

Worksheet 7 (Solved)

HoTTEST Summer School 2022

The HoTTEST TAs 30 July 2022

1 (*)

Consider two embeddings $f:A\hookrightarrow B$ and $g:B\hookrightarrow C$. Construct a function

$$is-equiv(g \circ f) \rightarrow (is-equiv(f) \times is-equiv(g))$$
.

Suppose that $g \circ f$ is an equivalence. By the 3-for-2 property of equivalences, it suffices to prove that f is an equivalence. Define $\psi: B \to A$ by

$$\psi(b) := (g \circ f)^{-1} (g(b)).$$

For every b:B,

$$g(f(\psi(b))) = g(b).$$

Since g is an embedding, this implies that

$$f(\psi(b)) = b.$$

Moreover, $\psi(f(a)) = a$ for all a : A. Thus, ψ is an inverse of f.

2 (**)

- 1. Let A be a type. Prove that the canonical map $\emptyset \xrightarrow{!_A} A$ is an embedding.
- 2. Let A and B be types. Prove that the inclusions in $A \to A + B$ and in $B \to A + B$ are embeddings.
- 3. Let A and B be types. Prove that in $A \to A + B$ is an equivalence if and only if $B \simeq \emptyset$.

Conclude that if both A and B are contractible, then A + B is not contractible.

1. For every $x, y : \emptyset$, we must prove that

$$\mathsf{ap}_{!_A}(x,y): (x=y) \to (!_A(x) = !_A(y))$$

is an equivalence. This follows directly from induction on \emptyset .

2. Let x, y : A and consider the map

$$\mathsf{ap}_{\mathsf{inl}}(x,y):(x=y)\to(\mathsf{inl}(x)=\mathsf{inl}(y))$$
 .

Recall from Lecture 7 the family

$$\operatorname{\mathsf{eq-id}}: \prod_{s,t:A+B} (s=t) \to \operatorname{\mathsf{Eq+}}_{a,b}(s,t)$$

of equivalences, defined by path induction. It is easy to check that $\mathsf{ap}_{\mathsf{inl}}(x,y)$ is a section of $\mathsf{eq}\text{-}\mathsf{id}_{\mathsf{inl}(x),\mathsf{inl}(y)}$. Since the latter is an equivalence, it follows that $\mathsf{ap}_{\mathsf{inl}}(x,y)$ is actually an equivalence with inverse $\mathsf{eq}\text{-}\mathsf{id}_{\mathsf{inl}(x),\mathsf{inl}(y)}$. This proves that inl is an embedding.

Similarly, for any x, y : B, the map $\operatorname{\mathsf{ap}}_{\mathsf{inr}}(x, y)$ is an equivalence with inverse $\operatorname{\mathsf{eq}-\mathsf{id}}_{\mathsf{inr}(x),\mathsf{inr}(y)}$. Thus, inr is also an embedding.

3. Suppose that inl is an equivalence with inverse $\psi: A+B \to A$. Let b: B. Then $\mathsf{inl}(\psi(\mathsf{inr}(b))) = \mathsf{inr}(b)$. But recall that

$$(\operatorname{inl}(\psi(\operatorname{inr}(b))) = \operatorname{inr}(b)) \simeq \emptyset.$$

This gives us an element of \emptyset and thus an element of $\neg B$.

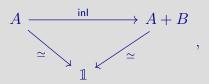
Conversely, consider an equivalence $e:B\to\emptyset$. Define the function $\varphi:A+B\to A$ by

$$\varphi(\mathsf{inl}(a)) \coloneqq a$$

 $\varphi(\mathsf{inr}(b)) \coloneqq \mathsf{ind}_{\emptyset}(e(b)).$

It is easy to check that φ is an inverse of inl.

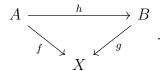
Now, suppose that both A and B are contractible. Also, suppose that A + B is contractible. We have a commuting triangle



so that in is an equivalence. This implies that $B \simeq \emptyset$. Since B is contractible, this gives us an element of \emptyset . Therefore, A+B is not contractible.

3 (**)

Consider a commuting triangle



- 1. Suppose that g is an embedding. Prove that f is an embedding if and only if h is one.
- 2. Suppose that h is an equivalence. Prove that f is an embedding if and only if g is one.

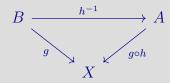
Note that for every x, y : A, we have a commuting square

$$\begin{aligned} x &= y & \xrightarrow{\mathrm{ap}_h(x,y)} & h(x) &= h(y) \\ \mathrm{ap}_f(x,y) \Big\downarrow & & \Big\downarrow \mathrm{ap}_g(h(x),h(y)) \cdot \\ f(x) &= f(y) & \xrightarrow{\sim} & g(h(x)) &= g(h(y)) \end{aligned}$$

1. We have that $\operatorname{ap}_g(h(x),h(y))$ is an equivalence. Suppose that f is an embedding. Then $\operatorname{ap}_h(x,y)$ is an equivalence, and thus h is an embedding.

Conversely, suppose that h is an embedding. Then $\operatorname{\sf ap}_f(x,y)$ is an equivalence, and thus f is an embedding.

2. Suppose that f is an embedding. Then $\operatorname{\sf ap}_{g\circ h}(x,y)$ is an equivalence. Further, the triangle



commutes, and h^{-1} is an embedding because it's an equivalence. By part (1), it follows that g is an embedding.

Conversely, suppose that g is an embedding. Then $\operatorname{\sf ap}_f(x,y)$ is an equivalence. As h is also an embedding, so is f.

4 (**)

Let A, B, and C be types and let $f: A \to C$ and $g: B \to C$ be maps. Prove that the following are logically equivalent.

- 1. The map $[f,g]:A+B\to C$ is an embedding.
- 2. Both f and g are embeddings, and $f(a) \neq g(b)$ for all a: A and b: B.

Suppose that [f,g] is an embedding. Then $f \doteq [f,g] \circ \text{inl}$ is an embedding as the composite of two embeddings. Likewise, g is an embedding. Let a:A and b:B and suppose that f(a)=g(b). Since [f,g] is an embedding,

$$inl(a) = inr(b).$$

But $(\mathsf{inl}(a) = \mathsf{inr}(b)) \simeq \emptyset$, and thus $f(a) \neq g(b)$.

Conversely, suppose that both f and g are embeddings and that

$$\tau: \prod_{a:A} \prod_{b:B} f(a) \neq g(b).$$

We must show that

$$\mathsf{ap}_{[f,g]}(s,t):(s=t)\to ([f,g]\,(s)=[f,g]\,(t))$$

is an equivalence for all s, t: A+B. Define $\psi_{s,t}: ([f,g](s)=[f,g](t)) \to (s=t)$ by double induction on A+B:

$$\begin{split} \psi_{\mathrm{inl}(a),\mathrm{inl}(a')}(p) &\coloneqq \mathrm{ap}_f(a,a')^{-1}(p) \\ \psi_{\mathrm{inl}(a),\mathrm{inr}(b)}(p) &\coloneqq \mathrm{ind}_{\emptyset}(\tau_{a,b}(p)) \\ \psi_{\mathrm{inr}(b),\mathrm{inr}(b')}(p) &\coloneqq \mathrm{ap}_g(b,b')^{-1}(p) \\ \psi_{\mathrm{inr}(b),\mathrm{inl}(a)}(p) &\coloneqq \mathrm{ind}_{\emptyset}(\tau_{a,b}(p^{-1})). \end{split}$$

By double induction on A+B, it's easy to prove that $\psi_{s,t}$ is an inverse of $\mathsf{ap}_{[f,g]}(s,t)$.

1. Let $f, g: \prod_{x:A} B(x) \to C(x)$. Construct a function

$$\left(\prod_{x:A} f(x) \sim g(x)\right) \to \left(\mathsf{tot}(f) \sim \mathsf{tot}(g)\right).$$

- 2. Let $f: \prod_{x:A} B(x) \to C(x)$ and $g: \prod_{x:A} C(x) \to D(x)$. Construct a homotopy $tot(\lambda x.g(x) \circ f(x)) \sim tot(g) \circ tot(f)$.
- 3. For any type family B over A, construct a homotopy

$$tot(\lambda x.id_{B(x)}) \sim id_{\sum_{x:A} B(x)}.$$

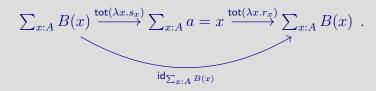
- 4. Let a:A and let B be a type family over A. Prove that if B(x) is a retract of a=x for each x:A, then $(a=x) \simeq B(x)$ for each x:A.
- 5. Let $f: \prod_{x:A} (a=x) \to B(x)$. Prove that if each f(x) has a section, then f is a family of equivalences.

As a consequence, for any function $k: X \to Y$, if

$$ap_k(x, y) : (x = y) \to (k(x) = k(y))$$

has a section for every x, y : X, then k is an embedding.

- 1. Let $H: \prod_{x:A} f(x) \sim g(x)$. For each $(x,y): \sum_{x:A} B(x)$, we have a term pair = $(\text{refl}_x, H_x(y)): \text{tot}(f)(x,y) = \text{tot}(g)(x,y)$.
- 2. For each $(x,y):\sum_{x:A}B(x)$, we have a term $\operatorname{refl}_{(x,g(x,f(x,y)))}: \operatorname{tot}(\lambda x.g(x)\circ f(x))(x,y)=(\operatorname{tot}(g)\circ\operatorname{tot}(f))(x,y).$
- 3. For each $(x,y):\sum_{x:A}B(x)$, we have a term $\mathsf{refl}_{(x,y)}\ :\ \mathsf{tot}(\lambda x.\mathsf{id}_{B(x)})(x,y)=\mathsf{id}_{\sum_{x:A}B(x)}(x,y).$
- 4. For each x:A, suppose that we have maps $B(x) \xrightarrow{s_x} (a=x)$ and $(a=x) \xrightarrow{r_x} B(x)$ such that $r_x \circ s_x \sim \operatorname{id}_{B(x)}$. Let us show that $\lambda x.r_x$ is a family of equivalences. By combining parts (1), (2), and (3), we get a commuting diagram



Moreover, $\sum_{x:A} a = x$ is contractible. As a retract of a contractible type is itself contractible, we see that $tot(\lambda x.r_x)$ is an equivalence. Hence $\lambda x.r_x$ is a family of equivalences.

5. This follows immediately from our proof of part (4).

$$6 \quad (\star \star \star)$$

We say that a map $f: A \to B$ is path-split if

- 1. f has a section and
- 2. the map $\operatorname{\mathsf{ap}}_f(x,y): (x=y) \to (f(x)=f(y))$ has a section for each x,y:A.

Prove that a map $f: A \to B$ is an equivalence if and only if it is path-split.

Suppose that f is an equivalence. Then f has a section. It's also an embedding, so that $\operatorname{ap}_f(x,y)$ has a section.

Conversely, suppose that f is path-split with section $s_f: B \to A$. By Problem 5, f is an embedding. Let b: B and note that

$$s_f(b)$$
 : $fib_f(b)$.

To see that $fib_f(b)$ is contractible, we must show that any two elements (a,p),(a',u): $fib_f(b)$ of the fiber of f over b are equal. Since f is an embedding, we have a term

$$q := \mathsf{ap}_f(a, a')^{-1}(p \cdot u^{-1}) \ : \ a = a',$$

with

$$\mathrm{tr}_{f(x)=b}(q,p) = \mathrm{ap}_f(a,a')(q)^{-1} \cdot p = \left(p \cdot u^{-1}\right)^{-1} \cdot p = u.$$

It follows that

$$(a,p) = (a',u).$$