CHAPTER

THIRTEEN

PÓLYA'S ENUMERATION THEOREM

In this chapter, we introduce a powerful enumeration technique generally referred to as Pólya's enumeration theorem[1]. Pólya's approach to counting allows us to use symmetries (such as those of geometric objects like polygons) to form generating functions. These generating functions can then be used to answer combinatorial questions such as

1. How many different necklaces of six beads can be formed using red, blue and green beads? What about 500-bead necklaces?
2. How many musical scales consisting of 6 notes are there?
3. How many isomers of the compound xylenol, $C_6H_3(CH_3)_2(OH)$, are there? What about C_nH_{2n+2}? (In chemistry, *isomers* are chemical compounds with the same number of molecules of each element but with different arrangements of those molecules.)
4. How many nonisomorphic graphs are there on four vertices? How many of them have three edges? What about on 1000 vertices with $257,000$ edges? How many r-regular graphs are there on 40 vertices? (A graph is *r-regular* if every vertex has degree r.)

To use Pólya's techniques, we will require the idea of a permutation group. However, our treatment will be self-contained and driven by examples. We begin with a simplified version of the first question above.

[1]Like so many results of mathematics, the crux of the result was originally discovered by someone other than the mathematician whose name is associated with it. J.H. Redfield published this result in 1927, 10 years prior to Pólya's work. It would take until 1960 for Redfield's work to be discovered, by which time Pólya's name was firmly attached to the technique.

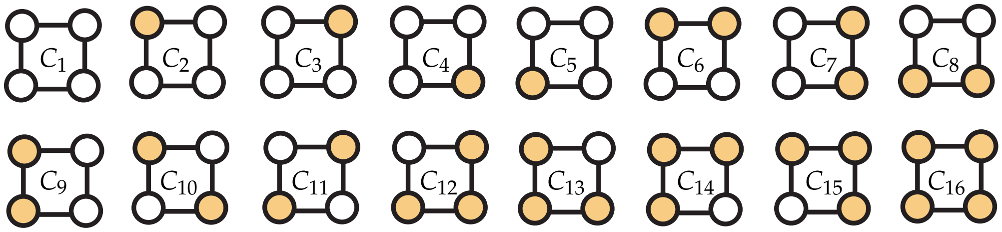

Figure 13.1: The 16 colorings of the vertices of a square.

13.1 Coloring the Vertices of a Square

Let's begin by coloring the vertices of a square using white and gold. If we fix the position of the square in the plane, there are $2^4 = 16$ different colorings. These colorings are shown in Figure 13.1. However, if we think of the square as a metal frame with a white bead or a gold bead at each corner and allow the frame to be rotated and flipped over, we realize that many of these colorings are equivalent. For instance, if we flip coloring C_7 over about the vertical line dividing the square in half, we obtain coloring C_9. If we rotate coloring C_2 clockwise by 90°, we obtain coloring C_3. In many cases, we want to consider such equivalent colorings as a single coloring. (Recall our motivating example of necklaces made of colored beads. It makes little sense to differentiate between two necklaces if one can be rotated and flipped to become the other.)

To systematically determine how many of the colorings shown in Figure 13.1 are not equivalent, we must think about the transformations we can apply to the square and what each does to the colorings. Before examining the transformations' effects on the colorings, let's take a moment to see how they rearrange the vertices. To do this, we consider the upper-left vertex to be 1, the upper-right vertex to be 2, the lower-right vertex to be 3, and the lower-left vertex to be 4. We denote the clockwise rotation by 90° by r_1 and see that r_1 sends the vertex in position 1 to position 2, the vertex in position 2 to position 3, the vertex in position 3 to position 4, and the vertex in position 4 to position 1. For brevity, we will write $r_1(1) = 2$, $r_1(2) = 3$, etc. We can also rotate the square clockwise by 180° and denote that rotation by r_2. In this case, we find that $r_2(1) = 3$, $r_2(2) = 4$, $r_2(3) = 1$, and $r_2(4) = 2$. Notice that we can achieve the transformation r_2 by doing r_1 twice in succession. Furthermore, the clockwise rotation by 270°, r_3, can be achieved by doing r_1 three times in succession. (Counterclockwise rotations can be avoided by noting that they have the same effect as a clockwise rotation, although by a different angle.)

When it comes to flipping the square, there are four axes about which we can flip it: vertical, horizontal, positive-slope diagonal, and negative-slope diagonal. We denote these flips by v, h, p, and n, respectively. Now notice that $v(1) = 2$, $v(2) = 1$, $v(3) = 4$,

Transformation	Fixed colorings
ι	All 16
r_1	C_1, C_{16}
r_2	C_1, C_{10}, C_{11}, C_{16}
r_3	C_1, C_{16}
v	C_1, C_6, C_8, C_{16}
h	C_1, C_7, C_9, C_{16}
p	C_1, C_3, C_5, C_{10}, C_{11}, C_{13}, C_{15}, C_{16}
n	C_1, C_2, C_4, C_{10}, C_{11}, C_{12}, C_{14}, C_{16}

Table 13.1: Colorings fixed by transformations of the square

and $v(4) = 3$. For the flip about the horizontal axis, we have $h(1) = 4$, $h(2) = 3$, $h(3) = 2$, and $h(4) = 1$. For p, we have $p(1) = 3$, $p(2) = 2$, $p(3) = 1$, and $p(4) = 4$. Finally, for n we find $n(1) = 1$, $n(2) = 4$, $n(3) = 3$, and $n(4) = 2$. There is one more transformation that we must mention. The transformation that does nothing to the square is called the *identity transformation*, denoted ι. It has $\iota(1) = 1$, $\iota(2) = 2$, $\iota(3) = 3$, and $\iota(4) = 4$.

Now that we've identified the eight transformations of the square, let's make a table showing which colorings from Figure 13.1 are left unchanged by the application of each transformation. Not surprisingly, the identity transformation leaves all of the colorings unchanged. Because r_1 moves the vertices cyclically, we see that only C_1 and C_{16} remain unchanged when it is applied. Any coloring with more than one color would have a vertex of one color moved to one of the other color. Let's consider which colorings are fixed by v, the flip about the vertical axis. For this to happen, the color at position 1 must be the same as the color at position 2, and the color at position 3 must be the same as the color at position 4. Thus, we would expect to find $2 \cdot 2 = 4$ colorings unchanged by v. Examining Figure 13.1, we see that these colorings are C_1, C_6, C_8, and C_{16}. Performing a similar analysis for the remaining five transformations leads to Table 13.1.

At this point, it's natural to ask where this is going. After all, we're trying to count the number of *nonequivalent* colorings, and Table 13.1 makes no effort to group colorings based on how a transformation changes one coloring to another. It turns out that there is a useful connection between counting the nonequivalent colorings and determining the number of colorings fixed by each transformation. To develop this connection, we first need to discuss the equivalence relation created by the action of the transformations of the square on the set $\mathcal{C}$ of all 2-colorings of the square. (Refer to **??** for a refresher on the definition of equivalence relation.) To do this, notice that applying a transformation to a square with colored vertices results in another square with colored vertices. For instance, applying the transformation r_1 to a square colored as in C_{12} results in a square colored as in C_{13}. We say that the transformations of the square *act* on the set $\mathcal{C}$ of colorings. We denote this action by adding a star to the transformation name. For instance, $r_1^*(C_{12}) = C_{13}$ and $v^*(C_{10}) = C_{11}$.

If τ is a transformation of the square with $\tau^*(C_i) = C_j$, then we say colorings C_i and C_j are *equivalent* and write $C_i \sim C_j$. Since $\iota^*(C) = C$ for all $C \in \mathcal{C}$, $\sim$ is reflexive. If $\tau_1^*(C_i) = C_j$ and $\tau_2^*(C_j) = C_k$, then $\tau_2^*(\tau_1^*(C_i)) = C_k$, so $\sim$ is transitive. To complete our verification that $\sim$ is an equivalence relation, we must establish that it is symmetric. For this, we require the notion of the *inverse* of a transformation τ, which is simply the transformation τ^{-1} that undoes whatever τ did. For instance, the inverse of r_1 is the *counter*clockwise rotation by 90°, which has the same effect on the location of the vertices as r_3. If $\tau^*(C_i) = C_j$, then ${\tau^{-1}}^*(C_j) = C_i$, so $\sim$ is symmetric.

Before proceeding to establish the connection between the number of nonequivalent colorings (equivalence classes under $\sim$) and the number of colorings fixed by a transformation in full generality, let's see how it looks for our example. In looking at Figure 13.1, you should notice that $\sim$ partitions $\mathcal{C}$ into six equivalence classes. Two contain one coloring each (the all white and all gold colorings). One contains two colorings (C_{10} and C_{11}). Finally, three contain four colorings each (one gold vertex, one white vertex, and the remaining four with two vertices of each color). Now look again at Table 13.1 and add up the number of colorings fixed by each transformation. In doing this, we obtain 48, and when 48 is divided by the number of transformations (8), we get 6 (the number of equivalence classes)! It turns out that this is far from a fluke, as we will soon see. First, however, we introduce the concept of a permutation group to generalize our set of transformations of the square.

13.2 Permutation Groups

Entire books have been written on the theory of the mathematical structures known as *groups*. However, our study of Pólya's enumeration theorem requires only a few facts about a particular class of groups that we introduce in this section. First, recall that a bijection from a set X to itself is called a *permutation*. A *permutation group* is a set P of permutations of a set X so that

1. the identity permutation ι is in P;
2. if $\pi_1, \pi_2 \in P$, then $\pi_2 \circ \pi_1 \in P$; and
3. if $\pi_1 \in P$, then $\pi_1^{-1} \in P$.

For our purposes, X will always be finite and we will usually take $X = [n]$ for some positive integer n. The *symmetric group on n elements*, denoted S_n, is the set of all permutations of $[n]$. Every finite permutation group (and more generally every finite group) is a subgroup of S_n for some positive integer n.

As our first example of a permutation group, consider the set of permutations we discussed in section 13.1, called the *dihedral group of the square*. We will denote this group by D_8. We denote by D_{2n} the similar group of transformations for a regular n-gon, using $2n$ as the subscript because there are $2n$ permutations in this group.[2] The first criterion to be a permutation group is clearly satisfied by D_8. Verifying the other two is quite tedious, so we only present a couple of examples. First, notice that $r_2 \circ r_1 = r_3$. This can be determined by carrying out the composition of these functions as permutations or by noting that rotating 90° clockwise and then 180° clockwise is the same as rotating 270° clockwise. For $v \circ r$, we find $v \circ r(1) = 1$, $v \circ r(3) = 3$, $v \circ r(2) = 4$, and $v \circ r(4) = 2$, so $v \circ r = n$. For inverses, we have already discussed that $r_1^{-1} = r_3$. Also, $v^{-1} = v$, and more generally, the inverse of *any* flip is that same flip.

13.2.1 Representing permutations

The way a permutation rearranges the elements of X is central to Pólya's enumeration theorem. A proper choice of representation for a permutation is very important here, so let's discuss how permutations can be represented. One way to represent a permutation π of $[n]$ is as a $2 \times n$ matrix in which the first row represents the domain and the second row represents π by putting $\pi(i)$ in position i. For example,

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 3 & 5 & 1 \end{pmatrix}$$

is the permutation of $[5]$ with $\pi(1) = 2$, $\pi(2) = 4$, $\pi(3) = 3$, $\pi(4) = 5$, and $\pi(5) = 1$. This notation is rather awkward and provides only the most basic information about the permutation. A more compact (and more useful for our purposes) notation is known as *cycle notation*. One way to visualize how the cycle notation is constructed is by constructing a digraph from a permutation π of $[n]$. The digraph has $[n]$ as its vertex set and a directed edge from i to j if and only if $\pi(i) = j$. (Here we allow a directed edge from a vertex to itself if $\pi(i) = i$.) The digraph corresponding to the permutation π from above is shown in Figure 13.2. Since π is a permutation, every component of such a digraph is a directed cycle. We can then use these cycles to write down the

[2] Some authors and computer algebra systems use D_n as the notation for the dihedral group of the n-gon.

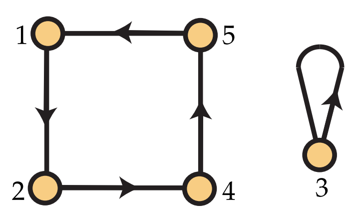

Figure 13.2: The digraph corresponding to permutation $\pi = (1245)(3)$

permutation in a compact manner. For each cycle, we start at the vertex with smallest label and go around the cycle in the direction of the edges, writing down the vertices' labels in order. We place this sequence of integers in parentheses. For the 4-cycle in Figure 13.2, we thus obtain (1245). (If $n \geq 10$, we place spaces or commas between the integers.) The component with a single vertex is denoted simply as (3), and thus we may write $\pi = (1245)(3)$. By convention, the disjoint cycles of a permutation are listed so that their first entries are in increasing order.

Example 13.1. The permutation $\pi = (1483)(27)(56)$ has $\pi(1) = 4$, $\pi(8) = 3$, $\pi(3) = 1$, and $\pi(5) = 6$. The permutation $\pi' = (13)(2)(478)(56)$ has $\pi'(1) = 3$, $\pi'(2) = 2$, and $\pi'(8) = 4$. We say that π consists of two cycles of length 2 and one cycle of length 4. For π', we have one cycle of length 1, two cycles of length 2, and one cycle of length 3. A cycle of length k will also called a k-cycle in this chapter.

13.2.2 Multiplying permutations

Because the operation in an arbitrary group is frequently called multiplication, it is common to refer to the composition of permutations as multiplication and write $\pi_2\pi_1$ instead of $\pi_2 \circ \pi_1$. The important thing to remember here, however, is that the operation is simply function composition. Let's see a couple of examples.

Example 13.2. Let $\pi_1 = (1234)$ and $\pi_2 = (12)(34)$. (Notice that these are the permutations r_1 and v, respectively, from D_8.) Let $\pi_3 = \pi_2\pi_1$. To determine π_3, we start by finding $\pi_3(1) = \pi_2\pi_1(1) = \pi_2(2) = 1$. We next find that $\pi_3(2) = \pi_2\pi_1(2) = \pi_2(3) = 4$. Similarly, $\pi_3(3) = 3$ and $\pi_3(4) = 2$. Thus, $\pi_3 = (1)(24)(3)$, which we called n earlier.

Now let $\pi_4 = \pi_1\pi_2$. Then $\pi_4(1) = 3$, $\pi_4(2) = 2$, $\pi_4(3) = 1$, and $\pi_4(4) = 4$. Therefore, $\pi_4 = (13)(2)(4)$, which we called p earlier. It's important to note that $\pi_1\pi_2 \neq \pi_2\pi_1$, which hopefully does not surprise you, since function composition is not in general commutative. To further illustrate the lack of commutativity in permutation groups, pick up a book (Not this one! You need to keep reading directions here.) so that cover is up and the spine is to the left. First, flip the book over from left to right. Then rotate it 90° clockwise. Where is the spine? Now return the book to the cover-up, spine-left position. Rotate the book 90° clockwise and then flip it over from left to right. Where is the spine this time?

It quickly gets tedious to write down where the product of two (or more) permutations sends each element. A more efficient approach would be to draw the digraph and then write down the cycle structure. With some practice, however, you can build the cycle notation as you go along, as we demonstrate in the following example.

Example 13.3. Let $\pi_1 = (123)(487)(5)(6)$ and $\pi_2 = (18765)(234)$. Let $\pi_3 = \pi_2\pi_1$. To start constructing the cycle notation for π_3, we must determine where π_3 sends 1. We find that it sends it to 3, since π_1 sends 1 to 2 and π_2 sends 2 to 3. Thus, the first cycle begins 13. Now where is 3 sent? It's sent to 8, which goes to 6, which goes to 5, which goes to 1, completing our first cycle as (13865). The first integer not in this cycle is 2, which we use to start our next cycle. We find that 2 is sent to 4, which is set to 7, which is set to 2. Thus, the second cycle is (247). Now all elements of 8 are represented in these cycles, so we know that $\pi_3 = (13865)(247)$.

We conclude this section with one more example.

Example 13.4. Let's find [(123456)][(165432)], where we've written the two permutations being multiplied inside brackets. Since we work from *right* to *left*, we find that the first permutation applied sends 1 to 6, and the second sends 6 to 1, so our first cycle is (1). Next, we find that the product sends 2 to 2. It also sends i to i for every other $i \leq 6$. Thus, the product is (1)(2)(3)(4)(5)(6), which is better known as the identity permutation. Thus, (123456) and (165432) are inverses.

In the next section, we will use standard counting techniques we've seen before in this book to prove results about groups acting ons ets. We will state the results for arbitrary groups, but you may safely replace "group" by "permutation group" without losing any understanding required for the remainder of the chapter.

13.3 Burnside's Lemma

Burnside's lemma[3] relates the number of equivalence classes of the action of a group on a finite set to the number of elements of the set fixed by the elements of the group. Before stating and proving it, we need some notation and a proposition. If a group G acts on a finite set $\mathcal{C}$, let $\sim$ be the equivalence relation induced by this action. (As before, the action of $\pi \in G$ on $\mathcal{C}$ will be denoted π^*.) Denote the equivalence class containing $C \in \mathcal{C}$ by $\langle C \rangle$. For $\pi \in G$, let $\mathrm{fix}_{\mathcal{C}}(\pi) = \{C \in \mathcal{C} \colon \pi^*(C) = C\}$, the set of colorings fixed by π. For $C \in \mathcal{C}$, let $\mathrm{stab}_G(C) = \{\pi \in G \colon \pi(C) = C\}$ be the *stabilizer* of C in G, the permutations in G that fix C.

To illustrate these concepts before applying them, refer back to Table 13.1. Using that information, we can determine that $\mathrm{fix}_{\mathcal{C}}(r_2) = \{C_1, C_{10}, C_{11}, C_{16}\}$. Determining the stabilizer of a coloring requires finding the rows of the table in which it appears. Thus, $\mathrm{stab}_{D_8}(C_7) = \{\iota, h\}$ and $\mathrm{stab}_{D_8}(C_{11}) = \{\iota, r_2, p, n\}$.

Proposition 13.5. *Let a group G act on a finite set $\mathcal{C}$. Then for all $C \in \mathcal{C}$,*

$$\sum_{C' \in \langle C \rangle} |\,\mathrm{stab}_G(C')| = |G|.$$

Proof. Let $\mathrm{stab}_G(C) = \{\pi_1, \ldots, \pi_k\}$ and $T(C, C') = \{\pi \in G \colon \pi^*(C) = C'\}$. (Note that $T(C, C) = \mathrm{stab}_G(C)$.) Take $\pi \in T(C, C')$. Then $\pi \circ \pi_i \in T(C, C')$ for $1 \leq i \leq k$. Furthermore, if $\pi \circ \pi_i = \pi \circ \pi_j$, then $\pi^{-1} \circ \pi \circ \pi_i = \pi^{-1} \circ \pi \circ \pi_j$. Thus $\pi_i = \pi_j$ and

[3] Again, not originally proved by Burnside. It was known to Frobenius and for the most part by Cauchy. However, it was most easily found in Burnside's book, and thus his name came to be attached.

$i = j$. If $\pi' \in T(C, C')$, then $\pi^{-1} \circ \pi' \in T(C, C)$. Thus, $\pi^{-1} \circ \pi' = \pi_i$ for some i, and hence $\pi' = \pi \circ \pi_i$. Therefore $T(C, C') = \{\pi \circ \pi_1, \ldots, \pi \circ \pi_k\}$. Additionally, we observe that $T(C', C) = \{\pi^{-1} \colon \pi \in T(C, C')\}$. Now for all $C' \in \langle C \rangle$,

$$|\operatorname{stab}_G(C')| = |T(C', C')| = |T(C', C)| = |T(C, C')| = |T(C, C)| = |\operatorname{stab}_G(C)|.$$

Therefore,

$$\sum_{C' \in \langle C \rangle} |\operatorname{stab}_G(C')| = \sum_{C' \in \langle C \rangle} |T(C, C')|.$$

Now notice that each element of G appears in $T(C, C')$ for precisely one $C' \in \langle C \rangle$, and the proposition follows. □

With Proposition 13.5 established, we are now prepared for Burnside's lemma.

Lemma 13.6 (Burnside's Lemma)**.** *Let a group G act on a finite set $\mathcal{C}$ and let N be the number of equivalence classes of $\mathcal{C}$ induced by this action. Then*

$$N = \frac{1}{|G|} \sum_{\pi \in G} |\operatorname{fix}_{\mathcal{C}}(\pi)|.$$

Before we proceed to the proof, note that the calculation in Burnside's lemma for the example of 2-coloring the vertices of a square is exactly the calculation we performed at the end of section 13.1.

Proof. Let $X = \{(\pi, C) \in G \times \mathcal{C} \colon \pi(C) = C\}$. Notice that $\sum_{\pi \in G} |\operatorname{fix}_{\mathcal{C}}(\pi)| = |X|$, since each term in the sum counts how many ordered pairs of X have π in their first coordinate. Similarly, $\sum_{C \in \mathcal{C}} |\operatorname{stab}_G(C)| = |X|$, with each term of this sum counting how many ordered pairs of X have C as their second coordinate. Thus, $\sum_{\pi \in G} |\operatorname{fix}_{\mathcal{C}}(\pi)| = \sum_{C \in \mathcal{C}} |\operatorname{stab}_G(C)|$. Now note that the latter sum may be rewritten as

$$\sum_{\substack{\text{equivalence} \\ \text{classes } \langle C \rangle}} \left(\sum_{C' \in \langle C \rangle} |\operatorname{stab}_G(C')| \right).$$

By Proposition 13.5, the inner sum is $|G|$. Therefore, the total sum is $N \cdot |G|$, so solving for N gives the desired equation. □

Burnside's lemma helpfully validates the computations we did in the previous section. However, what if instead of a square we were working with a hexagon and instead of two colors we allowed four? Then there would be $4^6 = 4096$ different colorings and the dihedral group of the hexagon has 12 elements. Assembling the analogue of Table 13.1 in this situation would be a nightmare! This is where the genius of Pólya's approach comes into play, as we see in the next section.

Transformation	Monomial	Fixed colorings
$\iota = (1)(2)(3)(4)$	x_1^4	16
$r_1 = (1234)$	x_4^1	2
$r_2 = (13)(24)$	x_2^2	4
$r_3 = (1432)$	x_4^1	2
$v = (12)(34)$	x_2^2	4
$h = (14)(23)$	x_2^2	4
$p = (14)(2)(3)$	$x_1^2 x_2^1$	8
$n = (1)(24)(3)$	$x_1^2 x_2^1$	8

Table 13.2: Monomials arising from the dihedral group of the square

13.4 Pólya's Theorem

Before getting to the full version of Pólya's formula, we must develop a generating function as promised at the beginning of the chapter. To do this, we will return to our example of section 13.1.

13.4.1 The cycle index

Unlike the generating functions we encountered in **??**, the generating functions we will develop in this chapter will have more than one variable. We begin by associating a monomial with each element of the permutation group involved. In this case, it is D_8, the dihedral group of the square. To determine the monomial associated to a permutation, we need to write the permutation in cycle notation and then determine the monomial based on the number of cycles of each length. Specifically, if π is a permutation of $[n]$ with j_k cycles of length k for $1 \leq k \leq n$, then the monomial associated to π is $x_1^{j_1} x_2^{j_2} \cdots x_n^{j_n}$. Note that $j_1 + 2j_2 + 3j_3 + \cdots + nj_n = n$. For example, the permutation $r_1 = (1234)$ is associated with the monomial x_4^1 since it consists of a single cycle of length 4. The permutation $r_2 = (13)(24)$ has two cycles of length 2, and thus its monomial is x_2^2. For $p = (14)(2)(3)$, we have two 1-cycles and one 2-cycle, yielding the monomial $x_1^2 x_2^1$. In Table 13.2, we show all eight permutations in D_8 along with their associated monomials.

Now let's see how the number of 2-colorings of the square fixed by a permutation can be determined from its cycle structure and associated monomial. If $\pi(i) = j$, then we know that for π to fix a coloring C, vertices i and j must be colored the same in C. Thus, the second vertex in a cycle must have the same color as the first. But then the third vertex must have the same color as the second, which is the same color as the first.

In fact, all vertices appearing in a cycle of π must have the same color in C if π fixes C! Since we are coloring with the two colors white and gold, we can choose to color the points of each cycle uniformly white or gold. For example, for the permutation $v = (12)(34)$ to fix a coloring of the square, vertices 1 and 2 must be colored the same color (2 choices) and vertices 3 and 4 must be colored the same color (2 choices). Thus, there are $2 \cdot 2 = 4$ colorings fixed by v. Since there are two choices for how to uniformly color the elements of a cycle, letting $x_i = 2$ for all i in the monomial associated with π gives the number of colorings fixed by π. In Table 13.2, the "Fixed colorings" column gives the number of 2-colorings of the square fixed by each permutation. Before, we obtained this manually by considering the action of D_8 on the set of all 16 colorings. Now we only need the cycle notation and the monomials that result from it to derive this!

Recall that Burnside's lemma (13.6) states that the number of colorings fixed by the action of a group can be obtained by adding up the number fixed by each permutation and dividing by the number of permutations in the group. If we do that instead for the monomials arising from the permutations in a permutation group G in which every cycle of every permutation has at most n entries, we obtain a polynomial known as the *cycle index* $P_G(x_1, x_2, \ldots, x_n)$. For our running example, we find

$$P_{D_8}(x_1, x_2, x_3, x_4) = \frac{1}{8}\left(x_1^4 + 2x_1^2x_2^1 + 3x_2^2 + 2x_4^1\right).$$

To find the number of distinct 2-colorings of the square, we thus let $x_i = 2$ for all i and obtain $P_{D_8}(2,2,2,2) = 6$ as before. Notice, however, that we have something more powerful than Burnside's lemma here. We may substitute *any* positive integer m for each x_i to find out how many nonequivalent m-colorings of the square exist. We no longer have to analyze how many colorings each permutation fixes. For instance, $P_{D_8}(3,3,3,3) = 21$, meaning that 21 of the 81 colorings of the vertices of the square using three colors are distinct.

13.4.2 The full enumeration formula

Hopefully the power of the cycle index to count colorings that are distinct when symmetries are considered is becoming apparent. In the next section, we will provide additional examples of how it can be used. However, we still haven't seen the full power of Pólya's technique. From the cycle index alone, we can determine how many colorings of the vertices of the square are distinct. However, what if we want to know how many of them have two white vertices and two gold vertices? This is where Pólya's enumeration formula truly plays the role of a generating function.

Let's again consider the cycle index for the dihedral group D_8:

$$P_{D_8}(x_1, x_2, x_3, x_4) = \frac{1}{8}\left(x_1^4 + 2x_1^2x_2^1 + 3x_2^2 + 2x_4^1\right).$$

Instead of substituting integers for the x_i, let's consider what happens if we substitute something that allows us to track the colors used. Since x_1 represents a cycle of length 1 in a permutation, the choice of white or gold for the vertex in such a cycle amounts to a single vertex receiving that color. What happens if we substitute $w+g$ for x_1? The first term in P_{D_8} corresponds to the identity permutation ι, which fixes all colorings of the square. Letting $x_1 = w+g$ in this term gives

$$(w+g)^4 = g^4 + 4g^3w + 6g^2w^2 + 4gw^3 + w^4,$$

which tells us that ι fixes one coloring with four gold vertices, four colorings with three gold vertices and one white vertex, six colorings with two gold vertices and two white vertices, four colorings with one gold vertex and three white vertices, and one coloring with four white vertices.

Let's continue establishing a pattern here by considering the variable x_2. It represents the cycles of length 2 in a permutation. Such a cycle must be colored uniformly white or gold to be fixed by the permutation. Thus, choosing white or gold for the vertices in that cycle results in two white vertices or two gold vertices in the coloring. Since this happens for every cycle of length 2, we want to substitute w^2+g^2 for x_2 in the cycle index. The $x_1^2x_2^1$ terms in P_{D_8} are associated with the flips p and n. Letting $x_1 = w+g$ and $x_2 = w^2+g^2$, we find

$$x_1^2x_2^1 = g^4 + 2g^3w + 2g^2w^2 + 2gw^3 + w^4,$$

from which we are able to deduce that p and n each fix one coloring with four gold vertices, two colorings with three gold vertices and one white vertex, and so on. Comparing this with Table 13.1 shows that the generating function is right on.

By now the pattern is becoming apparent. If we substitute w^i+g^i for x_i in the cycle index for each i, we then keep track of how many vertices are colored white and how many are colored gold. The simplification of the cycle index in this case is then a generating function in which the coefficient on g^sw^t is the number of distinct colorings of the vertices of the square with s vertices colored gold and t vertices colored white. Doing this and simplifying gives

$$P_{D_8}(w+g, w^2+g^2, w^3+g^3, w^4+g^4) = g^4 + g^3w + 2g^2w^2 + gw^3 + w^4.$$

From this we find one coloring with all vertices gold, one coloring with all vertices white, one coloring with three gold vertices and one white vertex, one coloring with one gold vertex and three white vertices, and two colorings with two vertices of each color.

As with the other results we've discovered in this chapter, this property of the cycle index holds up beyond the case of coloring the vertices of the square with two colors. The full version is Pólya's enumeration theorem:

Theorem 13.7 (Pólya's Enumeration Theorem)**.** *Let S be a set with $|S| = r$ and $\mathcal{C}$ the set of colorings of S using the colors $c_1, \ldots, c_m$. Let a permutation group G act on S to induce an equivalence relation on $\mathcal{C}$. Then*

$$P_G\left(\sum_{i=1}^{m} c_i, \sum_{i=1}^{m} c_i^2, \ldots, \sum_{i=1}^{m} c_i^r\right)$$

is the generating function for the number of nonequivalent colorings of S in $\mathcal{C}$.

If we return to coloring the vertices of the square but now allow the color blue as well, we find

$$\begin{aligned} P_{D_8}(w+g+b, w^2+g^2+b^2, w^3+g^3+b^3, w^3+g^3+b^3) &= b^4 + b^3g + 2b^2g^2 + bg^3 + g^4 \\ &+ b^3w + 2b^2gw + 2bg^2w + g^3w + 2b^2w^2 + 2bgw^2 + 2g^2w^2 + bw^3 + gw^3 + w^4. \end{aligned}$$

From this generating function, we can readily determine the number of nonequivalent colorings with two blue vertices, one gold vertex, and one white vertex to be 2. Because the generating function of Pólya's enumeration theorem records the number of nonequivalent patterns, it is sometimes called the *pattern inventory*.

What if we were interested in making necklaces with 500 (very small) beads colored white, gold, and blue? This would be equivalent to coloring the vertices of a regular 500-gon, and the dihedral group D_{1000} would give the appropriate transformations. With a computer algebra system[4] such as *Mathematica*®, it is possible to quickly produce the pattern inventory for such a problem. In doing so, we find that there are

$$\begin{aligned} &36360291795869936842385267079543319118023385026001623040346035832580600 \\ &19158389548419850826297938878330817970253440404662728779643042527149927 \\ &03135653472347417085467453334179308247819807028526921872536424412922 \\ &797565759360408045671032 29 \approx 3.6 \times 10^{235} \end{aligned}$$

possible necklaces. Of them,

$$\begin{aligned} &2529491842340460773490413186201010487791417294078808662803638965678244 \\ &7138833704326875393229442323085905838200071479575905731776660508802696 \\ &8640797415175535033372572682057214340157297357996345021733060 \approx 2.5 \times 10^{200} \end{aligned}$$

have 225 white beads, 225 gold beads, and 50 blue beads.

The remainder of this chapter will focus on applications of Pólya's enumeration theorem and the pattern inventory in a variety of settings.

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[4]With some more experience in group theory, it is possible to give a general formula for the cycle index of the dihedral group D_{2n}, so the computer algebra system is a nice tool, but not required.

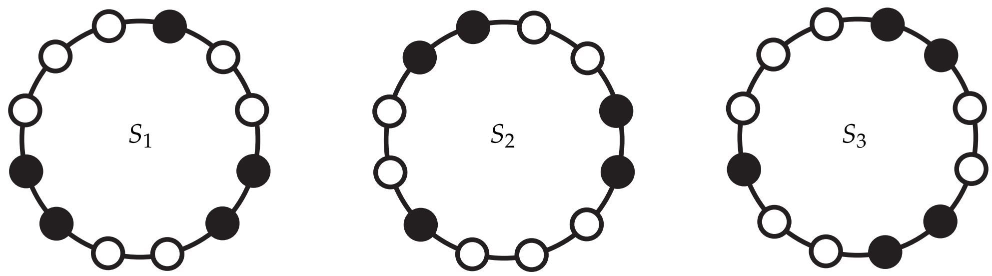

Figure 13.3: Three scales depicted by coloring

13.5 Applications of Pólya's Enumeration Formula

This section explores a number of situations in which Pólya's enumeration formula can be used. The applications are from a variety of domains and are arranged in increasing order of complexity, beginning with an example from music theory and concluding with counting nonisomorphic graphs.

13.5.1 Counting musical scales

Western music is generally based on a system of 12 equally-spaced *notes*. Although these notes are usually named by letters of the alphabet (with modifiers), for our purposes it will suffice to number them as $0, 1, \ldots, 11$. These notes are arranged into *octaves* so that the next pitch after 11 is again named 0 and the pitch before 0 is named 11. For this reason, we may consider the system of notes to correspond to the integers modulo 12. With these definitions, a *scale* is a subset of $\{0, 1, \ldots, 11\}$ arranged in increasing order. A *transposition* of a scale is a uniform transformation that replaces each note x of the scale by $x + a \pmod 1 2$ for some constant a. Musicians consider two scales to be equivalent if one is a transposition of the other. Since a scale is a subset, no regard is paid to which note starts the scale, either. The question we investigate in this section is "How many nonequivalent scales are there consisting of precisely k notes?"

Because of the cyclic nature of the note names, we may consider arranging them in order clockwise around a circle. Selecting the notes for a scale then becomes a coloring problem if we say that selected notes are colored black and unselected notes are colored white. In Figure 13.3, we show three 5-note scales using this convention. Notice that since S_2 can be obtained from S_1 by rotating it forward seven positions, S_1 and S_2 are equivalent by the transposition of adding 7. However, S_3 is not equivalent to S_1 or S_2, as it cannot be obtained from them by rotation. (Note that S_3 could be obtained from S_1 if we allowed flips in addition to rotations. Since the only operation allowed is the transposition, which corresponds to rotation, they are inequivalent.)

We have now mathematically modeled musical scales as discrete structures in a way that we can use Pólya's enumeration theorem. What is the group acting on our black/white colorings of the vertices of a regular 12-gon? One permutation in the group is $\tau = (0\ 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11)$, which corresponds to the transposition by one note. In fact, every element of the group can be realized as some power of τ since only rotations are allowed and τ is the smallest possible rotation. Thus, the group acting on the colorings is the *cyclic group of order* 12, denoted $C_{12} = \{\iota, \tau, \tau^2, \ldots, \tau^{11}\}$. Exercise 5 asks you to write all the elements of this group in cycle notation. The best way to do this is by multiplying τ^{i-1} by τ (i.e., compute $\tau\tau^{i-1}$) to find τ. Once you've done this, you will be able to easily verify that the cycle index is

$$P_{C_{12}}(x_1, \ldots, x_{12}) = \frac{x_1^{12}}{12} + \frac{x_2^6}{12} + \frac{x_3^4}{6} + \frac{x_4^3}{6} + \frac{x_6^2}{6} + \frac{x_{12}}{3}.$$

Since we've chosen colorings using black and white, it would make sense to substitute $x_i = b^i + w^i$ for all i in $P_{C_{12}}$ now to find the number of k-note scales. However, there is a convenient shortcut we may take to make the resulting generating function look more like those to which we grew accustomed in **??**. The information about how many notes are *not* included in our scale (the number colored white) can be deduced from the number that are included. Thus, we may eliminate the use of the variable w, replacing it by 1. We now find

$$\begin{aligned} P_{C_{12}}(1+b, 1+b^2, \ldots, 1+b^{12}) = b^{12} + b^{11} + 6b^{10} + 19b^9 + 43b^8 + 66b^7 + 80b^6 \\ + 66b^5 + 43b^4 + 19b^3 + 6b^2 + b + 1. \end{aligned}$$

From this, we are able to deduce that the number of scales with k notes is the coefficient on b^k. Therefore, the answer to our question at the beginning of the chapter about the number of 6-note scales is 80.

13.5.2 Enumerating isomers

Benzene is a chemical compound with formula C_6H_6, meaning it consists of six carbon atoms and six hydrogen atoms. These atoms are bonded in such a way that the six carbon atoms form a hexagonal ring with alternating single and double bonds. A hydrogen atom is bonded to each carbon atom (on the outside of the ring). From benzene it is possible to form other chemical compounds that are part of a family known as *aromatic hydrocarbons*. These compounds are formed by replacing one or more of the hydrogen atoms by atoms of other elements or functional groups such as CH_3 (methyl group) or OH (hydroxyl group). Because there are six choices for which hydrogen atoms to replace, molecules with the same chemical formula but different structures can be formed in this manner. Such molecules are called *isomers*. In this subsection, we will see how Pólya's enumeration theorem can be used to determine the number of isomers of the aromatic hydrocarbon xylenol (also known as dimethylphenol).

Permutation	Monomial	Permutation	Monomial
$\iota = (1)(2)(3)(4)(5)(6)$	x_1^6	$f = (16)(25)(34)$	x_2^3
$r = (123456)$	x_6^1	$fr = (15)(24)(3)(6)$	$x_1^2x_2^2$
$r^2 = (135)(246)$	x_3^2	$fr^2 = (14)(23)(56)$	x_2^3
$r^3 = (14)(25)(36)$	x_2^3	$fr^3 = (13)(2)(46)(5)$	$x_1^2x_2^2$
$r^4 = (153)(264)$	x_3^2	$fr^4 = (12)(36)(45)$	x_2^3
$r^5 = (165432)$	x_6^1	$fr^5 = (1)(26)(35)(4)$	$x_1^2x_2^2$

Table 13.3: CYCLE REPRESENTATION OF PERMUTATIONS IN D_{12}

Before we get into the molecular structure of xylenol, we need to discuss the permutation group that will act on a benzene ring. Much like with our example of coloring the vertices of the square, we find that there are rotations and flips at play here. In fact, the group we require is the dihedral group of the hexagon, D_{12}. If we number the six carbon atoms in clockwise order as $1, 2, \ldots, 6$, then we find that the clockwise rotation by $60°$ corresponds to the permutation $r = (123456)$. The other rotations are the higher powers of r, as shown in Table 13.3. The flip across the vertical axis is the permutation $f = (16)(25)(34)$. The remaining elements of D_{12} (other than the identity ι) can all be realized as some rotation followed by this flip. The full list of permutations is shown in Table 13.3, where each permutation is accompanied by the monomial it contributes to the cycle index.

With the monomials associated to the permutations in D_{12} identified, we are able to write down the cycle index

$$P_{D_{12}}(x_1, \ldots, x_6) = \frac{1}{12}(x_1^6 + 2x_6^1 + 2x_3^2 + 4x_2^3 + 3x_1^2x_2^2).$$

With the cycle index determined, we now turn our attention to using it to find the number of isomers of xylenol. This aromatic hydrocarbon has three hydrogen molecules, two methyl groups, and a hydroxyl group attached to the carbon atoms. Recalling that hydrogen atoms are the default from benzene, we can more or less ignore them when choosing the appropriate substitution for the x_i in the cycle index. If we let m denote methyl groups and h hydroxyl groups, we can then substitute $x_i = 1 + m^i + h^i$ in $P_{D_{12}}$. This substitution gives the generating function

$$\begin{aligned}1 &+ h + 3h^2 + 3h^3 + 3h^4 + h^5 + h^6 + m + 3hm + 6h^2m + 6h^3m \\ &+ 3h^4m + h^5m + 3m^2 + 6hm^2 + 11h^2m^2 + 6h^3m^2 + 3h^4m^2 + 3m^3 + 6hm^3 \\ &+ 6h^2m^3 + 3h^3m^3 + 3m^4 + 3hm^4 + 3h^2m^4 + m^5 + hm^5 + m^6.\end{aligned}$$

Since xylenol has one hydroxyl group and two methyl groups, we are looking for the

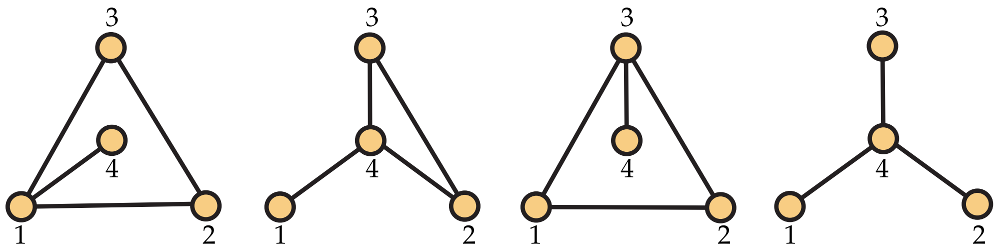

Figure 13.4: Four lalbeled graphs on four vertices

coefficient on hm^2 in this generating function. The coefficient is 6, so there are six isomers of xylenol.

In his original paper, Pólya used his techniques to enumerate the number of isomers of the alkanes C_nH_{2n+2}. When modeled as graphs, these chemical compounds are special types of trees. Since that time, Pólya's enumeration theorem has been used to enumerate isomers for many different chemical compounds.

13.5.3 Counting nonisomorphic graphs

Counting the graphs with vertex set $[n]$ is not difficult. There are $C(n,2)$ possible edges, each of which can be included or excluded. Thus, there are $2^{C(n,2)}$ *labeled* graphs on n vertices. It's only a bit of extra thought to determine that if you only want to count the labeled graphs on n vertices with k edges, you simply must choose a k-element subset of the set of all $C(n,2)$ possible edges. Thus, there are

$$\binom{\binom{n}{2}}{k}$$

graphs with vertex set $[n]$ and exactly k edges.

A more difficult problem arises when we want to start counting *nonisomorphic* graphs on n vertices. (One can think of these as *unlabeled* graphs as well.) For example, in Figure 13.4, we show four different labeled graphs on four vertices. The first three graphs shown there, however, are isomorphic to each other. Thus, only two nonisomorphic graphs on four vertices are illustrated in the figure. To account for isomorphisms, we need to bring Pólya's enumeration theorem into play.

We begin by considering all $2^{C(n,2)}$ graphs with vertex set $[n]$ and choosing an appropriate permutation group to act in the situation. Since any vertex can be mapped to any other vertex, the symmetric group S_4 acts on the vertices. However, we have to be careful about how we find the cycle index here. When we were working with colorings of the vertices of the square, we realized that all the vertices appearing in the same cycle of a permutation π had to be colored the same color. Since we're concerned with edges here and not vertex colorings, what we really need for a permutation to fix a graph is that every edge be sent to an edge and every non-edge be sent to a non-edge. To be specific, if $\{1,2\}$ is an edge of some $\mathbf{G}$ and $\pi \in S_4$ fixes $\mathbf{G}$, then $\{\pi(1), \pi(2)\}$ must also be an edge of $\mathbf{G}$. Similarly, if vertices 3 and 4 are not adjacent in $\mathbf{G}$, then $\pi(3)$ and $\pi(4)$ must also be nonadjacent in $\mathbf{G}$.

To account for edges, we move from the symmetric group S_4 to its *pair group* $S_4^{(2)}$. The objects that $S_4^{(2)}$ permutes are the 2-element subsets of $\{1,2,3,4\}$. For ease of notation, we will denote the 2-element subset $\{i,j\}$ by e_{ij}. To find the permutations in $S_4^{(2)}$, we consider the vertex permutations in S_4 and see how they permute the e_{ij}. The identity permutation $\iota = (1)(2)(3)(4)$ of S_4 corresponds to the identity permutation $\iota = (e_{12})(e_{13})(e_{14})(e_{23})(e_{24})(e_{34})$ of $S_4^{(2)}$. Now let's consider the permutation $(12)(3)(4)$. It fixes e_{12} since it sends 1 to 2 and 2 to 1. It also fixeds e_{34} by fixing 3 and 4. However, it interchanges e_{13} with e_{23} (3 is fixed and 1 is swapped with 2) and e_{14} with e_{24} (1 is sent to 2 and 4 is fixed). Thus, the corresponding permutation of pairs is $(e_{12})(e_{13}e_{23})(e_{14}e_{24})(e_{34})$. For another example, consider the permutation $(123)(4)$. It corresponds to the permutation $(e_{12}e_{23}e_{13})(e_{14}e_{24}e_{34})$ in $S_4^{(2)}$.

Since we're only after the cycle index of $S_4^{(2)}$, we don't need to find all 24 permutations in the pair group. However, we do need to know the types of those permutations in terms of cycle lengths so we can associate the appropriate monomials. For the three examples we've considered, the cycle structure of the permutation in the pair group doesn't depend on the original permutation in S_4 other than for *its* cycle structure. Any permutation in S_4 consisting of a 2-cycle and two 1-cycles will correspond to a permutation with two 2-cycles and two 1-cycles in $S_4^{(2)}$. A permutation in S_4 with one 3-cycle and one 1-cycle will correspond to a permutation with two 3-cycles in the pair group. By considering an example of a permutation in S_4 consisting of a single 4-cycle, we find that the corresponding permutation in the pair group has a 4-cycle and a 2-cycle. Finally, a permutation of S_4 consisting of two 2-cycles corresponds to a permutation in $S_4^{(2)}$ having two 2-cycles and two 1-cycles. (Exercise 8 asks you to verify these claims using specific permutations.)

Now that we know the cycle structure of the permutations in $S_4^{(2)}$, the only task remaining before we can find its cycle index of is to determine how many permutations have each of the possible cycle structures. For this, we again refer back to permutations of the symmetric group S_4. A permutation consisting of a single 4-cycle begins with 1 and then has 2, 3, and 4 in any of the $3! = 6$ possible orders, so there are 6 such permutations. For permutations consisting of a 1-cycle and a 3-cycle, there are 4 ways to choose the element for the 1-cycle and then 2 ways to arrange the other three as a 3-cycle. (Remember the smallest of them must be placed first, so there are then 2 ways to arrange the remaining two.) Thus, there are 8 such permutations. For a permutation consisting of two 1-cycles and a 2-cycle, there are $C(4,2) = 6$ ways to choose the two elements for the 2-cycle. Thus, there are 6 such permutations. For a permutation to consist of two 2-cycles, there are $C(4,2) = 6$ ways to choose two elements for the first 2-cycle. The other two are then put in the second 2-cycle. However, this counts each permutation twice, once for when the first 2-cycle is the chosen pair and once for when it is the "other two." Thus, there are 3 permutations consisting of two 2-cycles. Finally, only ι consists of four 1-cycles.

Now we're prepared to write down the cycle index of the pair group

$$P_{S_4^{(2)}}(x_1,\ldots,x_6) = \frac{1}{24}\left(x_6^1 + 9x_1^2x_2^2 + 8x_3^2 + 6x_2x_4\right).$$

To use this to enumerate graphs, we can now make the substitution $x_i = 1 + x^i$ for $1 \leq i \leq 6$. This allows us to account for the two options of an edge not being present or being present. In doing so, we find

$$P_{S_4^{(2)}}(1+x,\ldots,1+x^6) = 1 + x + 2x^2 + 3x^3 + 2x^4 + x^5 + x^6$$

is the generating function for the number of 4-vertex graphs with m edges, $0 \leq m \leq 6$. To find the total number of nonisomorphic graphs on four vertices, we substitute $x = 1$ into this polynomial. This allows us to conclude there are 11 nonisomorphic graphs on four vertices, a marked reduction from the 64 labeled graphs.

The techniques of this subsection can be used, given enough computing power, to find the number of nonisomorphic graphs on any number of vertices. For 30 vertices, there are

$$\begin{aligned}&334494316309257669249439569928080028956631479935393064329967834887217\\&734534880582749030521599504384 \approx 3.3 \times 10^{98}\end{aligned}$$

nonisomorphic graphs, as compared to $2^{435} \approx 8.9 \times 10^{130}$ labeled graphs on 30 vertices. The number of nonisomorphic graphs with precisely 200 edges is

$$\begin{aligned}&313382480997072627625877247573364018544676703365501785583608267705079\\&9699893512219821910360979601 \approx 3.1 \times 10^{96}.\end{aligned}$$

The last part of the question about graph enumeration at the beginning of the chapter was about enumerating the graphs on some number of vertices in which every vertex has degree r. While this might seem like it could be approached using the techniques of this chapter, it turns out that it cannot because of the increased dependency between where vertices are mapped.

13.6 Exercises

1. Write the permutations shown below in cycle notation.

$$\pi_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 2 & 5 & 6 & 3 & 1 \end{pmatrix} \qquad \pi_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 6 & 1 & 3 & 4 & 2 \end{pmatrix}$$

$$\pi_3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 1 & 5 & 8 & 2 & 6 & 4 & 7 \end{pmatrix} \qquad \pi_4 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 7 & 1 & 6 & 8 & 4 & 2 & 5 \end{pmatrix}$$

2. Compute $\pi_1\pi_2$, $\pi_2\pi_1$, $\pi_3\pi_4$, and $\pi_4\pi_3$ for the permutations π_i in exercise 1.

3. Find $\text{stab}_{D_8}(C_3)$ and $\text{stab}_{D_8}(C_{16})$ for the colorings of the vertices of the square shown in Figure 13.1 by referring to Table 13.1.

4. In Figure 13.5, we show a regular pentagon with its vertices labeled. Use this labeling to complete this exercise.

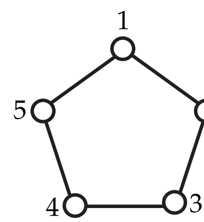

Figure 13.5: A pentagon with labeled vertices

   a) The dihedral group of the pentagon, D_{10}, contains 10 permutations. Let $r_1 = (12345)$ be the clockwise rotation by 72° and $f_1 = (1)(25)(34)$ be the flip about the line passing through 1 and perpendicular to the opposite side. Let r_2, r_3, and r_4 be the other rotations in D_{10}. Denote the flip about the line passing through vertex i and perpendicular to the other side by f_i, $1 \leq i \leq 5$. Write all 10 elements of D_{10} in cycle notation.

   b) Suppose we are coloring the vertices of the pentagon using black and white. Draw the colorings fixed by r_1. Draw the colorings fixed by f_1.

   c) Find $\text{stab}_{D_{10}}(C)$ where C is the coloring of the vertices of the pentagon in which vertices 1, 2, and 5 are colored black and vertices 3 and 4 are colored white.

   d) Find the cycle index of D_{10}.

   e) Use the cycle index to determine the number of nonequivalent colorings of vertices of the pentagon using black and white.

f) Making an appropriate substitution for the x_i in the cycle index, find the number of nonequivalent colorings of the vertices of the pentagon in which two vertices are colored black and three vertices are colored white. Draw these colorings.

5. Write all permutations in C_{12}, the cyclic group of order 12, in cycle notation.

6. The 12-note western scale is not the only system on which music is based. In classical Thai music, a scale with seven equally-spaced notes per octave is used. As in western music, a scale is a subset of these seven notes, and two scales are equivalent if they are transpositions of each other. Find the number of k-note scales in classical Thai music for $1 \leq k \leq 7$.

7. Xylene is an aromatic hydrocarbon having two methyl groups (and four hydrogen atoms) attached to the hexagonal carbon ring. How many isomers are there of xylene?

8. Find the permutations in $S_4^{(2)}$ corresponding to the permutations (1234) and $(12)(34)$ in S_4. Confirm that the first consists of a 4-cycle and a 2-cycle and the second consists of two 2-cycles and two 1-cycles.

9. Draw the three nonisomorphic graphs on four vertices with 3 edges and the two nonisomorphic graphs on four vertices with 4 edges.

10. a) Use the method of subsection 13.5.3 to find the cycle index of the pair group $S_5^{(2)}$ of the symmetric group on five elements.

    b) Use the cycle index from 10a to determine the number of nonisomorphic graphs on five vertices. How many of them have 6 edges?

11. Tic-tac-toe is a two-player game played on a 9×9 grid. The players mark the squares of the grid with the symbols X and O. This exercise uses Pólya's enumeration theorem to investigate the number of different tic-tac-toe boards. (The analysis of *games* is more complex, since it requires attention to the order the squares are marked and stopping when one player has won the game.)

    a) Two tic-tac-toe boards are equivalent if one may be obtained from the other by rotating the board or flipping it over. (Imagine that it is drawn on a clear piece of plastic.) Since the 9×9 grid is a square, the group that acts on it in this manner is the dihedral group D_8 that we have studied in this chapter. However, as with counting nonisomorphic graphs, we have to be careful to choose the way this group is represented in terms of cycles. Here we are interested in how permutations rearrange the nine squares of the tic-tac-toe board as numbered in Figure 13.6. For example, the effect of the

1	2	3
4	5	6
7	8	9

Figure 13.6: Numbered squares of a tic-tac-toe board

transformation r_1, which rotates the board 90° clockwise, can be represented as a permutation of the nine squares as $(13971)(2684)(5)$.

Write each of the eight elements of D_8 as permutations of the nine squares of a tic-tac-toe board.

b) Find the cycle index of D_8 in terms of these permutations.

c) Make an appropriate substitution for x_i in the cycle index to find a generating function $t(X, O)$ in which the coefficient on $X^i O^j$ is the number of nonequivalent tic-tac-toe boards having i squares filled by symbol X and j squares filled by symbol O. (Notice that some squares might be blank!)

d) How many nonequivalent tic-tac-toe boards are there?

e) How many nonequivalent tic-tac-toe boards have three X's and three O's?

f) When playing tic-tac-toe, the players alternate turns, each drawing their symbol in a single unoccupied square during a turn. Assuming the first player marks her squares with X and the second marks his with O, then at each stage of the game there are either the same number of X's and O's or one more X than there are O's. Use this fact and $t(X, O)$ to determine the number of nonequivalent tic-tac-toe boards that can actually be obtained in playing a game, assuming the players continue until the board is full, regardless of whether one of them has won the game.

12. Suppose you are painting the faces of a cube and you have white, gold, and blue paint available. Two painted cubes are equivalent if you can rotate one of them so that all corresponding faces are painted the same color. Determine the number of nonequivalent ways you can paint the faces of the cube as well as the number having two faces of each color. *Hint*: It may be helpful to label the faces as U ("up"), D ("down"), F ("front"), B ("back"), L ("left"), and R ("right") instead of using integers. Working with a three-dimensional model of a cube will also aid in identifying the permutations you require.