Priors, Part 2

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Outline

- Priors Review
- Eliciting Priors – Placenta Previa Example
- Eliciting Priors - Kidney Cancer Example

Bayes' Theorem

Likelihood
(Probability of Data Given Model)

Posterior Distribution
(Probability of Model Given Data)

$$\boxed{P(\theta|X)} = \frac{\boxed{P(X|\theta)}\,\boxed{P(\theta)}}{\boxed{P(X)}}$$

Prior Distribution
(Probability of Model)

$$P(X) = \sum_i P(X \cap \theta_i) = \sum_i P(X|\theta_i)P(\theta_i)$$

Probability of ***your data*** occurring for any choice of θ
Note: Possible θ_i are constrained by the model type.

Marginal Distribution
(Probability of Data)

Priors Review

Priors represent our prior knowledge about the phenomena under study.

- More specifically, they are probability distributions assigned to each parameter in a model that represent our prior beliefs about the *location* and *uncertainty* of the parameters.

There are many "classifications" of priors

- Conjugate priors – priors that result in the posterior being from the same distributional family
- Informative priors – priors that have something to say
- Weakly/uninformative priors – priors without much information given

<u>Key Point: A posterior represents a compromise between data and the priors</u>

Placenta Previa

Placenta Previa is a material condition in which the placenta is located low in the uterus.

- This blocks normal vaginal delivery, and if not accounted for, would result in hemorrhaging.

The question here is: Are male or female births more likely to have placenta previa?

Early study from Germany found that out of 980 PP births, 437 were female.

Placenta Previa – The likelihood

437/980 female births. We are interested in the proportion of female births relative to male births. Is it higher than .485, which is the proportion of female births in the overall population.

What is a reasonable likelihood for this data?

- Our parameter of interest is a proportion.
- Our data is # of female births out of # of total births.

An appropriate likelihood for this is the *binomial distribution.*

This example is taken from BDA3 by Gelman et al.

Binomial Likelihood

$$f(k|n,p) = \binom{n}{k} p^k (1-p)^{n-k}$$

- n – Total number of births
- k – Number of female births
- p – The probability of female births.

We want to perform inference on p. So what do we need to specify?

Prior Choice

$$f(k|n,p) = \binom{n}{k} p^k (1-p)^{n-k}$$

We need to specify our prior for p. This represents our prior (to seeing the data) belief about the probability of a female birth given placenta previa.

p is a probability/proportion.

- Bound by 0/1.

What would make for a good prior distribution?

Prior Choice - Beta

The Beta distribution is the conjugate prior for a binomial likelihood.

$$g(p|\alpha,\beta) = \frac{1}{B(\alpha,\beta)} p^{\alpha-1}(1-p)^{\beta-1}$$

Two *hyperparameters*, α and β

- What do these mean?

Prior Choice – Hyperparameters

$$g(p|\alpha,\beta) = \frac{1}{B(\alpha,\beta)} p^{\alpha-1}(1-p)^{\beta-1}$$

$$E[p|\alpha,\beta] = \frac{\alpha}{\alpha+\beta}$$

$$Var[p|\alpha,\beta] = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

- What are some informative priors, let say we believe that male births have more prevalence of placenta previa?
- What is an "uninformative" prior choice?

The Posterior

The Beta distribution is a conjugate prior for the binomial likelihood. This means that the posterior will be distributed as a???

- Beta distribution! Specifically this one: $Beta(\alpha + k, n - k + \beta)$

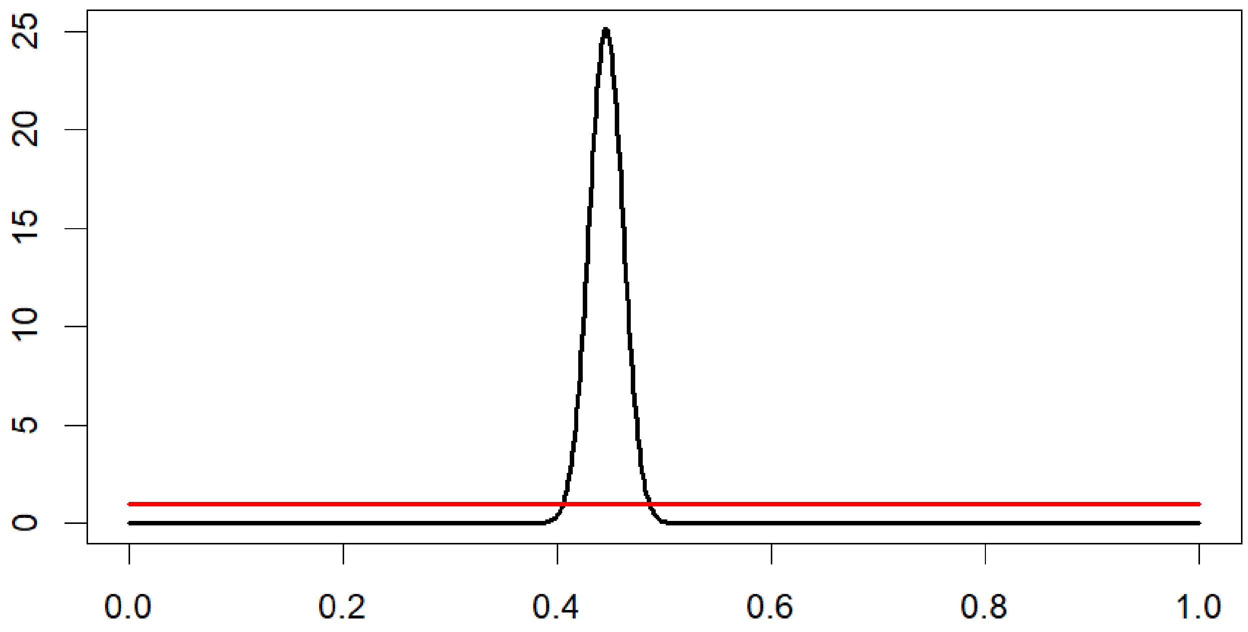
What do you think is going to happen as the information content in the prior increases?

- What about if we had even more data?

The Posterior

Using a non-informative prior $Beta(1,1)$

- Posterior - $Beta(438, 544)$
- EAP: .446
- 95% Credible Interval: [.415, .476]

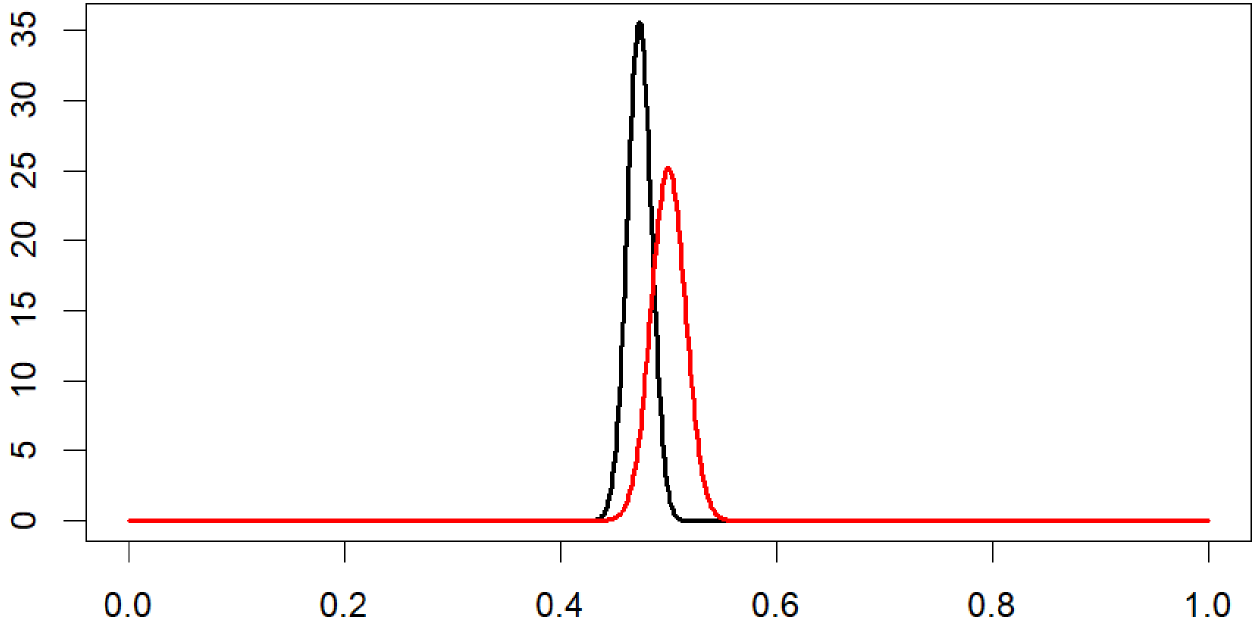


If we use a non-informative prior, we could conclude that the rate of female births with placenta previa is lower than the general rate

The Posterior

Using an informative prior $Beta(500,500)$

- Posterior - $Beta(937, 1043)$
- EAP: .473
- 95% Credible Interval: [.451, .495]



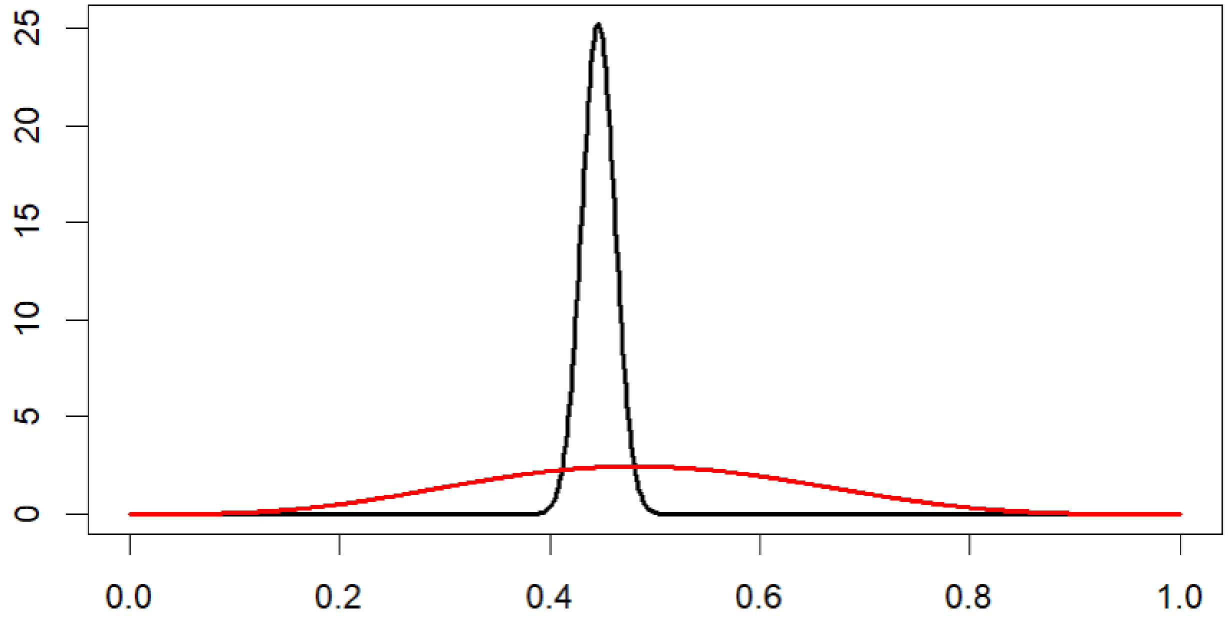
If we use an informative prior, we wouldn't conclude the same…

- What's wrong with this prior?

The Posterior

Using a weakly informative prior $Beta(4.85, 5.15)$

- Posterior - $Beta(441.85, 548.15)$
- EAP: .446
- 95% Credible Interval: [.415, .477]



Placenta Previa

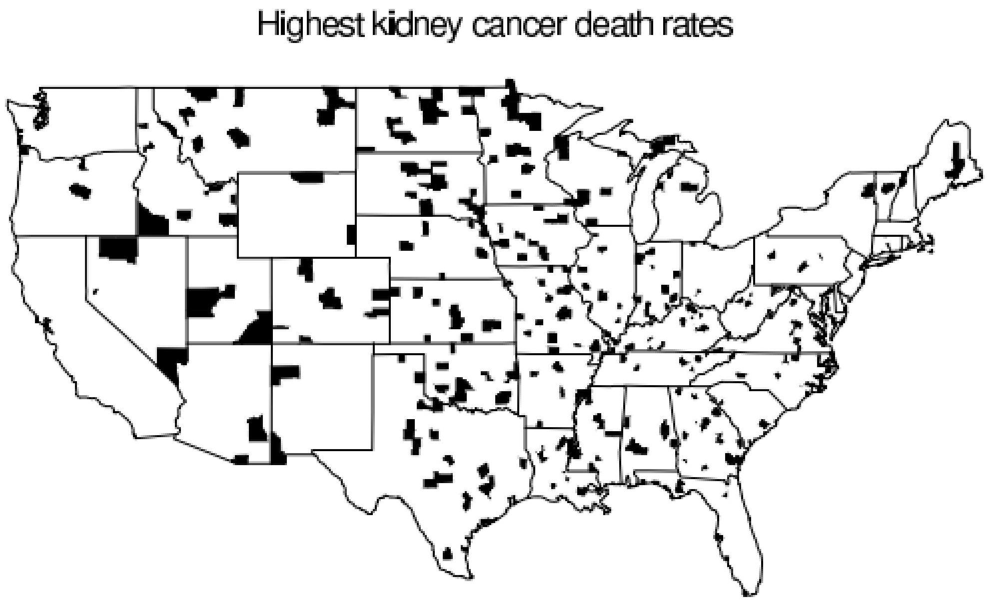
Under a very simple model (binomial likelihood) with uninformative or weakly informative priors, we can conclude that female births are slightly less likely in placenta previa than in the general population.

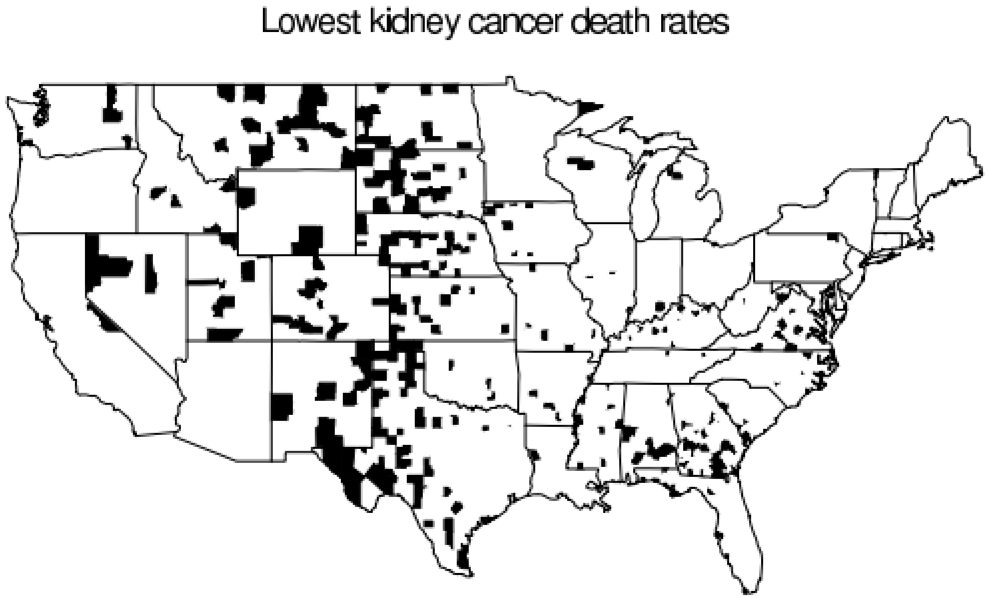
- So many caveats to this point…

Take away points:

- **Like all statistics, you can think about a prior in terms of *location* and *uncertainty.***
  - The informativeness of a prior is based on its uncertainty, not it's location.
- As data increases, the prior is overwhelmed.

Kidney Cancer in the 1980s





This example is taken from BDA3 by Gelman et al.

Kidney Cancer in the 1980s

As before, we start by defining our model (the likelihood):

$$y_j \sim Poisson(10 n_j \theta_j)$$

Poisson distributions are useful for modeling rates of events.

- Why not use a binomial here?

Data:

- y_j – Number of deaths in the county between 1980-1989
- n_j - Population size of the county

Parameter-

- θ_j- Rate of kidney cancer death in the county per year.

Kidney Cancer in the 1980s

$$y_j \sim Poisson(10 n_j \theta_j)$$

Now we need a prior for θ_j

- It's actually more accurate to say we need a prior for $n_j \theta_j$. Why?

The conjugate prior for a Poisson is a Gamma:

$$Gamma(\alpha, \beta)$$

Hyperparameters:

- α – Prior count of the events
- β – Prior time interval

This example is taken from BDA3 by Gelman et al.

Kidney Cancer in the 1980s

$$y_j \sim Poisson(n_j \theta_j)$$
$$\theta_j \sim Gamma(\alpha, \beta)$$

Hyperparameters:

- α – Prior count of the events
- β – Prior population size

Due to conjugacy:

$$\theta_j | y_j, n_j \sim Gamma(\alpha + y_j, \beta + 10 n_j)$$

Kidney Cancer in the 1980s

$$\theta_j | y_j, n_j \sim Gamma(\alpha + y_j, \beta + 10n_j)$$

Recall that Gamma distributions have:

- $E[X] = \frac{\alpha}{\beta}, Var[X] = \frac{\alpha}{\beta^2}$

What is a non-informative prior?

- Balance variance with expected value.

How about an informative prior, when we know that on average the rates of kidney cancer deaths are .0005?

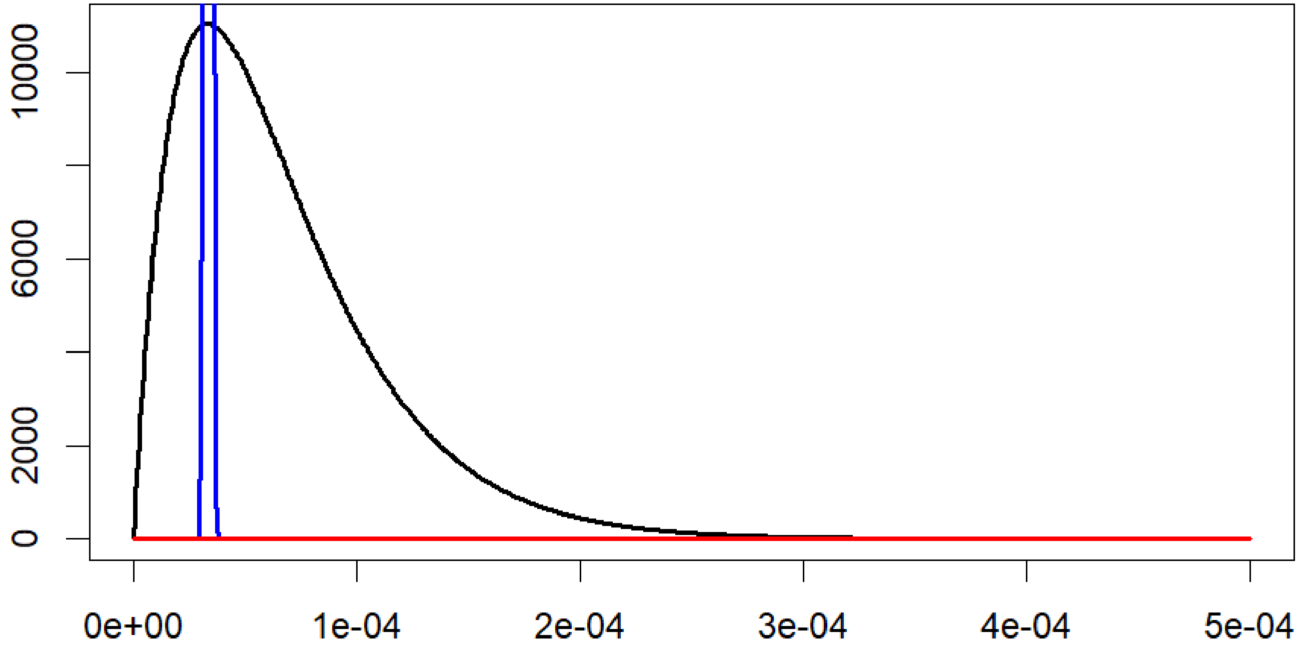
How do we make this informative prior highly informative?

Non-Informative Prior

$$\theta_j | y_j, n_j \sim Gamma(\alpha + y_j, \beta + 10n_j)$$

- How about a Gamma(1,1)?
- For a low population county
  - 1 death out of 3000
- For a high population county
  - 1000 deaths out of 3 million

Is the prior really impacting the posterior?

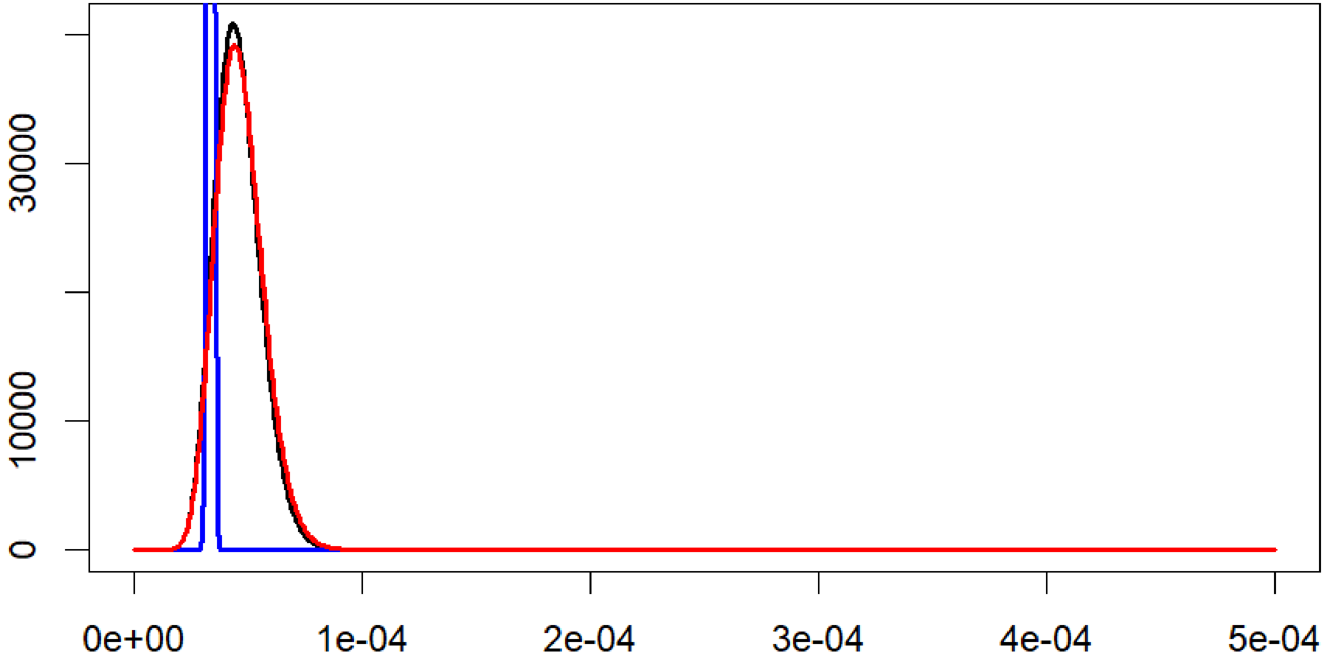


Non-Informative Prior

$$\theta_j | y_j, n_j \sim Gamma(\alpha + y_j, \beta + 10n_j)$$

How about a Gamma(20,430000)?

- Derived from *moment matching*



New Concept: Predictive Distribution

We can calculate/provide the following:

- Posterior – $P(\theta|X)$
- Likelihood - $P(X|\theta)$
- Prior - $P(\theta)$
- Marginal - $P(X)$

But what about the distribution for new observations of X, given our data and priors?

$$P(X|X^*,\theta)$$

Predictive Distribution

For any given parameter value, we can use the likelihood to predict new values of X. But, in Bayesian Statistics, parameters are not single values.

So, we need to integrate over our posterior distribution.

$$P(X|X^*) = \int P(X|\theta)P(\theta|X^*)d\theta$$

In some cases, this distribution has an easy form. But in most cases, you need to sample from this distribution.

1. Sample a value from the posterior of θ
2. Sample a value from the likelihood given the θ you sampled.

This example is taken from BDA3 by Gelman et al.

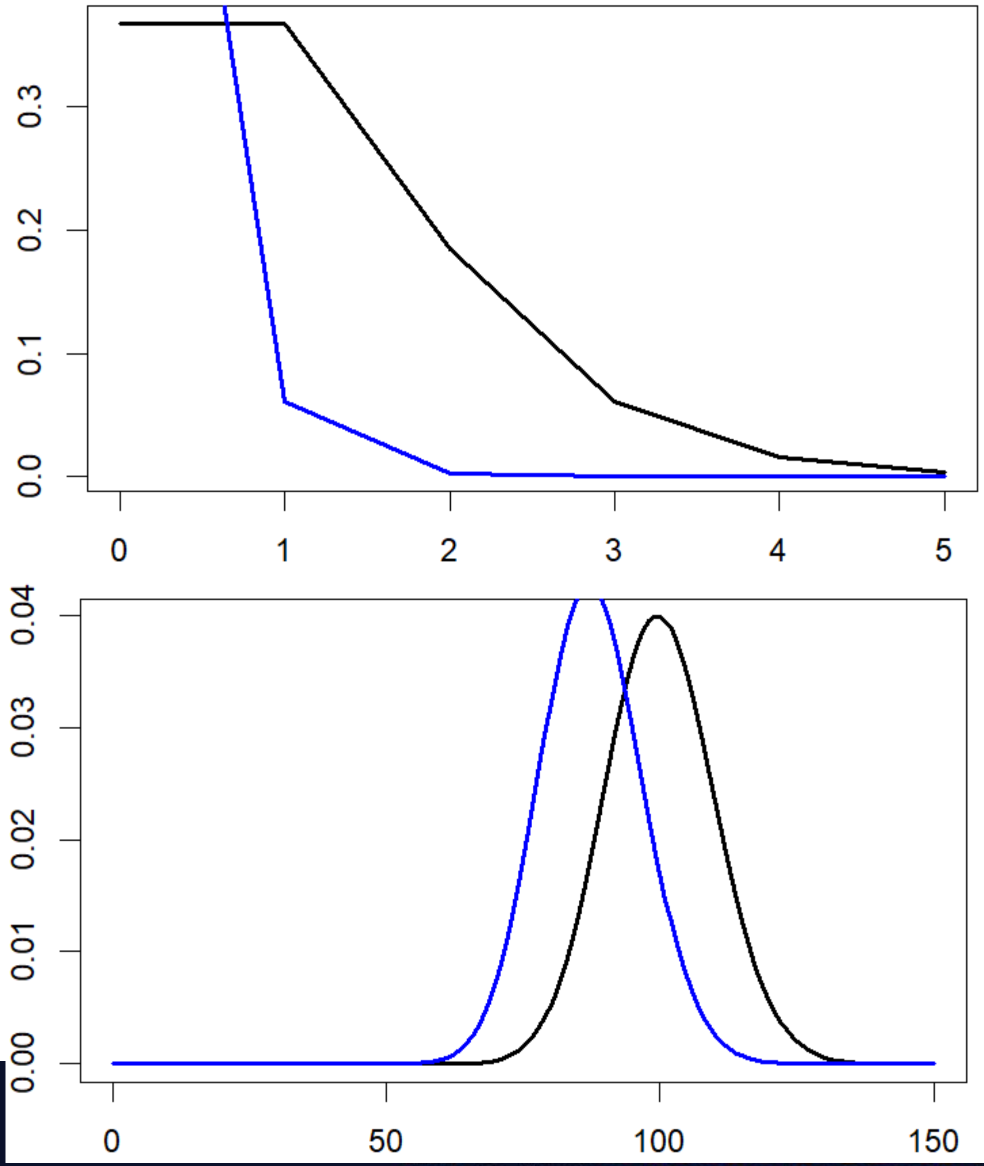
Back to Kidney Cancer

As it turns out, the predictive distribution for a Poisson Likelihood with Gamma Prior is a Negative Binomial

$$NegBin\left(\alpha + 10n_j y_j, \frac{\beta + 10n_j}{\beta + 10n_j + 1}\right)$$

Top: 1 death, 3000 n, $\alpha: 1{,}20$ $\beta: 1{,}4300000$

Bot: 100 deaths, 300000 n



Next Week on Bayes ML

Tuesday

- Bayesian Classification
- Conjugate Bayesian Regression

Thursday

- Bayesian Inference and Samplers!