Worksheet 4 (Solved)

HoTTEST Summer School 2022

The HoTTEST TAs
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1 $(\star)$

We define the standard finite types $\mathsf{Fin} : \mathbb{N} \to \mathcal{U}_0$ inductively with constructors

$$\mathsf{pt} : \Pi_{n:\mathbb{N}}\mathsf{Fin}(\mathsf{suc}(n))$$
$$\mathsf{i} : \Pi_{n:\mathbb{N}}\mathsf{Fin}(n) \to \mathsf{Fin}(\mathsf{suc}(n)).$$

Spell out all elements of $\mathsf{Fin}(3)$.

> **The elements are**
>
> $$\mathbf{pt}(2),$$
> $$\mathbf{i}(2, \mathbf{pt}(1)),$$
> $$\mathbf{i}(2, \mathbf{i}(1, \mathbf{pt}(0))).$$

It is common practice to leave the argument n of the constructors implicit. Then the induction principle states that a dependent function

$$f : \Pi_{n:\mathbb{N}}\Pi_{x:\mathsf{Fin}(n)}P_n(x)$$

is determined by

$$g_n : \Pi_{x:\mathsf{Fin}(n)}P_n(x) \to P_{\mathsf{suc}(n)}(\mathsf{i}(x))$$

and

$$p_n : P_{\mathsf{suc}(n)}(\mathsf{pt}).$$

The function f satisfies the judgemental equalities

$$f_{\mathsf{suc}(n)}(\mathsf{i}(x)) \doteq g_n(x, f_n(x))$$
$$f_{\mathsf{suc}(n)}(\mathsf{pt}) \doteq p_n.$$

2 $(\star\star\star)$

It is also possible to define the standard finite types $\mathsf{Fin'} : \mathbb{N} \to \mathcal{U}_0$ recursively as a type family over $\mathbb{N}$,

$$\mathsf{Fin'}(0) \doteq \emptyset$$
$$\mathsf{Fin'}(\mathsf{suc}(n)) \doteq \mathsf{Fin'}(n) + \mathbb{1}.$$

We suggestively use the notation i' : $\mathsf{Fin'}_n \to \mathsf{Fin'}_{\mathsf{suc}(n)}$ and pt' : $\mathsf{Fin'}_{\mathsf{suc}(n)}$ for the inclusions inl and inr into the coproduct $\mathsf{Fin'}(n) + \mathbb{1}$. Formulate the induction principle of $\mathsf{Fin'}$.

> **The induction principle given to Fin' is exactly (the primed version of) the induction principle carried by Fin, described above.**

3 $(\star\star)$

Choose your favourite version of the finite types. Use pattern matching to define two different inclusions $\iota, \hat{\iota} : \Pi_{n:\mathbb{N}}\mathsf{Fin}(n) \to \mathbb{N}$, such that the images of $\iota_{\mathsf{suc}(n)}$ and $\hat{\iota}_{\mathsf{suc}(n)}$ are the first $n+1$ natural numbers.

> **We define**
>
> $$\iota_{\mathsf{suc}(n)}(\mathbf{i}(x)) \doteq \iota_n(x)$$
> $$\iota_{\mathsf{suc}(n)}(\mathbf{pt}) \doteq n$$
> $$\hat{\iota}_{\mathsf{suc}(n)}(\mathbf{i}(x)) \doteq \mathbf{suc}(\iota'(x))$$
> $$\hat{\iota}_{\mathsf{suc}(n)}(\mathbf{pt}) \doteq 0.$$
>
> **It is not necessary to define ι_0 because Fin(0) is empty.**

4 $(\star)$

Give a recursive definition of the ordering relation $\leq: \mathbb{N} \to \mathbb{N} \to \mathcal{U}_0$.

> **Using induction on $\mathbb{N}$ twice we may define**
>
> $$0 \leq 0 \doteq \mathbb{1}$$
> $$m +_{\mathbb{N}} 1 \leq 0 \doteq \emptyset$$
> $$0 \leq n +_{\mathbb{N}} 1 \doteq \mathbb{1}$$
> $$m +_{\mathbb{N}} 1 \leq n +_{\mathbb{N}} 1 \doteq m \leq n$$

5 ($\star\star$)

Define is-prime $: \mathbb{N} \to$ Type.

> **There are various ways of defining this property. The one implemented in the repository is**
>
> $$\textbf{is-prime}(n) \doteq (2 \leq n) \times (\Pi_{x,y:\mathbb{N}}(x *_{\mathbb{N}} y = n) \to (x = 1) + (x = n)).$$
>
> **Egbert's book uses**
>
> $$\textbf{is-prime'}(n) \doteq \Pi_{d:\mathbb{N}}((d \neq n) \times (d \mid n)) \leftrightarrow (d = 1).$$

6 ($\star\star$)

State the twin prime conjecture and Goldbach's conjecture in HoTT.

> **The twin prime conjecture is**
>
> $$\Pi_{n:\mathbb{N}}\Sigma_{p:\mathbb{N}}((n \leq p) \times \textbf{is-prime}(p) \times \textbf{is-prime}(p +_{\mathbb{N}} 2)).$$
>
> **Goldbach's conjecture can be phrased**
>
> $$\Pi_{n:\mathbb{N}} \left(((4 \leq n) \times \textbf{is-even}(n)) \to \Sigma_{p,q:\mathbb{N}}(\textbf{is-prime}(p) \times \textbf{is-prime}(q) \times (n = p +_{\mathbb{N}} q))\right).$$

7 ($\star\star$)

Suppose we had constructed a proof

$$\textsf{infinitude-of-primes} : \Pi_{n:\mathbb{N}}\Sigma_{p:\mathbb{N}}(\textsf{is-prime}(p) \times (n \leq p)).$$

Further assume that the prime p returned by this program is the least prime above n. A definition of such a term can be found in the Agda UniMath library[1]. Construct a function prime $: \mathbb{N} \to \mathbb{N}$ which computes the n-th prime.

> **We inductively define**
>
> $$\textbf{prime}(0) \doteq 2$$
> $$\textbf{prime}(\textbf{suc}(n)) \doteq \textbf{pr}_1(\textbf{infinitude-of-primes}(\textbf{prime}(n)).$$

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[1] https://unimath.github.io/agda-unimath/elementary-number-theory.infinitude-of-primes.html

8 ($\star\star$)

We define the predicate

$$\text{is-decidable}(A) \doteq A + \neg A$$

for an arbitrary type A. Do we expect

$$\Pi_{n:\mathbb{N}}\text{is-decidable}(\text{is-prime}(n))$$

to be true (inhabited)? Why or why not?

> **We expect this to be true because it's easy to write down an algorithm which checks if a number is prime on paper. In fact, a proof in Agda is referenced on the same UniMath docs page.**

9 ($\star\star\star$)

Suppose we had a proof

$$\text{is-decidable-is-prime} : \Pi_{n:\mathbb{N}}\text{is-decidable}(\text{is-prime}(n)).$$

Construct a function

$$\text{prime-counting} : \mathbb{N} \to \mathbb{N}$$

which computes the number of primes less than or equal to its input.

> **As usual, we define this function inductively. We put**
>
> $$\textbf{prime-counting}(0) \doteq 0.$$
>
> **For the inductive step, is-decidable-is-prime allows us to proceed by case analysis on whether or not $n + 1$ is a prime number. In other words, we may define**
>
> $$\textbf{if-prime} : \textbf{is-decidable}(\textbf{is-prime}(\textbf{suc}(n))) \to \mathbb{N}$$
> $$\textbf{if-prime}(\textbf{inl}(x)) \doteq \textbf{suc}(\textbf{prime-counting}(n))$$
> $$\textbf{if-prime}(\textbf{inr}(x)) \doteq \textbf{prime-counting}(n)$$
>
> **and put**
>
> $$\textbf{prime-counting}(\textbf{suc}(n)) \doteq \textbf{if-prime}(\textbf{is-decidable-is-prime}(\textbf{suc}(n))).$$

10 $(\star\star\star)$

Show that adding k is an injective function which respects equality, i.e. that

$$(m = n) \leftrightarrow (m +_{\mathbb{N}} k = n +_{\mathbb{N}} k)$$

for all $m, n, k : \mathbb{N}$.

> **A proof of $(m = n) \to (m +_{\mathbb{N}} k = n +_{\mathbb{N}} k)$ is given by the action of the function**
>
> $$\lambda x.x +_{\mathbb{N}} k : \mathbb{N} \to \mathbb{N}$$
>
> **on paths $p : (m = n)$.**
> **For the converse direction we induct on k. In the base case we need to show that $(m +_{\mathbb{N}} 0 = n +_{\mathbb{N}} 0) \to (m = n)$. Assume we have $p : m +_{\mathbb{N}} 0 = n +_{\mathbb{N}} 0$. By two applications of**
>
> $$\mathsf{concat} : \Pi_{x,y,z:A}(x = y) \to ((y = z) \to (x = z)),$$
>
> **a sequence of identifications**
>
> $$m = (m +_{\mathbb{N}} 0) = (n +_{\mathbb{N}} 0) = n$$
>
> **implies $m = n$. The identification in the middle is proved by p. Since addition was defined by induction on the right argument, the outer identities hold judgementally. If $+$ had been defined by induction on the first argument, $m = m +_{\mathbb{N}} 0$ can be proved inductively.**
> **In the inductive step we need to prove**
>
> $$((m +_{\mathbb{N}} \mathsf{suc}(k)) = (n +_{\mathbb{N}} \mathsf{suc}(k))) \to (m = n).$$
>
> **The induction hypothesis is of type**
>
> $$((m +_{\mathbb{N}} k) = (n +_{\mathbb{N}} k)) \to (m = n),$$
>
> **so by function composition it suffices to construct a proof of**
>
> $$((m +_{\mathbb{N}} \mathsf{suc}(k)) = (n +_{\mathbb{N}} \mathsf{suc}(k))) \to ((m +_{\mathbb{N}} k) = (n +_{\mathbb{N}} k)).$$
>
> **Application of the predecessor function proves that suc is injective. This gives us a function of type**
>
> $$(\mathsf{suc}(m +_{\mathbb{N}} k) = \mathsf{suc}(n +_{\mathbb{N}} k)) \to ((m +_{\mathbb{N}} k) = (n +_{\mathbb{N}} k)).$$

Again, by function application, we have reduced our goal to

$$((m +_{\mathbb{N}} \mathsf{suc}(k)) = (n +_{\mathbb{N}} \mathsf{suc}(k))) \to (\mathsf{suc}(m +_{\mathbb{N}} k) = \mathsf{suc}(n +_{\mathbb{N}} k)).$$

Assuming $q : ((m +_{\mathbb{N}} \mathsf{suc}(k)) = (n +_{\mathbb{N}} \mathsf{suc}(k)))$, we can form a sequence of identifications

$$\mathsf{suc}(m +_{\mathbb{N}} k) = m +_{\mathbb{N}} \mathsf{suc}(k) = n +_{\mathbb{N}} \mathsf{suc}(k) = \mathsf{suc}(n +_{\mathbb{N}} k),$$

where the outer equalities are judgemental.

Remark: These two proofs are formalized and in the course repository. They're called plus-on-paths and plus-is-injective in the module natural-numbers-functions.