



Worksheet 10 (Solved)

HoTTEST Summer School 2022

The HoTTEST TAs

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1 (★)

Let A be a type. Show that

- (a) $|||A||| \leftrightarrow ||A||$
- (b) $\exists_{(x:A)} ||B(x)|| \leftrightarrow ||\Sigma_{(x:A)} B(x)||$
- (c) $\neg\neg ||A|| \leftrightarrow \neg\neg A$
- (d) $\text{is-decidable}(A) \rightarrow (||A|| \rightarrow A)$

(a) For the left-to-right direction, we need to construct a map $f : |||A||| \rightarrow ||A||$. Since the codomain is a proposition, it suffices, by the recursion principle for propositional truncation, to define $f(|x|)$, where $x : ||A||$. We simply let, $f(|x|) = x$. For the other direction, we simply map $x : ||A||$ to $|x| : |||A|||$.

(b) For the left-to-right direction, we need to define a map

$$f : ||\Sigma_{(x:A)} ||B(x)||| \rightarrow ||\Sigma_{(x:A)} B(x)||$$

The codomain is a proposition, so it suffices to define $f(|x, b_x|)$ for $x : A$ and $b_x : ||B(x)||$. Now, the codomain is still a proposition, so we may apply the recursion principle again, this time to b_x . Thus, it suffices to define $f(|x, |b'_x|||)$ for $x : A$ and $b'_x : B(x)$. We define $f(|x, |b'_x|||) = |x, b'_x|$, and we are done. For the other direction, we need a map

$$g : ||\Sigma_{(x:A)} B(x)|| \rightarrow ||\Sigma_{(x:A)} ||B(x)|||$$

The recursion rule applies here too, so we only need to define $g(|x, b_x|)$ for $x : A$ and $b_x : \Sigma_{(x:A)} B(x)$. We define $g(|x, b_x|) = |x, |b_x||$, and we are done.

(c) Let us spell out the types involved. What we need is a bi-implication

$$((\|A\| \rightarrow \emptyset) \rightarrow \emptyset) \leftrightarrow ((A \rightarrow \emptyset) \rightarrow \emptyset)$$

For the left-to-right direction, we are given $p : ((\|A\| \rightarrow \emptyset) \rightarrow \emptyset)$ and $q : A \rightarrow \emptyset$. The goal is to produce an element of type \emptyset . If we can construct an element of type $r : \|A\| \rightarrow \emptyset$, then $p(r) : \emptyset$. Since \emptyset is a proposition, it suffices to define $r(| a |)$ for $a : A$. Defining $r(| a |) := q(a)$ does the job. The other direction is proved similarly.

(d) Recall, $\text{is-decidable}(A) := A + \neg A$. We want to define $f(x, p) : A$ for $x : \text{is-decidable}(A)$ and $p : \|A\|$. Let us do this by induction on x . When x is $\text{inl}(a)$ with $a : A$, it's easy: we define $f(\text{inl}(a), p) := a$. When x is $\text{inr}(b)$ with $b : \neg A$, we cannot directly produce an element of A . However, we can produce an element of \emptyset , which gives us an element of A via \emptyset -recursion. We have $p : \|A\|$, which immediately gives us a term $q : \neg \neg \|A\|$. We apply part (c), and get a term $r : \neg \neg A$. Now, $r(b) : \emptyset$ and we are done.

2 (★★)

Consider two maps $f : A \rightarrow P$ and $g : B \rightarrow Q$ into propositions P and Q .

- (a) Show that if f and g are propositional truncations, then $f \times g : A \times B \rightarrow P \times Q$ is also a propositional truncation
- (b) Conclude that $\|A \times B\| \simeq \|A\| \times \|B\|$

(a) Let R be a proposition, We need to show that precomposition

$$- \circ (f \times g) : (P \times Q \rightarrow R) \rightarrow (A \times B \rightarrow R)$$

is an equivalence. Since both sides are propositions, it suffices to construct a map of type $(A \times B \rightarrow R) \rightarrow (P \times Q \rightarrow R)$. To this end, let $F : A \times B \rightarrow R$, and let $(p, q) : P \times Q$. We need to construct an element of R . Consider the type $(Q \rightarrow R)$. This is a proposition, so let us instantiate the universal property of f with it. We get

$$- \circ f : (P \rightarrow (Q \rightarrow R)) \simeq (A \rightarrow (Q \rightarrow R))$$

If we can construct an element $h : A \rightarrow (Q \rightarrow R)$, we are done, since then $((- \circ f)^{-1}(h))(p, q) : R$. In order to construct h , we introduce $a : A$ and $q' : Q$. Our goal is again to construct an element of R , but this time, we have managed to introduce an $a : A$ in our context! Let's repeat the procedure, this time using the universal property of g . We instantiate it with the proposition $(P \rightarrow R)$, which gives us an equivalence.

$$- \circ g : (Q \rightarrow (P \rightarrow R)) \simeq (B \rightarrow (P \rightarrow R))$$

By the above, it again suffices to construct a function $\ell : B \rightarrow (P \rightarrow R)$, since $((- \circ g)^{-1}(\ell))(q, p) : R$. We define ℓ by $\ell(b, p) = F(a, b)$ and we are done.

(b) For any type C , the map $\lambda x. |x| : C \rightarrow \|C\|$ is a propositional truncation. By part (a), this implies that the canonical map $A \times B \rightarrow \|A\| \times \|B\|$ is a propositional truncation. But so is the map $\lambda x. |x| : A \times B \rightarrow \|A \times B\|$. Hence we get, by the two-out-of-three property of propositional truncations (Proposition 14.1.4) that $\|A\| \times \|B\| \simeq \|A \times B\|$

3 (★★)

Consider a map $f : A \rightarrow B$. Show that the following are equivalent:

- (i) f is an equivalence
- (ii) f is both surjective and an embedding

For the left-to-right direction, there is not much to prove. We know that all equivalences are embeddings, and clearly, since f is an equivalence, it must be surjective.

For the other direction, let us assume that f is both surjective and an embedding. Let us consider the family of propositions $P(b) := \text{isContr}(\text{fib}_f(b))$ over B . We are done if we can construct a section $\Pi_{(b:B)} P(b)$. Since f is surjective, we get, by Proposition 15.2.3, an equivalence

$$(\Pi_{(b:B)} P(b)) \simeq (\Pi_{(a:A)} P(f(a)))$$

Applying the inverse of the above equivalence, we only need to show that $\text{fib}_f(f(a))$ is contractible for each $a : A$. Clearly, $\text{fib}_f(f(a))$ is pointed (has an element) since $(a, \text{refl}_{f(a)}) : \text{fib}_f(f(a))$. We now need to show that

$$(a, \text{refl}_{f(a)}) =_{\text{fib}_f(f(a))} (x, p) \quad (1)$$

for each $x : A$ and $p : f(x) = f(a)$. Since f is an embedding, we get $(\text{ap}_f)^{-1}(p^{-1}) : a = x$. This shows that the first components in (1) agree. We now need to show that the second components agree, with respect to $(\text{ap}_f)^{-1}(p^{-1})$. That is, we need to show that

$$\text{tr}_{\lambda x. f(x)=f(a)}((\text{ap}_f)^{-1}(p^{-1}), \text{refl}_{f(a)}) = p$$

It is an easy lemma that the left-hand-side is equal to

$$(\text{ap}_f((\text{ap}_f)^{-1}(p^{-1})))^{-1} \cdot \text{refl}_{f(a)}$$

ap_f and ap_f^{-1} cancel out, and thus it is equal to

$$(p^{-1})^{-1} \cdot \text{refl}_{f(a)} = p$$

and we are done.

4 (★★)

Prove **Lawvere's fixed point theorem**: For any two types A and B , if there is a surjective map $f : A \rightarrow B^A$, then for any $h : B \rightarrow B$, there (merely) exists an $x : B$ such that $h(x) = x$. In other words, show that

$$(\exists_{(f:A \rightarrow (A \rightarrow B))} \text{is-surj}(f)) \rightarrow (\forall_{(h:B \rightarrow B)} \exists_{(b:B)} h(b) = b)$$

Let $x : (\exists_{(f:A \rightarrow (A \rightarrow B))} \text{is-surj}(f))$ and $h : B \rightarrow B$. The goal is to show that there merely exists an element $b : B$ s.t. $h(b) = b$. This is a proposition, so we may assume $x := | f, p |$, where $f : A \rightarrow B^A$ and $p : \text{is-surj}(f)$. Given $a : A$, define, for ease of notation, $f_a : A \rightarrow B$ by $f_a = f(a)$. We get a function $F : A \rightarrow B$, given by

$$F(a) := h(f_a(a))$$

Surjectivity of f tells us that there is some $a : A$ such that $f_a = F$. We claim that $f_a(a)$ is a fixed point. We need to show that $h(f_a(a)) = f_a(a)$. Since $f_a = F$, it is enough to show that $h(f_a(a)) = F(a)$. But this is precisely how we defined F , so we are done.

Disclaimer In the following exercises, we will use $\{0, \dots, n\}$ to denote the elements of Fin_{n+1} , the finite type of $n + 1$ elements.

5 (★)

- (a) Construct an equivalence $\mathbf{Fin}_{n^m} \simeq (\mathbf{Fin}_m \rightarrow \mathbf{Fin}_n)$. Conclude that if A and B are finite, then $(A \rightarrow B)$ is finite.
- (b) Construct an equivalence $\mathbf{Fin}_{n!} \simeq (\mathbf{Fin}_n \simeq \mathbf{Fin}_n)$. Conclude that if A is finite, then $A \simeq A$ is finite.

(a) We proceed by induction on m . For $m := 0$, we need to show that

$$\mathbf{Fin}_1 \simeq (\mathbf{Fin}_0 \rightarrow \mathbf{Fin}_n)$$

Now, \mathbf{Fin}_1 contains precisely 1 element, and so does $\mathbf{Fin}_0 \rightarrow \mathbf{Fin}_n$ since $\mathbf{Fin}_0 \simeq \emptyset$. Hence, the two types must be equivalent. For the inductive step, assume that $\mathbf{Fin}_{n^m} \simeq (\mathbf{Fin}_m \rightarrow \mathbf{Fin}_n)$ holds. The goal is to show that

$$\mathbf{Fin}_{n^{m+1}} \simeq (\mathbf{Fin}_{m+1} \rightarrow \mathbf{Fin}_n)$$

First, note that

$$\mathbf{Fin}_{n^{m+1}} = \mathbf{Fin}_{n^m \cdot n} = \mathbf{Fin}_{\underbrace{n^m + n^m + \dots + n^m}_{n \text{ times}}} \simeq \underbrace{\mathbf{Fin}_{n^m} + \dots + \mathbf{Fin}_{n^m}}_{n \text{ times}}$$

It is an easy lemma that for any type A , we have $\underbrace{A + \dots + A}_{n \text{ times}} \simeq A \times \mathbf{Fin}_n$.

Consequently, we get

$$\mathbf{Fin}_{n^{m+1}} \simeq \mathbf{Fin}_{n^m} \times \mathbf{Fin}_n$$

By the inductive hypothesis, we have

$$\mathbf{Fin}_{n^m} \times \mathbf{Fin}_n \simeq (\mathbf{Fin}_m \rightarrow \mathbf{Fin}_n) \times \mathbf{Fin}_n$$

Hence, it suffices to construct an equivalence

$$f : (\mathbf{Fin}_m \rightarrow \mathbf{Fin}_n) \times \mathbf{Fin}_n \simeq (\mathbf{Fin}_{m+1} \rightarrow \mathbf{Fin}_n)$$

We define $f(g, x)$ by

$$f(g, x)(y) = \begin{cases} g(y) & \text{if } y < m \\ x & \text{otherwise} \end{cases}$$

and its inverse f^{-1} by

$$f^{-1}(g) = (g_m, g(m))$$

where $g_m : \mathbf{Fin}_m \rightarrow \mathbf{Fin}_n$ is the restriction of g to \mathbf{Fin}_m . The fact that f and f^{-1} cancel out is immediate.

For the second part of the question, let us assume that A and B are finite types. We want to show that $(A \rightarrow B)$ is finite. This is a proposition, so we may assume that we have equivalences $A \simeq \mathbf{Fin}_m$ and $B \simeq \mathbf{Fin}_n$ for some $n, m : \mathbb{N}$. We then have

$$(A \rightarrow B) \simeq (\mathbf{Fin}_m \rightarrow \mathbf{Fin}_n) \simeq \mathbf{Fin}_{n^m}$$

and thus $(A \rightarrow B)$ is also a finite type.

(b) We proceed by induction on n . For $n = 0$, we immediately get

$$\mathbf{Fin}_{0!} = \mathbf{Fin}_1 \simeq (\mathbf{Fin}_0 \simeq \mathbf{Fin}_0)$$

since there is precisely one equivalence $\mathbf{Fin}_0 \simeq \mathbf{Fin}_0$ (recall, $\mathbf{Fin}_0 \simeq \emptyset$). Assume as inductive hypothesis that $\mathbf{Fin}_{n!} \simeq (\mathbf{Fin}_n \simeq \mathbf{Fin}_n)$. We need to construct an equivalence

$$\mathbf{Fin}_{(n+1)!} \simeq (\mathbf{Fin}_{n+1} \simeq \mathbf{Fin}_{n+1})$$

We have

$$\mathbf{Fin}_{(n+1)!} = \mathbf{Fin}_{n!} \times \mathbf{Fin}_{n+1}$$

using similar reasoning as in (a). Hence, by the inductive hypothesis, we have

$$\mathbf{Fin}_{(n+1)!} \simeq (\mathbf{Fin}_n \simeq \mathbf{Fin}_n) \times \mathbf{Fin}_{n+1}$$

Proving that

$$(\mathbf{Fin}_n \simeq \mathbf{Fin}_n) \times \mathbf{Fin}_{n+1} \simeq (\mathbf{Fin}_{n+1} \simeq \mathbf{Fin}_{n+1})$$

is now just straightforward combinatorics. The idea is that given an element $x : \mathbf{Fin}_{n+1}$, any equivalence $f : \mathbf{Fin}_n \simeq \mathbf{Fin}_n$ can be extended to an equivalence $f_x : \mathbf{Fin}_{n+1} \simeq \mathbf{Fin}_{n+1}$ by

$$f_x(y) = \begin{cases} f(y) & \text{if } y < x \\ n & \text{if } y = x \\ f(y-1) & \text{otherwise} \end{cases}$$

We won't do this here, but it's easy to verify that $(f, x) \mapsto f_x$ is an equivalence. Hence

$$\mathbf{Fin}_{(n+1)!} \simeq (\mathbf{Fin}_{n+1} \simeq \mathbf{Fin}_{n+1})$$

and we are done.

Using an almost identical argument to that in the second part of (a); we get that $A \simeq A$ is finite for any finite type A .

6 (★★)

Consider a map $f : X \rightarrow Y$, and suppose that X is finite.

- (a) For $y : Y$, define $\text{inlm}_f(y) := \exists_{x:X}(f(x) = y)$. Show that, if type the Y has decidable equality, then inlm_f is decidable.
- (b) Suppose that f is surjective. Show that the following two statements are equivalent:
 - (i) The type Y has decidable equality
 - (ii) The type Y is finite

Hint for (i) \implies (ii): Induct on the size of X . If $f : X \simeq \mathbf{Fin}_{n+1} \rightarrow Y$, consider its restriction $f_n : \mathbf{Fin}_n \rightarrow Y$. Use (a) to do a case distinction on whether or not $\text{inlm}_{f_n}(f(n))$ holds.

Note that for all problems here, we are concerned with proving *propositions*. Thus, since X is finite, we may assume simply let $X := \mathbf{Fin}_n$ for some n in what follows.

(a) Assume Y has decidable equality and let $y : Y$. The goal is to show that for any $f : \mathbf{Fin}_n \rightarrow Y$, we have a term of type $(\text{inlm}_f(y) + \neg \text{inlm}_f(y))$. We proceed by induction on n , the size of the domain of f . For $n = 0$, the lemma is trivial, since the image of a map $\mathbf{Fin}_0 \rightarrow Y$ is empty. Assume that the statment holds for all maps $\mathbf{Fin}_n \rightarrow Y$. Let us prove it for $f : \mathbf{Fin}_{n+1} \rightarrow Y$. Let $f_n : \mathbf{Fin}_n \rightarrow Y$ be the restriction of f to \mathbf{Fin}_n . Using the inductive hypothesis on f_n , we get a term of type $(\text{inlm}_{f_n}(y) + \neg \text{inlm}_{f_n}(y))$. This allows us to do a case distinction:

- If y is in the image of f_n , it must be in the image of f .
- If not, consider instead the two cases $y = f(n)$ and $y \neq f(n)$, which we get from the decidable equality of Y .
 - If $y = f(n)$, then y is in the image of f .
 - If not, y cannot be in the image of f , since $y \neq f(n)$ and $y \neq f_n(x)$ for any $x : \mathbf{Fin}_n$.

This covers all cases, and hence we have a term of type $(\text{inIm}_f(y) + \neg \text{inIm}_f(y))$.

(b)

(i) \implies (ii) Assume Y has decidable equality. We proceed by induction on n . For $n = 0$, the problem is trivial, since surjectivity of f implies that Y is empty. For the inductive step, let $f_n : \mathbf{Fin}_n \rightarrow Y$ be the restriction of $f : \mathbf{Fin}_{n+1} \rightarrow Y$ to \mathbf{Fin}_n . Using (a), we may consider two cases:

Case 1: $f(n)$ is in the image of f_n . In this case, f_n is surjective and thus Y is finite by the inductive hypothesis.

Case 2: $f(n)$ is not in the image of f_n . In this case, one can construct an equivalence

$$e : Y \simeq \text{im}(f_n) + \mathbb{1}$$

defined (informally) by

$$e(y) = \begin{cases} \mathbf{inr}(\star) & \text{if } y = f(n) \\ \mathbf{inl}(y) & \text{otherwise} \end{cases}$$

Note that $\text{im}(f_n)$ inherits decidable equality from Y . Now, since f_n surjects onto its own image, we may conclude that $\text{im}(f_n)$ is finite. Furthermore, $\mathbb{1}$ is finite, and thus Y is finite.

(ii) \implies (i) Since having decidable equality is a proposition and Y is finite, we may assume that $Y \simeq \mathbf{Fin}_n$ for some n . But then we are done, since \mathbf{Fin}_n has decidable equality.