Bernoulli Distribution :-

Consider a random experiment whose sample space has been divided into two categories [success/failure]. Then, any performance of the random experiment is known as Bernoulli trail. (Trail is an attempt to produce event, the event is neither impossible or certain)

Let X denotes the outcome of a Bernoulli trail such that

$$X = \begin{cases} 1 & \text{if success occurs} \\ 0 & \text{if failure occurs} \end{cases}$$

Here X is a Bernoulli random variable, $P(S)=p$.

So, $X = \begin{cases} 1 & \text{with probability } p \\ 0 & \text{with probability } (1-p)=q \end{cases}$

pmf $f_X(x) = \begin{cases} p^x (1-p)^{1-x}, & x=0,1 \\ 0 & \text{otherwise} \end{cases}$

$E(X)=p$

$V(X) = p-p^2 = p(1-p) = pq \leq \frac{1}{4}$.

AM $\geq$ GM

$\frac{p+q}{2} \geq \sqrt{pq}$

$\frac{1}{2} \geq \sqrt{pq}$

$pq \leq \frac{1}{4}$

1) If $P(X=0) = 1-P(X=1)$ and $E(X) = 3V(X)$. Find $P(X=0)$.

X is Bernoulli RV.

$E(X) = 3V(X)$

$\Rightarrow p = 3p(1-p) = 3p-3p^2$

$\Rightarrow 2p-3p^2 = 0$

$\Rightarrow p(2-3p)=0$

$p=0, \quad 2=3p \Rightarrow p = 2/3$

$P(X=0) = p^0 (1-p)^1 = (1-p) = \begin{cases} 1, & p=0 \\ \frac{1}{3}, & p=2/3 \end{cases}$

2) Let X have pdf $f_X(x) = \frac{1}{\pi(1+x^2)}$, $x \in \mathbb{R}$.

$Y = \begin{cases} 1, & |x| \geq 1 \\ 0 & \text{ow} \end{cases}$ Find the distribution of Y.

$P(Y=1) = P(|x| \geq 1)$

$= 1 - P(|x| < 1)$

$= P(-1 < x < 1)$

$$= 1 - P(-1 < X < 1)$$

$$= 1 - \int_{-1}^{1} \frac{1}{\pi} \frac{1}{(1+x^2)}\, dx$$

$$= 1 - \frac{1}{\pi} \left[\tan^{-1} x\right]_{-1}^{1}$$

$$= 1 - \frac{2}{\pi} \tan^{-1}(1)$$

$$= 1 - \frac{2}{\pi} \frac{\pi}{4} = 1 - \frac{1}{2} = \frac{1}{2}$$

$$Y = \begin{cases} 1 & p = 1/2 \\ 0 & q = 1/2 \end{cases}$$

$Y \sim \text{Bernoulli}\left(p = \frac{1}{2}\right)$

Two Point Distribution :

$$X = \begin{cases} a & \text{with probability } p \\ b & \text{with probability } q = 1-p \end{cases}$$

then, X is a RV having two point distribution.

$$Y = \frac{X-b}{a-b} = \begin{cases} 1 & \text{wp } p \\ 0 & \text{wp } q \end{cases}$$

Y is a Bernoulli random variable.

Every two point distribution can be converted into Bernoulli Distribution