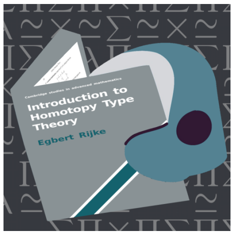

Worksheet 10 (Solved)

HoTTEST Summer School 2022

The HoTTEST TAs

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1 $(\star)$

Let A be a type. Show that

(a) $\|\|A\|\| \leftrightarrow \|A\|$

(b) $\exists_{(x:A)} \|B(x)\| \leftrightarrow \left\|\Sigma_{(x:A)}B(x)\right\|$

(c) $\neg\neg\|A\| \leftrightarrow \neg\neg A$

(d) $\mathsf{is\text{-}decidable}(A) \to (\|A\| \to A)$

(a) **For the left-to-right direction, we need to construct a map $f : \|\|A\|\| \to \|A\|$. Since the codomain is a proposition, is suffices, by the recursion principle for propositional truncation, to define $f(|x|)$, where $x : \|A\|$. We simply let, $f(|x|) = x$. For the other direction, we simply map $x : \|A\|$ to $|x| : \|\|A\|\|$.**

(b) **For the left-to-right direction, we need to define a map**

$$f : \left\|\Sigma_{(x:A)} \|B(x)\|\right\| \to \left\|\Sigma_{(x:A)}B(x)\right\|$$

The cocodomain is a proposition, so it suffices to define $f(| x, b_x |)$ for $x : A$ and $b_x : \|B(x)\|$ Now, the codomain is still a proposition, so we may apply the recursion principle again, this time to b_x. Thus, it suffices to define $f(| x, | b'_x ||)$ for $x : A$ and $b'_x : B(x)$. We define $f(| x, | b'_x ||) =| x, b'_x |$, and we are done. For the other direction, we need a map

$$g : \left\|\Sigma_{(x:A)}B(x)\right\| \to \left\|\Sigma_{(x:A)} \|B(x)\|\right\|$$

The recursion rule applies here too, so we only need to define $g(| x, b_x |)$ for $x : A$ and $b_x : \Sigma_{(x:A)}B(x)$. We define $g(| x, b_x |) =| x, | b_x ||$, and we are done.

(c) **Let us spell out the types involved. What we need is a bi-implication**

$$((\|A\| \to \emptyset) \to \emptyset) \leftrightarrow ((A \to \emptyset) \to \emptyset)$$

For the left-to-right direction, we are given $p : ((\|A\| \to \emptyset) \to \emptyset)$ **and** $q : A \to \emptyset$**. The goal is to produce an element of type** $\emptyset$**. If we can construct an element of type** $r : \|A\| \to \emptyset$**, then** $p(r) : \emptyset$**. Since** $\emptyset$ **is a proposition, it suffices to define** $r(|\, a\,|)$ **for** $a : A$**. Defining** $r(|\, a\,|) := q(a)$ **does the job. The other direction is proved similarly.**

(d) **Recall,** $\textsf{is-decidable}(A) := A + \neg A$**. We want to define** $f(x, p) : A$ **for** $x : \textsf{is-decidable}(A)$ **and** $p : \|A\|$**. Let us do this by induction on** x**. When** x **is** $\mathsf{inl}(a)$ **with** $a : A$**, it's easy: we define** $f(\mathsf{inl}(a), p) := a$**. When** x **is** $\mathsf{inr}(b)$ **with** $b : \neg A$**, we cannot directly produce an element of** A**. However, we can product an element of** $\emptyset$**, which gives us an element of** A **via** $\emptyset$**-recursion. We have** $p : \|A\|$**, which immediately gives us a term** $q : \neg\neg\,\|A\|$**. We apply part (c), and get a term** $r : \neg\neg A$**. Now,** $r(b) : \emptyset$ **and we are done.**

2 $(\star\star)$

Consider two maps $f : A \to P$ and $g : B \to Q$ into propositions P and Q.

(a) Show that if f and g are propositional truncations, then $f \times g : A \times B \to P \times Q$ is also a propositional truncation

(b) Conclude that $\|A \times B\| \simeq \|A\| \times \|B\|$

(a) **Let** R **be a proposition, We need to show that precomposition**

$$- \circ (f \times g) : (P \times Q \to R) \to (A \times B \to R)$$

is an equivalence. Since both sides are propositions, it suffices to construct a map of type $(A \times B \to R) \to (P \times Q \to R)$**. To this end, let** $F : A \times B \to R$**, and let** $(p, q) : P \times Q$**. We need to construct an element of** R**. Consider the type** $(Q \to R)$**. This is a proposition, so let us instantiate the universal property of** f **with it. We get**

$$- \circ f : (P \to (Q \to R)) \simeq (A \to (Q \to R))$$

If we can construct and elment $h : A \to (Q \to R)$, we are done, since then $((- \circ f)^{-1}(h))(p, q) : R$. In order to construct h, we introduce $a : A$ and $q' : Q$. Our goal is again to construct an element of R, but this time, we have managed to introduce $a : A$ to our context! Let's repeat the procedure, this time using the universal property of g. We instantiate it with the proposition $(P \to R)$, which gives us an equivalence.

$$- \circ g : (Q \to (P \to R)) \simeq (B \to (P \to R))$$

By the above, it again suffices to construct a function $\ell : B \to (P \to R)$, since $((- \circ g)^{-1}(\ell))(q, p) : R$. We define ℓ by $\ell(b, p) = F(a, b)$ and we are done.

(b) For any type C, the map $\lambda x.\ |x| : C \to \|C\|$ is a propositional truncation. By part (a), this implies that the canonical map $A \times B \to \|A\| \times \|B\|$ is a propositional truncation. But so is the map $\lambda x.\,|x| : A \times B \to \|A \times B\|$. Hence we get, by the two-out-of-three property of propositional truncations (Proposition 14.1.4) that $\|A\| \times \|B\| \simeq \|A \times B\|$

3 $(\star\star)$

Consider a map $f : A \to B$. Show that the following are equivalent:

(i) f is an equivalence

(ii) f is both surjective and an embedding

For the left-to-right direction, there is not much to prove. We know that all equivalences are embeddings, and clearly, since f is an equivalence, it must be surjective.

For the other direction, let us assume that f is both surjective and an embedding. Let us consider the family of propositions $P(b) := \mathsf{isContr}(\mathsf{fib}_f(b))$ over B. We are done if we can construct a section $\Pi_{(b:B)}P(b)$. Since f is surjective, we get, by Proposition 15.2.3, an equivalence

$$\left(\Pi_{(b:B)}P(b)\right) \simeq \left(\Pi_{(a:A)}P(f(a))\right)$$

Applying the inverse of the above equivalence, we only need to show that $\mathsf{fib}_f(f(a))$ is contractible for each $a : A$. Clearly, $\mathsf{fib}_f(f(a))$ is pointed (has an element) since $(a, \mathsf{refl}_{f(a)}) : \mathsf{fib}_f(f(a))$. We now need to show that

$$(a, \mathsf{refl}_{f(a)}) =_{\mathsf{fib}_f(f(a))} (x, p) \tag{1}$$

for each $x : A$ and $p : f(x) = f(a)$. Since f is an embedding, we get $(\mathsf{ap}_f)^{-1}(p^{-1}) : a = x$. This shows that the first components in (1) agree. We now need to show that the second components agree with respect to $(\mathsf{ap}_f)^{-1}(p^{-1})$. That is, we need to show that

$$\mathsf{tr}_{\lambda x.f(x)=f(a)}((\mathsf{ap}_f)^{-1}(p^{-1}), \mathsf{refl}_{f(a)}) = p$$

It is an easy lemma that the left-hand-side is equal to

$$(\mathsf{ap}_f((\mathsf{ap}_f)^{-1}(p^{-1})))^{-1} \cdot \mathsf{refl}_{f(a)}$$

ap_f and ap_f^{-1} cancel out, an thus it is equal to

$$(p^{-1})^{-1} \cdot \mathsf{refl}_{f(a)} = p$$

and we are done.

4 $(\star\star\star)$

Prove **Lawvere's fixed point theorem:** For any two types A and B, if there is a surjective map $f : A \to B^A$, then for any $h : B \to B$, there (merely) exists an $x : B$ such that $h(x) = x$. In other words, show that

$$\left(\exists_{(f:A\to(A\to B))}\mathsf{is\text{-}surj}(f)\right) \to \left(\forall_{(h:B\to B)}\exists_{(b:B)}h(b) = b\right)$$

Let $x : \left(\exists_{(f:A\to(A\to B))}\mathsf{is\text{-}surj}(f)\right)$ and $h : B \to B$. The goal is to show that there merely exists an element $b : B$ s.t. $h(b) = b$. This is a proposition, so we may assume $x :=\mid f, p \mid$, where $f : A \to B^A$ and $p : \mathsf{is\text{-}surj}(f)$. Given $a : A$, define, for ease of notation, $f_a : A \to B$ by $f_a = f(a)$. We get a function $F : A \to B$, given by

$$F(a) := h(f_a(a))$$

Surjectivity of f tells us that there is some $a : A$ such that $f_a = F$. We claim that $f_a(a)$ is a fixed point. We need to show that $h(f_a(a)) = f_a(a)$. Since $f_a = F$, it is enough to show that $h(f_a(a)) = F(a)$. But this is precisely how we defined F, so we are done.

Disclaimer In the following exercises, we will use $\{0, \ldots, n\}$ to denote the elements of Fin_{n+1}, the finite type of $n + 1$ elements.

5 $(\star)$

(a) Construct an equivalence $\mathsf{Fin}_{n^m} \simeq (\mathsf{Fin}_m \to \mathsf{Fin}_n)$. Conclude that if A and B are finite, then $(A \to B)$ is finite.

(b) Construct an equivalence $\mathsf{Fin}_{n!} \simeq (\mathsf{Fin}_n \simeq \mathsf{Fin}_n)$. Conclude that if A is finite, then $A \simeq A$ is finite.

(a) We proceed by induction on m. For $m := 0$, we need to show that

$$\mathsf{Fin}_1 \simeq (\mathsf{Fin}_0 \to \mathsf{Fin}_n)$$

Now, Fin_1 contains precisely 1 element, and so does $\mathsf{Fin}_0 \to \mathsf{Fin}_n$ since $\mathsf{Fin}_0 \simeq \emptyset$. Hence, the two types must be equivalent. For the inductive step, assume that $\mathsf{Fin}_{n^m} \simeq (\mathsf{Fin}_m \to \mathsf{Fin}_n)$ holds. The goal is to show that

$$\mathsf{Fin}_{n^{m+1}} \simeq (\mathsf{Fin}_{m+1} \to \mathsf{Fin}_n)$$

First, note that

$$\mathsf{Fin}_{n^{m+1}} = \mathsf{Fin}_{n^m \cdot n} = \mathsf{Fin}_{\underbrace{n^m + n^m + \cdots + n^m}_{n \text{ times}}} \simeq \underbrace{\mathsf{Fin}_{n^m} + \cdots + \mathsf{Fin}_{n^m}}_{n \text{ times}}$$

It is an easy lemma that for any type A, we have $\underbrace{A + \cdots + A}_{n \text{ times}} \simeq A \times \mathsf{Fin}_n$.

Consequently, we get

$$\mathsf{Fin}_{n^{m+1}} \simeq \mathsf{Fin}_{n^m} \times \mathsf{Fin}_n$$

By the inductive hypothesis, we have

$$\mathsf{Fin}_{n^m} \times \mathsf{Fin}_n \simeq (\mathsf{Fin}_m \to \mathsf{Fin}_n) \times \mathsf{Fin}_n$$

Hence, it suffices to construct an equivalence

$$f : (\mathsf{Fin}_m \to \mathsf{Fin}_n) \times \mathsf{Fin}_n \simeq (\mathsf{Fin}_{m+1} \to \mathsf{Fin}_n)$$

We define $f(g, x)$ by

$$f(g,x)(y) = \begin{cases} g(y) & \text{if } y < m \\ x & \text{otherwise} \end{cases}$$

and its inverse f^{-1} by

$$f^{-1}(g) = (g_m, g(m))$$

where $g_m : \mathsf{Fin}_m \to \mathsf{Fin}_n$ is the restriction of g to Fin_m. The fact that f and f^{-1} cancel out is immediate.

For the second part of the question, let us assume that A and B are finite types. We want to show that $(A \to B)$ is finite. This is a proposition, so we may assume that we have equivalences $A \simeq \mathsf{Fin}_m$ and $B \simeq \mathsf{Fin}_n$ for some $n, m : \mathbb{N}$. We then have

$$(A \to B) \simeq (\mathsf{Fin}_m \to \mathsf{Fin}_n) \simeq \mathsf{Fin}_{n^m}$$

and thus $(A \to B)$ is also a finite type.

(b) We proceed by induction on n. For $n = 0$, we immediately get

$$\mathsf{Fin}_{0!} = \mathsf{Fin}_1 \simeq (\mathsf{Fin}_0 \simeq \mathsf{Fin}_0)$$

since there is precisely one equivalence $\mathsf{Fin}_0 \simeq \mathsf{Fin}_0$ (recall, $\mathsf{Fin}_0 \simeq \emptyset$). Assume as inductive hypothesis that $\mathsf{Fin}_{n!} \simeq (\mathsf{Fin}_n \simeq \mathsf{Fin}_n)$. We need to construct an equivalence

$$\mathsf{Fin}_{(n+1)!} \simeq (\mathsf{Fin}_{n+1} \simeq \mathsf{Fin}_{n+1})$$

We have

$$\mathsf{Fin}_{(n+1)!} = \mathsf{Fin}_{n!} \times \mathsf{Fin}_{n+1}$$

using similar reasoning as in (a). Hence, by the inductive hypothesis, we have

$$\mathsf{Fin}_{(n+1)!} \simeq (\mathsf{Fin}_n \simeq \mathsf{Fin}_n) \times \mathsf{Fin}_{n+1}$$

Proving that

$$(\mathsf{Fin}_n \simeq \mathsf{Fin}_n) \times \mathsf{Fin}_{n+1} \simeq (\mathsf{Fin}_{n+1} \simeq \mathsf{Fin}_{n+1})$$

is now just straightforward combinatorics. The idea is that given an element $x : \mathsf{Fin}_{n+1}$, any equivalence $f : \mathsf{Fin}_n \simeq \mathsf{Fin}_n$ can be extended to an equivalence $f_x : \mathsf{Fin}_{n+1} \simeq \mathsf{Fin}_{n+1}$ by

$$f_x(y) = \begin{cases} f(y) & \text{if } y < x \\ n & \text{if } y = x \\ f(y-1) & \text{otherwise} \end{cases}$$

We won't do this here, but it's easy to verify that $(f, x) \mapsto f_x$ is an equivalence. Hence

$$\mathsf{Fin}_{(n+1)!} \simeq (\mathsf{Fin}_{n+1} \simeq \mathsf{Fin}_{n+1})$$

and we are done.

Using an almost identical argument to that in the second part of (a); we get that $A \simeq A$ is finite for any finite type A.

6 $(\star\star\star)$

Consider a map $f : X \to Y$, and suppose that X is finite.

(a) For $y : Y$, define $\mathsf{inIm}_f(y) := \exists_{x:X}(f(x) = y)$. Show that, if type the Y has decidable equality, then inIm_f is decidable.

(b) Suppose that f is surjective. Show that the following two statements are equivalent:

(i) The type Y has decidable equality

(ii) The type Y is finite

Hint for (i) $\implies$ (ii): Induct on the size of X. If $f : X \simeq \mathsf{Fin}_{n+1} \to Y$, consider its restriction $f_n : \mathsf{Fin}_n \to Y$. Use (a) to do a case distinction on whether or not $\mathsf{inIm}_{f_n}(f(n))$ holds.

Note that since X is finite, we have an element of type $e' : \|X \simeq \mathsf{Fin}_n\|$ for some $n : \mathbb{N}$. For all problems here, we are concerned with proving *propositions*, so we may use the recursion rule on e' to get an element $e : X \simeq \mathsf{Fin}_n$. It is an easy lemma that

- **$f : X \to Y$ has decidable image (question (a)) iff $f \circ e^{-1} : \mathsf{Fin}_n \to Y$ has decidable image**
- **$f : X \to Y$ is surjective (question (b)) iff $f \circ e^{-1} : \mathsf{Fin}_n \to Y$ is surjective**

Therefore, let's be informal and pretend that $f : \mathsf{Fin}_n \to Y$ in what follows (even though, under the hood, we are working with $f \circ e^{-1}$) and forget about X altogether.

(a) **Assume Y has decidable equality and let $y : Y$. The goal is to show that for any $f : \mathsf{Fin}_n \to Y$, we have a term of type $(\mathsf{inIm}_f(y) + \neg\,\mathsf{inIm}_f(y))$. We proceed by induction on n, the size of the domain of f. For $n = 0$, the lemma is trivial, since the image of a map $\mathsf{Fin}_0 \to Y$ is empty. Assume that the statment holds for all maps $\mathsf{Fin}_n \to Y$. Let us prove it for $f : \mathsf{Fin}_{n+1} \to Y$. Let $f_n : \mathsf{Fin}_n \to Y$ be the restriction of f to Fin_n. Using the induction hypothesis on f_n, we get a term of type $(\mathsf{inIm}_{f_n}(y) + \neg\,\mathsf{inIm}_{f_n}(y))$. This allows us to do a case distinction:**

- **If y is in the image of f_n, it must be in the image of f.**
- **If not, consider instead the two cases $y = f(n)$ and $y \neq f(n)$, which we get from the decidable equality of Y.**
  - **If $y = f(n)$, then y is in the image of f.**
  - **If not, y cannot be in the image of f, since $y \neq f(n)$ and $y \neq f_n(x)$ for any $x : \mathsf{Fin}_n$.**

This covers all cases, and hence we have a term of type $(\mathsf{inIm}_f(y) + \neg\,\mathsf{inIm}_f(y))$.

(b)

(i) $\implies$ (ii) **Assume Y has decidable equality. We proceed by induction on n. For $n = 0$, the problem is trivial, since surjectivity of f implies that Y is empty. For the inductive step, let $f_n : \mathsf{Fin}_n \to Y$ be the restriction of $f : \mathsf{Fin}_{n+1} \to Y$ to Fin_n. Using (a), we may consider two cases:**

Case 1: $f(n)$ is in the image of f_n. In this case, f_n is surjective and thus Y is finite by the induction hypothesis.

Case 2: $f(n)$ is not in the image of f_n. In this case, one can construct an equivalence

$$e : Y \simeq \mathsf{im}(f_n) + \mathbb{1}$$

defined (informally) by

$$e(y) = \begin{cases} \mathsf{inr}(\star) & \text{if } y = f(n) \\ \mathsf{inl}(y) & \text{otherwise} \end{cases}$$

Note that $\mathsf{im}(f_n)$ inherits decidable equality from Y. Now, since f_n surjects onto its own image, we may conclude that $\mathsf{im}(f_n)$ is finite. Furthermore, $\mathbb{1}$ is finite, and thus Y is finite.

(ii) $\implies$ (i) **Since having decidable equality is a proposition and Y is finite, we may assume that $Y \simeq \mathsf{Fin}_n$ for some n. But then we are done, since Fin_n has decidable equality.**