Clase 10 - Análisis Matemático 1 - LC: Límites III

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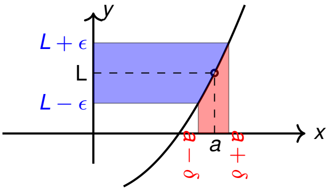
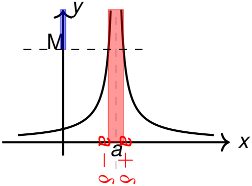
Índice

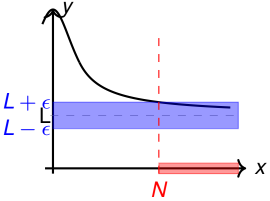
$$\lim_{x\to a} f(x) = L \text{ si } \forall \epsilon > 0, \exists \delta > 0 \;/$$
$$\text{si } 0 < |x-a| < \delta \;\Rightarrow\; |f(x) - L| < \epsilon$$



$$\lim_{x\to a} f(x) = \infty \text{ si } \forall M > 0, \exists \delta > 0 \;/$$
$$\text{si } 0 < |x-a| < \delta \;\Rightarrow\; f(x) > M$$



$$\lim_{x\to \infty} f(x) = L \text{ si } \forall \epsilon > 0, \exists\, N > 0 \;/$$
$$\text{si } x > N \Rightarrow\; |f(x) - L| < \epsilon$$



$$\lim_{x\to \infty} f(x) = \infty \text{ si } \forall M > 0, \exists\, N > 0 \;/$$
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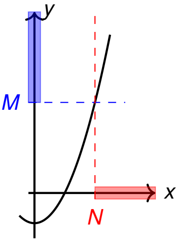

Acotación

Tipos de indeterminaciones

$$\frac{0}{0} \qquad \frac{\pm\infty}{\pm\infty} \qquad 0.\infty \qquad (\infty - \infty)$$

Por ahora...

¿Qué hago si es indeterminado? OPERAR!

Denoto Primero:

$\lim_{x \to a} F(x) \longrightarrow \text{ind}$

No Pongo =

Ejercicio 7d

$$\lim_{x \to \infty} \sqrt{18x^2 + 1} \frac{1}{\sqrt{32x^2 - 3}}$$

Ejercicio 7d

$$\lim_{x\to\infty} \underbrace{\sqrt{18x^2+1}}_{\to\infty} \underbrace{\frac{1}{\sqrt{32x^2-3}}}_{\to 0}$$

Ejercicio 7d

$$\lim_{x\to\infty} \underbrace{\underbrace{\sqrt{18x^2+1}}_{\to\infty} \underbrace{\frac{1}{\sqrt{32x^2-3}}}_{\to 0}}_{\to(\infty).(0)\ \textit{INDETERMINADO}}$$

$$\sqrt{18x^2+1}\,\frac{1}{\sqrt{32x^2-3}} = \frac{\sqrt{18x^2+1}}{\sqrt{32x^2-3}} = \sqrt{\frac{18x^2+1}{32x^2-3}}$$

$$\lim_{x\to\infty}\sqrt{\frac{18x^2+1}{32x^2-3}}\ \textit{si}\ \exists \Rightarrow = \sqrt{\lim_{x\to\infty}\frac{18x^2+1}{32x^2-3}} \qquad \to \frac{\infty}{\infty}\ \textit{INDETERMINADO}$$

$$\sqrt{\lim_{x\to\infty}\frac{18x^2+1}{32x^2-3}} = \sqrt{\lim_{x\to\infty}\frac{x^2(18+\frac{1}{x^2})}{x^2(32-\frac{3}{x^2})}} = \sqrt{\lim_{x\to\infty}\frac{(18+\frac{1}{x^2})}{(32-\frac{3}{x^2})}}$$

Ejercicio 7d

$$\underbrace{\lim_{x\to\infty} \underbrace{\sqrt{18x^2+1}}_{\to\infty} \underbrace{\frac{1}{\sqrt{32x^2-3}}}_{\to 0}}_{\to(\infty).(0)\ \textit{INDETERMINADO}}$$

$$\sqrt{18x^2+1}\frac{1}{\sqrt{32x^2-3}} = \frac{\sqrt{18x^2+1}}{\sqrt{32x^2-3}} = \sqrt{\frac{18x^2+1}{32x^2-3}}$$

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$$\sqrt{\lim_{x\to\infty}\frac{18x^2+1}{32x^2-3}} = \sqrt{\lim_{x\to\infty}\frac{x^2(18+\frac{1}{x^2})}{x^2(32-\frac{3}{x^2})}} = \sqrt{\lim_{x\to\infty}\frac{(18+\overbrace{\frac{1}{x^2}}^{\to 0})}{(32-\underbrace{\frac{3}{x^2}}_{\to 0})}}$$

Ejercicio 7d

$$\lim_{x\to\infty} \underbrace{\underbrace{\sqrt{18x^2+1}}_{\to\infty} \underbrace{\frac{1}{\sqrt{32x^2-3}}}_{\to 0}}_{\to(\infty).(0)\ \textit{INDETERMINADO}}$$

$$\sqrt{18x^2+1}\,\frac{1}{\sqrt{32x^2-3}} = \frac{\sqrt{18x^2+1}}{\sqrt{32x^2-3}} = \sqrt{\frac{18x^2+1}{32x^2-3}}$$

$$\lim_{x\to\infty} \sqrt{\frac{18x^2+1}{32x^2-3}} \ \textit{si}\ \exists \Rightarrow = \sqrt{\lim_{x\to\infty} \frac{18x^2+1}{32x^2-3}} \qquad \to \frac{\infty}{\infty}\ \textit{INDETERMINADO}$$

$$\sqrt{\lim_{x\to\infty} \frac{18x^2+1}{32x^2-3}} = \sqrt{\lim_{x\to\infty} \frac{x^2(18+\frac{1}{x^2})}{x^2(32-\frac{3}{x^2})}} = \sqrt{\lim_{x\to\infty} \frac{\overbrace{(18+\overbrace{\frac{1}{x^2}}^{\to 0})}^{\to 18}}{\underbrace{(32-\underbrace{\frac{3}{x^2}}_{\to 0})}_{\to 32}}} = \sqrt{\frac{18}{32}} = \sqrt{\frac{9}{16}} = \frac{\sqrt{9}}{\sqrt{16}} = \frac{3}{4}$$

Ejercicio

$$\lim_{x \to -2} \frac{x^2+x-2}{7(x+2)} = \frac{\overbrace{x^2+x-2}^{\to 0}}{\underbrace{7(x+2)}_{\to 0}} \to \frac{0}{0} \; \textit{INDETERMINADO}$$

Factorizar el num: Baskhara $x_{1,2} = \dfrac{-1 \pm \sqrt{1+8}}{2} = \dfrac{-1 \pm 3}{2} \Rightarrow \; x_1 = 1$ y $x_2 = -2$

$$\lim_{x \to -2} \frac{x^2+x-2}{7(x+2)} = \lim_{x \to -2} \frac{(x-1)\cancel{(x+2)}}{7\cancel{(x+2)}}$$

$$\lim_{x \to -2} \frac{x^2+x-2}{7(x+2)} = \lim_{x \to -2} \frac{\overbrace{x-1}^{\to -3}}{\underbrace{7}_{\to 7}} = -\frac{3}{7}$$

Ejercicio

Determinar las asíntotas verticales del gráfico de f (si tuviera)

$$f(x) = \frac{x^2 - x - 2}{x^2 - 2x - 3}$$

Puntos que no pertenecen al dominio? $Dom\ f = \{x \in \mathbb{R} \ / \ x^2 - 2x - 3 \neq 0\}$

Baskhara: $x_{1,2} = \dfrac{2 \pm \sqrt{4+12}}{2} = \dfrac{2 \pm 4}{2}$

$\Rightarrow \underbrace{x_1 = 3 \text{ y } x_2 = -1}_{potenciales\ A.V.}$ $\qquad$ $Dom\ f = \mathbb{R} - \{-1, 3\}$

Calcular $\lim\limits_{x \to -1^-} f(x)$, $\lim\limits_{x \to -1^+} f(x)$, $\lim\limits_{x \to 3^-} f(x)$ y $\lim\limits_{x \to 3^+} f(x)$

$$\lim_{x \to -1^-} \frac{\overbrace{x^2 - x - 2}^{\to 0}}{\underbrace{(x+1)(x-3)}_{\to 0.(-4) \to 0}} \to \frac{0}{0}\ INDETERMINADO$$

$Bskh.\ num:\ x_1 = 2 \text{ y } x_2 = -1$

$$\lim_{x \to -1^-} \frac{x^2 - x - 2}{(x+1)(x-3)} = \lim_{x \to -1^-} \frac{\cancel{(x+1)}(x-2)}{\cancel{(x+1)}(x-3)} = \lim_{x \to -1^-} \frac{\overbrace{x-2}^{\to -3}}{\underbrace{x-3}_{\to -4}} = \frac{3}{4}$$

$$\lim_{x \to -1^+} f(x) = \frac{3}{4}$$

$$\boxed{x = 1\ NO\ es\ A.V.}$$

Ejercicio

Determinar las asíntotas verticales del gráfico de f (si tuviera)

$$f(x) = \frac{x^2 - x - 2}{x^2 - 2x - 3}$$

$$\lim_{x \to 3^-} \frac{x^2 - x - 2}{(x+1)(x-3)} = \lim_{x \to 3^-} \frac{\overbrace{x-2}^{\to 1}}{\underbrace{x-3}_{\to 0}} \to \infty \; o \; -\infty$$

$$\lim_{\underbrace{x \to 3^-}_{x<3 \to x-3<0}} \frac{\overbrace{x-2}^{\to 1}}{\underbrace{x-3}_{\to 0,\ <0}} = -\infty$$

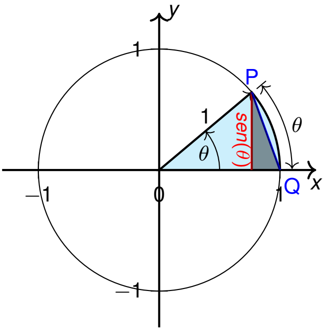
$$\boxed{x = 3 \; es \; A.V.}$$

Calcular $\lim_{x \to 3^+} f(x)$

Límites de funciones trigonométricas cerca del 0

$$\lim_{\theta \to 0} sen(\theta) \qquad \text{y} \qquad \lim_{\theta \to 0} cos(\theta)$$

$$\lim_{\theta \to 0} sen(\theta)$$



$$0 < \theta < \frac{\pi}{2}$$

$\overline{PQ} \leq \theta$

$sen(\theta) \leq \overline{PQ}$

$$sen(\theta) \leq \theta$$

$-\frac{\pi}{2} < \theta < 0$

$$sen(-\theta) \leq -\theta$$

$$-sen(\theta) \leq -\theta$$

$$-\frac{\pi}{2} < \theta < \frac{\pi}{2} \qquad -|\theta| \leq sen(\theta) \leq |\theta|$$

Teo del sandwich:

$$\underbrace{\lim_{\theta \to 0} -|\theta|}_{\to 0} \leq \lim_{\theta \to 0} sen(\theta) \leq \underbrace{\lim_{\theta \to 0} |\theta|}_{\to 0}$$

$$\boxed{\lim_{\theta \to 0} sen(\theta) = 0}$$

Límites de funciones trigonométricas cerca del 0

$$\lim_{\theta \to 0} sen(\theta) \qquad \text{y} \qquad \lim_{\theta \to 0} cos(\theta)$$

$$\lim_{\theta \to 0} sen(\theta) = 0$$

$$\lim_{\theta \to 0} \underbrace{cos(\theta)}_{=\sqrt{1-sen^2(\theta)}} = \lim_{\theta \to 0} \sqrt{1 - \underbrace{sen^2(\theta)}_{\to 0}} =$$

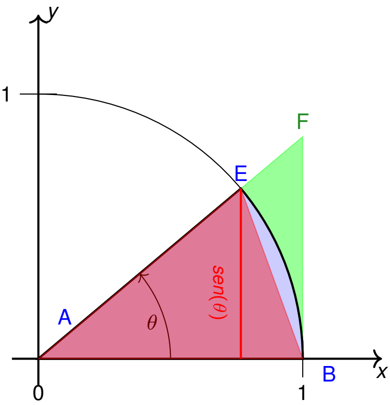
$$\lim_{\theta \to 0} cos(\theta) = 1$$

Límites notables

$$\lim_{\theta \to 0} \frac{sen(\theta)}{\theta} = 1 \qquad \lim_{\theta \to 0} \frac{cos(\theta) - 1}{\theta} = 0 \qquad \lim_{\theta \to 0} \frac{tan(\theta)}{\theta} = 1$$

Límites notables

$$\lim_{\theta \to 0} \frac{sen(\theta)}{\theta}$$



$$Area \triangle ABE \leq Area \triangleleft ABE \leq Area \triangle ABF$$

$$\text{Area} \triangle = \frac{\text{base.altura}}{2} \qquad \text{Area} \triangleleft = \frac{\theta}{2} r^2$$

$$Area \triangle ABE = \frac{\overline{AB}.sen(\theta)}{2} = \frac{sen(\theta)}{2}$$

$$Area \triangleleft ABE = \frac{\theta}{2}$$

$$Area \triangle ABF = \frac{\overline{AB}.\overline{BF}}{2} = \frac{1.tan(\theta)}{2}$$

$$sen(\theta) \leq \theta \leq tan(\theta)$$

$$\frac{sen(\theta)}{sen(\theta)} \leq \frac{\theta}{sen(\theta)} \leq \frac{tan(\theta)}{sen(\theta)} \Rightarrow 1 \leq \frac{\theta}{sen(\theta)} \leq \frac{1}{cos(\theta)}$$

$$\underbrace{\lim_{\theta \to 0^+} 1}_{\to 1} \leq \lim_{\theta \to 0^+} \frac{\theta}{sen(\theta)} \leq \underbrace{\lim_{\theta \to 0^+} \frac{1}{cos(\theta)}}_{\to 1}$$

$$\lim_{\theta \to 0^+} \frac{\theta}{sen(\theta)} = 1 \qquad \Rightarrow \boxed{\lim_{\theta \to 0^+} \frac{sen(\theta)}{\theta} = 1}$$

Límites notables - Ejercicios

$$\lim_{\theta\to 0}\frac{sen(\theta)}{\theta}=1 \qquad \lim_{\theta\to 0}\frac{cos(\theta)-1}{\theta}=0 \qquad \lim_{\theta\to 0}\frac{tan(\theta)}{\theta}=1$$

$$\lim_{x\to 0}\frac{sen(5x)}{2x}=\lim_{x\to 0}\frac{\overbrace{sen(5x)}^{\to 0}}{\underbrace{2x}_{\to 0}}\left(\to\frac{0}{0}\ IND\right)=\lim_{x\to 0}\frac{sen(5x)}{2x}\frac{5}{5}=\lim_{x\to 0}\frac{sen(5x)}{5x}\frac{5}{2}=$$

$$\frac{5}{2}\lim_{x\to 0}\frac{sen(5x)}{5x}$$

Cambio de variable $u=5x \Rightarrow x\to 0, u\to 0$

$$=\frac{5}{2}\underbrace{\lim_{u\to 0}\frac{sen(u)}{u}}_{Lim.Notable:\to 1}=\frac{5}{2}$$

Límites notables - Ejercicios

$$\lim_{\theta\to 0}\frac{sen(\theta)}{\theta}=1 \qquad \lim_{\theta\to 0}\frac{cos(\theta)-1}{\theta}=0 \qquad \lim_{\theta\to 0}\frac{tan(\theta)}{\theta}=1$$

$$\lim_{x\to 0}\frac{tan(x)}{sen(x)}=\lim_{x\to 0}\frac{\overbrace{tan(x)}^{\to 0}}{\underbrace{sen(x)}_{\to 0}}\left(\to\frac{0}{0}\ IND.\right)$$

$$\lim_{x\to 0}\frac{tan(x)}{sen(x)}=\lim_{x\to 0}\frac{tan(x)}{sen(x)}\frac{x}{x}=\lim_{x\to 0}\frac{tan(x)}{x}\frac{x}{sen(x)}=\lim_{x\to 0}\frac{tan(x)}{x}\frac{1}{\frac{sen(x)}{x}}$$

$$=\lim_{x\to 0}\frac{\overbrace{\frac{tan(x)}{x}}^{L.N.\to 1}}{\underbrace{\frac{sen(x)}{x}}_{L.N.\to 1}}=\frac{\lim_{x\to 0}\frac{tan(x)}{x}}{\lim_{x\to 0}\frac{sen(x)}{x}}=\frac{1}{1}=\boxed{1}$$

Límites notables - Ejercicios

$$\lim_{\theta \to 0} \frac{sen(\theta)}{\theta} = 1 \qquad \lim_{\theta \to 0} \frac{cos(\theta) - 1}{\theta} = 0 \qquad \lim_{\theta \to 0} \frac{tan(\theta)}{\theta} = 1$$

$$\lim_{x \to \frac{\pi}{2}} \frac{\frac{\pi}{2} - x}{cos(x)} = \lim_{x \to \frac{\pi}{2}} \frac{\overbrace{\frac{\pi}{2} - x}^{\to 0}}{\underbrace{cos(x)}_{\to 0}} \to \frac{0}{0} \; \mathit{INDETERMINADO}$$

Cambio de variable $u = \frac{\pi}{2} - x$ $\qquad x \to \frac{\pi}{2} \Rightarrow u \to 0$ $\qquad x = \frac{\pi}{2} - u$

$$cos(x) = cos\left(\frac{\pi}{2} - u\right) = \underbrace{cos\left(\frac{\pi}{2}\right)}_{=0} cos(u) + \underbrace{sen\left(\frac{\pi}{2}\right)}_{=1} sen(u) = sen(u)$$

$$\lim_{x \to \frac{\pi}{2}} \frac{\frac{\pi}{2} - x}{cos(x)} = \lim_{u \to 0} \frac{u}{sen(u)} = \lim_{u \to 0} \frac{1}{\frac{sen(u)}{u}} = \frac{\lim_{u \to 0} 1}{\lim_{u \to 0} \frac{sen(u)}{u}} = 1$$

FIN