1 Cross-Correlation of Disturbance Input and Measurement Noise

Contents of this proof are adopted from section 4.4 Correlated Disturbance Inputs and Measurement Noise, pp. 361-364 of [1] and reproduced following the notation introduced in "Non-Linear State Estimation for Humanoid Walking" by S. Piperakis, M. Koskinopoulou, and P. Trahanias.

Assume the linearized discrete-time system:

$$x_k = F_{x_{k-1}} x_{k-1} + F_{u_{k-1}} u_{k-1} + w_{k-1}$$
$$y_k = H_{x_k} x_k + H_{u_k} u_k + n_k$$

where x_k is the state, u_k is the input, w_k is zero mean Gaussian noise with covariance Q_k (process noise), y_k is the output, n_k is zero mean Gaussian noise with covariance R_k (measurement noise), and F_{x_k}, F_{u_k}, H_{x_k}, H_{u_k} are the linearizations of $f(x_k, u_k)$ and $h(x_k, u_k)$ with respect to x_k and u_k respectively.

A linear minimum-variance filter for this system takes the form:

$$\hat{x}_k(-) = F_{x_{k-1}} \hat{x}_{k-1}(+) + F_{u_{k-1}} u_{k-1}$$
$$\hat{x}_k(+) = \hat{x}_k(-) + K_k(y_k - H_{x_k} \hat{x}_k - H_{u_k} u_k)$$

where K_k is the optimal filter gain to be determined, $\hat{x}_k(-)$ is the state estimate before the measurement update and $\hat{x}_k(+)$ is the estimate after the update. Defining the equivalent state residuals as:

$$e_k(-) = x_k - \hat{x}_k(-)$$
$$e_k(+) = x_k - \hat{x}_k(+)$$

the dynamics of the estimation error can be described by:

$$e_k(-) = F_{x_{k-1}} e_{k-1}(+) + w_{k-1} \tag{1.1}$$
$$e_k(+) = e_k(-) - K_k(H_{x_k} e_k(-) + n_k) \tag{1.2}$$

Thus, the known control input has no effect on the estimation error, therefore H_{x_k} is rewritten as H_k for simplicity in the following.

The disturbance input and measurement noise are modeled as a time-skewed, white joint stochastic process such that:

$$E\left\{\begin{bmatrix} w_{k-1} \\ n_k \end{bmatrix} \begin{bmatrix} w_{k-1}^\top & n_k^\top \end{bmatrix}\right\} = \begin{bmatrix} Q_{k-1} & M_k \\ M_k^\top & R_k \end{bmatrix}$$

and

$$E\begin{bmatrix} w_{k-1} \\ n_k \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

M_k expresses the cross-correlation between the disturbances and the measurement at discrete time k.

The covariances of the state estimate error before and after the measurement update are expressed by forming the outer products of the corresponding state residuals (Eqs.1.1,1.2) and taking the expected values:

$$\begin{aligned}
E[e_k(-)e_k(-)^\top] &= P_k(-) \\
&= E[F_{x_{k-1}}e_{k-1}(+)e_{k-1}(+)^\top F_{x_{k-1}}^\top + w_{k-1}w_{k-1}^\top] \\
&= F_{x_{k-1}}P_{k-1}(+)F_{x_{k-1}}^\top + Q_{k-1} \qquad (1.3)\\
E[e_k(+)e_k(+)^\top] &= P_k(+) \\
&= E[(e_k(-) - K_k(H_k e_k(-) + n_k)(e_k(-) - K_k(H_k e_k(-) + n_k)^\top] \\
&= P_k - E[e_k(-)e_k(-)^\top H_k^\top K_k^\top + e_k(-)n_k^\top K_k^\top] \\
&- E[K_k H_k e_k(-)e_k(-)^\top + K_k n_k e_k(-)^\top] \\
&+ E[K_k H_k e_k(-)e_k(-)^\top H_k^\top K_k^\top \\
&+ K_k H_k e_k(-)n_k^\top K_k^\top + K_k n_k e_k^\top(-)H_k^\top K_k^\top \\
&+ K_k n_k n_k^\top K_k^\top] \qquad (1.4)
\end{aligned}$$

Substituting Eq.1.1 in Eq.1.4 and moving the expectation operation inside the deterministic system matrices, the postupdate covariance can be written in the Joseph form as:

$$\begin{aligned}
P_k(+) =&(I - K_k H_k)P_k(-)(I - K_k H_k)^\top + K_k R_k K_k^\top + K_k(H_k M_k + M_k^\top H_k^\top)K_k^\top \\
&- M_k K_k^\top - K_k M_k^\top \qquad (1.5)
\end{aligned}$$

The optimal gain matrix that minimizes the expected value of the state-residual squared at each step:

$$J_k = E[e_k(+)^\top e_k(+)] = Tr[P_k(+)] \qquad (1.6)$$

Consequently,

$$\frac{\partial J_k}{\partial K_k} = 2\left(K_k\left(H_k P_k(-)H_k^\top + H_k M_k + M_k^\top H_k^\top + R_k\right) - P_k(-)H_k^\top - M_k\right) = 0 \qquad (1.7)$$

and the optimal gain is:

$$K_k = \left(P_k(-)H_k^\top + M_k\right)\left(H_k P_k(-)H_k^\top + H_k M_k + M_k^\top H_k^\top + R_k\right)^{-1} \qquad (1.8)$$

Using this optimal gain in Eq.1.5, the updated state error covariance matrix is:

$$\begin{aligned}
P_k(+) =&P_k(-) - \left(P_k(-)H_k^\top + M_k\right)\left(H_k P_k(-)H_k^\top + H_k M_k + M_k^\top H_k^\top + R_k\right)^{-1} \\
&\left(H_k P_k(-) + M_k^\top\right) \\
=&P_k(-) - K_k\left(M_k^\top + H_k P_k(-)\right) \qquad (1.9)
\end{aligned}$$

Notice if M_k is zero the expression in Eq.1.8 reduces to the standard Kalman Gain of the Extended Kalman filter (EKF) and Eq.1.9 to the updated covariance estimate of the EKF.

REFERENCES

[1] Robert F. Stengel, "OPTIMAL CONTROL AND ESTIMATION", Dover Publications, New York, 1994. (originally published as STOCHASTIC OPTIMAL CONTROL; Theory and Application, J. Wiley & Sons, New York, 1986.)