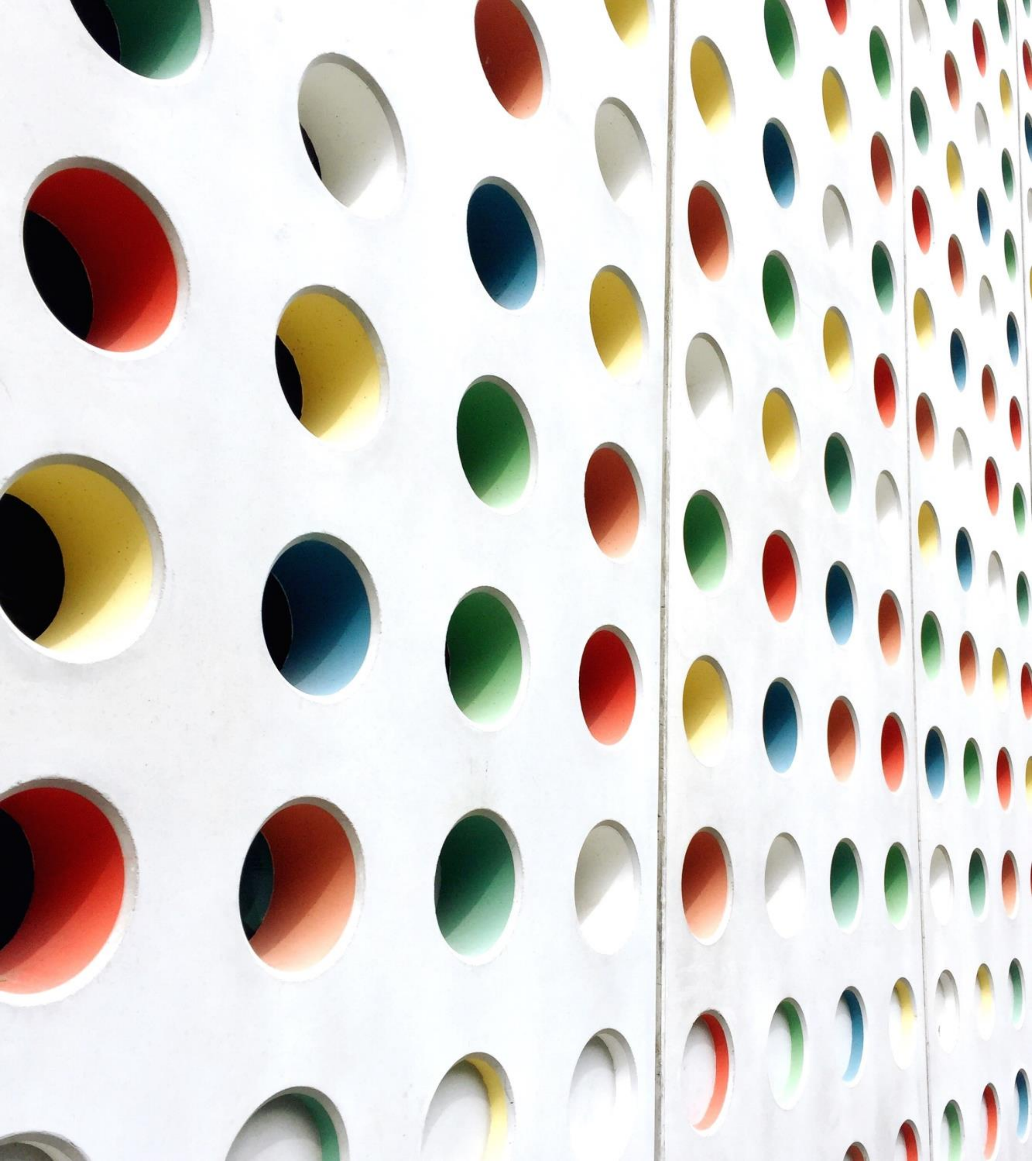

System of Linear Equations

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The Angelite Prayer

Almighty God, we glorify you for giving us the Angelite Charism. We thank you for the gift of your Son, Jesus Christ, who is the Way, Truth and Life. We bless you for the continuous guidance of the Holy Spirit.
Grant us, we pray, courage and strength that we may give perpetual praise to you in whatever we do.
We ask this through Christ, our Lord. Amen.

Oh, Holy Guardian Angels, guide us and protect us!
Laus Deo semper!

Linear Equations in n Variables

A **linear equation** in *n* variables can be written in the form $a_1x_1 + a_2x_2 + a_3x_3 + \ldots + a_nx_n = b$, where a_1, a_2, a_3, ... a_n, and b are real numbers.

Examples of Linear Equations

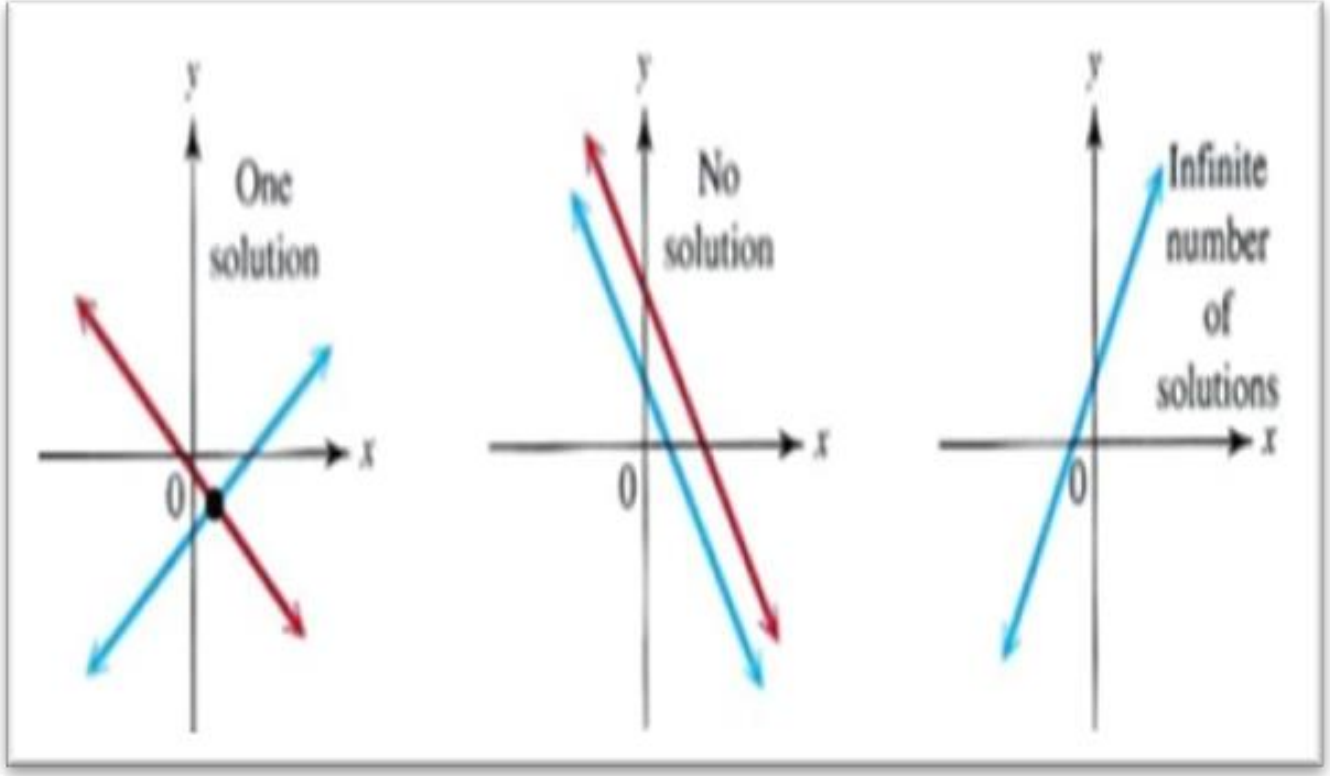
Linear equation in 2 variables	$2x + 4y = 19$
Linear equation in 3 variables	$3x - 5y + z = 1$
Linear equation in 4 variables	$-4w + 4x - 4y + 4z = 9$

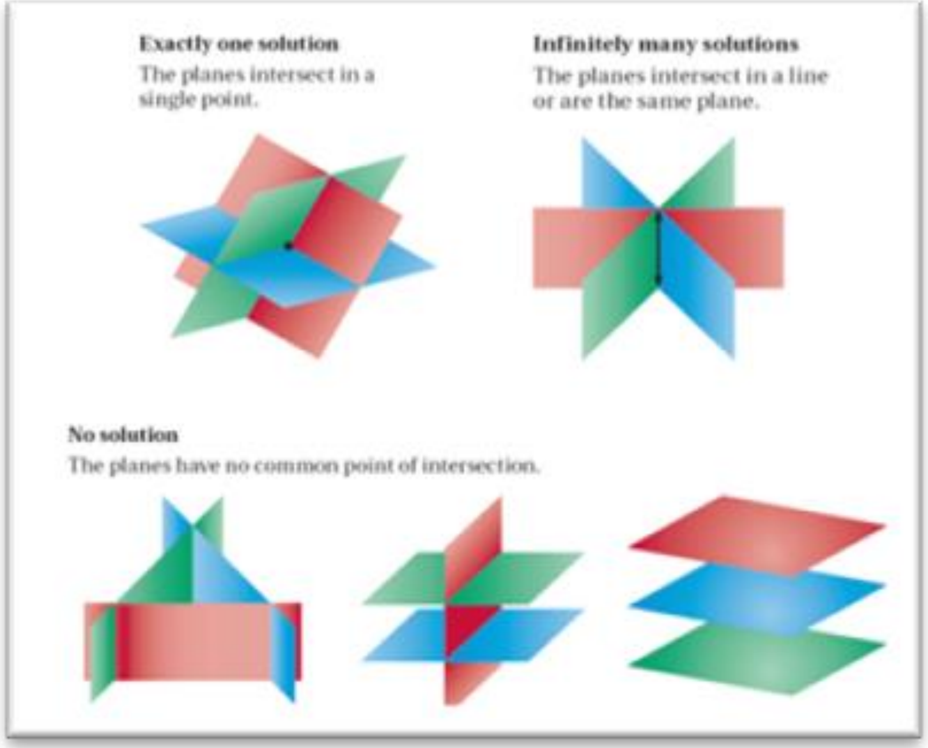
System of Linear Equations

A **system of linear equations** in n variables is a set of m linear equations, each of which has same variables. It also called as linear system.

Examples of System of Linear Equations

Systems of linear equations in 2 variables:	$2x + 3y = 3$ $x + 4y = 5$	$x + y = 10$ $5y = 6$
Systems of linear equations in 3 variables:	$x + y + z = 1$ $2x + 3y - 3z = 10$ $5x - 4y + 6z = 2$	$x + y = 3$ $4x + 5z = 10$ $6y - 10z = -7$
Systems of linear equations in 4 variables:	$-2w + 4x - 6y + 8z = 10$ $2w - 5x - 16y + 4z = 0$ $3w + 4x - 5y + 6z = 7$ $y - x + y + z = 1$	$w + x + y + z = 4$ $w - x + 2y = 8$ $w + 3x = 12$ $w = 16$





Linear System in 2 Variables

Linear System in 3 Variables

Systems of Linear Equations

Solution of Linear System

A **solution** is an ordered set of values of variables that satisfies the equations of a system of equations.

For a system of linear equations, precisely one of the statements below is true.

a. The system has exactly one solution (consistent system).

b. The system has infinitely many solutions (consistent system).

c. The system has no solution (inconsistent system).

LINEAR SYSTEM AND MATRICES

An **augmented matrix** is a matrix derived from the coefficient and constant terms of a system of linear equations.

A **coefficient matrix** is a matrix containing only the coefficients of the system of linear equations.

System of Linear Equations	*Augmented Matrix*	*Coefficient Matrix*
$2x + 5y = 10$ $3x + 4y = 8$	$\begin{bmatrix} 2 & 5 & 10 \\ 3 & 4 & 8 \end{bmatrix}$	$\begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix}$
$3x - 5y + 6z = 1$ $4x + 8y + 7z = 0$ $7x + y - 9z = 10$	$\begin{bmatrix} 3 & -5 & 6 & 1 \\ 4 & 8 & 7 & 0 \\ 7 & 1 & -9 & 10 \end{bmatrix}$	$\begin{bmatrix} 3 & -5 & 6 \\ 4 & 8 & 7 \\ 7 & 1 & -9 \end{bmatrix}$
$2x + 5y + z = 11$ $7x \quad + 4z = -6$ $y - z = 0$	$\begin{bmatrix} 2 & -5 & 1 & 11 \\ 7 & 0 & 4 & -6 \\ 0 & 1 & -1 & 0 \end{bmatrix}$	$\begin{bmatrix} 2 & -5 & 1 \\ 7 & 0 & 4 \\ 0 & 1 & -1 \end{bmatrix}$

Elementary Matrices

An n x n matrix is an **elementary matrix** when it can be obtained from the identity matrix I_n by a single elementary row operation.

Elementary Row Operations

1. Interchange two rows of a matrix.
2. Multiply the elements of a row by a nonzero constant.
3. Add a multiple of the elements of one row to the corresponding elements of another row.

1. 3 x 3 elementary matrix

$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ To obtain A from $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, multiply the second row of I_3 by 3.

2. 2 x 2 matrix

$B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ To obtain B from $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, interchange row 1 and row 2 of I_2.

Row-Echelon Form

A matrix is in **row echelon form** when it satisfies the following conditions:

a. The first nonzero element in each row is 1 (known as leading entry).
b. Each leading entry is in a column to the right of the leading entry in the previous row.
c. Rows with all zero elements, if any, are below rows having a nonzero element.

Examples

1. $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

2. $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

3. $\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

Conditions	Example 1	Example 2	Example 3
The first nonzero element in each row is 1	$\begin{bmatrix} \mathbf{1} & 2 & 3 & 4 \\ 0 & 0 & \mathbf{1} & 3 \\ 0 & 0 & 0 & \mathbf{1} \end{bmatrix}$	$\begin{bmatrix} \mathbf{1} & 2 & 3 & 4 \\ 0 & \mathbf{1} & 3 & 1 \\ 0 & 0 & \mathbf{1} & 5 \\ 0 & 0 & 0 & \mathbf{1} \end{bmatrix}$	$\begin{bmatrix} \mathbf{1} & 2 \\ 0 & \mathbf{1} \\ 0 & 0 \end{bmatrix}$
Each leading entry is in a column to the right of the leading entry in the previous row.	$\begin{bmatrix} \mathbf{1} & 2 & 3 & 4 \\ 0 & 0 & \mathbf{1} & 3 \\ 0 & 0 & 0 & \mathbf{1} \end{bmatrix}$	$\begin{bmatrix} \mathbf{1} & 2 & 3 & 4 \\ 0 & \mathbf{1} & 3 & 1 \\ 0 & 0 & \mathbf{1} & 5 \\ 0 & 0 & 0 & \mathbf{1} \end{bmatrix}$	$\begin{bmatrix} \mathbf{1} & 2 \\ 0 & \mathbf{1} \\ 0 & 0 \end{bmatrix}$
Rows with all zero elements, if any, are below rows having a nonzero element.	$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ *Not applicable - has no row with all 0 elements*	$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ *Not applicable - has no row with all 0 elements*	$\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$

4. $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

The matrix is not in row-echelon form because it does not satisfy condition 1. The leading entry in the second row is not 1, $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

5. $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

The matrix is not in row echelon form because it does not satisfy condition 2. The leading entry of the third row is not the right of the previous row (second row), $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Reduced Row-Echelon Form

A matrix is in **reduced row-echelon form** when it satisfies the following conditions:

a. The matrix satisfies conditions for row-echelon form.
b. The leading entry in each row is the only nonzero entry in its column.

Matrices in Reduced Row-Echelon Form

Examples

1. $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

2. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

3. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

The matrices are in reduced row-echelon form. They all satisfy the conditions for row-echelon form. The leading entry in each row is the only nonzero entry in its column.

Why?

Is the matrix below in its reduced row-echelon form? Why?

$$4.\ \begin{bmatrix} 1 & 2 & 5 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Transforming Matrices in REF

Elementary Row Operations

1. Interchange two rows of a matrix.
2. Multiply the elements of a row by a nonzero constant.
3. Add a multiple of the elements of one row to the corresponding elements of another row.

Take Note

In this process, we shall use the notation **Rn** to indicate n^{th} row of a matrix. For example, R3 means third row of the matrix. We use ↔ to indicate interchange of rows and → Rn to show the result in Rn. R2 ↔ R3 means we interchange the second row and the third row. → R3 means we will see the result in the 3^{rd} row.

TIPS

- To make a leading entry of a row equal to 1, multiply the elements of that row by the reciprocal of that element.
- To change an **element** of a row to zero, multiply the elements of the previous row by **its opposite** and add to the corresponding elements of the row.
- You can perform simultaneous elementary operations.
- There are times that you need to interchange rows to facilitate the process.

Example

1. Transform $\begin{bmatrix} 2 & 4 \\ 3 & 10 \end{bmatrix}$ in row-echelon form.

Condition	Elementary Row Operation	Output
The leading entry of R1 must be 1. You multiply the elements of R1 by $\frac{1}{2}$.	$\frac{1}{2}R1 \rightarrow R1$ $\frac{1}{2}\begin{bmatrix}2 & 4\end{bmatrix} \rightarrow \begin{bmatrix}\mathbf{1} & \mathbf{2}\end{bmatrix}$	$\begin{bmatrix}1 & 2\\3 & 10\end{bmatrix}$
The first element of R2 must be 0. We can a multiple of the elements of R1 to the corresponding elements of R2. Multiply the elements of R1 by -3, then add the products to the corresponding elements of R2.	$-3R1 + R2 \rightarrow R2$ $-3\begin{bmatrix}1 & 2\end{bmatrix} + \begin{bmatrix}3 & 10\end{bmatrix} =$ $\begin{bmatrix}-3 & -6\end{bmatrix} + \begin{bmatrix}3 & 10\end{bmatrix}$ $\rightarrow \begin{bmatrix}\mathbf{0} & \mathbf{4}\end{bmatrix}$	$\begin{bmatrix}1 & 2\\0 & 4\end{bmatrix}$
The leading entry of R2 must be 1. Multiply the elements of R2 by $\frac{1}{4}$.	$\frac{1}{4}R2 \rightarrow R2$ $\frac{1}{4}\begin{bmatrix}0 & 4\end{bmatrix} \rightarrow \begin{bmatrix}\mathbf{0} & \mathbf{1}\end{bmatrix}$	$\begin{bmatrix}\mathbf{1} & \mathbf{2}\\\mathbf{0} & \mathbf{1}\end{bmatrix}$

Example

2. Transform $\begin{bmatrix} 2 & 4 & 6 \\ 0 & -3 & 3 \\ 1 & 6 & -2 \end{bmatrix}$ in row-echelon form.

Condition	Elementary Row Operation	Output
The leading entry of R1 must be 1. We interchange R1 and R3.	R1 $\leftrightarrow$ R3	$\begin{bmatrix} 1 & 6 & -2 \\ 0 & -3 & 3 \\ 2 & 4 & 6 \end{bmatrix}$
The first element of R2 is already 0. The first element of R3 must be zero. We multiply the elements of R1 by -2, then add to the corresponding elements of R3.	-2R1 + R3 $\rightarrow$ R3 -2[1 6 -2] + [2 4 6] = [-2 -12 4] + [2 4 6] $\rightarrow$ **[0 -8 10]**	$\begin{bmatrix} 1 & 6 & -2 \\ 0 & -3 & 3 \\ 0 & -8 & 10 \end{bmatrix}$
The leading entry of R2 must be 1. Multiply the elements of R2 by $\frac{-1}{3}$.	$\frac{-1}{3}$R2 $\rightarrow$ R2 $\frac{-1}{3}$[0 – 3 3] $\rightarrow$ **[0 1 -1]**	$\begin{bmatrix} 1 & 6 & -2 \\ 0 & 1 & -1 \\ 0 & -8 & 10 \end{bmatrix}$
The second element of R3 must be 0. We multiply the elements of R2 by 8, then add to the corresponding elements of R3.	8R1 + R3 $\rightarrow$ R3 8[0 1 -1] + [0 -8 10] = [0 8 -8] + [0 -8 10] $\rightarrow$ **[0 0 2]**	$\begin{bmatrix} 1 & 6 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix}$
The leading entry of R3 must be 1. Multiply the elements of R3 by $\frac{1}{2}$.	$\frac{1}{2}$R3 $\rightarrow$ R3 $\frac{1}{2}$[0 0 2] $\rightarrow$ **[0 0 1]**	$\begin{bmatrix} 1 & 6 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

Transforming Matrices in RREF

Remember the conditions for RREF:

a. Row-echelon form

b. The leading entry in each row is the only nonzero entry in its column.

Example

1. Transform $\begin{bmatrix} 2 & 4 \\ 3 & 10 \end{bmatrix}$ in reduced row-echelon form.

Transform this matrix in row-echelon first. In the previous example, the row-echelon form of the matrix is $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$.

Condition	Elementary Row Operation	Output
The leading entry in R2 must be the only nonzero entry in its column (C2). To "eliminate" 2, multiply the elements of R2 by -2, then add to the corresponding elements of R1. We consider now $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$. Eliminate = make it 0	$-2R2 + R1 \rightarrow R1$ $-2[0 \quad 1] + [1 \quad 2] =$ $[0 \quad -2] + [1 \quad 2]$ $\rightarrow \mathbf{[1 \quad 0]}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Example

2. Transform $\begin{bmatrix} 2 & 4 & 6 \\ 0 & -3 & 3 \\ 1 & 6 & -2 \end{bmatrix}$ in reduced row-echelon form.

- Transform this matrix in row-echelon form first. In the previous example, the row-echelon form of the matrix is $\begin{bmatrix} 1 & 6 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$.

We continue the process.

Condition	Elementary Row Operation	Output
The leading entry in R2 must be the only nonzero entry in its column (C2). To eliminate 6, multiply the elements of R2 by -6, then add to the corresponding elements of R1. We consider now $\begin{bmatrix} 1 & 6 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$.	-6R2 + R1 → R1 -6[0 1 -1] + [1 6 -2] = [0 -6 6] + [1 6 2] → **[1 0 8]**	$\begin{bmatrix} 1 & 0 & 8 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$
The leading entry in R3 must be the only nonzero entry in its column (C3). To eliminate 8, multiply the elements of R3 by -8, then add to the corresponding elements of R1. To eliminate -1, add the corresponding elements of R2 and R3.	-8R3 + R1 → R1 -8[0 0 1] + [1 0 8] = [0 0 -8] + [1 0 8] → **[1 0 0]** R2 + R3 → R2 [0 1 -1] + [0 0 1] → **[0 1 0]**	$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$

Example

3. Transform the matrix $\begin{bmatrix} 1 & 3 & 4 \\ 2 & 5 & 6 \end{bmatrix}$ in reduced row-echelon form.

The matrix is not in row-echelon form.

Condition	Elementary Row Operation	Output
The first term of R2 must be 0.	-2R1 + R2 → R2 -2[1 3 4] + [2 5 6] = [-2 -6 -8] + [2 5 6] → **[0 -1 -2]**	$\begin{bmatrix} 1 & 3 & 4 \\ 0 & -1 & -2 \end{bmatrix}$
The leading entry of R2 must be 1.	-1R2 → R2 -1[0 -1 -2] → **[0 1 2]**	$\begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}$
(It is in row-echelon form.) Eliminate 3 in C2.	-3R2 + R1 → R1 -3[0 1 2] + [1 3 4] = [0 -3 -6] + [1 3 4] → **[1 0 -2]**	$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \end{bmatrix}$

Title Lorem Ipsum

LOREM IPSUM DOLOR SIT AMET,
CONSECTETUER ADIPISCING ELIT.

NUNC VIVERRA IMPERDIET ENIM.
FUSCE EST. VIVAMUS A TELLUS.

PELLENTESQUE HABITANT MORBI
TRISTIQUE SENECTUS ET NETUS.