply `[c] = a` and update
   - `u :: [c]`
2. apply `[[d]] = b` and update
   - `v :: [[d]]`
3. apply `[e] = [d]` to get `e = d` and update
   - `v :: [[e]]`
   - `concat :: [[e]] -> [e]`
4. apply `f = c` and update

2.5 Type inference

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Given `f u v = min (head u) (last (concat v))`
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1. apply `[c] = a` and update
   - `u :: [c]`
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   - `v :: [[d]]`
3. apply `[e] = [d]` to get `e = d` and update
   - `v :: [[e]]`
   - `concat :: [[e]] -> [e]`
4. apply `f = c` and update
   - `u :: [f]`

2.5 Type inference

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Given `f u v = min (head u) (last (concat v))`
Goal `f :: a -> b -> f`

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   - `u :: [c]`
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   - `v :: [[d]]`
3. apply `[e] = [d]` to get `e = d` and update
   - `v :: [[e]]`
   - `concat :: [[e]] -> [e]`
4. apply `f = c` and update
   - `u :: [f]`
   - `head :: [f] -> f`

2.5 Type inference

Example

Given `f u v = min (head u) (last (concat v))`

Goal `f :: a -> b -> f`

Step 4 (cont.)

2.5 Type inference

Example

Given `f u v = min (head u) (last (concat v))`

Goal `f :: a -> b -> f`

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1. apply `f = e` and update

2.5 Type inference

Example

Given `f u v = min (head u) (last (concat v))`

Goal `f :: a -> b -> f`

Step 4 (cont.)

1. apply `f = e` and update
   - `v :: [[f]]`

2.5 Type inference

Example

Given `f u v = min (head u) (last (concat v))`

Goal `f :: a -> b -> f`

Step 4 (cont.)

1. apply `f = e` and update
   - `v :: [[f]]`
   - `concat :: [[f]] -> [f]`

2.5 Type inference

Example

Given `f u v = min (head u) (last (concat v))`

Goal `f :: a -> b -> f`

Step 4 (cont.)

1. apply `f = e` and update
    - `v :: [[f]]`
    - `concat :: [[f]] -> [f]`
    - `last :: [[f]] -> [f]`

2.5 Type inference

Example

Given `f u v = min (head u) (last (concat v))`

Goal `f :: a -> b -> f`

Step 4 (cont.)

1. apply `f = e` and update
   - `v :: [[f]]`
   - `concat :: [[f]] -> [f]`
   - `last :: [[f]] -> [f]`
2. no further simplification possible,
   return `f :: Ord f => [f] -> [[f]] -> f`