Stable Populations Assignment

European Doctoral School of Demography 2021

Liliana Patricia Calderón Bernal
Gonzalo Daniel Garcia
Ainhoa-Elena Leger
Ozge Elif Ozer

26/3/2021

Exercise 1: Everybody loves penguins

Here are two stage-classified matrices for the Emperor penguin. The life cycle diagram is given on the next-to-last page of the (Jenouvrier et al. 2009) paper in the readings for the course. The first of these matrices was obtained under normal conditions, the other during an unusual warm event during which the penguin population declined by about 50%.

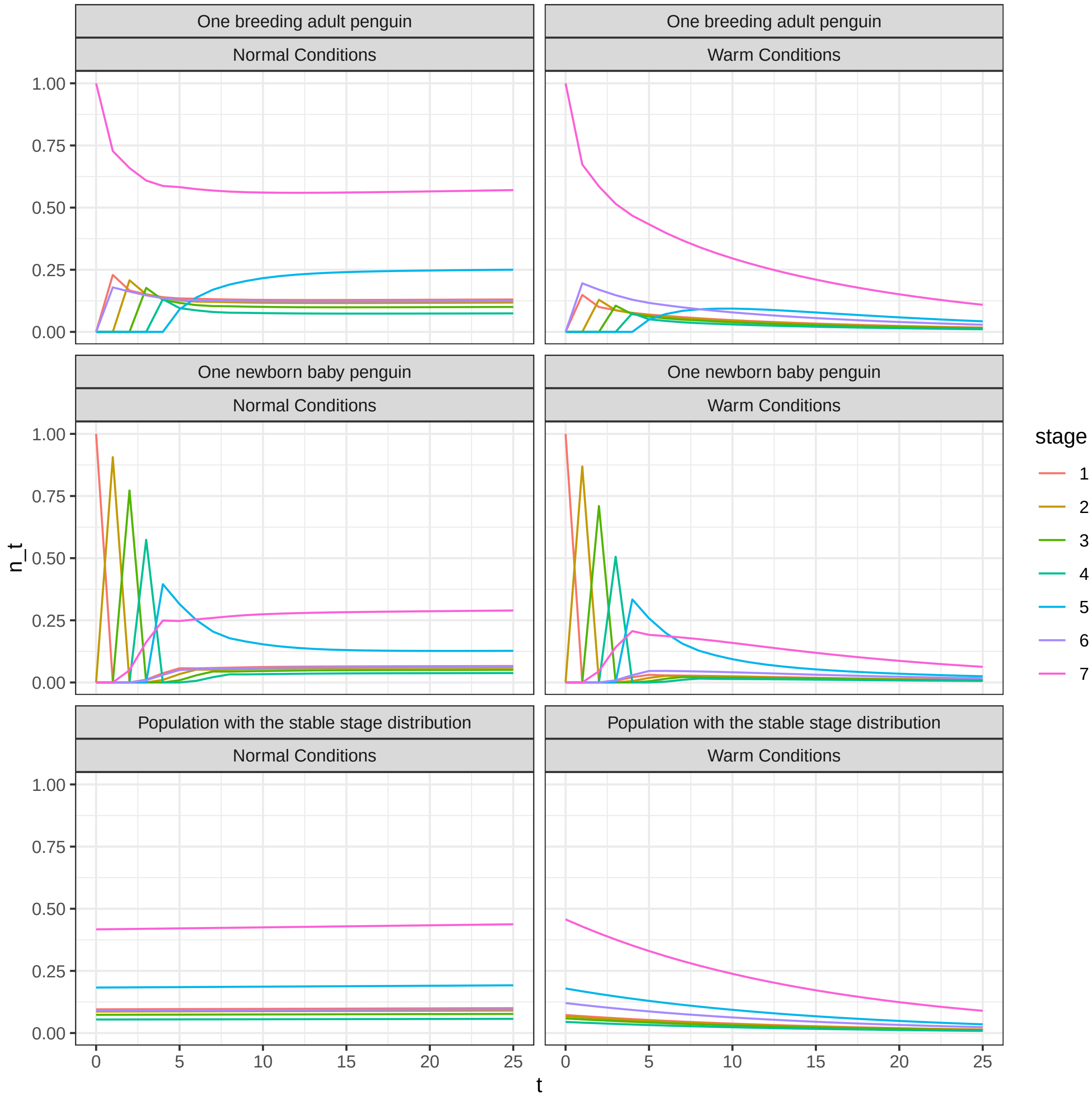
(a) Write a program to project the population (for what seems like an interesting length of time) starting from several different initial conditions:

$\mathbf{n}(0)$ = one newborn baby penguin
$\mathbf{n}(0)$ = one breeding adult penguin
$\mathbf{n}(0)$ = a population with the stable stage distribution

Plot the results (plotting on a log scale will probably be particularly informative). Compare the fates of the population under the two different environmental conditions.

[Sol.]

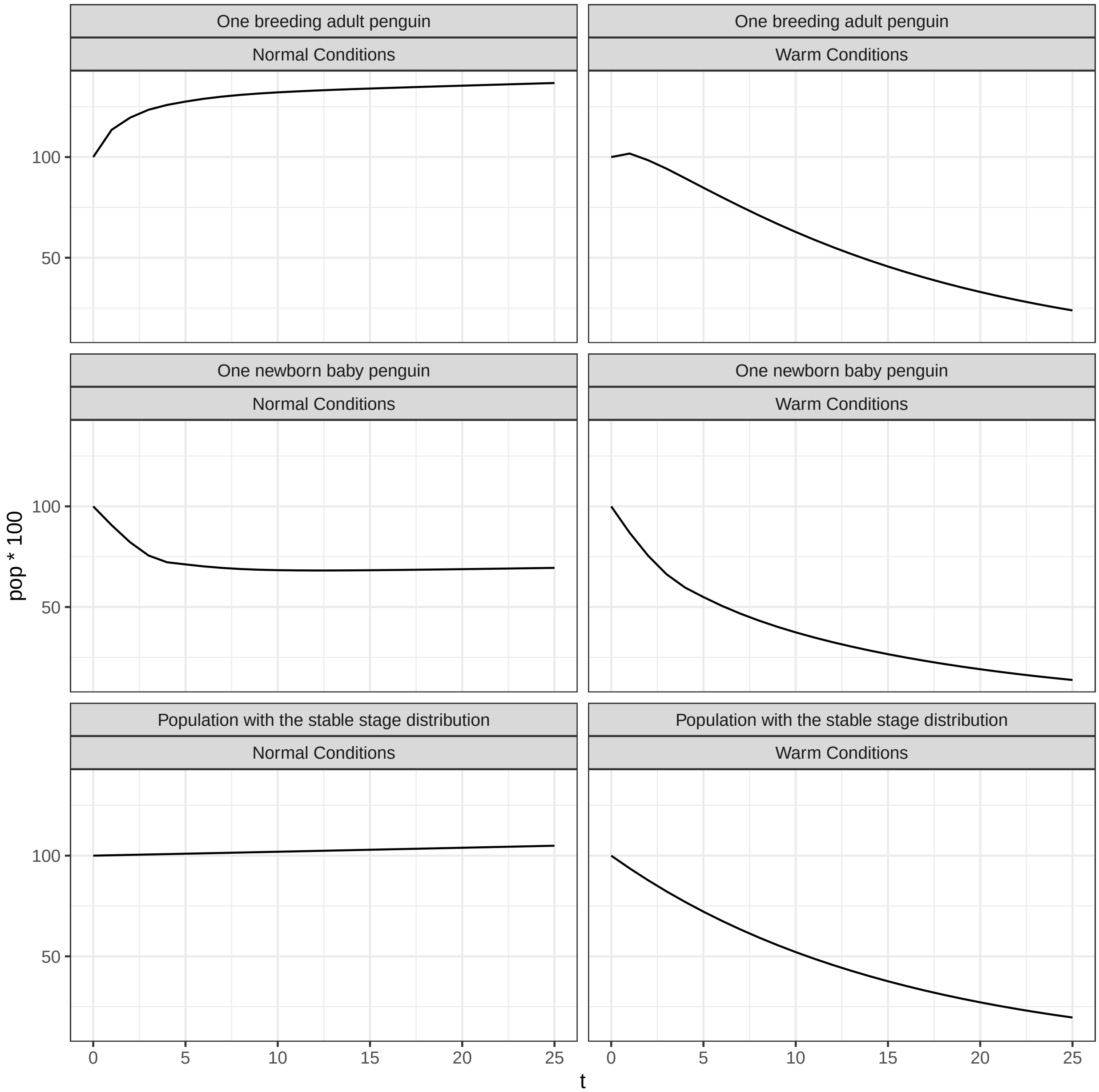
First we plot the distribution of each stage across time for the 3 different initial conditions and the 2 matrices:



As was expected, the penguin population in Warm conditions has a declining population projection in the long run. Compared to the population under Normal conditions, the stage 7 population does not stabilize in warm conditions.

Under Normal conditions, in both cases when we start with only newborns or when we start with a breeding adult penguin, the population stabilizes rather quickly around periods 5 and 10 (depending on each stage). What we found interesting about these populations is that under Normal conditions is the stage 5 the one that takes much longer to stabilize.

We can also plot the full population for each temperature and initial condition. We are representing here the sum of all stages for each period:



In the next section we will see the population growth rates that are linked to the matrices we were provided, but already from this plot we can appreciate that the Warm conditions matrix is one such that population is declining in the long run. While for the Normal conditions matrix, the rate of growth under stable stage distribution is positive but very low.

Another insight from the plots is that when the initial condition is the one where we have only newborns, then the population declines in the first periods until stabilizes (except when temperature conditions are warm).

(b) Find the population growth rate λ, the stable stage distribution $\mathbf{w}$, and the reproductive value distribution v for each matrix. Make some plots; make some comparisons.

[Sol.]

For the matrix on Normal conditions we have the following results:

- Rate of growth: 1.00192737796366+0i

- Stable Stage distribution: 0.194440489963298+0i, 0.175901830790301+0i, 0.149580062517039+0i, 0.110954052067729+0i, 0.373173973271385+0i, 0.184964925313631+0i, 0.850350284936257+0i
- Reproductive value distribution: 0.277988893150835+0i, 0.307286719789979+0i, 0.326397257241825+0i, 0.319780416181722+0i, 0.292201270312972+0i, 0.484018240807112+0i, 0.547582982944706+0i

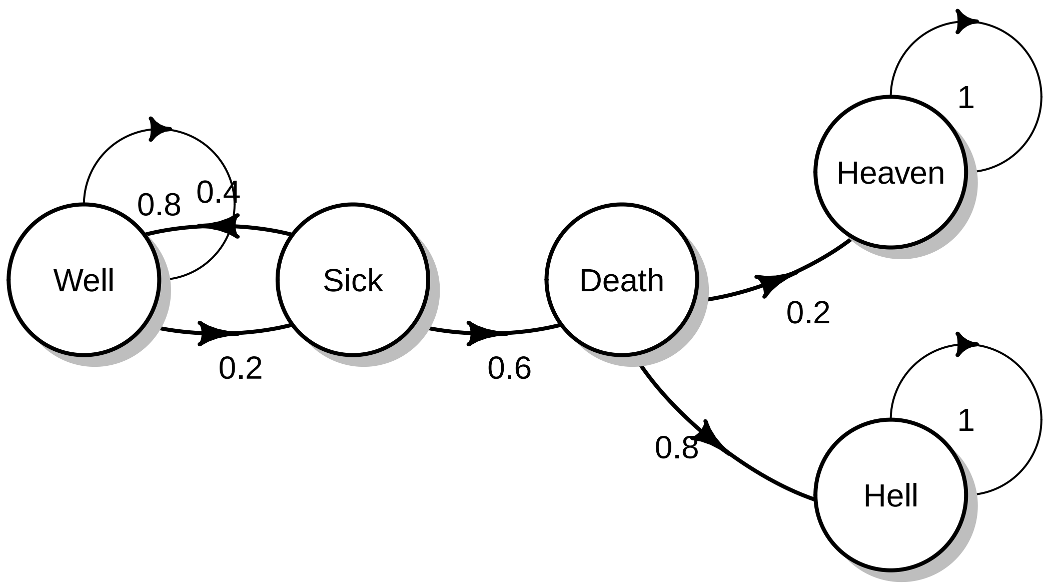
And for the matrix on Warm conditions:

- Rate of growth: 0.93688602955638+0i
- Stable Stage distribution: -0.139427539561691+0i, -0.12930984698589+0i, -0.112735465879544+0i, -0.0857351021417296+0i, -0.344220488555178+0i, -0.231497307704186+0i, -0.878464787832999+0i
- Reproductive value distribution: 0.299560273857171+0i, 0.322999005163846+0i, 0.337410300336863+0i, 0.329843613943409+0i, 0.304221577800046+0i, 0.471005898376633+0i, 0.518551289509912+0i

Exercise 2: The Irish tea-towel problem

In an airport in Belfast, I found a souveneir tea-towel, inscribed with a verse identified as "Irish philosophy". See the figure. It looks like an incidence-based health model to me.

(a) Create a life cycle graph for this system, based on your interpretation of the verse.



(b) Identify the transient and absorbing states.

[Sol.]

We make up the transition probabilities and plot the life cycle graph with them. In our life cycle, the individuals start as sick or well and then they can either change between these two states, or the sick ones can die. As it can be seen, similar to “Well” and “Sick”, “Death” is a transient state. After death, people can go to hell or heaven. “Hell” and “Heaven” as it can be seen from the graph are two absorbing states.Therefore, at the end we have an absorbing Markov Chain.

(c) Make up some transition probabilities (your choice) and calculate mean occupancy times and the probabilities of ending up in Heaven or Hell.

[Sol.]

Transition probabilities

We included these in the probability matrix from the previous question.

Mean occupancy time

Since this is an absorbing Markov chain with two absorbing states at the end of the lifecycle, we need to extract the transient matrix and mortality vector. We can calculate these with the transition probabilities matrix that we used for the Markov chain earlier.

Following the formulae for calculating the fundamental matrix, N, we end up with the a matrix that shows mean occupancy times at each stage.

By looking at this matrix, we can see that if your initial state is well, you can revisit this state 8.33 times on average, and this number is 3.33 for people whose initial state is sickness. For instance, death is a stage that can be visited only once. To calculate the probability of ending up in heaven or hell, we can make rows with the transient states, and columns with the absorbing states as follow;

```
      Hell Heaven
Well   0.0    0.0
Sick   0.0    0.0
Death  0.8    0.2
```

Exercise 3: An extra problem about Sweden

Sweden has an unusually long sequence of mortality and fertility data. There are two text files (`parray.txt` and `fertarray.txt`) in the Calculation Materials folder. One has survival probabilities as a function of age, the other has fertility as a function of age; one column for each year from 1891 to 2007.

(a) Write a program to use this information to create an age-classified projection matrix **A** for each year.

[Sol.]

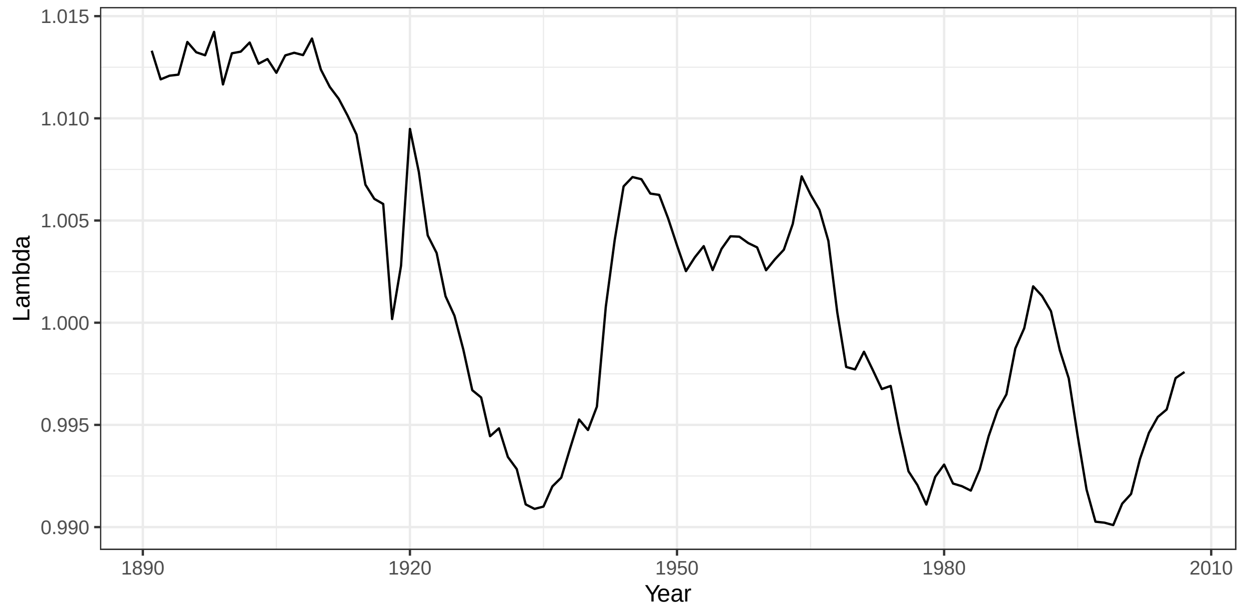
We created a list of 117 elements collecting annual **A** matrices. Each **A** matricx has fertility in the first row, survival probabilities on the subdiagonal.

```r
# Grab number of years
years <- c(1891:2007)
nby <- length(years)

# Initialize the list with annual A matrices as elements
A.y <- replicate(nby, diag(111), simplify=F)

for (i in 1:nby) {

  ## Extract each year survival and fertility
  surv.y <- surv[,i]
  fert.y <- fert[,i]

  ## Creating matrix A with
  A <- diag(surv.y)
  A <- cbind(A,rep(0,111))
  A <- rbind(t(fert.y), A)

  ## Saving A in the list
  A.y[[i]] <- A

}
```

(b) Compute the population growth rate λ_1 and the corresponding right and left eigenvectors **w1** and **v1** for each year.

[Sol.]

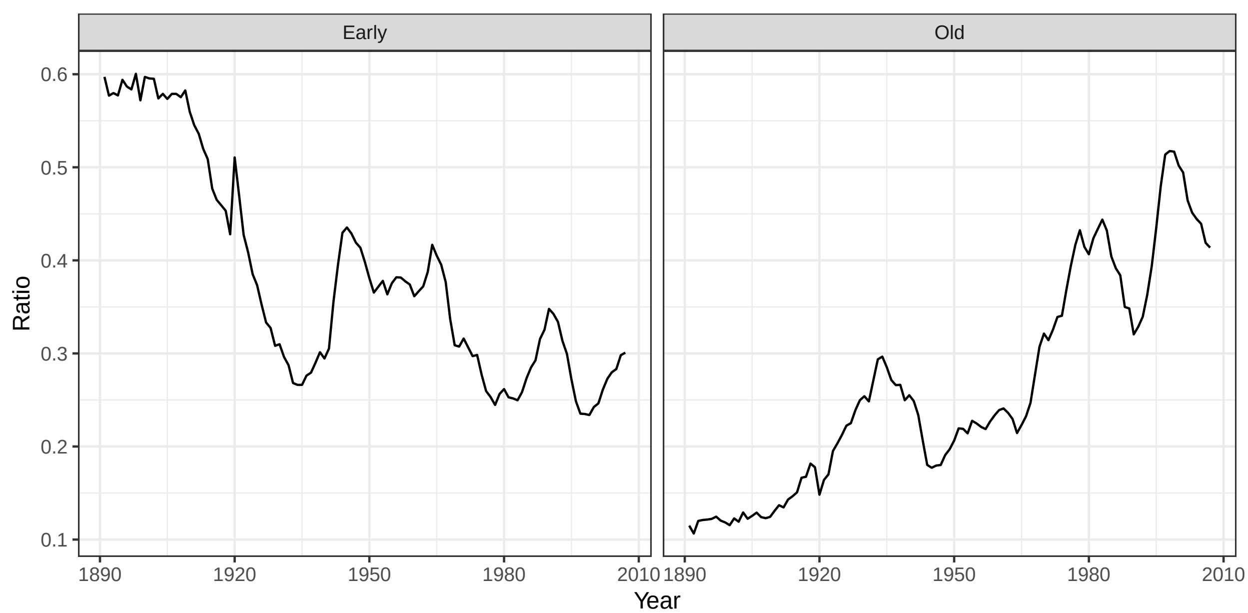
We plotted the λ values over time.



(c) This is a lot of information. To summarize the population structure, compute and plot the early-age dependency ratio and the old-age dependency ratio calculated from **w**.

[The dependency ratio is the ratio of population numbers during `dependent" ages (0-15 and older than 65) to the numbers in`productive" years (16-65). The early age and late age ratios just look at those portions of the dependent population.]

[Sol.]



References

Jenouvrier, Stéphanie, Hal Caswell, Christophe Barbraud, Marika Holland, Julienne Strœve, and Henri Weimerskirch. 2009. "Demographic Models and Ipcc Climate Projections Predict the Decline of an Emperor Penguin Population." *Proceedings of the National Academy of Sciences* 106 (6): 1844–7.