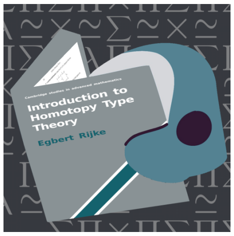

Worksheet 11

HoTTEST Summer School 2022

The HoTTEST TAs , and
15 August 2022

1 $(\star)$

Let A, B, C be types in a universe Type. Prove the following computation rules of $\mathsf{equiv{-}eq}$:

1. $\mathsf{equiv{-}eq}(\mathsf{refl}_A) = \mathsf{id}_A$;
2. for all paths $p : A = B$, $q : B = C$, we have $\mathsf{equiv{-}eq}(p \cdot q) = \mathsf{equiv{-}eq}(q) \circ \mathsf{equiv{-}eq}(p)$;
3. for any path $p : A = B$, we have $\mathsf{equiv{-}eq}(\bar{p}) = \mathsf{equiv{-}eq}(p)^{-1}$ where $(\bar{-})$ is path inversion.

Now suppose Type is a univalent universe. Do the analogous equations of (1–3) hold for $\mathsf{eq{-}equiv}$, the inverse of $\mathsf{equiv{-}eq}$?

2 $(\star)$

Consider a type familly $P : \mathsf{Type} \to \mathsf{Type}$, and let $p : A = B$ in Type. We can form $\mathsf{ap}_P(p) : P(A) = P(B)$, and we can transport along p to get an equivalence $p_* : P(A) \simeq P(B)$.

(a) Show that $\mathsf{equiv{-}eq}(\mathsf{ap}_P(p)) = p_*$. When Type is univalent, deduce that

$$\mathsf{ap}_P(p) = \mathsf{eq{-}equiv}(p_*).$$

(b) Let A, B, C : Type. Using the universal property of propositional truncations, construct functions $\|A = B\| \to \|B = C\| \to \|A = C\|$ and $\|A = B\| \to \|B = A\|$ corresponding to composition of truncated paths and inversion of truncated paths, respectively.

We will use the usual symbols for composition and inversion of truncated paths, since the operation is clear from the context.

Recall (or show) that in the family $X \mapsto (A = X) : \mathsf{Type} \to \mathsf{Type}$, a path $p : X = Y$ acts by post-composition: for any $q : A = X$, we have that $p_*(q) \equiv q \cdot p : (A = Y)$.

(c) Show that in the family $X \mapsto \|A = X\| : \mathsf{Type} \to \mathsf{Type}$, a path $p : X = Y$ acts by *truncated* post-composition: for any $q : \|A = X\|$, we have that

$$p_*(q) = q \cdot |p| : \|A = Y\| .$$

3 ($\star\star$)

Assume Type is univalent.

1. Show that the type $\Sigma_{A:\mathsf{Type}}\mathsf{is\text{–}contr}A$ of all contractible types in Type is contractible;
2. Show that the universe of k-types

$$\mathsf{Type}^{\leq k} := \Sigma_{A:\mathsf{Type}}\mathsf{is\text{–}trunc}_k(A)$$

   is a $(k+1)$-type, for any $k \geq -2$;
3. Show that the universe of propositions $\mathsf{Type}^{\leq -1}$ is not a proposition;
4. ($\star\star\star$) Show that the universe of sets $\mathsf{Type}^{\leq 0}$ is not a set.

(This is exercise 17.1 from the HoTT intro book.)

4 $(\star\star)$

Give an example of a type family $B : A \to \mathsf{Type}$ for which the implication

$$\neg\big(\Pi_{(a:A)} B(a)\big) \longrightarrow \big(\Sigma_{(a:A)} \neg B(a)\big)$$

is false. *(This is exercise 17.2 from the HoTT intro book.)*

5 $(\star\star)$

Let $A : \mathsf{Type}$. The type $\mathsf{BAut}(A) :\equiv \Sigma_{X:\mathsf{Type}} \|A = X\|$ is called **the path component of A in Type**.

(a) Show that for $(X,p),(Y,q) : \mathsf{BAut}(A)$, we have $\big((X,p) =_{\mathsf{BAut}(A)} (Y,q)\big) \simeq (X = Y)$.

Note that $(A, |\mathsf{refl}_A|) : \mathsf{BAut}(A)$, so that $\mathsf{BAut}(A)$ is pointed. Write pt for this base point, and denote $\mathsf{Aut}(A) :\equiv (A \simeq A)$.

(b) Assuming Type is univalent, deduce that $(\mathsf{pt} =_{\mathsf{BAut}(A)} \mathsf{pt}) \simeq \mathsf{Aut}(A)$.

Next, show that $\mathsf{BAut}(A)$ is **connected**:

(c) Show that for every $(X,p),(Y,q) : \mathsf{BAut}(A)$ we *merely* have a path:

$$\big\|(X,p) =_{\mathsf{BAut}(A)} (Y,q)\big\|.$$

The type $\mathsf{BAut}(\mathbb{2}) \equiv BS_2$ is also called *the universe of* 2*-element sets*.

(d) By combining the previous points and exercise 5 from worksheet 10, show that

$$(\mathsf{pt} =_{\mathsf{BAut}(\mathbb{2})} \mathsf{pt}) \simeq \mathbb{2}.$$

6 $(\star\star\star)$

Let Type be a univalent universe, and consider $A : \mathsf{Type}$. Recall (or show!) the *type-theoretic Yoneda lemma (Theorem 13.3.3):* for any $P : A \to \mathsf{Type}$ and $a : A$ we have an equivalence

$$\big(\Pi_{b:A}(a = b) \to P(b)\big) \simeq P(a).$$

(a) Suppose $\Sigma_{a:A}P(a)$ is contractible. Show that you then get an equivalence

$$\big(\Pi_{b:A}(a = b) \simeq P(b)\big) \simeq P(a).$$

(b) Show that the identity type, seen as a function $\mathsf{Id} : A \to (A \to \mathsf{Type})$ is an embedding. *(This is exercise 17.5 from the HoTT intro book, and it is due to Escardó.)*