Poisson Distribution :—

Note that, $e^{\lambda} = \sum_{x=0}^{\infty} \frac{\lambda^x}{x!}$

$$\Rightarrow \sum_{x=0}^{\infty} e^{-\lambda} \frac{\lambda^x}{x!} = e^{-\lambda} e^{\lambda} = 1 \quad , \text{ for } \lambda > 0.$$

$$\therefore \quad e^{-\lambda} \frac{\lambda^x}{x!} > 0 \quad \forall \; x = 0, 1, 2, \ldots$$

The terms of the series $\left\{ e^{-\lambda} \frac{\lambda^x}{x!} \right\}$, $x = 0, 1, \ldots$ forms a pmf. The pmf is known as Poisson Distribution.

Definition :-

A counting random variable X is said to have a Poisson Distribution if its pmf is given by,

$$f(x) = \begin{cases} \frac{e^{-\lambda} \lambda^x}{x!} & , x = 0, 1, \ldots \; , \lambda > 0 \\ 0 & , \text{otherwise} \end{cases}$$

Remark : For $x = 0$, $f(0) = e^{-\lambda} \frac{\lambda^0}{0!} = 1$

$P(X = 0) = 1$

$\therefore$ X is degenerated at 0.

Moments :

Factorial moments $\rightarrow$ $x! = (x)_r \, (x-r)!$

$E(X_r)$

$$= \sum_{x=0}^{\infty} (x)_r \, e^{-\lambda} \frac{\lambda^x}{x!} = \sum_{x=r}^{\infty} e^{-\lambda} \frac{\lambda^x}{(x-r)!}$$

$$= \lambda^r \sum_{y = x-r = 0}^{\infty} \frac{e^{-\lambda} \lambda^y}{y!}$$

$$= \lambda^r e^{-\lambda} e^{\lambda} = \lambda^r$$

$E(X) = E(X_1) = \lambda$

$\text{var}(X) = E(X^2) - E^2(X)$

$\quad = E(X(X-1) + X) - \lambda^2$

$$= E(X_2) + E(x) - \lambda^2$$
$$= \lambda^2 + \lambda - \lambda^2 = \lambda$$

NOTE : For $Bin(n, p) \rightarrow E(x) > var(x)$
For $NB(n, p) \rightarrow E(x) < var(x)$
For $Poi(\lambda) \rightarrow E(x) = var(x)$

Mode :-

$$\frac{f(x)}{f(x-1)} = \frac{e^{-\lambda}\lambda^x}{x!} \frac{(x-1)!}{e^{-\lambda}\lambda^{x-1}}$$

$$= \frac{\lambda}{x} \gtreqless 1 \text{ according as } \lambda \gtreqless x.$$

Case I — λ is an integer. Let $\lambda = k$.

$$\frac{f(1)}{f(0)} > 1 \Rightarrow f(1) > f(0)$$

$$\frac{f(2)}{f(1)} > 1 \Rightarrow f(2) > f(1)$$

$$\vdots$$

$$\frac{f(x)}{f(x-1)} = 1 \Rightarrow f(x) = f(x-1)$$

$$\frac{f(x+1)}{f(x)} < 1 \Rightarrow f(x+1) < f(x)$$

So, $f(0) < f(1) < \cdots < \underbrace{f(x-1) = f(x)} > f(x+1) > f(x+2) > \cdots$

Mode is at λ and $\lambda - 1$.

Case II — λ is not an integer, let $k = [\lambda]$

$f(0) < f(1) < \cdots < f(k-1) < f(k) > f(k+1) > \cdots$

$k = [\lambda]$ is the mode.

Mean Deviation :—

$$MD_\lambda = E|x-\lambda| = \sum_{x=0}^{\infty} |x-\lambda| \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= \sum_{x=0}^{[\lambda]} (\lambda - x)\frac{e^{-\lambda}\lambda^x}{x!} + \sum_{x=[\lambda]+1}^{\infty} (x-\lambda)\frac{e^{-\lambda}\lambda^x}{x!}$$

as $k=[\lambda]$ is the mode.

$$= 2\, e^{-\lambda} \sum_{x=k+1}^{\infty} (x-\lambda) \frac{\lambda^x}{x!}$$

$$= 2\, e^{-\lambda} \sum_{x=k+1}^{\infty} \left\{ \frac{\lambda^x}{(x-1)!} - \frac{\lambda^{x+1}}{x!} \right\}$$

$$= 2 e^{-\lambda} \sum_{x=k+1}^{\infty} (Y_x - Y_{x+1}) , \qquad Y_x = \frac{\lambda^x}{(x-1)!}$$

$$= 2 e^{-\lambda} \left(Y_{k+1} - \cancel{Y_{k+2}} + \cancel{Y_{k+2}} - \cancel{Y_{k+3}} + \cdots \right)$$

$$= 2 e^{-\lambda} \frac{\lambda^{k+1}}{k!}$$

$$= 2 e^{-\lambda} \frac{\lambda\, \lambda^k}{k!} = 2\lambda e^{-\lambda} \frac{\lambda^{[\lambda]}}{[\lambda]!}$$

Remark :—

$$\frac{MD\lambda}{SD} = \frac{2\lambda e^{-\lambda} \lambda^{[\lambda]}}{[\lambda]!} \frac{1}{\sqrt{\lambda}}$$

If $\lambda = 1$, $\dfrac{MD}{SD} = \dfrac{2}{e} = 2e^{-1}$.

Relationship with Compound Binomial Distribution :—

Suppose that, the probability of an insect laying n eggs is $\frac{e^{-\lambda}\lambda^n}{n!}$ and the probability that the egg developing is p. Assuming natural independence of the eggs, show that the probability of the total survivor is given by Poi(λp).

Let, N be the no of eggs laid down by an insect.
$N \sim$ Poi(λ).

Let, X be the no of survivor,

$$P(X=x \mid N=n) = \begin{cases} \binom{n}{x} p^x q^{n-x} , & x=0, \dots n \\ 0 & \text{ow} \end{cases}$$

$$P(X=x) = \sum_{n=0}^{\infty} P(X=x \mid N=n)\, P(N=n) \qquad , \begin{cases} n=0,1,\dots\infty \\ \dots \end{cases}$$

$$= \sum_{n=x}^{\infty} \binom{n}{x} p^x q^{n-x} e^{-\lambda} \frac{\lambda^n}{n!}$$

$$= \frac{e^{-\lambda}\lambda^x p^x}{x!} \sum_{n=x}^{\infty} \frac{\lambda^{n-x} q^{n-x}}{(n-x)!}$$

$$= \frac{e^{-\lambda}(\lambda p)^x}{x!} \sum_{n-x=0}^{\infty} \frac{(\lambda q)^{n-x}}{(n-x)!}$$

$$= e^{-\lambda} \frac{(\lambda p)^x}{x!} e^{\lambda(1-p)}$$

$$= \frac{e^{-\lambda p}(\lambda p)^x}{x!}, \quad x = 0, 1, \ldots$$

$X \sim Poi(\lambda p)$.

$0 < x < n < \infty$.

$\begin{cases} x < n < \infty \\ 0 < n < \infty. \end{cases}$

Find the value of $E\left(\frac{1}{1+X}\right)$

$$E\left(\frac{1}{1+X}\right) = \sum_{x=0}^{\infty} \frac{1}{1+x} e^{-\lambda} \frac{\lambda^x}{x!}$$

$$= \sum_{x=0}^{\infty} \frac{e^{-\lambda}}{\lambda} \frac{\lambda^{(x+1)}}{(x+1)!} = \frac{e^{-\lambda}}{\lambda} \sum_{x+1=y=1}^{\infty} \frac{\lambda^y}{y!}$$

$$= \frac{e^{-\lambda}}{\lambda}\left(1 - e^{-\lambda}\right)$$

Recursion Relation of Poisson Distribution :-

$$\mu_r = E(X-\lambda)^r = \sum_{x=0}^{\infty} (x-\lambda)^r e^{-\lambda} \frac{\lambda^x}{x!}$$

$$\frac{d\mu_r}{\partial\lambda} = -\sum_{x=0}^{\infty} r(x-\lambda)^{r-1} e^{-\lambda} \frac{\lambda^x}{x!} - \sum_{x=0}^{\infty} (x-\lambda)^r e^{-\lambda} \frac{\lambda^x}{x!} + \sum_{x=0}^{\infty} (x-\lambda)^r e^{-\lambda} \frac{x\lambda^{x-1}}{x!}$$

$$= (-r)\sum_{x=0}^{\infty} (x-\lambda)^{r-1} e^{-\lambda} \frac{\lambda^x}{x!} + \sum_{x=0}^{\infty} (x-\lambda)^r e^{-\lambda} \frac{\lambda^x}{x!}\left[-1 + \frac{x}{\lambda}\right]$$

$$= (-r)\mu_{r-1} + \sum_{x=0}^{\infty} (x-\lambda)^r e^{-\lambda} \frac{\lambda^x}{x!} \frac{x-\lambda}{\lambda}$$

$$= (-r)\mu_{r-1} + \frac{1}{\lambda} \sum_{x=0}^{\infty} (x-\lambda)^{r+1} e^{-\lambda} \frac{\lambda^x}{x!}$$

$$= (-r)\mu_{r-1} + \frac{1}{\lambda}\mu_{r+1}$$

$$\Rightarrow \mu_{r+1} = \lambda\left(\frac{d\mu_r}{d\lambda} + r\mu_{r-1}\right)$$

Corollary :

$\mu_0 = 1, \quad \mu_1 = 0.$

① $r = 1, \quad \mu_2 = \lambda\left(\frac{d\mu_1}{d\lambda} + \mu_0\right) = \lambda$ = variance

② $r = 2, \quad \mu_3 = \lambda\left(\frac{d\mu_2}{d\lambda} + 2\mu_1\right) = \lambda(1+0) = \lambda$

③ $r = 3, \quad \mu_4 = \lambda\left(\frac{d\mu_3}{d\lambda} + 3\mu_2\right) = \lambda(1+3\lambda) = 3\lambda^2 + \lambda$

④ Skewness $= \frac{\mu_3}{\mu_2^{3/2}} = \frac{\lambda}{\lambda^{3/2}} = \frac{1}{\sqrt{\lambda}} > 0$

$\therefore$ positively skewed.

⑤ Kurtosis $= \frac{\mu_4}{\mu_2^2} - 3 = \frac{3\lambda^2 + \lambda}{\lambda^2} - 3 = 3 + \frac{1}{\lambda} - 3 = \frac{1}{\lambda} > 0.$

$\therefore$ leptokurtic.

Poisson Approximation of Binomial Distribution :-

Suppose, $X \sim Bin(n, p)$. If the following conditions are satisfied.

① $n \to \infty$. (The no of trails are infinitely large)
② $p \to 0$ (probability of success in each trail is too small)
③ $\begin{cases} np \to \lambda & (\text{finite} > 0) \\ np = \lambda & (\text{finite} > 0) \end{cases}$

Ex: ① $p = \frac{1}{n}, \Rightarrow np = 1$, ($n$ = large), $\lambda = 1$

② $p = \frac{1}{n} + \frac{1}{n^2}$, ($n$ large), $n \to \infty$, $\Rightarrow p \to 0$

$np = 1 + \frac{1}{n} \to 1, \quad \lambda \to 1.$

The distribution of X will converge in a Poisson Distribution with parameter λ.

proof : $X \sim Bin(n, p)$.

$$p(x) = \binom{n}{x} p^x q^{n-x}$$

$$\lim_{\substack{n\to\infty \\ p\to 0 \\ np=\lambda}} \binom{n}{x} p^x q^{n-x} = \lim_{n\to\infty} \frac{n!}{x!\,(n-x)!} \left(\frac{\lambda}{n}\right)^x \left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$= \frac{\lambda^x}{x!} \lim_{n\to\infty} \frac{n!}{(n-x)!} \frac{1}{n^x} \left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$= \frac{\lambda^x}{x!} \lim_{n\to\infty} \left[\frac{n(n-1)(n-2)\cdots(n-x+1)}{n^x}\right] \left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$= \frac{\lambda^x}{x!} \lim_{n\to\infty} \underbrace{\left[1\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\cdots\left(1-\frac{x-1}{n}\right)\right] \frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^x}} \quad -①$$

Now, $\lim_{n\to\infty} \left[1\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{x-1}{n}\right)\right] = 1$

$$\lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^n = e^{-\lambda}$$

$$\lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^x = e^{-\infty} = 0.$$

By ①, $\lim_{\substack{n\to\infty \\ p\to 0 \\ np=\lambda}} \binom{n}{x} p^x q^{n-x} = \frac{\lambda^x}{x!} e^{-\lambda}$ (pmf of poisson distribution)

Poisson Approximation of Negative Binomial Distribution :-

Let $X \sim NB(r, p)$. Suppose the following conditions are satisfied,

1) $r \to \infty$
2) $q \to 0 \Leftrightarrow p \to 1$
3) $rq \to \lambda$ or $rq = \lambda$.

Then, the distribution of X will converge to a Poisson (λ) distribution.

Proof - $X \sim NB(r, p)$.

$$P(X=x) = \binom{x+r-1}{x} p^r q^x$$

$$\lim_{\substack{r\to\infty \\ q\to 0 \\ rq=\lambda}} \binom{x+r-1}{x} p^r q^x = \lim_{r\to\infty} \frac{(x+r-1)!}{x!\,(r-1)!} \left(1-\frac{\lambda}{r}\right)^r \left(\frac{\lambda}{r}\right)^x$$

$$= \frac{\lambda^x}{x!} \lim_{r\to\infty} \left[\frac{(x+r-1)(x+r-2)\cdots(r+1)r}{r^x}\right] \left(1-\frac{\lambda}{r}\right)^r$$

$$= \frac{\lambda^x}{x!} \lim_{r\to\infty} \left[\left(\frac{x-1}{r}+1\right)\left(\frac{x-2}{r}+1\right)\cdots\left(1+\frac{1}{r}\right)1\right] \left(1-\frac{\lambda}{r}\right)^r$$

Now, $\lim_{r\to\infty} \left[\left(\frac{x-1}{r}+1\right)\left(\frac{x-2}{r}+1\right)\left(1+\frac{1}{r}\right)1\right] = 1$

$$\lim \left(1-\frac{\lambda}{r}\right)^r = e^{-\lambda}$$

$\lim_{r\to\infty} \left(1-\frac{\lambda}{r}\right)^{r} = e^{-\lambda}$

$\therefore \lim_{\substack{r\to\infty \\ q\to 0 \\ rq=\lambda}} \binom{x+r-1}{x} p^{r} q^{x} = e^{-\lambda}\frac{\lambda^{x}}{x!}$ [pmf of Poisson Distribution]

Remark – If $X \sim Poi(\lambda)$ for any function $g(X)$

$$E[Xg(X)] = \sum_{x=0}^{\infty} x g(x) e^{-\lambda}\frac{\lambda^{x}}{x!}$$

$$= \sum_{x=1}^{\infty} g(x) e^{-\lambda}\frac{\lambda^{x}}{(x-1)!}$$

$$= \sum_{y=x-1=0}^{\infty} g(y+1) e^{-\lambda}\frac{\lambda^{y+1}}{y!}$$

$$= \lambda \sum_{y=0}^{\infty} g(y+1) e^{-\lambda}\frac{\lambda^{y}}{y!}$$

$$= \lambda E[g(Y+1)] \equiv \lambda E[g(X+1)]$$

Now, 1) $g(X) = c \; (\neq 0)$ let.

$E(Xc) = \lambda E(c) = \lambda c \Rightarrow E(X) = \lambda$

2) $g(X) = X$

$E(X^2) = \lambda E(X+1) = \lambda[\lambda+1] = \lambda^2+\lambda.$

$V(X) = E(X^2) - E^2(X) = \lambda^2+\lambda-\lambda^2 = \lambda$

1) If $X_i \sim Poi(\lambda_i)$ $i=1,2$ independently, then show that $X_1+X_2 \sim Poi(\lambda_1+\lambda_2)$ [Reproductive Property].

$Y = X_1+X_2$

$$P(Y=y) = P(X_1+X_2=y) = P(X_1=x, X_2=y-x)$$

$$= \sum_{x=0}^{\infty} P(X_1=x)\, P(X_2=y-x)$$

$$= \sum_{x=0}^{\infty} e^{-\lambda_1}\frac{\lambda_1^{x}}{x!} e^{-\lambda_2}\frac{\lambda_2^{y-x}}{(y-x)!}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{y!} \sum_{x=0}^{\infty} \lambda_1^{x} \lambda_2^{y-x} \frac{y!}{x!\,(y-x)!}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{y!} (\lambda_1+\lambda_2)^{y}$$

$\therefore Y = X_1+X_2 \sim Poi(\lambda_1+\lambda_2)$

2) Find the conditional distribution of $(X_1 | X_1 + X_2)$.

$$P[X_1 = x | X_1 + X_2 = y] = \frac{P(X_1 = x,\ X_1 + X_2 = y)}{P(X_1 + X_2 = y)}$$

$$= \frac{P(X_1 = x)\ P(X_2 = y - x)}{P(X_1 + X_2 = y)}$$

$$= \frac{e^{-\lambda_1} \lambda_1^{x} / x! \quad e^{-\lambda_2} \lambda_2^{y-x} / (y-x)!}{e^{-(\lambda_1+\lambda_2)} \frac{(\lambda_1+\lambda_2)^y}{y!}}$$

$$= \frac{y!}{x!\,(y-x)!} \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^{x} \left(1 - \frac{\lambda_1}{\lambda_1+\lambda_2}\right)^{y-x}$$

So, $(X_1 | X_1 + X_2 = y) \sim Bin\left(y, \frac{\lambda_1}{\lambda_1+\lambda_2}\right)$

3) $X \sim Poi(\lambda)$. Show that, a) $P(X \le \frac{\lambda}{2}) < \frac{4}{\lambda}$

b) $P(X > 2\lambda) < \frac{1}{\lambda}$

a) $P(X \le \frac{\lambda}{2}) = P(-X \geq -\frac{\lambda}{2}) = P(\lambda - X \geq \frac{\lambda}{2})$

$$= P\left((\lambda - X)^2 \geq \frac{\lambda^2}{4}\right)$$

$$\leq \frac{E(\lambda - X)^2}{\lambda^2/4} = \frac{4\lambda}{\lambda^2} = \frac{4}{\lambda}$$

b) $P(X > 2\lambda) = P(X - \lambda > \lambda)$

$$= P\left((X-\lambda)^2 > \lambda^2\right) \leq \frac{E(X-\lambda)^2}{\lambda^2} = \frac{\lambda}{\lambda^2} = \frac{1}{\lambda}.$$

Cumulative Distribution :

If $X \sim Poi(\lambda)$, for any non-negative integer k,

$$P(X \le k) = \frac{1}{\overline{|k+1}} \int_{\lambda}^{\infty} e^{-k} x^k \, dx$$

$P(X \le k) = \sum_{x=0}^{k} e^{-\lambda} \frac{\lambda^x}{x!} = g(\lambda)$ (say) $\rightarrow$ a continuous function of λ.

$$g'(\lambda) = -\sum_{x=0}^{k} e^{-\lambda} \frac{\lambda^x}{x!} + \sum_{x=0}^{k} e^{-\lambda} \frac{x \lambda^{x-1}}{x!}$$

$$= \sum_{x=0}^{k} \left\{ \frac{\lambda^{x-1}}{(x-1)!} e^{-\lambda} - e^{-\lambda} \frac{\lambda^x}{x!} \right\}$$

$$= e^{-\lambda} \{ \cancel{\lambda^0} \quad \cancel{\lambda^0} \quad \cancel{\lambda^1} . \quad \cancel{\lambda^1} \qquad \cancel{\lambda^{k-1}} \quad \cancel{\lambda^{k-1}} \quad \lambda^k \}$$

$$- \quad \left\{ \frac{}{\cancel{0!}} + \frac{}{\cancel{0!}} - \frac{}{\cancel{1!}} + \frac{}{\cancel{1!}} - \cdots - \frac{}{\cancel{(k-1)!}} + \frac{}{\cancel{(k-1)!}} - \frac{}{k!} \right\}$$

$$= - e^{-\lambda} \frac{\lambda^k}{k!}$$

we can say, $g'(u) = - e^{-u} \frac{u^k}{k!}$

$$\int_{\lambda}^{\infty} g'(u)\, du = - \int_{\lambda}^{\infty} e^{-u} \frac{u^k}{k!}\, du$$

$$\Rightarrow g(\infty) - g(\lambda) = - \frac{1}{\Gamma k+1} \int_{\lambda}^{\infty} e^{-u} u^k\, du$$

$$g(\lambda) = \frac{1}{\Gamma k+1} \int_{\lambda}^{\infty} e^{-u} u^k\, du$$

Note : $P(X \le k) = \sum_{x=0}^{k} f_X(x) = g(\lambda)$

$g(\lambda)$ is a decreasing function of λ.

The tail function of X is given by,

$$P(X > x) = 1 - P(X \le x)$$

$$= 1 - \frac{1}{\Gamma k+1} \int_{\lambda}^{\infty} e^{-u} u^k\, du$$

$$= \frac{1}{\Gamma k+1} \int_{0}^{\infty} e^{-u} u^k\, du - \frac{1}{\Gamma k+1} \int_{\lambda}^{\infty} e^{-u} u^k\, du$$

$$= \frac{1}{\Gamma k+1} \int_{0}^{\lambda} e^{-u} u^k\, du$$

$$P(X \le k) = \frac{1}{\Gamma k+1} \int_{\lambda}^{\infty} e^{-u} u^{\overline{k+1}-1}\, du$$

$$P(X > k) = \frac{1}{\Gamma k+1} \int_{0}^{\lambda} e^{-u} u^{\overline{k+1}-1}\, du = \Gamma_{\lambda}(k+1)$$ (Just a notation for incomplete gamma function)

4) If $X \sim Poi(\lambda)$ for any non-negative k show that, $P(X \ge k) < \frac{\lambda^k}{k!}$

$$P(X \ge k) = P(X > k-1) = \frac{1}{\Gamma k} \int_{0}^{\lambda} e^{-u} u^{k-1}\, du$$

$$< \frac{1}{\Gamma k} \int_{0}^{\lambda} u^{k-1}\, du \quad (\because e^{-u} > 0)$$

$$= \frac{1}{\Gamma k} \left[\frac{u^k}{k} \right]_0^{\lambda}$$

$$= \frac{1}{k(k-1)!} \lambda^k = \frac{\lambda^k}{k!}$$

5) The no of bacteria in a source of a liquid is known to be a Poisson random variable with mean λ per cc. If n one cc tubes are filled with liquid, what is the probability dist of the number of test tubes that show growth? (ie. have at least one bacteria)

X = no of test tubes that shows growth.
Y = no of bacteria in 1 cc.
given, $Y \sim Poi(\lambda)$

$$P(\text{at least one bacteria}) = 1 - P(\text{no bacteria})$$
$$= 1 - \frac{e^{-\lambda}\lambda^0}{0!} = 1 - e^{-\lambda}$$

P(one test tube will show a growth)
$$= P(Y \geq 1)$$
$$= 1 - P(Y=0) = 1 - \frac{e^{-\lambda}\lambda^0}{0!} = 1 - e^{-\lambda}$$

So, the distribution of $X \sim Bin(n, p)$ where $p = 1 - e^{-\lambda}$
$$P(X=x) = \binom{n}{x} p^x q^{n-x}$$
$$= \binom{n}{x} (1-e^{-\lambda})^x (e^{-\lambda})^{n-x}$$

6) If $X \sim Bin(n, p)$ and $np = \lambda$, show that
$$\frac{\lambda^k}{k!}\left(1-\frac{\lambda}{n}\right)^{n-k} \geq P(X=k) \geq \frac{\lambda^k}{k!}\left(1-\frac{k}{n}\right)^k \left(1-\frac{\lambda}{n}\right)^{n-k}$$

$$P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \frac{n(n-1)\dots(n-k+1)}{k!}\left(\frac{\lambda}{n}\right)^k \left(1-\frac{\lambda}{n}\right)^{n-k}$$

$$= \frac{\frac{n}{n}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\dots\left(1-\frac{k-1}{n}\right)}{k!}\lambda^k \left(1-\frac{\lambda}{n}\right)^{n-k}$$

Now, $\left(1-\frac{1}{n}\right) < 1$, $\left(1-\frac{2}{n}\right) < 1, \dots, \left(1-\frac{k-1}{n}\right) < 1$

$$\Rightarrow \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\dots\left(1-\frac{k-1}{n}\right) < 1$$

$$P(X=k) = \frac{\lambda^k}{k!}\left(1-\frac{\lambda}{n}\right)^{n-k} \leq \frac{\lambda^k}{k!}\left(1-\frac{\lambda}{n}\right)^{n-k}$$

Now, 1 − 1 > 1 − K   1 − 2 > 1 − K   1 − K−1 > 1 − K

[illegible]

$$\Rightarrow \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right) > \left(1-\frac{k}{n}\right)^k$$

$$\therefore\ P(X=k) \geqslant \frac{\lambda^k}{k!}\left(1-\frac{\lambda}{n}\right)^{n-k}\left(1-\frac{k}{n}\right)^k$$

Taking limits on both side,

$$\Rightarrow e^{-\lambda}\frac{\lambda^k}{k!} \geqslant \lim_{n\to\infty} P(X=k) \geqslant e^{-\lambda}\frac{\lambda^k}{k!}$$

By Sandwich theorm, we can say $\lim_{n\to\infty} P(X=k) = e^{-\lambda}\frac{\lambda^k}{k!}$

Poisson process:

Suppose, we are observing the occurrence of an event over time or space or length. Assuming that there is a quantity such that

1. the probability that exactly one happening will occur in a small time interval of length t. Is approximately.
2. the probability that more than one happening will occur in a small time interval of length this negligible
3. The number of happenings in non-overlapping intervals are independent

Under the above condition, it can be shown that the number of occurrence in time interval of length T has a distribution.

Also, the number of occurrence in time interval of length ‘kt’ has a distribution. Here, is the mean rate of occurrence of the event in a unit time interval.

It will follow Poi ($\lambda = wkt$) distribution.

Incomplete Gamma and Beta Function :-

Consider standard Gamma $(n, \theta = 1)$ dist with $f(x) = \begin{cases} \frac{e^{-x} x^{n-1}}{\Gamma n}, & 0<x<1 \\ 0, & \text{ow} \end{cases}$

For, $0<x<\infty$, $F(x) = \int_0^x \frac{e^{-t} t^{n-1}}{\Gamma n} dt = \frac{\int_0^x e^{-t} t^{n-1} dt}{\int_0^\infty e^{-t} t^{n-1} dt}$

$$= \Gamma_x(n)$$

Tabulated values are obtained from Karl-Pearson Table.

Use : If $X \sim Poi(\lambda)$, $P(X \geq K) = \sum_{x=K}^{\infty} \frac{e^{-\lambda} \lambda^x}{x!} = \int_0^\lambda \frac{e^{-t} t^{K-1}}{\Gamma K} dt$

converse :

If X is a discrete random variable with DF $F(K) = \int_\lambda^\infty \frac{e^{-t} t^{K-1}}{\Gamma K} dt$, $K = 0, 1, 2, \ldots$ show that $X \sim Poi(\lambda)$

Relationship between Poisson and Gamma Distribution :-

Let, the no of occurance of an event over time has a Poisson distribution. Let T_n be the waiting time for the nth occurance of the event, then T_n has a Gamma Distribution.

Examples of Poisson Distribution:

1. Under the condition of Poisson process, some random variables involving counts of happenings of an event over time, space or length, can be modelled realistically, by a Poisson distribution. Following are such examples of Poisson distribution.

a) The number of telephone calls in an hour in a large business house.

b) The number of defects in an unit of a material.

c) The number of natural deaths in a year in a given region.

2. Each of the following and numerous others random variables are approximately following Poisson distribution because of Poisson approximation to Binomial.

a) The number of misprints in a page of a book.

Here, we can assume that there is a small probability 'p' that each letter typed will be misprinted and the number of letters 'n' in a page is large. Hence, the number of misprints in a page will approximately follow a Poisson distribution. (np=lambda)

If the printing quality is poor or bad, the probability ‘p’ that each letter typed will be misprinted, is not a small quantity, then the Poisson approximation to Binomial is not applicable.

b) The number of individuals in a large community living upto 100 years of age.