Problem 1.1 - Manipulation of complex vectors

(a) Find the magnitude and direction of the vector $\left(4 - \sqrt{5}j\right)^3$.

(b) What is the real and imaginary part of

$$\frac{Ae^{j(\omega t+\pi/2)}}{4+5j}$$

assuming that A and ω are real?

(c) Write the following complex vectors Z in terms of $a + jb$ (a and b are real). Notice there may be more than one solution.

$$Z_1 = (j)^j \qquad\qquad Z_2 = (j)^{8.03}$$

Solution(a)

The vector is of the form $(z)^3$ and can be written as a complex exponential using Euler's formula, but first we find the magnitude and argument of z.

$$\operatorname{Arg} z = \tan^{-1}\left(\frac{-\sqrt{5}}{4}\right) = -0.5097, \quad |z| = \sqrt{(-\sqrt{5})^2 + (4)^2} = \sqrt{21}.$$

Thus,

$$(z)^3 = \left(\sqrt{21}e^{j(-0.5097)}\right)^3.$$

This means the direction of z is scaled up by a factor of three and magnitude is raised to the power of three.

$$\text{Direction of } (z)^3 = 3\tan^{-1}\left(\frac{-\sqrt{5}}{4}\right) = -1.529\,\text{rad} = -87.6°,$$

$$\text{Magnitude of } (z)^3 = \left(\sqrt{21}\right)^3 = 96.2.$$

Solution(b)

Let the given expression is $\zeta = \frac{Ae^{j(\omega t+\pi/2)}}{z}$ where $z = 4 + 5j$ and can be expressed a complex exponential in the following way

$$z = \sqrt{5^2+4^2}e^{j\tan^{-1}\left(\frac{5}{4}\right)} = \sqrt{41}e^{j\theta} \qquad \text{where} \qquad \theta = \tan^{-1}\left(\frac{5}{4}\right).$$

Now we have a simplified form of ζ.

$$\zeta = \frac{Ae^{j(\omega t+\pi/2)}}{\sqrt{41}e^{j\theta}} = \frac{A}{\sqrt{41}}e^{j(\omega t+\pi/2-\theta)} = \frac{A}{\sqrt{41}}(\cos(\omega t+\pi/2-\theta) + j\sin(\omega t+\pi/2-\theta)).$$

Finally,

$$\operatorname{Re}\zeta = \frac{A}{\sqrt{41}}\cos(\omega t+\pi/2-\theta) \qquad \text{and} \qquad \operatorname{Im}\zeta = \frac{A}{\sqrt{41}}\sin(\omega t+\pi/2-\theta).$$

Solution(c)

Imaginary number j can be represented as

$$\cos(\pi/2 \pm 2\pi n) + j\sin(\pi/2 \pm 2\pi n) = e^{j(\pi/2 \pm 2\pi n)} \tag{1}$$

where n is a positive integer. This representation makes our objective relatively straight forward.

$$Z_1 = (j)^j = \left(e^{j(\pi/2 \pm 2\pi n)}\right)^j = e^{j \times j(\pi/2 \pm 2\pi n)} = e^{-(\pi/2 \pm 2\pi n)}$$

Hence, surprisingly, Z_1 is a real number and has infinitely many solutions with each depending on the value of n

For Z_2, we can again use equation (1) to simplify the expression.

$$(j)^{8.03} = \left(e^{j(\pi/2 \pm 2\pi n)}\right)^{8.03} = e^{j(4.015\pi \pm 16.06\pi n)} = \cos(4.015\pi \pm 16.06\pi n) + j\sin(4.015\pi \pm 16.06\pi n).$$

This time around, Z_2 is complex but still has infinite number of solutions.

Problem 1.2 - Simple harmonic motion of y as a function of x

Verify that the differential equation $\frac{d^2y}{dx^2} = -k^2 y$ has as its solution

$$y = A\cos(kx) + B\sin(kx)$$

where A and B are arbitrary constants. Show also that this solution can be written in the form

$$y = C\cos(kx + \alpha) = C\,\mathrm{Re}\left[e^{j(kx+\alpha)}\right] = \mathrm{Re}\left[Ce^{j\alpha}e^{jkx}\right]$$

and express C and α as functions of A and B.

Solution

First and second derivatives of the given solution are

$$\frac{dy}{dx} = -Ak\sin(kx) + Bk\cos(kx)$$

$$\frac{d^2y}{dx^2} = -Ak^2\cos(kx) - Bk^2\sin(kx) = -k^2(A\cos(kx) + B\sin(kx)) = -k^2 y.$$

Hence, $y = A\cos(kx) + B\sin(kx)$ is indeed the solution of given differential equation.

This solution can be written in a different way using the trigonometric identity

$$\cos(P + Q) = \cos P\cos Q - \sin P\sin Q.$$

Thus, now we have

$$A\cos(kx) + B\sin(kx) = C\cos(kx)\cos(\alpha) - C\sin(kx)\sin(\alpha) = C\cos(kx + \alpha)$$

where $A = C\cos(\alpha)$ and $B = -C\sin(\alpha)$. We also know that

$$e^{j(kx+\alpha)} = \cos(kx+\alpha) + j\sin(kx+\alpha)$$

with $\cos(kx+\alpha)$ being the real and $\sin(kx+\alpha)$ being the imaginary part of $e^{j(kx+\alpha)}$.

$$y = C\cos(kx+\alpha) = C\,\mathrm{Re}\left[e^{j(kx+\alpha)}\right] = C\,\mathrm{Re}\left[e^{(jkx+j\alpha)}\right] = \mathrm{Re}\left[Ce^{j\alpha}e^{jkx}\right]$$

Dividing B with A results in

$$-\frac{C\sin(\alpha)}{C\cos(\alpha)} = \frac{B}{A} \Rightarrow \tan(\alpha) = -\frac{B}{A} \Rightarrow \alpha = \tan^{-1}\left(-\frac{B}{A}\right).$$

C can also be expressed as a function of A and B.

$$A^2 + B^2 = (C\cos(\alpha))^2 + (-C\sin(\alpha))^2 \Rightarrow C^2(\cos^2(\alpha)) + \sin^2(\alpha)) = A^2 + B^2$$
$$\Rightarrow C = \sqrt{A^2+B^2}.$$

Problem 1.3 - Oscillating springs

A mass on the end of a spring oscillates with an amplitude of 5 cm at a frequency of 1 Hz (cycles per second). At $t = 0$ the mass is at its equilibrium position ($x = 0$).

(a) Find the possible equations describing the position of the mass as a function of time, in the form $x = A\cos(\omega t + \alpha)$. What are the numerical values of A, ω, and α?

(b) What are the values of x, $\frac{dx}{dt}$, and $\frac{d^2x}{dt^2}$ at $t = \frac{8}{3}$ sec?

Solution(a)

Since the spring oscillates at a frequency of 1 cycle per second, its angular frequency is

$$\omega = \frac{2\pi}{1} = 2\pi\,\mathrm{rad\,s^{-1}}.$$

A being the amplitude has a value of 5 if x is measured in centimeters. To find α, we choose the equilibrium position of x ($x = 0$) at the start of the motion ($t = 0$) as α merely defines the initial angular position of the mass.

$$5\cos(\alpha) = 0 \Rightarrow \alpha = \pm\frac{\pi}{2}.$$

Hence, the two possible equations for this system are

$$x = 5\cos\left(2\pi t + \frac{\pi}{2}\right), \quad x = 5\cos\left(2\pi t - \frac{\pi}{2}\right).$$

Solution(b)

Differentiating x with respect to t twice gives

$$\frac{dx}{dt} = v = -10\pi\sin\left(2\pi t \pm \frac{\pi}{2}\right) \quad \text{and} \quad \frac{d^2x}{dt^2} = a = -20\pi^2\cos\left(2\pi t \pm \frac{\pi}{2}\right).$$

Substituting the given value of t into the above equations gives the values of displacement, velocity and acceleration at that instant.

$$x = \pm 4.33\,\mathrm{cm}, \quad v = \pm 15.71\,\mathrm{cms^{-1}}, \quad a = \pm 170.95\,\mathrm{cms^{-2}}.$$

Problem 1.4 - Floating cylinder

A cylinder of diameter d floats with l of its length submerged. The total height is L. Assume no damping. At time $t = 0$ the cylinder is pushed down a distance B and released.

(a) What is the frequency of oscillation?

(b) Draw a graph of velocity versus time from $t = 0$ to $t =$ one period. The correct amplitude and phase should be included.

Solution(a)

At equilibrium, the weight of floating cylinder is equal to buoyant force of water.

$$m_{\text{cyl}}g = \rho_{\text{w}}gV_{\text{sub}} \Rightarrow m_{\text{cyl}} = \frac{\rho_{\text{w}}\pi d^2 l}{4}.$$

When the cylinder is pushed a distance x in the water, the buoyant force increases and we have a resultant force.

$$F_{\text{net}} = m_{\text{cyl}}g - \frac{\rho_{\text{w}}\pi d^2 lg}{4} - \frac{\rho_{\text{w}}\pi d^2 xg}{4} \Rightarrow m_{\text{cyl}}\frac{d^2x}{dt^2} = -\frac{\rho_{\text{w}}\pi d^2 g}{4}x$$

$$\Rightarrow \frac{d^2x}{dt^2} + \frac{\rho_{\text{w}}\pi d^2 g}{4m_{\text{cyl}}}x = 0 \tag{2}$$

this equation shows a simple harmonic oscillation with angular frequency

$$\omega = \sqrt{\frac{\rho_{\text{w}}\pi d^2 g}{4m_{\text{cyl}}}} = \sqrt{\frac{\rho_{\text{w}}\pi d^2 g}{4} \times \frac{4}{\rho_{\text{w}}\pi d^2 l}} = \sqrt{\frac{g}{l}}.$$

Frequency of the oscillation is given by

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{g}{l}}.$$

Solution(b)

Equation for the position of the cylinder at time t is

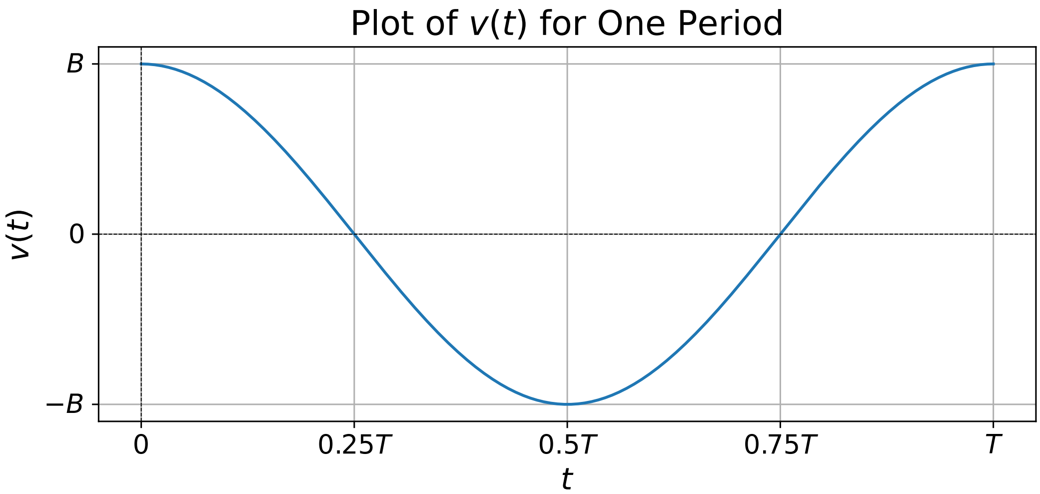
$$x = x_0 \cos\left(\sqrt{\frac{g}{l}}t + \phi\right). \tag{3}$$

At time $t = 0$, cylinder is at amplitude B ($-B$ at $t = 1T$). Substituting these values into equation (3)

$$B = B\cos\left(\sqrt{\frac{g}{l}} \times 0 + \phi\right) \Rightarrow \phi = 0$$

Differentiating equation (3) with respect to t gives velocity of the oscillations.

$$v = \frac{dx}{dt} = -B\sqrt{\frac{g}{l}}\sin\left(\sqrt{\frac{g}{l}}t\right).$$



Problem 1.5 - A damped oscillating spring

An object of mass 0.2 kg is hung from a spring whose spring constant is 80 N/m. The object is subject to a resistive force given by $-bv$, where v is its velocity in meters per second.

(a) Set up the differential equation of motion for free oscillations of the system.

(b) If the damped frequency is 0.995 of the undamped frequency, what is the value of the constant b?

(c) What is the Q of the system, and by what factor is the amplitude of the oscillation reduced after 4 complete cycles?

(d) Which fraction of the original energy is left after 4 oscillations?

Solution(a)

Applying Newton's second law to the system:

$$F_{\text{net}} = F_{\text{spring}} + F_{\text{resistive}} \Rightarrow ma = -kx - bv$$

$$\Rightarrow \frac{d^2x}{dt^2} = -kx - b\frac{dx}{dt} \Rightarrow \frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = 0$$

$$\Rightarrow \frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = 0$$

where $\gamma = b/m$ and $\omega_0^2 = k/m$.

Solution(b)

Damped frequency, ω, is given by the expression

$$\omega^2 = \omega_0^2 - \frac{\gamma^2}{4}. \tag{4}$$

Given that damped frequency is equal to $0.995\omega_0$ where ω_0 is the undamped frequency. Substituting this value, b/m for γ and k/m for ω_0 into equation (4) gives

$$\left(0.995\sqrt{\frac{k}{m}}\right)^2 = \frac{k}{m} - \frac{b^2}{4m^2} \Rightarrow 0.01k = \frac{b^2}{4m} \Rightarrow b = \sqrt{0.04km} = 0.8\,\mathrm{kgs^{-1}}.$$

Solution(c)

Quality Q is given by

$$Q = \frac{\omega_0}{\gamma} = \frac{\sqrt{k/m}}{b/m} = \frac{\sqrt{80/0.2}}{0.8/0.2} = 5.$$

Factor responsible for damping is $e^{-\frac{\gamma}{2}t}$. After 4 complete cycles, a time $t = \frac{8\pi}{\omega} = 1.263\,\mathrm{s}$ has passed. Thus, the amplitude after 4 complete cycles is

$$A_{4T} = A_0 e^{-(\frac{0.8/0.2}{2}\times 1.263)} \Rightarrow \frac{A_{4T}}{A_0} \approx 0.08.$$

The amplitude has decreased to 80% of the initial value.

Solution(d)

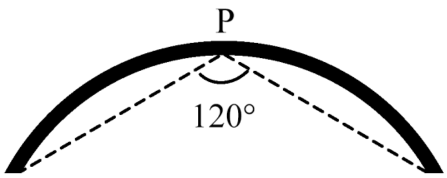
Energy after 4 oscillations is

$$E_{4T} = E_0 e^{-\gamma t} \Rightarrow \frac{E_{4T}}{E_0} = e^{-\frac{0.8}{0.2}\times 1.263} \approx 0.0064.$$

Energy has decreased to 0.64% of its initial value.

Problem 1.6 - A physical pendulum

A uniform rod of mass m is bent in a circular arc with radius R. It is suspended in the middle and it can freely swing about point P (see Figure). The length of the arc is $\frac{2}{3}\pi R$.



(a) What is the period of small angle oscillations about P?

(b) Compare your result with the period derived (and demonstrated) in lectures for a hoop with mass m and radius R.

Solution(a)

To solve this problem, we consider a simpler version in which the whole mass is equally divided and concentrated at both ends of this circular arc ($m/2$ at each end). Both masses (M_1 and M_2) subtend an angle β at P. The moment of inertia about point P of these masses would be

$$I_P = \frac{ml^2}{2} + \frac{ml^2}{2} = ml^2$$

where l is the distance of each mass from P. If we displace this system by a small angle *theta* in the anticlockwise direction, there will be a restoring torque on both masses due to their respective weights.

$$\begin{aligned}\vec{\tau}_{\text{net}} &= \vec{\tau_1} + \vec{\tau_2} = M_1\vec{l}\times\vec{g} - M_2\vec{l}\times\vec{g} \\ &= \frac{m}{2}lg\sin\left(\frac{\beta}{2} - \theta\right) - \frac{m}{2}lg\sin\left(\frac{\beta}{2} + \theta\right) \\ &= \frac{m}{2}lg\left(\sin\frac{\beta}{2}\cos\theta - \cos\frac{\beta}{2}\sin\theta - \sin\frac{\beta}{2}\cos\theta - \cos\frac{\beta}{2}\sin\theta\right) \\ &= -mlg\cos\frac{\beta}{2}\sin\theta. \end{aligned} \tag{5}$$

From geometry, we find that $\cos\frac{\beta}{2} = \frac{l}{D}$. Since θ is small, $\sin\theta \approx \theta$ is a very good approximation. Now, the resultant torque (which is into the page) found in equation (5) becomes

$$\vec{\tau}_{\text{net}} = -\frac{ml^2g}{D}\theta.$$

Given that this expression is also equal to $I_P\alpha$, where α is the angular acceleration.

$$\begin{aligned} I_P\alpha = -\frac{ml^2g}{D}\theta &\Rightarrow ml^2\ddot{\theta} = -\frac{ml^2g}{D}\theta \\ &\Rightarrow \ddot{\theta} + \frac{g}{D}\theta = 0. \end{aligned} \tag{6}$$

Equation (6) shows a simple harmonic oscillation with period

$$T = 2\pi\sqrt{\frac{D}{g}} = 2\pi\sqrt{\frac{2R}{g}}. \tag{7}$$

Since the period is independent of mass m and angle β, the period of actual system is same as found in equation (7).

Solution(b)

The period found in part (a) is exactly the same as of the hoop (derived in lecture 1).

Problem 1.7 - Damped oscillator and initial conditions

The displacement from equilibrium, $s(t)$, of the pen of a chart recorder can be modelled as a damped harmonic oscillator satisfying the homogeneous differential equation

$$\ddot{s}(t) + \gamma \dot{s}(t) + \omega_0^2 s(t) = 0$$

(a) Find the time evolution of the displacement if the pen is critically damped and subject to the initial conditions $s(t = 0) = 0$ and $\dot{s}(t = 0) = v_0$. Does $s(t)$ change sign before it settles to its equilibrium position at $s = 0$?

(b) Find the response of an overdamped pen subject to the initial conditions $s(t = 0) = s_0$ and $\dot{s}(t = 0) = 0$.

(c) Use your favorite mathematical tool to plot your solution for $s(t)$ in (b) as a function of time. Use $\omega_0 = 3/7 \times \pi$, $\gamma = 3$ and $s_0 = 1$ for the plot you turn in. Let time run from 0 to 10 seconds. For your own curiosity, once you have your code written, you can vary γ to see the effect of the damping on the response.

Solution(a)

For a critically damped system, the position of pen as a function of time is

$$s(t) = (A + Bt)e^{-\frac{\gamma}{2}t}$$

$$\dot{s}(t) = -\frac{\gamma}{2}(A + Bt)e^{-\frac{\gamma}{2}t} + Be^{-\frac{\gamma}{2}t}.$$

$s(t = 0) = 0$ means $A = 0$ and $\dot{s}(t = 0) = v_0$ leads to $B = v_0$. Thus, the time evolution of displacement in this scenario is

$$s(t) = v_0 t e^{-\frac{\gamma}{2}t}.$$

$s(t)$ never changes sign during the entire course of motion.

Solution(b)

The solution for an overdamped system is of the form

$$s(t) = Ae^{-(\gamma/2+\alpha)t} + Be^{-(\gamma/2-\alpha)t}$$

$$\dot{s}(t) = -(\gamma/2 + \alpha) Ae^{-(\gamma/2+\alpha)t} - (\gamma/2 - \alpha) Be^{-(\gamma/2-\alpha)}$$

using the initial conditions stated in the problem, we obtain

$$s(t = 0) = s_0 = A + B \Rightarrow B = s_0 - A$$

and

$$\dot{s}(t = 0) = 0 = -(\gamma/2 + \alpha) A - (\gamma/2 - \alpha) B$$

$$-(\gamma/2 + \alpha) A - (\gamma/2 - \alpha)(s_0 - A) = 0 \Rightarrow A\gamma/2 + A\alpha = \alpha s_0 - A\alpha - \gamma s_0/2 + A\gamma/2$$

$$2A\alpha = s_0\left(\alpha - \gamma/2\right) \Rightarrow A = s_0\left(\frac{\alpha - \gamma/2}{2\alpha}\right) = s_0\left(\frac{1}{2} - \frac{\gamma}{4\alpha}\right)$$

$$B = s_0 - s_0\left(\frac{1}{2} - \frac{\gamma}{4\alpha}\right) = s_0\left(\frac{1}{2} + \frac{\gamma}{4\alpha}\right).$$

Finally,

$$s(t) = s_0\left(\frac{1}{2} - \frac{\gamma}{4\alpha}\right)e^{-(\gamma/2+\alpha)t} + s_0\left(\frac{1}{2} + \frac{\gamma}{4\alpha}\right)e^{-(\gamma/2-\alpha)t}$$

where

$$\alpha = \sqrt{\frac{\gamma^2}{4} - \omega_0^2}.$$

Solution(c)