Formula Sheet 1

Bold type – time dependent quantities; normal type – time independent quantities

Differential operators (Cartesian co-ordinates)

$\nabla = \partial/\partial x\ \underline{i} + \partial/\partial y\ \underline{j} + \partial/\partial z\ \underline{k}$

$\nabla\phi = \partial\phi/\partial x\ \underline{i} + \partial\phi/\partial y\ \underline{j} + \partial\phi/\partial z\ \underline{k}$

$\nabla \cdot \underline{\mathbf{F}} = \partial\mathbf{F}_x/\partial x + \partial\mathbf{F}_y/\partial y + \partial\mathbf{F}_z/\partial z$

$\nabla \times \underline{\mathbf{F}} = \{\partial\mathbf{F}_z/\partial y - \partial\mathbf{F}_y/\partial z\}\ \underline{i} + \{\partial\mathbf{F}_x/\partial z - \partial\mathbf{F}_z/\partial x\}\ \underline{j} + \{\partial\mathbf{F}_y/\partial x - \partial\mathbf{F}_x/\partial y\}\ \underline{k}$

$\nabla^2\phi = \partial^2\phi/\partial x^2 + \partial^2\phi/\partial y^2 + \partial^2\phi/\partial z^2$

Identities

$\nabla \times \nabla \times \underline{\mathbf{F}} = \nabla(\nabla \cdot \underline{\mathbf{F}}) - \nabla^2\underline{\mathbf{F}}$

Integral theorems

$\iint_A \underline{\mathbf{F}} \cdot d\underline{a} = \iiint_V \nabla \cdot \underline{\mathbf{F}}\ dv$ (Gauss' theorem)

$\int_L \underline{\mathbf{F}} \cdot d\underline{L} = \iint_A (\nabla \times \underline{\mathbf{F}}) \cdot d\underline{a}$ (Stokes' theorem)

Formula Sheet 2

Maxwell's equations – integral form

$\iint_A \underline{\mathbf{D}} \cdot d\underline{a} = \iiint_V \rho \, dv$ (Gauss' law)

$\iint_A \underline{\mathbf{B}} \cdot d\underline{a} = 0$ (Magnetic equivalent of Gauss' law)

$\int_L \underline{\mathbf{E}} \cdot d\underline{L} = - \iint_A \partial\underline{\mathbf{B}}/\partial t \cdot d\underline{a}$ (Faraday's law)

$\int_L \underline{\mathbf{H}} \cdot d\underline{L} = \iint_A [\underline{\mathbf{J}} + \partial\underline{\mathbf{D}}/\partial t] \cdot d\underline{a}$ (Ampère's law)

Maxwell's equations – differential form

$\mathrm{div}(\underline{\mathbf{D}}) = \rho$

$\mathrm{div}(\underline{\mathbf{B}}) = 0$

$\mathrm{curl}(\underline{\mathbf{E}}) = -\partial\underline{\mathbf{B}}/\partial t$

$\mathrm{curl}(\underline{\mathbf{H}}) = \underline{\mathbf{J}} + \partial\underline{\mathbf{D}}/\partial t$

Constitutive relations

$\underline{\mathbf{J}} = \sigma \, \mathbf{E}$; $\mathbf{D} = \varepsilon \, \mathbf{E}$; $\mathbf{B} = \mu \, \underline{\mathbf{H}}$

Formula Sheet 3

Electromagnetic waves (pure dielectric media)

Time dependent vector wave equation $\nabla^2\underline{\mathbf{E}} = \mu_0\varepsilon\ \partial^2\underline{\mathbf{E}}/\partial t^2$

Time independent scalar wave equation $\nabla^2\underline{E} = -\omega^2\mu_0\varepsilon_0\varepsilon_r\ \underline{E}$

For z-going, x-polarized plane waves $d^2E_x/dz^2 + \omega^2\mu_0\varepsilon_0\varepsilon_r\ E_x = 0$

Where $\underline{E}$ is a time-independent vector field

Plane wave solutions have the form $E(z) = E_0\ \exp(-jk_0z)$

Where $k = \omega\sqrt{(\mu_0\varepsilon_0\varepsilon_r)}$ is the propagation constant

Impedance of free space $Z_0 = \sqrt{(\mu_0/\varepsilon_0)}$

Refractive index $n = \sqrt{\varepsilon_0}$

Power

Instantaneous power flow $\underline{\mathbf{S}} = \underline{\mathbf{E}} \times \underline{\mathbf{H}}$

Time-averaged power flow $\underline{S} = 1/2\ \mathrm{Re}\ (\underline{E} \times \underline{H}^*)$

Formula Sheet 4

Transmission line formulae

Transmission line equations for line with per unit length inductance and capacitance L_p and C_p

$dV/dz = -j\omega L_p I$

$dI/dz = -j\omega C_p V$

Phase velocity and impedance of line with per unit length inductance L_p and capacitance C_p

$v_{ph} = 1/\sqrt{(L_p C_p)}$

$Z_0 = \sqrt{(L_p/C_p)}$

Reflection and transmission coefficients at a junction between lines of impedance Z_1 and Z_2

$R_V = (Z_2 - Z_1) / (Z_2 + Z_1)$

$T_V = 2Z_2 / (Z_2 + Z_1)$

Input impedance for length d of line with properties (Z_0, k) terminated by load Z_L

$Z_{in} = Z_0 \{Z_L + jZ_0 \tan(kd)\} / \{Z_0 + jZ_L \tan(kd)\}$

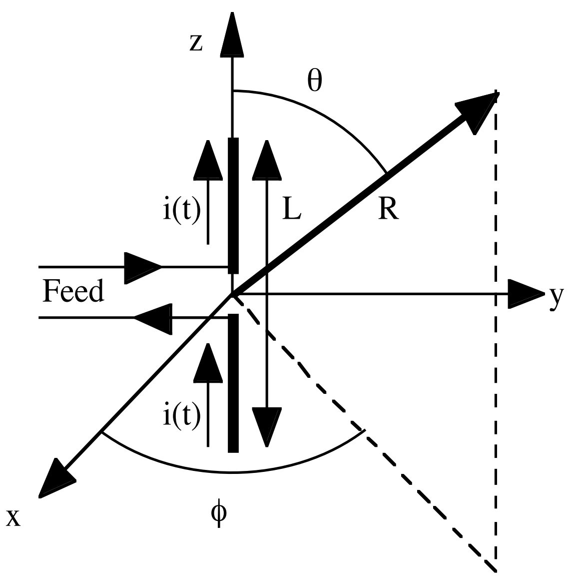
Formula Sheet 5

Antenna formulae

Far-field pattern of half-wave dipole

$E_\theta = j\ 60 I_0 \{\cos[(\pi/2)\cos(\theta)] / \sin(\theta)\} \exp(-jkR)/R$; $H_\phi = E_\theta/Z_0$

Here I_0 is peak current, R is range and $k = 2\pi/\lambda$



Formula Sheet 6

Antenna formulae

Time averaged power flow $\underline{S} = 1/2 \, \mathrm{Re} \, (\underline{E} \times \underline{H}^*) = S(R, \theta) \, \underline{r}$

Normalised radiation pattern $F(\theta, \phi) = S(R, \theta, \phi) / S_{max}$

Directivity $D = 1/ \{1/4\pi \iint_{4\pi} F(\theta, \phi) \sin(\theta) \, d\theta \, d\phi\}$

Gain $G = \eta D$ where η is antenna efficiency

Effective area $A_e = \lambda^2 D/4\pi$

Transmission formulae

Friis transmission equation $P_R = P_T \, \eta_T \eta_R A_T \, A_R/R^2\lambda^2 = P_T \, G_T G_R(\lambda/4\pi R)^2$

Radar equation $P_R = P_T \, (\eta_T{}^2 A_T{}^2 \sigma/4\pi R^4 \lambda^2)$

Microwave formulae

Skin depth $\delta = 1/\sqrt{(\pi f \mu_0 \sigma)}$

Formula Sheet 7

Optical formulae

Snell's law $n_1 \sin(\theta_1) = n_2 \sin(\theta_2)$

Fresnel reflection coefficient $\Gamma_E = \{n_1 \cos(\theta_1) - n_2 \cos(\theta_2)\}/\{n_1 \cos(\theta_1) + n_2 \cos(\theta_2)\}$

Imaging formulae

Spherical wave $E(r) = (E_0/r) \exp(-jk_0 r)$ where $r^2 = x^2 + y^2 + z^2$ in cartesian coordinates

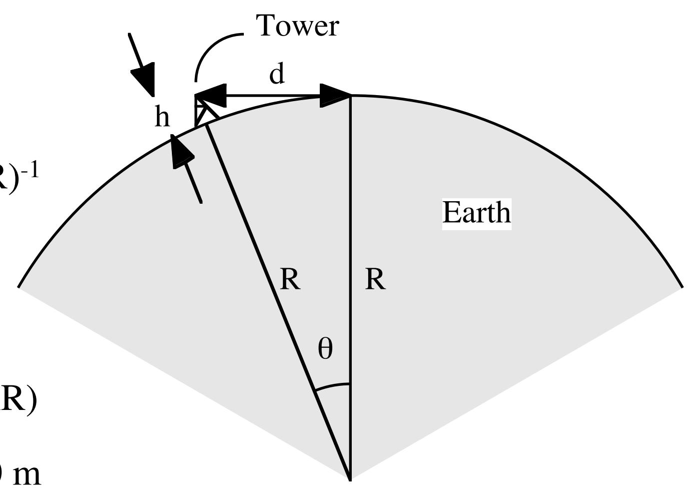
Paraxial wave $E(R, z) = A(z) \exp(-jk_0 R^2/2z)$ where $R^2 = x^2 + y^2$ in cartesian coordinates

The lens maker's formula $1/f = (n - 1) (1/R_1 + 1/R_2)$

The imaging equation $1/u + 1/v = 1/f$

Questions Lecture 1

1.1a) The diameter of the earth is approximately 12,756 km.The horizon is the furthest distance that can be viewed directly, standing on the Earth's surface. Calculate the distance of the horizon, seen from a tower 50 m tall.



To find the horizon distance, assume the Earth's radius

is R and that the observer is a h above the earth's surface.

Trigonometry gives $(R + h) \cos(\theta) = R$, so $\cos(\theta) = (1 + h/R)^{-1}$

Use binomial approximations for each side:

$(1 - \theta^2/2) \approx (1 - h/R)$ so $\theta \approx \sqrt{(2h/R)}$

The horizon distance is $d = (R + h) \sin(\theta)$, so $d \approx R\theta = \sqrt{(2hR)}$

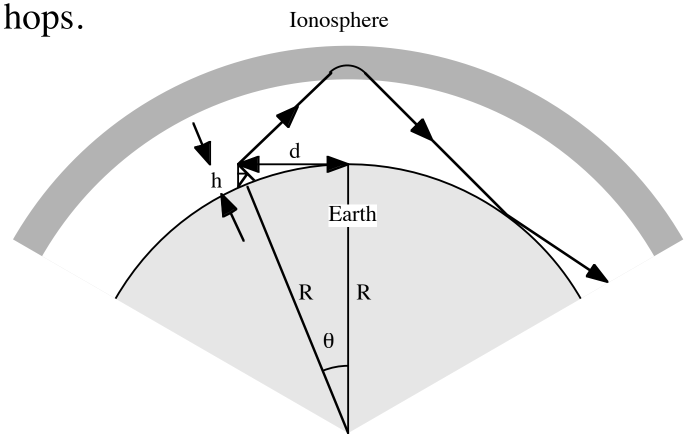
Assuming that $R = 12756 \times 10^3 / 2 = 6378000$ m and $h = 50$ m

We then get $d = \sqrt{(2 \times 6378000 \times 50)} = 25254.7$ m, or 25 km.

Questions Lecture 1

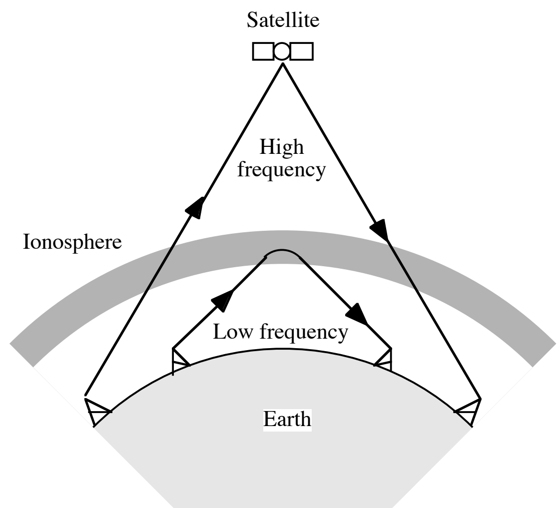
1.1b) How was Marconi able to transmit radio waves from Cornwall to Cape Cod? What are the disadvantages of his method, and how does modern satellite communication operate?

Clearly, the horizon distance is vastly smaller than transatlantic distances. Over-the horizon radio communication can be achieved at short wavelengths by using the ionosphere and the earth's surface (particularly the ocean, which contains sodium and chlorine ions) as reflectors, so that transmission is obtained in a series of short hops.



Questions Lecture 1

This low frequency transmission mechanism is subject to fading as the condition of the ionosphere and ocean alters with the weather (electrical storms, rough sea). Modern communication systems operate via one or more geostationary satellites, thus avoiding the horizon problem entirely. However, to reach the satellite, the radio wave must pass through the ionosphere. Fortunately the ionosphere is transparent at shorter wavelengths, so all that is required is to use a high operating frequency (VHF).



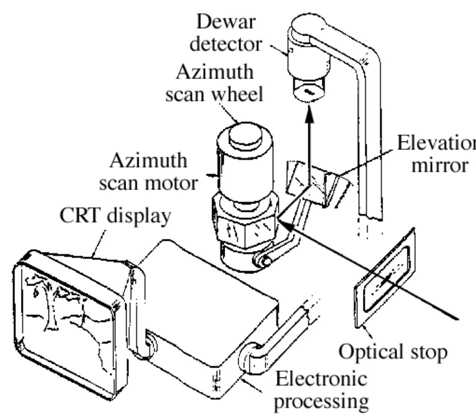
Questions Lecture 1

1.2a) What wavelengths are used for night vision, and why?

a) Daytime vision relies on the detection of light waves generated by the sun and scattered from the object into the eye or some other imaging device. At night, there is clearly no sunlight. Instead, thermal radiation emitted by the object by virtue of its internal heat must be detected. Because most objects (e.g. the human body) are relatively cool, this radiation is of longer wavelength than visible light, typically infrared.

Questions Lecture 1

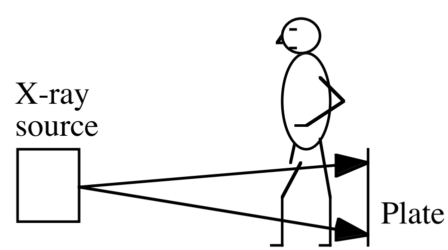
Detection involves conversion of the photon into an electron. However, IR photons have low energy. A photoconductive detector based on a low band-gap semiconductor can be used; however, such detectors are noisy, because band-to-band transitions may take place due to thermal energy. To minimise thermal noise, the largest practical bandgap is used, and the detector is cryogenically cooled. The most common wavelengths are near 8 μm and 12 μm, since these avoid the H_2O atmospheric absorption bands. The latter can be detected using a compound semiconductor, cadmium mercury telluride (CdHgTe). However, it is difficult to make a detector array and the image is often scanned over a single element.



Questions Lecture 1

1.2b) If it is almost impossible to form a lens for X-rays, how is the image formed when broken bones are X-rayed in the surgery? What is the magnification of the image?

An X-ray photograph is not an image in the same sense as a normal photograph, and it does not require a lens. It is effectively a shadow cast, relying on differences in absorption between tissue and bone to obtain contrast. Normally the X-ray plate is held close to or in contact with the patient, so the plate receives more or less exposure depending on the local integrated absorption as the wave passes through the tissue. The magnification is then effectively unity.



Questions Lecture 1

1.3a) What factors limit the possibilities for free-space point-to-point communications using electromagnetic waves?

The factors limiting free space propagation of a bounded beam are:

- Absorption, due to electronic transitions in the molecules of the earth's atmosphere at visible/UV wavelengths, and molecular vibrational transitions at infrared wavelengths. Absorption losses are concentrated near spectral bands known as absorption lines.
- Rayleigh scattering, due to inhomogeneities (soot particles, water droplets) and small-scale fluctuations (variations in air density) in the molecular arrangement of the atmosphere. Scattering losses rise rapidly at short wavelengths (rising as $1/\lambda^4$).
- Diffraction, due to the spreading of a bounded beam as it propagates. Diffraction effects increase rapidly as the the wavelength approaches the dimensions of the source aperture.

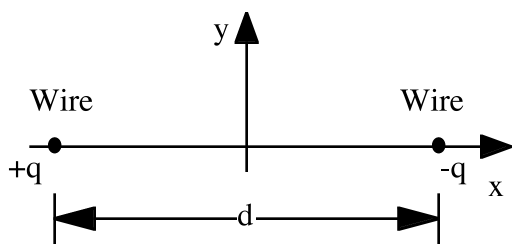
Questions Lecture 1

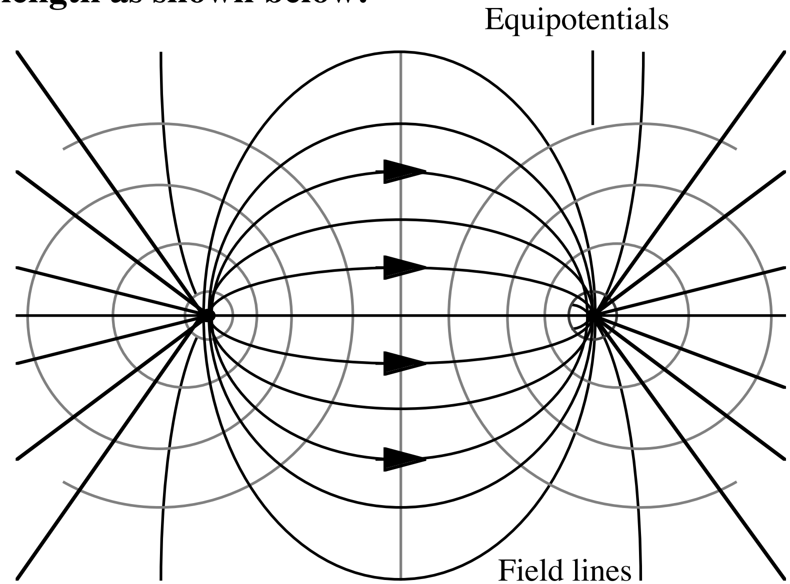
1.3b) What wavelengths are used in practise?

Microwaves are used for line-of-sight point to point communications, since the wavelength is then sufficiently long that scattering is small and the major atmospheric absorption bands are avoided, but sufficiently short that dish antennae may be many wavelengths in diameter and diffraction effects are minimised. Clearly, the absorption frequencies of water and other common molecules present in the atmosphere must be avoided.

Questions Lecture 2

2.1a) Sketch the electric field distribution created by two parallel cylindrical wires separated by a distance d and carrying charges ±q per unit length as shown below.





The distribution may be guessed, assuming it is the sum of contributions from two isolated line charges and that the field lines start and end on the charges. The equipotentials are perpendicular to the field lines.

Questions Lecture 2

2.1b) Show that the equipotential lines are circles, and find how the radii and origin of the circles depend on the value of the potential.

The field of a line charge q is $\underline{E} = q/2\pi\varepsilon r\ \underline{r}$

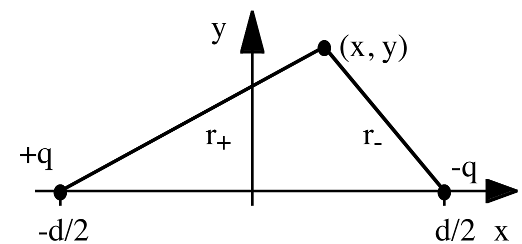
The potential is $V = - \int E_r\, dr = -(q/2\pi\varepsilon) \log_e(r) + A$

Or $V = -(q/2\pi\varepsilon) \log_e(Br)$

For two line charges, the potential at (x, y) is:

$V = -(q/2\pi\varepsilon) \{\log_e(Br_+) - \log_e(Br_-)\} = -(q/2\pi\varepsilon) \log_e(r_+/r_-)$

Where $r_\pm = \sqrt{[(x \pm d/2)^2 + y^2]}$



Hence:

$V = -(q/2\pi\varepsilon) \log_e\{ \sqrt{[(x + d/2)^2 + y^2]} / \sqrt{[(x - d/2)^2 + y^2]} \}$

Questions Lecture 2

Rearrange as:

$-2\pi\varepsilon V/q = \log_e\{ \sqrt{[(x + d/2)^2 + y^2]} / \sqrt{[(x - d/2)^2 + y^2]} \}$

Taking exponentials of both sides gives:

$\exp(-2\pi\varepsilon V/q) = \sqrt{[(x + d/2)^2 + y^2]} / \sqrt{[(x - d/2)^2 + y^2]}$

Putting $\exp(-2\pi\varepsilon V/q) = k$ and squaring gives:

$k^2 = [(x + d/2)^2 + y^2] / [(x - d/2)^2 + y^2]$

Multiplying out and re-arranging gives:

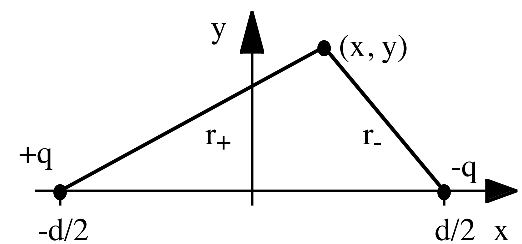
$x^2 + d + (d/2)^2 + y^2 = k^2 [x^2 - d + (d/2)^2 + y^2]$

$x^2(1 - k^2) + d (1 + k^2) + (d/2)^2 (1 - k^2) + y^2 (1 - k^2) = 0$

$x^2 + 2x(\alpha d/2) + (d/2)^2 + y^2 = 0$ where $\alpha = (1 + k^2)/(1 - k^2)$

Re-arranging gives $(x + \alpha d/2)^2 + y^2 = (d/2)^2 (\alpha^2 - 1)$

This is the equation of a circle, centred at $(-\alpha d/2, 0)$ with radius $(d/2) \sqrt{(\alpha^2 - 1)}$



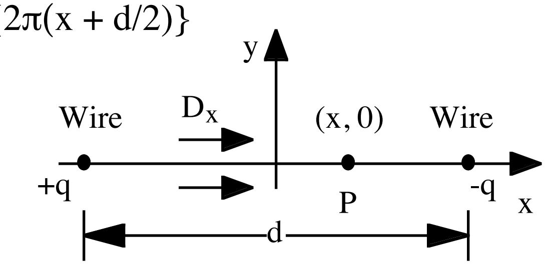
Questions Lecture 2

2.2 Assuming the diameters of the wires are small compared with their spacing, find the capacitance per metre between two parallel wires as above. Hence find the capacitance per metre between parallel wires of 1 mm diameter, spaced d = 2 cm apart in air.

Hint: Use Gauss' law to find the flux D_x at P, a distance x from one of the wires. Integrate the electric field E_x to find the potential V between the wires, and then find the capacitance pul as $C = q/V$.

For the LH wire, the flux density in the x-direction at P is $D_{1x} = q/\{2\pi(x + d/2)\}$

For the RH wire, the corresponding value is $D_{2x} = -q/\{2\pi(x - d/2)\}$



For both wires, the flux density in the x-direction at P can be found by superposition as:

$D_x = D_{1x} + D_{2x} = (q/2\pi) \{1/(x + d/2) - 1/(x - d/2)\}$

Questions Lecture 2

Since D = εE, the total electric field in the x-direction at P is

$E_x = (q/2\pi\varepsilon) \{1/(x + d/2) - 1/(x - d/2)\}$

Here ε is the dielectric constant of the surrounding medium.

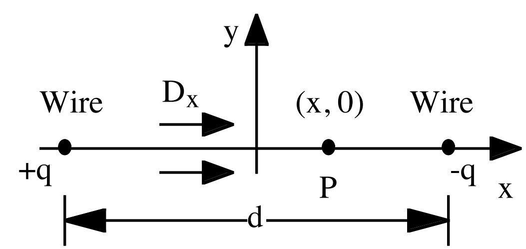
The voltage between the wires is then the integral of the field between the two, or

$V = \int_{-d/2+a}^{d/2-a} E_x \, dx$, or

$V = \int_{-d/2+a}^{d/2-a} (q/2\pi\varepsilon) \{1/(x + d/2) - 1/(x - d/2)\} \, dx$, or

$V = (q/2\pi\varepsilon) [\log_e\{ (x + d/2) / (x - d/2) \}]_{-d/2+a}^{d/2-a}$, or

$V = (q/2\pi\varepsilon) \log_e\{ [(d - a)/a]^2 \} = (q/\pi\varepsilon) \log_e\{(d - a)/a\}$



The capacitance per unit length is therefore $C = q/V = \pi\varepsilon / \log_e\{(d - a)/a\}$.

If $a = 5 \times 10^{-4}$ and $d = 2 \times 10^{-2}$ and $\varepsilon = \varepsilon_0 = 8.85 \times 10^{-12}$, C = 7.6 pF/m

Questions Lecture 2

2.3 Find the inductance per unit length for the two-wire transmission line in Q2.2. Hence, find the propagation velocity of the line.

Hint: Assume that the conductors carry currents ±I. Use Ampère's law to calculate the magnetic field H created by each wire at the point P. Integrate the corresponding magnetic flux density B to find the total flux Φ per unit length passing between the wires, and then find the inductance per unit length as $L = \Phi/I$.

The magnetic field radius r from a wire carrying a current I is found from Ampère's law as $\underline{H} = I/2\pi r\ \underline{\theta}$.

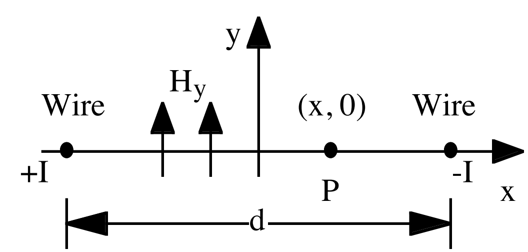
For the two-wire system, the field can be found by assuming the wires carry currents ±I.

For the LH wire, the magnetic field in the y-direction at P is found from Ampere's law as

$H_{y1} = (I/2\pi)\{1/(x + d/2)\}$

For the RH wire, the corresponding value is

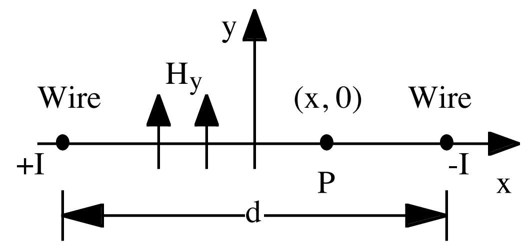
$H_{y2} = (-I/2\pi)\{1/(x - d/2)\}$



Questions Lecture 2

For both wires, the field in the y-direction at P can be found by superposition as:

$H_y = (I/2\pi)\{1/(x + d/2) - 1/(x - d/2)\}$



Since $\underline{B} = \mu_0\underline{H}$, the flux density in the y-direction at P is

$B_y = (\mu_0 I/2\pi)\{1/(x + d/2) - 1/(x - d/2)\}$

So the flux per unit length crossing the x-axis between the two wires is

$\Phi = \int_{-d/2+a}^{d/2-a} B_y \, dx$, or

$\Phi = (\mu_0 I/2\pi) \int_{-d/2+a}^{d/2-a} \{1/(x + d/2) - 1/(x - d/2)\} \, dx$

Notice the similarity between this integral and the one in Q2.2.

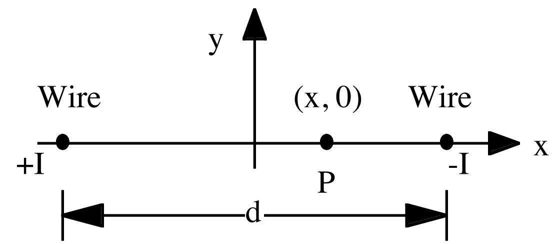
Questions Lecture 2

Integrating, we get

$\Phi = (\mu_0 I/2\pi)\ [\log_e(x + d/2) - \log_e(x - d/2)]_{-d/2+a}^{d/2-a}$ so

$\Phi = (\mu_0 I/\pi) \log_e\{(d - a)/a\}$

The inductance $L = \Phi/I$ is then $L = (\mu_0/\pi) \log_e\{(d - a)/a\}$ pul



If $a = 5 \times 10^{-4}$ and $d = 2 \times 10^{-2}$ and $\mu_0 = 4\pi \times 10^{-7}$, $L = 1.46\ \mu H/m$

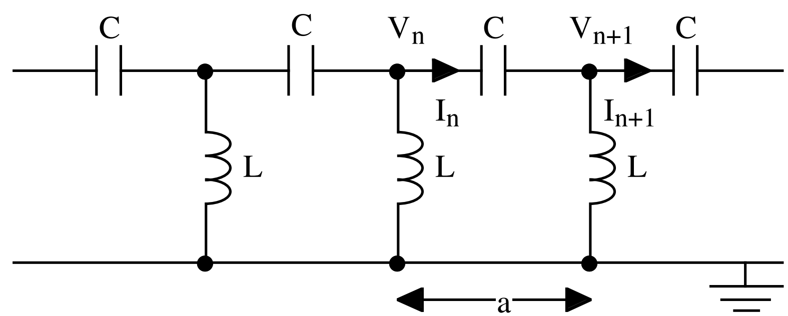
The propagation velocity is $v_{ph} = 1/\sqrt{(L_{pul}C_{pul})} = 1/(7.6 \times 10^{-12} \times 1.46 \times 10^{-6}) = 3.0 \times 10^8$ m/s

Notice that this result is the velocity of a light wave in free space, despite the fact that the wave is propagating here on a two-wire transmission line.

Questions Lecture 3

3.1 The figure below shows a C-L ladder network. Using Kirchhoff's laws, write down the relations between the nodal voltages and currents V_n and I_n in the n^{th} section and the corresponding values V_{n+1} and I_{n+1} in the $n+1^{th}$ section. Assuming travelling wave solutions, derive the dispersion relation.

NB: The lattice is not the one in the notes!



At angular frequency ω, KVL gives:

$V_{n+1} = V_n - I_n/j\omega C$

$I_{n+1} = I_n - V_{n+1}/j\omega L$

Substituting the travelling wave solutions $V_n = V_0 \exp(-jnka)$ and $I_n = I_0 \exp(-jkna)$ gives:

$V_0 \exp\{-j(n+1)ka\} = V_0 \exp(-jnka) - I_0 \exp(-jnka)/j\omega C$

$I_0 \exp\{-j(n+1)ka\} = I_0 \exp(-jnka) - V_0 \exp\{-j(n+1)ka\}/j\omega L$

Questions Lecture 3

Rearranging, we get:

$\{\exp(-jka) - 1\}V_0 + I_0/j\omega C = 0$

$V_0 \exp(-jka)/j\omega L + \{\exp(-jka) - 1\}I_0 = 0$

To avoid non-trivial solutions ($V_0 = 0$; $I_0 = 0$) we then require:

$\{\exp(-jka) - 1\}\{\exp(-jka) - 1\} + \exp(-jka)/\omega^2 LC = 0$

Expanding, we then get:

$\exp(-2jka) - 2 \exp(-jka) + 1 + \exp(-jka)/\omega^2 LC = 0$

$\exp(-jka) - 2 + \exp(+jka) + 1/\omega^2 LC = 0$

Grouping terms together we then get:

$2\{\cos(ka) - 1\} + 1/\omega^2 LC = 0$

$1/\omega^2 LC = 2\{1 - \cos(ka)\} = 4 \sin^2(ka/2)$

Finally we can obtain the dispersion relation as $\omega/\omega_0 = 1/\{2 \sin(ka/2)\}$ where $\omega_0^2 = 1/LC$

Questions Lecture 3

3.2 Plot the dispersion characteristic for the ladder network above. What is the cutoff frequency? What kind of filter does this network represent?

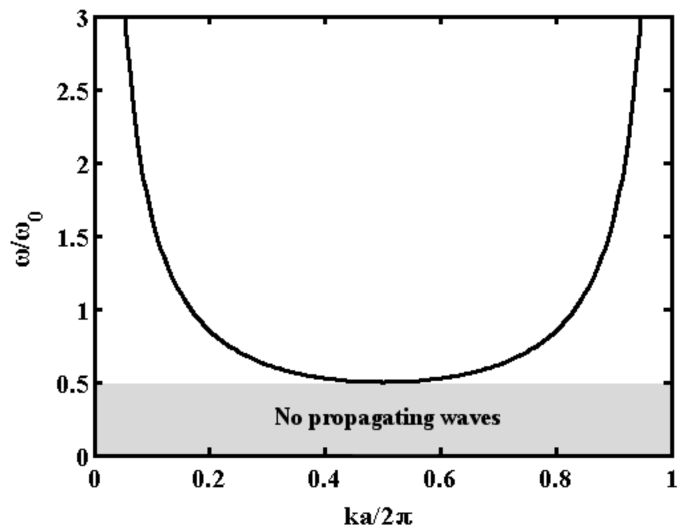
The dispersion characteristic $\omega/\omega_0 = 1/\{2 \sin(ka/2)\}$ can be sketched as follows:

First find the asymptotes:

When $ka = 0$ or 2π, ω tends to infinity

When $ka = \pi$, $\omega = \omega_0/2$

Then join the asymptotes with a smooth curve



The cutoff frequency is clearly $\omega_C = \omega_0/2$.

Since there can be no propagation below the cutoff frequency, the ladder is a high-pass filter.

Questions Lecture 3

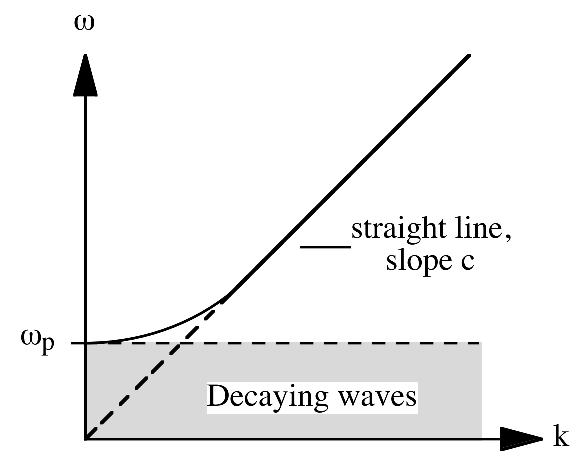
3.3a) The relation between ω and k for EM waves in the ionosphere has the form $\omega = \sqrt{(\omega_p^2 + c^2k^2)}$, where c is the velocity of light and ω_p is the plasma frequency. Sketch the dispersion diagram.

The dispersion diagram can be sketched as follows:

First find the asymptotes:

When k tends to zero, ω tends to ω_p
When k tends to infinity, ω tends ck

Then join the asymptotes with a smooth curve

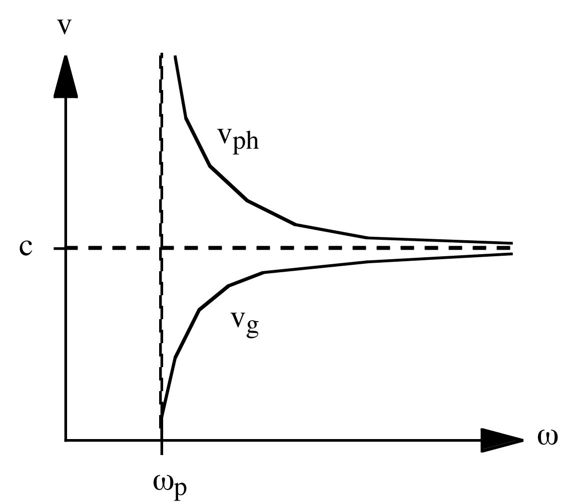


Questions Lecture 3

3.3b) The relation between ω and k for EM waves in the ionosphere has the form $\omega = \sqrt{(\omega_p^2 + c^2k^2)}$, where c is the velocity of light and ω_p is the plasma frequency. Sketch the variations of the phase velocity $v_{ph} = \omega/k$ and group velocity $v_g = d\omega/dk$ with ω.

The velocity variations can be sketched as follows:
First find $v_{ph} = \sqrt{\{(\omega_p^2/k^2) + c^2\}}$

Then find the asymptotes:
When ω tends to ω_p, k tends to zero so v_{ph} tends to infinity.
When ω tends to infinity, k tends to ω/c so v_{ph} tends to c
Then join with a smooth curve



Then find $v_g = c^2/\sqrt{\{(\omega_p^2/k^2) + c^2\}}$

Then find the asymptotes:
When ω tends to ω_p, k tends to zero so v_g tends to zero
When ω tends to infinity, k tends to ω/c so v_g tends to c, etc.

Questions Lecture 3

3.3c) Find the value of v_g when ω tends to ω_p. What is the significance of this result?

When v_g tends to zero, the velocity of information and energy propagation will also tend to zero.

d) Find a solution for k when $\omega < \omega_p$. What form of wave does the solution describe?

Rearranging the dispersion relation, $k = (1/c) \sqrt{(\omega^2 - \omega_p^2)}$

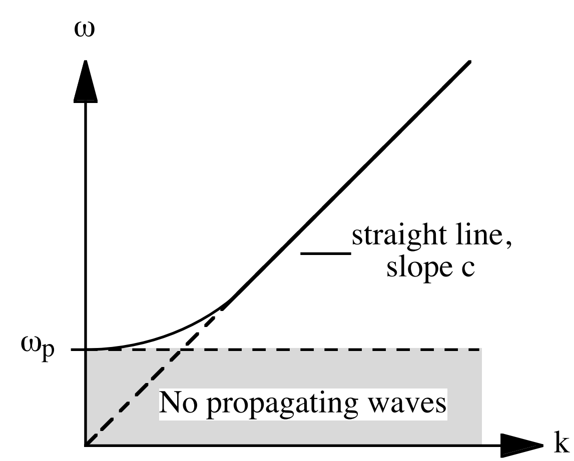
When $\omega < \omega_p$, k must be imaginary, so we can write: $k = -jk''$

Now, a z-propagating wave is $E = E_0 \exp\{j(\omega t - kz)\}$

This becomes $E = E_0 \exp(j\omega t) \exp(-k''z)$

This solution is not a travelling wave, but decays exponentially

At this frequency, the wave is reflected rather than transmitted.



Questions Lecture 4

4.1a) Find the characteristic impedance of a lossless transmission line with constants L = 4 mH/km and C = 0.1 μF/km. What is the phase velocity? What is the group velocity?

The characteristic impedance is $Z_0 = \sqrt{(L_{pul}/C_{pul})}$

In this case we get:

$Z_0 = \sqrt{(4 \times 10^{-6}/10^{-10})} = 200\ \Omega$.

The phase velocity is $v_{ph} = 1/\sqrt{(LC)}$

In this case we get:

$v_{ph} = 1/\sqrt{(4 \times 10^{-6}/10^{-10})} = 5 \times 10^7$ m/s

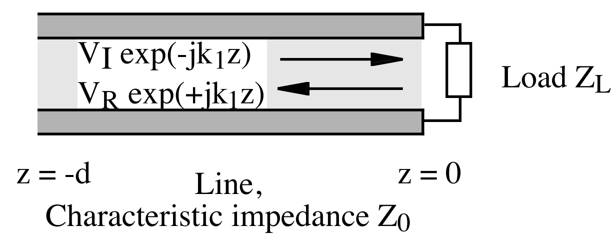
The group and phase velocities are the same for a lossless transmission line.

Questions Lecture 4

4.1b) Find the input impedance of 5 km of the line, with a terminating load impedance Z_L (unspecified) at frequencies of i) 5 kHz, ii) 2 kHz and iii) 50 Hz.

The general expression for the input impedance of a length L of line with parameters (Z_0, k) is:

$Z_{in} = Z_0 \{Z_L + jZ_0 \tan(kL)\} / \{Z_0 + jZ_L \tan(kL)\}$, so:



i) At 5 kHz frequency:

The propagation constant is $k = \omega/v_{ph} = 2\pi \times 5 \times 10^3/5 \times 10^7 = 2\pi \times 10^{-4}\ m^{-1}$

The phase shift along the line at this frequency is $kL = 2\pi \times 10^{-4} \times 5 \times 10^3 = \pi$.

If $kL = \pi$, tan(kL) is zero, so the input impedance is $Z_{in} = Z_0Z_L/Z_0 = Z_L$

Questions Lecture 4

ii) At 2 kHz frequency:

The propagation constant is now $k = 2\pi \times 2 \times 10^3/5 \times 10^7 = 8\pi \times 10^{-5}$ m^{-1}

The phase shift along the line at this frequency is $kL = 8\pi \times 10^{-5} \times 5 \times 10^3 = 0.4\pi$.

If $kL = 0.4\pi$, $Z_{in} = 200 \{Z_L + j200 \tan(0.4\pi)\} / \{200 + jZ_L \tan(0.4\pi)\}$

Hence, $Z_{in} = (Z_L + j615)/(1 + jZ_L/65)$

iii) At 50 Hz frequency:

The propagation constant is now $k = 2\pi \times 50/5 \times 10^7 = 2\pi \times 10^{-6}$ m^{-1}

The phase shift along the line at this frequency is $kL = 2\pi \times 10^{-6} \times 5 \times 10^3 = 0.01\pi$.

If $kL = 0.01\pi$, $Z_{in} = 200 \{Z_L + j200 \tan(0.01\pi)\} / \{200 + jZ_L \tan(0.01\pi)\}$

Hence, $Z_{in} = (Z_L + j6.28)/(1 + jZ_L/6364)$

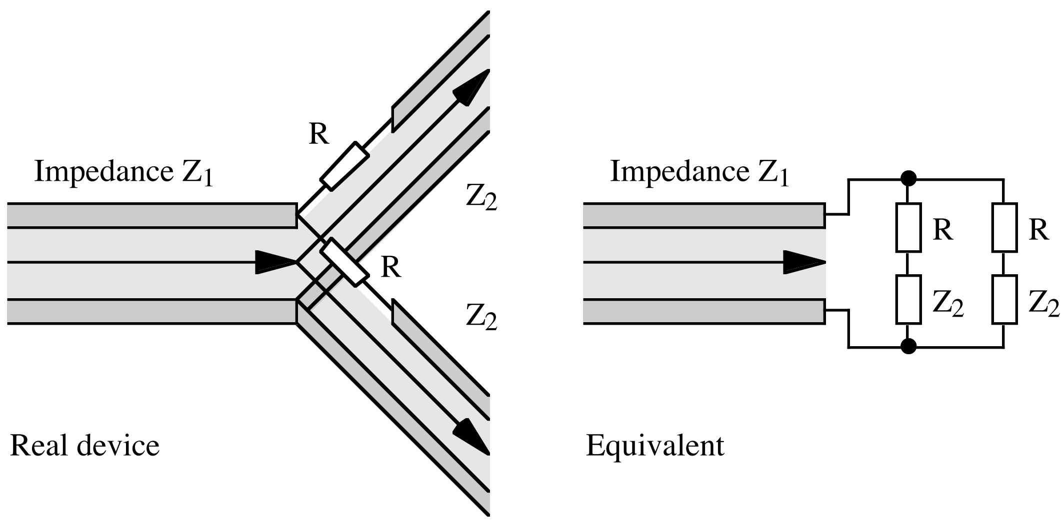
If $6.2 << Z_L << 6364\ \Omega$, Z_{in} is approximately equal to Z_L. NB: This is how 50 Hz mains avoids TL effects

Questions Lecture 4

4.2a) A 50 Ω transmission line is connected in a splitter topology to two parallel 75 Ω lines. Devise a circuit that will suppress reflections for signals incident from the 50 Ω line.

Referring to the figure below, resistors R must be inserted in series with the lines Z_2 so that the parallel combination has impedance Z_1. This can be achieved if $(R + Z_2)/2 = Z_1$.

If $Z_2 = 75\ \Omega$ and $Z_1 = 50\ \Omega$, we require $R = 25\ \Omega$.



Questions Lecture 4

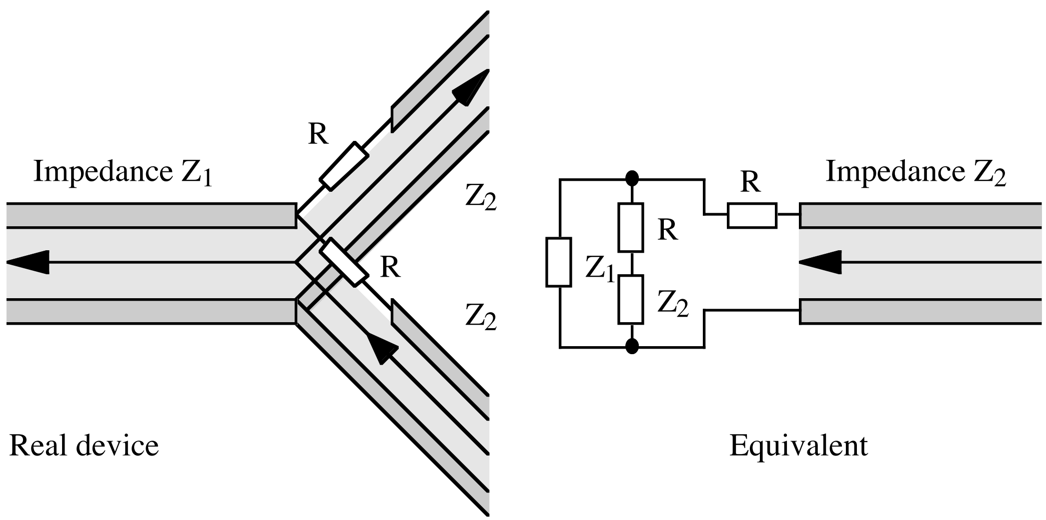
4.2b) What is the reflection coefficient for signals incident from one or other of the 75 Ω lines?

A wave incident from one of the lines with impedance Z_2 will then see the load shown below.

This load has an impedance of $Z_L = R + 1/\{1/Z_1 + 1/(R + Z_2)\} = 58.33\ \Omega$.

The voltage reflection coefficient is then

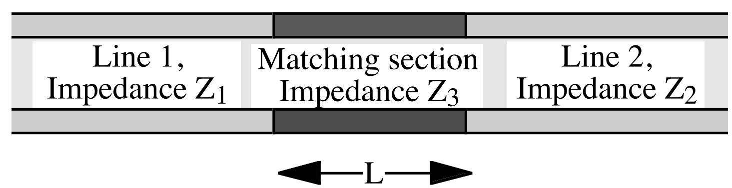
$R_V = (Z_L - Z_2)/(Z_L + Z_2) = (58.33 - 75)/(58.33 + 75) = -0.125$



Questions Lecture 4

4.3 The transmission line in Q4.1 is to be connected to a second lossless line with characteristic impedance $Z_2 = 100\ \Omega$. Find the characteristic impedance and length of of a quarter wave transformer that would match the lines together at a frequency of 100 MHz. Assume that the matching section has a relative dielectric constant of 4.

The input impedance of Lines 3 and 2 together is:



$Z_{in} = Z_3 \{Z_2 + jZ_3 \tan(k_3L)\} / \{Z_3 + jZ_2 \tan(k_3L)\}$

In a quarter wave transformer, $k_3L = \pi/2$

The tan functions become infinite, so $Z_{in} = Z_3^2/Z_2$

Z_{in} presents a matched load to Line 1 if $Z_{in} = Z_1$

Hence we require $Z_3 = \sqrt{(Z_1Z_2)}$ so Z_3 is the geometric mean of Z_1 and Z_2

If $Z_1 = 200\ \Omega$ and $Z_2 = 100\ \Omega$ then $Z_3 = 141\ \Omega$

Questions Lecture 4

The propagation constant is $k = \omega\sqrt{(L_{pul}C_{pul})}$

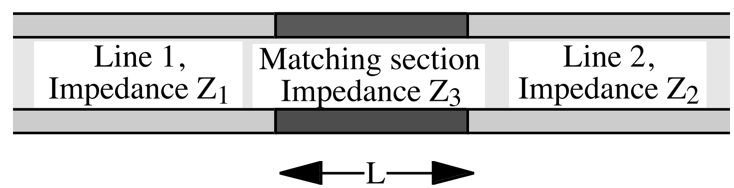
For a co-axial line, $L_{pul}C_{pul} = \mu_0\varepsilon = \mu_0\varepsilon_0\varepsilon_r$ so $k = \omega\sqrt{(\mu_0\varepsilon_0\varepsilon_r)}$

However, the velocity of light in vacuo is $c = 1/\sqrt{(\mu_0\varepsilon_0)}$ so $k = \omega\sqrt{\varepsilon_r}/c$

For quarter wave transformer, $k_3L = \pi/2$

Hence $2\pi f(\sqrt{\varepsilon_{r3}}/c)L = \pi/2$

So the length of line required is $L = c/4f\sqrt{\varepsilon_{r3}}$



At 100 MHz frequency, assuming $\sqrt{\varepsilon_{r3}} = 2$ we get :

$L = 3 \times 10^8/(4 \times 10^8 \times 2) = 0.375$ m

Since $\lambda_0 = 3 \times 10^8 / 1 \times 10^8 = 3$ m, $\lambda = \lambda_0/\sqrt{\varepsilon_{r3}} = 1.5$ m $= 4L$

L is indeed a quarter of a wavelength in a medium of relative dielectric constant $\sqrt{\varepsilon_{r3}}$.

Questions Lecture 5

5.1. Starting from the differential form of Maxwell's equations, derive a time-dependent vector wave equation, valid for a uniform dielectric medium and containing only the magnetic field $\underline{H}$.

Start with Maxwell's equations $\nabla \times \underline{\mathbf{E}} = -\partial\underline{\mathbf{B}}/\partial t$ and $\nabla \times \underline{\mathbf{H}} = \underline{\mathbf{J}} + \partial\underline{\mathbf{D}}/\partial t$

Then assume that in a dielectric medium $\underline{\mathbf{J}} = \underline{0}$, $\underline{\mathbf{B}} = \mu_0\underline{\mathbf{H}}$ and $\underline{\mathbf{D}} = \varepsilon\underline{\mathbf{E}}$

Hence, we can say that $\nabla \times \underline{\mathbf{E}} = -\mu_0\, \partial\underline{\mathbf{H}}/\partial t$ and $\nabla \times \underline{\mathbf{H}} = \varepsilon\, \partial\underline{\mathbf{E}}/\partial t$

Now take the curl of the second equation, to get: $\nabla \times (\nabla \times \underline{\mathbf{H}}) = \nabla \times (\varepsilon\, \partial\underline{\mathbf{E}}/\partial t)$

If ε is constant, $\nabla \times (\nabla \times \underline{\mathbf{H}}) = \varepsilon\{\nabla \times (\partial\underline{\mathbf{E}}/\partial t)\}$

Reversing the order of differentiation, we get $\nabla \times (\nabla \times \underline{\mathbf{H}}) = \varepsilon\{\partial(\nabla \times \underline{\mathbf{E}})/\partial t\}$

Substituting for $\nabla \times \underline{\mathbf{E}}$, we get $\nabla \times (\nabla \times \underline{\mathbf{H}}) = -\mu_0\varepsilon\, \partial^2\underline{\mathbf{H}}/\partial t^2$

Now, the following standard identity exists: $\nabla \times (\nabla \times \underline{\mathbf{H}}) = \nabla (\nabla \cdot \underline{\mathbf{H}}) - \nabla^2\underline{H}$

Since $\nabla \cdot \underline{\mathbf{B}} = 0$, it follows that $\nabla \cdot (\mu_0\underline{\mathbf{H}}) = 0$, so $\nabla \cdot \underline{\mathbf{H}} = 0$ and $\nabla \times (\nabla \times \underline{\mathbf{H}}) = -\nabla^2\underline{\mathbf{H}}$

Hence, the required vector wave equation in $\underline{\mathbf{H}}$ alone is $\nabla^2\underline{\mathbf{H}} = \mu_0\varepsilon\, \partial^2\underline{\mathbf{H}}/\partial t^2$.

Questions Lecture 5

5.2 A plane electromagnetic wave oscillates at an angular frequency of 2.978 x 10^{15} rad/sec. Find its wavelength in a) free space, and b) a material of relative dielectric constant 2.25.

The phase velocity is $v_{ph} = f\lambda$ so $\lambda = v_{ph}/f$

The angular frequency is $\omega = 2\pi f$ so $f = \omega/2\pi$

a) In free space, the velocity of light is $v_{ph} = c = 3 \times 10^8$ m/s

The wavelength is then $\lambda_0 = 3 \times 10^8 \times 2\pi / 2.978 \times 10^{15}$ m $= 0.633 \times 10^{-6}$ m (HeNe laser light)

b) In a medium of refractive index n, the wavelength is reduced to $\lambda = \lambda_0/n = \lambda_0/\sqrt{\varepsilon}$.

For the values here, $\lambda = 0.633 \times 10^{-6}/\sqrt{2.25} = 0.422 \times 10^{-6}$ m.

Questions Lecture 5

5.3 Show that a transverse electromagnetic wave cannot have a component of its electric or magnetic field parallel to its direction of propagation.

Hint. Assume propagation in a particular direction (say, z). Apply the conditions relevant to a plane wave to the field variations in directions perpendicular to the direction of propagation, and use the divergence conditions div($\underline{D}$) = 0 and div($\underline{B}$) = 0 to find the conditions parallel to the direction of propagation.

Maxwell's equations require div($\underline{D}$) = 0 in the absence of charges

In an isotropic medium, div($\underline{D}$) = ε div($\underline{E}$) so div($\underline{E}$) = $\partial E_x/\partial x + \partial E_y/\partial y + \partial E_z/\partial z = 0$

For a plane EM wave propagating in the z-direction, there is no variation in the perpendicular directions

Hence $\partial \underline{E}/\partial x$ and $\partial \underline{E}/\partial y = 0$. This means that $\partial E_x/\partial x = \partial E_y/\partial y = 0$

Substitution into the divergence condition implies that $\partial E_z/\partial z$ must be zero

Hence E_z must be a constant independent of z and there are no travelling-wave solutions for E_z

A similar argument can be used to show that there are no wave solutions for H_z.

Questions Lecture 6

6.1 An optical wave of wavelength λ_0 = 0.633 μm is travelling in a material of complex relative dielectric constant ε_r = 2.25 - j10^{-8}. What are the values of the propagation constant and the absorption coefficient? Find the distance the wave must travel in the medium before the power it carries decays to 1/e of its initial value.

The propagation constant is $k = \omega\sqrt{}(\mu_0\varepsilon_0\varepsilon_r)$

If $\varepsilon_r = \varepsilon_r' - j\varepsilon_r''$ is complex, $k = k' - jk''$ must be complex too

Write k as $k = \omega\sqrt{}(\mu_0\varepsilon_0\varepsilon_r') \sqrt{}(1 - j\varepsilon_r''/\varepsilon_r')\}$

Use binomial approximation to obtain $k \approx \omega\sqrt{}(\mu_0\varepsilon_0\varepsilon_r') (1 - j\varepsilon_r''/2\varepsilon_r')$

Hence

$k' \approx \omega\sqrt{}(\mu_0\varepsilon_0\varepsilon_r')$ Real part of propagation constant

$k'' \approx k'\varepsilon_r''/2\varepsilon_r'$ Attenuation coefficient

Questions Lecture 6

The propagation constant is $k' = \omega\sqrt{(\mu_0\varepsilon_0\varepsilon_r')} = 2\pi\sqrt{\varepsilon_r'}/\lambda_0$

Here we get $k' = 2\pi \sqrt{2.25} / 0.633 \times 10^{-6} = 14.89 \times 10^{6}$ m^{-1}

The attenuation coefficient is $k'' = k'\varepsilon_r''/(2\varepsilon_r')$

Here we get $k'' = 14.89 \times 10^{6} \times 10^{8} / (2 \times 2.25) = 0.0331$ m^{-1}

The wave propagates as $E_x = E_0 \exp(-k''z) \exp(-jk'z)$

The power carried by the wave is proportional to $|E_x|^2 = E_0^2 \exp(-2k''z)$

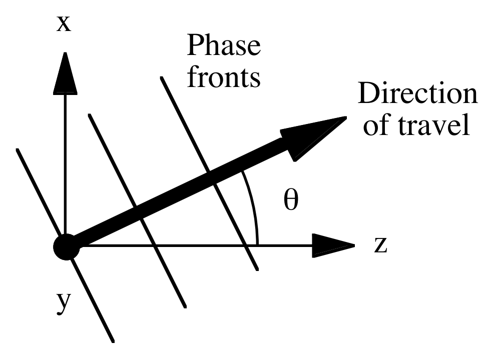
The power will therefore decay to 1/e of its initial value when $2k''z = 1$

Or when $z = 1/2k'' = 1/(2 \times 0.0331) = 15.11$ m

Questions Lecture 6

6.2 A plane electromagnetic wave is defined by $\underline{E} = E_{y0} \exp\{-jk_0(ax + bz)\}\, \underset{\sim}{j}$, where k_0 is the propagation constant of free space, and a = 1, b = √3. Find (a) the wave amplitude, (b) the direction of polarization, (c) the direction of travel, and (d) the refractive index of the medium.

Write the solution in standard form, as $\underline{E} = E_{y0} \exp\{-jk_0 n(x \sin\theta + z \cos\theta)\}\, \underset{\sim}{j}$



By inspection:

a) The wave amplitude is E_{y0}

b) The E-vector is in the $\underset{\sim}{j}$-direction, so the wave is y-polarized

c) There is no y-variation, so the wave is travelling in the x - z plane

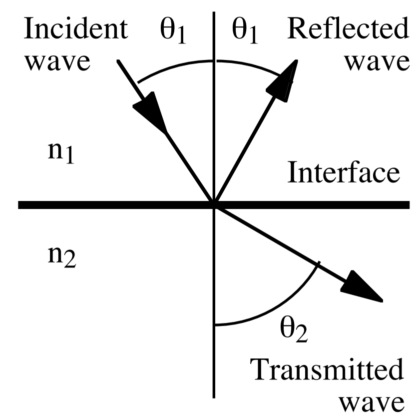
The angle of propagation relative to the x-axis is $\theta = \tan^{-1}(a/b) = \tan^{-1}(1/\sqrt{3}) = 30^o$

d) The refractive index is $n = a/\sin(\theta) = 1/0.5 = 2$

Questions Lecture 6

6.3a) Find the critical angle for an interface between a high-index glass (n_1 = 1.7) and air.

b) The critical angle for an interface between two media is 30°. Find the transmission coefficient, for normal incidence from the high-index side of the interface.



a) The critical angle is $\theta_c = \sin^{-1}(n_2/n_1)$

Here we get $\theta_c = \sin^{-1}(1/1.7) = 36.03^o$.

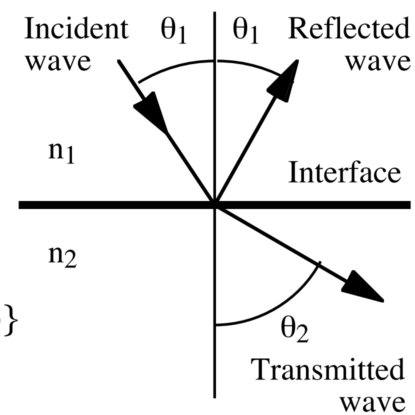
b) If the critical angle is $\theta_c = \sin^{-1}(n_2/n_1) = 30^o$ then $n_2/n_1 = 0.5$.

At normal incidence, the transmission coefficient is $T_E = 2n_1 / (n_1 + n_2) = 2 / (1 + n_2/n_1)$

Here we get $T_E = 2 \times 1/(1 + 0.5) = 1.333$.

Questions Lecture 6

6.3c) Find the power reflectivity $|\Gamma|^2$ and transmissivity $|T|^2$, for TE incidence at $\theta_1 = 30^o$ on an interface between air ($n_1 = 1$) and glass ($n_2 = 1.5$), from the air side.



c) The angle of the transmitted wave is $\theta_2 = \sin^{-1}\{(n_1/n_2) \sin(\theta_1)\}$

If the angle of incidence is $\theta_1 = 30^o$ then $\theta_2 = \sin^{-1}\{(1/1.5) \sin(30)\} = 19.47^o$.

Reflection coefficient is $\Gamma_E = \{n_1 \cos(\theta_1) - n_2 \cos(\theta_2)\} / \{n_1 \cos(\theta_1) + n_2 \cos(\theta_2)\}$

Here we get $\Gamma_E = \{0.866 - 1.5 \times 0.9428\} / \{0.866 + 1.5 \times 0.9428\} = -0.2404$.

Hence, the power reflectivity is $|\Gamma_E|^2 = 0.05779$

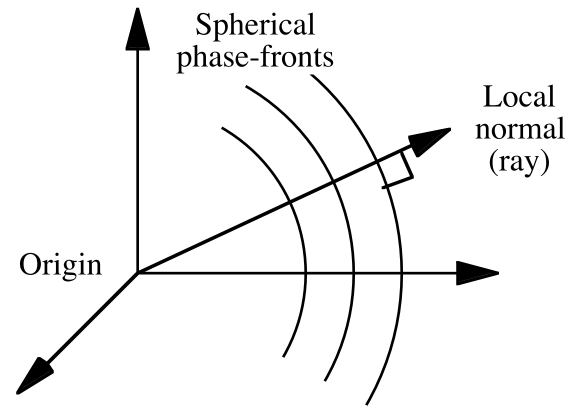
The transmission coefficient is $T_E = 2n_1 \cos(\theta_1) / \{n_1 \cos(\theta_1) + n_2 \cos(\theta_2)\}$

Here we get $T_E = 2 \times 0.866 / \{0.866 + 1.5 \times 0.9428\} = 0.7596$.

Hence, the power transmissivity is $|T_E|^2 = 0.57699$

Questions Lecture 7

7.1a) For symmetric solutions, the time-independent scalar wave equation can be written in spherical co-ordinates as $d^2E/dr^2 + (2/r)\, dE/dr + \omega^2\mu_0\varepsilon_0 E = 0$. Show that the spherical wave $E(r) = (E_0/r) \exp(-jk_0r)$ satisfies the equation, and find k_0.



If $E(r) = (E_0/r) \exp(-jk_0r)$ then:

$dE/dr = E_0 (-1/r^2 - jk_0/r) \exp(-jk_0r)$

$d^2E/dr^2 = E_0 (2/r^3 + jk_0/r^2 + jk_0/r^2 - k_0^2/r) \exp(-jk_0r)$

So:

$d^2E/dr^2 + (2/r)\, dE/dr + \omega^2\mu_0\varepsilon_0 E = E_0\{(2/r^3 + 2jk_0/r^2 - k_0^2/r) - (2/r^3 + 2jk_0/r^2) + \omega^2\mu_0\varepsilon_0/r\} \exp(-jk_0r)$

$d^2E/dr^2 + (2/r)\, dE/dr + \omega^2\mu_0\varepsilon_0 E = E_0\{-k_0^2/r + \omega^2\mu_0\varepsilon_0/r\}\exp(-jk_0r)$

The result yields zero if $k_0^2 = \omega^2\mu_0\varepsilon_0$, which requires $k_0 = \omega\sqrt{(\mu_0\varepsilon_0)}$

Questions Lecture 7

7.1b) Explain how this solution satisfies power conservation, and find the power density at a distance r due to a spherical wave source of power P.

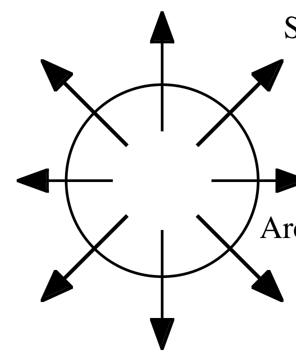
The power carried by an EM is defined by the time-averaged Poynting vector $\underline{S} = 1/2 \, \mathrm{Re} \{\underline{E} \times \underline{H}^*\}$

Where $\underline{E}$ and $\underline{H}$ are the time-independent electric and magnetic fields.

For a plane wave $\underline{S} = (E_0^2/2Z_0) \, \underline{k}$, where Z_0 is the characteristic impedance of free space.

By analogy, for a spherical wave, $\underline{S}(r) = (|E|^2/2Z_0) \, \underline{r}$.

The total power at a distance r is the integral of $\underline{S}$ over a spherical surface, or



$P = {}_A\int\int \underline{S} \, . \, d\underline{a}$

Since $\underline{S}(r)$ is entirely radial, and $|E| = E_0/r$ we then get:

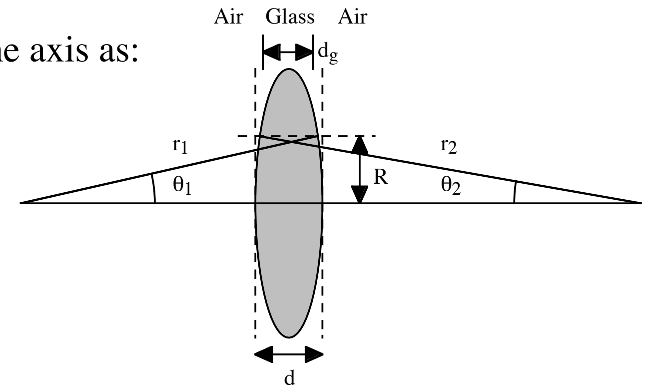
$P = (E_0^2/2Z_0 r^2) \, 4\pi r^2 = 2\pi E_0^2/Z_0$

Clearly this result is constant and must be equal to the source power.

Questions Lecture 7

7.2 Prove the lens-maker's formula $1/f = (n - 1) (1/r_1 + 1/r_2)$ for the focal length of a spherical lens.

The formula follows from the transmission function of a spherical lens.



First, work out the thickness of glass, at a distance R from the axis as:

$t = d - \{ r_1 [1 - \cos (\theta_1)] + r_2 [1- \cos (\theta_2)]\}$

Use small-angle approximations to get:

$t \approx d - (r_1\theta_1^2/2 + r_2\theta_2^2/2)$, or

$t \approx d - (R^2/2) (1/r_1 + 1/r_2)$

In travelling from z = 0 to z = d, a wave will cover a distance t in glass (refractive index n) and a distance d - t in air (unity refractive index). This yields a total optical thickness δ of:

$\delta = nt + (d - t) = (n - 1)t + d$

$\approx (n - 1) \{d - (R^2/2) (1/r_1 + 1/r_2)\} + d$

$\approx nd - (n - 1) (R^2/2) (1/r_1 + 1/r_2)$

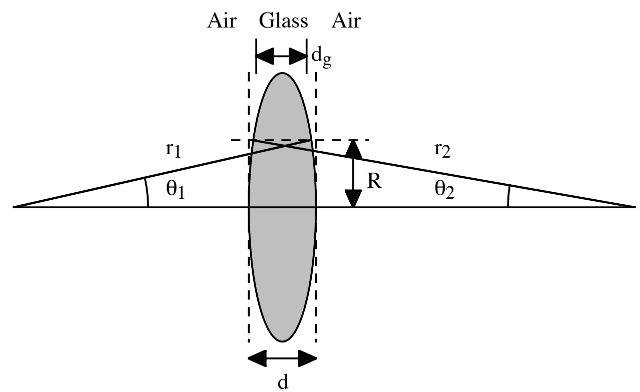
Questions Lecture 7

The lens transfer function τ_L can now be found as:

$\tau_L = \exp(-jk_0\delta)$ where $k_0 = 2\pi/\lambda$, or:

$\tau_L = \exp\{-jk_0[nd - (n - 1) (R^2/2) (1/r_1 + 1/r_2)]\}$, or:

$\tau_L = \exp(-jk_0 nd) \exp\{+jk_0(n - 1) (R^2/2) (1/r_1 + 1/r_2)]\}$



The first term $\exp(-jk_0 nd)$ is the transfer function τ_s of a slab

Hence $\tau_L = \tau_s \exp\{+jk_0(n - 1) (R^2/2) (1/r_1 + 1/r_2)\}$

If we write this as $\tau_L = \tau_s \exp(+jk_0R^2/2f)$

Then τ_L has the same form as a paraxial spherical wave, namely $E(R) \approx A \exp(+jk_0R^2/2z)$

and can describe the transformation of one paraxial wave into another.

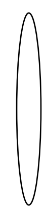
Comparing the two expressions, the focal length f satisfies $1/f = (n - 1) (1/r_1 + 1/r_2)$

Questions Lecture 7

7.3a) Find the focal length of a converging glass lens, having (i) two spherical surfaces, each of radius 200 mm, and (ii) one plane surface, and one spherical surface of radius 400 mm. Assume than n = 1.5 for the glass.

Use the lens maker's formula $1/f = (n - 1)(1/r_1 + 1/r_2)$ with $n = 1.5$.

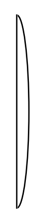
i) In this case we obtain:



$1/f = 0.5\ (1/200 + 1/200) = 1/200$

Hence the focal length is f = 200 mm.

ii) In this case we obtain:



$1/f = 0.5(1/\infty + 1/400) = 1/800$

Hence the focal length is f = 800 mm.

Questions Lecture 7

7.3b) The lenses are used in turn to form an image of a source positioned at a point 400 mm away. Find the image position in each case. What is the difference between the image types?

The imaging formula is $1/u + 1/v = 1/f$.

Hence $1/v = 1/f - 1/u$.

i) If $f = 200$ mm, $1/v = 1/200 - 1/400 = 1/400$

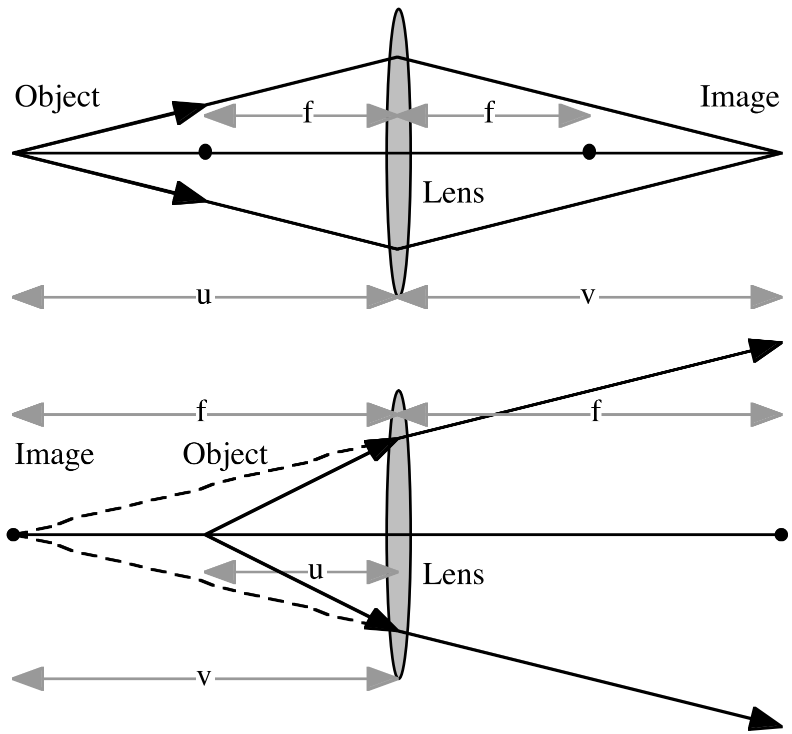
Hence the image position is $v = 400$ mm.

Since this value is positive the image is real.

ii) If $f = 800$ mm, $1/v = 1/800 - 1/400 = -1/800$

Hence the image position is $v = -800$ mm.

Since this value is negative the image is virtual.



Questions Lecture 8

8.1 a) An antenna radiates entirely isotropically in space. If the peak electric field is 50 mV m^{-1} at a distance of 1 km, estimate the power passing through a 1 m^2 area at this distance and the transmitter power.

a) The power density carried by a plane EM wave with a peak electric field E is $S = 1/2\ E^2/Z_0$

Where $Z_0 = 377\ \Omega$ is the impedance of free space

The power passing through an area A is then $P_r = S \times A$

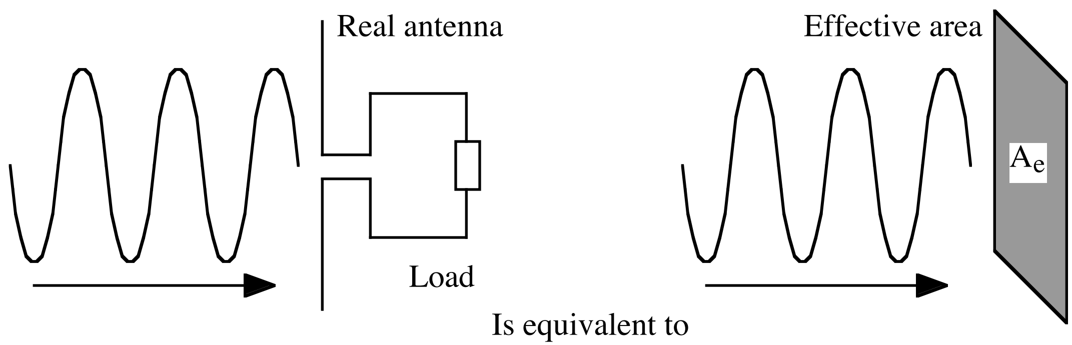
If E = 50 mV m^{-1} and A = 1 m^2, the received power is $P_r = 1/2 \times 0.05^2/377 = 3.32\ \mu W$.

Assuming the antenna radiates entirely isotropically, the power density S at a distance r is $P_t/4\pi r^2$, where P_t is the transmitter power. Re-arranging, we obtain $P_t = 4\pi r^2 S$

If $r = 10^3$ m, we then obtain $P_t = 4\pi \times (10^3)^2 \times 3.32 \times 10^{-6} = 41.7$ W.

Questions Lecture 8

8.1b) A communications link is based on microwave dish antennae with identical gains of 100 at 1 GHz. What is the effective area of the antennae, if they are 100% efficient?



The effective area A_e of an antenna is its equivalent receiving area

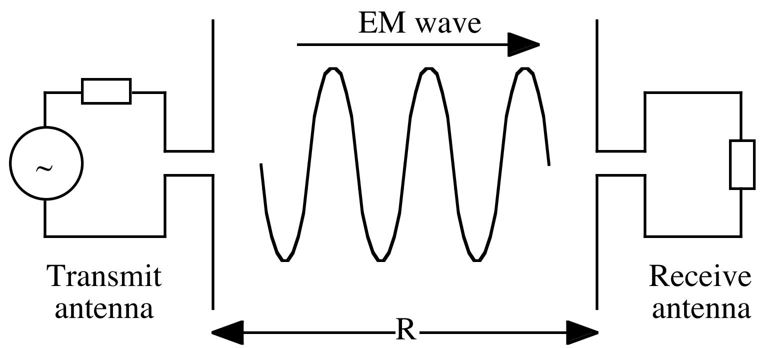
Its general expression is $A_e = \lambda^2 G/4\pi\eta$

Where λ is the wavelength, G is the antenna gain and η is the antenna efficiency.

At 1 GHz, $\lambda = 0.3$ m. If $\eta = 1$ and G = 100, the effective area is $A_e = 0.3^2 \times 100 / 4\pi = 0.72$ m^2.

Questions Lecture 8

8.1c) How much power will be received at a distance of 10 km, if the transmitter power is 10 kW?



The Friis formula is used in link budget calculations.

The received power P_r in a link is $P_r = P_t\, G_t G_r (\lambda/4\pi R)^2$

Where P_t is the transmit power, G_t and G_r are the gains of the transmit and receive antennae,

λ is the wavelength and R is the distance.

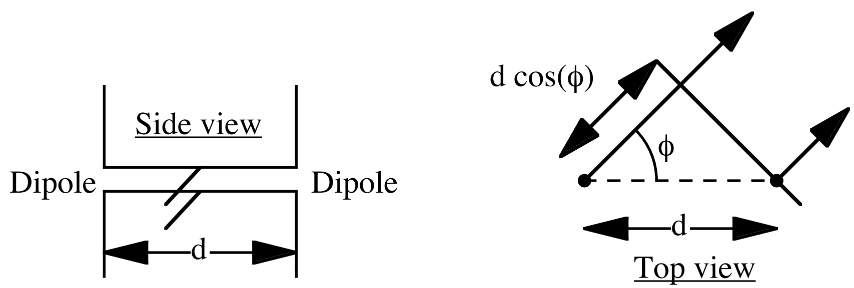
If $P_t = 10^4$ W, $G_t = G_r = 100$ and $R = 10^4$ m

Then $P_r = 10^4 \times 100^2 \times (0.3 / 4\pi \times 10^4)^2 = 0.57$ mW

Questions Lecture 8

8.2 A two-element dipole-based antenna array has an element spacing of 1 m. Calculate and sketch polar diagrams of the normalised radiation pattern in a plane perpendicular to the dipole elements at frequencies of a) 10 MHz, b) 100 MHz, c) 150 MHz, and d) 200 MHz.

For a two-element array, the field $E'(\phi)$ in a plane perpendicular to the dipole elements can be found from the individual dipole field $E(\phi)$ as $E'(\phi) = E(\phi) \{1 + \exp[-jkd \cos(\phi)]\}$, where d is the element spacing.



Rearranging, we get $E'(\phi) = 2E(\phi) \exp\{-jkd \cos(\phi)/2\} \cos\{kd \cos(\phi)/2\}$

Since $E(\phi)$ is constant for a dipole, the normalised radiation pattern is then $F'(\phi) = \cos^2\{kd \cos(\phi)/2\}$

The shape of the radiation pattern depends entirely on the value of $kd = 2\pi d/\lambda$.

Questions Lecture 8

The normalised radiation pattern is $F'(\phi) = \cos^2\{(2\pi d/\lambda)\cos(\phi)/2\}$

When $d/\lambda \ll 1$, the argument of the cosine is small, and $F'(\phi) \approx 1$. The radiation pattern is then isotropic.

Significant differences appear when $d/\lambda \approx 1$ and the radiation pattern forms main and side lobes.

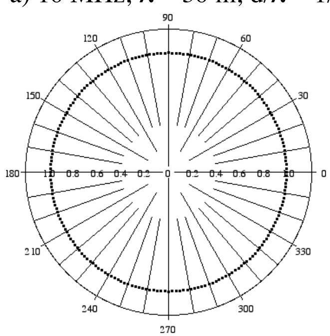
$F'(\phi)$ is always unity when $\phi = \pi/2$. When $d/\lambda = 1/2$, $F'(\phi)$ must be zero for $\phi = 0$.

Similarly, when $d/\lambda = 1$, $F(\phi)$ must be 1 for $\phi = 0$.

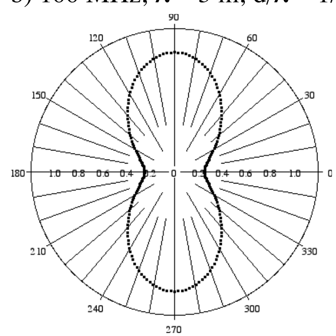
As the frequency rises, the array stares in the direction $\phi = \pi/2$, i.e. broadside on.

However if d/λ is too big, sidelobes appear

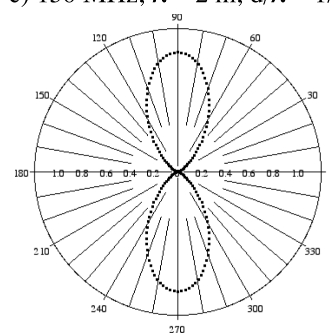
a) 10 MHz; λ = 30 m; $d/\lambda = 1/30$



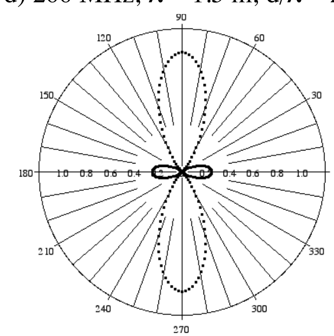
b) 100 MHz; λ = 3 m; $d/\lambda = 1/3$



c) 150 MHz; λ = 2 m; $d/\lambda = 1/2$



d) 200 MHz; λ = 1.5 m; $d/\lambda = 2/3$



Questions Lecture 8

8.3 The photograph below shows a WWII German night-fighter. Based on your assessment of the antenna, discuss the performance of the radar system and of the aircraft.

The antenna is cumbersome, with a number of relatively long dipoles forming each antenna array. Based on the size of the pilot, who is just visible, each dipole is around 2 m long. If these are half-wave dipoles, the wavelength is therefore around 4 m and the frequency is $3 \times 10^8 / 4 = 75$ MHz. Consequently we may assume that Germany failed to develop centimetric radar, probably due to a lack of a high-power magnetron source. Because the number of dipoles is so small, the directivity of the array will have been poor, and the forward gain small. The range will have been short, and the aircraft will have flown badly.

Questions Lecture 9

9.1 The conductivity of copper is 5.8×10^7 S/m. Calculate the skin depth at frequencies of a) 50 Hz, b) 50 kHz and c) 50 MHz. Hence, estimate the corresponding resistance per metre of 1 mm wire.

The skin depth is $\delta = 1/\sqrt{(\pi f \mu_0 \sigma)}$, where f is the frequency, σ is the conductivity and $\mu_0 = 4\pi \times 10^{-7}$.

DC resistance per metre can be calculated by assuming that the current is uniformly distributed over the cross-section as $R_{DC} = 1/\sigma\pi r^2$, where r is the radius of the wire.

AC resistance per metre can be estimated by assuming that the current is carried within a skin depth from the surface as $R_{AC} = 1/\sigma 2\pi r\delta$. The effective resistance is the larger of the two.

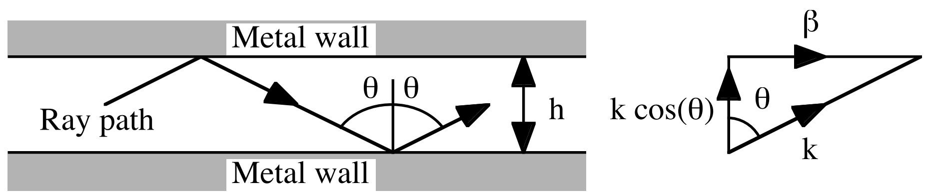
If $r = 5 \times 10^{-3}$ m and $\sigma = 5.8 \times 10^7$ S/m the following data are obtained.

Skin effects are negligible at the lower frequencies.

Frequency (Hz)	Skin depth δ	R_{DC} (Ω)	R_{AC} (Ω)
50	9.35 mm	0.022	0.00059
50×10^3	0.29 mm	0.022	0.019
50×10^6	9.35 μm	0.022	0.59

Questions Lecture 9

9.2a) Find the cutoff frequency of a planar microwave guide, formed from a metallised slab of dielectric with a thickness of 0.5 μm and a refractive index of 1.5.



For a ray angle of θ, the propagation constant is $\beta = k \sin(\theta)$

The eigenvalue equation is $2kh \cos(\theta) = 2\nu\pi$, where $k = k_0 n$ and $k_0 = 2\pi/\lambda$.

Cutoff occurs when θ tends to zero, so that the ray makes no forward progress. This occurs when $kh = \nu\pi$.

The cutoff wavelength of the ν^{th} mode is then $\lambda_\nu = 2nh/\nu$. The corresponding frequency is $f_\nu = c\nu/2nh$.

Since the lowest mode number is 1, all modes are cut off when $f < c/2nh$.

Here, the lowest operating frequency is $f = 3 \times 10^8 / (2 \times 1.5 \times 0.5 \times 10^{-6}) = 2 \times 10^{14}$ Hz, or 200,000 GHz.

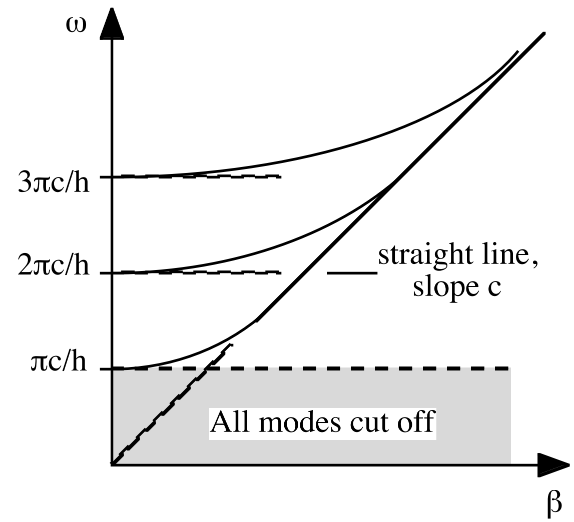
However, losses in the metal impose a practical limit far below this value.

Questions Lecture 9

9.2b) Sketch the dispersion diagram for modes in a metal-walled, air-spaced microwave waveguide.

The dispersion relation is $\beta = k_0 \sqrt{\{1 - \cos^2(\theta)\}}$

Since $\cos(\theta) = \nu\pi/k_0h$, and $k_0 = \omega/c$, $\beta = (\omega/c) \sqrt{\{1 - (\nu\pi c/h\omega)^2\}}$



The dispersion diagram can be sketched as follows.

First calculate the asymptotes

For large ω, β tends to ω/c.

A plot of ω against β is then a straight line with slope c

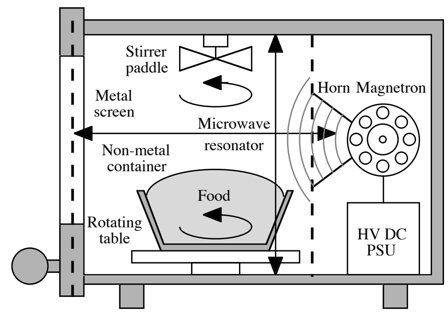
Modes cut off (i.e., β tends to zero) when $\omega = \nu\pi c/h$.

Curves therefore intercept the ω axis at $\pi c/h$, $2\pi c/h$, etc.

Then join the asymptotes with smooth curves

Questions Lecture 9

9.3a) Why is your food rotated on a turntable inside a microwave oven? What is the 'stirrer paddle' used for?



Food is rotated on a turntable in a microwave oven to try to ensure even heating. The oven is a cavity resonator, and so contains a standing wave pattern. This pattern must have nodes and anti-nodes, and consequently regions where the electric field intensity is zero. Rotation helps to ensure that none of the food spends the entire time in one of the field zeros. The metal stirrer paddle is used to try and disrupt the standing wave pattern, for exactly the same reason. (NB: It is not used to stir the air in the oven.)

Questions Lecture 9

9.3b) Why might it be dangerous to operate the oven with the door open? What might happen if a metal food container was used, or if the food was completely dry?

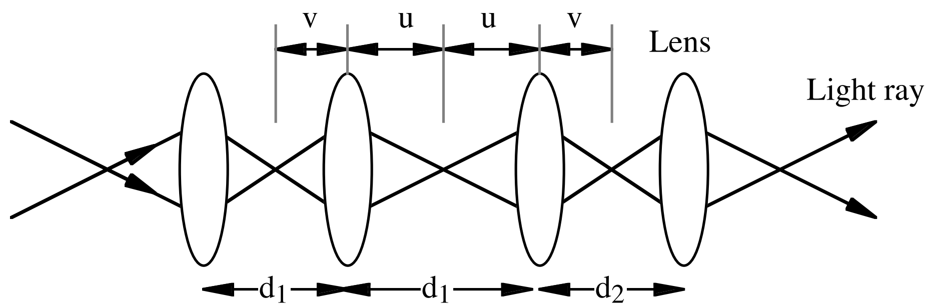
The microwave oven operates by absorption of RF energy by water molecules. Since the human body is mainly composed of water, the user will also be cooked if the oven is operated with the door open. Consequently all ovens have a door-interlock.

The RF energy produced by the magnetron source is clearly supposed to be dissipated in food during the cooking process. If a metal container is used, or if the food is dry, there is a chance that significant energy will be reflected back to the source instead, possibly resulting in a fire.

Questions Lecture 10

10.1 A sequence of identical lenses, each of focal length f, are separated by alternating distances d_1 and d_2. Find a relation between d_1 and d_2 that must be satisfied to obtain a lens waveguide. Hence find the value of d_2 needed if d_1 = 50 cm when using lenses of focal length 10 cm.

The lens waveguide geometry must be as shown below. Hence $d_1 = 2u$, $d_2 = 2v$.



Now, the imaging formula requires that $1/u + 1/v = 1/f$.

Consequently, the relation that must be satisfied is $2/d_1 + 2/d_2 = 1/f$ or $1/d_1 + 1/d_2 = 1/2f$.

If d_1 =50 cm and f = 10 cm, then $1/50 + 1/d_2 = 1/20$, so $1/d_2 = 3/100$ and d_2 = 33.3 cm.

Questions Lecture 10

10.2a) What are the advantages of fibre optic communications? What wavelengths are used for transmission in silica optical fibres, and why?

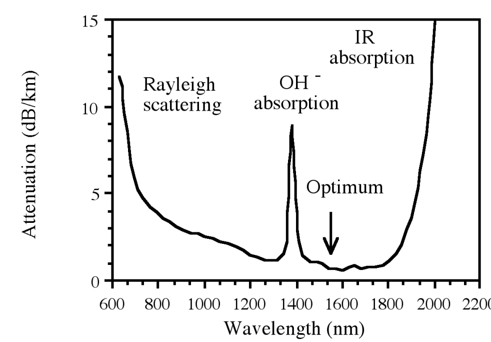
The advantage of optical fibre communication is that diffraction effects may be avoided entirely, even when the beam diameter is very small, since the beam is confined inside a waveguide.

The fibre material may be purified so that absorption lines due to contaminants may be largely eliminated. The main sources of loss are then Rayleigh scattering at short wavelengths and molecular absorption in the mid-infrared (around $\lambda = 10$ μm).

The former reduces as the wavelength increases from the visible

The latter reduces as the wavelength decreases from mid IR

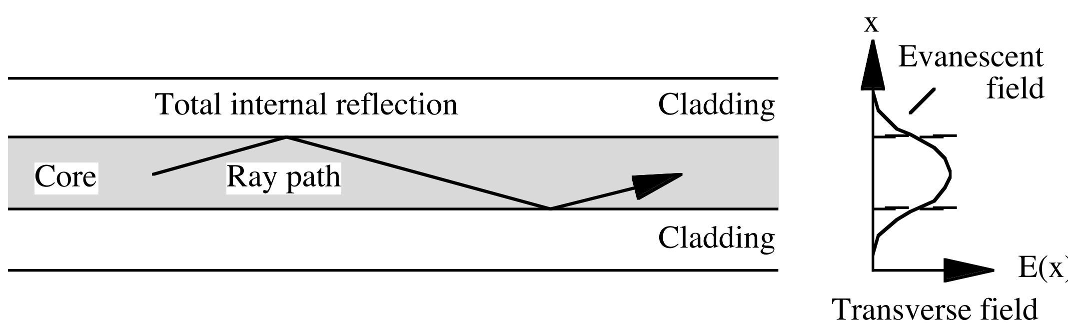
Silica-based fibres are operated at around 1.55 μm, where the combined effect of the two loss mechanisms is at a minimum.



Questions Lecture 10

10.2b) A multi-mode fibre has core and cladding indices of 1.5 and 1.499, respectively. Find the maximum data transmission rate that it can support (expressed as a bit-rate : length product).

Modes far from cutoff have most of their energy inside the core, so their group velocity will be roughly equal to the phase velocity in a medium of index n_{Core}, namely c/n_{Core}. Similarly, modes close to cutoff will have $v_g \approx c/n_{Clad}$



Questions Lecture 10

In a distance L, the time-spread of a signal is then:

$\Delta t = L(1/v_{gmin} - 1/v_{gmax}) \approx (L/c)\ (n_{Core} - n_{Clad})$.

In a digital communications system, pulses of duration T are normally separated by gaps of T. Intersymbol interference will occur when consecutive pulses start to overlap, i.e. when $\Delta t \approx T$. Consequently, intermodal dispersion limits the maximum bit rate to $B \approx 1/(2\Delta t)$.

Expressed as a bit-rate: length product, this is $BL \approx c/\{2(n_{Core} - n_{Clad})\}$.

For $n_1 = 1.5$, $n_2 = 1.499$, $BL \approx 150$ Mbit/s km.

Questions Lecture 10

10.3a) A semiconductor laser of length 250 μm is formed in a material of refractive index 3.5. What value of gain coefficient is needed to achieve lasing?

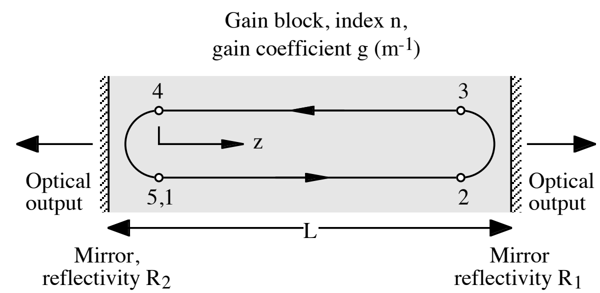
In the linear model of a laser, the condition for oscillation is:

$\exp(-j2\beta L) \exp(2gL) R_1R_2 = 1$

This result can be split into separate gain and phase conditions:

Gain: $\exp(2gL) R_1R_2 = 1$, so $g = (1/2L) \log_e(1/R_1R_2)$

Phase: $2\beta L = 2\nu\pi$, so $2(2\pi n/\lambda)L = 2\nu\pi$ and $\lambda = 2nL/\nu$



In a semiconductor laser, identical mirrors are formed by cleaving

The reflectivity is $R_1 = R_2 = (n - 1)/(n + 1)$, where n is the refractive index

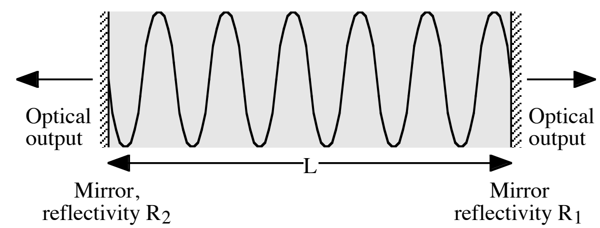
If $n = 3.5$, $R_1 = R_2 = 2.5/4.5 = 0.555$.

Using the gain condition $g = \{1/(2 \times 250 \times 10^{-6})\} \log_e(1/0.555^2) = 2351$ m^{-1}.

Questions Lecture 10

10.3b) What is the spectral separation of the emitted wavelengths, near 1.5 μm wavelength?

Using the phase condition $\lambda = 2nL/\nu$



If the wavelength of the mode with a given ν is $\lambda_\nu = 2nL/\nu$

Then the wavelength of the adjacent mode is $\lambda_{\nu+1} = 2nL/(\nu + 1)$

The mode separation is $\lambda_\nu - \lambda_{\nu+1} = 2nL\{1/\nu - 1/(\nu + 1)\} = 2nL/\{\nu(\nu + 1)\}$

If n >> 1 (true for L >> λ), we can approximate this as $\lambda_\nu - \lambda_{\nu+1} \approx 2nL/\nu^2 \approx \lambda_\nu^2/2nL$

If $\lambda_\nu = 1.5$ μm, n = 3.5 and L = 250 x 10^{-6}:

$\lambda_\nu - \lambda_{\nu+1} \approx (1.5 \times 10^{-6})^2 / (2 \times 3.5 \times 250 \times 10^{-6}) = 1.29 \times 10^{-9}$ m, or 1.29 nm