



## Worksheet 7 (Solved)

HoTTEST Summer School 2022

The HoTTEST TAs

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### 1 (★)

Consider two embeddings  $f : A \hookrightarrow B$  and  $g : B \hookrightarrow C$ . Construct a function

$$\text{is-equiv}(g \circ f) \rightarrow (\text{is-equiv}(f) \times \text{is-equiv}(g)).$$

**Suppose that  $g \circ f$  is an equivalence. By the 3-for-2 property of equivalences, it suffices to prove that  $f$  is an equivalence. Define  $\psi : B \rightarrow A$  by**

$$\psi(b) := (g \circ f)^{-1}(g(b)).$$

**For every  $b : B$ ,**

$$g(f(\psi(b))) = g(b).$$

**Since  $g$  is an embedding, this implies that**

$$f(\psi(b)) = b.$$

**Moreover,  $\psi(f(a)) = a$  for all  $a : A$ . Thus,  $\psi$  is an inverse of  $f$ .**

## 2 (★★)

1. Let  $A$  be a type. Prove that the canonical map  $\emptyset \xrightarrow{!_A} A$  is an embedding.
2. Let  $A$  and  $B$  be types. Prove that the inclusions  $\text{inl} : A \rightarrow A + B$  and  $\text{inr} : B \rightarrow A + B$  are embeddings.
3. Let  $A$  and  $B$  be types. Prove that  $\text{inl} : A \rightarrow A + B$  is an equivalence if and only if  $B \simeq \emptyset$ .

Conclude that if both  $A$  and  $B$  are contractible, then  $A + B$  is *not* contractible.

1. For every  $x, y : \emptyset$ , we must prove that

$$\text{ap}_{!_A}(x, y) : (x = y) \rightarrow (!_A(x) = !_A(y))$$

is an equivalence. This follows directly from induction on  $\emptyset$ .

2. Let  $x, y : A$  and consider the map

$$\text{ap}_{\text{inl}}(x, y) : (x = y) \rightarrow (\text{inl}(x) = \text{inl}(y)).$$

Recall from Lecture 7 the family

$$\text{eq-id} : \prod_{s, t : A+B} (s = t) \rightarrow \text{Eq}_{A+B}(s, t)$$

of equivalences, defined by path induction. It is easy to check that  $\text{ap}_{\text{inl}}(x, y)$  is a section of  $\text{eq-id}_{\text{inl}(x), \text{inl}(y)}$ . Since the latter is an equivalence, it follows that  $\text{ap}_{\text{inl}}(x, y)$  is actually an equivalence with inverse  $\text{eq-id}_{\text{inl}(x), \text{inl}(y)}$ . This proves that  $\text{inl}$  is an embedding.

Similarly, for any  $x, y : B$ , the map  $\text{ap}_{\text{inr}}(x, y)$  is an equivalence with inverse  $\text{eq-id}_{\text{inr}(x), \text{inr}(y)}$ . Thus,  $\text{inr}$  is also an embedding.

3. Suppose that  $\text{inl}$  is an equivalence with inverse  $\psi : A + B \rightarrow A$ . Let  $b : B$ . Then  $\text{inl}(\psi(\text{inr}(b))) = \text{inr}(b)$ . But recall that

$$(\text{inl}(\psi(\text{inr}(b))) = \text{inr}(b)) \simeq \emptyset.$$

This gives us an element of  $\emptyset$  and thus an element of  $\neg B$ .

Conversely, consider an equivalence  $e : B \rightarrow \emptyset$ . Define the function  $\varphi : A + B \rightarrow A$  by

$$\begin{aligned} \varphi(\text{inl}(a)) &:= a \\ \varphi(\text{inr}(b)) &:= \text{ind}_{\emptyset}(e(b)). \end{aligned}$$

It is easy to check that  $\varphi$  is an inverse of  $\text{inl}$ .

Now, suppose that both  $A$  and  $B$  are contractible. Also, suppose that  $A + B$  is contractible. We have a commuting triangle

$$\begin{array}{ccc} A & \xrightarrow{\text{inl}} & A + B \\ & \searrow \simeq & \swarrow \simeq \\ & \mathbb{1} & \end{array},$$

so that  $\text{inl}$  is an equivalence. This implies that  $B \simeq \emptyset$ . Since  $B$  is contractible, this gives us an element of  $\emptyset$ . Therefore,  $A + B$  is not contractible.

### 3 (★★)

Consider a commuting triangle

$$\begin{array}{ccc} A & \xrightarrow{h} & B \\ & \searrow f & \swarrow g \\ & X & \end{array} .$$

1. Suppose that  $g$  is an embedding. Prove that  $f$  is an embedding if and only if  $h$  is one.
2. Suppose that  $h$  is an equivalence. Prove that  $f$  is an embedding if and only if  $g$  is one.

**Note that for every  $x, y : A$ , we have a commuting square**

$$\begin{array}{ccc} x = y & \xrightarrow{\text{ap}_h(x, y)} & h(x) = h(y) \\ \text{ap}_f(x, y) \downarrow & & \downarrow \text{ap}_g(h(x), h(y)) \\ f(x) = f(y) & \xrightarrow{\simeq} & g(h(x)) = g(h(y)) \end{array} .$$

1. We have that  $\text{ap}_g(h(x), h(y))$  is an equivalence. Suppose that  $f$  is an embedding. Then  $\text{ap}_h(x, y)$  is an equivalence, and thus  $h$  is an embedding.

Conversely, suppose that  $h$  is an embedding. Then  $\text{ap}_f(x, y)$  is an equivalence, and thus  $f$  is an embedding.

2. Suppose that  $f$  is an embedding. Then  $\text{ap}_{g \circ h}(x, y)$  is an equivalence. Further, the triangle

$$\begin{array}{ccc} B & \xrightarrow{h^{-1}} & A \\ & \searrow g & \swarrow g \circ h \\ & X & \end{array}$$

commutes, and  $h^{-1}$  is an embedding because it's an equivalence. By part (1), it follows that  $g$  is an embedding.

Conversely, suppose that  $g$  is an embedding. Then  $\text{ap}_f(x, y)$  is an equivalence. As  $h$  is also an embedding, so is  $f$ .

## 4 (★★)

Let  $A$ ,  $B$ , and  $C$  be types and let  $f : A \rightarrow C$  and  $g : B \rightarrow C$  be maps. Prove that the following are logically equivalent.

1. The map  $[f, g] : A + B \rightarrow C$  is an embedding.
2. Both  $f$  and  $g$  are embeddings, and  $f(a) \neq g(b)$  for all  $a : A$  and  $b : B$ .

**Suppose that  $[f, g]$  is an embedding. Then  $f \doteq [f, g] \circ \text{inl}$  is an embedding as the composite of two embeddings. Likewise,  $g$  is an embedding. Let  $a : A$  and  $b : B$  and suppose that  $f(a) = g(b)$ . Since  $[f, g]$  is an embedding,**

$$\text{inl}(a) = \text{inr}(b).$$

**But  $(\text{inl}(a) = \text{inr}(b)) \simeq \emptyset$ , and thus  $f(a) \neq g(b)$ .**

**Conversely, suppose that both  $f$  and  $g$  are embeddings and that**

$$\tau : \prod_{a:A} \prod_{b:B} f(a) \neq g(b).$$

**We must show that**

$$\text{ap}_{[f,g]}(s, t) : (s = t) \rightarrow ([f, g](s) = [f, g](t))$$

**is an equivalence for all  $s, t : A + B$ . Define  $\psi_{s,t} : ([f, g](s) = [f, g](t)) \rightarrow (s = t)$  by double induction on  $A + B$ :**

$$\psi_{\text{inl}(a), \text{inl}(a')}(p) := \text{ap}_f(a, a')^{-1}(p)$$

$$\psi_{\text{inl}(a), \text{inr}(b)}(p) := \text{ind}_{\emptyset}(\tau_{a,b}(p))$$

$$\psi_{\text{inr}(b), \text{inr}(b')}(p) := \text{ap}_g(b, b')^{-1}(p)$$

$$\psi_{\text{inr}(b), \text{inl}(a)}(p) := \text{ind}_{\emptyset}(\tau_{a,b}(p^{-1})).$$

**By double induction on  $A + B$ , it's easy to prove that  $\psi_{s,t}$  is an inverse of  $\text{ap}_{[f,g]}(s, t)$ .**

**5**    **(★★)**

1. Let  $f, g : \prod_{x:A} B(x) \rightarrow C(x)$ . Construct a function

$$\left( \prod_{x:A} f(x) \sim g(x) \right) \rightarrow (\text{tot}(f) \sim \text{tot}(g)).$$

2. Let  $f : \prod_{x:A} B(x) \rightarrow C(x)$  and  $g : \prod_{x:A} C(x) \rightarrow D(x)$ . Construct a homotopy

$$\text{tot}(\lambda x. g(x) \circ f(x)) \sim \text{tot}(g) \circ \text{tot}(f).$$

3. For any type family  $B$  over  $A$ , construct a homotopy

$$\text{tot}(\lambda x. \text{id}_{B(x)}) \sim \text{id}_{\sum_{x:A} B(x)}.$$

4. Let  $a : A$  and let  $B$  be a type family over  $A$ . Prove that if  $B(x)$  is a retract of  $a = x$  for each  $x : A$ , then  $(a = x) \simeq B(x)$  for each  $x : A$ .
5. Let  $f : \prod_{x:A} (a = x) \rightarrow B(x)$ . Prove that if each  $f(x)$  has a section, then  $f$  is a family of equivalences.

As a consequence, for any function  $k : X \rightarrow Y$ , if

$$\text{ap}_k(x, y) : (x = y) \rightarrow (k(x) = k(y))$$

has a section for every  $x, y : X$ , then  $k$  is an embedding.

1. Let  $H : \prod_{x:A} f(x) \sim g(x)$ . For each  $(x, y) : \sum_{x:A} B(x)$ , we have a term

$$\text{pair}^=(\text{refl}_x, H_x(y)) : \text{tot}(f)(x, y) = \text{tot}(g)(x, y).$$

2. For each  $(x, y) : \sum_{x:A} B(x)$ , we have a term

$$\text{refl}_{(x, g(x, f(x, y)))} : \text{tot}(\lambda x. g(x) \circ f(x))(x, y) = (\text{tot}(g) \circ \text{tot}(f))(x, y).$$

3. For each  $(x, y) : \sum_{x:A} B(x)$ , we have a term

$$\text{refl}_{(x, y)} : \text{tot}(\lambda x. \text{id}_{B(x)})(x, y) = \text{id}_{\sum_{x:A} B(x)}(x, y).$$

4. For each  $x : A$ , suppose that we have maps  $B(x) \xrightarrow{s_x} (a = x)$  and  $(a = x) \xrightarrow{r_x} B(x)$  such that  $r_x \circ s_x \sim \text{id}_{B(x)}$ . Let us show that  $\lambda x. r_x$  is a family of equivalences. By combining parts (1), (2), and (3), we get a commuting diagram

$$\begin{array}{ccccc} \sum_{x:A} B(x) & \xrightarrow{\text{tot}(\lambda x. s_x)} & \sum_{x:A} a = x & \xrightarrow{\text{tot}(\lambda x. r_x)} & \sum_{x:A} B(x) \\ & \searrow & & \nearrow & \\ & & \text{id}_{\sum_{x:A} B(x)} & & \end{array}$$

Moreover,  $\sum_{x:A} a = x$  is contractible. As a retract of a contractible type is itself contractible, we see that  $\text{tot}(\lambda x. r_x)$  is an equivalence. Hence  $\lambda x. r_x$  is a family of equivalences.

5. This follows immediately from our proof of part (4).

## 6 $(\star \star \star)$

We say that a map  $f : A \rightarrow B$  is *path-split* if

1.  $f$  has a section and
2. the map  $\mathbf{ap}_f(x, y) : (x = y) \rightarrow (f(x) = f(y))$  has a section for each  $x, y : A$ .

Prove that a map  $f : A \rightarrow B$  is an equivalence if and only if it is path-split.

**Suppose that  $f$  is an equivalence. Then  $f$  has a section. It's also an embedding, so that  $\mathbf{ap}_f(x, y)$  has a section.**

**Conversely, suppose that  $f$  is path-split with section  $s_f : B \rightarrow A$ . By Problem 5,  $f$  is an embedding. Let  $b : B$  and note that**

$$s_f(b) : \mathbf{fib}_f(b).$$

**To see that  $\mathbf{fib}_f(b)$  is contractible, we must show that any two elements  $(a, p), (a', u) : \mathbf{fib}_f(b)$  of the fiber of  $f$  over  $b$  are equal. Since  $f$  is an embedding, we have a term**

$$q := \mathbf{ap}_f(a, a')^{-1}(p \cdot u^{-1}) : a = a',$$

**with**

$$\mathbf{tr}_{f(x)=b}(q, p) = \mathbf{ap}_f(a, a')(q)^{-1} \cdot p = (p \cdot u^{-1})^{-1} \cdot p = u.$$

**It follows that**

$$(a, p) = (a', u).$$