Assignment **8** (Lecture 26-28)

Problem 8.1 - An *LRC* Circuit

A circuit contains self-inductance L in series with a capacitor C and a resistor R. This circuit is driven by an alternating voltage $V = V_0 \sin \omega t$. We have $L = 15\,\text{mH}$, $R = 80\,\Omega$, $C = 5\,\mu\text{F}$, and $V_0 = 40$ volts.

(a) What is the resonance frequency, ω_0?

(b) Consider three separate cases for which $\omega = 0.25\omega_0$, $\omega = \omega_0$, and $\omega = 4\omega_0$ respectively. For each case, calculate the peak current I_0.

(c) Find the energy $U_C(t)$ and the energy $U_L(t)$ stored, respectively, in the capacitor and in the inductor as a function of time for $\omega = \omega_0$.

Solution(a)

The formula for the resonance frequency can be easily determined from the expression for the amplitude of current in an RLC circuit (when the inductive reactance cancels the capacitive reactance). Substituting the given values into the formula gives

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{15 \times 10^{-3} \times 5 \times 10^{-6}}} \approx 3650\,\text{rad}\,\text{s}^{-1}.$$

Solution(b)

Peak current is given by

$$I_0 = \frac{V_0}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}} \tag{1}$$

Thus, for each value of ω given in the problem, the peak current is given below.

$$\omega = 0.25\omega_0 \longrightarrow I_0 = 0.18\,\text{A}$$

$$\omega = \omega_0 \longrightarrow I_0 = 0.50\,\text{A}$$

$$\omega = 4\omega_0 \longrightarrow I_0 = 0.18\,\text{A}$$

Solution(c)

At $\omega = \omega_0$, the current in the circuit is given by

$$I(t) = \frac{V_0}{R} \sin \omega_0 t.$$

Integrating the above equation gives the charge in the circuit as a function of time at the same frequency.

$$Q(t) = \frac{V_0}{R} \int \sin \omega_0 t \, dt = -\frac{V_0}{\omega_0 R} \cos \omega_0 t.$$

Integration constant is zero since the problem has assumed that the voltage across the capacitor, and hence the charge on the capacitor is purely sinusoidal in time.

$$\begin{aligned} U_C(t) &= \frac{1}{2} QC = -\frac{V_0 C}{\omega_0 R} \cos \omega_0 t \sin \omega_0 t \\ U_L(t) &= \frac{1}{2} L I^2 = \frac{V_0^2 L}{2R^2} \sin^2(\omega_0 t). \end{aligned} \tag{2}$$

Problem 8.2 - Traveling waves on a string

The equation of a *transverse* wave traveling along a string is given by $y = 0.4\sin[\pi(0.5x - 200t)]$ where y and x are measured in cm and t in seconds.

(a) Find the amplitude, wavelength, wave number, frequency, period, and speed of the wave.

(b) Carefully draw the wave (y versus x) at $t = 0$, and at $t = 1/400\,\text{sec}$.

(c) Find the maximum *transverse* speed of any mass element of the string.

(d) Suppose that you clamp the string at two points $L\,\text{cm}$ apart and you observe a standing wave of the same wavelength as in part (a). For what values of L less than $10\,\text{cm}$ is this possible?

Solution(a)

In general, a standing wave is of the form

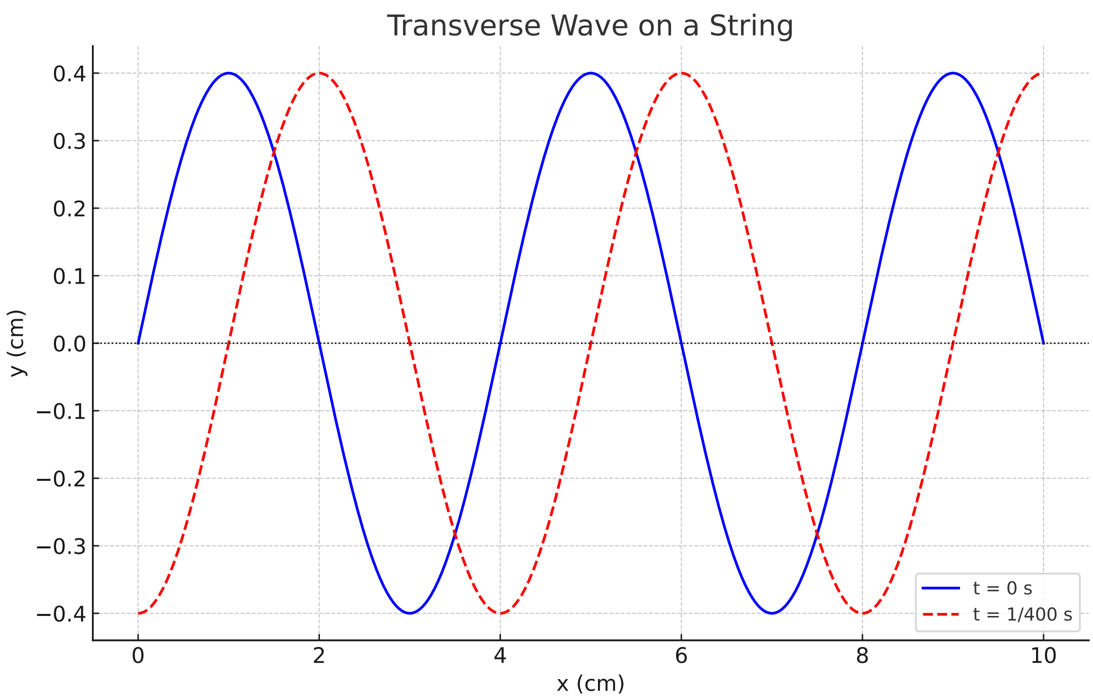
$$y(x,t) = A\sin(kx - \omega t), \tag{3}$$

where A is the amplitude, k is the wave number, and ω is the angular frequency. It is also given that $k = \frac{2\pi}{\lambda}$ and $v = \frac{\omega}{k}$. Thus, using all of these facts, we have

$$A = 0.4\,\text{cm}, \qquad k = 0.5\pi \approx 1.57\,\text{cm}^{-1}, \qquad \lambda = \frac{2\pi}{k} \approx 4\,\text{cm}$$

$$f = \frac{\omega}{2\pi} \approx 100\,\text{Hz}, \qquad T = \frac{2\pi}{\omega} \approx 0.01\,\text{s}, \quad \text{and} \quad v = \frac{\omega}{k} \approx 400\,\text{cm}\,\text{s}^{-1}.$$

Solution(b)



Solution(c)

The transverse speed of any point on a transverse wave is given by

$$v_y = \frac{\partial y}{\partial t} = -80\pi\cos[\pi(0.5x - 200t)]$$

v_y is maximum when the cosine function is equal to ± 1.

$$v_{y,\,\text{max}} = 80\pi \approx 251\,\text{cm}\,\text{s}^{-1}.$$

Solution(d)

For constant wavelengths of 4 cm, we can have the first, second, third, and fourth harmonics. As the length L is equal to $\frac{n\lambda}{2}$ (where $n = 1, 2, 3, 4, \ldots$) . Hence, the values of L for this specific case are 2, 4, 6, and 8 cm.

Problem 8.3 - Standing waves on a string

The equation of a transverse standing wave on a string is given by $y = 0.3(\sin 3x)(\cos 1200t)$ where y and x are in cm and t in seconds.

(a) What is the wavelength, wave number, frequency, and period of this wave?

(b) Carefully draw the wave (y versus x) at $t = 0$, at $t = 1.31 \times 10^{-3}$ s, and at $t = 2.62 \times 10^{-3}$ s.

(c) What is the maximum transverse speed?

(d) What is the speed of propagation (of a transverse disturbance) *along* the string?

Solution(a)

A traveling wave can be modeled by the function

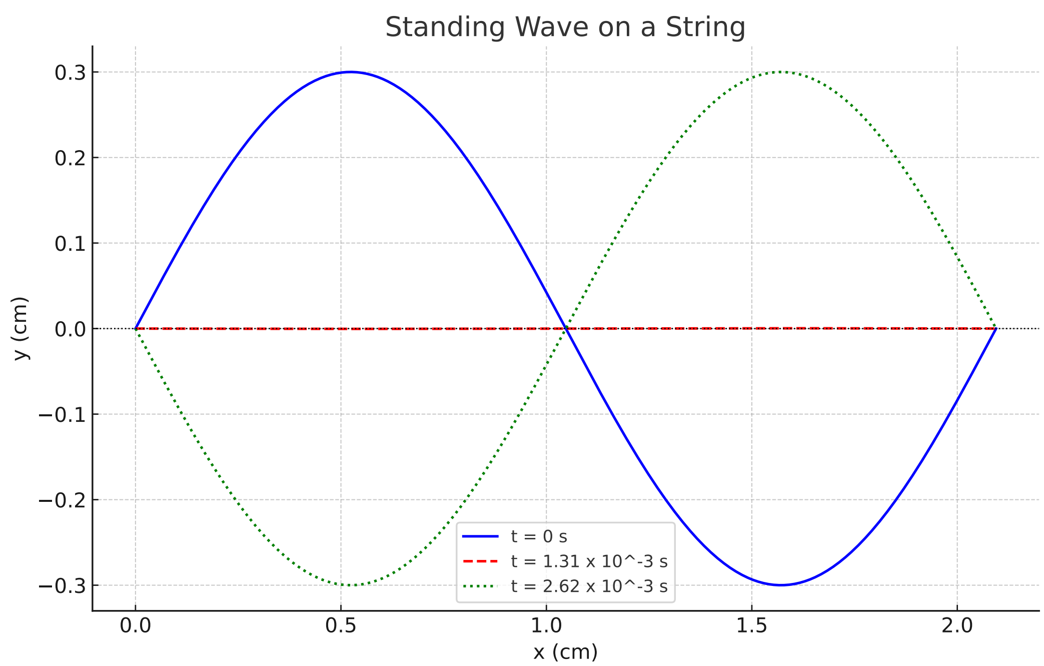
$$y(x, t) = A \sin kx \cos \omega t, \tag{4}$$

which follows the same analogy as discussed in problem 8.4 for the quantities A, k, and ω. Therefore, for the values given,

$$k = 3\,\mathrm{cm}^{-1}, \qquad \lambda = \frac{2\pi}{k} \approx 2.1\,\mathrm{cm}, \qquad \omega = 1200\,\mathrm{rad\,s}^{-1},$$

$$f = \frac{\omega}{2\pi} \approx 191\,\mathrm{Hz}, \quad \text{and} \quad T = \frac{2\pi}{\omega} \approx 5.24 \times 10^{-3}\,\mathrm{s}.$$

Solution(b)



Solution(c)

Again, using the same approach as in problem 8.2 (c), the transverse speed is

$$v_y = \frac{\partial y}{\partial t} = -360 \sin 3x \sin 1200t,$$

that has a maximum value of $360\,\mathrm{cm\,s}^{-1}$.

Solution(d)

The phase speed is $\frac{\omega}{k} \approx 400\mathrm{cm\,s^{-1}}$. However, it is important to note that this speed does not mean that the waveforms are traveling at this speed; since this is a standing wave, the waveforms have zero speed because they are merely changing their phase with the passage of time.

Problem 8.4 - Average dissipated in an LRC circuit

In an LRC circuit, suppose $I = I_0 \sin \omega t$ and $V = V_0 \sin(\omega t + \phi)$. Determine the instantaneous power dissipated in the circuit from $P = IV$ using the equations and show that on average, $\bar{P} = \frac{1}{2} V_0 I_0 \cos \phi$

Solution

Using the expressions given for I and V, we have

$$P = I_0 V_0 \sin \omega t \sin(\omega t + \phi) = I_0 V_0 \left(\sin^2 \omega t \cos \phi + \sin \omega t \cos \omega t \sin \phi \right).$$

For the average power, we use the mean values of the functions $\sin^2 \omega t$ and $\sin \omega t \cos \omega t$ over one period (which is $\frac{1}{2}$ and 0 respectively). Hence, on average,

$$\bar{P} = \frac{1}{2} I_0 V_0 \cos \phi. \tag{5}$$

Problem 8.5 - Width of resonance peak

(a) Determine a formula for the average power $\bar{P}$, dissipated in an LRC circuit in terms of L, R, C, ω, and V_0.

(b) At what frequency is the power a maximum?

(c) Find an approximate formula for the width of the resonance peak in average power, $\Delta\omega$, which is the difference in the two (angular) frequencies where $\bar{P}$ has half its maximum value. Assume a sharp peak.

Solution(a)

In an RLC circuit, the phase angle is given by

$$\tan \phi = \frac{\omega L - \frac{1}{\omega C}}{R}. \tag{6}$$

Thus, using the Pythagorean theorem on Eq. (6) gives

$$\cos \phi = \frac{R}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}. \tag{7}$$

Substituting equations 1 and 7 into Eq. (5) results in

$$\bar{P} = \frac{V_0^2 R}{2\left[R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2\right]}. \tag{8}$$

Solution(b)

The power is maximum when the terms ωL and $\frac{1}{\omega C}$ cancel each other, that is, when the inductive (X_L) and the capacitive (X_C) reactances are equal to each other.

$$X_L = X_C \Rightarrow \omega L = \frac{1}{\omega C} \Rightarrow \omega = \frac{1}{\sqrt{LC}}.$$

Thus the frequency for which the power is maximum is

$$f = \frac{1}{2\pi\sqrt{LC}}. \tag{9}$$

Solution(c)

At the frequency given by Eq. (9), the maximum average power is $\bar{P} = \frac{V_0^2}{2R}$. Thus, equating half the maximum power with Eq. (8) gives

$$\frac{V_0^2}{4R} = \frac{V_0^2 R}{2\left[R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2\right]} \Rightarrow \omega L - \frac{1}{\omega C} = \pm R.$$

For a sharp resonance peak,

$$\omega \approx \omega_0 + \delta\omega. \tag{10}$$

Substituting this value of ω into the relation we derived above gives

$$(\omega_0 + \delta\omega)L - \frac{1}{(\omega_0 + \delta\omega)C} = \pm R \tag{11}$$

where,

$$\frac{1}{(\omega_0 + \delta\omega)C} \approx \frac{1}{\omega_0 C} - \frac{\delta\omega}{\omega_0^2 C}$$

is a good approximation. Given that $\omega_0 L = \frac{1}{\omega_0 C}$ and hence $L = \frac{1}{\omega_0^2 C}$ (see part b). Now, Eq. (11) simplifies to

$$\omega_0 L + \delta\omega L - \frac{1}{\omega_0 C} + \frac{\delta\omega}{\omega_0^2 C} = \pm R \Rightarrow \delta\omega = \pm\frac{R}{2L}.$$

Eq. (10) now gives two frequencies at which the average power is half of its maximum value.

$$\omega_- = \omega_0 - \frac{R}{2L} \quad \text{and} \quad \omega_+ = \omega_0 + \frac{R}{2L}.$$

Finally, width of the resoncance peak is given by

$$\Delta\omega = \omega_+ - \omega_- = \frac{R}{L}.$$

Problem 8.6 - Distance sensing with sound

A bat can sense its distance from the wall of a cave (or whatever) by emitting a sharp ultrasonic pulse that reflects off the wall. The bat can tell the distance from the time the echo takes to return.

(a) If a bat is to determine the distance to a wall 8 m away with an error of less than ±0.2 m, how accurately must it sense the time interval between emission and return of the pulse?

(b) Suppose that a bat flies into a cave filled with methane (swamp gas). By what factor will this gas distort the bat's perception of distances? At 20° C, the speed of sound in methane is 432 m/s.

Solution(a)

Time between the emission and detection of the reflected pulse from the wall is given by $T = \frac{2L}{v}$, where L is the distance between the bat and wall and v is the speed of sound in air at standard conditions and has a value of $343\,\mathrm{ms}^{-1}$. Thus, the maximum error that bat can afford in the value of time is

$$\Delta T = \frac{2\Delta l}{v} \approx 1.2 \times 10^{-3}\,\mathrm{s}.$$

Solution(b)

Due to the presence of methane, the bat will receive the reflected signal more quickly, and therefore the perceived distance L_{apparent} would be smaller than the actual distance L_{real}. For example, the perceived distance is

$$L_{\text{apparent}} = \frac{v_a T}{2} = \frac{v_a}{v_m} L_{\text{real}}$$

where v_a is speed sound in air, v_m is speed of sound in methane, and T is the actual time between emission and reflection ($T = 2L_{\text{real}}/v_m$). Substituting the values gives

$$L_{\text{apparent}} = 0.8 L_{\text{real}}.$$

Bat will interpret the actual distance to be 20% less than its true value.