Problem Solutions

Chapter 4

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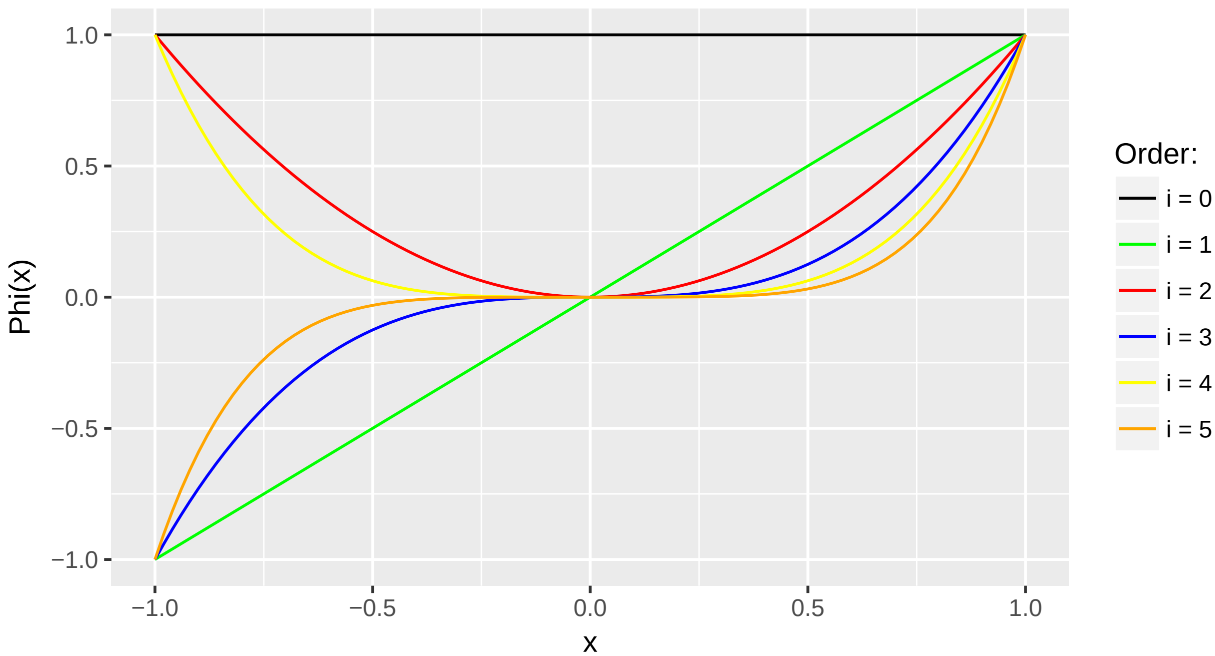
Problem 4.1

Below we plot the monomials of order i, $\phi_i(x) = x^i$.



It is easy to see that as the order i increases, so does the complexity of the curve (in the sense that it is able to fit more complex target functions).

Problem 4.2

We may write

$$
\begin{aligned}
h(x) &= \begin{pmatrix} 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} L_0(x) \\ L_1(x) \\ L_2(x) \end{pmatrix} \\
&= L_0(x) - L_1(x) + L_2(x) \\
&= \frac{3}{2}x^2 - x + \frac{1}{2}
\end{aligned}
$$

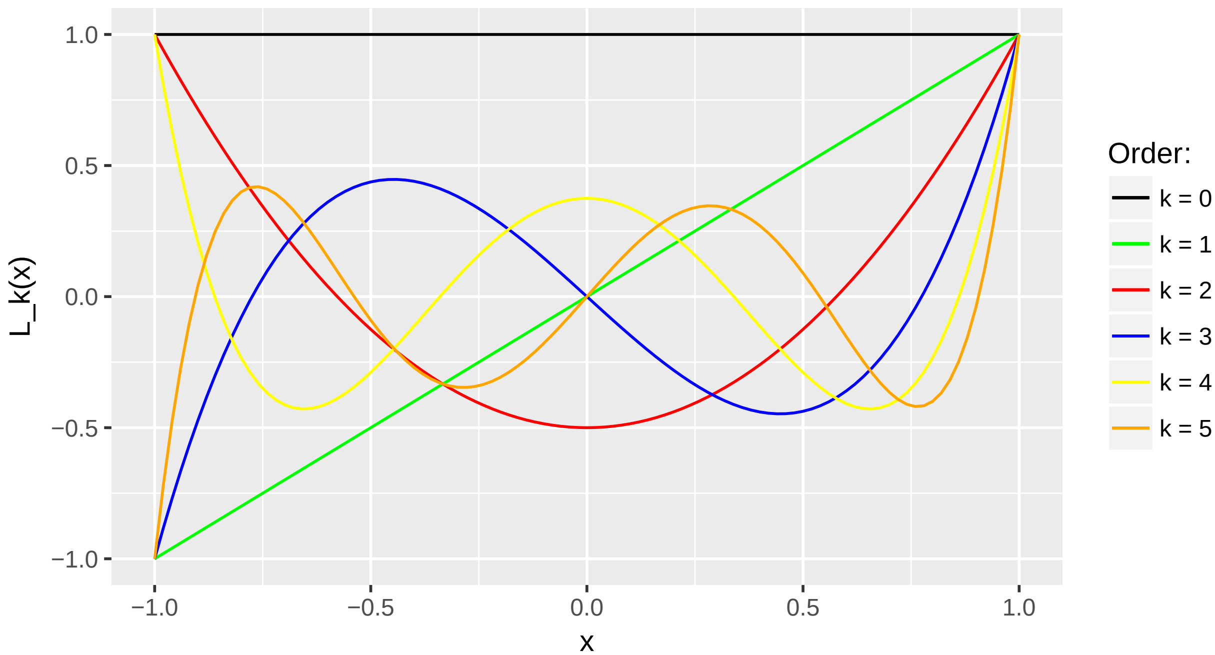
So we get a degree 2 polynomial.

Problem 4.3

(a) We use the recursive definition of the Legendre polynomials to develop an algorithm to compute $L_k(x)$ given x.

```
Legendre <- function(x, k) {
  if (k == 0)
    return(1)
  if (k == 1)
    return(x)
  else
    return(((2 * k - 1) / k) * x * Legendre(x, k - 1) - ((k - 1) / k) * Legendre(x, k - 2))
}
```

Now we plot the first six Legendre polynomials below.



(b) We prove this fact by induction. For $k=0$, we have $L_0(x)=1$ which is a monomial of order 0. For $k=1$, we have $L_1(x)=x$ which is a monomial of order 1. Now we assume that the result is true for all order less than $k+2$, and we will prove it is still true for order $k+2$. We will also assume that k is even (the case when it is odd is proved in the same way). We have

$$\begin{aligned} L_{k+2}(x) &= \underbrace{\frac{2k+3}{k+2}}_{=c_1} x \cdot \underbrace{L_{k+1}(x)}_{=a_{k+1}x^{k+1}+a_{k-1}x^{k-1}+\cdots+a_1x} - \underbrace{\frac{k+1}{k+2}}_{=c_0} \cdot \underbrace{L_k(x)}_{=b_kx^k+b_{k-2}x^{k-2}+\cdots+b_0} \\ &= c_1a_{k+1}x^{k+2}+(c_1a_{k-1}-c_0b_k)x^k+\cdots+(c_1a_1-c_0b_2)x^2-c_0b_0 \end{aligned}$$

which is actually a linear combination of monomials all of even order with highest order $k+2$. In this case we obviously have

$$L_k(-x)=(-1)^kL_k(x).$$

(c) Once again we proceed by induction on k. For $k=1$, we have

$$\frac{x^2-1}{1}\underbrace{\frac{dL_1(x)}{dx}}_{=1}=x^2-1=xL_1(x)-L_0(x).$$

Now we assume that the result is true for all order less than k, and we prove it is still true for k. We have that

$$
\begin{aligned}
&\frac{x^2-1}{k}\frac{dL_k(x)}{dx} \\
&= \frac{x^2-1}{k}\left(\frac{2k-1}{k}L_{k-1}(x)+\frac{(2k-1)x}{k}\frac{dL_{k-1}(x)}{dx}-\frac{k-1}{k}\frac{dL_{k-2}(x)}{dx}\right) \\
&= \frac{(x^2-1)(2k-1)}{k^2}L_{k-1}(x)+\frac{(2k-1)(k-1)x}{k^2}\underbrace{\frac{x^2-1}{k-1}\frac{dL_{k-1}(x)}{dx}}_{=xL_{k-1}(x)-L_{k-2}(x)}-\frac{(k-1)(k-2)}{k^2}\underbrace{\frac{x^2-1}{k-2}\frac{dL_{k-2}(x)}{dx}}_{=xL_{k-2}(x)-L_{k-3}(x)} \\
&= \frac{(2k-1)(kx^2-1)}{k^2}L_{k-1}(x)-\frac{(k-1)(3kx-3x)}{k^2}L_{k-2}(x)+\frac{(k-1)(k-2)}{k^2}L_{k-3}(x) \\
&= \underbrace{x\left(\frac{2k-1}{k}xL_{k-1}(x)-\frac{k-1}{k}L_{k-2}(x)\right)}_{=L_k(x)}-\frac{2k-1}{k^2}L_{k-1}(x)-\frac{(k-1)^2}{k^2}\underbrace{\left(\frac{2k-3}{k-1}xL_{k-2}(x)-\frac{k-2}{k-1}L_{k-3}(x)\right)}_{=L_{k-1}(x)} \\
&= xL_k(x)-\frac{(2k-1)+(k-1)^2}{k^2}L_{k-1}(x) \\
&= xL_k(x)-L_{k-1}(x).
\end{aligned}
$$

(d) We may write that

$$
\begin{aligned}
\frac{d}{dx}\left((x^2-1)\frac{dL_k(x)}{dx}\right) &= \frac{d}{dx}\left(xkL_k(x)-kL_{k-1}(x)\right) \\
&= kL_k(x)+xk\frac{dL_k(x)}{dx}-k\frac{dL_{k-1}(x)}{dx} \\
&= kL_k(x)+\frac{k^2x^2}{x^2-1}L_k(x)-\frac{k^2x}{x^2-1}L_{k-1}(x)-\frac{k(k-1)}{x^2-1}xL_{k-1}(x)+\frac{k(k-1)}{x^2-1}L_{k-2(x)} \\
&= \frac{kx^2-k+k^2x^2}{x^2-1}L_k(x)-\frac{k}{x^2-1}[(2k-1)xL_{k-1}(x)-(k-1)L_{k-2}(x)] \\
&= \frac{kx^2-k+k^2x^2}{x^2-1}L_k(x)-\frac{k^2}{x^2-1}L_k(x) \\
&= \frac{k}{x^2-1}[(x^2-1)+kx^2-k]L_k(x) \\
&= k(k+1)L_k(x).
\end{aligned}
$$

(e) We will first consider the case where $l \neq k$. We have that

$$\frac{d}{dx}\left((1-x^2)\frac{dL_k(x)}{dx}\right)+k(k+1)L_k(x)=0$$

and

$$\frac{d}{dx}\left((1-x^2)\frac{dL_l(x)}{dx}\right)+l(l+1)L_l(x)=0,$$

now we multiply the first identity by $L_l(x)$ and the second by $L_k(x)$, if we substract and integrate the two identities obtained, we get

$$\int_{-1}^{1} L_l(x)\frac{d}{dx}\left((1-x^2)\frac{dL_k(x)}{dx}\right)-L_k(x)\frac{d}{dx}\left((1-x^2)\frac{dL_l(x)}{dx}\right)dx+[k(k+1)-l(l+1)]\int_{-1}^{1} L_k(x)L_l(x)dx=0.$$

Using integration by parts for the first integral, we get

$$\Bigg(\underbrace{L_l(x)(1-x^2)\frac{dL_k(x)}{dx}\Bigg|_{-1}^{1}}_{=0} - \underbrace{L_k(x)(1-x^2)\frac{dL_l(x)}{dx}\Bigg|_{-1}^{1}}_{=0}\Bigg) - \underbrace{\int_{-1}^{1} \frac{dL_l(x)}{dx}(1-x^2)\frac{dL_k(x)}{dx} - \frac{dL_k(x)}{dx}(1-x^2)\frac{dL_l(x)}{dx}dx}_{=0} = 0.$$

Finally, we obtain

$$\int_{-1}^{1} L_k(x)L_l(x)dx = 0.$$

Now, we consider the case where $l = k$. We have that

$$\begin{aligned}
A_k = \int_{-1}^{1} L_k^2(x) &= \frac{2k-1}{k}\int_{-1}^{1} xL_k(x)L_{k-1}(x)dx - \frac{k-1}{k}\underbrace{\int_{-1}^{1} L_k(x)L_{k-2}(x)dx}_{=0} \\
&= \frac{(2k-1)(k+1)}{k(2k+1)}\underbrace{\int_{-1}^{1} L_{k+1}(x)L_{k-1}(x)dx}_{=0} + \frac{(2k-1)k}{k(2k+1)}\int_{-1}^{1} L_{k-1}^2(x)dx \\
&= \frac{2k-1}{2k+1}\int_{-1}^{1} L_{k-1}^2(x)dx.
\end{aligned}$$

Finally, we are able to obtain that

$$\begin{aligned}
A_k &= \frac{2k-1}{2k+1}A_{k-1} \\
&= \frac{2k-1}{2k+1}\cdot\frac{2k-3}{2k-1}A_{k-2} \\
&= \frac{2k-1}{2k+1}\cdot\frac{2k-3}{2k-1}\cdots\frac{3}{5}\frac{1}{3}\underbrace{A_0}_{=2} \\
&= \frac{2}{2k+1}.
\end{aligned}$$