Alpha-Level Aggregation: A Practical Approach to Type-1 OWA Operation for Aggregating Uncertain Information with Applications to Breast Cancer Treatments

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Abstract—Type-1 Ordered Weighted Averaging (OWA) operator provides us with a new technique for directly aggregating uncertain information with uncertain weights via OWA mechanism in soft decision making and data mining, in which uncertain objects are modeled by fuzzy sets. The *Direct Approach* to performing type-1 OWA operation involves high computational overhead. In this paper, we define a type-1 OWA operator based on the α-cuts of fuzzy sets. Then, we prove a *Representation Theorem* of type-1 OWA operators, by which type-1 OWA operators can be decomposed into a series of α-level type-1 OWA operators. Furthermore, we suggest a fast approach, called *Alpha-Level Approach*, to implementing the type-1 OWA operator. A practical application of type-1 OWA operators to breast cancer treatments is addressed. Experimental results and theoretical analyses show that: 1) the *Alpha-Level Approach* with linear order complexity can achieve much higher computing efficiency in performing type-1 OWA operation than the existing *Direct Approach*, 2) the type-1 OWA operators exhibit different aggregation behaviors from the existing fuzzy weighted averaging (FWA) operators, and 3) the type-1 OWA operators demonstrate the ability to efficiently aggregate uncertain information with uncertain weights in solving real-world soft decision-making problems.

Index Terms—OWA operators, aggregation, fuzzy sets, type-1 OWA operators, Alpha-cuts, Alpha level, uncertain information, soft decision making, breast cancer treatments.

1 Introduction

Aggregation operation is not only an important research topic in knowledge and data engineering [1], [2], [3], [4], [5], but also one of the most important steps in dealing with multiexpert decision making, multicriteria decision making, and multiexpert multicriteria decision making [6], [7], [8]. The objective of aggregation is to combine individual sources of information into an overall one in a proper way, so that the final result of aggregation can take into account all the individual contributions [9]. Currently, at least 90 different families of aggregation operators have been studied [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19]. Among them, the Ordered Weighted Averaging (OWA) operator proposed by Yager [18] is one of the most widely used, with many successful applications achieved in areas, such as: decision making [6], [8], [12], [21], [22], fuzzy control [23], [24], market analysis [25], and image compression [26]. However, the majority of the existing aggregation operators, including the OWA one, focus exclusively on aggregating crisp numbers. As a matter of fact, inherent subjectivity, imprecision, and vagueness in the articulation of opinions in real-world decision applications make human experts exhibit remarkable capability to manipulate perceptions without any measurements [20]. In these cases, the use of linguistic terms instead of precise numerical values seems to be more adequate in dealing with vague or imprecise information or to express experts' opinions on qualitative aspects that cannot be assessed by means of quantitative values [6], [21]. Thus, techniques for aggregating uncertain information rather than precise crisp values are in high demand, which motivated us to suggest a new OWA operator, called *type-1 OWA operator* [27]. The *type-1 OWA operator* is able to aggregate linguistic terms represented as fuzzy sets via OWA mechanism, and a *Direct Approach* has been suggested to perform type-1 OWA operation [27]. Interestingly, some well-known existing aggregation operators, such as Yager's OWA operator, the *join* and the *meet* operators of fuzzy sets [41], [42] are special cases of this type-1 OWA operator [28].

Different ways of aggregating linguistic assessments, including the ones that follow the way of fuzzifying Yager's OWA operators, have been proposed in literature [13], [21], [29], [30], [31], [32], [33], [34], [35]. A detailed review of the state-of-the-art research in this topic can be found in [27] and [28]. The type-1 OWA operator is different from these existing methods. For example, an approach to OWA aggregation with interval weights and interval inputs was suggested in [32], in which two definitions of aggregating

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interval arguments with interval weights based on the rank of intervals via probabilistic measures were given. However, different probabilistic distributions could lead to different reorderings of the inputs and consequently different outputs could be derived using this approach. Ahn's method focused on the use of the uniform distribution, although no evidence is provided to support that this type of distribution should always be used [32]. The type-1 OWA operator does not suffer from the aforementioned drawback as it is defined according to Zadeh's Extension Principle, only the issues of reordering of crisp values are involved, and therefore, it avoids dealing with the ranking of fuzzy sets/intervals. Moreover, in this paper, we propose an α-level type-1 OWA operator and prove that the *Alpha-Level Approach* can lead to its equivalence one obtained by the Extension Principle. There is no evidence to support that Ahn's method has such property.

To the best of our knowledge, the research work by Mitchell and Schaefer [33], and the research on fuzzified Choquet integral [34], [35] may be the most relevant to our research on type-1 OWA operators. Mitchell and Schaefer also applied Zadeh's Extension Principle to Yager's OWA operator, but their approach focused on the ordering of fuzzy sets during the aggregation process. The type-1 OWA operator avoids ordering fuzzy sets. The Yager's OWA operator is treated as a nonlinear function and is fuzzified to the case of having fuzzy sets as inputs in a type-1 OWA operator. As for the research on fuzzified Choquet integrals, the existing approaches only consider the aggregation of fuzzy sets with crisp weights, while the type-1 OWA operator is able to aggregate fuzzy sets with fuzzy weights as well.

Another widely investigated fuzzified aggregation operators, the fuzzy weighted averaging (FWA) operators [36], [37], [38], can also be applied to the aggregation of fuzzy sets with fuzzy weights. Noteworthily, Yager's OWA operator is a nonlinear aggregation operator, while the weighted averaging operator is linear. Therefore, the type-1 OWA operator is significantly different from the FWA operator [27], [28].

However, the *Direct Approach* to performing type-1 OWA operation suggested in [27] involves high computational load, which inevitably curtails further applications of the type-1 OWA operator to real-world decision making. This paper focuses on how to achieve a high computing efficiency in performing type-1 OWA operations for aggregating uncertain information with uncertain weights, where these uncertain objects are modeled by fuzzy sets. To this end, the α-level type-1 OWA operator is defined using the α-cuts of fuzzy sets. Moreover, a fast approach to type-1 OWA operation, called *Alpha-Level Approach*, with detailed theoretical analyses is addressed. Promisingly, the complexity of this *Alpha-Level Approach* is of linear order, so it can be used in real-time soft decision making, database integration and information fusion that involve aggregation of uncertain information.

This paper is organized as follows: Section 2 describes the definition of α-level type-1 OWA operator. Section 3 proposes the fast approach to implementing the type-1 OWA operation. The complexity of the *Direct Approach* and the fast *Alpha-Level Approach* are analyzed in Section 4. Section 5 extensively evaluates the computing efficiency of the proposed approach including a practical application of type-1 OWA operators to breast cancer treatments. Finally, conclusions and discussion are presented in Section 6.

2 Definition of Type-1 OWA Operators Based on α-Cuts of Fuzzy Sets

As a generalization of Yager's OWA operator and based on Zadeh's Extension Principle, the type-1 OWA operator is defined to aggregate uncertain information with uncertain weights, when both are modeled as fuzzy sets.

First, let $F(X)$ be the set of fuzzy sets with domain of discourse X, a type-1 OWA operator is defined as follows [27], [28]:

Definition 1. *Given n linguistic weights $\{W^i\}_{i=1}^n$ in the form of fuzzy sets defined on the domain of discourse $U = [0, 1]$, a type-1 OWA operator is a mapping Φ*

$$\begin{aligned} \Phi : F(X) \times \cdots \times F(X) &\longrightarrow F(X) \\ (A^1, \ldots, A^n) &\mapsto Y \end{aligned} \tag{1}$$

such that

$$\mu_Y(y) = \sup_{\substack{\sum_{k=1}^n \bar{w}_i a_{\sigma(i)} = y \\ w_i \in U,\, a_i \in X}} \begin{pmatrix} \mu_{W^1}(w_1) \wedge \cdots \wedge \mu_{W^n}(w_n) \\ \wedge \mu_{A^1}(a_1) \wedge \cdots \wedge \mu_{A^n}(a_n) \end{pmatrix}, \tag{2}$$

where

$$\bar{w}_i = \frac{w_i}{\sum_{i=1}^n w_i}$$

and $\sigma : \{1, \ldots, n\} \longrightarrow \{1, \ldots, n\}$ is a permutation function such that $a_{\sigma(i)} \geq a_{\sigma(i+1)}$, $\forall i = 1, \ldots, n-1$, i.e., $a_{\sigma(i)}$ is the ith highest element in the set $\{a_1, \ldots, a_n\}$.

From the above definition, it can be seen that the aggregation result $\Phi(A^1, \ldots, A^n) = Y \in F(X)$ is a fuzzy set defined on X. However, implementation of type-1 OWA operation in aggregating a group of fuzzy sets is not straightforward and easy. A *Direct Approach* to performing type-1 OWA operation has been suggested in [27], but it involves high computational load.

In the interests of improving computing efficiency of type-1 OWA aggregation, in this section, we describe an alternative way of defining type-1 OWA operators based on α-cuts of fuzzy sets. To do this, we first introduce the concept of the α-level type-1 OWA operator guided by α-cuts of fuzzy weights.

Definition 2. *Given the n linguistic weights $\{W^i\}_{i=1}^n$ in the form of fuzzy sets defined on the domain of discourse $U = [0,\ 1]$, then for each $\alpha \in [0,\ 1]$, an α-level type-1 OWA operator with α-level sets $\{W_\alpha^i\}_{i=1}^n$ to aggregate the α-cuts of fuzzy sets $\{A^i\}_{i=1}^n$ is given as*

$$\begin{aligned} &\Phi_\alpha(A_\alpha^1, \ldots, A_\alpha^n) \\ &\quad = \left\{ \frac{\sum_{i=1}^n w_i a_{\sigma(i)}}{\sum_{i=1}^n w_i} \middle| w_i \in W_\alpha^i,\ a_i \in A_\alpha^i,\ i = 1, \ldots, n \right\}, \end{aligned} \tag{3}$$

where $W_\alpha^i = \{w | \mu_{W_i}(w) \geq \alpha\}$, $A_\alpha^i = \{x | \mu_{A_i}(x) \geq \alpha\}$, and $\sigma : \{1, \ldots, n\} \rightarrow \{1, \ldots, n\}$ is a permutation function such that $a_{\sigma(i)} \geq a_{\sigma(i+1)}$, $\forall\, i = 1, \ldots, n-1$, i.e., $a_{\sigma(i)}$ is the ith largest element in the set $\{a_1, \ldots, a_n\}$.

According to the *Representation Theorem* of fuzzy set [40], the α-level sets $\Phi_\alpha(A_\alpha^1, \ldots, A_\alpha^n)$ obtained via Definition 2 can be used to construct the following fuzzy set:

$$G = \bigcup_{0 \le \alpha \le 1} \alpha \Phi_\alpha\left(A^1_\alpha, \ldots, A^n_\alpha\right) \tag{4}$$

with membership function

$$\mu_G(x) = \bigvee_{\alpha : x \in \Phi_\alpha\left(A^1_\alpha, \ldots, A^n_\alpha\right)_\alpha} \alpha \tag{5}$$

From the above definition, it can be seen that the aim of the α-level type-1 OWA operator is to aggregate the α-cuts of fuzzy sets $\{A^i\}_{i=1}^n$ with the α-cuts of fuzzy set weights $\{W^i\}_{i=1}^n$. Given the fact that the α-cuts of fuzzy numbers (i.e., normal and convex fuzzy sets on the domain of real numbers $\mathbb{R}$) are intervals, the α-level type-1 OWA operator actually provides a way of aggregating uncertain arguments with uncertain weights to some extent as Ahn's method did [32]. However, we proceed further to aggregate uncertain information modeled by fuzzy sets.

First, the two apparently different aggregation operators in (2) and (5), defined according to Zadeh's Extension Principle and the α-cut of fuzzy sets, respectively, are equivalent as it is proved in the following:

Theorem 1. *Given the n linguistic weights $\{W^i\}_{i=1}^n$ in the form of fuzzy sets defined on the domain of discourse $U = [0,\ 1]$, and the fuzzy sets $A^1, \ldots, A^n$, then we have that*

$$Y = G,$$

where Y is the aggregation result defined in (2) and G is the result defined in (4).

Proof. We need to prove that for any fuzzy sets $A^1, \ldots, A^n$ and $\alpha \in [0,1]$

$$Y_\alpha = \Phi_\alpha\left(A^1_\alpha, \ldots, A^n_\alpha\right),$$

To prove $Y_\alpha \subseteq \Phi_\alpha(A^1_\alpha, \ldots, A^n_\alpha)$, we note that $\forall y \in Y_\alpha$, there exist $w_1, \ldots, w_n \in U$, and $a_1, \ldots, a_n \in X$ such that $y = \sum_{i=1}^n \bar{w}_i a_{\sigma(i)}$, where $\bar{w}_i = \frac{w_i}{\sum_{i=1}^n w_i}$, and $\alpha \le \mu_{W^1}(w_1) \wedge \cdots \wedge \mu_{W^n}(w_n) \wedge \mu_{A^1}(a_1) \wedge \cdots \wedge \mu_{A^n}(a_n)$. Thus, we have that $\alpha \le \mu_{W^i}(w_i)$ and $\alpha \le \mu_{A^i}(a_i) \forall i$, i.e, $w_i \in W^i_\alpha$, $a_i \in A^i_\alpha$, $i = 1, \ldots, n$. As a result, $y \in \Phi_\alpha(A^1_\alpha, \ldots, A^n_\alpha)$ according to Definition 2.

To prove that $\Phi_\alpha(A^1_\alpha, \ldots, A^n_\alpha) \subseteq Y_\alpha$, we note that $\forall y \in \Phi_\alpha(A^1_\alpha, \ldots, A^n_\alpha)$, there exist $\hat{w}_1 \in W^i_\alpha, \ldots, \hat{w}_n \in W^n_\alpha$ and $\hat{a}_1 \in A^1_\alpha, \ldots, \hat{a}_n \in A^n_\alpha$ such that $y = \sum_{i=1}^n \hat{\bar{w}}_i \hat{a}_{\sigma(i)}$, where $\hat{\bar{w}}_i = \frac{\hat{w}_i}{\sum_{i=1}^n \hat{w}_i}$. Because $\alpha \le \mu_{W^i}(\hat{w}_i)$ and $\alpha \le \mu_{A^i}(\hat{a}_i)\ \forall i$, then

$$\alpha \le \mu_{W^1}(\hat{w}_1) \wedge \cdots \wedge \mu_{W^n}(\hat{w}_n) \wedge \mu_{A^1}(\hat{a}_1) \wedge \cdots \wedge \mu_{A^n}(\hat{a}_n).$$

As a result

$$\alpha \le \sup_{\substack{\sum_{k=1}^n \bar{w}_i a_{\sigma(i)} = y \\ w_i \in U \\ a_i \in X}} \begin{pmatrix} \mu_{W^1}(w_1) \wedge \cdots \wedge \mu_{W^n}(w_n) \\ \wedge\ \mu_{A^1}(a_1) \wedge \cdots \wedge \mu_{A^n}(a_n) \end{pmatrix} = \mu_Y(y).$$

Hence, $y \in Y_\alpha$. □

Theorem 1 is called the *Representation Theorem* of type-1 OWA operators. According to this *Representation Theorem*, type-1 OWA operators can be decomposed into a series of α-level type-1 OWA operators. It provides an effective tool for performing type-1 OWA operations.

It is noted that in fuzzy sets-based soft decision making, linguistic terms are commonly modeled by fuzzy numbers. In what follows, we will focus on these type of fuzzy sets, unless otherwise stated.

When the linguistic weights and the aggregated objects are fuzzy number, the α-level type-1 OWA operator produces closed intervals, as the following theorem states:

Theorem 2. *Let $\{W^i\}_{i=1}^n$ be fuzzy numbers on $U = [0,1]$ and $\{A^i\}_{i=1}^n$ be fuzzy numbers on $\mathbb{R}$. Then, for each $\alpha \in [0,1]$, $\Phi_\alpha(A^1_\alpha, \ldots, A^n_\alpha)$ is a closed interval.*

Proof. First, we have that

$$y(w_1, \ldots, w_n, a_1, \ldots, a_n) = \frac{\sum_{i=1}^n w_i a_{\sigma(i)}}{\sum_{i=1}^n w_i}$$

is a continuous function of $w_1, \ldots, w_n, a_1, \ldots, a_n$. Because

$$a_{\sigma(1)} \ge \frac{\sum_{i=1}^n w_i a_{\sigma(i)}}{\sum_{i=1}^n w_i} \ge a_{\sigma(n)},$$

we have that $y(w_1, \ldots, w_n, a_1, \ldots, a_n)$ is also a bounded function.

Second, because $\{W^i\}_{i=1}^n$ and $\{A^i\}_{i=1}^n$ are fuzzy numbers on $U = [0,1]$, their α-level sets are of the form $W^i_\alpha = [W^i_{\alpha-}, W^i_{\alpha+}], A^i_\alpha = [A^i_{\alpha-}, A^i_{\alpha+}]$ $(i = 1, \ldots, n)$, and therefore compact sets of $\mathbb{R}$ (closed and bounded). The Cartesian product of W^i_α and A^i_α is a compact subset of $\mathbb{R}^{2n}$. Function $y(w_1, \ldots, w_n, a_1, \ldots, a_n)$ is continuous and therefore the image of the Cartesian product of W^i_α and A^i_α is also a compact subset of $\mathbb{R}$.

It is well known that a closed interval of $\mathbb{R}$ is a connected set, and that the Cartesian product of two closed intervals of $\mathbb{R}$ is a connected set of $\mathbb{R}^2$. Consequently, the Cartesian product of W^i_α and A^i_α is a connected subset of $\mathbb{R}^{2n}$. As a result, the image of the Cartesian product of W^i_α and A^i_α is a connected subset of $\mathbb{R}$. Because the only connected subsets of $\mathbb{R}$ are intervals, we conclude that the image of the Cartesian product of W^i_α and A^i_α by the continuous function $y(w_1, \ldots, w_n, a_1, \ldots, a_n)$ is a closed interval [39]. Hence, $\Phi_\alpha(A^1_\alpha, \ldots, A^n_\alpha)$ is a closed interval. □

Based on this theorem, the computation of the type-1 OWA output according to (4), G, reduces to compute the left endpoints and right endpoints of the intervals $\Phi_\alpha(A^1_\alpha, \ldots, A^n_\alpha)$

$$\Phi_\alpha\left(A^1_\alpha, \ldots, A^n_\alpha\right)_- \text{ and } \Phi_\alpha\left(A^1_\alpha, \ldots, A^n_\alpha\right)_+,$$

where $A^i_\alpha = [A^i_{\alpha-}, A^i_{\alpha+}], W^i_\alpha = [W^i_{\alpha-}, W^i_{\alpha+}]$.

For the left endpoints, we have

$$\Phi_\alpha\left(A^1_\alpha, \ldots, A^n_\alpha\right)_- = \min_{\substack{W^i_{\alpha-} \le w_i \le W^i_{\alpha+} \\ A^i_{\alpha-} \le a_i \le A^i_{\alpha+}}} \sum_{i=1}^n w_i a_{\sigma(i)} \Big/ \sum_{i=1}^n w_i, \tag{6}$$

while for the right endpoints, we have

$$\Phi_\alpha\left(A_\alpha^1, \ldots, A_\alpha^n\right)_+ = \max_{\substack{lW_{\alpha-}^i \leq w_i \leq W_{\alpha+}^i \\ A_{\alpha-}^i \leq a_i \leq A_{\alpha+}^i}} \sum_{i=1}^n w_i a_{\sigma(i)} \Big/ \sum_{i=1}^n w_i. \tag{7}$$

It can be seen that (6) and (7) are programming problems. In the next section, we will address how to solve these problems so that the type-1 OWA aggregation operation can be performed efficiently.

3 Fast Implementation of Type-1 OWA Operation

The objective of type-1 OWA operators is to aggregate uncertain information modeled as fuzzy sets. In this section, we propose a fast algorithm for type-1 OWA operations, which can be used in real-time applications. The idea behind this algorithm hails from the above α-level type-1 OWA aggregations. For the type-1 OWA operations, we only need to calculate all the necessary α-level aggregations in (6) and (7), then based on the Representation Theorem of fuzzy set, the final aggregation result can be constructed as shown in (4). This fast algorithm is called the *Alpha-Level Approach* in this paper.

First in the following lemma, we list some basic inequalities as described in some textbooks that will be used later in the paper.

Lemma 1. *1) For $a \geq 0,\ c \geq 0$, if $\frac{b}{a} \geq \frac{d}{c}$, then*

$$\frac{b}{a} \geq \frac{b+d}{a+c} \geq \frac{d}{c}.$$

2) If $a \geq c$, $\frac{b}{a} \geq \frac{d}{c}$, then

$$\frac{b-d}{a-c} \geq \frac{b}{a}.$$

3) If $a \geq c$, $\frac{b}{a} \leq \frac{d}{c}$, then

$$\frac{b-d}{a-c} \leq \frac{b}{a}.$$

Note that for the left endpoints in (6), the function

$$f(w_i, a_i) = \sum_{i=1}^n w_i a_{\sigma(i)} \Big/ \sum_{i=1}^n w_i, \tag{8}$$

is a monotonically nondecreasing function of a_i. So,

$$\begin{aligned} \Phi_\alpha\left(A_\alpha^1, \ldots, A_\alpha^n\right)_- &= \min_{W_{\alpha-}^i \leq w_i \leq W_{\alpha+}^i} \sum_{i=1}^n w_i A_{\alpha-}^{\sigma(i)} \Big/ \sum_{i=1}^n w_i, \\ &= \min_{W_{\alpha-}^i \leq w_i \leq W_{\alpha+}^i} h(w_1, \ldots, w_n), \end{aligned} \tag{9}$$

where $A_{\alpha-}^{\sigma(1)} \geq \cdots \geq A_{\alpha-}^{\sigma(n)}$, and

$$h(w_1, \ldots, w_n) = \frac{\sum_{i=1}^n w_i A_{\alpha-}^{\sigma(i)}}{\sum_{i=1}^n w_i}. \tag{10}$$

Now we construct a new function of endpoints of intervals W_α^i as follows:

$$\rho_{\alpha-}^{i_0} \triangleq \frac{\sum_{i=1}^{i_0-1} W_{\alpha-}^i A_{\alpha-}^{\sigma(i)} + \sum_{i=i_0}^n W_{\alpha+}^i A_{\alpha-}^{\sigma(i)}}{J_{i_0}}, \tag{11}$$

where

$$J_{i_0} \triangleq \sum_{i=1}^{i_0-1} W_{\alpha-}^i + \sum_{i=i_0}^n W_{\alpha+}^i. \tag{12}$$

In particular, we have

$$\rho_{\alpha-}^1 \triangleq \frac{\sum_{i=1}^n W_{\alpha+}^i A_{\alpha-}^{\sigma(i)}}{J_1}, \tag{13}$$

where

$$J_1 \triangleq \sum_{i=1}^n W_{\alpha+}^i. \tag{14}$$

Then, we have the following theorem:

Theorem 3. *1) If $\rho_{\alpha-}^{i_0} \geq A_{\alpha-}^{\sigma(i_0)}$, then*

$$\rho_{\alpha-}^{i_0+1} \geq \rho_{\alpha-}^{i_0} \geq A_{\alpha-}^{\sigma(i_0)}.$$

2) If $\rho_{\alpha-}^{i_0} \leq A_{\alpha-}^{\sigma(i_0)}$, then

$$A_{\alpha-}^{\sigma(i_0)} \geq \rho_{\alpha-}^{i_0} \geq \rho_{\alpha-}^{i_0+1}.$$

Proof. Denoting

$$E = \sum_{i=1}^{i_0-1} W_{\alpha-}^i A_{\alpha-}^{\sigma(i)},$$

and

$$F = \sum_{i=i_0}^n W_{\alpha+}^i A_{\alpha-}^{\sigma(i)},$$

then,

$$\rho_{\alpha-}^{i_0} = \frac{E+F}{J_{i_0}}$$

and

$$\begin{aligned} \rho_{\alpha-}^{i_0+1} &= \frac{E + W_{\alpha-}^{i_0} A_{\alpha-}^{\sigma(i_0)} + F - W_{\alpha+}^{i_0} A_{\alpha-}^{\sigma(i_0)}}{J_{i_0} + \left(W_{\alpha-}^{i_0} - W_{\alpha+}^{i_0}\right)}, \\ &= \frac{E + F - \left(W_{\alpha+}^{i_0} - W_{\alpha-}^{i_0}\right) A_{\alpha-}^{\sigma(i_0)}}{J_{i_0} - \left(W_{\alpha+}^{i_0} - W_{\alpha-}^{i_0}\right)}. \end{aligned}$$

Because

$$J_{i_0} \geq W_{\alpha+}^{i_0} \geq W_{\alpha+}^{i_0} - W_{\alpha-}^{i_0}$$

then according to statements 2 and 3 in Lemma 1, results 1 and 2 can be derived. □

The solution to problem (9), and thus (6) is given in the following theorem:

Theorem 4. *Let i_0^* be the minimum number in $\{1, \ldots, n\}$ satisfying $\rho_{\alpha-}^{i_0^*} \geq A_{\alpha-}^{\sigma(i_0^*)}$, then $\rho_{\alpha-}^{i_0^*}$ is the minimum of (9).*

Proof. Starting with $i_0 = 1$, we check the relation between $\rho_{\alpha-}^{i_0}$ and $A_{\alpha-}^{\sigma(i_0)}$ until the first pair $\{\rho_{\alpha-}^{i_0^*}, A_{\alpha-}^{\sigma(i_0^*)}\}$ satisfying $\rho_{\alpha-}^{i_0^*} \geq A_{\alpha-}^{\sigma(i_0^*)}$ is found. This search process is guaranteed to produce such a first pair because

$$\rho^n_{\alpha-} = \frac{\sum_{i=1}^{n-1} W^i_{\alpha-} A^{\sigma(i)}_{\alpha-} + W^n_{\alpha+} A^{\sigma(n)}_{\alpha-}}{J_{i_0}} \geq A^{\sigma(n)}_{\alpha-}.$$

Next, we prove that $\rho^{i_0^*}_{\alpha-}$ is the minimum of (9).

According to the above search process, for any $j \in \{1, \ldots, i_0^* - 1\}$ we have that $\rho^j_{\alpha-} \leq A^{\sigma(j)}_{\alpha-}$. Theorem 3 implies that

$$\rho^{i_0^*}_{\alpha-} \leq \rho^{i_0^*-1}_{\alpha-} \leq \cdots \leq \rho^2_{\alpha-} \leq \rho^1_{\alpha-}.$$

On the other hand, the application of Theorem 3 to $\rho^{i_0^*}_{\alpha-} \geq A^{\sigma(i_0^*)}_{\alpha-}$ leads to

$$\rho^{i_0^*+1}_{\alpha-} \geq \rho^{i_0^*}_{\alpha-} \geq A^{\sigma(i_0^*)}_{\alpha-}.$$

Because $A^{\sigma(i_0^*)}_{\alpha-} \geq A^{\sigma(i_0^*+1)}_{\alpha-}$ then we have that $\rho^{i_0^*+1}_{\alpha-} \geq A^{\sigma(i_0^*+1)}_{\alpha-}$, and therefore

$$\rho^{i_0^*+2}_{\alpha-} \geq \rho^{i_0^*+1}_{\alpha-} \geq A^{\sigma(i_0^*+1)}_{\alpha-}.$$

Following a similar reasoning, we get

$$\vdots$$
$$\rho^n_{\alpha-} \geq \rho^{n-1}_{\alpha-} \geq A^{\sigma(n-1)}_{\alpha-}.$$

So,

$$\rho^n_{\alpha-} \geq \cdots \geq \rho^{i_0^*+1}_{\alpha-} \geq \rho^{i_0^*}_{\alpha-}$$

and therefore $\rho^{i_0^*}_{\alpha-}$ is the minimum of $\{\rho^1_{\alpha-}, \ldots, \rho^n_{-}\}$. In the following, we prove the minimum of $h(w_1, \ldots, w_n)$ is in the form of $\rho^{i_0}_{\alpha-}$.

Because

$$\frac{\partial h(w_1, \ldots, w_n)}{\partial w_i} = \frac{A^{\sigma(i)}_{\alpha-}\left(\sum_{i=1}^n w_i\right) - \sum_{i=1}^n w_i A^{\sigma(i)}_{\alpha-}}{\left(\sum_{i=1}^n w_i\right)^2} = \frac{A^{\sigma(i)}_{\alpha-} - h(w_1, \ldots, w_n)}{\sum_{i=1}^n w_i} \tag{15}$$

so, if $A^{\sigma(i)}_{\alpha-} \geq h(w_1, \ldots, w_n)$, then $\frac{\partial h(w_1, \ldots, w_n)}{\partial w_i} \geq 0$, i.e., if $A^{\sigma(i)}_{\alpha-} \geq h(w_1, \ldots, w_n)$, then $h(w_1, \ldots, w_n)$ is monotonically nondecreasing on each one of its arguments w_i. As a result, $A^{\sigma(i)}_{\alpha-} \geq h(w_1, \ldots, w_n)$ leads to minimizing $h(w_1, \ldots, w_n)$ at $W^i_{\alpha-}$ in the direction of w_i, i.e.,

$$h\left(w_1, \ldots, w_{i-1}, W^i_{\alpha-}, w_{i+1}, \ldots, w_n\right) \leq h(w_1, \ldots, w_n).$$

Similarly, $A^{\sigma(i)}_{\alpha-} \leq h(w_1, \ldots, w_n)$ leads to minimizing $h(w_1, \ldots, w_n)$ at $W^i_{\alpha+}$ in the direction of w_i.

Assume that $A^{\sigma(i_0-1)}_{\alpha-} \geq h(w_1, \ldots, w_n) \geq A^{\sigma(i_0)}_{\alpha-}$. Because $A^{\sigma(1)}_{\alpha-} \geq \cdots \geq A^{\sigma(n)}_{\alpha-}$, then $h(w_1, \ldots, w_n)$ reaches its minimum at $w_1 = W^1_{\alpha-}, \ldots, w_{i_0-1} = W^{i_0-1}_{\alpha-}, w_{i_0} = W^{i_0}_{\alpha+}, \ldots, w_n = W^n_{\alpha+}$, that is to say, the minimum of $h(w_1, \ldots, w_n)$ can be expressed in the form of $\rho^{i_0}_{\alpha-}$. Hence, $\rho^{i_0^*}_{\alpha-}$ is the solution of (9). □

For the right endpoints, the monotonicity of function (8) implies that

$$\Phi_\alpha\left(A^1_\alpha, \ldots, A^n_\alpha\right)_+ = \max_{W^i_{\alpha-} \leq w_i \leq W^i_{\alpha+}} \sum_{i=1}^n w_i A^{\sigma(i)}_{\alpha+} \Big/ \sum_{i=1}^n w_i, = \max_{W^i_{\alpha-} \leq w_i \leq W^i_{\alpha+}} g(w_1, \ldots, w_n), \tag{16}$$

where $A^{\sigma(1)}_{\alpha+} \geq \cdots \geq A^{\sigma(n)}_{\alpha+}$, and

$$g(w_1, \ldots, w_n) = \frac{\sum_{i=1}^n w_i A^{\sigma(i)}_{\alpha+}}{\sum_{i=1}^n w_i}. \tag{17}$$

In order to find the solution of (7) and (16), we construct a new function of endpoints of intervals W^i_α as follows:

$$\rho^{i_0}_{\alpha+} \triangleq \frac{\sum_{i=1}^{i_0-1} W^i_{\alpha+} A^{\sigma(i)}_{\alpha+} + \sum_{i=i_0}^n W^i_{\alpha-} A^{\sigma(i)}_{\alpha+}}{H_{i_0}}, \tag{18}$$

where

$$H_{i_0} \triangleq \sum_{i=1}^{i_0-1} W^i_{\alpha+} + \sum_{i=i_0}^n W^i_{\alpha-} \tag{19}$$

in particular,

$$\rho^1_{\alpha+} \triangleq \frac{\sum_{i=1}^n W^i_{\alpha-} A^{\sigma(i)}_{\alpha+}}{H_1}, \tag{20}$$

where

$$H_1 \triangleq \sum_{i=1}^n W^i_{\alpha-}. \tag{21}$$

Then, we have the following theorem:

Theorem 5. *1) If $\rho^{i_0}_{\alpha+} \geq A^{\sigma(i_0)}_{\alpha+}$, then*

$$\rho^{i_0}_{\alpha+} \geq \rho^{i_0+1}_{\alpha+} \geq A^{\sigma(i_0)}_{\alpha+}.$$

2) If $\rho^{i_0}_{\alpha+} \leq A^{\sigma(i_0)}_{\alpha+}$, then

$$A^{\sigma(i_0)}_{\alpha+} \geq \rho^{i_0+1}_{\alpha+} \geq \rho^{i_0}_{\alpha+}.$$

Proof. Let

$$C = \sum_{i=1}^{i_0-1} W^i_{\alpha+} A^{\sigma(i)}_{\alpha+},$$

and

$$D = \sum_{i=i_0}^n W^i_{\alpha-} A^{\sigma(i)}_{\alpha+},$$

then

$$\rho^{i_0}_{\alpha+} = \frac{C + D}{H_{i_0}}$$

and

$$\rho^{i_0+1}_{\alpha+} = \frac{C + W^{i_0}_{\alpha+} A^{\sigma(i_0)}_{\alpha+} + D - W^{i_0}_{\alpha-} A^{\sigma(i_0)}_{\alpha+}}{H_{i_0} + \left(W^{i_0}_{\alpha+} - W^{i_0}_{\alpha-}\right)}, = \frac{C + D + \left(W^{i_0}_{\alpha+} - W^{i_0}_{\alpha-}\right) A^{\sigma(i_0)}_{\alpha+}}{H_{i_0} + \left(W^{i_0}_{\alpha+} - W^{i_0}_{\alpha-}\right)}.$$

Step 1. To set up the α- level resolution in [0, 1].
Step 2. For each $\alpha \in [0,1]$,
Step 2.1. **To calculate** $\rho_{\alpha+}^{i_0^*}$
1) Let $i_0 = 1$;
2) If $\rho_{\alpha+}^{i_0} \geq A_{\alpha+}^{\sigma(i_0)}$, stop, $\rho_{\alpha+}^{i_0}$ is the solution; otherwise go to ***Step 2.1-3***.
3) $i_0 \leftarrow i_0 + 1$, go to ***Step 2.1-2***.

Step 2.2. **To calculate** $\rho_{\alpha-}^{i_0^*}$
1) Let $i_0 = 1$;
2) If $\rho_{\alpha-}^{i_0} \geq A_{\alpha-}^{\sigma(i_0)}$, stop, $\rho_{\alpha-}^{i_0}$ is the solution; otherwise go to ***Step 2.2-3.***
3) $i_0 \leftarrow i_0 + 1$, go to step ***Step 2.2-2.***

Step 3. To construct the aggregation resulting fuzzy set G based on all the available intervals $\left[\rho_{\alpha-}^{i_0^*},\ \rho_{\alpha+}^{i_0^*}\right]$:

$$\mu_G(x) = \bigvee_{\alpha : x \in \left[\rho_{\alpha-}^{i_0^*},\ \rho_{\alpha+}^{i_0^*}\right]} \alpha$$

Fig. 1. Procedure of the *Alpha-Level Approach* to type-1 OWA operation.

Because $H_{i_0} \geq 0$, then according to the statement 1 in Lemma 1, results 1 and 2 can be derived. □

The solution to problems (7) and (16) is given in the following theorem:

Theorem 6. *Let i_0^* be the minimum number in $\{1, \ldots, n\}$ satisfying $\rho_{\alpha+}^{i_0^*} \geq A_{\alpha+}^{\sigma(i_0^*)}$, then $\rho_{\alpha+}^{i_0^*}$ is the maximum of (17), and thus the solution of (7).*

Proof. Starting with $i_0 = 1$ we check the relation between $\rho_{\alpha+}^{i_0}$ and $A_{\alpha+}^{\sigma(i_0)}$ until the first pair $\{\rho_{\alpha+}^{i_0^*},\ A_{\alpha+}^{\sigma(i_0^*)}\}$ satisfying $\rho_{\alpha+}^{i_0^*} \geq A_{\alpha+}^{\sigma(i_0^*)}$ is found. This search process is guarantee to produce such a first pair because

$$\rho_{\alpha+}^{n} = \frac{\sum_{i=1}^{n-1} W_{\alpha+}^{i} A_{\alpha+}^{\sigma(i)} + W_{\alpha-}^{n} A_{\alpha+}^{\sigma(n)}}{H_{i_0}} \geq A_{\alpha+}^{\sigma(n)}.$$

Next, we prove $\rho_{+}^{i_0^*}$ is the maximum of (17).

According to the above search process, for any $j \in \{1, \ldots, i_0^* - 1\}$, we have that $\rho_{\alpha+}^{j} \leq A_{\alpha+}^{\sigma(j)}$. Theorem 5 implies

$$\rho_{\alpha+}^{j} \leq \rho_{\alpha+}^{j+1} \leq A_{\alpha+}^{\sigma(j)}.$$

So,

$$\rho_{\alpha+}^{1} \leq \rho_{\alpha+}^{2} \leq \cdots \leq \rho_{\alpha+}^{i_0^*}.$$

On the other hand, the application of Theorem 5 to $\rho_{\alpha+}^{i_0^*} \geq A_{\alpha+}^{\sigma(i_0^*)}$ leads to

$$\rho_{\alpha+}^{i_0^*} \geq \rho_{\alpha+}^{i_0^*+1} \geq A_{\alpha+}^{\sigma(i_0^*)}.$$

Because $A_{\alpha+}^{\sigma(i_0^*)} \geq A_{\alpha+}^{\sigma(i_0^*+1)}$, then we have that $\rho_{\alpha+}^{i_0^*+1} \geq A_{\alpha+}^{\sigma(i_0^*+1)}$, and therefore

$$\rho_{\alpha+}^{i_0^*+1} \geq \rho_{\alpha+}^{i_0^*+2} \geq A_{\alpha+}^{\sigma(i_0^*+1)}.$$

Following a similar reasoning, we get

$$\vdots$$
$$\rho_{\alpha+}^{n-1} \geq \rho_{\alpha+}^{n} \geq A_{\alpha+}^{\sigma(n)}.$$

So,

$$\rho_{\alpha+}^{i_0^*} \geq \rho_{\alpha+}^{i_0^*+1} \geq \cdots \geq \rho_{+}^{n}$$

and therefore $\rho_{\alpha+}^{i_0^*}$ is the maximum of $\{\rho_{\alpha+}^{1}, \ldots, \rho_{\alpha+}^{n}\}$. In the following, we prove the maximum of $g(w_1, \ldots, w_n)$ is in the form of (18).

An analysis of function $g(w_1, \ldots, w_n)$ similar to the one applied to function $h(w_1, \ldots, w_n)$ in Theorem 3 produces the following: 1) If $A_{\alpha+}^{\sigma(i)} \geq g(w_1, \ldots, w_n)$ then function $g(w_1, \ldots, w_n)$ is monotonically nondecreasing on each of its arguments w_i and the maximum of $g(w_1, \ldots, w_n)$ in the direction of w_i is achieved at $W_{\alpha+}^{i}$

$$g\big(w_1, \ldots, w_{i-1}, W_{\alpha+}^{i}, w_{i+1}, \ldots, w_n\big) \geq g(w_1, \ldots, w_n).$$

2) If $A_{\alpha+}^{\sigma(i)} \leq g(w_1, \ldots, w_n)$ then function $g(w_1, \ldots, w_n)$ is monotonically nonincreasing on each of its arguments w_i and the maximum of $g(w_1, \ldots, w_n)$ in the direction of w_i is achieved at $W_{\alpha-}^{i}$

$$g\big(w_1, \ldots, w_{i-1}, W_{\alpha-}^{i}, w_{i+1}, \ldots, w_n\big) \geq g(w_1, \ldots, w_n).$$

Assume that $A_{\alpha+}^{\sigma(i_0-1)} \geq g(w_1, \ldots, w_n) \geq A_{\alpha+}^{\sigma(i_0)}$. Because $A_{\alpha+}^{\sigma(1)} \geq \cdots \geq A_{\alpha+}^{\sigma(n)}$, then $g(w_1, \ldots, w_n)$ reaches the maximum at $w_1 = W_{\alpha+}^{1}, \ldots, w_{i_0-1} = W_{\alpha+}^{i_0-1}, w_{i_0} = W_{\alpha-}^{i_0}, \ldots, w_n = W_{\alpha-}^{n}$, that is to say, this maximum can be expressed in the form of (18). Hence, $\rho_{\alpha+}^{i_0^*}$ is the maximum of $g(w_1, \ldots, w_n)$, i.e., the solution of (7) and (16). □

Theorems 4 and 6 and their proofs actually indicate the procedures for finding the values $\rho_{\alpha-}^{i_0^*}$ and $\rho_{\alpha+}^{i_0^*}$, respectively. Given n linguistic weights $\{W^i\}_{i=1}^{n}$, the procedure to aggregate $\{A^i\}_{i=1}^{n}$ by a type-1 OWA operator via the α-level aggregation scheme is given in Fig. 1, in which the α values are required to cover all the available membership grades $\{\mu_{W^i}(w_i)\}$ and $\{\mu_{A^i}(a_i)\}$.

Example 1. Assume the following numerical domains $U = \{0.0, 0.5, 1.0\}$ and $X = \{0.0, 1.0, 2.0\}$. Let the given linguistic weights $W = \begin{pmatrix} u_i \\ \mu_W(u_i) \end{pmatrix}_{u_i \in U}$ on U be

$$W^1 = \begin{pmatrix} 0.0 & 0.5 & 1.0 \\ 1.0 & 0.5 & 0.0 \end{pmatrix}; \quad W^2 = \begin{pmatrix} 0.0 & 0.5 & 1.0 \\ 0.0 & 1.0 & 0.0 \end{pmatrix};$$
$$W^3 = \begin{pmatrix} 0.0 & 0.5 & 1.0 \\ 0.0 & 0.5 & 1.0 \end{pmatrix};$$

and the aggregated objects on X be

$$A^1 = \begin{pmatrix} 0.0 & 1.0 & 2.0 \\ 0.0 & 0.5 & 1.0 \end{pmatrix}; \quad A^2 = \begin{pmatrix} 0.0 & 1.0 & 2.0 \\ 1.0 & 0.5 & 0.0 \end{pmatrix};$$
$$A^3 = \begin{pmatrix} 0.0 & 1.0 & 2.0 \\ 0.0 & 1.0 & 0.0 \end{pmatrix}.$$

To calculate the α-cuts of W^i and $A^i (i = 1, 2, 3)$, the following set of α values will be used: $\{0, 0.5, 1.0\}$. We use the type-1 OWA operator Φ_{W^1,W^2,W^3} to aggregate the sets A^1, A^2, A^3 according to the procedure in Fig. 1

$$G = \Phi_{W^1,W^2,W^3}(A^1, A^2, A^3).$$

So, we need to get the α-levels of G at $\alpha = 0, 0.5$ and 1.0, respectively.

Case I. $\alpha = 0.0$

Obviously, the α-levels of A^i and $W^i (i = 1, 2, 3)$ are

$$A^1_\alpha = A^2_\alpha = A^3_\alpha = \{0.0, 1.0, 2.0\}$$

and

$$W^1_\alpha = W^2_\alpha = W^3_\alpha = \{0.0, 0.5, 1.0\},$$

respectively. Thus, we have

$$\begin{aligned} A^1_{\alpha-} &= A^3_{\alpha-} = A^3_{\alpha-} = 0.0, \\ A^1_{\alpha+} &= A^2_{\alpha+} = A^3_{\alpha+} = 2.0, \\ W^1_{\alpha-} &= W^2_{\alpha-} = W^3_{\alpha-} = 0.0, \\ W^1_{\alpha+} &= W^2_{\alpha+} = W^3_{\alpha+} = 1.0. \end{aligned}$$

- Computation of $\rho^{i_0^*}_{\alpha-}$. Because $A^1_{\alpha-} = A^2_{\alpha-} = A^3_{\alpha-}$, the permutation operator is $\sigma = (1, 2, 3)$. Then,

  1. $i_0 = 1$. According to the (13), we have

$$\begin{aligned} \rho^{i_0}_{\alpha-} &= \frac{W^1_{\alpha+} A^{\sigma(1)}_{\alpha-} + W^2_{\alpha+} A^{\sigma(2)}_{\alpha-} + W^3_{\alpha+} A^{\sigma(3)}_{\alpha-}}{W^1_{\alpha+} + W^2_{\alpha+} + W^3_{\alpha+}} \\ &= 0.0 \\ &\geq A^{\sigma(i_0)}_{\alpha-} \\ &= A^1_{\alpha-}. \end{aligned}$$

  So, we get $\rho^{i_0^*}_{\alpha-} = 0.0$.
- Computation of $\rho^{i_0^*}_{\alpha+}$. Because $A^1_{\alpha+} = A^2_{\alpha+} = A^3_{\alpha+}$, the permutation operator is $\sigma = (1, 2, 3)$. Then,

  1. $i_0 = 1$. According to (20), we have

$$\begin{aligned} \rho^{i_0}_{\alpha+} &= \frac{W^1_{\alpha-} A^{\sigma(1)}_{\alpha+} + W^2_{\alpha-} A^{\sigma(2)}_{\alpha+} + W^3_{\alpha-} A^{\sigma(3)}_{\alpha+}}{W^1_{\alpha-} + W^2_{\alpha-} + W^3_{\alpha-}} \\ &= 0.0 \\ &< A^{\sigma(i_0)}_{\alpha+} \\ &= A^1_{\alpha+}. \end{aligned}$$

  So, we should continue this procedure by letting $i_0 = 2$.

  2. $i_0 = 2$. According to (18), we have

$$\begin{aligned} \rho^{i_0}_{\alpha+} &= \frac{W^1_{\alpha+} A^{\sigma(1)}_{\alpha+} + W^2_{\alpha-} A^{\sigma(2)}_{\alpha+} + W^3_{\alpha-} A^{\sigma(3)}_{\alpha+}}{W^1_{\alpha+} + W^2_{\alpha-} + W^3_{\alpha-}} \\ &= \frac{1.0 \times 2.0 + 0.0 \times 2.0 + 0.0 \times 2.0}{1.0 + 0.0 + 0.0} \\ &= 2.0 \\ &\geq A^{\sigma(i_0)}_{\alpha+} \\ &= A^2_{\alpha+}. \end{aligned}$$

  So, we get $\rho^{i_0^*}_{\alpha+} = 2.0$. As a result, $G_\alpha = [0.0, 2.0] \cap X = \{0.0, 1.0, 2.0\}$.

Case II. $\alpha = 0.5$

The α-levels of A^i and $W^i (i = 1, 2, 3)$ are

$$A^1_\alpha = \{1.0, 2.0\}, A^2_\alpha = \{0.0, 1.0\}, A^3_\alpha = \{1.0\}$$

and

$$W^1_\alpha = \{0.0, 0.5\}, W^2_\alpha = \{0.5\}, W^3_\alpha = \{0.5, 1.0\},$$

respectively. Thus, we have

$$\begin{aligned} A^1_{\alpha-} &= 1.0, A^1_{\alpha+} = 2.0; \\ A^2_{\alpha-} &= 0.0, A^2_{\alpha+} = 1.0; \\ A^3_{\alpha-} &= 1.0, A^3_{\alpha+} = 1.0; \end{aligned}$$

and

$$\begin{aligned} W^1_{\alpha-} &= 0.0, W^1_{\alpha+} = 0.5; \\ W^2_{\alpha-} &= 0.5, W^2_{\alpha+} = 0.5; \\ W^3_{\alpha-} &= 0.5, W^3_{\alpha+} = 1.0. \end{aligned}$$

- Computation of $\rho^{i_0^*}_{\alpha-}$. Because $A^1_{\alpha-} \geq A^3_{\alpha-} \geq A^2_{\alpha-}$, the permutation operator is $\sigma = (1, 3, 2)$. Then,

  1. $i_0 = 1$. According to (13), we have

$$\begin{aligned} \rho^{i_0}_{\alpha-} &= \frac{W^1_{\alpha+} A^{\sigma(1)}_{\alpha-} + W^2_{\alpha+} A^{\sigma(2)}_{\alpha-} + W^3_{\alpha+} A^{\sigma(3)}_{\alpha-}}{W^1_{\alpha+} + W^2_{\alpha+} + W^3_{\alpha+}} \\ &= \frac{0.5 \times 1.0 + 0.5 \times 1.0 + 1.0 \times 0.0}{0.5 + 0.5 + 1.0} \\ &= 0.5 \\ &< A^{\sigma(i_0)}_{\alpha-} \\ &= A^1_{\alpha-}. \end{aligned}$$

  So, we should continue this procedure by letting $i_0 = 2$.

  2. $i_0 = 2$. According to (11), we have

$$\begin{aligned}\rho^{i_0}_{\alpha-} &= \frac{W^1_{\alpha-}A^{\sigma(1)}_{\alpha-} + W^2_{\alpha+}A^{\sigma(2)}_{\alpha-} + W^3_{\alpha+}A^{\sigma(3)}_{\alpha-}}{W^1_{\alpha-} + W^2_{\alpha+} + W^3_{\alpha+}}\\ &= \frac{0.0\times 1.0 + 0.5\times 1.0 + 1.0\times 0.0}{0.0+0.5+1.0}\\ &= \frac{1}{3}\\ &< A^{\sigma(i_0)}_{\alpha-}\\ &= A^3_{\alpha-}.\end{aligned}$$

So, we should continue this procedure by letting $i_0 = 3$.

3. $i_0 = 3$. According to (11), we have

$$\begin{aligned}\rho^{i_0}_{\alpha-} &= \frac{W^1_{\alpha-}A^{\sigma(1)}_{\alpha-} + W^2_{\alpha-}A^{\sigma(2)}_{\alpha-} + W^3_{\alpha+}A^{\sigma(3)}_{\alpha-}}{W^1_{\alpha-} + W^2_{\alpha-} + W^3_{\alpha+}}\\ &= \frac{0.0\times 1.0 + 0.5\times 1.0 + 1.0\times 0.0}{0.0+0.5+1.0}\\ &= \frac{1}{3}\\ &> A^{\sigma(i_0)}_{\alpha-}\\ &= A^2_{\alpha-}.\end{aligned}$$

So, we get $\rho^{i_0^*}_{\alpha-} = \frac{1}{3}$.

- Computation of $\rho^{i_0^*}_{\alpha+}$. Because $A^1_{\alpha+} > A^2_{\alpha+} \geq A^3_{\alpha+}$, the permutation operator is $\sigma = (1,2,3)$. Then,

1. $i_0 = 1$. According to (20), we have

$$\begin{aligned}\rho^{i_0}_{\alpha+} &= \frac{W^1_{\alpha-}A^{\sigma(1)}_{\alpha+} + W^2_{\alpha-}A^{\sigma(2)}_{\alpha+} + W^3_{\alpha-}A^{\sigma(3)}_{\alpha+}}{W^1_{\alpha-} + W^2_{\alpha-} + W^3_{\alpha-}}\\ &= \frac{0.0\times 2.0 + 0.5\times 1.0 + 0.5\times 1.0}{0.0+0.5+0.5}\\ &= 1.0\\ &< A^{\sigma(i_0)}_{\alpha+}\\ &= A^1_{\alpha+}.\end{aligned}$$

So, we should continue this procedure by letting $i_0 = 2$.

2. $i_0 = 2$. According to (18), we have

$$\begin{aligned}\rho^{i_0}_{\alpha+} &= \frac{W^1_{\alpha+}A^{\sigma(1)}_{\alpha+} + W^2_{\alpha-}A^{\sigma(2)}_{\alpha+} + W^3_{\alpha-}A^{\sigma(3)}_{\alpha+}}{W^1_{\alpha+} + W^2_{\alpha-} + W^3_{\alpha-}}\\ &= \frac{0.5\times 2.0 + 0.5\times 1.0 + 0.5\times 1.0}{0.5+0.5+0.5}\\ &= \frac{4}{3}\\ &\geq A^{\sigma(i_0)}_{\alpha+}\\ &= A^2_{\alpha+}.\end{aligned}$$

So, we get $\rho^{i_0^*}_{\alpha+} = \frac{4}{3}$. As a result, $G_\alpha = [\frac{1}{3}, \frac{4}{3}] \cap X = \{1.0\}$.

Case III. $\alpha = 1.0$

The α-levels of A^i and $W^i (i = 1,2,3)$ are

$$A^1_\alpha = \{2.0\}, A^2_\alpha = \{0.0\}, A^3_\alpha = \{1.0\}$$

and

$$W^1_\alpha = \{0.0\}, W^2_\alpha = \{0.5\}, W^3_\alpha = \{1.0\},$$

respectively. Thus, we have

$$\begin{aligned}A^1_{\alpha-} &= A^1_{\alpha+} = 2.0;\\ A^2_{\alpha-} &= A^2_{\alpha+} = 0.0;\\ A^3_{\alpha-} &= A^3_{\alpha+} = 1.0;\end{aligned}$$

and

$$\begin{aligned}W^1_{\alpha-} &= W^1_{\alpha+} = 0.0;\\ W^2_{\alpha-} &= W^2_{\alpha+} = 0.5;\\ W^3_{\alpha-} &= W^3_{\alpha+} = 1.0.\end{aligned}$$

Following a similar computation process as in the two previous cases, we get $\rho^{i_0^*}_{\alpha-} = \rho^{i_0^*}_{\alpha+} = \frac{1}{3}$. As a result, $G_\alpha = \{\frac{1}{3}\} \cap X = \emptyset$.

Now we proceed to compute the membership grades of G according to the (5)

$$\begin{aligned}\mu_G(0) &= \bigvee_{\alpha:0.0\in G_\alpha} \alpha = 0.0,\\ \mu_G(1.0) &= \bigvee_{\alpha:1.0\in G_\alpha} \alpha = 0.0 \vee 0.5 = 0.5,\\ \mu_G(2.0) &= \bigvee_{\alpha:2.0\in G_\alpha} \alpha = 0.0.\end{aligned}$$

Hence, the result of aggregating the fuzzy sets A^1, A^2, A^3 by the type-1 OWA operator Φ_{W^1,W^2,W^3} is

$$G = \begin{pmatrix} 0.0 & 1.0 & 2.0\\ 0.0 & 0.5 & 0.0\end{pmatrix}.$$

4 Complexity Analyses of the *Direct Approach* and the Proposed *Alpha-Level Approach* to Type-1 OWA Operations

Given n fuzzy set $\{A^i\}_{i=1}^n$ to be aggregated by a type-1 OWA operator associated with n uncertain weights $\{W^i\}_{i=1}^n$, in this section, we analyze the complexity of the *Direct Approach* [27] and *Alpha-Level Approach* to type-1 OWA operations, which was not addressed yet in [27].

In the *Direct Approach*, assume the domain $U = [0, 1]$ be discretized with n_u points and the domain X with n_x points. For each combination of $w_1 \in U, \ldots, w_n \in U, a_1 \in X, \ldots, a_n \in X$, the type-1 OWA aggregation in the *Direct Approach* will involve $2(n-1)$ additions, n multiplications, 1 division, $2n-1$ t-norm operations and 1 maximum operation. Hence, the total operations for each combination of $w_1, \ldots, w_n, a_1, \ldots, a_n$ is

$$2(n-1) + n + 1 + 2n - 1 + 1 = 5n - 1. \tag{22}$$

Then, $(n_u)^n(n_x)^n$ combinations of $w_1, \ldots, w_n, a_1, \ldots, a_n$ lead to the number of operations involved in a *Direct Approach* to type-1 OWA operator to aggregate $\{A^i\}_{i=1}^n$ to be

$$(n_u n_x)^n(5n-1) = O(K^n), \tag{23}$$

where K is a constant. Hence, the complexity of the *Direct Approach* to type-1 OWA operation is in exponential order.

In the proposed *Alpha-Level Approach*, assume the number of α values in $[0,\ 1]$ be n_α, and the domain X be discretized with n_x points. For each α value, the operations in each round of the total i_0^* involved in the computation of each right endpoint $\rho_{\alpha+}^{i_0}$ of an α-cut include $2(n-1)$ additions, n multiplications, and 1 division. So, the total number of operations to compute the right endpoint $\rho_{\alpha+}^{i_0}$ is

$$i_0^*(2(n-1)+n+1) = i_0^*(3n-1). \tag{24}$$

Similarly, the total number of operations to compute the left endpoint $\rho_{\alpha-}^{i_0}$ is $i_0^{'}(3n-1)$. Therefore, the computation of each α-cut $[\rho_{\alpha-}^{i_0^{'}},\ \rho_{\alpha+}^{i_0^*}]$ involves $(i_0^*+i_0^{'})(3n-1)$ times of operations. Considering there exist $n_x(n_\alpha-1)$ operations to obtain the membership grades of the n_x points in X, the total number of operations involved in the *Alpha-Level Approach* is

$$n_\alpha(i_0^*+i_0^{'})(3n-1)+n_x(n_\alpha-1) = O(n). \tag{25}$$

That is to say, the complexity of the *Alpha-Level Approach* is in linear order. Hence, the *Alpha-Level Approach* achieves much higher computing efficiency than the *Direct Approach*.

5 Experimental Results

In this section, we first evaluate the computing efficiency of the proposed scheme in comparison with the *Direct Approach* [27], in which eight different kinds of type-1 OWA operators are designed to aggregate a group of fuzzy sets. Then, we provide a practical example for breast cancer treatment in which type-1 OWA operators are used. In these examples, the proposed type-1 OWA operators are compared with another widely investigated aggregation operator, the FWA operator [36], [37], [38].

5.1 Evaluation of Computing Efficiency and Comparisons with *Direct Approach*

As Yager's OWA operators do, type-1 OWA operators also depend on the choices of linguistic weights $\{W^i\}_{i=1}^n$. By choosing appropriate uncertain weights modeled as fuzzy sets, we can obtain a type-1 OWA operator with desired properties. In this section, eight different type-1 OWA operators are designed to aggregate the fuzzy sets shown in Fig. 2. These eight type-1 OWA operators are the meet operator, two meet-like operators, the join operator, two join-like operators, the mean operator, and a mean-like operator.

The meet and join operators of fuzzy sets were proposed by Zadeh [41] and named in [42]. Interestingly, as indicated in [27] and [28], the meet and join operations of fuzzy sets can be performed by type-1 OWA operators with singleton weights. For example, a type-1 OWA operator of dimension 3 becomes a meet operator if the following singleton weights are used: $W^i = \dot{0}\ (i \neq 3)$, $W^3 = \dot{1}$, i.e.,

$$\mu_{W^3}(w) = \begin{cases} 1, & w = 1, \\ 0, & others, \end{cases} \tag{26}$$

$$\mu_{W^i}(w) = \begin{cases} 1, & w = 0, \\ 0, & others, \end{cases} \quad (i \neq 3) \tag{27}$$

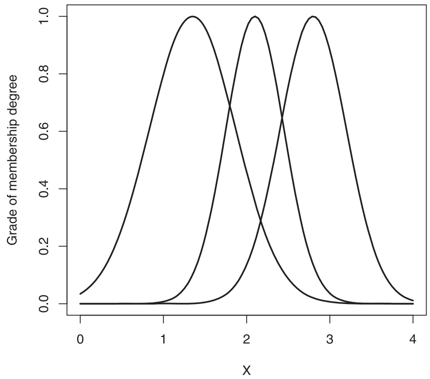


Fig. 2. Three aggregated fuzzy sets (from left to right): A^1, A^2, and A^3.

while the singleton weights $W^i = \dot{0}\ (i \neq 1), W^1 = \dot{1}$ make the type-1 OWA operator into a join operator. As a matter of fact, the meet of fuzzy sets yields the fuzzified minimum whereas the join of fuzzy sets yields the fuzzified maximum [27].

The traditional mean operator is a particular type of Yager's OWA operator with weights all equal to $1/n$. Therefore, the type-1 OWA operator with all weights in the form of singleton fuzzy sets $\dot{1/n}$

$$\mu_G(y) = \sup_{\substack{\frac{1}{n}\sum_{i=1}^{n} a_i = y \\ a_i \in X}} \mu_{A_1}(a_1) * \cdots * \mu_{A_n}(a_n) \tag{28}$$

can be seen as an extended mean operation on fuzzy sets [27], [28].

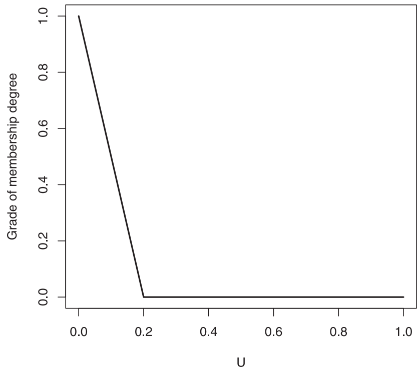
Meet-like type-1 OWA (MLT1OWA) operators [27], [28] can be obtained by selecting appropriate linguistic weights: the last linguistic weight is to approach to the singleton fuzzy set $\dot{1}$, and the rest of linguistic weights are to approach to the singleton fuzzy set $\dot{0}$ in turn. The MLT1OWA operator of dimension 3 with linguistic weights $W^1 = W^2 = L_0$, $W^3 = L_1$ depicted in Fig. 3 is denoted as MLT1OWA 1. Fig. 4 shows linguistic weights $\{W^i\}_{i=1}^3$ that guide another meet-like type-1 OWA operation, which is denoted as MLT1OWA2.

Join-like type-1 OWA (JLT1OWA) operators can also be obtained by selecting appropriate linguistic weights [27], [28]. Indeed, this is the case when the first linguistic weight is close to the singleton fuzzy set $\dot{1}$, and the rest are close to the singleton fuzzy set $\dot{0}$ in turn. One example of linguistic weights chosen for JLT1OWA operator is to set $W^1 = L_1$, $W^2 = W^3 = L_0$, in which the L_0 and L_1 are depicted in Fig. 3. This JLT1OWA is denoted as JLT1OWA1, whereas Fig. 5 illustrates another case of linguistic weights chosen for JLT1OWA operator, which is denoted as JLT1OWA2.

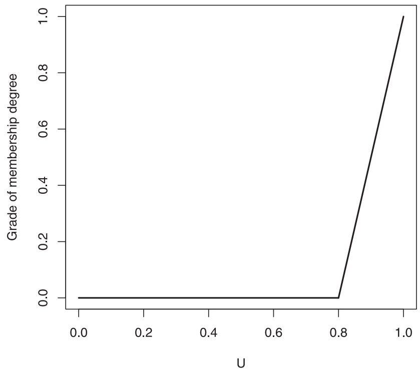
Mean-like type-1 OWA (MALT1OWA) operators can be obtained by selecting the linguistic weights appropriately. For example, Fig. 6 shows three linguistic weights in the forms of triangular fuzzy numbers whose cores locate at $1/3$ as follows,

$$\mu_{W^i}(u) = \max\{0,\ \min(3u,\ 2-3u)\}. \tag{29}$$

After choosing the above associated weights, respectively, we can use the proposed *Alpha-Level Approach* to implement these eight type-1 OWA operators for aggregating the fuzzy sets depicted in Fig. 2, and compare with the



(a)



(b)

Fig. 3. Linguistic weights. (a) L_0 and (b) L_1.

Direct Approach [27] in terms of computing efficiency, respectively. Table 1 shows the corresponding time costs of the proposed *Alpha-Level Approach* and the *Direct Approach* in completing these operations. It can be seen that the computing efficiency achieved by the *Alpha-Level Approach* is much higher than the one achieved by the *Direct Approach*.

5.2 Comparisons of the Type-1 OWA Operators with the FWA Operators

In this section, we further compare type-1 OWA operators using the proposed α-level approach with FWA operators [36], [37], [38] in aggregating fuzzy sets. In our experiments, the type-1 OWA operators and FWA operators use the same uncertain weights to aggregate the same groups of fuzzy sets, then we evaluate what different aggregation results can be achieved.

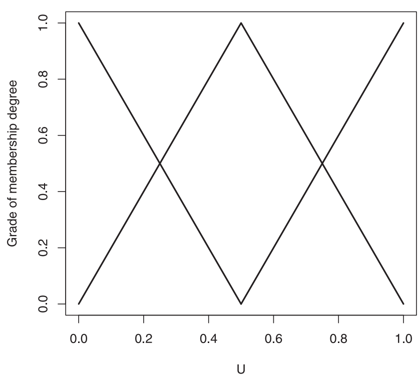


Fig. 4. Linguistic weights for MLT1OWA2 (from left to right): W^1, W^2, and W^3.

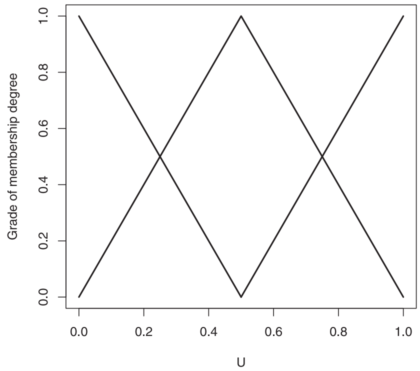


Fig. 5. Linguistic weights for JLT1OWA2 (from right to left): W^1, W^2, and W^3.

In the first example, a FWA operator with linguistic weights W^1, W^2, and W^3 being the fuzzy sets from right to left given in Fig. 5 is used to aggregate the three fuzzy sets depicted in Fig. 2. Fig. 7 illustrates the aggregation results obtained with the FWA and the corresponding type-1 OWA operator for the same set of weights, the JLT1OWA2 operator.

In the second example, Fig. 9 shows the corresponding aggregation results obtained using the FWA and type-1 OWA operator associated with the same linguistic weights depicted in Fig. 8b to aggregate the same group of fuzzy sets shown in Fig. 8a.

From the above examples, it can be seen that type-1 OWA operators and the FWA operators exhibit different aggregation behaviors, which resembles the different behaviors Yager's OWA operators and the weighted averaging operators have associated when data are crisp.

5.3 Type-1 OWA-Based Fuzzy Inferences for Breast Cancer Treatments

In this section, we further apply type-1 OWA operators to the aggregation of nonstationary fuzzy sets for diagnoses of breast cancer patients.

Nonstationary fuzzy sets [43], [44] have been proposed to model intraexpert variability and interexpert variability exhibited in multiexpert decision making, in which the membership function of a nonstationary fuzzy set may alter over time. As a result, given a problem, a nonstationary fuzzy system may generate different output fuzzy sets in different runs [45]. This means that some

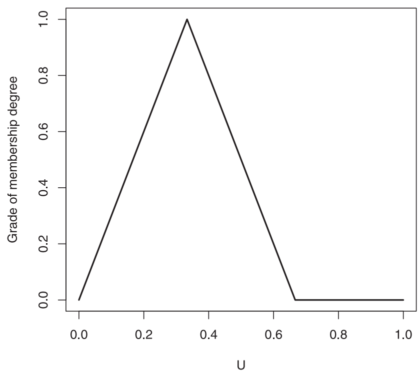


Fig. 6. Linguistic weights with cores locating at 1/3: W^i $(\mathrm{i} = 1, 2, 3)$.

TABLE 1
Comparison of Computing Efficiency of *Alpha-Level Approach* and *Direct Approach* to Type-1 OWA Operations

Type-1 OWA operators	Alpha-Level Approach	Direct Approach
Meet	0.13 seconds	200.81 seconds
MELT1OWA1	0.16 seconds	8313.72 seconds
MELT1OWA2	0.16 seconds	10824.67 seconds
Join	0.13 seconds	208.61 seconds
JLT1OWA1	0.14 seconds	7671.46 seconds
JLT1OWA2	0.14 seconds	11270.19 seconds
Mean	0.12 seconds	52.75 seconds
MALT1OWA	0.17 seconds	11552.68 seconds

additional components become necessary besides the commonly used in the standard fuzzy system: fuzzifier, rule base, rule engine, and defuzzifier. Among them, an important additional component is to aggregate these output sets into an overall one. In the following, we use the type-1 OWA operator as uncertain operator to aggregate the output sets, which leads to a type-1 OWA-based nonstationary fuzzy system (T1ONFS) as depicted in Fig. 10.

Generally speaking, the T1ONFS works as follows: In each run, crisp input values first feed into the system through the fuzzifier by which the fuzzification of these inputs is carried out in a singleton or nonsingleton way. The fuzzified nonstationary fuzzy sets then activate the inference engine and rule base to yield an output set by performing the union and intersection operations of fuzzy sets and compositions of relations. This process repeats n times. So n output sets are generated. Then, a type-1 OWA operator is applied to these output sets to generate an overall set. Finally, this overall fuzzy set is defuzzified to produce a crisp output.

In our study toward the design of a nonstationary fuzzy expert system for breast cancer treatments, 12 initial fuzzy rules are acquired [46] according to the professional clinical guidelines provided by Nottingham University Hospitals (NHS) Trust Breast Directorate, i.e., the fuzzy rule base is obtained from human experts' knowledge, which is different from the scheme of inducing fuzzy rules from a data set [52]. These guidelines include various treatment decisions based on many patients' assessment results. In our study, 1,310 breast cancer cases are considered. Each cancer case is to be diagnosed by the nonstationary fuzzy system that runs 10 times, then the diagnosis result is to be

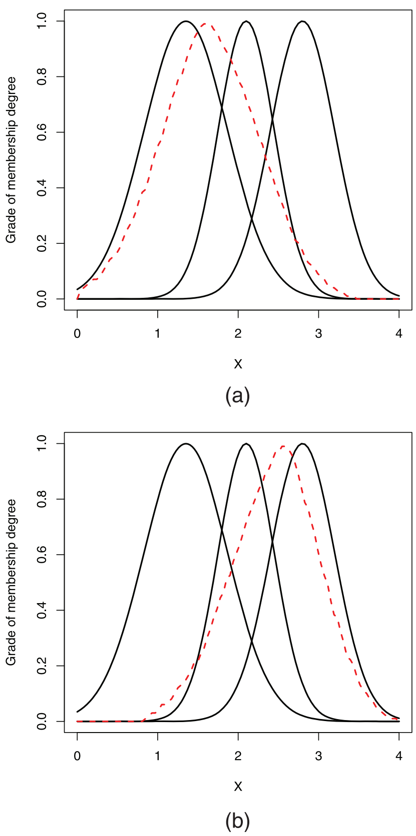


Fig. 7. Comparison of type-1 OWA operator with FWA operator: solid lines represent aggregated fuzzy sets, dashed line represents the aggregation results. (a) FWA aggregation result and (b) Type-1 OWA aggregation result.

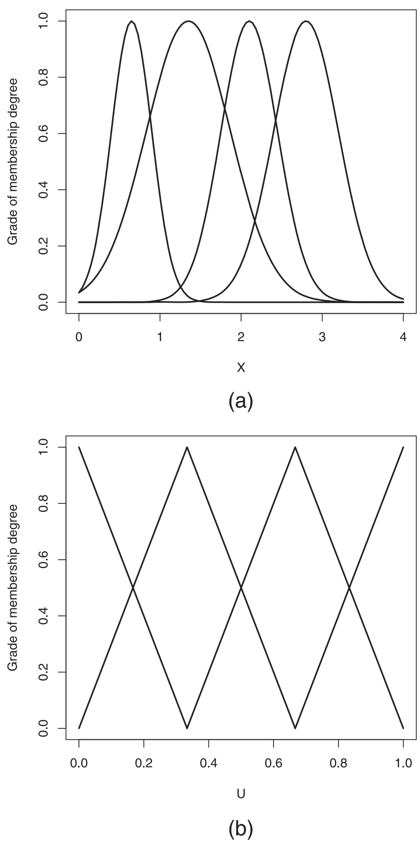


Fig. 8. (a) Four aggregated fuzzy sets (from left to right): A^1, A^2, A^3, and A^4; (b) Four linguistic weights (from left to right): W^1, W^2, W^3, and W^4.

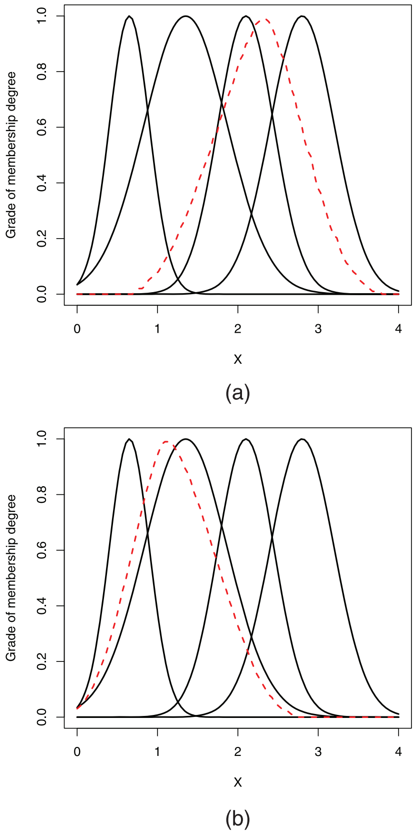


Fig. 9. Comparison of type-1 OWA operator with FWA operator: solid lines represent aggregated fuzzy sets, dashed line represents the aggregation results. (a) FWA aggregation result and (b) Type-1 OWA aggregation result.

compared with the doctor's recommendations. The system performance will be evaluated in terms of the rate of agreement with the doctor's judgments. Also, the proposed method will further compare with the FWA operator.

In this study, we use the meet-like type-1 OWA operator with $W^{10} = L_1$, $W^i = L_0$ $(i = 1, \ldots, 9)$, as depicted in Fig. 3, to aggregate the 10 output sets for breast cancer treatments. This meet-like type-1 OWA operator is denoted as MLT1OWA3. Tables 2 and 3 are the confusion matrices of the agreements of the different aggregation operators-based nonstationary fuzzy systems with doctor's judgments, in which the MLT1OWA3 and FWA-based nonstationary fuzzy systems are used to provide soft decision supports for breast cancer treatments, respectively. It can be seen that the nonstationary fuzzy system with type-1 OWA operator MLT1OWA3 can achieve better performance. However, like in the case of Yager's OWA operator [47], [48], [49], [50],

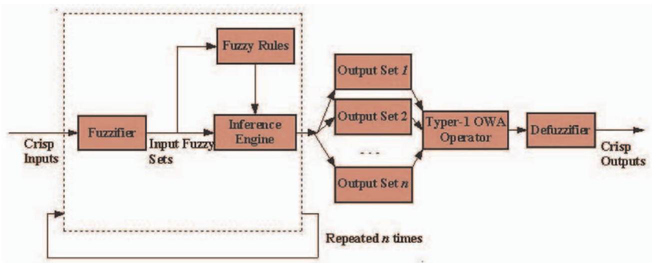


Fig. 10. Type-1 OWA-based nonstationary fuzzy system.

TABLE 2
Confusion Matrix Obtained by MLT1OWA3-Based Fuzzy Decision

Confusion Matrix		Clinician Decision		
		No	Maybe	Yes
Model Decision	No	79%	4.1%	14.6%
	Maybe	0.2%	0.0%	0.0%
	Yes	1.8%	0.0%	0.3%

[51], the identification of appropriate weights for type-1 operators is an important research topic.

All computations in these experiments were carried out using the *R*-software environment in version 2.4.0 [55]. The source codes of type-1 OWA operations in this paper are available upon request.

6 Discussion and Conclusions

This paper first defined the α-level type-1 OWA operator to aggregate the α-cuts of fuzzy sets. The *Representation Theorem* of type-1 OWA operators was proved. According to the *Representation Theorem*, type-1 OWA operators can be decomposed into its α-level type-1 OWA operators, which led to the proposal and development of a fast approach to implementing type-1 OWA operations. Promisingly, the complexity of the *Alpha-Level Approach* is in linear order, it can achieve much higher computing efficiency in performing type-1 OWA operation than the *Direct Approach*, and therefore it provides an efficient way of aggregating uncertain information via OWA mechanism in real-time applications.

It is known that in Yager's OWA aggregation, the identification of appropriate OWA weights is a very active research topic [47], [48], [49], [50], [51]. We have a similar issue in the case of the type-1 OWA operators, i.e., how to determine type-1 OWA weights to reflect the decision makers' desired agenda for aggregating the criteria/preferences. Type-2 linguistic quantifiers have been proposed for this purpose [27], although further schemes are worth investigating for different situations. Other interesting issues include the possibility of applying type-1 OWAs to the merging of similar fuzzy sets for improving fuzzy model interpretability/transparency and parsimony [52], [53], [54], as well as their applications to multiexpert decision making and multicriteria decision making.

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TABLE 3
Confusion Matrix Obtained by FWA-Based Fuzzy Decision

Confusion Matrix		Clinician Decision		
		No	Maybe	Yes
Model Decision	No	75%	3.8%	13.9%
	Maybe	1.6%	0.0%	0.2%
	Yes	4.5%	0.3%	0.8%

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