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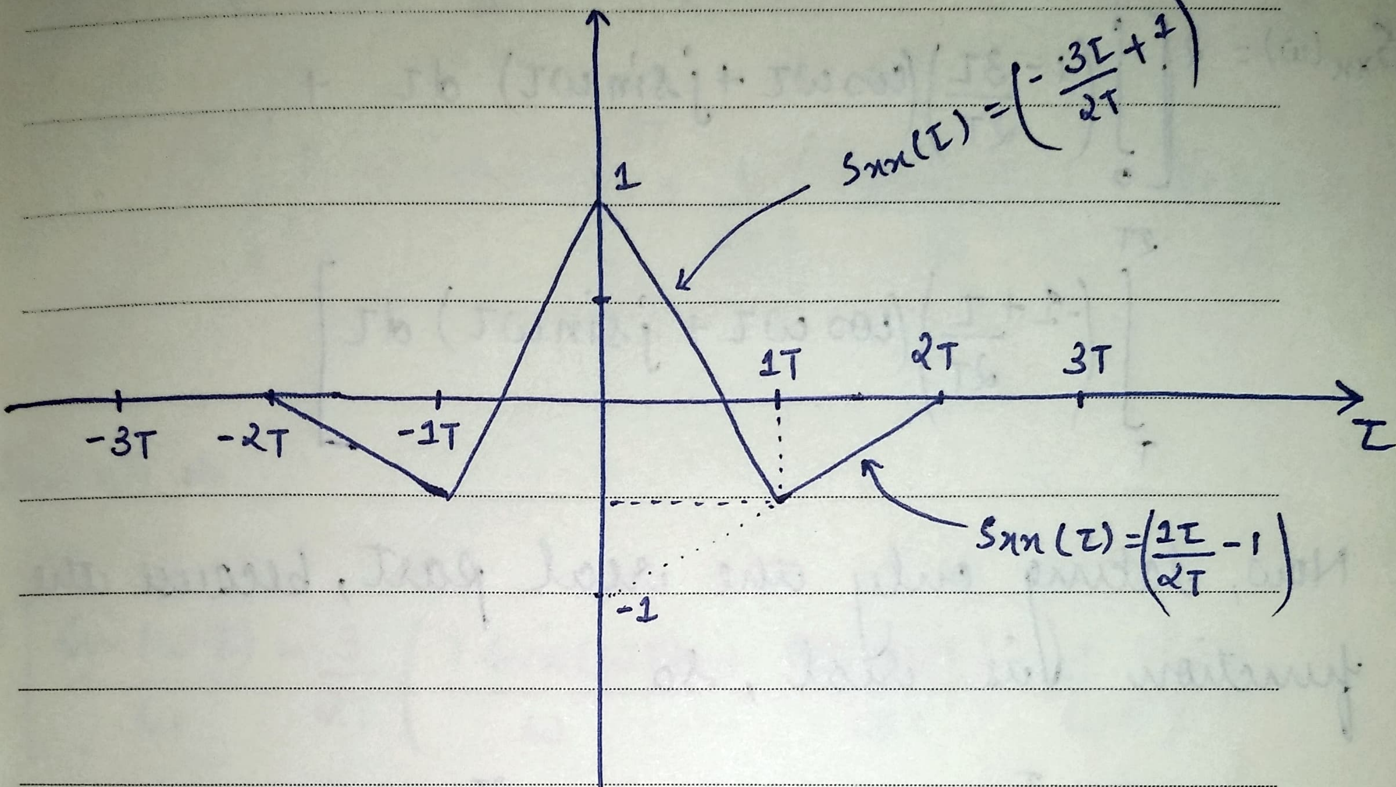
EXERCISE-4



Power spectral density $S_{xx}(\omega) = ?$

Solution :-

$$\because \quad S_{xx}(\omega) = \int_{-\infty}^{\infty} S_{xx}(\tau)\, e^{-j\omega\tau}\, d\tau$$

$$S_{xx}(\omega) = 2\int_{0}^{\infty} S_{xx}(\tau)\, e^{-j\omega\tau}\, d\tau = 2\int_{0}^{2T} S_{xx}(\tau)\, e^{-j\omega\tau}\, d\tau$$

$$S_{xx}(\omega) = 2\left[\int_{0}^{T}\left(1-\frac{3\tau}{2T}\right) e^{-j\omega\tau}\, d\tau + \int_{T}^{2T}\left(-1+\frac{\tau}{2T}\right) e^{-j\omega\tau}\, d\tau\right]$$

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$\left\{ \because e^{-j\omega\tau} = \cos\omega\tau + j\sin\omega\tau \right\}$

$$S_{xx}(\omega) = 2\left[\int_0^T \left(1-\frac{3\tau}{2T}\right)(\cos\omega\tau + j\sin\omega\tau)\,d\tau + \int_T^{2T}\left(-1+\frac{\tau}{2T}\right)(\cos\omega\tau + j\sin\omega\tau)\,d\tau\right]$$

Now, taking only the real part, because the function is real, so

$$S_{xx}(\omega) = 2\left[\int_0^T \left(1-\frac{3\tau}{2T}\right)\cos\omega\tau\,d\tau + \int_T^{2T}\left(\frac{\tau}{2T}-1\right)\cos\omega\tau\,d\tau\right]$$

$$S_{xx}(\omega) = 2\left[\int_0^T \cos\omega\tau\,d\tau - \frac{3}{2T}\int_0^T \tau\cos(\omega\tau)\,d\tau + \frac{1}{2T}\int_T^{2T}\tau\cos(\omega\tau)\,d\tau - \int_T^{2T}\cos(\omega\tau)\,d\tau\right]$$

using integration by parts :-

$$\because \int uv\,dx = u\int v\,dx - \int\left[\frac{d(u)}{dx}\int v\,dx\right]dx$$

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$$S_{xx}(\omega) = 2\left[\frac{\sin(\omega\tau)}{\omega}\Big|_0^T - \frac{3}{2T}\left(\frac{\tau\sin(\omega\tau)}{\omega} + \frac{\cos(\omega\tau)}{\omega^2}\right)\Big|_0^T\right.$$

$$\left. - \left(\frac{1}{\omega}\sin(\omega\tau)\right)_T^{2T} + \frac{1}{2T}\left(\frac{\tau\sin(\omega\tau)}{\omega} + \frac{\cos(\omega\tau)}{\omega^2}\right)_T^{2T}\right]$$

$S_{xx}(\omega) \Rightarrow$

$$2\left[\frac{\sin(\omega T)}{\omega} - \frac{3}{2T}\left(\frac{T\sin(\omega T)}{\omega} + \frac{\cos(\omega T)}{\omega^2} - \frac{1}{\omega^2}\right) - \right.$$

$$\left(\frac{\sin(2\omega T)}{\omega} - \frac{\sin(\omega T)}{\omega}\right) + \frac{1}{2T}\left(\frac{2T\sin(2\omega T)}{\omega} - \right.$$

$$\left.\left.\frac{T\sin(\omega T)}{\omega} + \frac{\cos(2\omega T)}{\omega^2}\right) - \frac{\cos(\omega T)}{\omega^2}\right]$$

$S_{xx}(\omega) \Rightarrow$

$$\frac{2}{\omega}\left[\sin(\omega T) - \frac{3}{2}\sin(\omega T) + \frac{3}{2\omega T} - \frac{3\cos\omega T}{2\omega T} + \sin\omega T\right.$$

$$\left. + \frac{\cos(2\omega T)}{2\omega T} + \frac{\cos(\omega T)}{2\omega T}\right]$$

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$S_{xx}(\omega) \Rightarrow$

$$\frac{2}{\omega}\left[2\sin\omega T - 2\sin\omega T + \frac{3}{2\omega T} - \frac{3\cos\omega T}{2\omega T} + \frac{\cos 2\omega T}{2\omega T} - \frac{\cos\omega T}{2\omega T}\right]$$

$$S_{xx}(\omega) \Rightarrow \frac{2}{\omega^2 T}\left[-2\cos\omega T + \frac{3}{2} + \frac{\cos 2\omega T}{2}\right]$$

$$\left\{\begin{array}{l} \because \cos 2A = \cos^2 A - \sin^2 A \\ \& \sin^2 A = 1 - \cos^2 A \\ \Rightarrow \cos 2A = 2\cos^2 A - 1 \end{array}\right\}$$

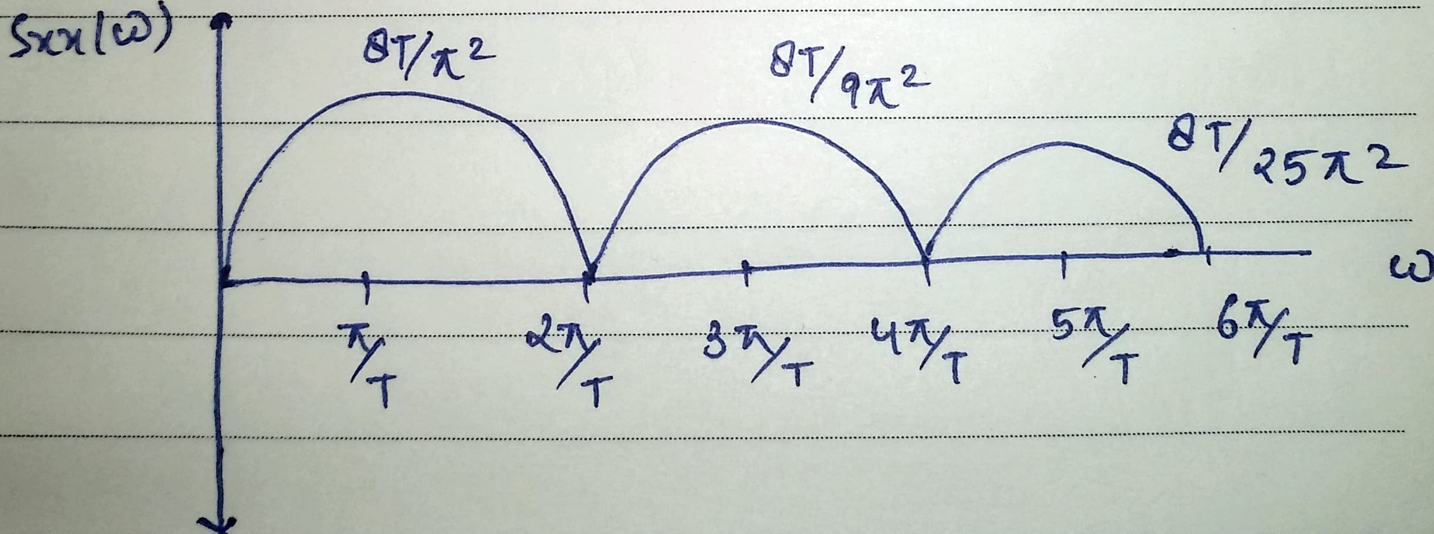
$$\therefore S_{xx}(\omega) = \frac{2}{\omega^2 T}\left[-2\cos\omega T + \frac{3}{2} + \cos^2\omega T - \frac{1}{2}\right]$$

$$S_{xx}(\omega) = \frac{2}{\omega^2 T}\left[-2\cos\omega T + 1 + \cos^2\omega T\right]$$

$$S_{xx}(\omega) = \frac{2}{\omega^2 T}\left[1 - \cos\omega T\right]^2 \quad \text{——— } (1)$$

$\therefore$ $S_{xx}(\omega)$ is

ω	$S_{xx}(\omega)$
0	0
π/T	$8T/\pi^2$
$2\pi/T$	0
$3\pi/T$	$8T/9\pi^2$
$4\pi/T$	0
$5\pi/T$	$8T/25\pi^2$



4.3

Matlab Solution:

```
%% Pre Processor

clc;
clear all;

N = 2000; % Number of Samples
Fs = 100; % Sampling Frequency
syms Sxx(tau) T;
T = 1;
Sxx(tau) = piecewise(tau >= 0 & tau <= T, (1 - 1.5*tau/T), tau >= T &
tau <= 2*T, (-1 + 0.5*tau/T), tau <= 0 & tau >= -T, (1 + 1.5*tau/T), tau
<= -T & tau >= -2*T, (-1 - 0.5*tau/T), 0);

tau = -N/(2*Fs):1/Fs:(N-1)/(2*Fs); % Time difference tau

acf = zeros(1,length(tau));
for i=1:length(tau)
    acf(i) = Sxx((i-length(tau)/2)/Fs);
end
figure('Name','Auto-Correlation Function and Power Spectral Density');
subplot(1,2,1)
plot(tau,acf), title('Auto-Correlation Function'), xlim([-5 5]),
ylim([-1.5 1.5]), xlabel('Time Difference \tau (in sec)'),
ylabel('Amplitude')
grid on

%% Power Spectral Density

Rxxdft= fftshift(abs(fft(acf))/Fs);
freq = (-Fs/2:Fs/length(acf):Fs/2-(Fs/length(acf)))*2*pi;
subplot(1,2,2)
plot(freq,Rxxdft), title('Power Spectral Density'), xlim([-30 30]),
ylim([-0.5 1.5]), xlabel('Angular Frequency \omega(in radians)'),
ylabel('Spectral Power')
grid on
```

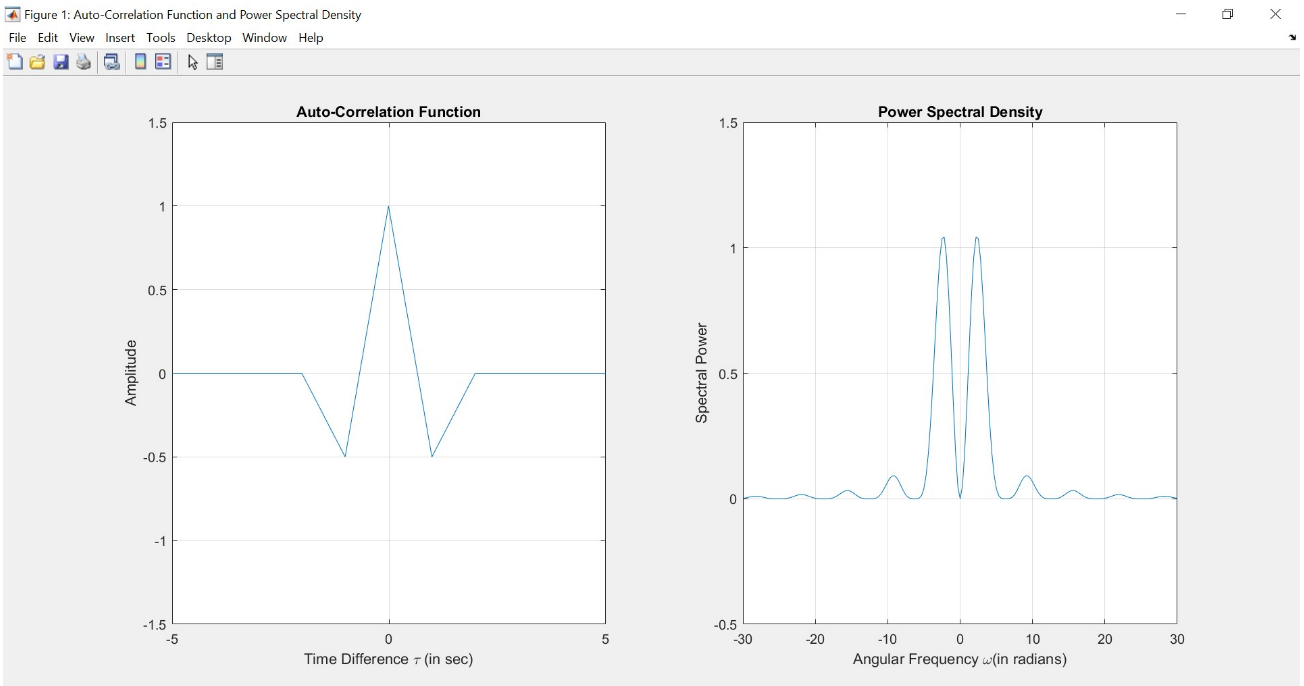


Figure 1: Auto-Correlation Function and Power Spectral Density
File Edit View Insert Tools Desktop Window Help
Auto-Correlation Function
Amplitude
Time Difference τ (in sec)
Power Spectral Density
Spectral Power
Angular Frequency ω(in radians)