Type-1 OWA Operators in Aggregating Multiple Sources of Uncertain Information: Properties and Real-World Applications in Integrated Diagnosis

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***Abstract*— The type-1 ordered weighted averaging (T1OWA) operator has demonstrated the capacity for directly aggregating multiple sources of linguistic information modeled by fuzzy sets rather than crisp values. Yager's ordered weighted averaging (OWA) operators possess the properties of *idempotence, monotonicity, compensativeness,* and *commutativity*. This article aims to address whether or not T1OWA operators possess these properties when the inputs and associated weights are fuzzy sets instead of crisp numbers. To this end, a partially ordered relation of fuzzy sets is defined based on the fuzzy maximum (*join*) and fuzzy minimum (*meet*) operators of fuzzy sets, and an *alpha-equivalently-ordered* relation of groups of fuzzy sets is proposed. Moreover, as the extension of *orness* and *andness* of an Yager's OWA operator, *joinness* and *meetness* of a T1OWA operator are formalized, respectively. Then, based on these concepts and the *representation theorem of T1OWA operators*, we prove that T1OWA operators hold the same properties as Yager's OWA operators possess, i.e., *idempotence, monotonicity, compensativeness,* and *commutativity*. Various numerical examples and a case study of diabetes diagnosis are provided to validate the theoretical analyses of these properties in aggregating multiple sources of uncertain information and improving integrated diagnosis, respectively.**

***Index Terms*—Aggregation, diabetes, fuzzy sets, integrated diagnosis, linguistic aggregation, ordered weighted averaging (OWA) operator, soft decision making, type-1 ordered weighted averaging (T1OWA) operator.**

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I. Introduction

IN DOMAINS where information fusion/integration or multifactorial evaluation is needed, an aggregation process is necessary to combine multiple sources of information into a global result so that in the final decision, all the individual sources of information are taken into account [1]. For example, in medicine, diagnosis or measurement can rarely be decided based on an individual criterion. Particularly, in the age of big data, the use of information aggregation is rapidly increasing, both because data are easily collected by ubiquitous information technologies and because the availability of cost-effective computational power allows combining information from multiple sources to be readily feasible.

Yager's ordered weighted averaging (OWA) operators [2], [3] have become a popular tool to aggregate information from multiple sources due to their flexibility for modeling a wide variety of aggregation scenarios via the appropriate definition/selection of the OWA operator's weighting vector [3]. However, Yager's OWA operators exclusively aggregate crisp numbers, while in real-world decisions, one is often not certain about the exact value of a crisp attribute. For example, in medicine, patients often find it difficult to describe how they feel, and doctors/nurses often find it difficult to describe what they observed. Thus, it is desirable to develop a technique that can aggregate multiple sources of uncertain information of attributes. The type-1 ordered weighted averaging (T1OWA) operators and the associated α-level T1OWA aggregations are such a technique [4], [5], in which uncertain information is modeled by fuzzy sets. In this way, with appropriately definitions of uncertain weights, T1OWA operators extend Yager's OWA operator [2], the *meet* operator of fuzzy sets and the *join* operators of fuzzy sets [7], [8].

Since their appearance, the T1OWA operators have received increasing attention in scientific applications [9]–[14]. To select optimal routes under uncertain environments, T1OWA operators have been designed to guide human decision making in a fuzzy weighted graph [10]. In addition to the α-level approach to fast implementation of T1OWA operators, another new method of calculating T1OWA was proposed via an opposite direction searching [11]. A T1OWA unbalanced fuzzy linguistic aggregation method has been applied to credit risk evaluation [12]. In group decision making, T1OWA operators can be used to combine multiple granular linguistic information and improve consensus reaching processes [13]. In type-2 fuzzy logic system modeling, the type-reduction of general type-2 fuzzy sets can be efficiently implemented via T1OWA operators [14].

Despite the above-mentioned advances on the development and applications of T1OWA operators, one issue remains unclear regarding aggregation mechanism properties. Yager's OWA operators are *idempotent*, *monotonic*, *compensative*, and *commutative* [2]. The question to be answered in our case is whether or not T1OWA operators hold these same properties when the inputs and associated weights become uncertain, being expressed as fuzzy sets instead of crisp numbers in soft decision making. This question is not trivial at all, because the mechanisms of operations on a group of (fuzzy) sets are completely different from those on crisp values, with more advanced computing techniques to be required. In this article, we aim to answer this important question.

To this end, based on the α-cuts of fuzzy sets, we suggest a new relation of fuzzy sets, named the *alpha-equivalently-ordered* relation of a group of fuzzy sets, and address the *join* and the *meet* based *partial order relation* of fuzzy sets. Then, we prove that the T1OWA operation is *commutative*, *idempotent*, *monotonic*, and *compensative* with respect to the fuzzy set partial order relation.

The rest of this article is organized as follows. In Section II, we briefly review two definitions of the T1OWA operator: one based on the *extension principle*, the other based on the α-cuts of fuzzy sets. Section III defines a fuzzy set partial order relation based on the meet and join operators of fuzzy sets. As the extension of the *andness* and *orness* of Yager's OWA operators. Section IV defines the *meetness* and *joinness* of T1OWA operators. Section V then analyzes and proves the properties of a T1OWA operator. Section VI provides a case study of diabetes diagnosis and further validation of computing efficiency of α-level T1OWA aggregation. Finally, Section VII concludes this article.

II. Preliminaries

Although the T1OWA operators can be defined either via Zadeh's extension principle or via the α-cuts of fuzzy sets [4], [5], their final aggregation results coincide.

Let $F(X)$ be the power set of fuzzy subsets on the domain of discourse X. One can define the T1OWA operator via the extension principle [4] as follows.

Definition 1: "Given n linguistic weights $\{\widetilde{W}^i\}_{i=1}^n$ in the form of fuzzy sets defined on the domain of discourse $U = [0,1]$, a T1OWA operator is a mapping Φ

$$\begin{aligned}\Phi : F(X) \times \cdots \times F(X) &\longrightarrow F(X)\\ (\widetilde{A}^1, \ldots, \widetilde{A}^n) &\mapsto \widetilde{Y}. \qquad (1)\end{aligned}$$

The membership function of outcome fuzzy set $\widetilde{Y}$ (aggregation result) is

$$\mu_{\widetilde{Y}}(y) = \sup_{\substack{\sum_{i=1}^{n} \bar{w}_i a_{\sigma(i)} = y \\ w_i \in U, a_i \in X}} \begin{pmatrix} \mu_{\widetilde{W}^1}(\omega_1) \wedge \cdots \wedge \mu_{\widetilde{W}^n}(\omega_n) \\ \wedge \mu_{\widetilde{A}^1}(a_1) \wedge \cdots \wedge \mu_{\widetilde{A}^n}(a_n) \end{pmatrix} \qquad (2)$$

where $\bar{\omega}_i = \frac{\omega_i}{\sum_{i=1}^n \omega_i}$, and $\sigma : \{1, \ldots, n\} \longrightarrow \{1, \ldots, n\}$ is a permutation function such that $a_{\sigma(i)} \geq a_{\sigma(i+1)}$, $\forall i = 1, \ldots, n-1$, i.e., $a_{\sigma(i)}$ is the ith largest element in the set $\{a_1, \ldots, a_n\}$."

Definition (2) can lead to a procedure for implementing T1OWA operations, called the *direct approach* [4]. Alternatively, one can define T1OWA operators using the α-cuts of a fuzzy set [5] as follows.

Definition 2: "Let $\{\widetilde{W}^i\}_{i=1}^n$ be a set of linguistic weights characterized by fuzzy sets on the domain of discourse $U = [0,1]$, and $\alpha \in [0,1]$. The α-level type-1 OWA operator with α-cuts $\{\widetilde{W}_\alpha^i\}_{i=1}^n$ is the operator that aggregates the α-cuts of the fuzzy sets $\{\widetilde{A}^1, \ldots, \widetilde{A}^n\}$ as follows:

$$\Phi_\alpha\left(\widetilde{A}_\alpha^1, \ldots, \widetilde{A}_\alpha^n\right) = \left\{ \frac{\sum_{i=1}^n \omega_i a_{\sigma(i)}}{\sum_{i=1}^n \omega_i} \middle| \omega_i \in \widetilde{W}_\alpha^i, a_i \in \widetilde{A}_\alpha^i, i = 1, \ldots, n \right\} \qquad (3)$$

where σ is a permutation function such that $a_{\sigma(i)} \geq a_{\sigma(i+1)}, \forall i = 1, \ldots, n-1$, $\widetilde{W}_\alpha^i = \{\omega | \mu_{\widetilde{W}^i}(\omega) \geq \alpha\}$, and $\widetilde{A}_\alpha^i = \{x | \mu_{\widetilde{A}^i}(x) \geq \alpha\}$."

In fact, one can use the α-level sets $\Phi_\alpha(\widetilde{A}_\alpha^1, \ldots, \widetilde{A}_\alpha^n)$ to create a fuzzy set as follows:

$$\widetilde{G} = \bigcup_{0 \leq \alpha \leq 1} \alpha \Phi_\alpha\left(\widetilde{A}_\alpha^1, \cdots, \widetilde{A}_\alpha^n\right) \qquad (4)$$

where the membership function is

$$\mu_{\widetilde{G}}(x) = \bigvee_{\alpha : x \in \Phi_\alpha\left(\widetilde{A}_\alpha^1, \cdots, \widetilde{A}_\alpha^n\right)} \alpha. \qquad (5)$$

The above-mentioned two methods of aggregating fuzzy sets via Yager's OWA mechanism are equivalent [5] as stated in the following.

Theorem 1 (Representation Theorem of T1OWA Operators): "Given a set of linguistic weights $\{\widetilde{W}^i\}_{i=1}^n$ in the form of fuzzy sets on U. For any fuzzy sets $\widetilde{A}^1, \ldots, \widetilde{A}^n$ on $F(X)$, let Y be the outcome aggregation result defined in (2) and $\widetilde{G}$ be the result defined in (4), then $\widetilde{Y} = \widetilde{G}$."

According to this *representation theorem*, one can implement the T1OWA aggregation through a series of α-level T1OWA operators. This provides a new way of theoretically analyzing the properties of T1OWA operators. The procedure for implementing the T1OWA aggregation through a series of α-level T1OWA operators is called the *alpha-level approach* [5].

III. Join and Meet Based Partial Order Relation of Fuzzy Sets

Zadeh [6] defined the *meet* and the *join* of fuzzy sets to aggregate linguistic variables $\widetilde{A}$ and $\widetilde{B}$ for the statements $\widetilde{A}$ *and* $\widetilde{B}$ and $\widetilde{A}$ *or* $\widetilde{B}$, respectively. The *meet* and the *join* of fuzzy sets now become fundamental operators in developing type-2 fuzzy systems [7], [8].

Definition 3: Given two fuzzy sets $\widetilde{S}$ and $\widetilde{T}$, the join ($\cup$) and meet ($\cap$) operators are defined as

$$\mu_{\widetilde{S} \cup \widetilde{T}}(v) = \sup_{\substack{s \vee t = v \\ s, t \in X}} \left(\mu_{\widetilde{S}}(s) \wedge \mu_{\widetilde{T}}(t)\right) \qquad (6)$$

$$\mu_{\widetilde{S}\cap\widetilde{T}}(v) = \sup_{\substack{s \wedge t = v \\ s,t \in X}} \left(\mu_{\widetilde{S}}(s) \wedge \mu_{\widetilde{T}}(t)\right) \tag{7}$$

where sup is a t-conorm, $\wedge$ is the minimum operator, and $\vee$ is the maximum operator.

It should be noted that the *join* (6) and the *meet* (7) operators can aggregate a set of criteria based on an imperative, such as *one of the criteria should be satisfied* and *all the criteria should be satisfied*, respectively [6].

A. Join and Meet are T1OWA Operators

By appropriately choosing linguistic weights in a T1OWA operator, the *join* operator (6) of fuzzy sets is, in fact, a special T1OWA operator.

Theorem 2: Let a T1OWA operator, *J*, be defined by the first linguistic weight being the singleton weight $\tilde{1}$: $\widetilde{W}_1 = \tilde{1}$, all other weights being the singleton weight $\tilde{0}$: $\widetilde{W}_i = \tilde{0}\,(i \neq 1)$, where

$$\mu_{\tilde{1}}(\omega) = \begin{cases} 1 & \text{for } \omega = 1 \\ 0 & \text{for } \omega \neq 1 \end{cases} \tag{8}$$

$$\mu_{\tilde{0}}(\omega) = \begin{cases} 1 & \text{for } \omega = 0 \\ 0 & \text{for } \omega \neq 0 \end{cases}. \tag{9}$$

For any groups of fuzzy sets $\{\widetilde{A}^i\}_{i=1}^n$

$$J\left(\widetilde{A}^1, \widetilde{A}^2, \ldots, \widetilde{A}^n\right) = \bigcup\nolimits_{i=1}^n \widetilde{A}^i. \tag{10}$$

Proof: Omitted ■

Example 2 in the *supplemental material* shows the *join* operation as a T1OWA operator in nature.

Interestingly, in T1OWA aggregation, if the first linguistic weight moves toward $\tilde{1}$, all the others toward $\tilde{0}$ (see *Example 3* in the *supplemental material*), then this operator demonstrates a *join-like* behavior. This type of operator is called a *join-like* T1OWA operator.

Similarly, the meet operation (7) of fuzzy sets is also a special T1OWA operator.

Theorem 3: Let a T1OWA operator *M* be defined by the last linguistic weight being the singleton weight $\tilde{1}$: $\widetilde{W}_n = \tilde{1}$, all the others being the singleton weight $\tilde{0}$: $\widetilde{W}_i = \tilde{0}\,(i \neq n)$. For any groups of fuzzy sets $\{\widetilde{A}^i\}_{i=1}^n$

$$M\left(\widetilde{A}^1, \widetilde{A}^2, \ldots, \widetilde{A}^n\right) = \bigcap\nolimits_{i=1}^n \widetilde{A}^i. \tag{11}$$

Proof: Omitted ■

Example 5 in the *supplemental material* shows the results of the *meet* operator to aggregate three fuzzy aggregated objects.

Correspondingly, in T1OWA aggregation, if the last linguistic weight moves toward $\tilde{1}$, all the other weights toward $\tilde{0}$, then this operator demonstrates *meet-like* type behavior (see *Example 6* in the *supplemental material*). We call it a *meet-like* T1OWA operator.

B. Partial Order Relation of Fuzzy Sets

The set of real numbers $\mathbb{R}$ is linearly ordered, and the $(\mathbb{R}, \wedge, \vee)$ forms a lattice. Then, for any $a, b \in \mathbb{R}$, a partially ordered relation "$\geq$"($\leq$) can be defined as follows:

$$\begin{aligned} s \geq t &\iff s \vee t = s \\ &\iff s \wedge t = t. \end{aligned} \tag{12}$$

As a matter of fact, according to Zadeh' extension principle, the *meet* ($\cap$) and *join* ($\cup$) operators are just fuzzification of the *min* ($\wedge$) and *max* ($\vee$) operators of crisp numbers, respectively. In this way, $\widetilde{S} \cap \widetilde{T}$ and $\widetilde{S} \cup \widetilde{T}$ are no other than the fuzzified minimum $\widetilde{S}$ and fuzzified maximum $\widetilde{T}$ of the fuzzy sets. It can be proved that $(F(\mathbb{R}), \cap, \cup)$ is a distributive lattice [15], with partial order relation defined as follows.

Definition 4: Given two fuzzy numbers $\widetilde{S}$ and $\widetilde{T}$, a partially ordered relation "$\succcurlyeq$" is defined as follows:

$$\begin{aligned} \widetilde{S} \succcurlyeq \widetilde{T} &\iff \widetilde{S} \cup \widetilde{T} = \widetilde{S} \\ &\iff \widetilde{S} \cap \widetilde{T} = \widetilde{T}. \end{aligned} \tag{13}$$

We have the following theorem.

Theorem 4: Let $\widetilde{S}$ and $\widetilde{T} \in F(R)$ be fuzzy numbers with core centers v_1 and v_2, respectively, and $v_1 \geq v_2$, then based on the t-conorm and t-norm,

$\widetilde{S} \succcurlyeq \widetilde{T} \iff \mu_{\widetilde{S}}(s) \leq \mu_{\widetilde{T}}(s)$ for $s \leq v_2$ and $\mu_{\widetilde{S}}(s) \geq \mu_{\widetilde{T}}(s)$ for $s \geq v_1$.

Proof:

1) First, if $\widetilde{S} \succcurlyeq \widetilde{T}$, then according to (13), for any $s \leq v_2$, we have $\mu_{\widetilde{T}}(v_2) = 1$ and

$$\begin{aligned} \mu_{\widetilde{T}}(s) &= \mu_{\widetilde{S}\cap\widetilde{T}}(s) \\ &= \sup_{\substack{s_1 \wedge s_2 = s \\ s_1, s_2 \in X.}} \left(\mu_{\widetilde{S}}(s_1) \wedge \mu_{\widetilde{T}}(s_2)\right) \end{aligned}$$

Because $x \wedge v_2 = s$

$$\begin{aligned} \mu_{\widetilde{T}}(s) &\geq \mu_{\widetilde{S}}(s) \wedge \mu_{\widetilde{T}}(v_2) \\ &= \mu_{\widetilde{S}}(s) \wedge 1 \\ &= \mu_{\widetilde{S}}(s). \end{aligned}$$

For any $s \geq v_1$, we have $\mu_{\widetilde{S}}(v_1) = 1$ and

$$\begin{aligned} \mu_{\widetilde{S}}(s) &= \mu_{\widetilde{S}\cup\widetilde{T}}(s) \\ &= \sup_{\substack{s_1 \vee s_2 = s \\ s_1, s_2 \in X.}} \left(\mu_{\widetilde{S}}(s_1) \wedge \mu_{\widetilde{T}}(s_2)\right) \end{aligned}$$

Because $v_1 \vee s = s$

$$\begin{aligned} \mu_{\widetilde{S}}(s) &\geq \mu_{\widetilde{S}}(v_1) \wedge \mu_{\widetilde{T}}(s) \\ &= 1 \wedge \mu_{\widetilde{T}}(s) \\ &= \mu_{\widetilde{T}}(s). \end{aligned}$$

2) If $\mu_{\widetilde{S}}(s) \leq \mu_{\widetilde{T}}(s)$ for any $x \leq v_2$ and $\mu_{\widetilde{S}}(s) \geq \mu_{\widetilde{T}}(s)$ for any $s \geq v_1$, we prove $\widetilde{S} \cup \widetilde{T} = \widetilde{S}$ in the following.
Let us denote $\widetilde{C} \equiv \widetilde{S} \cup \widetilde{T}$. For any s, s_1 and $s_2 \in X$ with $s_1 \wedge s_2 = s$, then $s = s_1$ or $s = s_2$. Hence, the membership function of fuzzy set $\widetilde{C}$ can be decomposed as follows:

$$\mu_{\widetilde{C}}(s) = u_1(s) \vee u_2(s)$$

where

$$u_1(s) = \bigvee_{s_1: s_1 \le s} \left(\mu_{\widetilde{S}}(s_1) \wedge \mu_{\widetilde{T}}(s)\right) = \mu_{\widetilde{T}}(s) \wedge \left(\bigvee_{s_1: s_1 \le s} \mu_{\widetilde{S}}(s_1)\right)$$

$$u_2(s) = \bigvee_{s_1: s_1 \le s} \left(\mu_{\widetilde{S}}(s) \wedge \mu_{\widetilde{T}}(s_1)\right) = \mu_{\widetilde{S}}(s) \wedge \left(\bigvee_{s_1: s_1 \le s} \mu_{\widetilde{T}}(s_1)\right).$$

Then, if $s \le v_2$, the $\mu_{\widetilde{S}}(\cdot)$ and $\mu_{\widetilde{T}}(\cdot)$ are both nondecreasing functions. So, we have $u_1(s) = \mu_{\widetilde{T}}(s) \wedge \mu_{\widetilde{S}}(s)$, and $u_2(s) = \mu_{\widetilde{S}}(s) \wedge \mu_{\widetilde{T}}(s)$, which lead to

$$\mu_{\widetilde{C}}(s) = \mu_{\widetilde{S}}(s) \wedge \mu_{\widetilde{T}}(s) = \mu_{\widetilde{S}}(s).$$

If $v_2 \le s \le v_1$, $\mu_{\widetilde{T}}(\cdot)$ is nonincreasing, and $\mu_{\widetilde{S}}(\cdot)$ is nondecreasing. Then, we have $u_1(s) = \mu_{\widetilde{T}}(s) \wedge \mu_{\widetilde{S}}(s)$, and $u_2(s) = \mu_{\widetilde{S}}(s) \wedge 1 = \mu_{\widetilde{S}}(s)$, which lead to

$$\mu_{\widetilde{C}}(s) = \mu_{\widetilde{S}}(s) \vee \left(\mu_{\widetilde{T}}(s) \wedge \mu_{\widetilde{S}}(s)\right) = \mu_{\widetilde{S}}(s).$$

If $v_1 \le x$, $\mu_{\widetilde{T}}(\cdot)$ is nonincreasing, and $\mu_{\widetilde{S}}(\cdot)$ is nonincreasing. So, we have $\sup_{s_1: s_1 \le s} \mu_{\widetilde{S}}(s) = 1$, and $\sup_{s_1: s_1 \le x} \mu_{\widetilde{T}}(s) = 1$. Then,

$$\mu_{\widetilde{C}}(s) = \mu_{\widetilde{S}}(s) \vee \mu_{\widetilde{T}}(s) = \mu_{\widetilde{S}}(s).$$

Hence, $\widetilde{S} \cup \widetilde{T} = \widetilde{S}$. ■

Theorem 4 provides a more strict finding than that investigated by Ramik and Rimanex [15] in the context of fuzzification of the min and max operators, which states that $\widetilde{S} \succcurlyeq \widetilde{T} \iff$ there must be v_1, u_*, and v_2 with $v_1 \ge u_* \ge v_2$, $\mu_{\widetilde{S}}(v_1) = \mu_{\widetilde{T}}(v_2) = 1$, $\mu_{\widetilde{S}}(s) \le \mu_{\widetilde{T}}(s)$ for any $s < u_*$ and $\mu_{\widetilde{S}}(s) \ge \mu_{\widetilde{T}}(s)$ for any $s > u_*$.

Based on the α-cuts of fuzzy sets, the following order relation has been defined [15].

Definition 5: "For any fuzzy numbers $\widetilde{S}$ and $\widetilde{T}$, an ordering relation $\widetilde{\ge}$ is defined as

$$\widetilde{S} \widetilde{\ge} \widetilde{T} \iff \widetilde{S}_{\alpha+} \ge \widetilde{T}_{\alpha+} \, and \, \widetilde{S}_{\alpha-} \ge \widetilde{T}_{\alpha-} \forall \alpha \in [0,\ 1]$$

where $\widetilde{S}_\alpha = [\widetilde{S}_{\alpha-}, \widetilde{S}_{\alpha+}]$ and $T_\alpha = [\widetilde{T}_{\alpha-}, \widetilde{T}_{\alpha+}]$ are the α-cuts of $\widetilde{S}$ and $\widetilde{T}$, respectively."

The following example shows an ordering relation between two fuzzy sets.

Example 1 (Ordering Relation): Fig. 1 illustrates two fuzzy sets such that $\widetilde{A} \widetilde{\ge} \widetilde{B}$.

The relation $\widetilde{\ge}$ is a partially ordered relation on $F(\mathbb{R})$, known as the *fuzzy max* order [15]. Interestingly, the two apparently different order relations $\succcurlyeq$ and $\widetilde{\ge}$ are equivalent on $F(\mathbb{R})$ as it was proved in [15].

Lemma 1: The following three relations are equivalent for any fuzzy numbers $\widetilde{S}$ and $\widetilde{T}$.

1) $\widetilde{S} \widetilde{\ge} \widetilde{T}$.
2) $\widetilde{S} \cup \widetilde{T} = \widetilde{S}$.
3) $\widetilde{S} \cap \widetilde{T} = \widetilde{T}$.

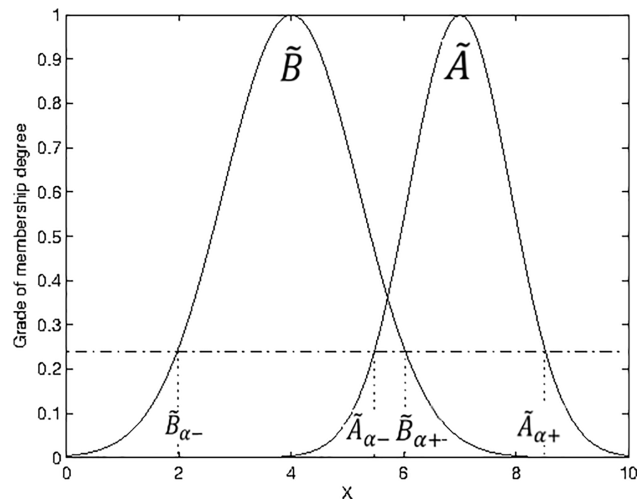

Fig. 1. Two fuzzy sets $\widetilde{A}$ and $\widetilde{B}$ having ordering relation.

IV. Joinness and Meetness of a T1OWA Operator

A popular way to evaluate the behavior of an OWA operator is to use the measure of *orness* and its dual *andness* proposed by Yager [2] and Kishor [3]. The two measures aim to assess the similarity of an OWA operator with the maximum and minimum operators, respectively, based on the associated weighting vector.

Similarly, in T1OWA aggregation, the following definitions of *joinness* and *meetness* associated with the linguistic weights evaluate how the T1OWA aggregation behaves like the operations of *join* and *meet*, respectively.

Definition 6: For a T1OWA operator with fuzzy set weights $\{\widetilde{W}_i\}_{i=1}^n$ on $U \subseteq [0,\ 1]$, its *joinness* is

$$\mu_{joinness}(u) = \sup_{j_{\omega_1,\ldots,\omega_n} = u} \mu_{\widetilde{W}_1}(\omega_1) * \cdots * \mu_{\widetilde{W}_n}(\omega_n) \quad (14)$$

where $*$ is a t-norm operator, and

$$j_{\omega_1,\ldots,\omega_n} = \frac{1}{(n-1)\sum_{i=1}^n \omega_i} \sum_{i=1}^n (n-i)\omega_i. \quad (15)$$

The corresponding *meetness* of the T1OWA is

$$\mu_{meetness}(u) = \sup_{m_{\omega_1,\ldots,\omega_n} = u} \mu_{\widetilde{W}_1}(\omega_1) * \cdots * \mu_{\widetilde{W}_n}(\omega_n) \quad (16)$$

where

$$m_{\omega_1,\ldots,\omega_n} = 1 - \frac{1}{(n-1)\sum_{i=1}^n \omega_i} \sum_{i=1}^n (n-i)\omega_i. \quad (17)$$

Clearly, the defined *joinness* and *meetness* of a T1OWA are fuzzy sets describing the linguistic expressions of aggregations behaving like the *join* and *meet*, respectively.

It is not difficult to calculate that the *joinness* and *meetness* of the *join* operator as a particular T1OWA operator (see Theorem 2), are $joinness(\{\widetilde{W}_i\}_{i=1}^n) = \tilde{1}$ and $meetness(\{\widetilde{W}_i\}_{i=1}^n) =$

$\tilde{0}$, which further confirms that this particular T1OWA operator is the *join* operator of fuzzy sets. Correspondingly, the *joinness* and *meetness* of the *meet* operator, as a particular T1OWA operator (see Theorem 3), are $joinness(\{\widetilde{W}_i\}_{i=1}^n) = \tilde{0}$ and $meetness(\{\widetilde{W}_i\}_{i=1}^n) = \tilde{1}$, confirming that this particular T1OWA operator is the *meet* operator of fuzzy sets.

Moreover, *Example 4* in the *supplemental material* depicts the *joinness* of the T1OWA operator shown in *Example 3* in the *supplemental material*.

V. Properties of T1OWA Operators

Yager's OWA operators possess the properties of *idempotence, monotonicity, compensativeness,* and *commutativity* [2]. In this section, we investigate the conditions for these properties to be verified by T1OWA operators.

First, because Yager's OWA operators and the *sup* operators in (6) and (7) are commutative, the T1OWA operator is *commutative* as well according to its definition in (2).

Theorem 5: For any T1OWA operator Φ and $\widetilde{A}^1, \ldots, \widetilde{A}^n \in F(\mathbb{R})$

$$\Phi\left(\widetilde{A}^1, \ldots, \widetilde{A}^n\right) = \Phi\left(\widetilde{A}^{p_1}, \ldots, \widetilde{A}^{p_n}\right)$$

where the sequence $\{p_1, \ldots, p_n\}$ is a permutation of the sequence $\{1, \ldots, n\}$.

The T1OWA operators with linguistic weights also verify the property of *idempotence* as addressed by the following theorem.

Theorem 6: For any fuzzy number $\widetilde{A}$, the T1OWA operators Φ with fuzzy number weights verify

$$\Phi\left(\widetilde{A}, \ldots, \widetilde{A}\right) = \widetilde{A}.$$

Proof: Let $y \in \mathbb{R}$ and $w_1, \ldots, w_n, a_1, \ldots, a_n \in \mathbb{R}$ such that $y = \sum_{i=1}^n \bar{w}_i a_{\sigma(i)}$ with $\bar{w}_i = w_i \big/ \sum_{i=1}^n w_i$. Convexity of $\widetilde{A} \in F(\mathbb{R})$ implies that

$$\begin{aligned}
&\mu_{\widetilde{A}}(y) = \mu_{\widetilde{A}}\left(\sum_{i=1}^n \bar{w}_i a_{\sigma(i)}\right) \\
&\geq \mu_{\widetilde{A}}(a_{\sigma(1)}) \wedge \cdots \wedge \mu_{\widetilde{A}}(a_{\sigma(n)}) \\
&= \mu_{\widetilde{A}}(a_1) \wedge \cdots \wedge \mu_{\widetilde{A}}(a_n) \\
&\geq \mu_{\widetilde{W}^1}(w_1) \wedge \cdots \wedge \mu_{\widetilde{W}^n}(w_n) \wedge \mu_{\widetilde{A}}(a_1) \wedge \cdots \wedge \mu_{\widetilde{A}}(a_n).
\end{aligned}$$

The above-mentioned inequality is true for any possible set of values $w_1, \cdots, w_n$, $a_1, \ldots, a_n \in \mathbb{R}$ such that $y = \sum_{i=1}^n \bar{w}_i a_{\sigma(i)}$ and therefore it is true that

$$\begin{aligned}
\mu_{\widetilde{A}}(y) &\geq \sup_{\substack{\sum_{k=1}^n \bar{w}_i a_{\sigma(i)} = y \\ w_i \in U, a_i \in \mathbb{R}}} \begin{pmatrix} \mu_{\widetilde{W}^1}(w_1) \wedge \cdots \wedge \mu_{\widetilde{W}^n}(w_n) \\ * \mu_{\widetilde{A}^1}(a_1) \wedge \cdots \wedge \mu_{\widetilde{A}^n}(a_n) \end{pmatrix} \\
&= \mu_{\widetilde{Y}}(y)
\end{aligned}$$

where $\widetilde{Y} = \Phi\,(\widetilde{A}^1, \ldots, \widetilde{A}^n)$.

In order to prove $\mu_{\widetilde{Y}}(y) = \mu_{\widetilde{A}}(y)$, we only need to find a specific combination of $\hat{w}_1, \ldots, \hat{w}_n$, $\hat{a}_1, \ldots, \hat{a}_n \in \mathbb{R}$ such that $\mu_{\widetilde{W}^1}(\hat{w}_1) \wedge \cdots \wedge \mu_{\widetilde{W}^n}(\hat{w}_n) \wedge \mu_{\widetilde{A}}(\hat{a}_1) \wedge \cdots \wedge \mu_{\widetilde{A}}(\hat{a}_n)$ reaches $\mu_{\widetilde{A}}(y)$. For $\widetilde{W}^i$ $(\forall i)$ being a fuzzy number, there exists at least one value $\hat{w}_i$ such that $\mu_{\widetilde{W}^i}(\hat{w}_i) = 1$ $(\forall i)$. Taking $\hat{a}_i = y$ $(\forall i)$, we have

$$\begin{aligned}
&\mu_{\widetilde{W}^1}(\hat{w}_1) \wedge \cdots \wedge \mu_{\widetilde{W}^n}(\hat{w}_n) \wedge \mu_A(\hat{a}_1) \wedge \cdots \wedge \mu_{\widetilde{A}}(\hat{a}_n) \\
&= \mu_{\widetilde{A}}(y) \wedge \cdots \wedge \mu_{\widetilde{A}}(y) \\
&= \mu_{\widetilde{A}}(y).
\end{aligned}$$

Consequently, $\mu_{\widetilde{G}}(y) = \mu_{\widetilde{A}}(y)$. ■

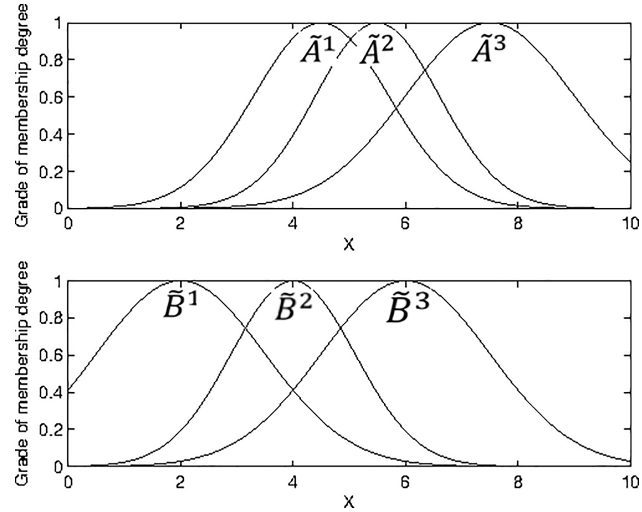

Fig. 2. *Alpha-equivalently-ordered* fuzzy numbers $\widetilde{B}^1, \widetilde{B}^2, \widetilde{B}^3$ (bottom) with $\widetilde{A}^1, \widetilde{A}^2, \widetilde{A}^3$ (up).

In what follows, we investigate how monotonicity is verified by T1OWA operators. First, we propose the *alpha-equivalently-ordered* relation between two sets of fuzzy numbers.

Definition 7: Let $\{\widetilde{A}^i\}_{i=1}^n$ and $\{\widetilde{B}^i\}_{i=1}^n$ be two sets of fuzzy numbers. The σ and η represent permutations of $\{1, \ldots, n\}$ defined by $\{\widetilde{A}^i_{\alpha+}\}_{i=1}^n$ and $\{\widetilde{A}^i_{\alpha-}\}_{i=1}^n$, respectively. If for any $\alpha \in [0, 1]$

$$\begin{aligned}
&\widetilde{A}^{\sigma(1)}_{\alpha+} \geq \widetilde{A}^{\sigma(2)}_{\alpha+} \geq \cdots \geq \widetilde{A}^{\sigma(n)}_{\alpha+} \Longrightarrow \\
&\widetilde{B}^{\sigma(1)}_{\alpha+} \geq \widetilde{B}^{\sigma(2)}_{\alpha+} \geq \cdots \geq \widetilde{B}^{\sigma(n)}_{\alpha+}
\end{aligned}$$

and

$$\begin{aligned}
&\widetilde{A}^{\eta(1)}_{\alpha-} \geq \widetilde{A}^{\eta(2)}_{\alpha-} \geq \cdots \geq \widetilde{A}^{\eta(n)}_{\alpha-} \Longrightarrow \\
&\widetilde{B}^{\eta(1)}_{\alpha-} \geq \widetilde{B}^{\eta(2)}_{\alpha-} \geq \cdots \geq \widetilde{B}^{\eta(n)}_{\alpha-}
\end{aligned}$$

then the fuzzy sets $\{\widetilde{B}^i\}_{i=1}^n$ are said to be *alpha-equivalently-ordered* with the sets $\{\widetilde{A}^i\}_{i=1}^n$.

The following example illustrates the *alpha-equivalently-ordered* relation, whereas *Example 7* in the *supplemental material* gives a counterexample of the *alpha-equivalently-ordered* relation.

Example 2 (Alpha-equivalently-ordered Relation): Fig. 2 illustrates a group of three fuzzy numbers $\{\widetilde{B}^1, \widetilde{B}^2, \widetilde{B}^3\}$ being *alpha-equivalently-ordered* with the group of three fuzzy numbers $\{\widetilde{A}^1, \widetilde{A}^2, \widetilde{A}^3\}$.

The following theorem states the conditions under which T1OWA operators are *monotonic* in the sense of partial order relation of fuzzy sets.

Theorem 7: Let Φ be a T1OWA operator. Supposing the two sets of fuzzy numbers $\{\widetilde{A}^i\}_{i=1}^n$ and $\{\widetilde{B}^i\}_{i=1}^n$ be *alpha-equivalently-ordered*. If $\forall i$, $\widetilde{A}^i \succcurlyeq \widetilde{B}^i$, then

$$\Phi\left(\widetilde{A}^1,\ldots,\widetilde{A}^n\right) \succcurlyeq \Phi\left(\widetilde{B}^1,\ldots,\widetilde{B}^n\right).$$

Proof: As defined in (3), for each $\alpha \in [0,1]$, the α-level aggregation of $\{\widetilde{A}^i\}_{i=1}^n$ by Φ is

$$\begin{aligned}&\Phi_\alpha\left(\widetilde{A}^1_\alpha,\ldots,\widetilde{A}^n_\alpha\right)\\ &\quad= \left\{ \frac{\sum_{i=1}^n w_i a_{\sigma(i)}}{\sum_{i=1}^n w_i} \,\middle|\, w_i \in \widetilde{W}^i_\alpha, a_i \in \widetilde{A}^i_\alpha, i=1,\ldots,n \right\}.\end{aligned}$$

We know from Theorem 1 that $\Phi(\widetilde{A}^1,\ldots,\widetilde{A}^n)_\alpha = \Phi_\alpha(\widetilde{A}^1_\alpha,\ldots,\widetilde{A}^n_\alpha)$, therefore

$$\begin{aligned}\left(\Phi\left(\widetilde{A}^1,\ldots,\widetilde{A}^n\right)\right)_{\alpha+} &= \Phi_\alpha\left(\widetilde{A}^1_\alpha,\ldots,\widetilde{A}^n_\alpha\right)_+\\ &= \max_{\substack{\widetilde{W}^i_{\alpha-} \le w_i \le \widetilde{W}^i_{\alpha+}\\ \widetilde{A}^i_{\alpha-} \le a_i \le \widetilde{A}^i_{\alpha+}}} \frac{\sum_{i=1}^n w_i a_{\sigma(i)}}{\sum_{i=1}^n w_i}\\ &= \max_{\widetilde{W}^i_{\alpha-} \le w_i \le \widetilde{W}^i_{\alpha+}} \frac{\sum_{i=1}^n w_i \widetilde{A}^{\sigma(i)}_{\alpha+}}{\sum_{i=1}^n w_i}.\end{aligned}$$

Because $\widetilde{A}^i \succcurlyeq \widetilde{B}^i$, and $\{\widetilde{B}^i\}_{i=1}^n$ is *alpha-equivalently-ordered* with $\{\widetilde{A}^i\}_{i=1}^n$, then we have that $\widetilde{A}^{\sigma(1)}_{\alpha+} \ge \widetilde{A}^{\sigma(2)}_{\alpha+} \ge \cdots \ge \widetilde{A}^{\sigma(n)}_{\alpha+}$ implies $\widetilde{B}^{\sigma(1)}_{\alpha+} \ge B^{\sigma(2)}_{\alpha+} \ge \cdots \ge \widetilde{B}^{\sigma(n)}_{\alpha+}$. Thus,

$$\begin{aligned}\left(\Phi\left(\widetilde{A}^1,\ldots,\widetilde{A}^n\right)\right)_{\alpha+} &\ge \max_{\widetilde{W}^i_{\alpha-} \le w_i \le \widetilde{W}^i_{\alpha+}} \frac{\sum_{i=1}^n w_i \widetilde{B}^{\sigma(i)}_{\alpha+}}{\sum_{i=1}^n w_i}\\ &= \Phi_\alpha\left(\widetilde{B}^1_\alpha,\ldots,\widetilde{B}^n_\alpha\right)_+.\end{aligned}$$

Because $\Phi(\widetilde{B}^1,\ldots,\widetilde{B}^n)_\alpha = \Phi_\alpha(\widetilde{B}^1_\alpha,\cdots,\widetilde{B}^n_\alpha)$, we conclude that $(\Phi(\widetilde{A}^1,\ldots,\widetilde{A}^n))_{\alpha+} \ge (\Phi(\widetilde{B}^1,\ldots,\widetilde{B}^n))_{\alpha+}$. A similar reasoning leads to $(\Phi(\widetilde{A}^1,\ldots,\widetilde{A}^n))_{\alpha-} \ge (\Phi(\widetilde{B}^1,\ldots,\widetilde{B}^n))_{\alpha-}$. Hence,

$$\Phi\left(\widetilde{A}^1,\ldots,\widetilde{A}^n\right) \succcurlyeq \Phi\left(\widetilde{B}^1,\ldots,\widetilde{B}^n\right).$$

■

The following example illustrates how the monotonic relation of aggregation in terms of $\succcurlyeq$ can be maintained for the aggregated objects which are *alpha-equivalently-ordered*.

Example 3 (Monotonic Relation): The fuzzy sets $\{\widetilde{B}^i\}_{i=1}^3$ depicted in Fig. 2 are *alpha-equivalently-ordered* with $\{\widetilde{A}^i\}_{i=1}^3$, and $\widetilde{A}^i \succcurlyeq \widetilde{B}^i$ $(i=1,2,3)$. Fig. 4 illustrates the results of aggregating the fuzzy numbers in Fig. 2 by a T1OWA operator Φ with the linguistic weights defined in Fig. 3, respectively,

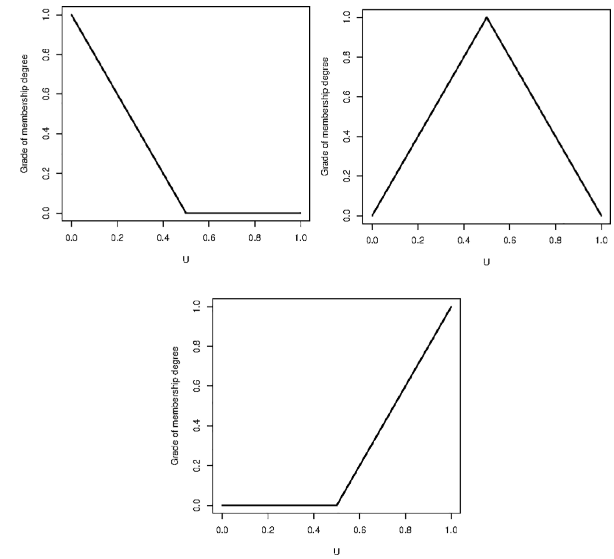

Fig. 3. Linguistic weights of a T1OWA operator: $\widetilde{W}^1$ (top-left), $\widetilde{W}^2$ (top-right), and $\widetilde{W}_3$ (bottom).

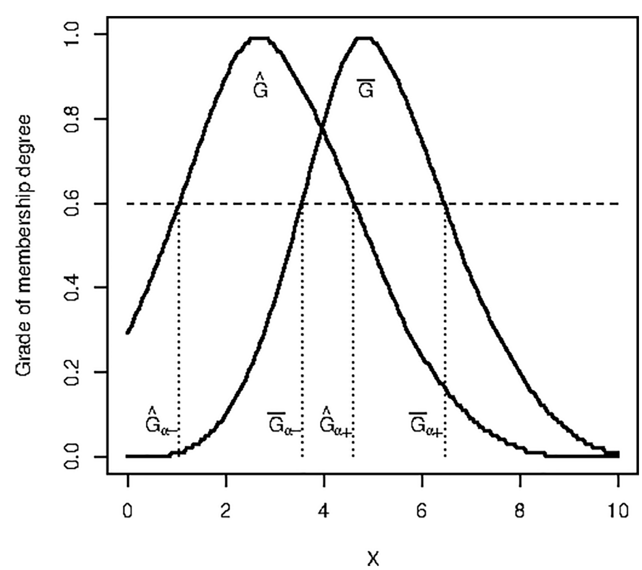

Fig. 4. Monotonic relation preserved in the results of aggregating the fuzzy numbers in Fig. 2 by a T1OWA operator defined by the linguistic weights in Fig. 3.

$\bar{G} = \Phi(\widetilde{A}^1,\widetilde{A}^2,\widetilde{A}^3), \hat{G} = \Phi(\widetilde{B}^1,\widetilde{B}^2,\widetilde{B}^3)$. It is clear that for each $\alpha \in [0,1], \bar{G}_{\alpha-} \ge \hat{G}_{\alpha-}$, and $\bar{G}_{\alpha+} \ge \hat{G}_{\alpha+}$, i.e., $\bar{G} \succcurlyeq \hat{G}$.

The next theorem states that the *meet* and *join* operators are the lower bound and upper bound of T1OWA aggregation in the sense of *partial order relation*.

Theorem 8: Any T1OWA operator Ψ is between the join J and the meet M

$$J\left(\widetilde{A}^1,\cdots\widetilde{A}^n\right) \succcurlyeq \Psi\left(\widetilde{A}^1,\ldots,\widetilde{A}^n\right) \succcurlyeq M\left(\widetilde{A}^1,\cdots\widetilde{A}^n\right).$$

Proof: According to (3), for each $\alpha \in [0,1]$, the α-level aggregation of $\{\widetilde{A}^i\}_{i=1}^n$ by the T1OWA operator

J is

$$J_\alpha\left(\widetilde{A}^1_\alpha,\ldots,\widetilde{A}^n_\alpha\right) = \left\{\frac{\sum_{i=1}^n w_i a_{\sigma(i)}}{\sum_{i=1}^n w_i}\middle| w_1 \in \tilde{1}_\alpha,\ w_i \in \tilde{0}_\alpha\ (i \neq 1),\ a_i \in \widetilde{A}^i_\alpha(\forall i)\right\}.$$

We have that $\tilde{1}_\alpha = \{1\}$, $\tilde{0}_\alpha = \{0\}$. Thus,

$$\begin{aligned} J_\alpha\left(\widetilde{A}^1_\alpha,\ldots,\widetilde{A}^n_\alpha\right) &= \left\{a_{\sigma(1)} | a_i \in \widetilde{A}^i_\alpha,\ i = 1,\cdots n\right\} \\ &= \left\{\max\{a_1,\ldots,a_n\} | a_i \in \widetilde{A}^i_\alpha(\forall i)\right\}. \end{aligned}$$

As a result, the end points of the α-cut intervals are

$$\begin{aligned} J_\alpha\left(\widetilde{A}^1_\alpha,\ldots,\widetilde{A}^n_\alpha\right)_+ &= \max\left(\widetilde{A}^1_{\alpha+},\cdots,\widetilde{A}^n_{\alpha+}\right) \\ J_\alpha\left(\widetilde{A}^1_\alpha,\ldots,\widetilde{A}^n_\alpha\right)_- &= \max\left(\widetilde{A}^1_{\alpha-},\cdots,\widetilde{A}^n_{\alpha-}\right). \end{aligned}$$

The α-level aggregation of $\{\widetilde{A}^i\}_{i=1}^n$ by a general T1OWA operator Ψ is

$$\Psi_\alpha\left(\widetilde{A}^1_\alpha,\ldots,\widetilde{A}^n_\alpha\right) = \left\{\frac{\sum_{i=1}^n w_i a_{\sigma(i)}}{\sum_{i=1}^n w_i}\middle| w_i \in \widetilde{W}^i_\alpha, a_i \in \widetilde{A}^i_\alpha(\forall i)\right\}.$$

Furthermore,

$$\begin{aligned} &\left(\Psi\left(\widetilde{A}^1,\ldots,\widetilde{A}^n\right)\right)_{\alpha+} \\ &= \Psi_\alpha\left(\widetilde{A}^1_\alpha,\ldots,\widetilde{A}^n_\alpha\right)_+ \\ &= \max_{\substack{\widetilde{W}^i_{\alpha-} \leq w_i \leq \widetilde{W}^i_{\alpha+} \\ \widetilde{A}^i_{\alpha-} \leq a_i \leq \widetilde{A}^i_{\alpha+}}} \frac{\sum_{i=1}^n w_i a_{\sigma(i)}}{\sum_{i=1}^n w_i} \\ &= \max_{\widetilde{W}^i_{\alpha-} \leq w_i \leq \widetilde{W}^i_{\alpha+}} \frac{\sum_{i=1}^n w_i \widetilde{A}^{\sigma(i)}_{\alpha+}}{\sum_{i=1}^n w_i} \\ &\leq \max_{\widetilde{W}^i_{\alpha-} \leq w_i \leq \widetilde{W}^i_{\alpha+}} \frac{\sum_{i=1}^n w_i \max\left(\widetilde{A}^1_{\alpha+},\ldots,\widetilde{A}^n_{\alpha+}\right)}{\sum_{i=1}^n w_i} \\ &= \max\left(\widetilde{A}^1_{\alpha+},\ldots,\widetilde{A}^n_{\alpha+}\right). \end{aligned}$$

We have proved that $(\Psi(\widetilde{A}^1,\ldots,\widetilde{A}^n))_{\alpha+} \leq J_\alpha(\widetilde{A}^1_\alpha,\ldots,\widetilde{A}^n_\alpha)_+$. Similarly, we have $(\Psi(\widetilde{A}^1,\ldots,\widetilde{A}^n))_{\alpha-} \leq J_\alpha(\widetilde{A}^1_\alpha,\ldots,\widetilde{A}^n_\alpha)_-$. So, we prove that

$$J\left(\widetilde{A}^1,\ldots,\widetilde{A}^n\right) \succcurlyeq \Psi\left(\widetilde{A}^1,\ldots,\widetilde{A}^n\right).$$

We omit the proof of the other inequality: $\Psi(\widetilde{A}^1,\ldots,\widetilde{A}^n) \succcurlyeq M(\widetilde{A}^1,\ldots,\widetilde{A}^n)$, because it follows the same above-mentioned line of reasoning. ■

According to Theorem 8, the *join* and *meet* operators are two extreme cases of T1OWA operators. T1OWA aggregation is located between the *meet* and the *join* of all the individual operands, i.e., T1OWA operators are *compensative:* low aggregation in the sense of approaching the *meet* operation is compensated by high aggregation in the sense of approaching the *join* operation.

Example 8 in the *supplemental material* illustrates the validation of how a T1OWA aggregation maintains the compensative property in terms of *partial order relations* of fuzzy numbers.

VI. Case Study and Experimental Results

A. Diabetes Diagnosis by T1OWA Based Fuzzy Inference System

T1OWA operators have gained many real-world applications in different domains [10], [12], [13]. In this section, we further provide a practical application of T1OWA operators to "Pima Indian Diabetes" for integrated patient diagnosis.

The "Pima Indian Diabetes" data set [18] describes the clinical conditions of 768 females who develop type-2 diabetes. All patients in this data set were women (≥ 21 years old): 500 (65.1%) healthy and 268 (34.9%) with diabetes. It can be seen that this is an imbalanced data set. Eight attributes describe the patients: age (years), plasma glucose concentration (*plaGlu*), number of times pregnant, triceps skin fold thickness (mm), diastolic blood pressure (mmHg), body mass index (*BMI*) [(weight in kg)/(height in m^2)], 2–h serum insulin ($mmol/L$), and diabetes pedigree function. The outcome is a class variable (0 or 1): *1=diabetic*, *0=nondiabetetic*.

In contrast to other studies using the "Pima Indian Diabetes" data set, we take into account two further underlying issues with the data. One issue is that clearly this is an imbalanced data set; hence, the widely used assessment metric, *classification rate* (CR) (also known as *accuracy*), is not appropriate and not reliable to assess such a real clinical scenario. For imbalanced data sets, which are very common in clinical studies, the F_1-score and balanced CR (BCR) are a preferred metric, as they makes more sense than others: $F_1 - score = (2 \times recall \times precision)/(recall + precision)$; $BCR = (sensitivity + specificity)/2$.

The second issue is that the majority of the existing studies using this data set did not consider its underlying missing value problem. Indeed, there are no specifically labeled missing values in the data set. But this cannot be the case, because so many zeros are used to represent the status of attributes where they are biologically impossible, such as the attributes of glucose concentration (5 records of zeros), triceps skin fold thickness (227 records of zeros), blood pressure (35 records of zeros), insulin (374 records of zeros), and body mass index (11 records of zeros). It is highly plausible that these zero values were actually originally used to encode missing values in these fields. In our study, we considered these zeros as missing values and used the nearest-neighbor method to impute them. Then, we apply different T1OWA aggregations to nonstationary fuzzy sets [16], [17] to find the optimal diagnoses of diabetes.

Given a standard fuzzy system as a baseline system, the T1OWA based nonstationary fuzzy system (T1OWANFS) (see Fig. 8 in the *supplemental material*) proceeds as follows. First, in each run, the crisp numbers of each input variable are fuzzified by a fuzzifier function, such as singleton or nonsingleton function. These fuzzified input sets then feed into the inference engine with the given rulebase to conduct operations of union and intersection on these fuzzy sets, and perform composition of the relations. Such a process is repeated n runs, so n fuzzy set outputs are produced. Then, a T1OWA aggregation operation is applied to these sets to achieve an overall solution. Finally, a crisp output is generated via defuzzification of this overall output fuzzy set.

The rulebase in this study consists of the following four rules based on two attributes of *plaGlu* and *BMI*.

1) *Rule 1: if (plaGlu is high), then diabetic.*
2) *Rule 2: if (plaGlu is medium) and (BMI is high), then diabetic.*
3) *Rule 3: if (plaGlu is low), then nondiabetic.*
4) *Rule 4: if (plaGlu is medium) and (BMI is low), then nondiabetic.*

where the variables *plaGlu*, *BMI*, and *outcome* are described by baseline fuzzy sets (see Fig. 9 of the *supplemental material*). Their corresponding nonstationary fuzzy sets were generated based on these baseline sets [16], [17]. In our study, the nonstationary fuzzy system ran ten times to generate the diagnoses for each patient (Fig. 10 in the *supplemental material* shows an example of ten fuzzy decision outputs for a patient). The system performance is evaluated in terms of F_1-score and BCR.

We use five different types of T1OWA operators in this case study to aggregate the fuzzy diagnosis from nonstationary fuzzy inference engine (see Fig. 8 in the *supplemental material*).

1) The standard *join* operator: denoted as *join_NFS*.
2) The standard *meet* operator: denoted as *meet_NFS*.
3) Join-like T1OWA operators with the linguistic weight $\widetilde{W}_1$ as in Fig. 3(a) and others $\widetilde{W}_i (i \neq 1)$ as in Fig. 3(b) in the *supplemental material* to aggregate the 10 output sets for diabetes diagnosis: denoted as *JLT1OWA_NFS*.
4) Meet-like T1OWA operators with the last linguistic weight $\widetilde{W}_{10}$ as in Fig. 3(a) and others $\widetilde{W}_i (i \neq 10)$ as in Fig. 3(b) in the *supplemental material:* denoted as *MLT1OWA_NFS*.
5) A T1OWA operator with linguistic weights implementing the fuzzy majority represented by the type-2 quantifier *most* [4]: denoted as *T2MT1OWA_NFS*.

Fig. 5 depicts an example of corresponding results of aggregating ten fuzzy decisions from nonstationary fuzzy inference engine for one patient (see Fig. 10 in the *supplemental material*) by the above-mentioned five T1OWA operators. These aggregated fuzzy outputs are then defuzzified to generate crisp values as final outputs.

The above-mentioned examples demonstrate the advantages of these T1OWA operators to aggregate the uncertain information modeled by fuzzy sets. The *T2MT1OWA_NFS* operator implements the *soft* majority in aggregating a group of uncertain decisions (perhaps expressed linguistically as *most of the decisions should be satisfied*), which is much closer to the real human perception in decision making than traditional aggregation methods. Fig. 11 in the *supplemental material* illustrates the *joinness* of the operator, *T2MT1OWA_NFS*, which clearly shows that the quantifier *most* guided operator approaches the meet operation (expressed linguistically as *all decisions should be satisfied*). As a matter of fact, such a linguistic quantifier based aggregation can be treated as a manifestation of a semantically guided aggregation [2], [4].

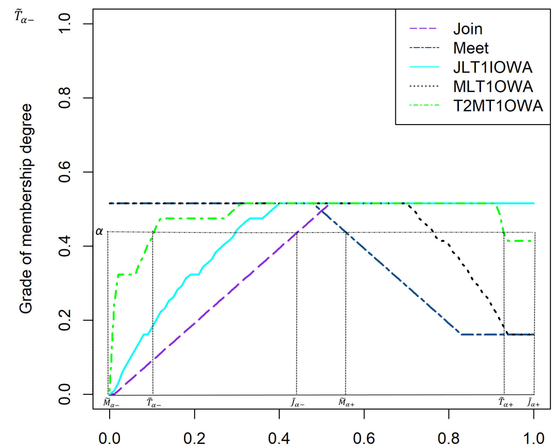

Fig. 5. Example of aggregation results of 10 fuzzy output decisions (from nonstationary fuzzy inference engine for a patient) by the five different T1OWA operators.

To validate the *compensative* property of these T1OWA operators, let us assume the aggregation results of the five operators shown in Fig. 5 (*join_NFS*, *meet_NFS*, *JLT1OWA_NFS*, *MLT1OWA_NFS*, and *T2MT1OWA_NFS*) be represented as $\widetilde{J}$, $\widetilde{M}$, $\widetilde{JL}$, $\widetilde{ML}$, and $\widetilde{T}$, respectively. Taking *T2MT1OWA_NFS* as an example, for any α level in Fig. 5, it can be seen that

$$\widetilde{J}_{\alpha+} \geq \widetilde{T}_{\alpha+} \geq \widetilde{M}_{\alpha+} \text{ and } \widetilde{J}_{\alpha-} \geq \widetilde{T}_{\alpha-} \geq \widetilde{M}_{\alpha-} \forall\alpha \in [0, 1].$$

Therefore, according to Definition 5, and because the relations $\succcurlyeq$ and $\widetilde{\geq}$ are equivalent, we get $\widetilde{J} \succcurlyeq \widetilde{T} \succcurlyeq \widetilde{M}$, i.e., the T2MT1OWA operator verifies *Theorem* 8, and the *compensative* property holds in this case study.

Furthermore, we show a comparison (in terms of F_1-score and BCR) with the following existing methods: first, standard fuzzy weighted average (FWA) operators [19]; and second, a zero-order Takagi–Sugeno fuzzy system with two rules (TS-FSTR) [20]. Table I summarizes the performances of these approaches. It can be seen that the *JLT1OWA_NFS* and *join* achieved the best performance in terms of F_1-score and BCR. The JLT1OWA_NFS significantly improved the recall without sacrificing much precision, so that a better F_1-score was achieved.

B. Validation of Computing Efficiencies of Alpha-Level Approach *to T1OWA Operations in Real-World Applications*

The *direct approach* [4] and *alpha-level approach* [5] generate the same results of aggregating fuzzy sets as shown in the *representation theorem of type-1 OWA operators* (see Theorem 1). However, the *direct approach* is an exponential-time algorithm that takes $O(K^n)$ operations [5], in which the constant K depends on $n_u \cdot n_x$, where n is the number of fuzzy sets to be aggregated, n_u is the number of sampling points on the domain $[0, 1]$ of the T1OWA operator's linguistic weights, and n_x is the number of sampling points on the domain of the fuzzy sets

TABLE I
PERFORMANCES OF DIFFERENT APPROACHES TO DIABETES DIAGNOSIS

Approach	**CR**	**Recall**	**Specificity**	**Precision**	**F_1-score**	**BCR**
Meet_NFS	0.760	0.519	0.890	0.716	0.602	0.704
MLT1OWA_NFS	0.760	0.519	0.890	0.716	0.602	0.704
T2MT1OWA_NFS	0.759	0.586	0.852	0.680	0.629	0.719
JLT1OWA_NFS	0.746	0.683	0.780	0.625	**0.652**	**0.731**
Join_NFS	0.746	0.683	0.780	0.625	**0.652**	**0.731**
FWA	0.751	0.619	0.822	0.651	0.635	0.721
TSFSTR	0.716	0.455	0.856	0.629	0.528	0.656

TABLE II
TIME-COSTS OF TYPE-1 OWA AGGREGATIONS FOR DIABETES DIAGNOSES (IN MINUTES)

Setting	$n_x = 100, n_u = 30, n = 10$		$n_x = 10, n_u = 5, n = 3$	
Method	**Alpha-Level Approach**	**Direct Approach**	**Alpha-Level Approach**	**Direct Approach**
join_NFS	2.927	Infeasible	0.295	8.786
JLT1OWA_NFS	2.861	Infeasible	0.294	123640.3
MLT1OWA_NFS	2.919	Infeasible	0.302	109579.8
meet_NFS	2.982	Infeasible	0.297	8.589

to be aggregated. In comparison, the *alpha-level approach* is a linear-time algorithm, taking $O(n)$ operations [5]. Therefore, the *alpha-level approach* can be used to implement T1OWA aggregations in real-time applications.

In Section VI-A, the T1OWA based fuzzy decision making for diabetes was implemented by the *alpha-level approach*. The domains of the linguistic weights and fuzzy sets have to be discretized. The default settings are: $n_u = 30$ and $n_x = 100$, while $n = 10$ (i.e., ten fuzzy decisions). However, such settings are unworkable for the *direct approach* in implementation due to the oversized vectors which need to be created by the computer. For comparison, therefore, simplified settings are used, such that $n_u = 5$ and $n_x = 10$, while $n = 3$. Even under such simplified settings, it is estimated that the *direct approach* still takes days to complete the diagnoses for all 768 patients using the *meet-like* or *join-like* T1OWA operators. Our solution to calculation of time-cost was: first, the time-cost tc_1 of the *direct approach* to aggregating three fuzzy decisions for only one patient is calculated; then, the total time-cost is $768 \cdot tc_1$. Table II shows the time-costs of the two approaches to diagnosing diabetic patients by the T1OWA operators, *join_NFS*, *meet_NFS*, *JLT1OWA_NFS*, and *MLT1OWA_NFS*. This validated that the *alpha-level approach* can achieve much higher computing efficiency than the *direct approach* to aggregating fuzzy sets in the manner of OWA operation in real-world applications.

The experimental results were generated in *R* on a computer with Intel(R) Core i5-4440@3.10 GHz and 16 GB memory. The *R* codes for type-1 OWA aggregations are available upon request.

VII. CONCLUSION

As a generalization of Yager's OWA operator, T1OWA operators provided an efficient tool to aggregate uncertain information modeled by fuzzy sets in soft decision making. By appropriately selecting fuzzy sets for the weights, various forms of T1OWA operators can be created to fulfill different tasks under multi-granular linguistic contexts. This was demonstrated in the earlier case study of diabetic diagnosis, which was an imbalanced data problem. By appropriately selecting the uncertain weights to favor the rare class, such as the *join-like* T1OWA operator for the diabetic class, the T1OWA aggregation approach has the potential to enable standard classifiers to be a cost-sensitive approach, whereby the cost of misclassifying the rare class was higher than the cost of misclassifying the other class. This topic merits further research. In addition, to date, T1OWA operators only consider the t-norm (minimum) and t-conorm (maximum); but how T1OWA aggregations and properties vary by using different forms of t-norm and t-conorm is an interesting research problem.

In summary, this article proved that T1OWA operators verify the same properties which hold for Yager's OWA operator, namely *idempotence*, *monotonicity*, *compensativeness*, and *commutativity*. Such theoretical analyses provide a solid foundation for T1OWA operators to be applied widely in different scenarios.

REFERENCES

[1] D. H. Hong and S. Han, "The general least square deviation OWA operator problem," *Mathematics*, vol. 7, no. 4, pp. 326–345, 2019.

[2] R. R. Yager, "On ordered weighted averaging aggregation operators in multicriteria decisionmaking," *IEEE Trans. Syst., Man, Cybern.*, vol. 18, no. 1, pp. 183–190, Jan./Feb. 1988.

[3] A. Kishor, S. Sonam, N. Pal, "A New Family of OWA Operators Featuring Constant Orness," *IEEE Transactions on Fuzzy Systems*, 2019, PP. 1-1. 10.1109/TFUZZ.2019.2928519.

[4] S.-M. Zhou, F. Chiclana, R. I. John, and J. M. Garibaldi, "Type-1 OWA operators for aggregating uncertain information with uncertain weights induced by type-2 linguistic quantifiers," *Fuzzy Sets Syst.*, vol. 159, no. 24, pp. 3281–3296, 2008.

[5] S.-M. Zhou, F. Chiclana, R. I. John, and J. M. Garibaldi, "Alpha-level aggregation: A practical approach to type-1 OWA operation for aggregating uncertain information with applications to breast cancer treatments," *IEEE Trans. Knowl. Data Eng.*, vol. 23, no. 10, pp. 1455–1468, Oct. 2011.

[6] L. A. Zadeh, “The concept of a linguistic variable and its application to approximate reasoning-2,” *Inf. Sci.*, vol. 8, pp. 301–357, 1975.
[7] J. M. Mendel and R. I. John, “Type-2 fuzzy sets made simple,” *IEEE Trans. Fuzzy Syst.*, vol. 10, no. 2, pp. 117–127, Apr. 2002.
[8] S.-M. Zhou, J. M. Garibaldi, R. I. John, and F. Chiclana, “On constructing parsimonious type-2 fuzzy logic systems via influential rule selection,” *IEEE Trans. Fuzzy Syst.*, vol. 17, no. 3, pp. 654–667, Jun. 2009.
[9] D. Wu and J. Huang, “Ordered novel weighted averages,” in *Type-2 Fuzzy Logic and Systems* (Studies in Fuzziness and Soft Computing), John *et al.*, Eds., vol. 362. Berlin, Germany: Springer, 2018.
[10] A. R. Buck, J. M. Keller, and M. Popescu, “An α-level OWA implementation of bounded rationality for fuzzy route selection,” in *Advance Trends in Soft Computing* (Studies in Fuzziness and Soft Computing), M. Jamshidi, V. Kreinovich, and J. Kacprzyk, Eds., vol. 312. Berlin, Germany: Springer, 2014.
[11] H. Hu, Q. Yang, and Y. Cai, “An opposite direction searching algorithm for calculating the type-1 ordered weighted average,” *Knowl.-Based Syst.* vol. 52, pp. 176–180, 2013.
[12] F. Chiclana, F. Mata, L. G. Perez, and E. Herrera-Viedma, “Type-1 OWA unbalanced fuzzy linguistic aggregation methodology: Application to eurobonds credit risk evaluation,” *Int. J. Intell. Syst.*, vol. 33, pp. 1071–1088, 2018.
[13] F. Mata, L. G. Perez, S.-M. Zhou, and F. Chiclana, “Type-1 OWA methodology to consensus reaching processes in multi-granular linguistic contexts,” *Knowl.-Based Syst.*, vol. 58, pp. 11–22, Mar. 2014
[14] F. Chiclana and S.-M. Zhou, “Type-reduction of general type-2 fuzzy sets: The type-1 OWA method,” *Int. J. Intell. Syst.*, vol. 28, pp. 505–522, 2013
[15] J. Ramik and J. Rimanek, “Inequality relation between fuzzy numbers and its use in fuzzy operation,” *Fuzzy Sets Syst.*, vol. 16, pp. 123–138, 1985.
[16] J. M. Garibaldi and T. Ozen, “Uncertain fuzzy reasoning: A case study in modelling expert decision making,” *IEEE Trans. Fuzzy Syst.*, vol. 15, no. 1, pp. 16–30, Feb. 2007.
[17] J. M. Garibaldi, M. Jaroszewski, and S. Musikasuwan, “Nonstationary fuzzy sets,” *IEEE Trans. Fuzzy Syst.*, vol. 16, no. 4, pp. 1072–1086, Aug. 2008.
[18] D. Dua., C. Graff, “UCI Machine Learning Repository,” 2019, [http://archive.ics.uci.edu/ml]. Irvine, CA: University of California, School of Information and Computer Science.
[19] F. Liu and J. M. Mendel, “Aggregation using the fuzzy weighted average as computed by the Karnik–Mendel algorithms,” *IEEE Trans. Fuzzy Syst.*, vol. 16, no. 1, pp. 1–12, Feb. 2008.
[20] N. Settouti, M. A. Chikh, and M. Saidi, “Generating fuzzy rules for constructing interpretable classifier of diabetes disease,” *Australas. Phys. Eng. Sci. Med.*, vol. 35, no. 3, pp. 257–270, Aug. 2012.