Reading Summary 1–A-optimality

Jingning Yao

Problem discussed

- A-optimality for a linear regression model $y = X\beta + \epsilon$.
- Consider the problem of determining optimal block designs in which ν treatments are to be applied to experimental units arranged in b blocks of size k. And A-optimality for such block design.

A-optimality for linear regression model

Consider the linear regression model

$$y = \mathbf{X}\beta + \epsilon$$

where y is an $m \times 1$ vector of observations, $\mathbf{X}$ is an $m \times n$ design matrix, β is an $n \times 1$ vector of unknown parameters. ϵ is a an $m \times 1$ vector of random variables with mean the $m \times 1$ zero vector and known covariance matrix Λ. Assume that $m \geqslant n$ and denote the eigenvalues of Λ in increasing order:

$$\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n \cdots \leqslant \lambda_m$$

.

For a design matrix $\mathbf{X}$ if rank n, based on observation y, an unbiased simple least squares estimation of parameter β is

$$(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'y$$

and covariance matrix is given by

$$(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\Lambda\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}$$

To choose $\mathbf{X}$ from a given experimental region, A-optimality try to find the minimum of the trace of covariance matrix.

The experimental region under consideration is taken to be the set H of all $m \times n$ real matrices of rank n whose ith column has a Euclidean norm not exceeding given positive numbers $c_i, i = 1, \dots, n$.

a basic inequality

For matrix Λ there exists an orthogonal matrix P and a diagonal matrix Λ_m such that

$$\Lambda = P' \Lambda_m P$$

and for any X in H

$$tr\{(X'X)^{-1} X' \Lambda X (X'X)^{-1}\} \geqslant (\sum_{i=1}^{n} c_i^2)^{-1} (\sum_{i=1}^{n} \lambda_i^{1/2})^2$$

and by the definition of the set H

$$tr\{X'X\} \leqslant \sum_{i=1}^{n} c_i^2$$

(Chan 1982)

A-Optimal designs

So the main goal is to obtain the matrix in H such that the lower bond is attained. the article only gives the theorem of existence:

Suppose that the positive numbers $c_i, i = 1, \dots, n$ are arranged in ascending order of magnitude and that the smallest eigenvalue λ_1 of the covariance matrix Λ is positive. Then there is an X in H such that

$$tr\{(X'X)^{-1} X' \Lambda X (X'X)^{-1}\} = (\sum_{i=1}^{n} c_i^2)^{-1} (\sum_{i=1}^{n} \lambda_i^{1/2})^2$$

if and only if

$$(\sum_{i=1}^{n} c_i^2)^{-1} \sum_{i=1}^{k} c_i^2 \geqslant (\sum_{i=1}^{n} \lambda_i^2)^{-1} \sum_{i=1}^{k} \lambda_i^2, \quad k = 1, \dots, n-1$$

(Chan 1982)

A-optimality for block design

Here size of blocks $k = \nu - 1$ or $k = \nu + 1$

Some notatinos

Use d to denote some particular block design that can be used. Let $N_d = (n_{dij})$ denote the treatment block incidence matrix of d; and n_{dij} is the number of experimental units to which treatment i is applied in block j. The ith row sum of N_d is denoted by r_{di} and represents the number of times treatment i is replicated in d. $N_d N_d'$ is called the concurrence matrix of d.(Jacroux 1989).

Now for observations Y_{ijp} obtained by applying treatment i to an experimental unit occurring in block j. Assumed model is:

$$Y_{ijp} = \alpha_i + \beta_j + E_{ijp}, \quad 0 \leq i \leq \nu, \quad 1 \leq j \leq b, \quad 0 \leq p \leq n_{dij}$$

where α_i is the effect of the ith treatment, β_j in the effect of the jth block, and the E_{ijp} are independent random variables having expectation 0 and constant variance σ^2

The coefficient matrix of the reduced normal equation for obtaining the least squares estimates of the treatment effects in d can be written as

$$C_d = diag(r_{d0}, r_{d1}, \dots, r_{d\nu}) - (1/k) N_d N_d'$$

where $diag(r_{d0}, r_{d1}, \dots, r_{d\nu})$ denotes a $(\nu + 1) \times (\nu + 1)$ diagonal matrix. The matrix C_d is called the C-matrix of d and is positive-semidefinite with zero row sums, having rank $\nu - 1$

Use $D(\nu, b, k)$ to denote the class of all connected block designs having ν treatments arranged in b blocks of size k.

For $d \in D(\nu, b, k)$ let $0 = z_{d0} < z_{d1} \leqslant \cdots \leqslant z_{d,\nu-1}$ denote the nonzero eigen values of C_d(Jacroux 1992)

References

Chan, NN. 1982. "A-Optimality for Regression Designs." *Journal of Mathematical Analysis and Applications* 87 (1): 45–50.

Jacroux, Mike. 1989. "The a-Optimality of Block Designs for Comparing Test Treatments with a Control." *Journal of the American Statistical Association* 84 (405): 310–17.

———. 1992. "On the a-Optimality of Block Designs." *Journal of Statistical Planning and Inference* 32 (3): 401–15.