Thursday, August 31, 2017

General Outline

Probability Experiment

- Well designed results
  - you know the result (IE: Flipping a Coin)
  - Results = Outcome
  - Trails

"Sample Space" = Set of all outcome

"Event" = Set of outcomes

- Flipping a Coin:

  S = { H, T } or Ω = { H, T }

  E = { H }

Classical Definition of Probability

- Equally likely outcome

Probability of Event = $P(\{E\}) = \dfrac{\text{number of outcomes in the event}}{\text{number of outcome in the sample space}} = \dfrac{n(e)}{n(s)}$

EX: Roll a Fair Die

S = {1, 2, 3, 4, 5, 6}

P(rolling a tree) = 1/6

P(rolling an even) = 3/6 = 1/2

P(five or two) = P(five) + P(two)

- Mutually Exclusive = Cannot occur at the same time
- Mutually Exclusive events of outcome = sum of both events
- P (A or B) = P(A) + P(B) if A & B are Mutually Exclusive

Complement of a Set = Makes up the whole set

Probability of Your sample space = 1

- Like saying what is the probability of getting Head or Tail on a coin flip

Probability of the Event = 1 - Probability of Compliment of the Event(1 - P(!E))

EX: Standard Deck of Cards

Sample Space = 52

P(3 of ♦) = 1/52

P(K or ♥) = 4/52 + 13/52 = 17/52 **INCORRECT** (Double Count)

- Both can happen at the same time (Not Mutually Exclusive)

EX: The probability a student owns a laptop is 80%. The probability a student owns a tablet is 30%. The probability a student owns both laptop and tablet is 25%. What is the probability of a student owns neither a laptop nor a tablet?

P(L or T) = P(L) + P(T) - P(L and T)

.8 + .3 - .25

1.1 - .25

.85

1 - P(L or T) = 1 - P(!L and !T)

1 - .85 = .15

<u>Lab</u>

Pseudo Random Number Generator

- Pseudo = False
- Uniform random = Individual have same probability of happening
    - Opposite from Deterministic Probability

Classical Probability

Review:

- Experiment
- Sample Space
- Event
- Equally Likely Outcome
- Mutually Exclusive

$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$

EX: Roll a fair die

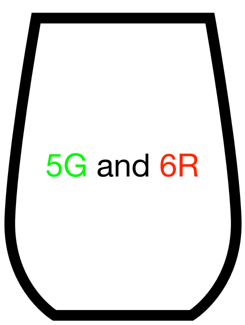
|

Conditioning:

$$P(\text{four given even}) = 1/3 = \frac{1/6}{3/6} = \frac{P(\text{four and even})}{P(\text{even})}$$

$$P(\text{ A given B}) = \frac{P(A \cap B)}{P(B)} = P(A \mid B) = \frac{P(A \quad B)}{P(A)} = P(A \cap B) = P(A)P(B|A)$$

EX: In an URN:



Remove one random chip: $P(\{G\}) = 5/11$

Remove two chips in succession (without replacement)

$$= P(G_1 \cap G_2) = P(G_1)\, P(G_2 \mid G_1) = 5/11 * 4/10 = 2/11$$

EX: A Company makes two models α and β of a microwave oven. 3% of the α model are defective and 7% of the β model are defective. As a charitable act, the company gives away these microwave to survivors of Harvey. The probability a survivor gets a β microwave is 40%. What is the probability it is defective?

• Survivor can't get both microwaves at the same time

$P(\alpha \cap D)$ and $P(\beta \cap D)$ are mutually exclusive

$P(D) = P(\alpha \cap D) + P(\beta \cap D)$

$P(D) = P(\alpha)\ P(D \mid \alpha) + P(\beta)\ P(D \mid \beta)$

$P(D) = (0.6)(0.03) + (0.4)(0.07) = 0.018 + 0.028 = 0.046 = 4.6\%$

What is the probability it is an α model given it is defective?

- Bayes rules = reverse of $P(D \mid \alpha)$

$$P(\alpha \mid D) = \frac{P(\alpha \cap D)}{P(D)} = 0.018 / 0.046 = 0.38 = 39\%$$

Tuesday, September 5, 2017

Independence vs. Dependence

Independent Event

If $P(B \mid A) = P(B)$ then A & B one said to be independent and we have

$P(A \cap B) = P(A)P(B)$

IE: Consider Flipping two fair coins

$S = \{ H\ H \}, \{H\ T\}, \{T\ H\}, \{T\ T\}$

$P(\{H\ H\}) = 1/4 = 1/2 * 1/2 = P(\{H\}) * P(\{H\})$

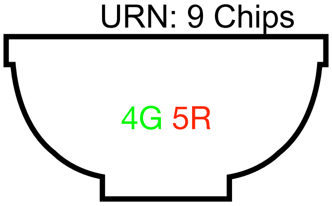
- IN general for independence events

$P(A \cap B \cap C \cap D) = P(A)*P(B)*P(C)*P(D)$

P({ Summation of i = 1 to n of A_i })

IE:

URN: 9 Chips



Remove two chips in succession. What is the probability they are both red?

i) without replacement (Conditional)

$- P(\{R_1 \cap R_1\}) = P(\{R_1\})P(\{R_2 \mid R_1\}) =$

$= 5/9 * 4/8 = 5/18$

ii) with replacement (Independent)

$-P(\{R_1 \cap R_2\}) = P(\{R_1\})P(\{R_2\}) =$

$= 5/9 * 5/9 = 25/81$

Bernoulli Trial

- Has two outcomes

- Often refers to Success and Failure

$P(\{S\}) = 1 - P(\{F\})$

$P(\{S\}) = p$

$P(\{F\}) = q$ $\quad\quad$ $p + q = 1$

Multiplication Rule

IE: 5 pants

4 Shirts $\quad\quad$ # of outfits = 5x4x6 = 120 outfits

6 Ties

Thursday, September 7, 2017

Population & Sample

Arithmetic Average

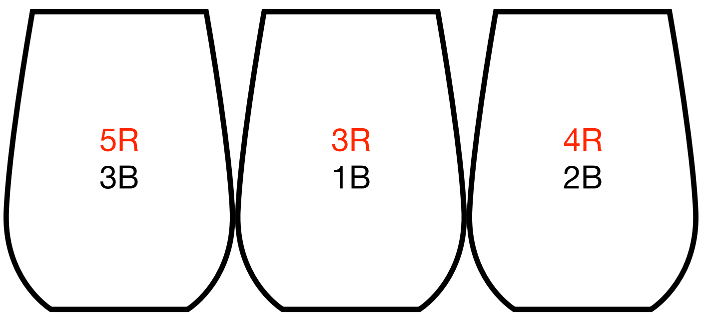
= Mean

Measure of Central Tendency

- Mean = (ε * x) / N
- Median = middle
- Mode = most frequent
    - if there are 2 numbers that are most frequent = mode is both

Practice Quiz:

> 2.) (5 pt.) Urn 1 contains 5 red balls and 3 black balls. Urn 2 contains 3 red balls and 1 black ball. Urn 3 contains 4 red balls and 2 black balls. If an urn is selected at random and a ball is drawn, find the probability it will be red.



1/3 (5/8 + 3/4 + 4/6) = 49/72

3.) (1 pt.) If $P(A) = \frac{1}{3}$, $P(B) = \frac{1}{2}$, and $P(A \cup B) = \frac{3}{4}$ what is the probability of $P(A \cap B)$?

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Measures of Variation

Tuesday, September 12, 2017

Permutation and Combination

Probability of Permutation & Combination

Four Distinct Object:

A, B, C, D

- ${}_4P_2$ = 4! / (4-2)! = 4! / 2! = 12
- ${}_4C_2$ = 4! / (4-2)! * 2! = 6

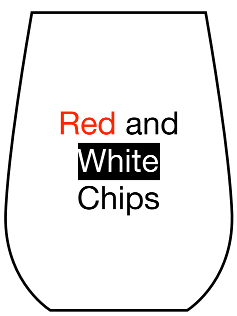
General Formula

$${}_nP_r = \frac{n!}{(n-r)!}$$

$${}_nC_r = \frac{n!}{(n-r)!r!}$$

EX:

URN:



r = # of Red Chips

w = # of white Chips

r + w = N

remove a sample of size N

(At the same time)

Y = # of red chips in the sample

Suppose that an urn contains r Red chips and w White chips (r + w = N). If n chips are drawn out at random, without replacement, and Y denotes the total number of red chips selected, Y is said to have “Hyper-geometric distribution” and P({Y=K}) = $\frac{{}_rC_k * {}_wC_{(n-k)}}{{}_NC_n}$

Thursday, September 14, 2017

Binomial Distribution

Probability

Motivation:

Toss a coin three times (Not necessarily fair)

$P(\{H\}) = p$ "X" is a random variable

$P(\{T\}) = q$ X = number of heads

$p+q = 1$ p^2q = probability of 2 heads (in 3 toss)

- $3(p^2q)$ = happens 3 times out of the 15 possible outcomes

X = x	P({X = x})
3	p^3
2	$3(p^2q)$
1	$3(pq^2)$
0	q^3

$P(\{X=x\}) = {}_nC_x\ p^xq^{(n-x)}$

n = Number of Trials

p = Probability of success

x = Number of successes

4 Requirements: Distribution (each distribution have its own requirements)

1. Finite number of trails
2. Two outcomes each trials
3. Each trial is independent
4. The Probability of Success "p" is constant
   - Since "p" is constant, replacement is implied to keep same success rate

Thursday, September 14, 2017

There are three tests in an inspection process. The probability of passing any one of these tests is the same and equal to $\frac{1}{3}$. (Each test is independent of the other) Determine the following…..

i) All test are failed.

ii). One test is passed $P(\{X=x\}) = {}_nC_x\, p^x q^{(n-x)}$

iii). Two tests are passed

iv). Complete Success $q = p - 1$ since $p+q = 1$

$n = 3$
$p = \frac{1}{3}$
$x = 0, 1, 2, 3$

i). For x = 0

$$P(\{X=x\}) = {}_3C_0\, (1/3)^0(2/3)^{(3-0)} = \frac{8}{27}$$

ii). For x = 1

$$P(\{X=x\}) = {}_3C_1\, (1/3)^1(2/3)^{(3-1)} = \frac{12}{27}$$

Explicitly

iii). For x = 2

$$\frac{8}{27} + \frac{12}{27} + \frac{6}{27} + \frac{1}{27} = 1$$

$$P(\{X=x\}) = {}_3C_2\, (1/3)^2(2/3)^{(3-2)} = \frac{6}{27}$$

iv). For x = 3

$$P(\{X=x\}) = {}_3C_3\, (1/3)^3(2/3)^{(3-3)} = \frac{1}{27}$$

P({passing at least one test}) = 1 - P(none)

In LAB:

C = have the Disease

B = Positive Diagnostic

P(C) = 0.0001 P(C') = 0.9999

P(B|C) = 0.90

P(B|C') = 0.001 = false positive

P(C|B) = $\frac{P(BandC)}{P(B)}$ =

$$\frac{P(C)P(B\mid C)}{P(C)P(B\mid C)+P(C')P(B\mid C')} = \frac{(0.0001)(0.9)}{0.0001(0.9)+0.9999(0.001)} =$$

$$\frac{0.00009}{0.00009+0.0009999} \approx 0.0825764 \approx 8.26\%$$

•False positive = P(B|C')

Tuesday, September 19, 2017

Random Variable

Expectation

Rolling two fair dice

- Equally likely Outcome

There are 36 Outcomes and $P(11) = \dfrac{1}{36}$

- Example Random Variable S_1

$S = D_1 + D_2$

$S \in \{ 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 \}$ Purple = Random Variables

$P(2) = 1/36$ Green = Probability $\neq$ R.V.

$P(3) = 2/36$

$P(4) = 3/36$

$P(5) = 4/36$

$P(6) = 5/36$

$P(7) = 6/36$ = Diagonal where it is the highest chance

$P(8) = 5/36$

$P(9) = 4/36$

$P(10) = 3/36$

$P(11) = 2/36$

$P(12) = 1/36$

Random Variable:

- A real value function where domain is the same space S(n**Ω**)

(Usual Notation is Upper case Latin: W,X,Y,Z)

Probability Mass Function (pmf) Such That

For the RV $X = F_x (X) = P(\{s \in S \mid X(s) = x\})$

X is binomially distributed

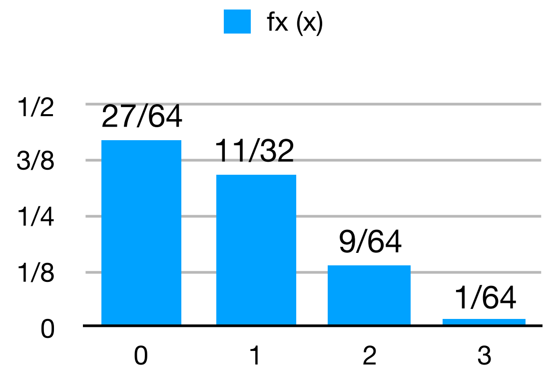
with n = 3, p = 1/4

$P(\{ x = 0 \}) = {}_3C_0(\frac{1}{4})^0(\frac{3}{4})^3 = \frac{27}{64}$

$P(\{ x = 1 \}) = {}_3C_1(\frac{1}{4})^1(\frac{3}{4})^2 = \frac{22}{64}$

$P(\{ x = 2 \}) = {}_3C_2(\frac{1}{4})^2(\frac{3}{4})^1 = \frac{9}{64}$

$P(\{ x = 3 \}) = {}_3C_3(\frac{1}{4})^3(\frac{3}{4})^0 = \frac{1}{64}$



Let X be a RV defined on a sample space S with probability function P. For any real number x the cumulative distribution function for X (cdf) $F_x(X)$ is the probability associated with the set of sample points on S that get mapped log X into value on the real line less than or equal to x.

$F_x(X) = P(\{s \in S \mid X(s) \leq x\})$

$F_x(X) = P(\{X \leq x\})$

Thursday, September 21, 2017

Expectation

Technical & Stipulation

$E(x) = \sum x_i \, f_x(x)$

Motivation:

Consider a Fair Die

$\boldsymbol{\Omega} = \{1,2,3,4,5,6\}$ x = face value (RV)

$$u = \frac{\sum x_i}{N} = \frac{1+2+3+4+5+6}{6} = \frac{21}{6} = 3.5$$

i = [1,6]

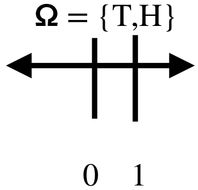
P({X=5}) = 1/6

1/6 (1+2+3+4+5+6)

p.m.f

Coin Fair

$\boldsymbol{\Omega} = \{T,H\}$



$f_Y(y_i) = P(\{Y=y_i\})$

$f_y(0) = 1/2$ $f_y(1) = 1/2$

Binomial Distribution Mean of X

Binomial Demonstration

$$u = E(x) = \sum_{all\ \mathrm{x}} x f_X(x) = \sum_{j=0}^{n} j_n C_j p^j q^{n-j} = \sum_{j=0}^{n} j \frac{n}{(n-j)!j!} p^j q^{n-j} = j \frac{n!}{(n-j)!j!} p^j q^{n-j} + \sum_{j=1}^{n} j \frac{n!}{(n-j)!j} p^j q^{n-j}$$

$$=> \sum_{j=1}^{n} j \frac{n!}{(n-j)!j(j-1)!} p^j q^{n-j} = \sum_{j=1}^{n} \frac{n!}{(n-j)!(j-1)!} p^j q^{n-j} = \sum_{j=1}^{n} j \frac{n(n-1)!}{(n-j)!(j-1)!} p * p^{j-1} q^{n-j}$$

$$=> np \sum_{j=1}^{n} \frac{(n-1)!}{(n-j)!(j-1)!} p^{j-1} q^{n-j} = np \sum_{j=1}^{n} j \frac{(n-1)!}{[(n-1)-(j-1)]!(j-1)!} p^{j-1} q^{[(n-1)-(j-n)]}$$

Digression:

$${}_N c_i = \frac{N!}{(N-i)!i!} \text{ Let k = j-1 } => np \sum_{k=0}^{n-1} \frac{(n-1)!}{[(n-1)-k]!k!} p^k q^{[(n-1)-k]}$$

Digression:

$$(a+b)^n = \sum_{k=0}^{n} {}_n C_k a^{(n-k)} b^k \text{ (The Binomial Theorem)}$$

$$^* np(q+q)^{n-1} = np \text{ (QuadErat Demonstration)}$$

Expected Values & Motivation

Properties & Examples

For any random variable X and Y and any numbers a & b

- E(aX + bY) = aE(X) + bE(Y) (E = expectation)

  provided both E(X) & E(Y) exists

Explanation:

$$E(x) = \sum_{all\ \text{x}} x f_x(x)$$

$$E(aX + bY) = E(aX) + E(bY) = \sum_{\text{all x}} axf_x(x) + \sum_{\text{all y}} byf_y(y)$$

$$a \sum_{\text{all x}} xf_x(x) + b \sum_{\text{all y}} yf_y(y)$$ (This is NOT A PROOF!!!)

In General:

$$E(\sum_{j=1}^{n} ajX_j) = \sum_{j=1}^{n} ajE(X_j)$$

(The expectation of the sum is the sum of the expectations)

Be able to do this on the TEST!! :

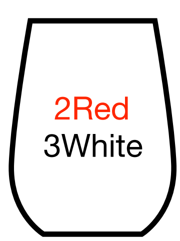
$$Var(X) = E[(X - \mu_x)^2]$$

$$E[X^2 - 2\mu_x X + \mu_x^2] = E(X^2) + E(-2\mu_x X) + E(\mu_x^2) => $$

$$=> E(X^2) - 2\mu_x E(X) + \mu_x^2 => E(X^2) - 2\mu_x \mu_x + \mu_x^2 =>$$

$$=> E(X^2) - 2\mu_x^2 + \mu_x^2 => E(X^2) - \mu_x^2$$ (This is a Proof!!)

EX: An urn contains 5 Chips, 2 red & 3 white. Suppose that two are drawn out at random, without replacement. Let X denote the number of red chips in the sample.



- Two chips
- X = number of red

Determine the expectation (mean) & variance of X

$$E(X) = \sum_{x=0}^{n} x f_x(x) = (0)\frac{{}_2C_0 *_3 C_2}{{}_5C_2} + (1)\frac{{}_2C_1 *_3 C_1}{{}_5C_2} + (2)\frac{{}_2C_2 *_3 C_0}{{}_5C_2} => $$

$$=> 0 + (1)\frac{2*3}{10} + (2)\frac{1*1}{10} = \frac{4}{5}$$

$$Var(X) = E[(X-u_x)^2] = \sum_{x=0}^{2}(x-u_x)^2 f_x(x) => (0-\frac{4}{5})^2 * \frac{{}_2C_0 *_3 C_2}{{}_5C_2} + (1-\frac{6}{5})^2 * \frac{{}_2C_1 *_3 C_1}{{}_5C_2} + (2-\frac{4}{5})^2 * \frac{{}_2C_2 *_3 C_0}{{}_5C_2}$$

$$=> 0 + (1)(\frac{1}{25})(\frac{6}{10}) + (2)(\frac{26}{25})(\frac{1}{10}) = \frac{39}{125}$$

Other way of doing it: $u_x = \frac{4}{5}$

$$Var(X) = E(X^2) - u_x^2 = (\sum_{x=0}^{2} x^2 f_x(x)) - \frac{16}{25} = (0)(\frac{{}_2C_0 *_3 C_2}{{}_5C_2}) + (1)(\frac{{}_2C_1 *_3 C_1}{{}_5C_2}) + (4)(\frac{{}_2C_2 *_3 C_0}{{}_5C_2}) - \frac{16}{25}$$

$$=> 0 + \frac{6}{10} + \frac{4}{10} - \frac{16}{25} = \frac{9}{25}$$

Thursday, September 28, 2017

Motivation:

Example 1:

Flip a fair coin three times. Let X equal the number of heads. Let Y equal the number of the flip on which the first head occurred (or zero if no head occurs)

- Y = when, in position, the first head occurs

Outcomes	HHH	HHT	HTH	THH	TTH	THT	HTT	TTT
X	3	2	2	2	1	1	1	0
Y	1	1	1	2	3	2	1	0
Probability	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$

		X				
		0	1	2	3	$f_y(n)$
Y	0	$\frac{1}{8}$	0	0	0	$\frac{1}{8}$
	1	0	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{2}$
	2	0	$\frac{1}{8}$	$\frac{1}{8}$	0	$\frac{1}{4}$
	3	0	$\frac{1}{8}$	0	0	$\frac{1}{8}$
	Sum	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$	1

In math notation $f_{xy}(2,1) = \frac{1}{4}$

Questions:

P({$X \leq 2$} ∩{Y=1}) = $\frac{1}{8} + \frac{1}{4} = \frac{3}{8}$

P({X=0}∩{Y=1})+P({X=1}∩{Y=1})+P({X=2)∩{Y=1} = 0 + 1/8 + 1/4

P({$X \leq 2$} ∩ {$Y \leq 1$} = 1/2

P({$\{X \leq 2\} \cup \{Y \leq 1\} = 1$

$$=> P(\{X \leq 2\}) + P(\{Y \leq 1\} - P(\{X \leq 2\} \cap \{Y \leq 1\} => \frac{7}{8} + \frac{5}{8} - \frac{1}{2} = \frac{12}{8} - \frac{4}{8} = \frac{8}{8} = 1$$

Co-Variance

$$Cvar(W_1 Z) = E[(W - \mu_W)(Z - \mu_Z)]$$

E(X) = np = # of trial * Probability of X = 3 * 1/2 = 3/2

$$E(Y) = \sum_{y=0}^{3} y f_y(y) = 0 * \frac{1}{8} + 1 * \frac{1}{2} + 2 * \frac{1}{4} + 3 * \frac{1}{8}$$

$$Cvar(X, Y) = E[(X - \mu_X)(Y - \mu_y)] = E[(X - \frac{3}{2})(Y - \frac{11}{8})] = \sum_{x=0}^{3} \sum_{y=0}^{3} (X - \frac{3}{2})(Y - \frac{11}{8}) f_{XY}(X, Y) =>$$

$$=> (-\frac{3}{2})(-\frac{11}{8})(\frac{1}{8}) + (1 - \frac{3}{2})(1 - \frac{11}{8})(\frac{1}{8}) + (2 - \frac{3}{2})(1 - \frac{11}{8})(\frac{1}{4}) + (3 - \frac{3}{2})(1 - \frac{11}{8})(\frac{1}{8})$$

$$+(1 - \frac{3}{2})(2 - \frac{11}{8})(\frac{1}{8}) + (2 - \frac{3}{2})(2 - \frac{11}{8})(\frac{1}{8}) + (1 - \frac{3}{2})(3 - \frac{11}{8})(\frac{1}{8})$$

Poisson

What is it?

Poisson took the limit of the Binomial Distribution

IE: Let λ be a fixed number & n is an arbitrary positive integer. For each non-negative integer X

$$\lim_{n\to\infty} {}_nC_x p^x(1-p)^{(n-x)} = \frac{e^{-\lambda}\lambda^x}{x!}$$ λ = n*p

where p = $\frac{\lambda}{n}$

Justification "Proof"

$$\lim_{n\to\infty} {}_nC_x p^x(1-p)^{(n-x)} = \lim_{n\to\infty} {}_nC_x(\frac{\lambda}{n})^x(1-\frac{\lambda}{n})^{n-x} \Rightarrow$$

$$\Rightarrow lim_{n\to\infty}\frac{n!}{(n-x)!x!} * \frac{\lambda^x}{1} * \frac{1}{n^x} * (1-\frac{\lambda}{n})^n \Rightarrow$$

$$\Rightarrow \frac{\lambda^x}{x!}\lim_{n\to\infty}\frac{n!}{(n-x)!}\frac{1}{n^x}(1-\frac{\lambda}{n})^x(1-\frac{\lambda}{n})^n \Rightarrow$$

$$\Rightarrow \frac{\lambda^x}{x!}\lim_{n\to\infty}\frac{n!}{(n-\lambda)!}\frac{1}{n^x}\frac{n^x}{(n-\lambda)^x}(1-\frac{\lambda}{n})^n \Rightarrow$$

$$\Rightarrow e^{-\lambda}\frac{\lambda^x}{x!}$$

Tuesday, October 10, 2017

Exam I

Review

10 Problems:

Theory: 10pt

- Anything that he Justified
- Two theory "justified"

Computation: 30pt

Not Limited to: (Multiple Choice)

- Bayes
- General Probability
- Binomial
- Hypergeometric
- Poisson
- Geometric

Simulation: 10pt

Don't FORGET: Conditional & Independence

- Money Game with Expectation
- Covariance
- Joint Distribution
- Expectation

Summary Statistic:

- Mean
- Median
- Mode
- Standard Deviation