Uniform / Rectangular Distribution :

A continuous random variable X is said to have a uniform (or rectangular) distribution over (a,b) if its pdf is given by,

$$f(x) = \text{constant, say } k, \quad x \in (a,b)$$

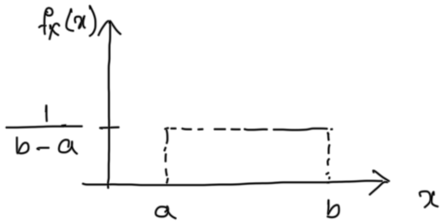
$$1 = \int_a^b f(x)\,dx \Rightarrow k\int_a^b dx = 1$$

$$\Rightarrow k = \frac{1}{b-a}$$

$$f(x) = \frac{1}{b-a} = \frac{1}{\text{length of interval } I}$$

Definition – A continuous random variable X is said to follow a uniform distribution over (a,b) if its PDF is given by

$$f_X(x) = \begin{cases} \dfrac{1}{b-a}, & a<x<b \\ 0, & \text{otherwise} \end{cases}$$



The graph of the pdf f(x) looks like a rectangle, that's why the distribution is known as Rectangular Distribution.

Notation – $X \sim U(a,b)$ or $X \sim \text{Rect}(a,b)$

Density $f_X(x)$ can be greater than 1.

CDF : $F(x) = \int_{-\infty}^{x} f(t)\,dt.$

$$= \begin{cases} 0 & x<a \\ \int_a^x \frac{1}{b-a}\,dt & a \le x < b \\ \int_a^b \frac{1}{b-a}\,dt & x \ge b \end{cases} = \begin{cases} 0 & x<a \\ \dfrac{x-a}{b-a} & a \le x < b \\ 1 & x \ge b \end{cases}$$

1) Let $X \sim U(0,1)$. does $E\left(\frac{1}{X}\right)$ exist ?

$$E\left(\frac{1}{X}\right) = \int_0^1 \frac{1}{x}\,dx = \lim_{a\to 0^+} \int_a^1 \frac{1}{x}\,dx$$

$$= \lim_{a\to 0^+} \left[\log|x|\right]_a^1$$

$$= \lim_{a\to 0^+} (\ln 1 - \ln a)$$

$$= \lim_{a\to 0^+} \ln \frac{1}{a}$$

$$= \infty$$

$E\left(\frac{1}{X}\right)$ does not exist.

2) Let $X \sim U(0,n)$ $\exists$ $n \in \mathbb{N}$, let $Y = X - [X]$. show that $Y \sim U(0,1)$

$$F_Y(y) = P(Y \le y) = P[X - [X] \le y]$$

$$= \sum_{k=0}^{n-1} P\left[X \le [x] + y,\ k \le X < k+1\right], \quad [x] = k, \text{ say}$$

$$= \sum_{k=0}^{n-1} P\left[X \le k+y,\ k \le X < k+1\right]$$

$$= \sum_{k=0}^{n-1} P\left[k \le X \le k+y\right]$$

$$= \sum_{k=0}^{n-1} \int_k^{k+y} \frac{1}{n}\,dx, \quad 0<y<1$$

$$= \sum_{k=0}^{n-1} \frac{y}{n}$$

$$= y, \quad 0<y<1$$

$$F_Y(y) = \begin{cases} 0, & y<0 \\ y, & 0 \le y < 1 \\ 1, & y \ge 1 \end{cases} \qquad f_Y(y) = \begin{cases} 1 & 0<y<1 \\ 0 & \text{o.w.} \end{cases}$$

$$\Rightarrow Y = X - [X] \sim U(0,1)$$

3) If $X \sim U\{0, 1, \ldots, n\}$ and $Y \sim U(0,1)$. Find the CDF of $Z = X+Y$ and find ρ_{XZ}.

Z can take any value in interval $(0, n+1)$

$$F_Z(z) = P(Z \le z) = P(X + Y \le z)$$

$$= \begin{cases} 0 & , z \le 0 \\ P(X=0, Y \le z) & , 0 \le z < 1 \end{cases}$$

$$= \begin{cases} P(X=0, Y\le 1) + P(X=1, Y\le z-1) & , 1\le z<2 \\ P(X=0,Y\le 1)+P(X=1,Y\le 1)+P(X=2,Y\le z-2) & , 2\le z<3 \\ P(X=0,Y\le 1)+P(X=1,Y\le 1)+\cdots+P(X=k-1,Y\le 1) & \\ \quad + P(X=k, Y\le z-k) & , k\le z<k+1 \end{cases}$$

$$= \begin{cases} 0 & , z<0 \\ \dfrac{k}{n+1} + \dfrac{1}{n+1}(z-k) & , k\le z<k+1,\ k=0,1,\dots n \\ 1 & , z\geq n+1 \end{cases}$$

$$f_Z(z) = \begin{cases} \dfrac{1}{n+1} & , 0\le z<n+1 \\ 0 & , \text{otherwise} \end{cases}$$

$$Z = X+Y \sim U(0, n+1)$$

4) Prajwel and his girlfriend decide to meet at a certain location. If each of them arrives at a time uniformly distributed between 12 noon to 1 pm. Find the probability that the first one to arrive has to wait larger than 10 minutes.

Let, X and Y be the arrival time in minutes of Prajwel and his gf after 12 noon.

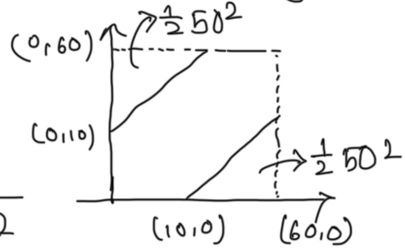
$$X \sim U(0,60)$$
$$Y \sim U(0,60)$$

$$P(|X-Y|>10) = \iint_{|X-Y|>10} f_{X,Y}(x,y)\, dx\, dy$$

$$= \iint_A \frac{1}{60^2}\, dx\, dy$$

$A = \{(x,y) : |x-y|>10,\ 0<x,y<60\}$

$$= \frac{\iint_A dx\,dy}{60^2}$$



$$= \frac{\text{Bounded by the region } A}{\text{The area bounded by } \Omega}$$

$$= \frac{50^2}{60^2} = \frac{5}{6}$$

$$\int_a^b\int_c^d f(x,y)\,dx\,dy = \text{base area} \times h$$

, b, d

$$\Rightarrow h\int_a \int_c dx\,dy = \text{base area} \times h$$

$$\Rightarrow \int_a^b \int_c^d dx\,dy = \text{Area}$$

Median :

Let M_e be the median of the distribution.

By definition, $\int_\alpha^{Me} f(x)\,dx = \int_{Me}^{\beta} f(x)\,dx = \frac{1}{2}$

$$\Rightarrow \int_{Me}^{\beta} f(x)\,dx = \frac{1}{2}$$

$$\Rightarrow \frac{\beta - Me}{\beta - \alpha} = \frac{1}{2}$$

$$\Rightarrow \beta - Me = \frac{\beta - \alpha}{2}$$

$$\Rightarrow Me = \beta - \frac{\beta - \alpha}{2} = \frac{\beta + \alpha}{2}$$

$$\text{Mean} = \text{median} = \frac{\alpha + \beta}{2}$$

Mode does not exist for uniform distribution.

Quartile Deviation :

Let Q_1, Q_3 be the 1st and 3rd quantile of the distribution,

By definition,

$$\int_\alpha^{Q_1} f(x)\,dx = 1/4 \qquad\qquad \int_\alpha^{Q_3} f(x)\,dx = \frac{3}{4}$$

$$\Rightarrow \frac{Q_1 - \alpha}{\beta - \alpha} = \frac{1}{4} \qquad\qquad \Rightarrow \frac{Q_3 - \alpha}{\beta - \alpha} = \frac{3}{4}$$

$$\Rightarrow Q_1 = \frac{\beta - \alpha}{4} + \alpha \qquad\qquad \Rightarrow Q_3 = \frac{3}{4}(\beta - \alpha) + \alpha$$

$$= \frac{3\alpha + \beta}{4} \qquad\qquad = \frac{3\beta + \alpha}{4}$$

$$QD = \frac{Q_3 - Q_1}{2} = \frac{3\beta + \alpha - 3\alpha - \beta}{8}$$

$$= \frac{2\beta - 2\alpha}{} = \frac{\beta - \alpha}{}$$

$$\frac{\qquad}{8} - \frac{\qquad}{4}$$

Remark :

Consider a uniform distribution over the region $\Omega \in \mathbb{R}^n$, then the PDF of X is,

$$f(\underset{\sim}{x}) = \begin{cases} c & \underset{\sim}{x} \in \Omega \\ 0 & \underset{\sim}{x} \in \mathbb{R}^n - \Omega \end{cases}$$

Then, $1 = \int_{\Omega} f(\underset{\sim}{x})\, d\underset{\sim}{x} \Rightarrow c \int_{\Omega} f(\underset{\sim}{x})\, d\underset{\sim}{x} = 1$

$$\Rightarrow c = \frac{1}{\text{Measure of } \Omega}$$

The probability that one randomly selected point $\underset{\sim}{x}$ falls in a region $A \subset \mathbb{R}^n$ $= P(\underset{\sim}{x} \in A) = \int_A f(\underset{\sim}{x})\, d\underset{\sim}{x}$

$$= c \int_A d\underset{\sim}{x}$$

$$= \frac{\text{The measure of } A}{\text{The measure of } \Omega}$$

Moments :

$$\mu_r' = E(x^r) = \int_a^b x^r \frac{1}{b-a}\, dx = \left.\frac{x^{r+1}}{(r+1)(b-a)}\right]_a^b$$

$$= \frac{b^{r+1} - a^{r+1}}{(b-a)\ (r+1)}$$

$$E(X) = \mu_1' = \frac{b+a}{2}$$

$$V(X) = \mu_2' - \mu_1'^{\,2} = \frac{b^3 - a^3}{3(b-a)} - \frac{(b+a)^2}{4}$$

$$= \frac{(b-a)^2}{12}$$

5) Let X be a RV denifed on $[0,1]$ if $P(x < X < y) \propto (y-x)\ \forall\ 0 < x < y < 1$, show that $X \sim U(0,1)$

Let $F(x)$ be the DF of X, $\quad , 0 \leq x \leq y \leq 1$

$$P(x \leq X \leq y) = k(y-x)$$

$$\Rightarrow F(y) - F(x) = k(y-x)$$

$$\Rightarrow \frac{F(y) - F(x)}{\qquad} = k \quad \forall\ x < y$$

$y - x$

$$\Rightarrow \lim_{y \to x} \frac{F(y) - F(x)}{y - x} = K$$

$$\Rightarrow F'(x) = K$$

$$\Rightarrow F(x) = Kx + C, \qquad 0 \leq x \leq .$$

$F(0) = 0 \Rightarrow C = 0$

$F(1) = 1 \Rightarrow K = 1$.

The CDF of X is $F_X(x) =$