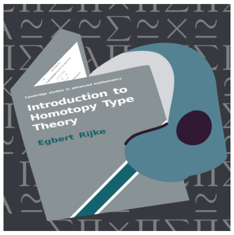

Worksheet 10

HoTTEST Summer School 2022

The HoTTEST TAs

10 August 2022

1 $(\star)$

Let A be a type. Show that

(a) $\|\|A\|\| \leftrightarrow \|A\|$

(b) $\exists_{(x:A)} \|B(x)\| \leftrightarrow \left\|\Sigma_{(x:A)} B(x)\right\|$

(c) $\neg\neg \|A\| \leftrightarrow \neg\neg A$

(d) $\mathsf{is\text{-}decidable}(A) \to (\|A\| \to A)$

2 $(\star\star)$

Consider two maps $f : A \to P$ and $g : B \to Q$ into propositions P and Q.

(a) Show that if f and g are propositional truncations, then $f \times g : A \times B \to P \times Q$ is also a propositional truncation

(b) Conclude that $\|A \times B\| \simeq \|A\| \times \|B\|$

3 $(\star\star)$

Consider a map $f : A \to B$. Show that the following are equivalent:

(i) f is an equivalence

(ii) f is both surjective and an embedding

4 $(\star\star)$

Prove **Lawvere's fixed point theorem:** For any two types A and B, if there is a surjective map $f : A \to B^A$, then for any $h : B \to B$, there (merely) exists an $x : B$ such that $h(x) = x$. In other words, show that

$$\left(\exists_{(f:A\to(A\to B))}\mathsf{is\text{-}surj}(f)\right) \to \left(\forall_{(h:B\to B)}\exists_{(b:B)} h(b) = b\right)$$

Disclaimer In the following exercises, we will use $\{0, \ldots, n\}$ to denote the elements of Fin_{n+1}, the finite type of $n+1$ elements.

5 $(\star)$

(a) Construct an equivalence $\mathsf{Fin}_{n^m} \simeq (\mathsf{Fin}_m \to \mathsf{Fin}_n)$. Conclude that if A and B are finite, then $(A \to B)$ is finite.

(b) Construct an equivalence $\mathsf{Fin}_{n!} \simeq (\mathsf{Fin}_n \simeq \mathsf{Fin}_n)$. Conclude that if A is finite, then $A \simeq A$ is finite.

6 $(\star\star)$

Consider a map $f : X \to Y$, and suppose that X is finite.

(a) For $y : Y$, define $\mathsf{inIm}_f(y) := \exists_{x:X}(f(x) = y)$. Show that, if type the Y has decidable equality, then inIm_f is decidable.

(b) Suppose that f is surjective. Show that the following two statements are equivalent:

   (i) The type Y has decidable equality

   (ii) The type Y is finite

   Hint for (i) $\implies$ (ii): Induct on the size of X. If $f : X \simeq \mathsf{Fin}_{n+1} \to Y$, consider its restriction $f_n : \mathsf{Fin}_n \to Y$. Use (a) to do a case distinction on whether or not $\mathsf{inIm}_{f_n}(f(n))$ holds.