CHAPTER

NINE

Ramsey Theory

9.1 Basic Notation and Terminology

For a positive integer n, we let $[n] = \{1, 2, \ldots, n\}$. For a set X and a non-negative integer k, $\binom{X}{k}$ denotes the set of all k-element subsets of X. In Ramsey theoretic settings, it is common to refer to a map $\phi : \binom{X}{k} :\longrightarrow R$ as a *coloring* of the k-element subsets of X, and the elements of R will be referred to as *colors*. Typically, we will just take $R = [r]$ for some positive integer r, so ϕ will also be called an r-coloring.

When $\phi : \binom{X}{k} \longrightarrow [r]$ is a r-coloring, a subset $H \subseteq S$ is called a *homogeneous* set (also a *monochromatic* set) when there exists a color $\alpha \in [r]$ so that $\phi(A) = \alpha$ for every $A \in \binom{H}{k}$.

Theorem 9.1 (Ramsey's theorem)**.** *If k and r are positive integers, and $(h_1, h_2, \ldots, h_r)$ are integers with $h_i \geq k$ for $i = 1, 2, \ldots, r$, then there exists a least positive integer $t_0 = R(k : h_1, h_2, \ldots, h_r)$ so that if X is any set with $|X| \geq t_0$, then for every r-coloring $\phi : \binom{S}{k} \longrightarrow [r]$ of the k-element subssets of X, there exists an $\alpha \in X$ and a subset $H \subseteq X$ with $|H| \geq h_i$ so that $\phi(A) = \alpha$ for every $A \in \binom{H}{k}$.*

Proof. We use a double induction. The first induction is on k, and the second is on r. When $k = 1$, the result holds for all r as the theorem reduces to a restatement of the Pigeonhole Principle. As an aside, we note that

$$R(1 : h_1, h_2, \ldots, h_r) = 1 + \sum_{i=1}^{r} h_i - 1.$$

However, in general, we will not be able to say much about the exact value of ramsey numbers. Instead, the emphasis is on the fact that they *exist*!

Now assume validity for some $k \geq 1$ and consider the next value of k. Now the induction is on r. When $r = 1$, it is easy to see that $R(k : h_1) = h_1$. Now consider the case $r = 2$.

Let $q = R(1; h_1, h_2)$ and define a sequence of numbers $(s_0, s_1, s_2, \ldots, s_q$ as follows. First set $s_0 = k - 1$. If s_i has been defined, and $1 \leq i < q$, set $s_{i+1} = 1 + R(k-1, s_i, s_i)$. Note that $s_1 = 1 + R(k-1; k-1, k-1) = 1 + (k-1) = k$. We show that $R(k : h_1, h_2)$ exists and it at most s_q.

Now let X be any set with $|X| \geq s_q$ and let $\phi : \binom{X}{k} \longrightarrow [2]$ be a 2-coloring of the k-element subsets of X. Without loss of generality, we may assume that X is a set of positive integers. Let x_1 be the least integer in X and set $X_1 = X - \{x_1\}$. Then ϕ determines a coloring ϕ_1 of the $k-1$-element subsets of X_1 by setting $\phi_1(B) = \phi(B \cup \{x_1\}$. It follows that there is some $\alpha_1 \in X$ and a subset $H_1 \subset X_1$ with $|H_1| = s_{q-1}$ so that $\phi_1(B) = \alpha_1$ for every $B \in \binom{X_1}{k-1}$. The let x_2 be the least integer in H_1 and set $X_2 = H_1 - \{x_2\}$. As before, ϕ determines a coloring ϕ_2 of the $k-1$ element subsets of X_2 by setting $\phi_2(B) = \phi(B \cup \{x_2\}$. Again, there is some $\alpha_2 \in X$ and a subset $H_2 \subset X_2$ so that $\phi_2(B) = \alpha_2$ for every $B \in \binom{H_2}{k-1}$. Note that α_2 may in fact be the same as α_1.

Repeat this process q times and consider the set $\{x_1, x_2, \ldots, x_q\}$ obtained as an end result. Also consider the elements $\{\alpha_i : 1 \leq i \leq q\}$. It follows from the Pigeon-hole Principle that there is an element $\alpha \in \{1, 2, \ldots, r\}$ and subset $S \subseteq \{1, 2, \ldots, q\}$ with $|S| = h_\alpha$ so that $\alpha_i = \alpha$ for every $i \in S$. Then set $H = \{x_i : i \in S\}$. Then $|H| = h_\alpha$. Furthermore $\phi(A) = \alpha$ for every k-element subset of H. This completes the argument when $r = 2$.

Now suppose that $r > 2$. We derive a 2-coloring from an r-coloring by considering the last $r - 1$ colors collectively as just one color. Now it follows easily that the ramsey number $R(k; h_1, h_2, \ldots, h_r)$ exists and satisfies:

$$R(k; h_1, h_2, \ldots, h_r) \leq R(k; h_1, R(k; h_2, h_3, \ldots, h_r))$$

□

Corollary 9.2. $R(2; m, n) \leq \binom{m+n-2}{m-1} = \binom{m+n-2}{n-1}$.

Proof. The argument when $r = 2$ gives the inequality

$$R(2; m, n) \leq R(2; m-1, n) + R(2; m, n-1).$$

However, the binomial coefficients satisfy

$$\binom{m+n-2}{m-1} = \binom{m+n-3}{m-2} + \binom{m+n-3}{m-1} = \binom{m+n-3}{m-2} + \binom{m+n-3}{n-2}$$

So the desired inequality follows by induction after noting that it holds for $m = 1$ and for $n = 1$. □

9.2 Small Ramsey Numbers

In the following table, we provide information about the ramsey numbers $R(2; m, n)$ when m and n are at most 7. When a cell contains a single number, that is the precise answer. When there are two numbers, they represent upper and lower bounds.

n / m	3	4	5	6	7	8	9
3	6	9	14	18	23	36	39
4		18	25	35, 41	49, 61	56, 84	69, 115
5			43, 49	58, 87	80, 143	95, 216	121, 316
6				102, 165	111, 298	127, 495	153, 780
7					205, 540	216, 1031	216, 1713
8						282, 1870	282, 3583
9							565, 6588

For additional data, do a web search and look for Stanley Radziszowski, who maintains the most current information on his web site.

9.3 Estimating Ramsey Numbers

We will find it convenient to utilize the following approximation due to Stirling. You can find a proof in almost any advanced calculus book.

$$n! \equiv \sqrt{2\pi n}\big(\frac{n}{e}\big)^n\big(1 + \frac{1}{12n} + \frac{1}{288n^2} - \frac{139}{51840n^3} + O(\frac{1}{n^4})\big).$$

Of course, we will normally be satisfied with the first term:

$$n! \equiv \sqrt{2\pi n}\big(\frac{n}{e}\big)^n$$

Using Stirling's approximation, we have the following upper bound:

$$R(2; n, n) \leq \binom{2n-2}{n-1} \equiv \frac{2^{2n}}{4\sqrt{\pi n}}$$

Here is an exponential lower bound.

Theorem 9.3.

$$R(2; n, n) \geq \big(1 + o(1)\big)\frac{n}{e\sqrt{2}}2^{\frac{1}{2}n}$$

Proof. Let t be an integer with $t > n$ and consider the following probability space. The outcomes in the probability space are graphs with vertex set $\{1, 2, \ldots, t\}$. For each i and j with $1 \leq i < j \leq t$, edge ij is present in the graph with probability $1/2$. Furthermore, the events for distinct pairs are independent.

Let X_1 denote the random variable which counts the number of n-element subsets of $\{1, 2, \ldots, t\}$ for which all $\binom{n}{2}$ pairs are edges in the graph. Similarly, X_2 is the random variable which counts the number of n-element subsets of $\{1, 2, \ldots, t\}$ for which all $\binom{n}{2}$ pairs are edges are *not* in the graph. Then set $X = X_1 + X_2$.

By linearity of expectation, $E(X) = E(X_1) + E(X_2)$ while

$$E(X_1) = E(X_2) = \binom{t}{n} \frac{1}{2^{\binom{n}{2}}}.$$

If $E(X) < 1$, then there must exist a graph with vertex set $\{1, 2, \ldots, t\}$ without a K_n or an I_n. We then consider the inequality

$$2\binom{t}{n} \frac{1}{2^{\binom{n}{2}}} < 1$$

Using the Stirling approximation, we see that this inequality holds when

$$t \geq \big(1 + o(1)\big) \frac{n}{e\sqrt{2}} 2^{\frac{1}{2}n}$$

$\square$