CHAPTER

**FIVE**

# GENERATING FUNCTIONS

A standard topic of study in first-year calculus is the representation of functions as infinite sums called power series; such a representation has the form $F(x) = \sum_{n=0}^{\infty} a_n x^n$. Perhaps surprisingly these power series can also serve as very powerful enumerative tools. In a combinatorial setting, we consider such power series of this type as another way of encoding the values of a sequence $\{a_n : n \geq 0\}$ indexed by the non-negative integers. The strength of power series as an enumerative technique is that they can be manipulated just like ordinary functions, i.e., they can be added, subtracted and multiplied, and for our purposes, we generally will not care if the power series converges, which anyone who might have found all of the convergence tests studied in calculus daunting will likely find reassuring. However, when we find it convenient to do so, we will use the familiar techniques from calculus and differentiate or integrate them term by term, and for those familiar series that do converge, we will use their representations as functions to facilitate manipulation of the series.

## 5.1 Basic Notation and Terminology

With a sequence $\sigma = \{a_n : n \geq 0\}$ of real numbers, we associate a "function" $F(x)$ defined by

$$F(x) = \sum_{n=0}^{\infty} a_n x^n.$$

The word "function" is put in quotes as we do not necessarily care about substituting a value of $x$ and obtaining a specific value for $F(x)$. In other words, we consider $F(x)$ as a formal power series and frequently ignore issues of convergence.

It is customary to refer to $F(x)$ as the *generating function* of the sequence $\sigma$. As we have already remarked, we are not necessarily interested in calculating $F(x)$ for specific values of $x$. However, by convention, we take $F(0) = a_0$.

*Example* 5.1. Consider the constant sequence $\sigma = \{a_n : n \geq 0\}$ with $a_n = 1$ for every $n \geq 0$. Then the generating function $F(x)$ of $\sigma$ is given by

$$F(x) = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + \cdots.$$

You probably remember that this last expression is the Maclaurin series for the function $F(x) = 1/(1-x)$ and that the series converges when $|x| < 1$. Since we want to think in terms of formal power series, let's see that we can justify the expression

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + \cdots = \sum_{n=0}^{\infty} x^n$$

without any calculus techniques. Consider the product

$$(1-x)(1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + \cdots)$$

and notice that, since we multiply formal power series just like we multiply polynomials (power series are pretty much polynomials that go on forever), we have that this product is

$$(1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + \cdots) - x(1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + \cdots) = 1.$$

Now we have that

$$(1-x)(1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + \cdots) = 1,$$

or, more usefully, after dividing through by $1 - x$,

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.$$

*Example* 5.2. Just like you learned in calculus for Maclaurin series, formal power series can be differentiated and integrated term by term. The rigorous mathematical framework that underlies such operations is not our focus here, so take us at our word that this can be done for formal power series without concern about issues of convergence.

To see this in action, consider differentiating the power series of the previous example. This gives

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + 7x^6 + \cdots = \sum_{n=1}^{\infty} nx^{n-1}.$$

Integration of the series represented by $1/(1+x) = 1/(1-(-x))$ yields (after a bit of algebraic manipulation)

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \cdots = \sum_{n=1}^{\infty} \frac{x^n}{n}$$

Before you become convinced that we're only going to concern ourselves with generating functions that actually converge, let's see that we can talk about the formal power series

$$F(x) = \sum_{n=0}^{\infty} n!x^n,$$

even though it has radius of convergence 0, i.e., the series $F(x)$ converges only for $x = 0$, so that $F(0) = 1$. Nevertheless, it makes sense to speak of the formal power series $F(x)$ as the generating function for the sequence $\{a_n : n \geq 0\}$, $a_0 = 1$ and $a_n$ is the number of permutations of $\{1, 2, \ldots, n\}$ when $n \geq 1$.

For reference, we state the following elementary result, which emphasizes the form of a product of two power series.

**Proposition 5.3.** *Let $A(x) = \sum_{n=0}^{\infty} a_n x^n$ and $B(x) = \sum_{n=0}^{\infty} b_n x^n$ be generating functions. Then $A(x)B(x)$ is the generating function of the sequence whose $n^{th}$ term is given by*

$$a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_n b_0 = \sum_{k=0}^{n} a_k b_{n-k}.$$

## 5.2 Another look at distributing apples or folders

A recurring problem so far in this book has been to consider problems that ask about distributing indistinguishable objects (say apples) to distinct entities (say children). We started in **??** by asking how many ways there were to distribute 40 apples to 5 children so that each child is guaranteed to get at least one apple and saw that the answer was $C(39, 4)$. We even saw how to restrict the situation so that one of the children was limited and could receive at most 10 apples. In **??**, we learned how to extend the restrictions so that more than one child had restrictions on the number of apples allowed by taking advantage of the Principle of Inclusion-Exclusion. Before moving on to see how generating functions can allow us to get even more creative with our restrictions, let's take a moment to see how generating functions would allow us to solve the most basic problem at hand.

*Example* 5.4. We already know that the number of ways to distribute $n$ apples to 5 children so that each child gets at least one apple is $C(n-1, 4)$, but it will be instructive to see how we can derive this result using generating functions. Let's start with an even simpler problem: how many ways are there to distribute $n$ apples to *one* child so that each child receives at least one apple? Well, this isn't too hard, there's only one way to do it—give all the apples to the lucky kid! Thus the *sequence* that enumerates the number of ways to do this is $\{a_n : n \geq 1\}$ with $a_n = 1$ for all $n \geq 1$. Then the generating function for this sequence is

$$x + x^2 + x^3 + \cdots = x(1 + x + x^2 + x^3 + \cdots) = \frac{x}{1-x}.$$

How can we get from this fact to the question of five children? Notice what happens when we multiply

$$(x + x^2 + \cdots)(x + x^2 + \cdots)(x + x^2 + \cdots)(x + x^2 + \cdots)(x + x^2 + \cdots).$$

To see what this product represents, first consider how many ways can we get an $x^6$? We could use the $x^2$ from the first factor and $x$ from each of the other four, or $x^2$ from the second factor and $x$ from each of the other four, etc., meaning that the coefficient on $x^6$ is $5 = C(5,4)$. More generally, what's the coefficient on $x^n$ in the product? In the expansion, we get an $x^n$ for every product of the form $x^{k_1}x^{k_2}x^{k_3}x^{k_4}x^{k_5}$ where $k_1 + k_2 + k_3 + k_4 + k_5 = n$. Returning to the general question here, we're really dealing with distributing $n$ apples to 5 children, and since $k_i > 0$ for $i = 1, 2, \ldots, 5$, we also have the guarantee that each child receives at least one apple, so the product of the generating function for *one* child gives the generating function for *five* children.

Let's pretend for a minute that we didn't know that the coefficients must be $C(n-1, 4)$. How could we figure out the coefficients just from the generating function? The generating function we're interested in is $x^5/(1-x)^5$, which you should be able to pretty quickly see satisfies

$$\begin{aligned}\frac{x^5}{(1-x)^5} &= \frac{x^5}{4!}\frac{d^4}{dx^4}\left(\frac{1}{1-x}\right) = \frac{x^5}{4!}\sum_{n=0}^{\infty} n(n-1)(n-2)(n-3)x^{n-4} \\ &= \sum_{n=0}^{\infty} \frac{n(n-1)(n-2)(n-3)}{4!}x^{n+1} = \sum_{n=0}^{\infty} \binom{n}{4} x^{n+1}.\end{aligned}$$

The coefficient on $x^n$ in this series $C(n-1, 4)$, just as we expected.

We could revisit an example from **??** to see that if we wanted to limit a child to receive at most 4 apples, we would use $(x + x^2 + x^3 + x^4)$ as its generating function instead of $x/(1-x)$, but rather than belabor that here, let's try something a bit more exotic.

*Example* 5.5. A grocery store is preparing holiday fruit baskets for sale. Each fruit basket will have 20 pieces of fruit in it, chosen from apples, pears, oranges, and grapefruit. How many different ways can such a basket be prepared if there must be at least one apple in a basket, a basket cannot contain more than three pears, and the number of oranges must be a multiple of four?

In order to get at the number of baskets consisting of 20 pieces of fruit, let's solve the more general problem where each basket has $n$ pieces of fruit. Our method is simple: find the generating function for how to do this with each type of fruit individually and then multiply them. As in the previous example, the product will contain the term $x^n$ for every way of assembling a basket of $n$ pieces of fruit subject to our restrictions. The apple generating function is $x/(1-x)$, since we only want positive powers of $x$ (corresponding to ensuring at least one apple). The generating function for pears is $(1+x+x^2+x^3)$, since we can have only zero, one, two, or three pears in basket. For oranges we have $1/(1-x^4) = 1+x^4+x^8+\cdots$, and the unrestricted grapefruit give us a factor of $1/(1-x)$. Multiplying, we have

$$\frac{x}{1-x}(1+x+x^2+x^3)\frac{1}{1-x^4}\frac{1}{1-x} = \frac{x}{(1-x)^2(1-x^4)}(1+x+x^2+x^3).$$

Now we want to make use of the fact that $(1+x+x^2+x^3) = (1-x^4)/(1-x)$ to see that our generating function is

$$\frac{x}{(1-x)^3} = \frac{x}{2}\sum_{n=0}^{\infty} n(n-1)x^{n-2} = \sum_{n=0}^{\infty}\frac{n(n-1)}{2}x^{n-1} = \sum_{n=0}^{\infty}\binom{n}{2}x^{n-1} = \sum_{n=0}^{\infty}\binom{n+1}{2}x^n.$$

Thus, there are $C(n+1,2)$ possible fruit baskets containing $n$ pieces of fruit, meaning that the answer to the question we originally asked is $C(21,2) = 210$.

*Example* 5.6. Find the number of integer solutions to the equation

$$x_1 + x_2 + x_3 = n$$

($n \geq 0$ an integer) with $x_1 \geq 0$ even, $x_2 \geq 0$, and $0 \leq x_3 \leq 2$.

Again, we want to look at the generating function we would have if each variable existed individually and take their product. For $x_1$, we get a factor of $1/(1-x^2)$; for $x_2$, we have $1/(1-x)$; and for $x_3$ our factor is $(1+x+x^2)$. Therefore, the generating function for the number of solutions to the equation above is

$$\frac{1+x+x^2}{(1-x)(1-x^2)} = \frac{1+x+x^2}{(1+x)(1-x)^2}.$$

In calculus, when we wanted to integrate a rational function of this form, we would use the method of partial fractions to write it as a sum of "simpler" rational functions whose antiderivatives we recognized. Here, our technique is the same, as we can readily recognize the formal power series for many rational functions. Our goal is to write

$$\frac{1+x+x^2}{(1+x)(1-x)^2} = \frac{A}{1+x} + \frac{B}{1-x} + \frac{C}{(1-x)^2}$$

for appropriate constants, $A$, $B$, and $C$. To find the constants, we clear the denominators, giving

$$1 + x + x^2 = A(1-x)^2 + B(1-x^2) + C(1+x).$$

Equating coefficients on terms of equal degree, we have:

$$\begin{aligned} 1 &= A + B + C \\ 1 &= -2A + C \\ 1 &= A - B \end{aligned}$$

Solving the system, we find $A = 1/4$, $B = -3/4$, and $C = 3/2$. Therefore, our generating function is

$$\frac{1}{4}\frac{1}{1+x} - \frac{3}{4}\frac{1}{1-x} + \frac{3}{2}\frac{1}{(1-x)^2} = \frac{1}{4}\sum_{n=0}^{\infty}(-1)^n x^n - \frac{3}{4}\sum_{n=0}^{\infty} x^n + \frac{3}{2}\sum_{n=0}^{\infty} n x^{n-1}.$$

The solution to our question is thus the coefficient on $x^n$ in the above generating function, which is

$$\frac{(-1)^n}{4} - \frac{3}{4} + \frac{3(n+1)}{2},$$

a surprising answer that would not be too easy to come up with via other methods!

## 5.3 Newton's Binomial Theorem

In **??**, we discussed the binomial theorem and saw that the following formula holds for all integers $p \geq 1$:

$$(1+x)^p = \sum_{n=0}^{p} \binom{p}{n} x^n.$$

You should quickly realize that this formula implies that the generating function for the number of $n$-element subsets of a $p$-element set is $(1+x)^p$. The topic of generating functions is what leads us to consider what happens if we encounter $(1+x)^p$ as a generating function with $p$ not a positive integer. It turns out that, by suitably extending the definition of the binomial coefficients to real numbers, we can also extend the binomial theorem in a manner originally discovered by Sir Isaac Newton.

We've seen several expressions that can be used to calculate the binomial coefficients, but in order to extend $C(p,k)$ to real values of $p$, we will utilize the form

$$\binom{p}{k} = \frac{P(p,k)}{k!},$$

recalling that we've defined $P(p,k)$ recursively as $P(p,0) = 1$ for all integers $p \geq 0$ and $P(p,k) = pP(p-1,k-1)$ when $p \geq k > 0$ ($k$ an integer). Notice here, however, that the expression for $P(p,k)$ makes sense for any real number $p$, so long as $k$ is a non-negative integer. We make this definition formal.

**Definition 5.7.** For all real numbers $p$ and nonnegative integers $k$, the number $P(p,k)$ is defined by

1. $P(p,0) = 1$ for all real numbers $p$ and
2. $P(p,k) = pP(p-1,k-1)$ for all real numbers $p$ and integers $k > 0$.

(Notice that this definition does not require $p \geq k$ as we did with integers.)

We are now prepared to extend the definition of binomial coefficient so that $C(p,k)$ is defined for all real $p$ and nonnegative integer values of $k$. We do this as follows.

**Definition 5.8.** For all real numbers $p$ and nonnegative integers $k$,

$$\binom{p}{k} = \frac{P(p,k)}{k!}.$$

Note that $P(p,k) = C(p,k) = 0$ when $p$ and $k$ are integers with $0 \leq p < k$. On the other hand, we have some interesting new concepts such as $P(-5,4) = (-5)(-6)(-7)(-8)$ and

$$\binom{-7/2}{5} = \frac{(-7/2)(-9/2)(-11/2)(-13/2)(-15/2)}{5!}.$$

With this more general definition of binomial coefficients in hand, we're ready to state Newton's Binomial Theorem for all non-zero real numbers. The proof of this theorem can be found in most advanced calculus books.

**Theorem 5.9.** *For all real $p$ with $p \neq 0$,*

$$(1+x)^p = \sum_{n=0}^{\infty} \binom{p}{n} x^n.$$

Note that the general form reduces to the original version of the binomial theorem when $p$ is a positive integer.

## 5.4 An Application of the Binomial Theorem

In this section, we see how Newton's Binomial Theorem can be used to derive another useful identity. We begin by establishing a different recursive formula for $P(p,k)$ than was used in our definition of it.

**Lemma 5.10.** *For each $k \geq 0$, $P(p,k+1) = P(p,k)(p-k)$.*

*Proof.* When $k = 0$, both sides evaluate to $p$. Now assume validity when $k = m$ for some non-negative integer $m$. Then

$$\begin{aligned} P(p, m+2) &= pP(p-1, m+1) \\ &= p[P(p-1, m)(p-1-m)] \\ &= [pP(p-1, m)](p-1-m) \\ &= P(p, m+1)[p-(m+1)]. \end{aligned}$$

□

Our goal in this section will be to invoke Newton's Binomial Theorem with the exponent $p = -1/2$. To do so in a meaningful manner, we need a simplified expression for $C(-1/2, k)$, which the next lemma provides.

**Lemma 5.11.** *For each $k \geq 0$, $\binom{-1/2}{k} = (-1)^k \frac{\binom{2k}{k}}{2^{2k}}$.*

*Proof.* We proceed by induction on $k$. Both sides reduce to 1 when $k = 0$. Now assume validity when $k = m$ for some non-negative integer $m$. Then

$$\begin{aligned} \binom{-1/2}{m+1} &= \frac{P(-1/2, m+1)}{(m+1)!} = \frac{P(-1/2, m)(-1/2-m)}{(m+1)m!} \\ &= \frac{-1/2-m}{m+1}\binom{-1/2}{m} = (-1)\frac{2m+1}{2(m+1)}(-1)^m\frac{\binom{2m}{m}}{2^{2m}} \\ &= (-1)^{m+1}\frac{1}{2^{2m}}\frac{(2m+2)(2m+1)}{(2m+2)2(m+1)}\binom{2m}{m} = (-1)^{m+1}\frac{\binom{2m+2}{m+2}}{2^{2m+2}}. \end{aligned}$$

□

**Theorem 5.12.** *The function $f(x) = (1-4x)^{-1/2}$ is the generating function of the sequence $\{\binom{2n}{n} : n \geq 0\}$.*

*Proof.* By Newton's Binomial Theorem and Lemma 5.11, we know that

$$\begin{aligned} (1-4x)^{-1/2} &= \sum_{n=0}^{\infty}\binom{-1/2}{n}(-4x)^n \\ &= \sum_{n=0}^{\infty}(-1)^n 2^{2n}\binom{-1/2}{n}x^n \\ &= \sum_{n=0}^{\infty}\binom{2n}{n}x^n. \end{aligned}$$

□

Now recalling Proposition 5.3 about the coefficients in the product of two generating functions, we are able to deduce the following corollary of Theorem 5.12 by squaring the function $f(x) = (1-4x)^{-1/2}$.

**Corollary 5.13.** *For all $n \geq 0$,*

$$2^{2n} = \sum_{k=0}^{n} \binom{2k}{k} \binom{2n-2k}{k}.$$

## 5.5 Partitions of an Integer

A recurring theme in this course has been to count the number of integer solutions to an equation of the form $x_1 + x_2 + \cdots + x_k = n$. What if we wanted to count the number of such solutions but didn't care what $k$ was? How about if we took this new question and required that the $x_i$ be *distinct* (i.e., $x_i \neq x_j$ for $i \neq j$)? What about if we required that each $x_i$ be odd? These certainly don't seem like easy questions to answer at first, but generating functions will allow us to say something very interesting about the answers to the last two questions.

By a *partition* $P$ of an integer, we mean a collection of (not necessarily distinct) positive integers such that $\sum_{i \in P} i = n$. (By convention, we will write the elements of $P$ from largest to smallest.) For example, $2+2+1$ is a partition of 5. For each $n \geq 0$, let $p_n$ denote the number of partitions of the integer $n$ (with $p_0 = 1$ by convention). Note that $p_8 = 22$ as evidenced by the list in Table 5.1. Note that there are 6 partitions of 8 into *distinct* parts. Also there are 6 partitions of 8 into *odd* parts. While it might seem that this is a coincidence, it in fact is always the case as the following theorem states.

**Theorem 5.14.** *For each $n \geq 1$, the number of partitions of $n$ into distinct parts is equal to the number of partitions of $n$ into odd parts.*

| | | |
|---|---|---|
| 8 distinct parts | 7+1 distinct parts, odd parts | 6+2 distinct parts |
| 6+1+1 | 5+3 distinct parts, odd parts | 5+2+1 distinct parts |
| 5+1+1+1 odd parts | 4+4 | 4+3+1 distinct parts |
| 4+2+2 | 4+2+1+1 | 4+1+1+1+1 |
| 3+3+2 | 3+3+1+1 odd parts | 3+2+2+1 |
| 3+2+1+1+1 | 3+1+1+1+1+1 odd parts | 2+2+2+2 |
| 2+2+2+1+1 | 2+2+1+1+1+1 | 2+1+1+1+1+1+1 |
| | 1+1+1+1+1+1+1+1 odd parts | |

**Table 5.1:** The partitions of 8, noting those into distinct parts and those into odd parts.

*Proof.* The generating function $D(x)$ for the number of partitions of $n$ into distinct parts is

$$D(x) = \prod_{n=1}^{\infty}(1+x^n).$$

On the other hand, the generating function $O(x)$ for the number of partitions of $n$ into odd parts is

$$O(x) = \prod_{n=1}^{\infty}\frac{1}{1-x^{2n-1}}.$$

To see that $D(x) = O(x)$, we note that $1 - x^{2n} = (1-x^n)(1+x^n)$ for all $n \geq 1$. Therefore,

$$\begin{aligned}
D(x) &= \prod_{n=1}^{\infty}(1+x^n) = \prod_{n=1}^{\infty}\frac{1-x^{2n}}{1-x^n} = \frac{\prod_{n=1}^{\infty}(1-x^{2n})}{\prod_{n=1}^{\infty}(1-x^n)} \\
&= \frac{\prod_{n=1}^{\infty}(1-x^{2n})}{\prod_{n=1}^{\infty}(1-x^{2n-1})\prod_{n=1}^{\infty}(1-x^{2n})} = \prod_{n=1}^{\infty}\frac{1}{1-x^{2n-1}} \\
&= O(x).
\end{aligned}$$

□

## 5.6 Exponential generating functions

If we had wanted to be absolutely precise earlier in the chapter, we would have referred to the generating functions we studied as *ordinary generating functions* or even *ordinary power series generating functions*. This is because there are other types of generating functions, based on other types of power series. In this section, we briefly introduce another type of generating function, the *exponential generating function*. While an ordinary generating function has the form $\sum_n a_n x^n$, an exponential generating function is based on the power series for the exponential function $e^x$. Thus, the exponential generating function for the sequence $\{a_n \colon n \geq 0\}$ is $\sum_n a_n x^n/n!$. In this section, we will see some ways we can use exponential generating functions to solve problems that we could not tackle with ordinary generating functions. However, we will only scratch the surface of the potential of this type of generating function. We begin with the most fundamental exponential generating function, in analogy with the ordinary generating function $1/(1-x)$ of Example 5.1.

*Example* 5.15. Consider the constant sequence $1, 1, 1, 1, \ldots$. Then the exponential generating function for this sequence is

$$E(x) = \sum_{n=0}^{\infty}\frac{x^n}{n!}.$$

From calculus, you probably recall that this is the power series for the exponential function $e^x$, which is why we call this type of generating function an exponential generating function. From this example, we can quickly recognize that the exponential generating function for the number of binary strings of length $n$ is $e^{2x}$ since

$$e^{2x} = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} = \sum_{n=0}^{\infty} 2^n \frac{x^n}{n!}.$$

In our study of ordinary generating functions earlier in this chapter, we considered examples where quantity (number of apples, etc.) mattered but order did not. One of the areas where exponential generating functions are preferable to ordinary generating functions is in applications where order matters, such as counting strings. For instance, although the bit strings 10001 and 011000 both contain three zeros and two ones, they are not the same strings. On the other hand, two fruit baskets containing two apples and three oranges would be considered equivalent, regardless of how you arranged the fruit. We now consider a couple of examples to illustrate this technique.

*Example* 5.16. Suppose we wish to find the number of ternary strings in which the number of 0s is even. (There are no restrictions on the number of 1s and 2s.) As with ordinary generating functions, we determine a generating function for each of the digits and multiply them. For 1s and 2s, since we may have any number of each of them, we introduce a factor of $e^x$ for each. For an even number of 0s, we need

$$1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}.$$

Unlike with ordinary generating functions, we cannot represent this series in a more compact form by simply substituting a function of $x$ into the series for $e^y$. However, with a small amount of cleverness, we are able to achieve the desired result. To do this, first notice that

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}.$$

Thus, when we add the series for $e^{-x}$ to the series for $e^x$ all of the terms with odd powers of $x$ will cancel! We thus find

$$e^x + e^{-x} = 2 + 2\frac{x^2}{2!} + 2\frac{x^4}{4!} + \cdots,$$

which is exactly twice what we need. Therefore, the factor we introduce for 0s is $(e^x + e^{-x})/2$.

Now we have an exponential generating function of

$$\frac{e^x + e^{-x}}{2} e^x e^x = \frac{e^{3x} + e^x}{2} = \frac{1}{2}\left(\sum_{n=0}^{\infty} \frac{3^n x^n}{n!} + \sum_{n=0}^{\infty} \frac{x^n}{n!}\right).$$

To find the number of ternary strings in which the number of 0s is even, we thus need to look at the coefficient on $x^n/n!$ in the series expansion. In doing this, we find that the number of ternary strings with an even number of 0s is $(3^n + 1)/2$.

We can also use exponential generating functions when there are bounds on the number of times a symbol appears, such as in the following example.

*Example* 5.17. How many ternary strings of length $n$ have at least one 0 and at least one 1? To ensure that a symbol appears at least once, we need the following exponential generating function

$$x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=1}^{\infty} \frac{x^n}{n!}.$$

You should notice that this is almost the series for $e^x$, except it's missing the first term. Thus, $\sum_{n=1}^{\infty} x^n/n! = e^x - 1$. Using this, we now have

$$(e^x - 1)(e^x - 1)e^x = e^{3x} - 2e^{2x} + e^x$$

as the exponential generating function for this problem. Finding the series expansion, we have

$$\sum_{n=0}^{\infty} \frac{3^n x^n}{n!} - 2\sum_{n=0}^{\infty} \frac{2^n x^n}{n!} + \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Now we can answer the question by reading off the coefficient on $x^n/n!$, which is $3^n - 2 \cdot 2^n + 1$.

Before proceeding to an additional example, let's take a minute to look at another way to answer the question from the previous example. To count the number of ternary strings of length $n$ with at least one 0 and at least one 1, we can count all ternary strings of length $n$ and use the principle of inclusion-exclusion to eliminate the undesirable strings lacking a 0 and/or a 1. If a ternary string lacks a 0, we're counting all strings made up of 1s and 2s, so there are $2^n$ strings. Similarly for lacking a 1. However, if we subtract $2 \cdot 2^n$, then we've subtracted the strings that lack both a 0 *and* a 1 twice. A ternary string that has no 0s and no 1s consists only of 2s. There is a single ternary string of length $n$ satisfying this criterion. Thus, we obtain $3^n - 2 \cdot 2^n + 1$ in another way.

*Example* 5.18. Alice needs to set an eight-digit passcode for her mobile phone. The restrictions on the passcode are a little peculiar. Specifically, it must contain an even number of 0s, at least one 1, and at most three 2s. Bob remarks that although the restrictions are unusual, they don't do much to reduce the number of possible passcodes from the total number of $10^8$ eight-digit strings. Carlos isn't convinced that's the case, so he works up an exponential generating function as follows. For the seven digits on which there are no restrictions, a factor of $e^{7x}$ is introduced. To account for an even number of 0s, he uses $(e^x + e^{-x})/2$. For at least one 1, a factor of $e^x - 1$ is required. Finally, $1 + x + x^2/2! + x^3/3!$ accounts for the restriction of at most three 2s. The exponential generating function for the number of $n$-digit passcodes is thus

$$e^{7x}\frac{e^x + e^{-x}}{2}(e^x - 1)\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}\right).$$

Dave sees this mess written on the whiteboard and groans. He figures they'll be there all day multiplying and making algebra mistakes in trying to find the desired coefficient. Alice points out that they don't really need to find the coefficient on $x^n/n!$ for all $n$. Instead, she suggests they use a computer algebra system to just find the coefficient on $x^8/8!$. After doing this, they find that there are 33847837 valid passcodes for the mobile phone. A quick calculation shows that Bob was totally off base in claiming that there was no significant reduction in the number of possible strings to use as a passcode. The total number of valid passcodes is only 33.85% of the total number of eight-digit strings!

Exponential generating functions are useful in many other situations beyond enumerating strings. For instance, they can be used to count the number of $n$-vertex, connected, labeled graphs. However, doing so is beyond the scope of this book. If you are interested in learning much more about generating functions, the book *generatingfunctionology* by Herbert S. Wilf is available online at `http://www.math.upenn.edu/~wilf/DownldGF.html`.

## 5.7 Exercises

Computer algebra systems can be powerful tools for working with generating functions. In addition to stand-alone applications that run on your computer, the free website Wolfram|Alpha (`http://www.wolframalpha.com`) is capable of finding general forms of some power series representations and specific coefficients for many more. However, unless an exercise specifically suggests that you use a computer algebra system, we strongly encourage you to solve the problem by hand. This will help you develop a better understanding of how generating functions can be used.

For all exercises in this section, "generating function" should be taken to mean "ordinary generating function." Exponential generating functions are only required in exercises specifically mentioning them.

1. For each *finite* sequence below, give its generating function.

   a) $1, 4, 6, 4, 1$
   b) $1, 1, 1, 1, 1, 0, 0, 1$
   c) $0, 0, 0, 1, 2, 3, 4, 5$
   d) $1, 1, 1, 1, 1, 1, 1$
   e) $3, 0, 0, 1, -4, 7$
   f) $0, 0, 0, 0, 1, 2, -3, 0, 1$

2. For each *infinite* sequence suggested below, give its generating function in closed form, i.e., *not* as an infinite sum. (Use the most obvious choice of form for the general term of each sequence.)

   a) $0, 1, 1, 1, 1, 1, \ldots$
   b) $1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, \ldots$
   c) $1, 2, 4, 8, 16, 32, \ldots$
   d) $0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, \ldots$
   e) $1, -1, 1, -1, 1, -1, 1, -1, 1, -1, \ldots$
   f) $2^8, 2^7\binom{8}{1}, 2^6\binom{8}{2}, \ldots, \binom{8}{8}, 0, 0, 0, \ldots$
   g) $1, 1, 1, 0, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, \ldots$
   h) $0, 0, 0, 1, 2, 3, 4, 5, 6, \ldots$
   i) $3, 2, 4, 1, 1, 1, 1, 1, 1, \ldots$
   j) $0, 2, 0, 0, 2, 0, 0, 2, 0, 0, 2, 0, 0, 2, \ldots$
   k) $6, 0, -6, 0, 6, 0, -6, 0, 6, \ldots$
   l) $1, 3, 6, 10, 15, \ldots, \binom{n+2}{2}, \ldots$

3. For each generating function below, give a closed form for the $n^{\text{th}}$ term of its associated sequence.

   a) $(1+x)^{10}$
   b) $\dfrac{1}{1-x^4}$
   c) $\dfrac{x^3}{1-x^4}$
   d) $\dfrac{1-x^4}{1-x}$
   e) $\dfrac{1+x^2-x^4}{1-x}$
   f) $\dfrac{1}{1-4x}$
   g) $\dfrac{1}{1+4x}$
   h) $\dfrac{x^5}{(1-x)^4}$
   i) $\dfrac{x^2+x+1}{1-x^7}$
   j) $3x^4 + 7x^3 - x^2 + 10 + \dfrac{1}{1-x^3}$

4. Find the coefficient on $x^{10}$ in each of the generating functions below.

   a) $(x^3 + x^5 + x^6)(x^4 + x^5 + x^7)(1 + x^5 + x^{10} + x^{15} + \cdots)$
   b) $(1 + x^3)(x^3 + x^4 + x^5 + \cdots)(x^4 + x^5 + x^6 + x^7 + x^8 + \cdots)$
   c) $(1+x)^{12}$
   d) $\dfrac{x^5}{1-3x^5}$
   e) $\dfrac{1}{(1-x)^3}$
   f) $\dfrac{1}{1-5x^4}$
   g) $\dfrac{x}{1-2x^3}$
   h) $\dfrac{1-x^{14}}{1-x}$

5. Find the generating function for the number of ways to create a bunch of $n$ balloons selected from white, gold, and blue balloons so that the bunch contains at least one white balloon, at least one gold balloon, and at most two blue balloons. How many ways are there to create a bunch of 10 balloons subject to these requirements? How about a bunch of $n$ balloons?