



Worksheet 7 (Solved)

HoTTEST Summer School 2022

The HoTTEST TAs

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1 (★)

Consider two embeddings $f : A \hookrightarrow B$ and $g : B \hookrightarrow C$. Construct a function

$$\text{is-equiv}(g \circ f) \rightarrow (\text{is-equiv}(f) \times \text{is-equiv}(g)).$$

Suppose that $g \circ f$ is an equivalence. By the 3-for-2 property of equivalences, it suffices to prove that f is an equivalence. Define $\psi : B \rightarrow A$ by

$$\psi(b) := (g \circ f)^{-1}(g(b)).$$

For every $b : B$,

$$g(f(\psi(b))) = g(b).$$

Since g is an embedding, this implies that

$$f(\psi(b)) = b.$$

Moreover, $\psi(f(a)) = a$ for all $a : A$. Thus, ψ is an inverse of f .

2 (★★)

1. Let A be a type. Prove that the canonical map $\emptyset \xrightarrow{!_A} A$ is an embedding.
2. Let A and B be types. Prove that the inclusions $\text{inl} : A \rightarrow A + B$ and $\text{inr} : B \rightarrow A + B$ are embeddings.
3. Let A and B be types. Prove that $\text{inl} : A \rightarrow A + B$ is an equivalence if and only if $B \simeq \emptyset$.

Conclude that if both A and B are contractible, then $A + B$ is *not* contractible.

1. For every $x, y : \emptyset$, we must prove that

$$\text{ap}_{!_A}(x, y) : (x = y) \rightarrow (!_A(x) = !_A(y))$$

is an equivalence. This follows directly from induction on \emptyset .

2. Let $x, y : A$ and consider the map

$$\text{ap}_{\text{inl}}(x, y) : (x = y) \rightarrow (\text{inl}(x) = \text{inl}(y)).$$

Recall from Lecture 7 the family

$$\text{eq-id} : \prod_{s, t : A+B} (s = t) \rightarrow \text{Eq}_{+_{a,b}}(s, t)$$

of equivalences, defined by path induction. It is easy to check that $\text{ap}_{\text{inl}}(x, y)$ is a section of $\text{eq-id}_{\text{inl}(x), \text{inl}(y)}$. Since the latter is an equivalence, it follows that $\text{ap}_{\text{inl}}(x, y)$ is actually an equivalence with inverse $\text{eq-id}_{\text{inl}(x), \text{inl}(y)}$. This proves that inl is an embedding.

Similarly, for any $x, y : B$, the map $\text{ap}_{\text{inr}}(x, y)$ is an equivalence with inverse $\text{eq-id}_{\text{inr}(x), \text{inr}(y)}$. Thus, inr is also an embedding.

3. Suppose that inl is an equivalence with inverse $\psi : A + B \rightarrow A$. Let $b : B$. Then $\text{inl}(\psi(\text{inr}(b))) = \text{inr}(b)$. But recall that

$$(\text{inl}(\psi(\text{inr}(b)))) = \text{inr}(b) \simeq \emptyset.$$

This gives us an element of \emptyset and thus an element of $\neg B$.

Conversely, consider an equivalence $e : B \rightarrow \emptyset$. Define the function $\varphi : A + B \rightarrow A$ by

$$\begin{aligned} \varphi(\text{inl}(a)) &:= a \\ \varphi(\text{inr}(b)) &:= \text{ind}_{\emptyset}(e(b)). \end{aligned}$$

It is easy to check that φ is an inverse of inl .

Now, suppose that both A and B are contractible. Also, suppose that $A + B$ is contractible. We have a commuting triangle

$$\begin{array}{ccc} A & \xrightarrow{\text{inl}} & A + B \\ & \searrow \simeq & \swarrow \simeq \\ & \mathbb{1} & \end{array},$$

so that inl is an equivalence. This implies that $B \simeq \emptyset$. Since B is contractible, this gives us an element of \emptyset . Therefore, $A + B$ is not contractible.

3 (★★)

Consider a commuting triangle

$$\begin{array}{ccc} A & \xrightarrow{h} & B \\ & \searrow f & \swarrow g \\ & X & \end{array} .$$

1. Suppose that g is an embedding. Prove that f is an embedding if and only if h is one.
2. Suppose that h is an equivalence. Prove that f is an embedding if and only if g is one.

Note that for every $x, y : A$, we have a commuting square

$$\begin{array}{ccc} x = y & \xrightarrow{\text{ap}_h(x, y)} & h(x) = h(y) \\ \text{ap}_f(x, y) \downarrow & & \downarrow \text{ap}_g(h(x), h(y)) \\ f(x) = f(y) & \xrightarrow{\simeq} & g(h(x)) = g(h(y)) \end{array} .$$

1. We have that $\text{ap}_g(h(x), h(y))$ is an equivalence. Suppose that f is an embedding. Then $\text{ap}_h(x, y)$ is an equivalence, and thus h is an embedding.

Conversely, suppose that h is an embedding. Then $\text{ap}_f(x, y)$ is an equivalence, and thus f is an embedding.

2. Suppose that f is an embedding. Then $\text{ap}_{g \circ h}(x, y)$ is an equivalence. Further, the triangle

$$\begin{array}{ccc} B & \xrightarrow{h^{-1}} & A \\ & \searrow g & \swarrow g \circ h \\ & X & \end{array}$$

commutes, and h^{-1} is an embedding because it's an equivalence. By part (1), it follows that g is an embedding.

Conversely, suppose that g is an embedding. Then $\text{ap}_f(x, y)$ is an equivalence. As h is also an embedding, so is f .

4 (★★)

Let A , B , and C be types and let $f : A \rightarrow C$ and $g : B \rightarrow C$ be maps. Prove that the following are logically equivalent.

1. The map $[f, g] : A + B \rightarrow C$ is an embedding.
2. Both f and g are embeddings, and $f(a) \neq g(b)$ for all $a : A$ and $b : B$.

Suppose that $[f, g]$ is an embedding. Then $f \doteq [f, g] \circ \text{inl}$ is an embedding as the composite of two embeddings. Likewise, g is an embedding. Let $a : A$ and $b : B$ and suppose that $f(a) = g(b)$. Since $[f, g]$ is an embedding,

$$\text{inl}(a) = \text{inr}(b).$$

But $(\text{inl}(a) = \text{inr}(b)) \simeq \emptyset$, and thus $f(a) \neq g(b)$.

Conversely, suppose that both f and g are embeddings and that

$$\tau : \prod_{a:A} \prod_{b:B} f(a) \neq g(b).$$

We must show that

$$\text{ap}_{[f,g]}(s, t) : (s = t) \rightarrow ([f, g](s) = [f, g](t))$$

is an equivalence for all $s, t : A + B$. Define $\psi_{s,t} : ([f, g](s) = [f, g](t)) \rightarrow (s = t)$ by double induction on $A + B$:

$$\psi_{\text{inl}(a), \text{inl}(a')}(p) := \text{ap}_f(a, a')^{-1}(p)$$

$$\psi_{\text{inl}(a), \text{inr}(b)}(p) := \text{ind}_{\emptyset}(\tau_{a,b}(p))$$

$$\psi_{\text{inr}(b), \text{inr}(b')}(p) := \text{ap}_g(b, b')^{-1}(p)$$

$$\psi_{\text{inr}(b), \text{inl}(a)}(p) := \text{ind}_{\emptyset}(\tau_{a,b}(p^{-1})).$$

By double induction on $A + B$, it's easy to prove that $\psi_{s,t}$ is an inverse of $\text{ap}_{[f,g]}(s, t)$.

5 **(★★)**

1. Let $f, g : \prod_{x:A} B(x) \rightarrow C(x)$. Construct a function

$$\left(\prod_{x:A} f(x) \sim g(x) \right) \rightarrow (\text{tot}(f) \sim \text{tot}(g)).$$

2. Let $f : \prod_{x:A} B(x) \rightarrow C(x)$ and $g : \prod_{x:A} C(x) \rightarrow D(x)$. Construct a homotopy

$$\text{tot}(\lambda x. g(x) \circ f(x)) \sim \text{tot}(g) \circ \text{tot}(f).$$

3. For any type family B over A , construct a homotopy

$$\text{tot}(\lambda x. \text{id}_{B(x)}) \sim \text{id}_{\sum_{x:A} B(x)}.$$

4. Let $a : A$ and let B be a type family over A . Prove that if $B(x)$ is a retract of $a = x$ for each $x : A$, then $(a = x) \simeq B(x)$ for each $x : A$.
5. Let $f : \prod_{x:A} (a = x) \rightarrow B(x)$. Prove that if each $f(x)$ has a section, then f is a family of equivalences.

As a consequence, for any function $k : X \rightarrow Y$, if

$$\text{ap}_k(x, y) : (x = y) \rightarrow (k(x) = k(y))$$

has a section for every $x, y : X$, then k is an embedding.

1. Let $H : \prod_{x:A} f(x) \sim g(x)$. For each $(x, y) : \sum_{x:A} B(x)$, we have a term

$$\text{pair}^=(\text{refl}_x, H_x(y)) : \text{tot}(f)(x, y) = \text{tot}(g)(x, y).$$

2. For each $(x, y) : \sum_{x:A} B(x)$, we have a term

$$\text{refl}_{(x, g(x, f(x, y)))} : \text{tot}(\lambda x. g(x) \circ f(x))(x, y) = (\text{tot}(g) \circ \text{tot}(f))(x, y).$$

3. For each $(x, y) : \sum_{x:A} B(x)$, we have a term

$$\text{refl}_{(x, y)} : \text{tot}(\lambda x. \text{id}_{B(x)})(x, y) = \text{id}_{\sum_{x:A} B(x)}(x, y).$$

4. For each $x : A$, suppose that we have maps $B(x) \xrightarrow{s_x} (a = x)$ and $(a = x) \xrightarrow{r_x} B(x)$ such that $r_x \circ s_x \sim \text{id}_{B(x)}$. Let us show that $\lambda x. r_x$ is a family of equivalences. By combining parts (1), (2), and (3), we get a commuting diagram

$$\begin{array}{ccccc} \sum_{x:A} B(x) & \xrightarrow{\text{tot}(\lambda x. s_x)} & \sum_{x:A} a = x & \xrightarrow{\text{tot}(\lambda x. r_x)} & \sum_{x:A} B(x) \\ & \searrow & & \nearrow & \\ & \text{id}_{\sum_{x:A} B(x)} & & & \end{array}$$

Moreover, $\sum_{x:A} a = x$ is contractible. As a retract of a contractible type is itself contractible, we see that $\text{tot}(\lambda x. r_x)$ is an equivalence. Hence $\lambda x. r_x$ is a family of equivalences.

5. This follows immediately from our proof of part (4).

6 $(\star \star \star)$

We say that a map $f : A \rightarrow B$ is *path-split* if

1. f has a section and
2. the map $\mathbf{ap}_f(x, y) : (x = y) \rightarrow (f(x) = f(y))$ has a section for each $x, y : A$.

Prove that a map $f : A \rightarrow B$ is an equivalence if and only if it is path-split.

Suppose that f is an equivalence. Then f has a section. It's also an embedding, so that $\mathbf{ap}_f(x, y)$ has a section.

Conversely, suppose that f is path-split with section $s_f : B \rightarrow A$. By Problem 5, f is an embedding. Let $b : B$ and note that

$$s_f(b) : \mathbf{fib}_f(b).$$

To see that $\mathbf{fib}_f(b)$ is contractible, we must show that any two elements $(a, p), (a', u) : \mathbf{fib}_f(b)$ of the fiber of f over b are equal. Since f is an embedding, we have a term

$$q := \mathbf{ap}_f(a, a')^{-1}(p \cdot u^{-1}) : a = a',$$

with

$$\mathbf{tr}_{f(x)=b}(q, p) = \mathbf{ap}_f(a, a')(q)^{-1} \cdot p = (p \cdot u^{-1})^{-1} \cdot p = u.$$

It follows that

$$(a, p) = (a', u).$$