

# Problem Solutions

## e-Chapter 7

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### Problem 7.1

To solve this problem, we first begin by separating the positive decision region into two components : the lower one corresponding to  $x_2 \in [-1, 1]$  and the upper one corresponding to  $x_2 \in [1, 2]$ . To define the decision region, we need 7 perceptrons, namely

$$h_1(x) = \text{sign}(x_2 - 2), \quad h_2(x) = \text{sign}(x_2 - 1), \quad h_3(x) = \text{sign}(x_2 + 1),$$

for the horizontal lines, and

$$h_4(x) = \text{sign}(x_1 + 2), \quad h_5(x) = \text{sign}(x_1 + 1), \quad h_6(x) = \text{sign}(x_1 - 1), \quad h_7(x) = \text{sign}(x_1 - 2)$$

for the vertical lines. We are now able to define the lower decision region by  $\overline{h_2}h_3h_4\overline{h_7}$ , and the upper decision region by  $\overline{h_1}h_2h_5\overline{h_6}$ , which means that the total decision region is defined by

$$f = \overline{h_2}h_3h_4\overline{h_7} + \overline{h_1}h_2h_5\overline{h_6}$$

which actually characterizes a 3-layer perceptron.

### Problem 7.2

(a) Let  $x$  and  $x'$  be two points from the same region. If we consider a set of  $M$  hyperplanes defined by  $\{x : w_i^T x = 0\}$ , we have that

$$(\text{sign}(w_1^T x), \dots, \text{sign}(w_M^T x)) = (\text{sign}(w_1^T x'), \dots, \text{sign}(w_M^T x'));$$

or put more simply that  $\text{sign}(w_i^T x) = \text{sign}(w_i^T x') = s_i$  for  $i = 1, \dots, M$  where  $s_i = \pm 1$ . We begin by the case where  $s_i = 1$ . Here, we know that  $w_i^T x > 0$  and  $w_i^T x' > 0$ , consequently we have that, for  $\lambda \in [0, 1]$ ,

$$w_i^T (\lambda x + (1 - \lambda)x') = \lambda w_i^T x + (1 - \lambda)w_i^T x' > 0$$

and

$$\text{sign}(w_i^T (\lambda x + (1 - \lambda)x')) = 1.$$

Now, we consider the case where  $s_i = -1$ . Here, we know that  $w_i^T x < 0$  and  $w_i^T x' < 0$ , consequently we have that, for  $\lambda \in [0, 1]$ ,

$$w_i^T (\lambda x + (1 - \lambda)x') = \lambda w_i^T x + (1 - \lambda)w_i^T x' < 0$$

and

$$\text{sign}(w_i^T (\lambda x + (1 - \lambda)x')) = -1.$$

So, in conclusion, the region is actually convex.

(b) A region is defined as the following set

$$\{x : (\text{sign}(w_1^T x), \dots, \text{sign}(w_M^T x)) = (s_1, \dots, s_M); s_i \in \{-1, 1\}\};$$

thus a region is characterized by a particular  $M$ -uple  $(s_1, \dots, s_M)$ . Since there are at most  $2^M$  of such  $M$ -uples, we have at most  $2^M$  different regions.

(c) Let  $B(N, d)$  be the maximum number of regions created by  $M$  hyperplanes in  $d$ -dimensional space. Now, consider adding an  $(M + 1)$ th hyperplane; this hyperplane can obviously be viewed as a  $(d - 1)$ -dimensional

space, so if we project the initial  $M$  hyperplanes into this space, we obtain  $M$  hyperplanes in a  $(d-1)$ -dimensional space. These hyperplanes can create at most  $B(M, d-1)$  regions in this space, and for each of these regions, we get two regions in the original  $d$ -dimensional space. Thus, this means that the  $(M+1)$ th hyperplane intersects at most  $B(M, d-1)$  of the regions created by the  $M$  hyperplanes in the  $d$ -dimensional space, and so

$$B(M+1, d) \leq B(M, d) + B(M, d-1).$$

Now, we will prove that

$$B(M, d) \leq \sum_{i=0}^d \binom{M}{i}$$

by induction. We begin by evaluating the boundary conditions, we have

$$B(M, 1) = M + 1 \leq \sum_{i=0}^1 \binom{M}{i} = \binom{M}{0} + \binom{M}{1} = M + 1$$

for all  $M$ , and

$$B(1, d) = 2 \leq \sum_{i=0}^d \binom{1}{i} = \binom{1}{0} + \binom{1}{1} = 2$$

for all  $d$ . Now, we assume the statement is true for  $M = M_0$  and all  $d$ , we will prove that the statement is still true for  $M = M_0 + 1$  and all  $d$ . We have that

$$\begin{aligned} B(M_0 + 1, d) &\leq B(M_0, d) + B(M_0, d-1) \\ &\leq \sum_{i=0}^d \binom{M_0}{i} + \sum_{i=0}^{d-1} \binom{M_0}{i} \\ &= \binom{M_0}{0} + \sum_{i=1}^d \binom{M_0}{i} + \sum_{i=1}^d \binom{M_0}{i-1} \\ &= 1 + \sum_{i=1}^d \underbrace{\left[ \binom{M_0}{i} + \binom{M_0}{i-1} \right]}_{= \binom{M_0+1}{i}} \\ &= \sum_{i=0}^d \binom{M_0+1}{i}. \end{aligned}$$

We have thus proved the induction step, so the statement is true for all  $M$  and  $d$ .

### Problem 7.3

We begin by proving the following equivalence relation

$$h_m(x) = c_m \Leftrightarrow h_m^{c_m}(x) = +1.$$

The condition is necessary because if  $c_m = +1$ , we have

$$h_m^{c_m}(x) = h_m(x) = c_m = +1;$$

and if  $c_m = -1$ , we have

$$h_m^{c_m}(x) = \bar{h}_m(x) = \bar{c}_m = +1.$$

Now the condition is also sufficient because if  $c_m = +1$ , we have

$$+1 = h_m^{c_m}(x) = h_m(x),$$

which means that  $h_m(x) = +1 = c_m$ ; and if  $c_m = -1$ , we have

$$+1 = h_m^{c_m}(x) = \bar{h}_m(x),$$

which implies that  $h_m(x) = -1 = c_m$ .

Now we are able to write that

$$\begin{aligned} & x \in r \\ \Leftrightarrow & (h_1(x), \dots, h_M(x)) = (c_1, \dots, c_M) \\ \Leftrightarrow & h_m^{c_m}(x) = +1, \forall m \\ \Leftrightarrow & \prod_{m=1}^M h_m^{c_m}(x) = +1 \\ \Leftrightarrow & t_r(x) = +1. \end{aligned}$$

The above relation also implies that

$$x \notin r \Leftrightarrow t_r(x) = -1.$$

Now if  $x$  is in a positive region ( $f(x) = +1$ ), we know that there exists  $i$  such that  $x \in r_i$ , and consequently that  $t_{r_i}(x) = +1$  which means that

$$t_{r_1}(x) + \dots + t_{r_k}(x) = +1 = f(x).$$

And if  $x$  is in a negative region ( $f(x) = -1$ ), we know that  $x \notin r_i$  for all  $i$ , so  $t_{r_i}(x) = -1$  for all  $i$  which means that

$$t_{r_1}(x) + \dots + t_{r_k}(x) = -1 = f(x).$$

## Problem 7.4

Since  $f = t_{r_1} + \dots + t_{r_k}$ , we may write that

$$f = \text{sign}\left(k - \frac{1}{2} + \sum_{i=1}^k t_{r_i}\right),$$

which characterizes the penultimate layer of our perceptron. For the layer before, we have that  $t_{r_i} = h_1^{c_1^{(i)}} \dots h_M^{c_M^{(i)}}$ , and consequently

$$t_{r_i} = \text{sign}\left(-M + \frac{1}{2} + \sum_{m=1}^M h_m^{c_m^{(i)}}\right);$$

moreover, the previous layer may be characterized with

$$h_m^{c_m^{(i)}} = \text{sign}(c_m^{(i)} w_m^T x).$$

Putting all this together, we obtain the following characterization of a 3-layer perceptron

$$f = \text{sign}\left(k - \frac{1}{2} + \sum_{i=1}^k \text{sign}\left(-M + \frac{1}{2} + \sum_{m=1}^M \text{sign}(c_m^{(i)} w_m^T x)\right)\right)$$

whose structure is given by  $[d, kM, k, 1]$ .

## Problem 7.5

First, we decompose the unit hypercube  $[0, 1]^d$  into  $1/\epsilon^d$   $\epsilon$ -hypercubes (hypercube whose sides have length equal to  $\epsilon$ ), thus we get a grid-like structure of our unit hypercube. Now, if we consider a decision region (which may be composed by disconnected regions) whose boundary surfaces are smooth, this decision region partition the unit hypercube into two regions : one labelled  $+1$  and one labelled  $-1$ . We now have  $k_\epsilon$   $\epsilon$ -hypercubes labelled  $+1$  which are formed by  $2d$  hyperplanes each defined by  $h_m^{(i)} = \text{sign}(w_m^{(i),T} x)$  where  $m = 1, \dots, 2d$  and  $i = 1, \dots, k_\epsilon$ . So, the first layer whose task is to activate the hyperplanes involved in the positive  $\epsilon$ -hypercubes is characterized by

$$h_m^{(i)} = \text{sign}(w_m^{(i),T} x).$$

Now to activate the positive  $\epsilon$ -hypercubes  $H_i$  themselves we characterize the second layer by

$$t_{H_i} = (h_1^{(i)})^{c_1^{(i)}} \dots (h_{2d}^{(i)})^{c_{2d}^{(i)}},$$

where the  $c_m^{(i)}$  are defined as in Problem 7.3 and 7.4; or

$$t_{H_i} = \text{sign}(-2d + \frac{1}{2} + \sum_{m=1}^{2d} (h_m^{(i)})^{c_m^{(i)}}).$$

And finally to activate all the positive  $\epsilon$ -hypercubes, we define the MLP output  $h$  by

$$h = t_{H_1} + \dots + t_{H_{k_\epsilon}};$$

or

$$h = \text{sign}(k_\epsilon - \frac{1}{2} + \sum_{i=1}^{k_\epsilon} t_{H_i}).$$

Putting all this together, we obtain the following characterization of a 3-layer perceptron

$$h = \text{sign}(k_\epsilon - \frac{1}{2} + \sum_{i=1}^{k_\epsilon} \text{sign}(-2d + \frac{1}{2} + \sum_{m=1}^{2d} \text{sign}(c_m^{(i)} w_m^{(i),T} x))).$$

Now, it remains to see that the above MLP can arbitrarily closely approximate the initial positive decision region  $D_+$  (and consequently the negative decision region also); to do so, we first note that

$$\text{Vol}(H_i) = \epsilon^d \rightarrow 0 \text{ and } k_\epsilon \rightarrow \infty$$

when  $\epsilon \rightarrow 0$ . So, the  $\epsilon$ -hypercubes can be made arbitrarily small, which obviously means that the total volume of the positive  $\epsilon$ -hypercubes can be made arbitrarily close to the volume of the positive decision region (because of its smoothness). Mathematically, we may write that

$$\text{Vol}(H_1 \cup \dots \cup H_{k_\epsilon}) = \sum_{i=1}^{k_\epsilon} \epsilon^d \rightarrow \text{Vol}(D_+)$$

when  $\epsilon \rightarrow 0$ . This means that the region where our 3-layer perceptron will output  $+1$  (resp.  $-1$ ) converges to the positive (resp. negative) decision region in our unit hypercube.

## Problem 7.6

For a specific layer  $l$ , if we replace the weight  $w_{ij}^{(l)}$  with  $w_{ij}^{(l)} + \epsilon$ , we need to recompute the corresponding node output of that layer and also the node outputs for the subsequent layers (which are the ones numbered from  $l+1$  to  $L$ ). Consequently, for each weight  $w_{ij}^{(l)}$ , we have

$$\sum_{k=l+1}^L d^{(l)}(d^{(l-1)} + 1) + 1 + \sum_{k=l+1}^L d^{(l)}$$

multiplications and  $\theta$ -evaluations respectively; this means that the computational complexity of obtaining the partial derivatives is overall equal to

$$\begin{aligned}
& 2 \underbrace{\sum_{l=1}^L d^{(l)}(d^{(l-1)} + 1)}_{=|W|} \left( \sum_{k=l+1}^L d^{(l)}(d^{(l-1)} + 1) + 1 + \sum_{k=l+1}^L d^{(l)} \right) \\
& \leq 2|W| \left( \underbrace{\sum_{k=1}^L d^{(l)}(d^{(l-1)} + 1)}_{=|W|} + 1 + \sum_{k=1}^L \underbrace{d^{(l)}}_{\leq d^{(l)}(d^{(l-1)} + 1)} \right) \\
& \leq 2|W|(2|W| + 1) = O(|W|^2)
\end{aligned}$$

since we need to compute the derivatives corresponding to  $w_{ij}^{(l)} + \epsilon$  and also to  $w_{ij}^{(l)} - \epsilon$ .

### Problem 7.7

(a) We know that

$$E_{in} = \frac{1}{N} \sum_{i=1}^N \|y_i - \hat{y}_i\|^2,$$

and also that

$$(Y - \hat{Y})(Y - \hat{Y})^T = \begin{pmatrix} y_1^T - \hat{y}_1^T \\ \vdots \\ y_N^T - \hat{y}_N^T \end{pmatrix} (y_1 - \hat{y}_1, \dots, y_N - \hat{y}_N) = \begin{pmatrix} \|y_1 - \hat{y}_1\|^2 & * & * \\ * & \ddots & * \\ * & * & \|y_N - \hat{y}_N\|^2 \end{pmatrix}.$$

Consequently, we get that

$$E_{in} = \frac{1}{N} \text{trace}((Y - \hat{Y})(Y - \hat{Y})^T).$$

(b) We may write that

$$\begin{aligned}
E_{in} &= \frac{1}{N} \text{trace}(YY^T - ZVY^T - YV^T Z^T + ZVV^T Z^T) \\
&= \frac{1}{N} \text{trace}(YY^T - 2ZVY^T + ZVV^T Z^T),
\end{aligned}$$

since  $\text{trace}(A) = \text{trace}(A^T)$ . We are now ready to compute the derivatives, we have

$$\begin{aligned}
\frac{\partial E_{in}}{\partial V} &= \frac{1}{N} \left( -2 \underbrace{\frac{\partial \text{trace}(ZVY^T)}{\partial V}}_{=Z^T Y} + \underbrace{\frac{\partial \text{trace}(ZVV^T Z^T)}{\partial V}}_{=Z^T ZV + Z^T ZV = 2Z^T ZV} \right) \\
&= \frac{1}{N} (2Z^T ZV - 2Z^T Y),
\end{aligned}$$

because of the following identities

$$\frac{\partial \text{trace}(AXB)}{\partial X} = A^T B^T \text{ and } \frac{\partial \text{trace}(AXX^T B)}{\partial X} = BAX + A^T B^T X.$$

We also have

$$\begin{aligned}
E_{in} &= \frac{1}{N} \text{trace}(YY^T - 2(V_0 + \theta(XW)V_1)Y^T + (V_0 + \theta(XW)V_1)(V_0^T + V_1^T\theta(XW)^T) \\
&= \frac{1}{N} \text{trace}(YY^T - 2V_0Y^T - 2\theta(XW)V_1Y^T + V_0V_0^T + V_0V_1^T\theta(XW)^T + \theta(XW)V_1V_0^T + \theta(XW)V_1V_1^T\theta(XW)^T) \\
&= \frac{1}{N} \text{trace}(YY^T - 2V_0Y^T - 2\theta(XW)V_1Y^T + V_0V_0^T + 2\theta(XW)V_1V_0^T + V_1V_1^T\theta(XW)^T\theta(XW)),
\end{aligned}$$

since the trace can be permuted in a cycle and  $\text{trace}(A) = \text{trace}(A^T)$ . The other derivative may be written as

$$\begin{aligned}
\frac{\partial E_{in}}{\partial W} &= \frac{1}{N} \left( -2 \frac{\partial \text{trace}(\theta(XW)V_1Y^T)}{\partial W} + 2 \frac{\partial \text{trace}(\theta(XW)V_1V_0^T)}{\partial W} + \frac{\partial \text{trace}(V_1V_1^T\theta(XW)^T\theta(XW))}{\partial W} \right) \\
&= \frac{1}{N} (-2X^T\theta'(XW) \otimes YV_1^T + 2X^T\theta'(XW) \otimes V_0V_1^T + X^T(\theta'(XW) \otimes [\theta(XW)2V_1V_1^T]) \\
&= 2X^T[\theta'(XW) \otimes (-YV_1^T + V_0V_1^T + \theta(XW)V_1V_1^T)]
\end{aligned}$$

because of the following identities

$$\frac{\partial \text{trace}(\theta(BX)A)}{\partial X} = B^T\theta'(BX) \otimes A^T \text{ and } \frac{\partial \text{trace}(A\theta(BX)^T\theta(BX))}{\partial X} = B^T[\theta'(BX) \otimes (\theta(BX)(A + A^T))].$$

## Problem 7.8

(a) By hypothesis, we know that  $\{\eta_1, \eta_2, \eta_3\}$  with  $\eta_1 < \eta_2 < \eta_3$  is an U-arrangement which means that

$$E(\eta_2) < \min\{E(\eta_1), E(\eta_3)\}.$$

Since  $E(\eta)$  is a quadratic curve, we know that it is decreasing (resp. increasing) to the left (resp. right) of its minimum  $\bar{\eta}$ . So if we assume that  $\bar{\eta} < \eta_1$ , we get that  $E(\eta_1) \leq E(\eta_2) \leq E(\eta_3)$  which is impossible by definition of an U-arrangement; and if we assume that  $\bar{\eta} > \eta_3$ , we get that  $E(\eta_1) \geq E(\eta_2) \geq E(\eta_3)$  which is also impossible by definition of an U-arrangement. Consequently, we have  $\bar{\eta} \in [\eta_1, \eta_3]$ .

(b) First, we solve the linear system in  $a$ ,  $b$ , and  $c$  below

$$\begin{cases} E(\eta_1) &= a\eta_1^2 + b\eta_1 + c = e_1 \\ E(\eta_2) &= a\eta_2^2 + b\eta_2 + c = e_2 \\ E(\eta_3) &= a\eta_3^2 + b\eta_3 + c = e_3 \end{cases}.$$

Let  $D$  be the determinant of the system, which is

$$D = \begin{vmatrix} \eta_1^2 & \eta_1 & 1 \\ \eta_2^2 & \eta_2 & 1 \\ \eta_3^2 & \eta_3 & 1 \end{vmatrix},$$

where  $D \neq 0$  since  $\eta_1 < \eta_2 < \eta_3$ ; now we easily get that

$$a = \begin{vmatrix} e_1 & \eta_1 & 1 \\ e_2 & \eta_2 & 1 \\ e_3 & \eta_3 & 1 \end{vmatrix} / D = \frac{(e_1 - e_2)(\eta_1 - \eta_3) - (e_1 - e_3)(\eta_1 - \eta_2)}{D}$$

and

$$b = \begin{vmatrix} \eta_1^2 & e_1 & 1 \\ \eta_2^2 & e_2 & 1 \\ \eta_3^2 & e_3 & 1 \end{vmatrix} / D = \frac{-(e_1 - e_2)(\eta_1^2 - \eta_3^2) + (e_1 - e_3)(\eta_1^2 - \eta_2^2)}{D}.$$

Since the minimum of such a quadratic function is given by  $-b/2a$ , we finally get

$$\bar{\eta} = \frac{1}{2} \left[ \frac{(e_1 - e_2)(\eta_1^2 - \eta_3^2) - (e_1 - e_3)(\eta_1^2 - \eta_2^2)}{(e_1 - e_2)(\eta_1 - \eta_3) - (e_1 - e_3)(\eta_1 - \eta_2)} \right].$$

(c) We enumerate the four cases below.

1. If  $\bar{\eta} < \eta_2$  :

- If  $E(\bar{\eta}) < E(\eta_2)$ , then  $\{\eta_1, \bar{\eta}, \eta_2\}$  is a new U-arrangement.
- If  $E(\bar{\eta}) > E(\eta_2)$ , then  $\{\bar{\eta}, \eta_2, \eta_3\}$  is a new U-arrangement.

2. If  $\bar{\eta} > \eta_2$  :

- If  $E(\bar{\eta}) < E(\eta_2)$ , then  $\{\eta_2, \bar{\eta}, \eta_3\}$  is a new U-arrangement.
- If  $E(\bar{\eta}) > E(\eta_2)$ , then  $\{\eta_1, \eta_2, \bar{\eta}\}$  is a new U-arrangement.

(d) If  $\bar{\eta} = \eta_2$ , by continuity we are always able to find another  $\eta'_2$  close to  $\eta_2$  such that

$$E(\eta'_2) < \min\{E(\eta_1), E(\eta_3)\}.$$

In this case, we can use this new  $\eta'_2$  in place of  $\eta_2$  and proceed with the algorithm.