# ON THE BUNDLE OF NULL CONES

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**Abstract**

We examine the bundle structure of the field of nowhere vanishing null vector fields on a (time-oriented) Lorentzian manifold. Sections of what we refer to as the null tangent, are by definition nowhere vanishing null vector fields. It is shown that the set of nowhere vanishing null vector fields comes equippedwith a para-associative ternary partial product. Moreover, the null tangent bundle is an example of a non-polynomial graded bundle.

**Keywords:** Lorentzian manifolds; null cones; fibre bundles; semi-heaps
**MSC 2020:** 20N10; 53B30; 53C50; 58A32; 83C99

## 1. INTRODUCTION

According to the theory of general relativity and closely related theories of gravity, spacetime is a four-dimensional Lorentzian manifold. An important aspect of Lorentzian geometry is the field of null cones due to their rôle in the causal nature of spacetime (see [6, 12]). In this work, we construct a fibre bundle of null cones (withthe zero section removed) over a connected time-oriented four-dimensional Lorentzian manifold - we refer to this bundle as the *null tangent bundle* (see Definition 2.1).

There is no Riemannian analogue of the null tangent bundle and such structures truly belong to Lorentzian geometry. Note that unit tangent bundles exist on any Lorentzian or Riemannian manifold. We show that the null tangent bundle is an example ofa natural bundle in the sense that it is canonically defined from the spacetime and requires no additional structure (see the proof of Proposition 2.2). Moreover, the null tangent bundle of a spacetime comes with an action of the multiplicative group of strictly positive real numbers: we refer to this action as the homothety (see Observation 2.5). We have a kind of graded manifold. However, the admissible coordinate transformations are not polynomial, so we do not have a graded bundle (see [5]). Sections of the null tangent bundle are, by definition, nowhere vanishing future or past directed null vector fields. Recall that we do not have a vector space on the set of null vectors at any given point on spacetime. For instance, the sum of two future-directed null vectors is a timelike vector, unless the two vectors are parallel. However, we will show that the sections of the null tangent bundle come with a partial ternary operation that is para-associative, i.e., we have a partial semiheap structure (see Theorem 2.13).

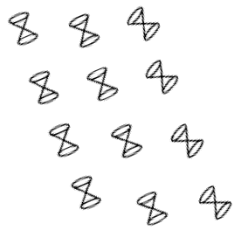

FIGURE 1. The null tangent bundle is constructed via attaching a split double cone to each point of a spacetime.

We remark that nowhere vanishing null vector fields, with other additional properties, are important in general relativity and supergravity. For example, Robinson manifolds come equipped with a nowhere vanishing vector field whose integral curved are null geodesics (see [15]). Other examples include pp-wave spacetimes, which are Lorentzian manifolds that admit a covariantly constant null vector field. All these examples fall under the umbrella of null G-structures (see [13]).

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**Conventions:** We will consider four-dimensional Lorentzian manifolds $(M, g)$ of signature $(-,+,+,+)$. Recall that a vector $v \in T_pM$ is

| | |
|---|---|
| timelike if | $g_p(v, v) < 0$, |
| null if | $g_p(v, v) = 0$, |
| spacelike if | $g_p(v, v) > 0$. |

The set of null vectors at $p \in M$ forms the *double cone* $N_p \subset T_pM$ (as a set). A vector field $\boldsymbol{v} \in \mathrm{Vect}(M)$ is said to be timelike/null/spacelike if at every point $p \in M$ the vector $\boldsymbol{v}|_p$ is timelike/null/spacelike.

**Definition 1.1.** A Lorentzian manifold $(M, g)$ is *time-orientable* if it admits a timelike vector field $\boldsymbol{\tau} \in \mathrm{Vect}(M)$.

We say that a vector $v \in T_pM$ is future directed if $g_p(v, \boldsymbol{\tau}|_p) < 0$ and past directed if $g_p(v, \boldsymbol{\tau}|_p) > 0$. Note that if $v$ is future/past directed then $-v$ is past/future directed.

**Definition 1.2.** A *spacetime* is a connected time-oriented four-dimensional Lorentzian manifold $(M, g, \boldsymbol{\tau})$.

Recall that by connected, we mean that $M$ is not homeomorphic to the union of two or more disjoint non-empty open subsets. A smooth manifold admits a Lorentzian metric if it admits a nowhere vanishing vector field. The existence of a nowhere vanishing vector field implies that $M$ must be non-compact or compact with zero Euler characteristic.

**Example 1.3.** A *globally hyperbolic spacetime* is of the form $M \cong \mathbb{R} \times \Sigma$, where $\Sigma$ is a three-dimensional smooth manifold. The vector field $\partial_t$, where $t$ is the global coordinate on $\mathbb{R}$, defines a timelike vector field and so $M$ is time-oriented.

Once a time-orientation has been chosen we can consistently decompose null cones at all points into the future and past directed components, $N_p = N_p^+ \cup N_p^-$. At each point, we can remove the zero vector $\mathbf{0}_p \in T_pM$ to get the *split null cone* $C_0 = C_0^+ \cup C_0^- \cong N_p \setminus \{\mathbf{0}_p\} \cong \mathbb{R}^3_* \sqcup \mathbb{R}^3_*$. Note that the split null cone is a three dimensional smooth manifold with an atlas consisting of two charts $\{(N_p^+, \varphi_+), (N_p^-, \varphi_-)\}$, with

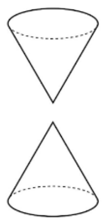

FIGURE 2. The split null cone $C_o$.

$$\begin{aligned} \varphi_\pm^{-1} : \mathbb{R}^3_* &\longrightarrow N_p^\pm \\ (x, y, z) &\longmapsto \left(\pm\sqrt{x^2 + y^2 + z^2}, x, y, z\right). \end{aligned} \tag{1.1}$$

We will employ vierbein fields in describing the tangent spaces. Let $\{(U_\alpha, \psi_\alpha)\}_{\in A}$ be an atlas on $M$ and let $TM$ be the tangent sheaf. Then for any $U \subset M$ in the chosen atlas we can employ vierbien fields, i.e., we have a basis $\{e_a = e_a^{\ \mu}(x)\partial_\mu\}$ of $TM(U)$. Recall that the vierbeins satisfy $g^{\mu\nu} = e_a^{\ \mu} e_b^{\ \nu} \eta^{ba}$ where $\eta$ is the Minkowski metric. The basis fields transform under the (restricted) Lorentz group, i.e., $e_{a'} = \Lambda_{a'}^{\ b} e_b$. The inverse vierbeins are defined via $\eta_{ab} = e_a^{\ \mu} e_b^{\ \nu} g_{\nu\mu}$. Thus, using the natural basis $\{e_a^{\ \mu}\partial_\mu|_p\}$ of the tangent space $T_pM$ we have that $g_p(v, w) = v^a w^b \eta_{ba}$. In particular, the components of any vector $v \in T_pM$ transform under Lorentz transformations as $v^a = v^b \Lambda_b^{\ a}$, where $\Lambda_b^{\ a} \Lambda_{a'}^{\ a} = \delta_b^{\ a}$.

**Remark.** One could further insist that a spacetime is parallelizable, i.e., we have a global basis for vector field. It is physically reasonable that spacetimes be parallelizable. However, we will not impose this condition in our definition of a spacetime.

The algebraic structure of *heaps* were introduced by Prüfer [14] and Baer [1] as a set equipped with a ternary operation satisfying some natural axioms. A heap can be thought of as a group in which the identity element is forgotten. Given a group, we can construct a heap by defining the ternary operation as $(a, b, c) \mapsto ab^{-1}c$. Conversely, by selecting an element in a heap, one can reduce the ternary operation to a group operation, such that the chosen element is the identity element.

There is a weaker notion of a *semiheap* as a non-empty set $H$, equipped with a ternary operation $[a, b, c] \in H$ that satisfies the *para-associative law*

$$[[a, b, c], d, e] = [a, [d, c, b], e] = [a, b, [c, d, e]] , \tag{1.2}$$

for all $a, b, c, d$ and $e \in H$. A semiheap is a *heap* when all its elements are *biunitary*, i.e., $[a, b, b] = a$ and $[b, b, a] = a$, for all $a$ and $b \in H$. For more details about heaps and related structures the reader my consult Hollings & Lawson [7]

## 2. The null tangent bundle

2.1. **Construction of the null tangent bundle.** We now proceed to the main definition of this paper. We build the null tangent bundle following the classical construction of the tangent bundle or unit tangent bundle but now replacing the tangent spaces with spilt null cones.

**Definition 2.1.** Let $(M, g, \boldsymbol{\tau})$ be a spacetime, then the *null tangent bundle* is defined as the disjoint union of split null cones, i.e.,

$$\begin{aligned} \mathrm{N}M &= \bigsqcup_{p\in M} \mathrm{N}_p \setminus \{\mathbf{0}_p\} \\ &= \big\{ (p, v) \mid p \in M,\ v \in \mathrm{N}_p \setminus \{\mathbf{0}_p\} \big\} , \end{aligned}$$

together with the natural projection

$$\begin{aligned} \pi : \mathrm{N}M &\longrightarrow M \\ (p, v) &\mapsto p\,. \end{aligned}$$

**Warning.** The null tangent bundle is *not* a vector bundle: this is clear as we do not have a zero section, and the sum of two (non-zero) null vectors is a timelike vector unless the pair of null vectors are linearly dependent.

We will on occasion use the fact that $\mathrm{N}M = \mathrm{N}M^+ \cup \mathrm{N}M^-$, where $\mathrm{N}M^\pm := \bigsqcup_{p\in M} \mathrm{N}_p^\pm \setminus \{\mathbf{0}_p\}$. Clearly, $\mathrm{N}M^+ \cap \mathrm{N}M^- = \emptyset$ and so $\mathrm{N}M$ is disconnected.

**Remark.** The null tangent bundle should be compared with the *unit tangent bundle* of a (pseudo-)Riemannian manifold $\mathrm{U}M := \bigsqcup_{p\in M} \{v \in \mathrm{T}_pM \mid g_p(v, v) = 1\}$. The fibres are diffeomorphic to $S^{n-1}$, assuming $M$ is of dimension $n$.

**Proposition 2.2.** *Let $(M, g, \boldsymbol{\tau})$ be a spacetime. The null tangent bundle $\mathrm{N}M$ is a smooth fibre bundle $\pi : \mathrm{N}M \to M$ with typical fibre $F_p \cong \mathrm{C}_0 = \mathrm{C}_0^+ \cup \mathrm{C}_0^-$.*

*Proof.* The only part of the proposition that is not immediate is that the total space $\mathrm{N}M$ is a smooth manifold. To show that the null tangent bundle of a spacetime is a (smooth) fibre bundle, we construct a natural bundle atlas of $\mathrm{N}M$ inherited from an atlas of $M$. We will consider $\mathrm{N}M$ as a natural bundle (see [11]). Let $\{(U_\alpha, \psi_\alpha)\}_{\alpha\in A}$ be an atlas of $M$ (not necessarily a maximal atlas). From the definition of the null tangent bundle $(p, v_\pm = v_\pm^a e_a) \in \pi^{-1}(U_\alpha)$, using vierbeins. Here "$\pm$" signifies

$$\begin{aligned} \pi^{-1}(U_\alpha) &\xrightarrow{\varphi_\alpha} \psi_\alpha(U_\alpha) \times \mathbb{R}^3_* \sqcup \mathbb{R}^3_* \\ \varphi_\alpha(p, v_\pm) &= (\psi_\alpha(p), v^i_\pm) = (x^\mu, v^i_\pm)\,, \end{aligned}$$

where $v^i_\pm$, now interpreted as coordinates on $\mathrm{N}_p^\pm \setminus \{\mathbf{0}_p\}$, are the spacial components of $v_\pm$. The inverse map is (see equation (1.1))

$$\varphi_\alpha^{-1}(x^\mu, v^i_\pm) = \Big(p,\ \pm\sqrt{v^j_\pm v^k_\pm \delta_{kj}},\ v^i_\pm\Big)\,.$$

## 3. Final remarks

We have constructed the null tangent explored some of its immediate mathematical properties. Interestingly, we have a generalisation of a graded bundle in which the coordinate transformations are not polynomial. Moreover, the sections of the null tangent bundle, so nowhere vanishing null vector field, come with the structure of a partial semiheap. These observations suggest that bundles of semiheaps and wider classes of graded manifolds should be studied. Particular focus should be on finding more geometric examples of ternary structures.

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