PROBABILISTIC INFERENCE, GENERATIVE MODELS AND HIDDEN VARIABLES

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Yandex School of Data Analysis

PROBABILISTIC INFERENCE

- Mr. White has two children. What is the probability that both children are boys?

EQUAL OPPORTUNITIES

- Mr. White has two children. What is the probability that both children are boys?
- Mr. Jones has two children. The older child is a boy. What is the probability that both children are boys?

EQUAL OPPORTUNITIES

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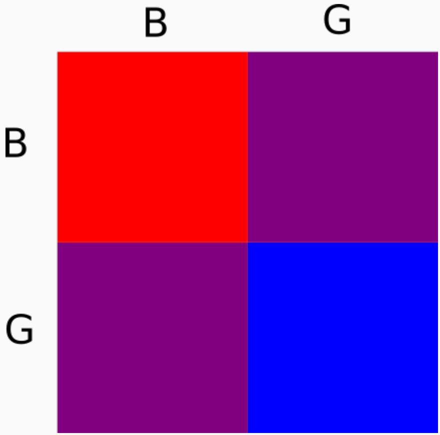
- Mr. Jones has two children. The older child is a boy. What is the probability that both children are boys?
- Mr. Smith has two children. One of them is a boy. What is the probability that both children are boys?

PRIOR PROBABILITY



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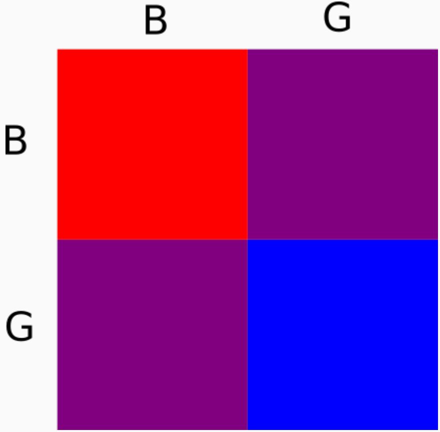

$\Pr(BB) = \frac{1}{4}$

CONDITION ON EVENT 'THE OLDER CHILD IS A BOY'

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$\Pr(BB|C_1 = B) = \frac{1}{2}$

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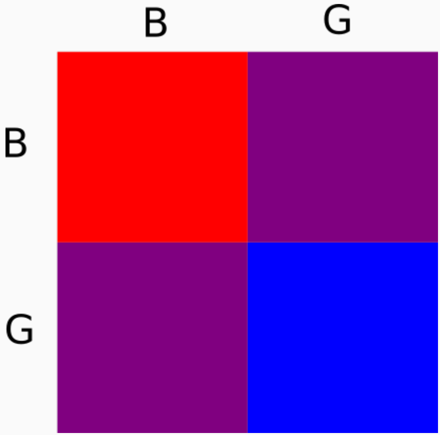

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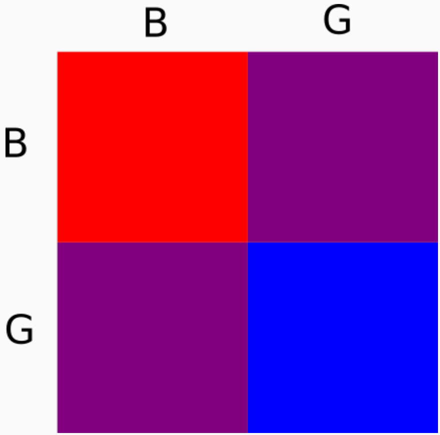

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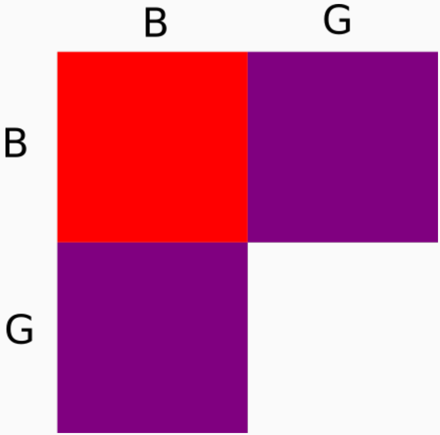

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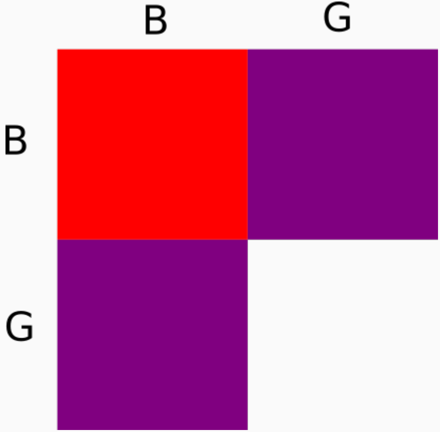

$\Pr(BB|C_1 = B \cup C_2 = B) = \frac{1}{3}$

Mr. Brown has two children. One of them is a boy born on a Tuesday. What is the probability that he has two boys?

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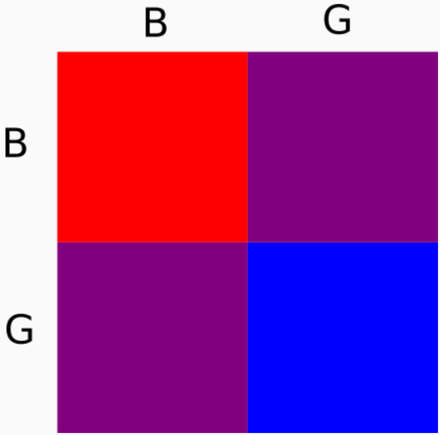

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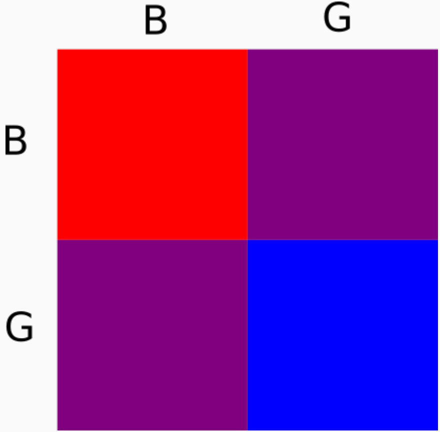

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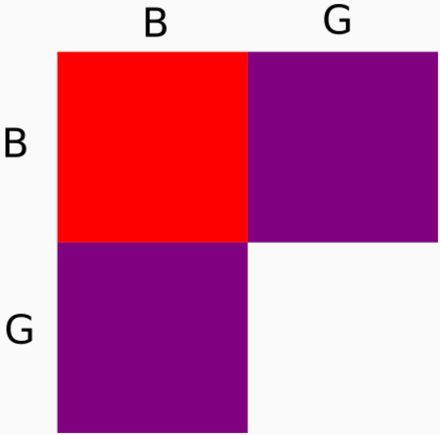

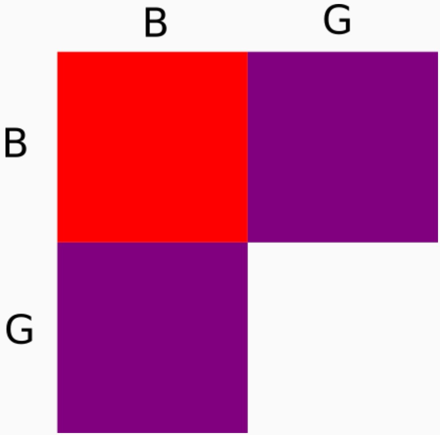

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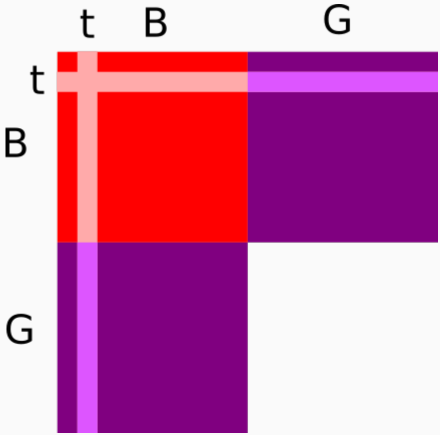

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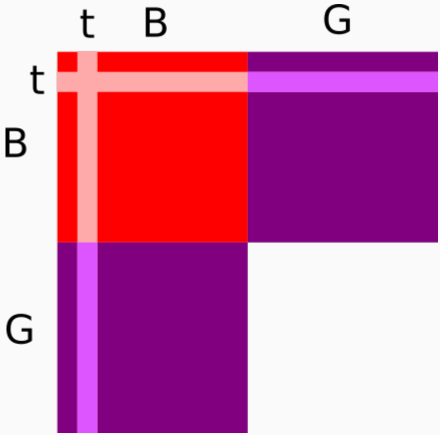
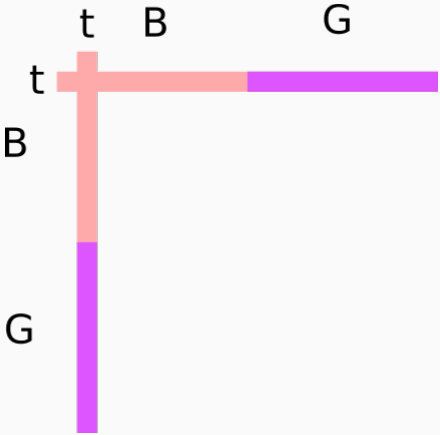


$$\Pr(BB|C_1 = B_{tue} \cup C_2 = B_{tue}) = \frac{13}{27} \approx \frac{1}{2}$$



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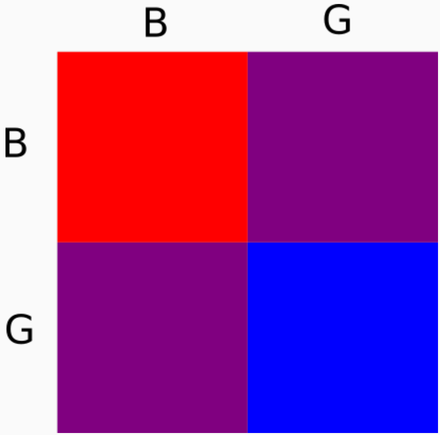

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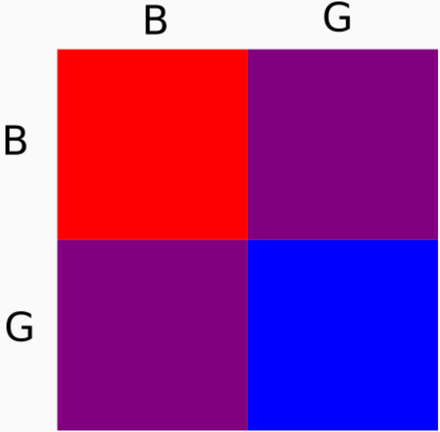

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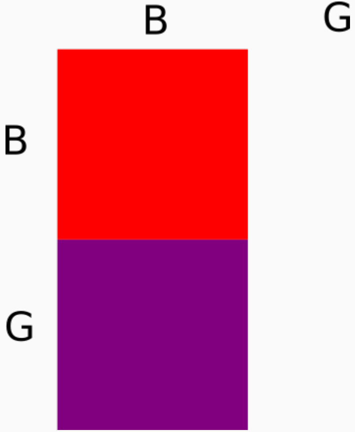

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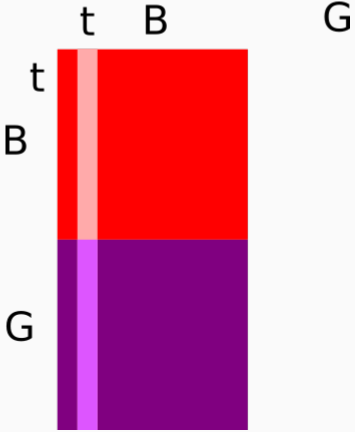

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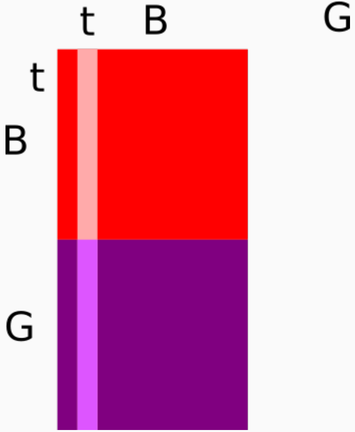

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SOME LESSONS

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- There are no paradoxes in probability, only badly posed questions :)

COINS

- You flip a coin 10 times and see 8 heads and 2 tails

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- What's the probability that on the next flip you see a head?

COINS

- You flip a coin 100 times and see 78 heads and 22 tails

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- You flip a coin 100 times and see 78 heads and 22 tails
- What's the probability that on the next flip you see a head?

- You flip a coin 1,000 times and see 811 heads and 189 tails

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- What assumptions are you making? Are they reasonable?

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Two events are *conditionally independent* given a third event E_3, if they are indepdendent when E_3 is observed.

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INDEPENDENCE AND CONDITIONAL INDEPENDENCE

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In otherwords, E_3 explains E_1 and E_2 such that they no longer provide any additional information about each other.

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- Flipping a single coin twice.
- Flipping two different coins once each.
- The time at which two students arrive at class.
- The amount of time you study and your results on the exam.

- You're given a bag with two coins in it C_1 and C_2. They look identical but ...

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while

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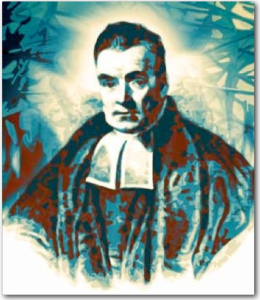
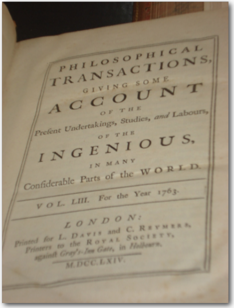
- You pick a coin at random from the bag. How would you determine which coin you had selected?
- How many times would you need to flip the coin?

REV. BAYES





How could this guy help you?

Given a prior $P(C_1)$ and an observation H_1 infer the posterior $P(C_1|H_1)$

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Flip the coin again and update your belief taking the posterior $P(C_1|H_1)$ as your new prior.

$$P(C_1|H_1, H_2) \propto P(C_1|H_1)P(H_2|C_1)$$

Take a look at `bayesian_updating.ipynb`

GENERATIVE MODELS

Generative models model a joint distribution over inputs X and outputs Y

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Describing a generative model often involves stating the parametric form and independence assumptions it makes.

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with parameters

$$\theta_{blue} := P(Y = blue) \qquad \text{etc.}$$

and

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Is this the *true* model for this process? How would you estimate these parameters from data?

MODELS IN NLP

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EXAMPLES OF GENERATIVE MODELS IN NLP

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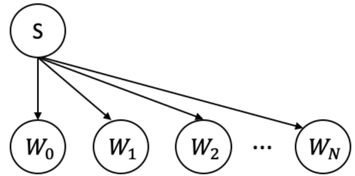
- Word alignment models in machine translation

$$P(W, S) = P(S) \prod_i P(W_i | S)$$



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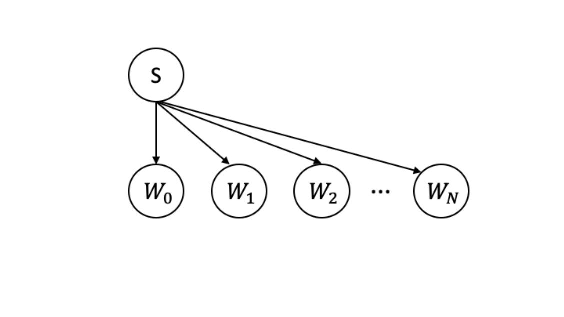

How can this possibly work for spam classification?

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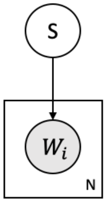

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BIGRAM MODEL (SLIGHTLY LESS NAIVE BAYES)

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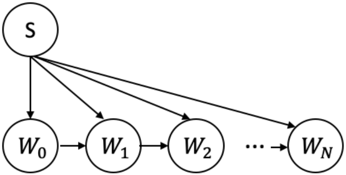

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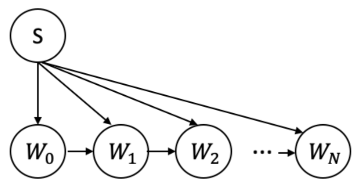

Why might this be even worse for spam classification?

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MAKING INDEPENDENCE ASSUMPTIONS

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  - probability of target sentence given source sentence

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Can you see any connection between *attention* and *independence assumptions*?

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Speech recognition, spelling correction, machine translation, swipe etc.

NOISY CHANNEL EXAMPLE: SPELLING CORRECTION

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Score hypotheses using prior $P(Y)$ and inverse model $P(X|Y)$

$$P(Y|X) \propto P(Y = \text{this is how})P(X = \text{htis is hw}|Y = \text{this is how})$$

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$$P(\text{htis is hw}|\text{this is how}) = P(\text{htis}|\text{this})P(\text{is}|\text{is})P(\text{hw}|\text{how})$$

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What could we train $P(Y)$ on? How about $P(X|Y)$?

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- Model of $P(Y)$ can compensate for these assumptions

HIDDEN VARIABLE MODELS

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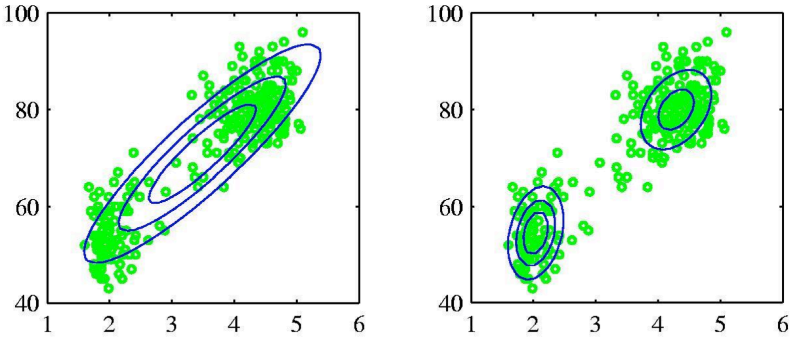
- Hidden variables can help cluster the data (e.g. share statistical strength)

GAUSSIAN MIXTURE MODELS



Single Gaussian

Mixture of two Gaussians

Generative model

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How does a mixture model improve on a single Gaussian model?

Discrete generative models for grouped data

- Bag of words model - each word is independent

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- All documents generated from a single distribution $P(W)$

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- The assumption $p(W|D, Z) = p(W|Z)$ forces topics to explain word cooccurences

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- Each document is conditionally independent given its hidden distribution over topics θ

TOPIC MIXTURE MODEL

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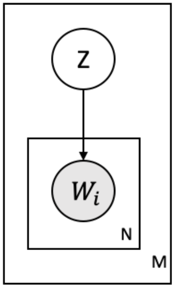
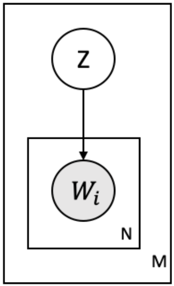
Each document has a single (hidden) topic

Model each *document* as having a single hidden topic Z



Each topic Z indexes a distribution over words.



LATENT DIRICHLET ALLOCATION

- Sample a distribution over topics for a document

$$\theta = (\theta_1, \theta_2, \ldots, \theta_K) \sim \text{Dirichlet}(\alpha)$$

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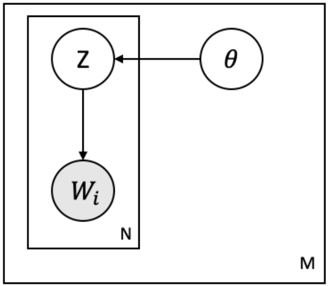
$$Z_i = \Pr(Z_i = z) = \theta_z$$

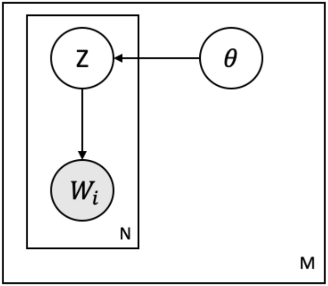
- Generate a word W according to the unigram distribution indexed by Z

$$W_i = \Pr(W_i = w|Z = z) = \beta_{z,w}$$

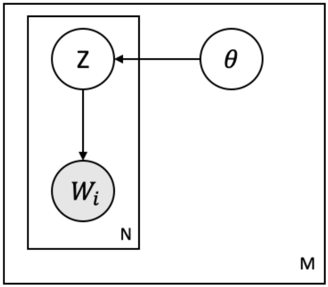
Model each *word* as having a single hidden topic.



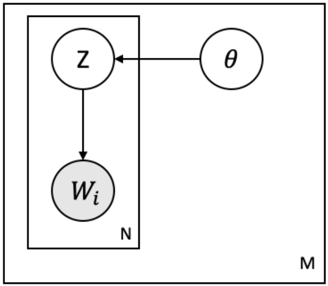
Each document has a hidden *distribution* over topics.



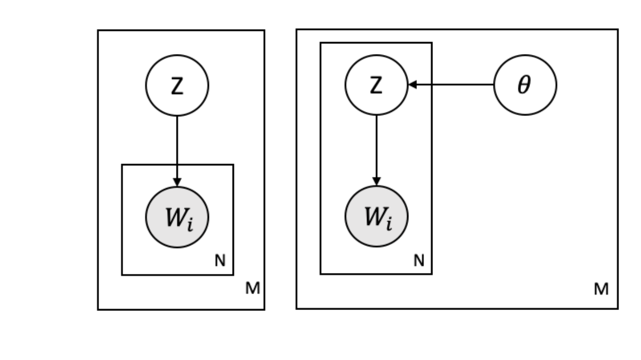
Topics can be shared across all documents



Documents are generated from multiple topics



MIXTURE VS. LDA MODEL



- Bigram language model (no hidden variables)

$$\Pr(w_t|w_{t-1},\ldots,w_0) \approx \Pr(w_t|w_{t-1})$$

- Class-based language model

$$\Pr(w_t|w_{t-1},\ldots,w_0) \approx \Pr(w_t|C(w_{t-1}))$$

Similar to a topic mixture model but trained on bigram data.

PARAMETER ESTIMATION

MAXIMUM LIKEHOOD PRINCIPLE

Choose parameters θ etc. s.t. *likelihood* of the data X, Y is maximized, i.e.

$$\theta^* = \operatorname*{argmax}_\theta \Pr(X, Y|\theta).$$

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How could you justify this method for choosing parameters?

We observed a sample D drawn from $(x, y) \in (X, Y)$ where $X \in \{H, T\}$, $Y = \{Red, Blue\}$. Each observation was labeled so

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$$\begin{aligned}\hat{\theta}_{mle} &= \underset{\theta}{\operatorname{argmax}} \sum_{(x,y)\in D} \log \Pr(X = x, Y = y|\theta) \\ &= \underset{\theta}{\operatorname{argmax}} \sum_{(x,y)\in(X,Y)} \#(X = x, Y = y) \log \Pr(X = x, Y = y|\theta)\end{aligned}$$

where we summarized the data using the *sufficient statistics*.

MAXIMIMUM LIKELIHOOD ESTIMATES FOR OUR MODEL

$$\hat{\theta}_{blue} = \frac{\#(B)}{\#(B) + \#(R)}$$

$$\hat{\theta}_{head\_blue} = \frac{\#(H, B)}{\#(B)}$$

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Sufficient statistics summarize all the information about a sample that can influence our estimate of the parameters.

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- You are given a bag with red and blue coins in it
- But the colours have washed off
- You don't know the proportions of red to blue
- Estimate the proportions of red and blue coins and the probability of heads for each coin

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MLE FOR MODELS WITH HIDDEN VARIABLES

- Two unknowns: hidden variables and parameters (Z, θ)
- If we knew Z, we could use MLE to estimate θ
- If we knew θ, we could use Bayes' rule to infer Z

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Intuition: if we knew θ we could just infer Z, likewise if we knew Z we could just estimate θ. Since we don't know either, just guess and iteratively improve.

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where $\delta(x, y) = 1 \iff x = y$ otherwise 0.

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This term is known as the *expected log-likelihood*.

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EXPECTATION MAXIMIZATION

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EM MAXIMIZES A BOUND ON THE OBSERVED LIKELIHOOD

$$
\begin{aligned}
\log \Pr(X|\theta) &= \log \sum_Z \Pr(X|\theta)\Pr(Z|X,\theta) \\
&= \log \sum_Z q(Z) \frac{\Pr(X|\theta)\Pr(Z|X,\theta)}{q(Z)} \\
&\geq \sum_Z q(Z) \log \frac{\Pr(X|\theta)\Pr(Z|X,\theta)}{q(Z)} \\
&= \sum_Z q(Z) \log \Pr(X|\theta) - \sum_Z q(Z) \log \frac{q(Z)}{\Pr(Z|,X,\theta)} \\
&= \log \Pr(X|\theta) - KL(q(Z)||\Pr(Z|,X,\theta))
\end{aligned}
$$

which implies that if $q(Z) = \Pr(Z|,X,\theta)$ the bound is tight.

Take a look at `mle_em_seminar.ipynb`