NEGATIVE BINOMIAL DISTRIBUTION :-

Note that, $(1-q)^{-r} = \sum_{x=0}^{\infty} \binom{x+r-1}{x} q^x$ for $|q|<1$ (Taylor's Expansion)
and $r>0$ (real no)

This is called Binomial Theorem for negative index.

$\sum_{x=0}^{\infty} \binom{x+r-1}{x} p^r q^x = 1$ and the terms of the series forms a pmf.

The probability distribution given by this pmf is known as Negative Binomial Distribution.

Definition – A counting random variable is said to follow a negative Binomial Distribution if its pmf is given by,

$$f(x) = \begin{cases} \binom{x+r-1}{x} p^r q^x, & x = 0,1,2\ldots \quad 0<p<1 \\ 0 & \text{ow} \quad q=1-p,\ r>0 \text{ (real no)} \end{cases}$$

Moments –

Note that, $(1-q)^{-r} = \sum_{x=0}^{\infty} \binom{x+r-1}{x} q^x$

Differentiating wrt q, $(-r)(1-q)^{-r-1}(-1) = \sum_{x=0}^{\infty} \binom{x+r-1}{x} x q^{x-1}$

$$\Rightarrow r p^{-r-1} = \sum_{x=1}^{\infty} \binom{x+r-1}{x} x q^{x-1}$$

Again, differentiating wrt q, $r(r+1)(1-q)^{-r-2} = \sum_{x=2}^{\infty} \binom{x+r-1}{x} x(x-1) q^{x-2}$

$$E(X) = \sum_{x=0}^{\infty} x \binom{x+r-1}{x} p^r q^x$$

$$= p^r q \sum_{x=1}^{\infty} x \binom{x+r-1}{x} q^{x-1}$$

$$= p^r q \, r p^{-r-1} = \frac{rq}{p}$$

(Derivative of sum = Sum of derivatives but it is not always true)

$$V(X) = E(X^2) - E^2(X) = E(X(X-1)) + E(X) - E^2(X)$$

$$= p^r q^2 \sum_{x=2}^{\infty} x(x-1) \binom{x+r-1}{x} q^{x-2} + \frac{rq}{p} - \frac{r^2 q^2}{p^2}$$

$$= p^r q^2 \, r(r+1) p^{-r-2} + \frac{rq}{p} - \frac{r^2 q^2}{p^2}$$

$$= r(r+1)\frac{q^2}{p^2} + \frac{rq}{p} - \frac{r^2q^2}{p^2}$$

$$= \frac{rq}{p}\left[\cancel{\frac{rq}{p}} + \frac{q}{p} + 1 - \cancel{\frac{rq}{p}}\right]$$

$$= \frac{rq}{p^2}$$

$$E(X) = \frac{rq}{p} < \frac{rq}{p^2} = var(X)$$

Recursion Relation of Negative Binomial Distribution :-

$$\mu_m = E[X - E(X)]^m = E\left[X - \frac{rq}{p}\right]^m$$

$$= \sum_{x=0}^{\infty}\left(x - \frac{rq}{p}\right)^m \binom{x+r-1}{x} p^r q^x$$

Differentiating wrt p, we get,

$$\frac{\partial \mu_m}{\partial p} = \sum_{x=0}^{\infty}\left(x - \frac{rq}{p}\right)^m \binom{x+r-1}{x} r p^{r-1} q^x + \sum_{x=0}^{\infty}\left(x - \frac{rq}{p}\right)^m \binom{x+r-1}{x} p^r x q^{x-1}(-1)$$

$$+ \sum_{x=0}^{\infty} m\left(x - \frac{rq}{p}\right)^{m-1} \binom{x+r-1}{x} p^r q^x (-r) \underbrace{\frac{p(-1) - (1-p)}{p^2}}_{= \frac{-1}{p^2}}$$

$$= \sum_{x=0}^{\infty}\left(x - \frac{rq}{p}\right)^{m-1} \binom{x+r-1}{x} p^r q^x \left(\frac{mr}{p^2}\right) +$$

$$\sum_{x=0}^{\infty}\left(x - \frac{rq}{p}\right)^m \binom{x+r-1}{x} p^r q^x \underbrace{\left[\frac{r}{p} - \frac{x}{q}\right]}$$

$$= \frac{rq - xp}{pq} = -\frac{xp - rq}{pq}$$

$$= -\left(x - \frac{rq}{p}\right)\frac{1}{q}$$

$$= \frac{mr}{p^2}\mu_{m-1} - \frac{1}{q}\mu_{m+1}$$

$$\Rightarrow \frac{1}{q}\mu_{m+1} = \frac{mr}{p^2}\mu_{m-1} - \frac{\partial \mu_m}{\partial p}$$

$$\Rightarrow \mu_{m+1} = q\left(\frac{mr}{p^2}\mu_{m-1} - \frac{\partial \mu_m}{\partial p}\right)$$

This is the recursion relation of negative binomial distribution.

Now, $\mu_0 = 1$, $\mu_1 = 0$

If $m = 1$, $\mu_2 = q\left(\frac{1r}{p^2}\mu_0 - 0\right) = \frac{qr}{p^2}$

$m_2 = 2$, $\mu_3 = q\left(\frac{2r}{p^2}\mu_1 - \frac{\partial \mu_2}{\partial p}\right)$

$$\frac{\partial}{\partial p}\frac{qr}{p^2} = r\,\frac{p^2(-1) - (1-p)2p}{p^4}$$

$$= q\left(\frac{2n}{p^2}\cdot 0 - n\frac{(p-2)}{p^3}\right) \qquad = n\frac{-p^2-2p+2p^2}{p^4}$$

$$= qn\frac{(2-p)}{p^3} \qquad = n\frac{p^2-2p}{p^4} \qquad = n\frac{(p-2)}{p^3}$$

$$\text{skewness} = \frac{\mu_3}{\mu_2^{3/2}} = \frac{qn\frac{(2-p)}{p^3}}{\left(\frac{qn}{p^2}\right)^{3/2}} = \frac{1+q}{\sqrt{qn}} > 0$$

$\Rightarrow$ the distribution is positively skewed.

$$\text{Kurtosis} = \frac{\mu_4}{\mu_2^2} - 3 = \frac{\frac{nq(1+4q+q^2)+3n^2q^2}{p^4}}{\frac{q^2n^2}{p^4}} - 3$$

$$= \frac{1+4q+q^2}{qn} + 3 - 3 > 0$$

$$\mu_4 = q\left\{\frac{3n}{p^2}\mu_2 - \frac{\partial \mu_3}{\partial q}\right\}$$

$$= q\left\{3q\left(\frac{n}{p^2}\right)^2 - \left\{-\frac{n}{p^2}\left(6p^{-2}-6p^{-1}+1\right)\right\}\right\}$$

$$= 3\left(\frac{qn}{p^2}\right)^2 + \frac{nq}{p^2}\left(\frac{6}{p^2}-\frac{6}{p}+1\right)$$

$$= \frac{nq(1+4q+q^2)+3n^2q^2}{p^4}$$

① If $X \sim NB(n,p)$, then show that $E(X^k) = \frac{nq}{p}E(Y+1)^{k-1}$ where $Y \sim NB(n+1,p)$. Hence, find $E(X)$, $Var(X)$.

$$E(X^k) = \sum_{x=0}^{\infty} x^k \binom{x+n-1}{x} p^n q^x$$

$$= \sum_{x=1}^{\infty} x^{k-1} \frac{(x+n-1)\cdots(n+1)n}{(x-1)!} p^n q^x$$

$$= \frac{nq}{p}\sum_{x=1}^{\infty} x^{k-1} \frac{(x-1+n+1-1)\cdots(n+1)}{(x-1)!} p^{n+1} q^{x-1}$$

$$= \frac{nq}{p}\sum_{x-1=y=0}^{\infty} (y+1)^{k-1} \frac{(y+n+1-1)\cdots(n+1)}{y!} p^{n+1} q^{y}$$

$$= \frac{nq}{p}\sum_{y=0}^{\infty} (y+1)^{k-1} \underbrace{\binom{y+n+1-1}{y} p^{n+1} q^{y}}_{\text{pmf of } NB(n+1,\,p)}$$

$$= \frac{nq}{p}E(Y+1)^{k-1}$$

then show that the mean and variance can be written as the corresponding formula for binomial distribution.

$$E(X) = \frac{rq}{p} E(Y+1) = \frac{rq}{p}.$$

$$E(X^2) = \frac{rq}{p}\left(E(Y)+1\right) = \frac{rq}{p}\left(\frac{(r+1)q}{p}+1\right)$$

$$= \frac{rq}{p}\,\frac{rq+q+p}{p} = \frac{rq(rq+1)}{p^2}$$

(2) Derive the PMF of NB distribution in the following form,

$$f(x) = \binom{-r}{x} P^{x} Q^{-r-x}, \quad x = 0, 1, 2, \dots$$

$$P + Q = 1$$

$$f(x) = \binom{x+r-1}{x} p^{r} q^{x}$$

$$= \frac{(x+r-1)\cdots(r+1)r}{x!}\left(\frac{1}{p}\right)^{-r} q^{x}$$

$$= \frac{(-r)(-r-1)\cdots(-x-r-1)}{x!}\left(\frac{1}{p}\right)^{-r} (-q)^{x}$$

$$= \binom{-r}{x}\left(-\frac{q}{p}\right)^{x}\left(\frac{1}{p}\right)^{-r-x}$$

$$= \binom{-r}{x}(P)^{x} Q^{-r-x} \qquad \text{where } P = -\frac{q}{p},\ Q = \frac{1}{p}$$

which is in the form of Binomial Distribution.

$$E(X) = nP = (-r)\left(-\frac{q}{p}\right) = \frac{rq}{p}.$$

$$V(X) = nPQ = (-r)\left(-\frac{q}{p}\right)\left(\frac{1}{p}\right) = \frac{rq}{p^2}$$

Probability Model:

Consider a sequence of independent Bernoulli trails with probability of success p. Let X be the no of failure before the 5th success.

So, then

P(X=x)
= P [that there are (r-1) success in the first (x-r+1) trails and the (x+r)th trail results in a success]

$$= P\left[(r-1) \text{ success in } (x+r-1) \text{ trails}\right] P(\text{success})$$

$$= \binom{x+r-1}{x} p^{r-1} q^{x}\, p$$

$$= \binom{x+r-1}{x} p^{r} q^{x}, \quad x = 0, 1, 2, \dots$$

This is the pmf of NB(r, p) distribution.

Let X be the no of trails needed to get the rth success, then

Let, Y be the no of trials needed to get the rth success. Then $Y = X + r$ with probability 1.

$$P(Y=y) = P(X = y-r) = \binom{y-r+r-1}{y-r} p^r q^{y-r}, \quad y = r, r+1, \dots$$

$$E(Y) = E(X) + r = \frac{rq}{p} + r = \frac{r}{p}$$

$$V(Y) = V(X) = \frac{rq}{p^2} \quad [\because \text{ no change due to linear shift}]$$

Cumulative Distribution Function (CDF):

$$P(X \leq k) = \sum_{x=0}^{k} \binom{x+r-1}{x} p^r q^x = g(p) \text{ (say)}$$

Differentiating wrt p,

$$g'(p) = \sum_{x=0}^{k} \binom{x+r-1}{x} r p^{r-1} (1-p)^x - \sum_{x=0}^{k} \binom{x+r-1}{x} p^r x (1-p)^{x-1}$$

$$= \sum_{x=0}^{k} \binom{x+r-1}{x} r p^{r-1} (1-p)^x - \sum_{x=0}^{k} \frac{(x+r-1)!}{(x-1)!\,(r-1)!} p^r (1-p)^{x-1}$$

$$= \sum_{x=0}^{k} \binom{x+r-1}{x} r p^{r-1} (1-p)^x - r \sum_{x=1}^{k} \binom{x+r-1}{x-1} p^r (1-p)^{x-1}$$

$$p^{r-1}\{1-(1-p)\}$$

$$- r \sum_{x=1}^{k} \binom{x+r-1}{x-1} (1-p)^{x-1} p^{r-1}$$

$$+ r \sum_{x=1}^{k} \binom{x+r-1}{x-1} p^{r-1} (1-p)^x$$

$$= \sum_{x=0}^{k} \left\{ \binom{x+r-1}{x} + \binom{x+r-1}{x-1} \right\} r p^{r-1} (1-p)^x - r \sum_{x=1}^{k} \binom{x+r-1}{x-1} (1-p)^{x-1} p^{r-1}$$

$$= \sum_{x=0}^{k} \binom{x+r}{x} r p^{r-1} (1-p)^x - r \sum_{x=1}^{k} \binom{x+r-1}{x-1} (1-p)^{x-1} p^{r-1}$$

$$= \sum_{x=0}^{k} \binom{x+r}{x} r p^{r-1} (1-p)^x - r \sum_{x-1=x'=0}^{k-1} \binom{x'+r}{x'} (1-p)^{x'} p^{r-1}$$

$$= \binom{k+r}{k} r p^{r-1} (1-p)^k \quad [= g'(p)]$$

$$g'(p) = \binom{r+k}{k} r p^{r-1} (1-p)^k$$

$$g'(t) = \binom{r+k}{k} r t^{r-1} (1-t)^k$$

$$\int_0^p g'(t)\,dt = \int_0^p r \binom{r+k}{k} t^{r-1} (1-t)^k \,dt$$

$$\Rightarrow g(p) - g(0) = \frac{\int_0^p t^{r-1} (1-t)^{k+1-1} \,dt}{B(r, k+1)} = I_p(r, k+1)$$

$$r\binom{r+k}{k} = \cancel{r}\frac{(r+k)!}{k!\,\cancel{r}\,(r-1)!} = \frac{1}{B(r, k+1)}$$

$$\Rightarrow g(p) - 0 = \sum_{x=0}^{k} \binom{x+r-1}{x} p^r q^x = g(p) = I_p(r, k+1)$$

It is an increasing function of p and decreasing function of q.

Mode :-

$$f(x) = \begin{cases} \binom{x+r-1}{x} p^r q^x, & x = 0, 1, \ldots \quad p+q=1, \ r>0 \text{ (real no)} \\ 0 & \text{ow} \end{cases}$$

$$\frac{f(x)}{f(x-1)} = \frac{\binom{x+r-1}{x} p^r q^x}{\binom{x+r-2}{x-1} p^r q^{x-1}} = q\,\frac{x+r-1}{x}$$

This is according as $\frac{x+r-1}{x} q \gtrless 1$

$$\Rightarrow (x+r-1)q \gtrless x$$

$$\Rightarrow x(q-1) \gtrless (r-1)q$$

$$\Rightarrow x \gtrless \frac{(r-1)q}{p}$$

Case I - $\frac{(r-1)q}{p}$ is not an integer, let $k = \left[\frac{(r-1)q}{p}\right]$ (say)

$$\frac{f(1)}{f(0)} > 1, \quad \frac{f(2)}{f(1)} > 1, \quad \frac{f(3)}{f(2)} > 1, \ \ldots, \ \frac{f(k)}{f(k-1)} > 1, \quad \frac{f(k+1)}{f(k)} < 1,$$

$\therefore f(0) < f(1) < \ldots < f(k-1) < f(k) > f(k+1) > f(k+2) > \ldots$

$\Rightarrow f(k)$ is the maximum.

$k = \left[\frac{(r-1)q}{p}\right]$ is the mode.

Case II : $\frac{(r-1)q}{p}$ is an integer. $(=k)$

$f(0) < f(1) < f(2) < \ldots < f(k-1) = f(k) > f(k+1) > f(k+2) > \ldots$

$\therefore f(k)$ or $f(k-1)$ be the maximum.

$k = \frac{(r-1)q}{p}$ is the mode.

Tail Function :-

$$P(X>k) = 1 - P(X \le k) = 1 - I_p(r, k+1)$$

$$\int_0^p t^{r-1} (1-t)^{k+1-1}\, dt$$

$$= 1 - \frac{\quad 0 \quad}{B(r, k+1)}$$

$$= \frac{\int_0^1 t^{r-1} (1-t)^{\overline{k+1}-1} dt - \int_0^p t^{r-1} (1-t)^{\overline{k+1}-1} dt}{\int_0^1 t^{r-1} (1-t)^{\overline{k+1}-1} dt}$$

$$= \frac{\int_p^1 (1-t)^{\overline{k+1}-1} t^{r-1} dt}{B(r, k+1)}$$

$$= \frac{\int_q^0 -z^{\overline{k+1}-1} (1-z)^{r-1} dz}{B(r, k+1)}$$

$1-t = z$

$-dt = dz$

t	1	p
z	0	q

$$= \frac{\int_0^q z^{\overline{k+1}-1} (1-z)^{r-1} dz}{B(r, k+1)}$$

$$= I_q(k+1, r)$$

③ Let X be the intensity of an accident proonness of an individual and let Y be the no of accident as the individual. Let,

$$f_X(x) = \begin{cases} \frac{\theta^\alpha}{\Gamma\alpha} e^{-\theta x} x^{\alpha-1} & , x \geq 1 \\ 0 & , \text{otherwise} \end{cases} , \theta > 0, \ \alpha > 0 \text{ (real no)}$$

$$f_{Y|X}(y|x) = \begin{cases} e^{-x} \frac{x^y}{y!} & , y = 0, 1, 2, \dots \\ 0 & \text{ow} \end{cases}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(y|x) f(x) dx , \ y \in \mathbb{R}.$$

$$= \int_0^\infty e^{-x} \frac{x^y}{y!} \frac{\theta^\alpha}{(\alpha-1)!} e^{-\theta x} x^{\alpha-1} dx$$

$$= \frac{\theta^\alpha}{y! (\alpha-1)!} \int_0^\infty e^{-x(1+\theta)} x^{y+\alpha-1} dx$$

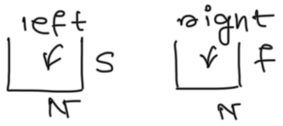
$$= \frac{\theta^\alpha}{y! (\alpha-1)!} \frac{(\alpha+y-1)!}{(1+\theta)^{\alpha+y}}$$

$$= \binom{y+\alpha-1}{y} \left(\frac{\theta}{1+\theta}\right)^\alpha \left(1 - \frac{\theta}{1+\theta}\right)^y , y = 0, 1, 2, \dots$$

$$Y \sim NB\left(\alpha, \frac{\theta}{1+\theta}\right)$$

A certain mathematician always carries one match box in his right pocket and one in his left. When he wants a match he selects a pocket at random and choose a match. Suppose initially each box contained exactly N matches and consider the moment when for the first time our mathematician discovers that one box is empty. At that point in time, find the probability that the other box contain exactly r matches. Find E(X).

matches. Find E(X).



let us identify the success with the choice of the left pocket. The left pocket box will be empty when the right pocket box contains r matches iff exactly (n-r) failure preceding (N+r) of success. A similar argument applies to the right pocket box.

$$P_r = 2\binom{N-r+\overline{N+1}-1}{N-r}\left(\frac{1}{2}\right)^{N+1}\left(\frac{1}{2}\right)^{N-r} \qquad [\text{left+right}]$$

$$= \binom{2N-r}{N-r}\left(\frac{1}{2}\right)^{2N-r}, \quad r=0,1,\dots N$$

clearly, P_r forms a pmf of a distribution.

$$\mu = \sum_{r=0}^{N} rP_r = \sum_{r=0}^{N} r\binom{2N-r}{N-r}\left(\frac{1}{2}\right)^{2N-r}$$

$$(N-\mu) = \sum_{r=0}^{N}(N-r)P_r$$

$$= \sum_{r=0}^{N}(N-r)\frac{(2N-r)!}{(N-r)!\;N!}\left(\frac{1}{2}\right)^{2N-r}$$

$$= \sum_{r=0}^{N-1}\frac{(2N-r)}{2}\,\frac{(2N-r-1)!}{(N-r-1)!\;N!}\left(\frac{1}{2}\right)^{2N-r-1}$$

$$= \sum_{r=0}^{N-1}\frac{2N-r}{2}\binom{2N-r-1}{N-r-1}\frac{1}{2^{2N-r-1}}$$

$$= \sum_{r=0}^{N-1}\frac{(2N+1)-(r+1)}{2}P_{r+1}$$

$$= \frac{2N+1}{2}\sum_{r=0}^{N-1}P_{r+1} - \frac{1}{2}\sum_{r=0}^{N-1}(r+1)P_{r+1}$$

$$= \frac{2N+1}{2}\sum_{r+1=K=1}^{N}P_K - \frac{1}{2}\sum_{r=0}^{N-1}(r+1)P_{r+1}$$

$$\Rightarrow (N-\mu) = \frac{2N+1}{2}(1-P_0) - \frac{\mu}{2}$$

$$\Rightarrow -\frac{\mu}{2} = \frac{2N+1}{2}\left\{1-\binom{2N}{N}\left(\frac{1}{2}\right)^{2N}\right\} - N$$

$$\Rightarrow \frac{\mu}{2} = N - \frac{2N+1}{2}\left\{1-\binom{2N}{N}\left(\frac{1}{2}\right)^{2N}\right\}$$

$$= N - \frac{2N+1}{2} + \frac{2N+1}{2}\binom{2N}{N}\left(\frac{1}{2}\right)^{2N}$$

$$= \frac{\cancel{2N} - \cancel{2N} + 1}{2} + \frac{2N+1}{2}\binom{2N}{N}\left(\frac{1}{2}\right)^{2N}$$

$$\Rightarrow \quad \omega = (2N+1)\binom{2N}{N}\left(\frac{1}{2}\right)^{2N} - 1$$

Relationship with Binomial Distribution :-

If $X \sim NB(r, p)$, $P(X \leq K) = P(U \geq r)$, $U \sim Bin(K+r, p)$

$= P(V \leq K)$, $V \sim Bin(K+1, q)$

X denotes the number of failure preceeding the rth success.
$(r+x)$ denotes the no of trails required to get the rth success.

$P(X \leq K) = P(X+r \leq K+r)$

$= P($ no of trails required to get rth success $\leq r+K)$

If we define U as the no of success in $(r+K)$ trails,

$= P(U \geq r)$ So, $U \sim Bin(K+r, p)$

$= P(-U \leq -r)$ $(K+r-U) \sim Bin(K+r, q)$

$= P(K+r-U \leq K)$ $\Rightarrow V \sim Bin(K+r, q)$

$= P(V \leq K)$

Let X be a random variable with pmf,

$$f(x) = \begin{cases} c\binom{2n-x}{n} 2^x & , x = 0, 1, \ldots n \\ 0 & , \text{ow} \end{cases}$$

Determine c such that it is a pmf.

$$\sum_{x=0}^{n} f(x) = 1$$

$n - x = y$

x	0	n
y	n	0

$$\Rightarrow c \sum_{x=0}^{n} \binom{2n-x}{n} 2^x = 1$$

$$\Rightarrow c \sum_{y=0}^{n} \binom{n+y}{n} 2^{n-y} = 1$$

$$\Rightarrow c\, 2^{2n+1} \underbrace{\sum_{y=0}^{n} \binom{y+(n+1)-1}{y} \left(\frac{1}{2}\right)^{n+1} \left(\frac{1}{2}\right)^y}_{} = 1 \quad \text{— (1)}$$

$= P(Y \leq n)$, $Y \sim NB(n+1, \frac{1}{2})$

$= P(V \leq n)$, $V \sim Bin(2n+1, \frac{1}{2})$

$$= \sum_{v=0}^{n} \binom{2n+1}{v} \left(\frac{1}{2}\right)^{2n+1}$$

$0, 1, \ldots\ldots 2n+1$

n = middle

$$= \frac{1}{2}$$

$$\Rightarrow c\, 2^{2n+1} \frac{1}{2} = 1$$

$$\Rightarrow c = 2^{-2n}$$

$$f(x) = 2^{-2n} \binom{2n-x}{n} 2^{x} = \binom{2n-x}{n} \left(\frac{1}{2}\right)^{2n-x}$$

Alternative, by ①, $c\,2^{2n+1} \underbrace{P(Y \le n)}_{I_{1/2}(n+1,\, n+1)} = 1$, $Y \sim NB(n+1, \frac{1}{2})$

$$\Rightarrow c\,2^{2n+1} I_{1/2}(n+1, n+1) = 1$$

$$\Rightarrow c\, 2^{2n+1} \frac{\int_0^{1/2} t^n (1-t)^n dt}{B(n+1, n+1)} = 1$$

$$\Rightarrow c\, 2^{2n+1} \frac{\int_0^{1/2} t^n (1-t)^n dt}{2\int_0^{1/2} t^n (1-t)^n dt} = 1$$

$$\Rightarrow c\, 2^{2n+1} \frac{1}{2} = 1$$

$$\Rightarrow c = \left(\frac{1}{2}\right)^{2n}$$

Putting, the value of c, we proof it is a pmf.

$$\sum_x f(x)$$

$$= 2^{-2n} \sum_{x=0}^{n} \binom{2n-x}{n} 2^{x} \qquad n-x=y$$

$$= 2^{-2n} \sum_{y=0}^{n} \binom{n+y}{y} 2^{n-y}$$

$$= 2^{-2n}\, 2^{2n+1} \sum_{y=0}^{n} \binom{y+(n+1)-1}{y} \left(\frac{1}{2}\right)^{n+1} \left(\frac{1}{2}\right)^{y}$$

$$= 2\, P(Y \le n)$$

$$= 2\, I_{1/2}(n+1, n+1)$$

$$= 2\, \frac{1}{2}$$

$$= 1.$$

Example:

In a series of indefinite Bernoulli trails with success probability p. Find the probability that ‘a’ successes will appear before ‘b’ failures. (Pascal format).

Let X denote the no of failures preceeding a th success.

$X \sim NB(a, p)$

$$P(X \leq b-1) = \sum_{x=0}^{b-1} \binom{x+a-1}{x} p^a q^x$$

Y denotes the no of failures in $(a+b-1)$ trails.

$Y \sim Bin(a+b-1, q)$

$$P(Y \leq b-1) = \sum_{y=0}^{b-1} \binom{a+b-1}{y} q^y p^{a+b-1-y}$$

So, $P(X \leq b-1) = P(Y \leq b-1)$

$$\Rightarrow \sum_{i=0}^{b-1} \binom{a+i-1}{a-i} \cancel{p^a} \cancel{q^i} = \sum_{i=0}^{b-1} \binom{a+b-1}{i} \cancel{q^i} p^{\cancel{a}+b-1-i}$$

$$\Rightarrow \sum_{i=0}^{b-1} \binom{a+i-1}{a-i} = \sum_{i=0}^{b-1} \binom{a+b-1}{i} p^{b-1-i}$$

Prove that $X_1 \sim NB(r_1, p)$, $X_2 \sim NB(r_2, p)$ independent

$X_1 + X_2 \sim NB(r_1 + r_2, p)$

$$P(X_1+X_2 = y) = \sum_x P(X_1 = x, X_2 = y-x)$$

$$= \sum_x P(X_1 = x) P(X_2 = y-x)$$

$$= \sum_x \binom{x+r_1-1}{r_1-1} p^{r_1} q^x \binom{y-x+r_2-1}{r_2-1} p^{r_2} q^{y-x}$$

$$= \sum_x \binom{\cancel{x}+r_1-1+y-\cancel{x}+r_2-1}{r_1+r_2-2} p^{r_1+r_2} q^y$$

$$= \sum_y \binom{y+\overline{r_1+r_2-2}+1}{r_1+r_2-2} p^{r_1+r_2} q^y$$

$$= \sum_y \binom{y+r_1+r_2-1}{y} p^{r_1+r_2} q^y$$

$Y \sim NB(r_1+r_2, p)$

$X_i \sim NB(r_i, p)$ $i = 1, 2$. Find the conditional distribution of $(X_1 \mid X_1 + X_2 = y)$

$$P(X_1 = x \mid X_1+X_2 = y) = \frac{P(X_1 = x)\, P(X_2 = y-x)}{P(X_1+X_2 = y)}$$

$$= \frac{\binom{x+r_1-1}{x} p^{r_1} q^{x} \binom{y-x+r_2-1}{y-x} p^{r_2} q^{y-x}}{\binom{y+r_1+r_2-1}{y} p^{r_1+r_2} q^{y}}$$

$$= \frac{\binom{x+r_1-1}{r_1-1} \binom{y-x+r_2-1}{r_2-1}}{\binom{y+r_1+r_2-1}{r_1+r_2-1}}$$

It is independent of parameters.

Remark:

In Bin(n,p) distribution we take a fixed number of trail, then we count no of success but in NB(r,p) distribution e fixed up the no of success and then we count the no of trail required to get the Roth success. Hence, the NB distribution (Pascal's) is also known as Inverse Binomial Sampling.

$X \sim Bin(n,p)$, show that $E\left(\frac{1}{1+X}\right) = \frac{p}{q(r-1)} \left\{1 - p^{r-1}\right\}$

$$E\left(\frac{1}{1+X}\right) = \sum_{x=0}^{\infty} \frac{1}{1+x} \binom{x+r-1}{x} p^{r} q^{x}$$

$$= \sum_{x=0}^{\infty} \frac{(x+r-1)!}{(x+1)!\,(r-1)!} p^{r} q^{x}$$

$$= \sum_{x=0}^{\infty} \frac{(\overline{x+1} + \overline{r-1} - 1)!}{(x+1)!\,(r-1)!} p^{r} q^{x}$$

$$= \frac{p}{q(r-1)} \sum_{x=0}^{\infty} \frac{(\overline{x+1} + \overline{r-1} - 1)!}{(x+1)!\,(r-2)!} p^{r-1} q^{x+1}$$

$$= \frac{p}{q(r-1)} \sum_{x+1=y=1}^{\infty} \frac{(y + \overline{r-1} - 1)!}{y!\,(r-2)!} p^{r-1} q^{y}$$

$$= \frac{p}{q(r-1)} \left[1 - P(X=0)\right]$$

$$= \frac{p}{q(r-1)} \left(1 - p^{r-1}\right)$$

Mean Deviation about mean :-

$$MD_{m}(x) = E\left[|X-m|\right]$$

$$= \sum_{x=0}^{\infty} |x-m| f(x)$$

$$= \sum_{x=0}^{[\mu]} (\mu - x) f(x) + \sum_{x=[\mu+1]}^{\infty} (x-\mu) f(x)$$

$$= 2 \sum_{x=0}^{[\mu]} (\mu - x) f(x)$$

$$= 2 \sum_{x=0}^{[\mu]} \left(\frac{rq}{p} - x \right) f(x)$$

$$= \frac{2}{p} \sum_{x=0}^{[\mu]} (rq - px) f(x)$$

$$= \frac{2}{p} \sum_{x=0}^{[\mu]} \left[(r+x) q f(x) - x f(x) \right]$$

Alternative, $g(x) = x f(x)$

$$g(x+1) = (x+1) f(x)$$

$$= (x+1) \binom{x+1+r-1}{x+1} p^r q^{x+1}$$

$$= (x+1) \frac{(x+r)!}{(x+1)!\,(r-1)!} p^r q^{x+1}$$

$$= (r+x)\, q \frac{(x+r-1)!}{x!\,(r-1)!} p^r q^x$$

$$= (x+r)\, q f(x)$$

$$MD_\mu(x) = \frac{2}{p} \sum_{x=0}^{[\mu]} \left\{ (x+r) q f(x) - x f(x) \right\}$$

$$= \frac{2}{p} \sum_{x=0}^{[\mu]} \left\{ g(x+1) - g(x) \right\}$$

$$= \frac{2}{p} \sum_{x=[\mu]+1}^{\infty} \left\{ g(x+1) - g(x) \right\}$$

$$= \frac{2}{p} \left\{ 1 - g[\mu] \right\}$$