Problem Solutions

Chapter 2

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Problem 2.1

Let us begin by extracting the value of N from the $\epsilon(M, N, \delta)$ expression. We have that

$$\sqrt{\frac{1}{2N}\ln\frac{2M}{\delta}} \leq \epsilon \Leftrightarrow N \geq \frac{1}{2\epsilon^2}\ln\frac{2M}{\delta}.$$

(a) So for $M = 1$ and $\delta = 0.03$, to have $\epsilon \leq 0.05$ we need

$$N \geq \frac{1}{2 \cdot 0.05^2}\ln\frac{2}{0.03} = 839.9410156.$$

(b) For $M = 100$ and $\delta = 0.03$, to have $\epsilon \leq 0.05$ we need

$$N \geq \frac{1}{2 \cdot 0.05^2}\ln\frac{2 \cdot 100}{0.03} = 1760.9750528.$$

(c) And for $M = 10000$ and $\delta = 0.03$, to have $\epsilon \leq 0.05$ we need

$$N \geq \frac{1}{2 \cdot 0.05^2}\ln\frac{2 \cdot 10000}{0.03} = 2682.00909.$$

Problem 2.2

For $N = 4$, if we consider four non aligned points, this $\mathcal{H}$ shatters these points (you only have to efectively enumerate them to see that all dichotomies can be generated), so in this case we have $m_{\mathcal{H}}(4) = 2^4$.

However, for $N = 5$, no matter how you place your five points, some point will be inside a rectangle defined by others. In this case, we are not able to generate all dichotomies and consequently $m_{\mathcal{H}}(5) < 2^5$.

From these two observations, we may conclude that, for poisitive rectangles, we have $d_{VC} = 4$, thus

$$m_{\mathcal{H}}(N) \leq N^4 + 1.$$

Problem 2.3

(a) We already know that the growth function for positive rays is equal to $N + 1$. If we enumerate the dichotomies added by negative rays, we get $N - 1$ new dichotomies (you get the opposite of the ones from positive rays and you have to substract the two dichotomies where all points are $+1$ and where all points are -1). So in total, we get that

$$m_{\mathcal{H}}(N) = 2N.$$

As the largest value of N for which $m_{\mathcal{H}}(N) = 2^N$ is 2 ($m_{\mathcal{H}}(3) = 6$), we have that $d_{VC} = 2$.

(b) Here, we already know that the growth function for positive intervals is equal to $N^2/2 + N/2 + 1$. If we add the new dichotomies generated by negative intervals, we get $N - 2$ new ones (for example for $N = 3$, we only add the $(+1, -1, +1)$ dichotomy, and for $N = 4$, we add the $(+1, -1, +1, +1)$ and $(+1, +1, -1, +1)$

dichotomies). Of course, this only holds if $N > 1$, in the case where $N = 1$ we already generate the two dichotomies with the positive intervals alone. In conclusion, we may write that

$$m_{\mathcal{H}}(N) = \frac{N^2}{2} + \frac{3N}{2} - 1 \text{ if } N > 1 \text{ and } 2 \text{ if } N = 1.$$

As the largest value of N for which $m_{\mathcal{H}}(N) = 2^N$ is 3 ($m_{\mathcal{H}}(4) = 13$), we have that $d_{VC} = 3$.

(c) To determine the growth function fo concentric circles, we have to map the problem from $\mathbb{R}^d$ to $[0, +\infty[$. To do this, we use the map ϕ defined as

$$\phi : (x_1, \cdots, x_d) \mapsto r = \sqrt{x_1^2 + \cdots + x_d^2}.$$

By doing that, we may see that the problem of concentric circles in $\mathbb{R}^d$ is equivalent to the problem of positive intervals in $\mathbb{R}$ (it is easy to see that ϕ maps points with the same radius to a unique point in $[0, +\infty[$), and consequently we may write that

$$m_{\mathcal{H}}(N) = \frac{N^2}{2} + \frac{N}{2} + 1$$

which is independant of d. As the largest value of N for which $m_{\mathcal{H}}(N) = 2^N$ is 2 ($m_{\mathcal{H}}(3) = 7$), we have that $d_{VC} = 2$.

Problem 2.4

Problem 2.5

To prove the inequality, we begin with the case $D = 0$. Here, it is easy to see that

$$1 = \binom{N}{0} \leq N^0 + 1 = 2.$$

Now, we assume the result is correct for D ($D \geq 1$), and we will prove it for $D + 1$. We may write that

$$\begin{aligned}
\sum_{i=0}^{D+1} \binom{N}{i} &= \sum_{i=0}^{D} \binom{N}{i} + \binom{N}{D+1} \\
&\leq N^D + 1 + \binom{N}{D+1} \\
&\leq N^D + 1 + \frac{N!}{(D+1)!(N-D-1)!}.
\end{aligned}$$

To continue, we have to prove that

$$\frac{N!}{(N-D-1)!} \leq N^{D+1},$$

which is equivalent to

$$\frac{1}{N^{D+1}} \cdot \frac{N!}{(N-D-1)!} \leq 1.$$

To see this, it suffices to note that

$$\frac{1}{N^{D+1}} \cdot \frac{N!}{(N-D-1)!} = \frac{1}{N^{D+1}} \prod_{i=0}^{D} (N-i) = \prod_{i=0}^{D} \frac{N-i}{N^{D+1}} \leq 1.$$

So, we are now able to write that

$$\begin{aligned}
\sum_{i=0}^{D+1} \binom{N}{i} &\leq N^D + 1 + \frac{N!}{(D+1)!(N-D-1)!} \\
&\leq N^D + 1 + \frac{N^{D+1}}{(D+1)!}.
\end{aligned}$$

As $D \geq 1$, we have $(D+1)! \geq 2$, and consequently

$$\frac{1}{(D+1)!} \leq \frac{1}{2},$$

which enables us to write that

$$\begin{aligned}
\sum_{i=0}^{D+1} \binom{N}{i} &\leq N^D + 1 + \frac{N^{D+1}}{(D+1)!} \\
&\leq N^D + 1 + \frac{N^{D+1}}{2}.
\end{aligned}$$

Moreover, as we assumed $N \geq D+1$ (if not, we trivially have the result, as in this case $\binom{N}{D+1} = 0$), we get $N \geq 2$ and consequently

$$\frac{1}{N} < \frac{1}{2} \Leftrightarrow \frac{N^D}{N^{D+1}} < \frac{1}{2} \Leftrightarrow N^D < \frac{N^{D+1}}{2},$$

which allows us to get our result as we now have

$$\begin{aligned}
\sum_{i=0}^{D+1} \binom{N}{i} &\leq N^D + 1 + \frac{N^{D+1}}{2} \\
&\leq \frac{N^{D+1}}{2} + 1 + \frac{N^{D+1}}{2} = N^{D+1} + 1.
\end{aligned}$$