SYSTEMS OF EQUATIONS

Aiman Shaikh

aimuu2310@gmail.com

INTRODUCTION

Linear algebra is a branch of mathematics that deals with the study of linear equations and their representations in vector spaces. It is a fundamental tool for solving a wide range of mathematical problems and has numerous applications in fields such as:

- computer graphics, machine learning, and quantum mechanics, to name a few.

In linear algebra, the main objects of study are vectors, matrices, and linear transformations. Vectors are ordered collections of numbers and can be thought of as mathematical objects that describe magnitudes and directions. Matrices are rectangular arrays of numbers and can be used to represent and manipulate linear transformations, which are operations that preserve the concept of "linearity."

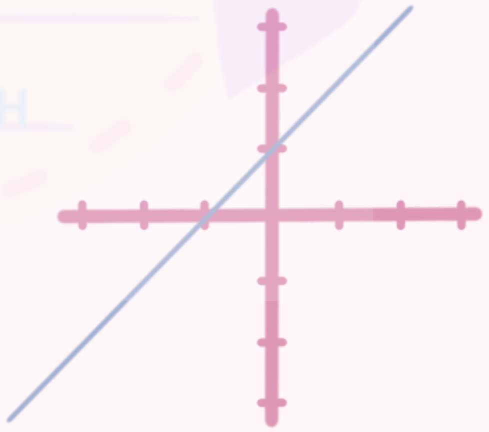
LINEAR ALGERA

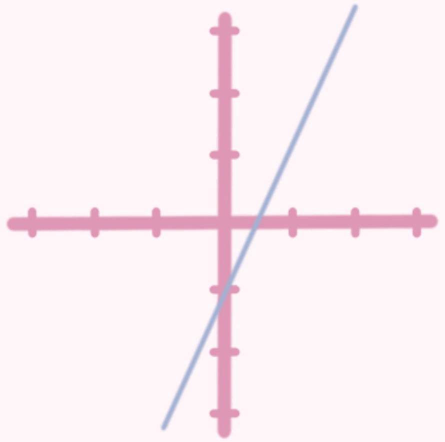
In particular, sets of two or more linear equations are what linear algebra is all about.

$$y = x + 1$$

$$y = 2x - 1$$

Additionally, we may graph the sets of equations' solutions, which enables us to see various patterns of activity.





A LINEAR SYSTEM

- In a system there are two variables in a pair (x,y)
- Linear equation is an algebraic equation with variables only to first power
- General linear equation: $c1x1 + c2x2 + \dots + cnxn = d$
- x's are variables and c's are constants
- Constants are sets of real numbers
- d is also a constant

TERMINOLOGY

SOLUTION

It is a value of the variables that makes LHS = RHS

SOLUTION SET

Set of all solutions to an equation

List: roster notation { (0,-1,0), ...}

LINEAR EQUATION

Is an algebraic equation with variables only to first power

TYPES OF EQUATIONS

1. Conditional: LHS = RHS on condition certain values of variable are used
    - Ex. $7x + 35 = 0$ only if $x = -5$ solution set is $\{-5\}$
2. Identity: LHS = RHS for any value of variable
    - Ex. $2x + 1 = 2x + 1$ any x value in the real numbers would work solution set is all values of x such that x is a real number $\{x \mid x \in R\}$
    - It has infinite number of solutions
3. Contradiction: no value of variables makes LHS = RHS
    - Is always false
    - Ex. $2x = 2x + 1$
    - No solutions, solution set is empty $\{\ \}$ or $\emptyset$ but not $\{\emptyset\}$

TYPES OF LINEAR EQUATIONS

CONSISTENT EQUATION

has at least one answer

(identities and conditions)

INCONSISTENT EQUATION

has no solution (contradiction)

Ex. $2x = 2x + 1$

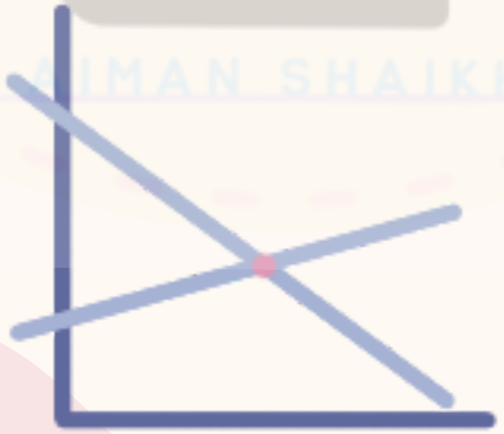
EQUIVALENT EQUATIONS

The two equations are not the same equations but they are equivalent because of the same solution set

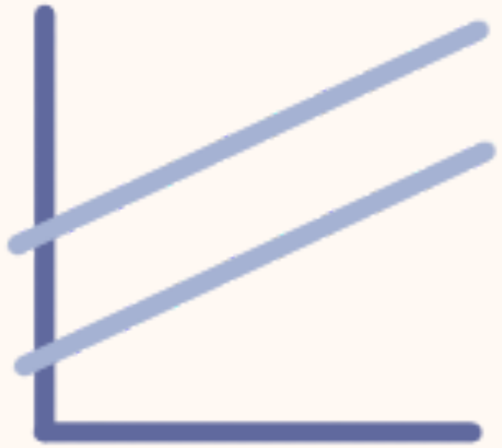
Ex.) $x + 35 = 0$ and $x + 5 = 0$

GEOMETRIC INTERPREATION OF LINEAR EQUATION

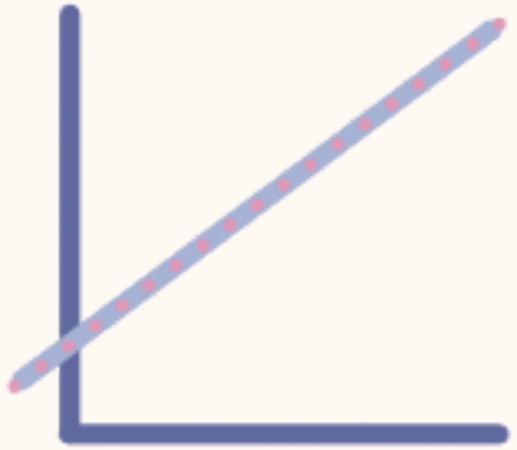
1 Solution



No Solution



Infinite Solns.



FORMS OF LINEAR SYSTEM

STANDARD FORM

$$\begin{cases} 8x_1 - x_2 = 4 \\ 5x_1 + 4x_2 = 1 \\ x_1 - 3x_2 = 2 \end{cases}$$

AUGMENTED FORM

$$[\vec{a}_1 + \vec{a}_2 \dots \vec{a}_n \mid \vec{b}]$$

$$\left[\begin{array}{cc|c} 8 & -1 & 4 \\ 5 & 4 & 1 \\ 1 & -3 & 2 \end{array}\right]$$

LINEAR COMBINATION VECTOR

$$x_1\vec{a}_1 + x_2\vec{a}_2 + \dots x_n\vec{a}_n = \vec{b}$$

$$x_1\begin{pmatrix} 8 \\ 5 \\ 1 \end{pmatrix} + x_2\begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}$$

MATRIX FORM

$$A\vec{x} = \vec{b}$$

$$\underbrace{\begin{pmatrix} 8 & -1 \\ 5 & 4 \\ 1 & -3 \end{pmatrix}}_{A}\underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{\vec{x}} = \underbrace{\begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}}_{\vec{b}}$$

VECTORS, MATRICES, AND LINEAR COMBINATIONS

Aiman Shaikh

aimuu2310@gmail.com

SPAN

Span is the set of all possible vectors that you can reach with a linear combination of a given pair of vectors. Here a and b could be any number to make new combinations.

$a\vec{u} + b\vec{v} = \vec{c}$

- one pivot position, the span is a line.
- two pivot positions, the span is a plane.
- three pivot positions, the span is R3.

SPAN

If the m X n matrix has a pivot position in every row, then the span of these vectors is m; that is, Span{v1,v2,...,vn} = R^m.

- m is the # of rows
- n is the # of columns

SPAN

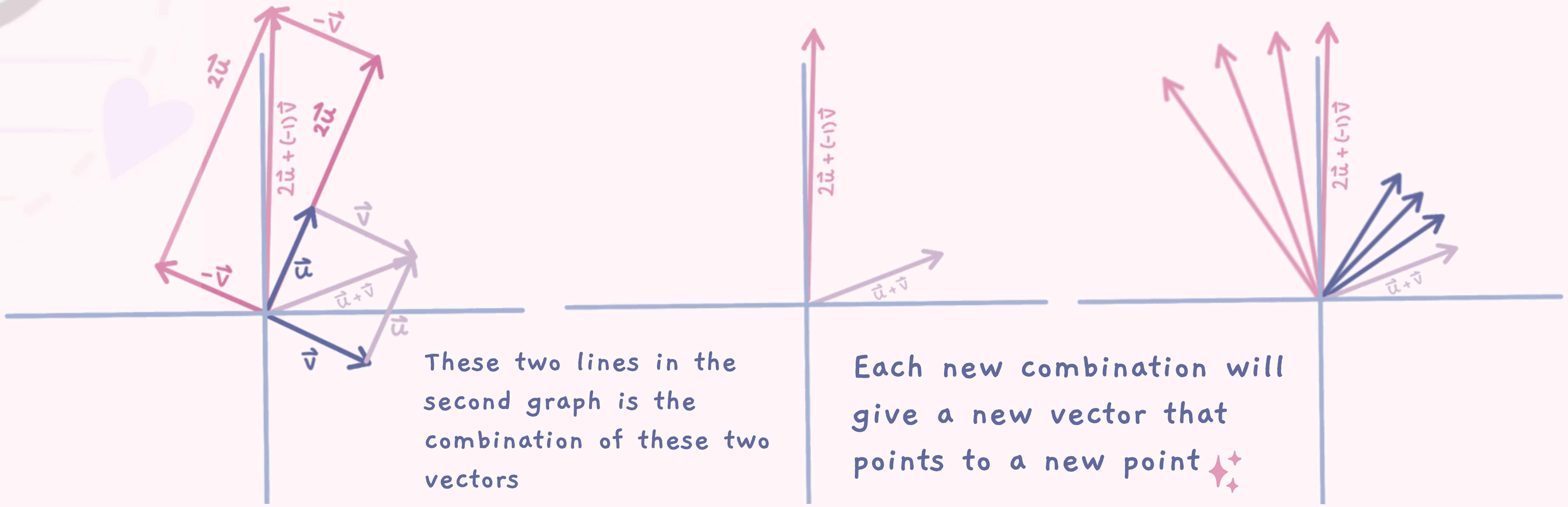
When two vectors are in different directions then we get infinite number of points

- The result we get is a plane like a sheet of paper filled with infinite number of points in R^2 (2D space).

$\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

If $a = 2$ $b = -1$

$2\vec{u} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ $-1\vec{v} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$



These two lines in the second graph is the combination of these two vectors

Each new combination will give a new vector that points to a new point

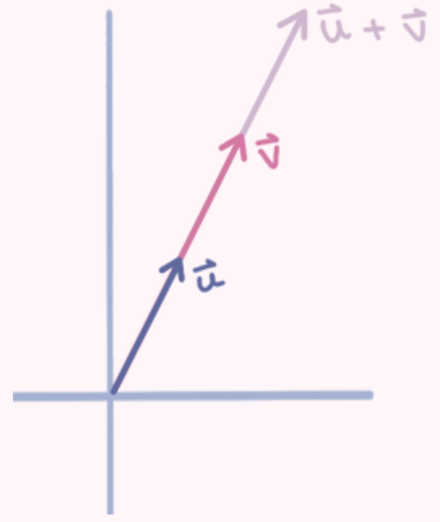
SPAN

Span of {u} is any vector making a combination of u which is just scaling the u.

- These vectors are multiples of u
- So the shape we get is a line
- In the example below v is just equal to 2u
- The result of these vector is equal to 3u

$\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

$\vec{u} + \vec{v} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$



SPAN

Note:

To span all of R^n:

- the system must have at least n columns and,
- pivot in every row

LINEAR DEPENDENCY

Linearly Dependent Vectors:

- Set of v1, v2, v3 ... vn is linearly dependent if anyone can be written as a linear combination of others
- ex:
  - If $v = 2u$ then v is a linear combination of u

$$\text{span}\{u, v\} = c_1 u + c_2 v$$
$$= c_1 u + c_2 2u$$
$$= (c_1 + 2c_2)u$$

- $\{u, v\}$ is linearly dependent since $v = 2u$
- v is in the same span of $\{u\}$

DEPENDENCY CHECK

Check the dependency by solving for $A\vec{x} = 0$.

- If A has any columns with no pivots:
  - Columns are linearly dependent
  - There are free variables and,
  - Infinite number of non trivial solutions

If a set $S = \{\vec{v}1, ... , \vec{v}p\}$ in Rn has more vectors (columns) than entries (rows) in each vector ($p > n$), then the set is linearly dependent

INDEPENDENCY CHECK

Check the independency by solving for $A\vec{x} = 0$.

- If A has pivots in every column:
  - Columns are linearly independent
  - No free variables
  - The system has a unique solution
  - $Ax = 0$ has only trivial solution
  - $c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n = 0$
  - Only if all c_s are 0

MATRIX-MATRIX RULES

- Addition of two matrices are possible only if dimensions of matrices are same
- To multiply one matrix with another, we need to check first, if the number of columns of the first matrix is equal to the number of rows of the second matrix.

MATRIX-MATRIX PRODUCT

CONCEPTUAL FORM

If A is an $m \times n$ matrix with columns $a1, \dots, an \in \mathbb{R}^m$ and if B is an $n \times p$ matrix with columns $b1, \dots, bp \in \mathbb{R}^n$, then the product of the matrices, denoted AB, is the $m \times p$ matrix whose columns are given by:

$AB = [Ab1 \ Ab2 \ \dots \ Abp]$

COMPUTATIONAL FORM

If A is an $m \times n$ matrix with rows $A1, \dots, An \in \mathbb{R}^n$ and if B is an $n \times p$ matrix with columns $b1, \dots, bp \in \mathbb{R}^n$, then the product of the matrices, denoted AB, is the $m \times p$ matrix

$$\boxed{AB = \begin{bmatrix} — A_1 — \\ \vdots \\ — A_m — \end{bmatrix} \begin{bmatrix} \vec{b}_1 & \vec{b}_2 \cdots \vec{b}_p \end{bmatrix} = \begin{bmatrix} A_1\cdot\vec{b}_1 & \cdots & A_1\cdot\vec{b}_p \\ \vdots & \cdots & \vdots \\ A_m\cdot\vec{b}_1 & \cdots & A_m\cdot\vec{b}_p \end{bmatrix}}$$

I AND E MATRICES

- Identity matrix is a square matrix composed of 0 and 1 and 1s are in the right spot.
- An n × n elementary matrix E is the result of applying one row operation to the n × n identity matrix I

IDENTITY AND ELEMENTARY MATRICES

I Matrix

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

E matrix

(2R1 + R2 on I)

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

IMPORTANT PROPERTIES

1. $A(BC) = (AB)C$
2. $A(B + C)A = BA + CA$
3. $r(AB) = (rA)B = A(rB)$
4. $IA = A = AI$

INVERTIBILITY AND DETERMINANTS

Aiman Shaikh

aimuu2310@gmail.com

INVERSE MATRIX

Algebraically:

- I is the multiplicative identity, any number multiplied by I is the number itself.
  - $x * I = x$
- Inverse is a value that you multiply by your original value to get back to the identity
  - $x * x^{-1} = I$

INVERSE MATRIX

In matrices:

- In is the multiplicative identity matrix, any matrix multiplied by In matrix is the matrix itself.
  - $A * I_n = A$
- Inverse of a matrix multiplied by its original form is equal to the identity matrix
  - $A * A^{-1} = I_n$ or $A^{-1} * A = I_n$

INVERTIBLE MATRIX

Conceptually:

- Let A be an $n \times n$ matrix. The matrix A is invertible if there exists an $n \times n$ matrix C such that $CA = I = AC$
- If A is an invertible $n \times n$ matrix, then for each $b \in Rn$, $Ax = b$ has a unique solution $x = A^{-1}b$
- An $n \times n$ matrix A is invertible if and only if A is row equivalent to I. In this case, any sequence of elementary row operations that reduces A to I also transforms I to A^{-1}
- If A is an invertible matrix, then A^{-1} is invertible and $(A^{-1})^{-1} = A$
- If A and B are invertible matrices, then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$

TO CALCULATE INVERSE

Note:

Augment the matrix A by the identity matrix I and transform A into I and whatever is on the right side is the A^{-1}

- $[A|I] \sim [I|A^{-1}]$

A MATRIX IS INVERTIBLE IF...

Given an $n \times n$ matrix A, the following are all equivalent statements.

- A is invertible.
- There exists a matrix C such that $CA = I \quad C = A^{-1}$.
- There exists a matrix D such that $AD = I \quad D = A^{-1}$.
- A is row equivalent to I.
- A has n pivot positions (pivot in every row and column).
- The equation $Ax = 0$ has only the trivial solution.
- The columns of A form a linearly independent set.
- The equation $Ax = b$ has a solution for every $b \in R^n$.
- The columns of A span R^n.

A MATRIX IS INVERTIBLE IF...

Another way to find if a matrix is invertible or not is to calculate its determinant which is the scalar value computed for a given square matrix.

- Any 2x2 matrix A formula is:
  - $ad - bc$ if this expression $\neq 0$ then A is invertible
- Any 3 x 3 matrix B formula is:
  - $a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$
- The part in the bracket is the determinant for (A_{11}), (A_{12}) and (A_{13}), if this expression $\neq 0$ then B is invertible

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$B = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

DETERMINANT

Conceptually:

- The determinant of a triangular matrix equals the product of its diagonal entries.
- If a matrix P is obtained by interchanging (swapping) two rows of the identity matrix, then $\det(P) = -1$.
  - The number of swaps done throughout rref will be the power to -1 so $(-1)^{\text{number of swaps}}$
- $\det(AB) = \det(A)\ \det(B)$
- The matrix A is invertible if and only if $\det(A) = 0$
- If A is an invertible matrix, then $\det(A-1)\quad 1/\det(A)$

EFFECT OF ROW OPERATIONS

Note:

- If A' is obtained from A by scaling a row by k, then $\det(A') = k\det(A)$.
- If A' is obtained from A by interchanging two rows, then $\det(A') = -\det(A)$.
- If A' is obtained from A by performing a replacement operation, then $\det(A') = \det(A)$

COFACTOR EXPANSIONS

Another way to calculate the determinant for n X n matrix

- We start by picking one row or column. No matter whatever option we select, the outcome will remain the same.
- By constructing a sum of terms, one for each element in the selected row, the determinant will be discovered. We multiply each entry in the row to create the term for that entry.
- The determinant of the entries left over when we have crossed out the row and column containing the entry.

EXAMPLE

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 3 & 1 & 2 \\ -2 & 4 & -3 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & -1 \\ 3 & 1 & 2 \\ -2 & 4 & -3 \end{bmatrix} \quad 3\begin{vmatrix} 0 & -1 \\ 4 & -3 \end{vmatrix} = -3((0\times-3)-(-1\times4)) = -3(0+4) = -12$$

$$\begin{bmatrix} 2 & 0 & -1 \\ 3 & 1 & 2 \\ -2 & 4 & -3 \end{bmatrix} \quad 1\begin{vmatrix} 2 & -1 \\ -2 & -3 \end{vmatrix} = 1((2\times-3)-(-1\times-2)) = 1(-6-2) = -8$$

$$\begin{bmatrix} 2 & 0 & -1 \\ 3 & 1 & 2 \\ -2 & 4 & -3 \end{bmatrix} \quad 2\begin{vmatrix} 2 & 0 \\ -2 & 4 \end{vmatrix} = -2((2\times4)-(0\times-2)) = -2(8-0) = -16$$

$$\det(A)\begin{bmatrix} 2 & 0 & -1 \\ 3 & 1 & 2 \\ -2 & 4 & -3 \end{bmatrix} = -12+(-8)-16 = -36$$

SIGN OF THE ENTRY

Note:

- The sign of the entries is determined in the alternate order

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

LINEAR ALGEBRA AND COMPUTING

Aiman Shaikh

aimuu2310@gmail.com

MATRIX FACTORIZATION

Introduction:

- It is a way to store some of the computation we perform in reducing a matrix A and reuse it in solving subsequent equations
- Same steps to solve these equations:
  - $Ax = b1$ and $Ax = b2$

MATRIX FACTORIZATION

Introduction (contd...):

- Through the use of two or more smaller matrices, a matrix can be represented mathematically as the result of matrix factorization.
- The LU decomposition is a typical illustration of matrix factorization, where a square matrix A is divided into two matrices, L (lower triangular) and U (upper triangular)
- L has ones along the diagonal and lower-triangular elements in this factorization, whereas U has upper-triangular elements.
- The initial matrix A is equivalent to the product of L and U.
- This factorization may be used to compute determinants and the inverses of matrices as well as to solve linear systems of equations.

STEPS TO FACTORIZE

Let A be a matrix and L be lower and U be upper matrix

1. Convert A to U form
2. Keep track of your elementary matrices while converting A to U.
3. Take inverse of all Elementary matrices $E_1^{-1} \ldots E_n^{-1}$.
4. The product of the inverse of all elementary matrix $(E_1^{-1} \ldots E_n^{-1})$ is equal to L.

STEPS TO FACTORIZE

Let A be a matrix and L be lower and U be upper matrix

1. $A = LU$ $\qquad\qquad Ax = b$
2. $LUx = b$ $\qquad\qquad$ compact U and x $\quad L(Ux) = b$
3. Write Ux as one entity: $Lc = b$
4. Solve for c by $[L \mid b]$
5. Solve for x in $Ux = b$ by $[U \mid c]$

MATRIX TRANSFORMATIONS

Aiman Shaikh

aimuu2310@gmail.com

LINEAR TRANSFORMATION

Introduction:

- A linear transformation is a function that maps one vector space to another while preserving the concept of linearity. This means that if you apply the linear transformation to a linear combination of vectors, the result will be equal to the linear combination of the images of those vectors under the transformation.
- Formally, a function T from a vector space R^n to a vector space R^m is a linear transformation if, for any scalars a and b, and any vectors $\vec{u}$ and $\vec{v}$ in R^n, it satisfies the following two properties:
  1. Linearity: $T(a * \vec{u} + b * \vec{v}) = a * T(\vec{u}) + b * T(\vec{v})$
  2. Additivity: $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$

SOLVING LINEAR TRANSFORMATION

To solve a linear transformation, you need to find the image of a given vector under the transformation. To do this, you simply apply the transformation to the vector, using the definition of the linear transformation.

- For example, consider the following linear transformation T:
- T: $R^2 \to R^2$ $T(\vec{x}, \vec{y}) = (2\vec{x} + 3\vec{y}, -\vec{x} + \vec{y})$
- Suppose you want to find the image of the vector (1, 2) under T. To do this, you simply substitute the values into the definition of the transformation:
- $T(1, 2) = (2 * 1 + 3 * 2, -1 + 2) = (5, 1)$
- So, the image of the vector (1, 2) under the linear transformation T is (5, 1).

RESULT

Note:

The result of a linear transformation is a vector in the target space, and you can think of the linear transformation as a function that maps the original vector to its image. In this example, the original vector is (1, 2) and its image is (5, 1)

ONE-TO-ONE

A linear transformation $T: V \to W$ is one-to-one (also called injective) if every distinct vector in the domain V maps to a distinct vector in the codomain W

- One way to check if a linear transformation is one-to-one is to check if its kernel (null space) contains only the zero vector. That is, if $T(u) = 0$, then u must be the zero vector. If the kernel contains any non-zero vectors, then the transformation is not one-to-one.
- The matrix will have independent columns which means there will be a pivot in every column.

ONTO

A linear transformation $T: V \to W$ is onto (also called surjective) if every vector in the codomain W is the image of at least one vector in the domain V

- One way to check if a linear transformation is onto is to check if its image (range) is equal to the codomain W. That is, if $T(V) = W$, where V is the domain of T, then T is onto.
- The matrix will have a pivot in every row.

STATE VECTORS

A state vector typically refers to a vector that represents the current state of a system that is being transformed by the linear transformation.

- $T(\vec{x})$ is the transition function
- $\vec{x}_0$ is the initial state
- $\vec{x}_1$ is the state next day
- $\vec{x}_n$ where n represents number of days
- Notation:
  - $\vec{x}_1 = T(\vec{x}_0) = A(\vec{x}_0)$ $\quad \vec{x}_1 = A(\vec{x}_0)$
  - $\vec{x}_n = T(\vec{x}_{n-1}) = A^n(\vec{x}_0)$

$$\vec{x}_1 = T(\vec{x}_0) = A\vec{x}_0$$

$$\begin{aligned}\vec{x}_2 &= T(\vec{x}_1) = A(\vec{x}_1)\\ &= A(A\vec{x}_0)\\ &= A^2\vec{x}_0\end{aligned}$$

$$\vec{x}_n = T(\vec{x}_{n-1}) = A^n\vec{x}_0$$

RANGE

To determine if a vector is in the range of the linear transformation T, you can use the following method:

1. Find the matrix representation of the linear transformation T. This is the matrix A that satisfies $T(x) = Ax$ for all input vectors x.
2. Solve the equation $Ax = b$ for x, where b is the vector you want to check if it is in the range of T.
3. If there is a solution for x, then b is in the range of T. If there is no solution for x, then b is not in the range of T.

KERNEL

To determine if a vector is in the kernel (also called the null space) of the linear transformation T, you can use the following method:

1. Find the matrix representation of the linear transformation T. This is the matrix A that satisfies $T(x) = Ax$ for all input vectors x.
2. Solve the equation $Ax = 0$ for x, where 0 is the zero vector. This means that you are looking for all solutions to the homogeneous equation $T(x) = 0$.
3. The set of all solutions to this equation is the kernel of T. If the vector v is a solution to the equation, then it is in the kernel of T.

COMPOSITION

To find the composition of two linear transformations, say T and S, you can apply the following steps:

1. Determine the domains and codomains of T and S. Let's say T maps vectors from a vector space R^n to a vector space R^m and S maps vectors from R^m to another vector space R^p.
2. Find the matrix representations of T and S. Let's say the matrix representation of T is A, and the matrix representation of S is B.
3. Multiply the matrices A and B in the correct order (BA) to obtain the matrix representation of the composition $S \circ T$. The resulting matrix will be the matrix representation of the linear transformation that results from first applying T to an input vector, and then applying S to the resulting output vector.

ORDER OF COMPOSITION

Note:

The order of the transformations matters when computing the composition. That is, $S \circ T$ is not the same as $T \circ S$ unless the domains and codomains of T and S happen to coincide.

INVERSE

General steps for finding the inverse of a linear transformation:

1. Find the matrix representation of the linear transformation: If the linear transformation is represented by a matrix, write down the matrix.
2. Form the augmented matrix: Append the identity matrix to the right of the matrix representation of the linear transformation to form an augmented matrix.
3. Use elementary row operations to transform the augmented matrix into the form $[I \mid A^{-1}]$, where I is the identity matrix and A^{-1} is the inverse of the matrix representation of the linear transformation.
4. Extract the inverse: The matrix A^{-1} is the inverse of the matrix representation of the linear transformation.

INVERTIBLE TRANSFORMATION

Note:

Not all linear transformations have matrix representations, and not all matrix representations correspond to invertible linear transformations.

IMPORTANT

Not all linear transformations have matrix representations, and not all matrix representations correspond to invertible linear transformations.

Check that the matrix A is invertible. A matrix is invertible if and only if its determinant is nonzero. If the determinant is zero, the matrix is not invertible and the transformation does not have an inverse.

Once you have found the inverse matrix A^{-1}, you can use it to apply the inverse transformation to a vector y as follows: $x = A^{-1} * y$

LINEAR TRANSFORMATION AND AREA

- The area of an object under a linear transformation is proportional to the determinant of the matrix that represents the transformation. Specifically, the determinant of a 2x2 matrix representing a linear transformation gives the factor by which the transformation scales areas.
- The determinant of this matrix is ad - bc. If the determinant is positive, then the transformation preserves orientation, and the area of a region under the transformation is scaled by a factor of |det[T]|. If the determinant is negative, then the transformation reverses orientation, and the area of a region under the transformation is scaled by a factor of |-det[T]|.

LINEAR TRANSFORMATION AND AREA

- A transformation is area preserving if and only if its determinant is equal to one.
- If the transformation is represented by a matrix A, then the determinant of A gives the factor by which the transformation changes the area. If det(A) = 1, then the transformation preserves the area, and if det(A) is not equal to 1, then the transformation changes the area by a factor equal to the absolute value of det(A).
- If det(A) is equal to 1 or -1, the transformation is area-preserving.

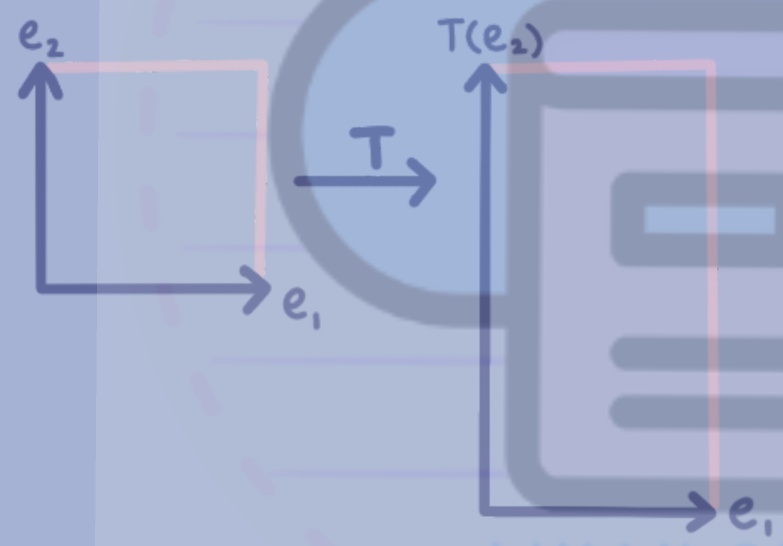
X AND Y STRETCH

In linear transformations, y-stretch and x-stretch refer to the amount by which the length of a vector in the y-direction and x-direction is changed under the transformation, respectively.

- Let T be a linear transformation from R^2 to R^2, and let v be a vector in R^2. If T stretches v by a factor of k in the y-direction and by a factor of h in the x-direction, then we say that T has a y-stretch of k and an x-stretch of h.

GEOMETRIC INTERPRETATION

Y-STRETCH

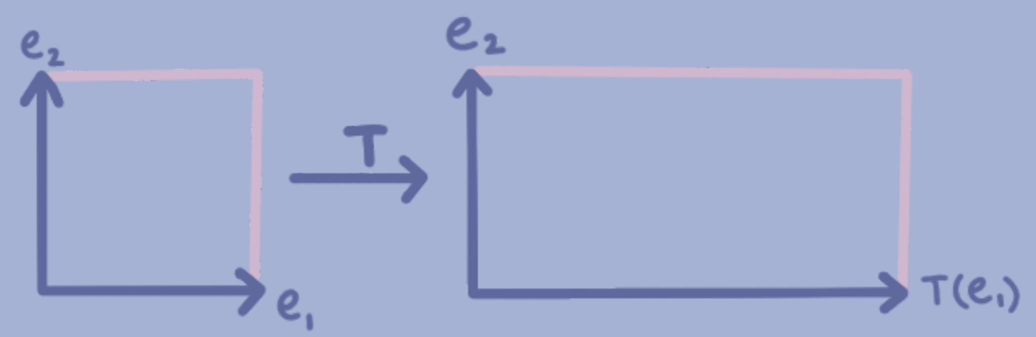


$$T(e1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$T(e2) = \begin{bmatrix} 0 \\ k \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$$

X-STRETCH



$$T(e1) = \begin{bmatrix} h \\ 0 \end{bmatrix} \qquad T(e2) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} h & 0 \\ 0 & 1 \end{bmatrix}$$

COMBINATION

$$\begin{bmatrix} h & 0 \\ 0 & k \end{bmatrix}$$

ROTATION

A rotation transformation is a linear transformation that rotates vectors in a plane by a certain angle around a fixed point called the center of rotation. The center of rotation is also referred to as the origin, and the angle of rotation is measured in degrees or radians.

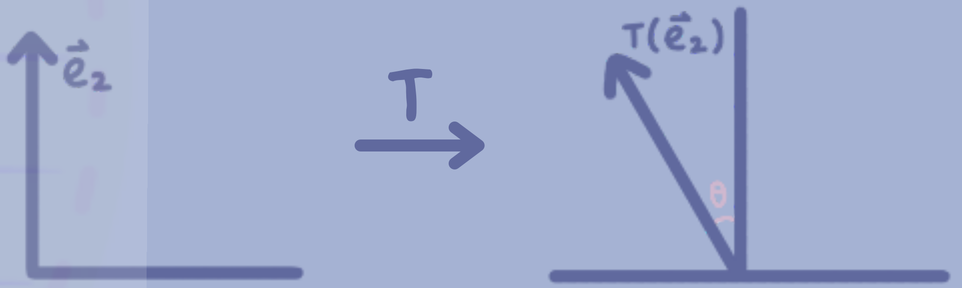
- Standard matrix is:
  - [cos(theta) -sin(theta); sin(theta) cos(theta)]

GEOMETRIC INTERPRETATION

X-ROTATION

$$T(e1) = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

Y-ROTATION



$$T(e2) = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

COMBINATION

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

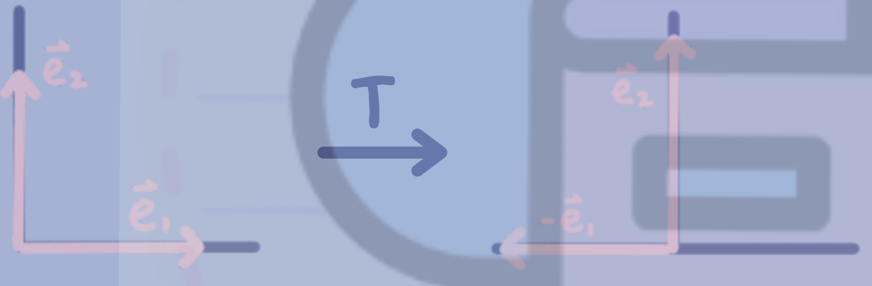
REFLECTION

A reflection is a type of linear transformation that involves flipping a vector or object over a line or plane.

- Geometrically, a reflection across a line or plane can be visualized as flipping an object or image over the line or plane, such that the distance between each point and the line or plane remains the same, but the direction of the vector from the line or plane is reversed.

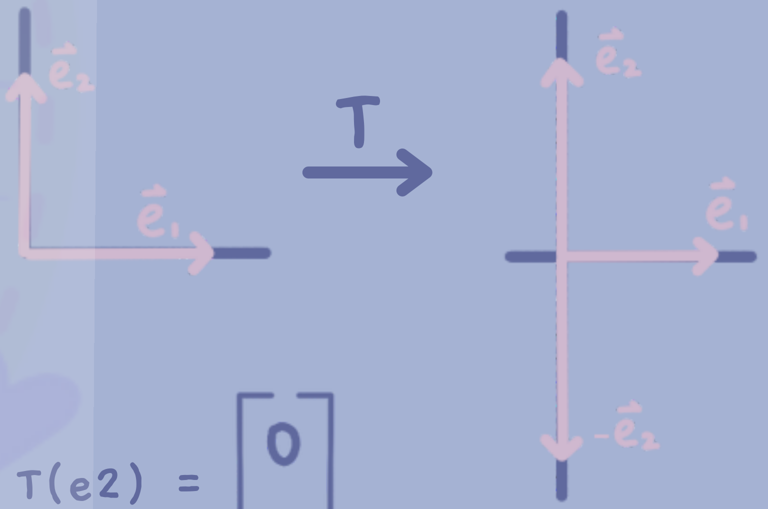
GEOMETRIC INTERPRETATION

X-AXIS FLIP



$T(e1) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ No change in e2

Y-ROTATION



$T(e2) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

No change in e1

COMBINATION

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

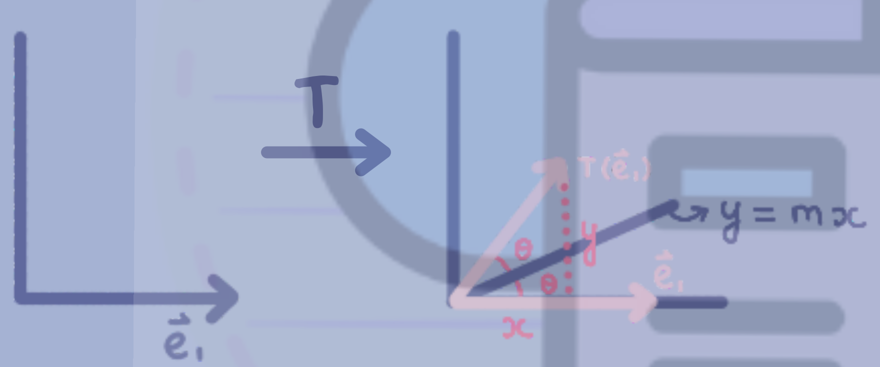
REFLECTION OVER A LINE

A reflection over a line is a linear transformation that involves flipping a vector or object over a given line in space

- Geometrically, the reflection of a vector u over a line L can be visualized as follows: First, draw a parallel line to L from u, and let p be the point where the two lines intersect. Then, the reflection of u over L is the vector that is equal in magnitude to u, but is on the other side of L from p.

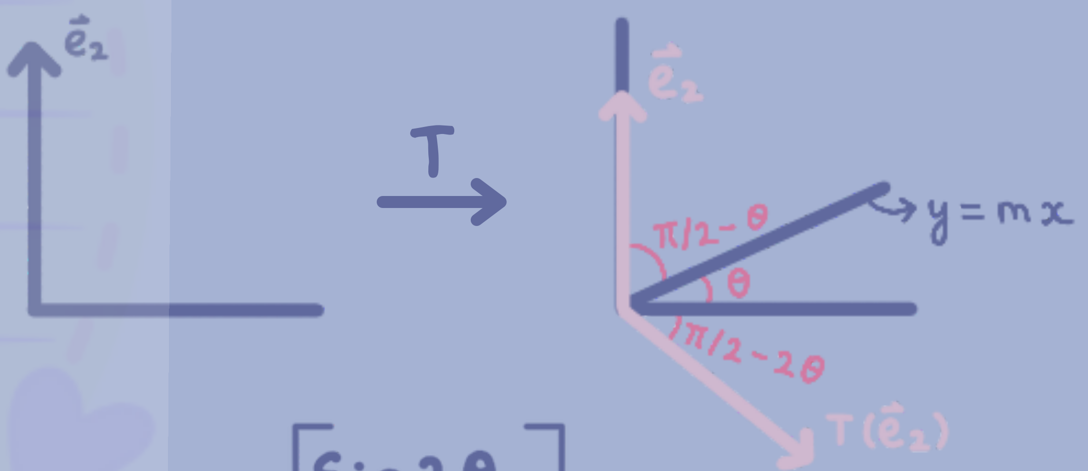
GEOMETRIC INTERPRETATION

E1 ROATION



$$T(e1) = \begin{bmatrix} \cos 2\theta \\ \sin 2\theta \end{bmatrix}$$

E2 ROATION



$$T(e2) = \begin{bmatrix} \sin 2\theta \\ -\cos 2\theta \end{bmatrix}$$

COMBINATION

$$\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$$

PROJECTION

A projection transformation is a linear transformation that maps vectors onto a subspace in a way that preserves their direction.

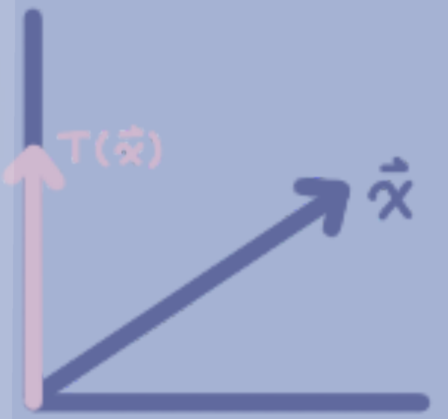
- The projection transformation can be thought of as "flattening" vectors onto a lower-dimensional subspace, while preserving their direction.
- Projection transformations are important in many areas of mathematics and physics, as they can be used to model phenomena such as shadows, reflections, and data compression. They also have applications in computer graphics, signal processing, and optimization.

GEOMETRIC INTERPRETATION

PROJECTION ON X-AXIS

$$T(e1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

PROJECTION ON Y-AXIS



$$T(e2) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

COMBINATION

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

DETERMINANT

The determinant of a square matrix represents the scaling factor of the transformation that the matrix represents

- In geometric terms, the determinant of a 2x2 matrix can be interpreted as the area of the parallelogram formed by the column vectors of the matrix. The sign of the determinant indicates whether the parallelogram is oriented clockwise or counterclockwise.
- the determinant of a 3x3 matrix can be interpreted as the volume of the parallelepiped formed by the column vectors of the matrix.

BASES AND SUBSPACES

Aiman Shaikh

aimuu2310@gmail.com

BASIS

A base (or basis) is a set of vectors that can be used to express any other vector in a given vector space as a linear combination of those basis vectors. In simpler terms, a basis is a set of vectors that "span" the space, meaning that any other vector in the space can be expressed as a combination of those basis vectors.

- For example, in a two-dimensional space, the vectors (1,0) and (0,1) form a basis. This means that any other vector in the space, such as (3,2), can be expressed as a linear combination of these basis vectors:(3,2) = 3(1,0) + 2(0,1)

BASIS

Representation of a basis in standard form where β represents the basis set:

- $\vec{x} = C_\beta\{\vec{x}\}_\beta$
- Columns of C_β are basis vectors hence $C_\beta \sim I$
- Because C_β can be reduced to the identity matrix C_β is invertible.
- If $\vec{x}$ is given and want $\{\vec{x}\}_\beta$ then solve the $C_\beta\{\vec{x}\}_\beta = \vec{x}$ by augmenting C_β by $\vec{x}$
- If $\{\vec{x}\}_\beta$ is given and want $\vec{x}$ then solve $C_\beta\{\vec{x}\}_\beta = \vec{x}$ by taking matrix-vector product of C_β and $\{\vec{x}\}_\beta$

BASIS

Note:

In general: $\beta = \{\vec{b}_1, \dots, \vec{b}_n\}$ for R^n

Any $\vec{x} \in R^n$: $\vec{x} = c_1\vec{b}_1 + \dots + c_n\vec{b}_n$

$$= \underbrace{[\vec{b}_1 \dots \vec{b}_n]}_{C_\beta}\underbrace{\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}}_{\{\vec{x}\}_\beta}$$

BASIS

Note:

A basis for R^n must consist of n linearly independent vectors.

SUBSPACE

An area within a vector space that is closed to addition and scalar multiplication is known as a subspace. A subspace is a non-empty subset W of a vector space V, where W contains the zero vector of V.

1. The zero vector of V is in W.
2. W is closed under addition, which means that if u and v are in W, then u + v is also in W.
3. W is closed under scalar multiplication, which means that if u is in W and c is a scalar, then cu is also in W.

BASIS

Note:

The fact that subspaces are themselves vector spaces with the same addition and scalar multiplication operations as the original space V is another special feature of subspaces. However, subspaces have smaller dimension than the original space, and can be used to study certain properties of the larger space in a more structured and manageable way.

BASIS

Note:

Subspace and subset both contain other subspaces and subsets, respectively. Any element in set B that is also in set A is said to be in set A if set B is a subset of set A. Any vector in the vector space V is also in the subspace W if the vector space V is a subspace of W.

CHANGE OF BASIS

If a vector $\vec{X}$ in a three-dimensional space represented by three coordinates (x,y,z) in a particular coordinate system. If we switch to another coordinate system, the same vector $\vec{X}$ will have different coordinate values (p, q, r).

- The process of finding these new coordinates is known as a change of basis. It involves finding a matrix that describes the relationship between the old and new coordinate systems, and applying this matrix to transform the original coordinates into the new ones.
- This matrix is called the "transition matrix" and it defines how each old basis vector can be expressed as a linear combination of the new basis vectors.

NULL SPACE

The collection of all vectors that, when multiplied by a matrix, result in the zero vector is known as a matrix's null space. It is the collection of solutions to the homogeneous equation $Ax = 0$, where A is a matrix and x is a column vector with the proper dimensions. The kernel of the matrix is another name for the null space. Geometrically speaking, the null space is the portion of the domain that the linear transformation represented by the matrix maps to the zero vector in the range. Null space is denoted by the notation "Null (A)," where A is the matrix $(m \times n)$.

CALCULATING NULL SPACE

Steps to find the null space:

1. Write the matrix A in augmented form [A | 0], where 0 is a column vector of zeros with the same number of rows as A.
2. Use row operations to reduce the augmented matrix to row echelon form.
3. Solve for the variables corresponding to the leading variables (i.e., the variables that correspond to the pivot positions in the row echelon form) in terms of the free variables (i.e., the variables that do not correspond to the pivot positions). This will give you a general solution to the equation $Ax = 0$ in terms of the free variables.
4. Write the general solution as a linear combination of vectors. Each free variable corresponds to a vector in the null space.

EXAMPLE

For example, consider the matrix A = [1 2 -1; 2 4 -2; 3 6 -3].

1. Writing this system in augmented form and row reducing, we get:
   - [1 2 -1 | 0] [0 0 0 | 0] [0 0 0 | 0]
2. The second and third rows are equivalent to the equation 0 = 0, which does not give us any information.
3. The first row gives us x1 = -2x2 + x3. So the general solution to the equation Ax = 0 is:
   - x = [-2t + s; t; s], where t and s are free variables.
4. To write this solution as a linear combination of vectors, we can rewrite it as:
   - x = t[-2; 1; 0] + s[1; 0; 1]
5. So the null space of A is the span of the vectors [-2; 1; 0] and [1; 0; 1].

IMPORTANT

- # of free variables = # of total columns - # of pivot columns
- # of pivot columns = # of total columns - # of free variables
- Dimension of Null(A) is the # of free variables in the solution set.
- Rank of matrix A (m x n) is the # of pivot columns
- If A is m x n:
  - n = Rank A + dimension of Null(A)

COLUMN SPACE

- The column space of a matrix is the set of all possible linear combinations of its column vectors. The column space is a subspace of the vector space in which the matrix operates. It is also called the range or image of the matrix.
- To find the column space of a matrix, we need to find the linearly independent column vectors that span the space. This can be done by performing row operations on the matrix and then identifying the pivot columns, which are the columns that correspond to the pivot positions in the row-echelon form of the matrix.
- Once we have identified the pivot columns, we can extract them from the original matrix and form a new matrix. The column space is then the span of the column vectors in this new matrix.

CALCULATING COLUMN SPACE

Steps to find the column space:

1. Write the matrix in augmented form [A | b].
2. Use row operations to obtain the row-echelon form of the matrix.
3. Identify the pivot columns, which are the columns that correspond to the pivot positions in the row-echelon form.
4. Extract the pivot columns from the original matrix to form a new matrix and not from the reduced form matrix.
5. The column space is the span of the column vectors in the new matrix.

EIGENVALUES AND EIGENVECTORS

Aiman Shaikh

aimuu2310@gmail.com

EIGENVALUE & EIGENVECTORS

Eigenvalues and eigenvectors are a pair of concepts that are used in linear algebra to describe how a linear transformation (or matrix) affects a given vector.

- An eigenvector is a vector that remains on the same line or direction after being multiplied by a matrix. More precisely, for a given matrix A, an eigenvector v is a nonzero vector that satisfies the equation:
  - $Av = \lambda v$
- Where λ is a scalar, known as the eigenvalue associated with the eigenvector v.

EIGENVALUE CALCULATION

To calculate the eigenvalues of a matrix A, we need to find the values of λ that satisfy the equation:

- det(A - λI) = 0
- Where det() denotes the determinant of a matrix and I is the identity matrix of the same size as A. The values of λ that satisfy this equation are the eigenvalues of A.

The process of finding the eigenvalues is called eigenvalue decomposition or diagonalization of the matrix A. Here are the steps to calculate the eigenvalues of a matrix:

- Start by writing the matrix A.
- Subtract λI from A, where λ is a scalar and I is the identity matrix of the same size as A.
- Calculate the determinant of the resulting matrix (A - λI).
- Set the determinant equal to zero and solve for λ. The values of λ that satisfy this equation are the eigenvalues of A.

EIGENVECTOR CALCULATION

Here are the steps to calculate the eigenvectors of a matrix:

1. Start with a square matrix A of size n x n.
2. Form the matrix equation $(A - \lambda I)v = 0$, where λ is an unknown scalar and I is the identity matrix of size n x n.
3. Solve the system of linear equations $(A - \lambda I)v = 0$ by finding the nullspace (also called the kernel) of the matrix $(A - \lambda I)$.
4. The non-trivial solutions of the system (i.e. the solutions that are not just the zero vector) will give you the eigenvectors of A.
5. The values of λ that make the system $(A - \lambda I)v = 0$ have non-trivial solutions are the eigenvalues of A.
6. Repeat steps 2-5 for each eigenvalue.

QUADRATIC FORMULA

Note:

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

REAL OR COMPLEX VALUE??

Note:

Discriminant of quadratic formula

- if $b^2 - 4ac > 0$ then 2 different real λ
- if $b^2 - 4ac = 0$ then 1 repeated real λ
- if $b^2 - 4ac < 0$ then 2 complex conjugate λ

EIGENSPACES

Note:

The sum of the dimensions of all the eigenspaces of a matrix A is equal to the dimension of the matrix, i.e., the number of rows (or columns) of A

INVERTIBILITY AND EIGENVALUES

Invertibility of a matrix A depends on whether or not its eigenvalues are zero. Specifically, A is invertible if and only if its eigenvalues are all nonzero.

- To understand why, we can use the fact that a matrix is invertible if and only if its determinant is nonzero.
  - The determinant of A is the product of its eigenvalues, so if any eigenvalue is zero, then the determinant is zero and the matrix is not invertible.
- On the other hand, if all eigenvalues are nonzero, then the determinant is nonzero and the matrix is invertible.

DIAGONALIZATION

Diagonalization of a matrix is the process of finding a diagonal matrix D and an invertible matrix P such that:

- $A = PDP^{-1}$
- Where A is the original matrix that we want to diagonalize, D is a diagonal matrix whose entries are the eigenvalues of A, and P is a matrix whose columns are the eigenvectors of A.

For n x n matrix A:

- If there are n distinct eigenvalues then A is always diagonalizable
- If there are repeated eigenvalues then A is diagonalizable if the sum of dimension of all eigenspaces equal to n.

WHAT IS A^n = ??

Note:

$$A = PDP^{-1}$$

$$A^2 = (PDP^{-1})(PDP^{-1})$$

$$= PD(P^{-1}P)DP^{-1}$$

$$= PDDP^{-1}$$

$$= PD^2 P$$

$$A^n = PD^n P^{-1}$$

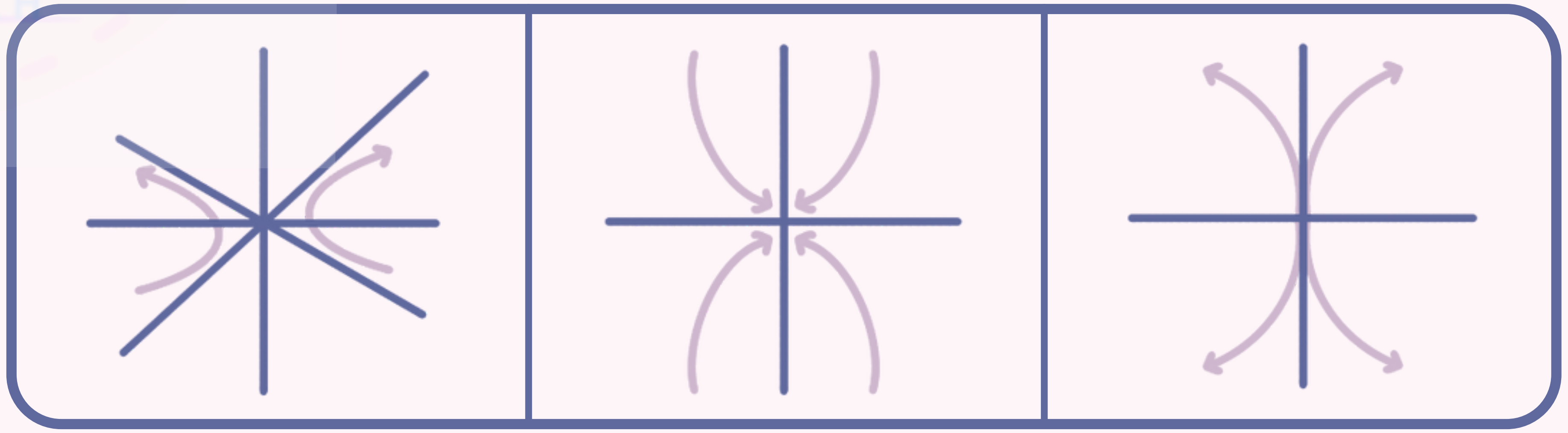
CHARACTERISTICS OF EIGENVALUES

1. Definition: An eigenvalue of a square matrix A is a scalar λ such that there exists a non-zero vector v satisfying the equation $Av = \lambda v$.
2. Multiplicity: Eigenvalues may have a multiplicity greater than one, which means that there can be multiple linearly independent eigenvectors corresponding to the same eigenvalue.
3. Determinant: The determinant of a matrix A is equal to the product of its eigenvalues.
4. Inverse: A matrix A is invertible if and only if all of its eigenvalues are non-zero.
5. Diagonalization: A matrix A is diagonalizable if and only if it has a full set of linearly independent eigenvectors, in which case it can be expressed as $A = PDP^{-1}$, where D is a diagonal matrix with the eigenvalues of A on the diagonal, and P is a matrix whose columns are the eigenvectors of A.
6. Eigenvectors: Eigenvectors associated with distinct eigenvalues of a matrix are linearly independent.
7. Geometric Interpretation: Eigenvalues can be thought of as scaling factors for the eigenvectors. Multiplication of a matrix by a vector with an associated eigenvalue scales that vector by that eigenvalue.

REAL E-VALUE TRAJECTORIES

The trajectories of the eigenvectors provide a way to visualize and understand the behavior of a dynamic system in terms of the scaling and motion of its individual components.

- Saddle: if $|\lambda_1| > 1 > |\lambda_2|$
- Attractor: if both $|\lambda_1| < |\lambda_2| < 1$ all trajectory approaches 0
- Repeller: if both $|\lambda_1| > |\lambda_2| > 1$



E-VECTORS IN COMPLEX FORM

Splitting e-vector $\vec{v}$ into real and imaginary parts

- $v = Re(\vec{v}) + Im(\vec{v})$

  $= \alpha + i\beta$

Example:

- $v = (-2 \ -4i;\ 5)$
- $v = \underbrace{(-2,\ 5)}_{\alpha} + \underbrace{i(-4,\ 0)}_{i\beta}$

Previously we saw $A = PDP^{-1}$, now $A = PCP^{-1}$ where A is similar to C

A SIMILAR TO C

A is similar to C if there exist an invertible matrix C such that:

$A = PCP'$

- $P = [\alpha \quad \beta]$
- $C = [a, \ -b; \quad b, \ a]$
    - a is the real part of eigenvalue and b is the imaginary part
- Each complex eigenvalue $\lambda = a + ib$ (where 'a' and 'b' are real numbers and 'i' is the imaginary unit) must occur in conjugate pairs, i.e., there must be another eigenvalue of the form $\lambda^* = a - ib$, which is the complex conjugate of λ.

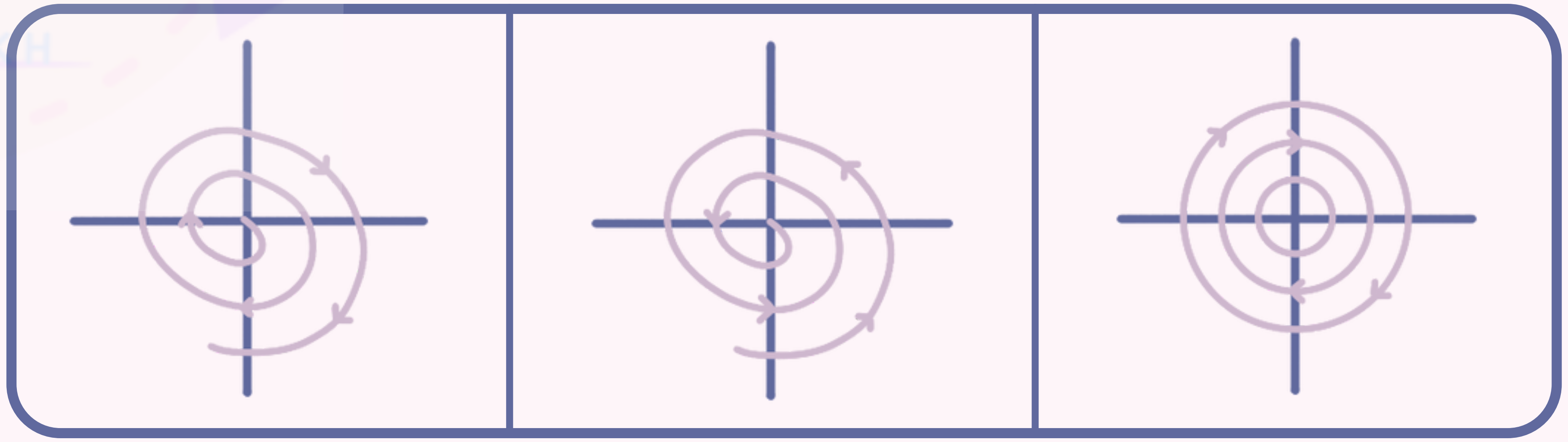
COMPLEX E-VALUE TRAJECTORIES

$|\lambda|$ is the size of e-value $|\lambda| = \sqrt{a + b}$

- Closed orbits "center": if $|\lambda| = 1$

- Spiral Attractor: if $|\lambda| < 1$ shrink for every application of A

- Spiral Repeller: if $|\lambda_1| > 1$ grow for every application of A



ORTHOGONALITY AND LEAST SQUARES

Aiman Shaikh

aimuu2310@gmail.com

DOT PRODUCT

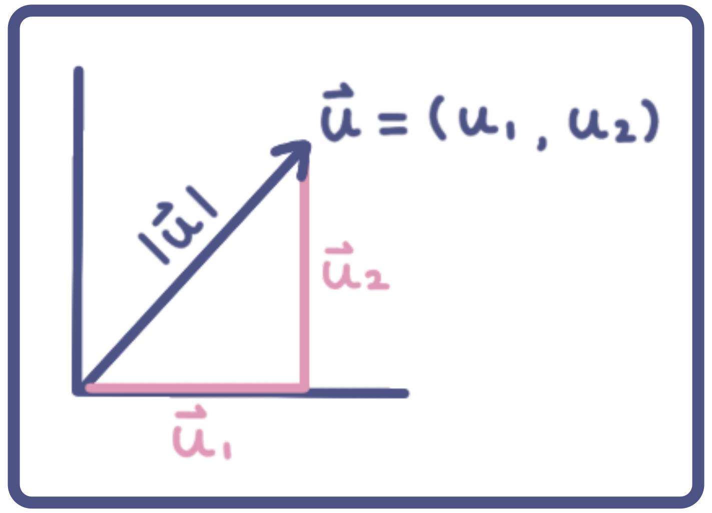
The dot product (also known as the scalar product or inner product) is a way of combining two vectors to obtain a scalar value.

- To calculate the dot product of two vectors, you multiply the corresponding components of the two vectors and then add up the results.
- For example, if you have two vectors u and u, each with three components, the dot product of these two vectors would be:
  - $\vec{u} \cdot \vec{v} = u1 * v1 + u2 * v2 + u3 * v3$
- The dot product can be used to calculate the length (magnitude) of a vector, as well as to find the angle between two vectors.

MAGNITUDE OF VECTOR

To calculate the length (magnitude) of a vector using the dot product, you can use the following formula:

- $|\vec{u}| = \sqrt{(u \cdot u)}$
- where $|\vec{u}|$ represents the length (magnitude) of vector u, and $u \cdot u$ represents the dot product of vector a with itself.
- For example, if you have a vector u = [3, 4, 5], you can calculate its length as follows:
    - $|\vec{u}| = \sqrt{(u \cdot u)}$
      $= \sqrt{(33 + 44 + 5*5)}$
      $= \sqrt{(9 + 16 + 25)}$
      $= \sqrt{50}$



TRANSPOSE

Note:

$$\vec{u} \cdot \vec{v} = \vec{u}^T \vec{v}$$

$$= (u_1 \; u_2 \dots u_n) \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

UNIT VECTOR

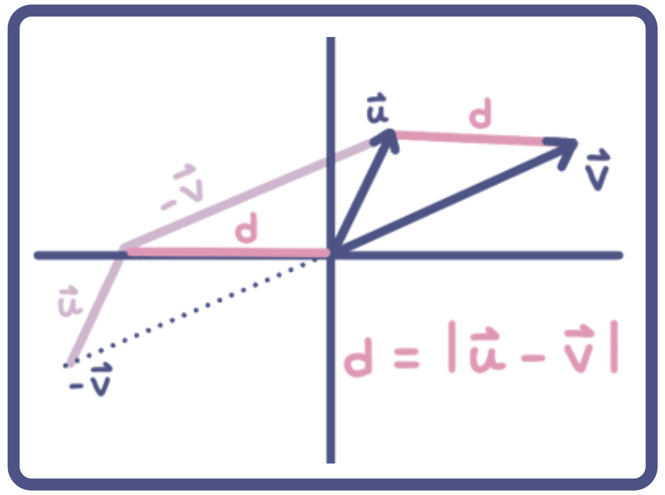
A vector with a length (magnitude) of one is referred to as a unit vector.

- A vector that has been normalised, or scaled down, to have a length of 1, is what is known as a unit vector. You can divide each component of the vector by its length to create a unit vector from the given vector (magnitude).
- For example, if you have a vector $\vec{v}$ = [3, 4], you can obtain a unit vector $\vec{u}$ by dividing each component of a by its length:
    - $|\vec{v}| = \sqrt{(33 + 44)} = \sqrt{(9 + 16)} = \sqrt{25} = \sqrt{5}$
    - $\vec{u} = \vec{v}/|\vec{v}| = [3/\sqrt{5}, 4/\sqrt{5}]$
    - So the resulting vector $\vec{u}$ is a unit vector in the direction of vector $\vec{v}$.

DISTANCE

To calculate the distance between two vectors:

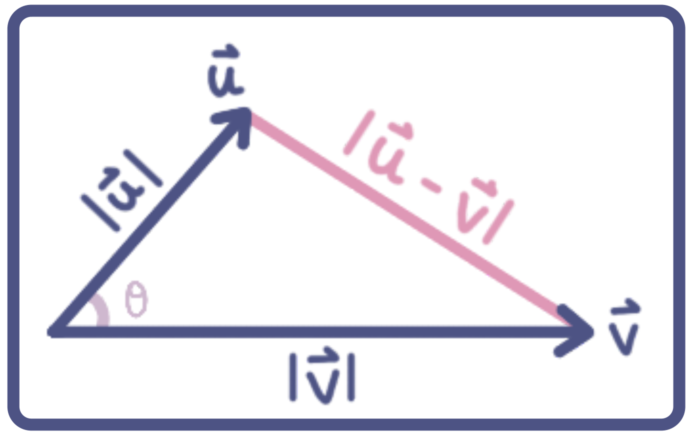
- Between two vectors, you subtract their corresponding components, square the differences, add up the squared differences, and take the square root of the result.
- $d(\vec{u}, \vec{v}) = ||\vec{u} - \vec{v}|| = \sqrt{((u1 - v1)^2 + \ldots + (un - vn)^2}$
- u1, ..., un and v1, ..., vn are the components of vectors $\vec{u}$ and $\vec{v}$, respectively.



ORIENTATION

The dot product can be used to determine the orientation of two vectors, which is the angle between them.

- Specifically, the dot product of two vectors a and b can be used to calculate the cosine of the angle θ between them using the following formula:
  - $\cos(\theta) = (\vec{u} \cdot \vec{v}) / (|\vec{u}|\ |\vec{v}|)$ $\quad \theta = \cos^{-1}((\vec{u} \cdot \vec{v}) / (|\vec{u}|\ |\vec{v}|))$
- Where $|\vec{u}|$ and $|\vec{v}|$ are the magnitudes (lengths) of vectors $\vec{u}$ and $\vec{v}$, respectively.



COSINE LAW

cosine law: $|\vec{u} - \vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2 - 2|\vec{u}||\vec{v}|\cos\theta$

$$
\begin{aligned}
|\vec{u} - \vec{v}|^2 &= \left(\sqrt{(\vec{u} - \vec{v}) \cdot (\vec{u} - \vec{v})}\right)^2 \\
&= (\vec{u} \cdot (\vec{u} - \vec{v}) - \vec{v}(\vec{u} - \vec{v})) \\
&= (\vec{u} \cdot \vec{u} - \vec{u} \cdot \vec{v} - \vec{v} \cdot \vec{u} + \vec{v} \cdot \vec{v}) \\
&= \vec{u} \cdot \vec{u} + \vec{v} \cdot \vec{v} - 2(\vec{u} \cdot \vec{v}) \\
&= |\vec{u}|^2 + |\vec{v}|^2 - 2(\vec{u} \cdot \vec{v})
\end{aligned}
$$

$$\cancel{|\vec{u}|^2} + \cancel{|\vec{v}|^2} - \cancel{2}(\vec{u} \cdot \vec{v}) = \cancel{|\vec{u}|^2} + \cancel{|\vec{v}|^2} - \cancel{2}|\vec{u}||\vec{v}|\cos\theta$$

$$(\vec{u} \cdot \vec{v}) = |\vec{u}||\vec{v}|\cos\theta$$

ORTHOGONALITY

Orthogonality refers to the concept of perpendicularity between vectors or subspaces. Two vectors are said to be orthogonal if they are perpendicular to each other, that is, their dot product is zero.

- Two vectors u and v are orthogonal if their dot product is zero:
  - $\vec{u} \cdot \vec{v} = 0$
- Geometrically, this means that the angle between the two vectors is 90 degrees (or $\pi/2$ radians), so they point in perpendicular directions.
- Largest $\vec{u} \cdot \vec{v} = \cos\theta = 1$
  - $\theta = 0$
  - The vectors $\vec{u}$ and $\vec{v}$ will be in the same direction

NORMAL VECTOR

- If you are given two vectors in three-dimensional space and you want to find a normal vector to the plane that they span, you can use the cross product of the two vectors. The cross product of two vectors is a vector that is perpendicular to both of them and has a magnitude equal to the product of their magnitudes times the sine of the angle between them.
- To find the normal vector to the plane spanned by two vectors u and v, you can take their cross product:
  - $\vec{n} = \vec{u} \times \vec{v}$
- Where x denotes the cross product. The resulting vector n is a normal vector to the plane that is spanned by $\vec{u}$ and $\vec{v}$.

NORMAL VECTOR

Note:

The order of the vectors matters, and the resulting normal vector will point in a direction determined by the right-hand rule. If you swap the order of $\vec{u}$ and $\vec{v}$, the resulting normal vector will point in the opposite direction.

NORMAL VECTOR

Note:

If the vectors $\vec{u}$ and $\vec{v}$ are linearly dependent (i.e., one is a scalar multiple of the other), then the plane that they span is actually a line, and there is no unique normal vector to that plane. In this case, you can choose any vector that is perpendicular to one of the given vectors as the normal vector to the plane.

GEOMETRIC INTERPRETATION OF CROSS PRODUCT

Let $A = \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix}$ $\det(A) = \hat{i}(\vec{u}_2\vec{v}_3 - \vec{u}_3\vec{v}_2) - \hat{j}(\vec{u}_1\vec{v}_3 - \vec{u}_3\vec{v}_1) + \hat{k}(\vec{u}_1\vec{v}_2 - \vec{v}_1\vec{u}_2)$

cross product of $\vec{u}$ and $\vec{v}$ defined as: (only apply for $\vec{u}$, $\vec{v}$ in R^3)

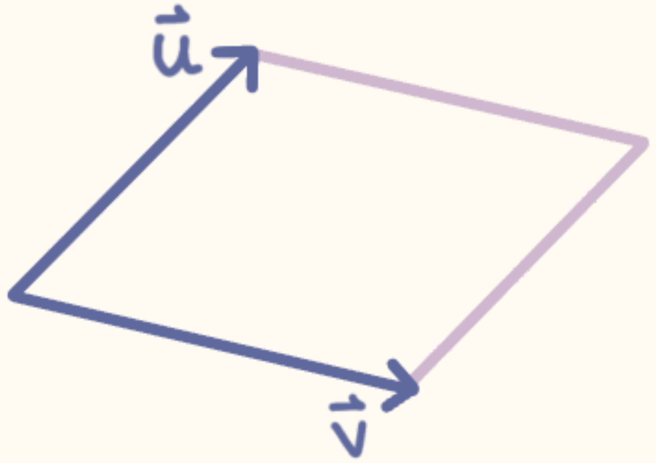
$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

NORMAL VECTOR

Note:

Turns out to be area of parallelogram

$= |\vec{u} \times \vec{v}|$



MATRIX TRANSPOSE

- The transpose of a matrix is a new matrix that is obtained by exchanging its rows and columns. The transpose of an $m \times n$ matrix A is denoted by A^T and is an $n \times m$ matrix.
- E.g.)
  - $A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$
  - $A^T = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

MATRIX TRANSPOSE

The transpose of a matrix has many useful properties, such as:

- $(A^T)^T = A$
- $(A + B)^T = A^T + B^T$
- $(cA)^T = cA^T$, where c is a scalar
- $(AB)^T = B^T A^T$ (the transpose of a product of matrices is the product of their transposes in reverse order)
- If A is invertible, then so is A^T and $(A^T)^{-1} = (A^{-1})^T$
- $\det(A^T) = \det(A)$
- $\mathrm{rank}(A^T) = \mathrm{rank}(A)$

MATRIX VECTOR TRANSPOSE

Note:

Matrix-Vector Product (Transpose Form): If A is a matrix whose columns are a1, a2, ..., am, then

$$A^T x = \begin{bmatrix} \vec{a}_1 \cdot \vec{x} \\ \vdots \\ \vec{a}_m \cdot \vec{x} \end{bmatrix} = \begin{bmatrix} \vec{a}_1^T \vec{x} \\ \vdots \\ \vec{a}_m^T \vec{x} \end{bmatrix}$$

ORTHOGONAL SUBSPACES

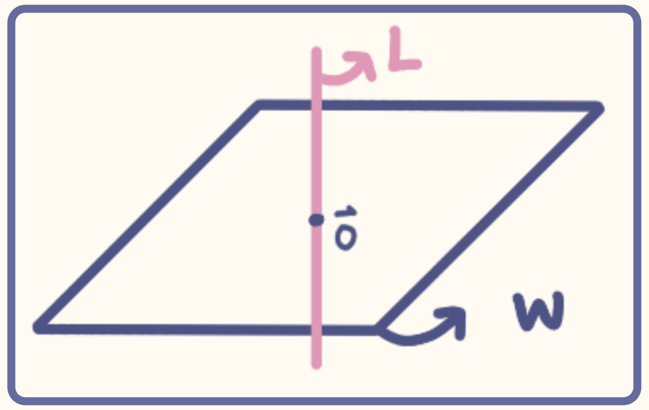
Two subspaces of a vector space are said to be orthogonal if every vector in one subspace is perpendicular (i.e., orthogonal) to every vector in the other subspace.

- This means that if you take any vector in one subspace and any vector in the other subspace, their dot product is zero.
- Geometrically, this means that the two subspaces are perpendicular to each other, just like two lines or two planes in three-dimensional space can be perpendicular to each other.

ORTHOGONAL SUBSPACES

Geometry

Here W and L are subspaces in R^n and L is orthogonal to W so every vector in L is going to be orthogonal to every vector in W



ORTHOGONAL COMPLEMENT

The orthogonal complement of a subspace is the set of all vectors in the vector space that are orthogonal to every vector in that subspace.

- let V be a vector space, and let W be a subspace of V. The orthogonal complement of W, denoted by $W^{\perp}$, is the set of all vectors in V that satisfy: $\vec{v} \cdot \vec{w} = 0$
- For every $\vec{w}$ in W, where · denotes the dot product (or inner product) of vectors in V.
- Geometrically, the orthogonal complement of a subspace W consists of all the vectors in the vector space that are perpendicular (i.e., orthogonal) to every vector in W.

ORTHOGONAL COMPLEMENT

If a basis for a subspace W of a vector space V is given and you want to find the orthogonal complement of W, you can follow these steps:

- Find a basis for W and write its basis vectors as column vectors in a matrix A.
- Take the transpose of the matrix $A \rightarrow A^T$
- Find the basis for $W^\perp$ by solving $A^T x = 0$
- The $Nul(A^T)$ is the orthogonal complement of W

ORTHOGONAL COMPLEMENT

Note:

The orthogonal complement of Col(A) is $Nul(A^T)$

- $(Col(A))^{\perp} = Nul(A^T)$
- $W^{\perp} = Nul(A^T)$

Let W be a k-dimensional subspace of R^n

- Then $\dim W + \dim W^{\perp} = n$.

$\dim Col(A) = \dim Col(A^T)$

RANK THEOREM OF A AND A^T

Rank Theorem: # of columns of A = # of pivot columns + # of free variables

A $(m \times n)$: $n = \dim \mathrm{Col}(A) + \dim \mathrm{Nul}(A)$ $\qquad \mathrm{Col}(A) \subseteq \mathbb{R}^m$

A^T $(n \times m)$: $m = \dim \mathrm{Col}(A^T) + \dim \mathrm{Nul}(A^T)$ $\qquad \mathrm{Nul}(A^T) \subseteq \mathbb{R}^m$

Pivots of A vs pivots of A^T

$$A = \begin{bmatrix} \times & o & \times \\ o & \times & o \end{bmatrix} \qquad A^T = \begin{bmatrix} \times & o \\ o & \times \\ \times & o \end{bmatrix}$$

$\mathrm{Rank}(A) = \dim \mathrm{Col}(A) = 2$ $\qquad \mathrm{Rank}(A^T) = \dim \mathrm{Col}(A^T) = 2$

$\therefore \mathrm{Rank}(A) = \mathrm{Rank}(A^T)$ $\qquad \therefore \dim \mathrm{Col}(A) = \dim \mathrm{Col}(A^T)$

$$A\ (n \times m):\ m = \dim \mathrm{Col}(A) + \dim \mathrm{Nul}(A^T)$$
$$= \dim(W) + \dim(W^\perp)$$

ROW SPACE & COLUMN SPACE

Note:

Given an $m \times n$ matrix A, the row space of A is the column space of A^T; that is,

- $\text{Row}(A) = \text{Col}(A^T)$
- $\text{Row}(A)^{\perp} = \text{Col}(A^T)^{\perp}$
- $\text{Row}(A)^{\perp} = \text{Nul}(A)$

ORTHOGONAL SET

A set of vectors {v1, v2, ..., vn} in a vector space V is called orthogonal if every pair of distinct vectors in the set is orthogonal, meaning that their dot product is zero.This means that if you take any vector in one subspace and any vector in the other subspace, their dot product is zero.

- Orthogonality always gives linearly independent vectors
- Orthogonal set of vectors always leads to trivial solution

ORTHOGONAL BASIS

If we have orthogonal basis $\{v1, \ldots, vn\}$ and a vector $b \in R^n$

- $b = c1v1 + \ldots + cnvn$

  $v1 \cdot b = v1 \cdot (c1v1 + \ldots + cnvn)$

  $v1 \cdot b = c1(v1 \cdot v1) + \ldots + cn(vn \cdot vn)$

  $c1 = (v1 \cdot b)/(v1 \cdot v1)$

In general: $ck = (b \cdot vk)/(vk \cdot vk)$

Basis representation: $b = \frac{(b \cdot v_1)v_1}{(v_1 \cdot v_1)} + \ldots + \frac{(b \cdot v_n)v_n}{(v_n \cdot v_n)}$

ORTHOGONAL MATRIX

An orthogonal matrix is a square matrix whose columns (and rows) are orthonormal vectors. That is, each column (and row) has unit length and is orthogonal to every other column (and row) in the matrix.

- An $n \times n$ matrix A is orthogonal if its transpose A^T satisfies the equation:
  - $A^T A = I$
  - $\therefore A^T = A^{-1}$
- The columns of A are orthonormal vectors, because the dot product of any two different columns of A is zero.
- Similarly, the rows of A are also orthonormal vectors, because the dot product of any two different rows of A is zero.

ORTHOGONAL PROJECTION

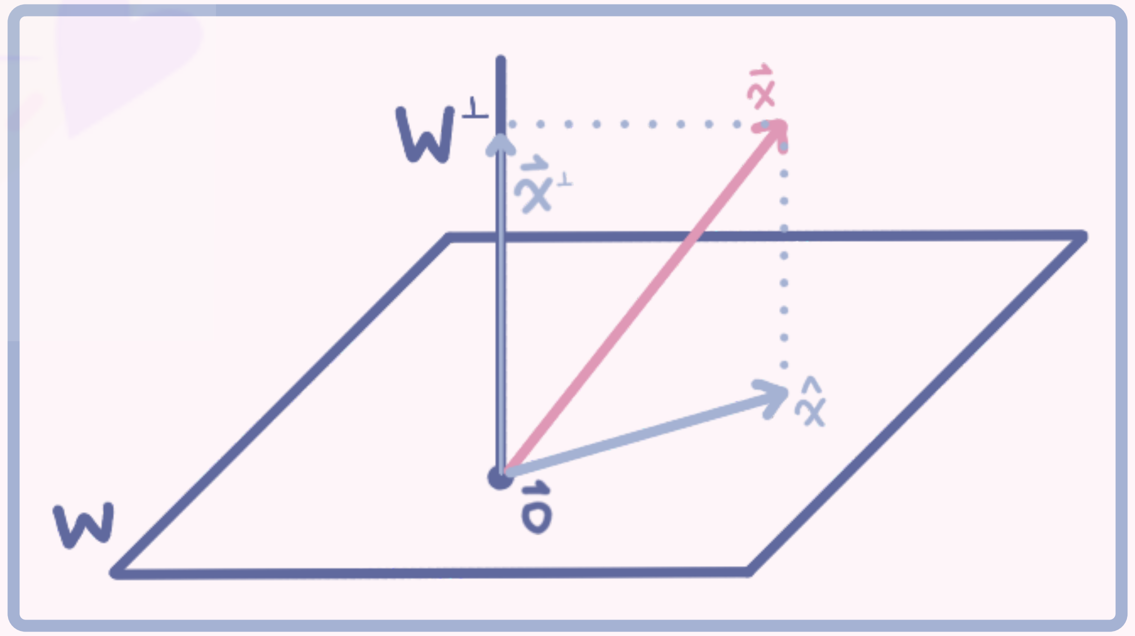
An orthogonal projection is a linear transformation that projects a vector onto a subspace in a way that preserves the perpendicularity of the vectors. equation:

- An orthogonal projection is a transformation that "flattens" a vector onto a subspace in such a way that the resulting projection is perpendicular to the subspace.
- Let W be a subspace of V, let y be any vector in V, and let $\hat{y}$ be the orthogonal projection of y onto W. Then $\hat{y}$ is the closest point in W to y in the sense that $|y - \hat{y}| \leqslant |y - v|$ for all v distinct from $\hat{y}$ in W

SUM OF ORTHOGONAL VECTORS

The sum of orthogonal vectors refers to the decomposition of a vector into the sum of two orthogonal vectors, where one of the vectors lies in a subspace being projected onto, and the other lies in its orthogonal complement.

- Given a subspace W of a vector space V, and a vector x in V, we can decompose x into the sum of two vectors, one of which lies in W, and the other in W's orthogonal complement, as follows:
  - $\vec{x} = \hat{x} + \vec{x}^{\perp}$
- Where $\hat{x}$ is the projection of $\vec{x}$ onto W, and $\vec{x}^{\perp}$ is the projection of $\vec{x}$ onto W's orthogonal complement. Since W and its orthogonal complement are orthogonal subspaces, $\hat{x}$ and $\vec{x}^{\perp}$ are orthogonal vectors.



COMPUTING $\hat{x}$

To calculate projection of $\vec{x}$ onto a subspace W:

- Method 1:

  1. Find the basis vectors for W
  2. Writing $\hat{x}$ as a linear combination of basis vectors of W
  3. Finding scalar values for each basis vector by the using the formula
  4. Adding up all the vectors

- Method 2: $\hat{x} = P_w \vec{x}$

  1. Find the basis vectors for W
  2. Normalize the basis vectors (change to unit length)
  3. Compute Projection matrix $P_w = QQ^T$
  4. Q consist of the orthonormal vectors (normalized basis vectors)

COMPUTING $\vec{x}^{\perp}$

To calculate projection of $\vec{x}^{\perp}$ onto a the orthogonal complement of W ($W^{\perp}$):

- Method 1:

  1. Subtract: $\vec{x}^{\perp} = \vec{x} - \hat{x}$

- Method 2: $\vec{x}^{\perp} = P_{W^{\perp}}\vec{x}$
  1. Find the basis vectors for W
     a. Make a matrix A that contains basis vectors of W
     b. Take transpose of A and compute the $\text{Nul}(A^T)$
  2. Normalize the basis vectors (change to unit length)
  3. Compute Projection matrix $P_{W^{\perp}} = QQ^T$
  4. Q consist of the orthonormal vectors (normalized basis vectors)

PROJECTIONS DISTANCE

Note:

- Distance between $\vec{x}$ and $\hat{x}$:
  - $|\vec{x} - \hat{x}|$
- Distance between $\vec{x}$ and $\vec{x}^{\perp}$:
  - $|\vec{x} - \vec{x}^{\perp}|$

GRAM-SCHMIDT ORTHOGONALITY

Gram-Schmidt orthogonality is a process that takes a set of linearly independent vectors in a vector space and produces an orthonormal set of vectors that span the same subspace.

- The process starts with a set of vectors $\{\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n\}$. The first vector a_1 is left unchanged. Then, for each subsequent vector a_i, the following steps are taken:
    - Subtract the projection of a_i onto the span of $\{\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_{i-1}\}$ from a_i to obtain a vector $\vec{v}_i$ that is orthogonal to $\{\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_{i-1}\}$.
    - Normalize $\vec{v}_i$ to obtain a unit vector $\vec{u}_i$.
    - Repeat steps 1 and 2 for all remaining vectors in the set.
- At the end of the process, the resulting set of vectors $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n\}$ is an orthonormal set that spans the same subspace as the original set $\{\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n\}$.

QR FACTORIZATION

It is a matrix factorization method that expresses a given matrix A as the product of an orthogonal matrix Q and an upper triangular matrix R

- The QR factorization of an m x n matrix A is given by:
  - A = QR
- Where Q is an m x m orthogonal matrix, and R is an m x n upper triangular matrix.
- The columns of Q are orthonormal vectors, and R contains the coefficients of the linear combination of the columns of Q that produce the columns of A.
- Go over the example in the practice question set

LEAST SQUARES

The least squares method is a way to approximate a system of equations that has no exact solution, by minimizing the sum of the squares of the residuals

- Given a system of m linear equations in n unknowns, we can represent it as:
  - $Ax = b$
- Where A is an m x n matrix of coefficients, x is an n x 1 column vector of unknowns, and b is an m x 1 column vector of constants.
- Always get a contradiction when solving for x so instead solve for:
  - $A\hat{x} = \hat{b}$ complicated to solve for b so instead use normal eq.
  - $A^T A\hat{x} = A^T \vec{b}$

LEAST SQUARE SOLUTION

There are two ways to solve of least square:

1. Normal equation: to avoid solving for $A\hat{x} = \hat{b}$
    - $A^T \vec{x} = \vec{0}$

      $A^T(\vec{b} - \hat{b}) = \vec{0}$

      $A^T \vec{b} - A^T \hat{b}$

      $A^T \vec{b} - A^T A\hat{x}$
    - $A^T A\hat{x} = A^T \vec{b}$
2. QR Factorization:
    - If columns of A are linearly independent ($A = QR$) then the least square solution $\hat{x}$ is found from:
        - $R\hat{x} = Q^T \vec{b}$

ORTHOGONAL DIAGONALIZATION

Aiman Shaikh

ORTHOGONAL DIAGONALIZATION

Orthogonal diagonalization is a process of transforming a matrix into a diagonal matrix using an orthogonal matrix.

- An orthogonal matrix is a special type of matrix whose columns are mutually perpendicular and have a length of one.
- When a matrix is diagonalized, it means that the matrix is transformed into a matrix where all the off-diagonal elements are zero, and the diagonal elements contain the eigenvalues of the matrix.

SYMMETRY

A matrix is said to be symmetric if it is equal to its transpose, i.e., if $A = A^T$.

- If a matrix is symmetric, then it can be orthogonally diagonalized, which means that it can be transformed into a diagonal matrix using an orthogonal matrix.
- The diagonal elements of the resulting diagonal matrix are the eigenvalues of the symmetric matrix, and the columns of the orthogonal matrix are the corresponding eigenvectors.

EXAMPLE OF A SYMMETRIC MATRIX

$$A = A^T$$

$$\begin{bmatrix} 2 & 1 & 4 \\ 1 & 5 & 6 \\ 4 & 6 & 3 \end{bmatrix}$$

EXAMPLE OF A SYMMETRIC MATRIX

Note:

Not all linear transformations have matrix representations, and not all matrix representations correspond to invertible linear transformations.

SPECTRAL THEOREM

It describes the properties of a matrix in terms of its eigenvalues and eigenvectors.

- It states that for any symmetric matrix (a square matrix that is equal to its transpose), there exists an orthonormal basis of eigenvectors that can be used to diagonalize the matrix.
- The spectral theorem tells us that any symmetric matrix can be represented as a sum of scalar multiples of its eigenvectors, with the corresponding eigenvalues serving as the scalar factors.
  - $A = \lambda_1 U_1 U_1^T + \ldots + \lambda_n U_n U_n^T$

WHAT IF A MATRIX IS NOT SQUARE??

Aiman Shaikh

https://digitalstudioatoz.etsy.com

SINGULAR VALUE DECOMPOSITION

Singular Value Decomposition (SVD) can be applied to matrices that are not square.

- If a matrix A (m x n) is not square, its SVD can still be computed by finding the eigenvalues and eigenvectors of the product AA^T and A^TA, which are both square and symmetric matrices
- The eigenvectors of these matrices correspond to the left and right singular vectors of A.
- While the square roots of the eigenvalues correspond to the singular values (σ) of A.
- The SVD of a non-square matrix A can be written as:
    - $A = U\Sigma V^T$

SINGULAR VALUE DECOMPOSITION

Note:

- Columns of U (output space) and V (input space) are orthonormal
- The singular values in Σ are non-negative and represent the amount of variation in the corresponding singular vectors.
- σ values are always non-zero

SINGULAR VALUE DECOMPOSITION

Note:

- The matrix V is formed by the eigenvectors of the matrix $A^T A$, normalized to have unit length. These eigenvectors correspond to the columns of V.
- The matrix U is formed by:
  - $u_i = 1/\sigma_i * Av_i$