Problem Solutions

Chapter 2

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Problem 2.1

Let us begin by extracting the value of N from the $\epsilon(M, N, \delta)$ expression. We have that

$$\sqrt{\frac{1}{2N}\ln\frac{2M}{\delta}} \leq \epsilon \Leftrightarrow N \geq \frac{1}{2\epsilon^2}\ln\frac{2M}{\delta}.$$

(a) So for $M = 1$ and $\delta = 0.03$, to have $\epsilon \leq 0.05$ we need

$$N \geq \frac{1}{2 \cdot 0.05^2}\ln\frac{2}{0.03} = 839.9410156.$$

(b) For $M = 100$ and $\delta = 0.03$, to have $\epsilon \leq 0.05$ we need

$$N \geq \frac{1}{2 \cdot 0.05^2}\ln\frac{2 \cdot 100}{0.03} = 1760.9750528.$$

(c) And for $M = 10000$ and $\delta = 0.03$, to have $\epsilon \leq 0.05$ we need

$$N \geq \frac{1}{2 \cdot 0.05^2}\ln\frac{2 \cdot 10000}{0.03} = 2682.00909.$$

Problem 2.2

For $N = 4$, if we consider four non aligned points, this $\mathcal{H}$ shatters these points (you only have to efectively enumerate them to see that all dichotomies can be generated), so in this case we have $m_{\mathcal{H}}(4) = 2^4$.

However, for $N = 5$, no matter how you place your five points, some point will be inside a rectangle defined by others. In this case, we are not able to generate all dichotomies and consequently $m_{\mathcal{H}}(5) < 2^5$.

From these two observations, we may conclude that, for poisitive rectangles, we have $d_{VC} = 4$, thus

$$m_{\mathcal{H}}(N) \leq N^4 + 1.$$

Problem 2.3

(a) We already know that the growth function for positive rays is equal to $N + 1$. If we enumerate the dichotomies added by negative rays, we get $N - 1$ new dichotomies (you get the opposite of the ones from positive rays and you have to substract the two dichotomies where all points are $+1$ and where all points are -1). So in total, we get that

$$m_{\mathcal{H}}(N) = 2N.$$

As the largest value of N for which $m_{\mathcal{H}}(N) = 2^N$ is 2 ($m_{\mathcal{H}}(3) = 6$), we have that $d_{VC} = 2$.

(b) Here, we already know that the growth function for positive intervals is equal to $N^2/2 + N/2 + 1$. If we add the new dichotomies generated by negative intervals, we get $N - 2$ new ones (for example for $N = 3$, we only add the $(+1, -1, +1)$ dichotomy, and for $N = 4$, we add the $(+1, -1, +1, +1)$ and $(+1, +1, -1, +1)$

dichotomies). Of course, this only holds if $N > 1$, in the case where $N = 1$ we already generate the two dichotomies with the positive intervals alone. In conclusion, we may write that

$$m_{\mathcal{H}}(N) = \frac{N^2}{2} + \frac{3N}{2} - 1 \text{ if } N > 1 \text{ and } 2 \text{ if } N = 1.$$

As the largest value of N for which $m_{\mathcal{H}}(N) = 2^N$ is 3 ($m_{\mathcal{H}}(4) = 13$), we have that $d_{VC} = 3$.

(c) To determine the growth function fo concentric circles, we have to map the problem from $\mathbb{R}^d$ to $[0, +\infty[$. To do this, we use the map ϕ defined as

$$\phi : (x_1, \cdots, x_d) \mapsto r = \sqrt{x_1^2 + \cdots + x_d^2}.$$

By doing that, we may see that the problem of concentric circles in $\mathbb{R}^d$ is equivalent to the problem of positive intervals in $\mathbb{R}$ (it is easy to see that ϕ maps points with the same radius to a unique point in $[0, +\infty[$), and consequently we may write that

$$m_{\mathcal{H}}(N) = \frac{N^2}{2} + \frac{N}{2} + 1$$

which is independant of d. As the largest value of N for which $m_{\mathcal{H}}(N) = 2^N$ is 2 ($m_{\mathcal{H}}(3) = 7$), we have that $d_{VC} = 2$.