CHAPTER

FIVE

Partially Ordered Sets

Alice was surfing the web and found a site listing top movies, grouped by categories (comedy, drama, family, etc) as well as by the decade in which they were released. The top seven dramas, at least according to the movie critic hosting the web site, were:

1. Saving Private Ryan
2. Life is Beautiful
3. Forrest Gump
4. Braveheart
5. Good Will Hunting
6. Titanic
7. Jurassic Park

Alice was intrigued by the listing and decided to make her own. For comparison purposes, she agreed to use the same set of films but she felt that the following list was a more accurate ranking:

1. Life is Beautiful
2. Saving Private Ryan
3. Good Will Hunting
4. Titanic

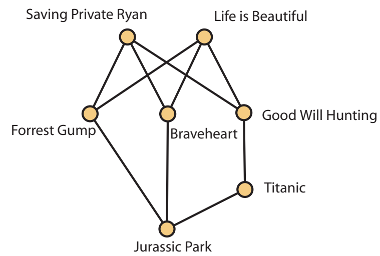


FIGURE 5.1: TOP MOVIES FROM THE 90'S

5. Braveheart
6. Forrest Gump
7. Jurassic Park

Bob listened carefully to Alice's rationale for her ranking, all the while scratching on his notepad. Eventually, he held up the following diagram and offered it as a statement of those comparisons on which both Alice and the movie critic were in agreement.

Do you see how Bob made up this diagram? Add your own rankings of these seven films and then draw the diagram that Bob would produce as a statement about the comparisons on which you, Alice and the movie critic were in agreement.

More generally, when humans are asked to express preferences among a set of options, they often report that establishing a totally ranked list is difficult if not impossible. Instead, they prefer to report a partial order—where comparisons are made between certain pairs of options but not between others. In this chapter, we make these observations more concrete by introducing the concept of a partially ordered set. Examples include (1) families of sets which are partially ordered by inclusion, and (2) a set of positive integers which is partially ordered by division. In computer science, file systems are modeled by trees which become partially ordered sets whenever links are added.

This chapter begins with a rapid survey of definitions and basic facts not explicitly marked as propositions, lemmas, or theorems. In fact, one could call them corollaries of the definitions. The validity of most of these facts is immediately clear, but readers are strongly encouraged to read *actively* to understand *why* each of these statements is true.

5.1 Basic Notation and Terminology

A *partially ordered set* or *poset* is a pair (X, P) where X is a set and P is a reflexive, antisymmetric, and transitive binary relation on X. (Refer to the appendix for a refresher

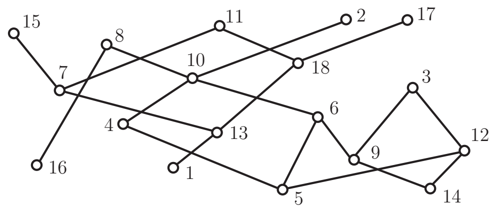


FIGURE 5.2: A POSET ON 17 POINTS

of what these properties are if you need to.) We call X the *ground set* while P is a *partial order* on X. Elements of the ground set X are also called *points*, and the poset is *finite* if the ground set is finite. In our class, we will be concerned almost exclusively with *finite* posets. To emphasize the order concept, we write $x \leq y$ in P and $y \geq x$ in P when $(x, y) \in P$. Of course, the notations $x < y$ in P and $y > x$ in P mean $x \leq y$ in P and $x \neq y$. When the poset remains fixed throughout a discussion, we will sometimes abbreviate $x \leq y$ in P by just writing $x \leq y$, etc. If $x, y \in X$ and either $x < y$ or $y < x$, we say x and y are *comparable* in P; else we say x and y are *incomparable* in P.

A partial order P is called a *total* order (also, a *linear* order) if for all $x, y \in X$, either $x \leq y$ in P or $y \leq x$ in P. For small finite sets, we can specify a linear order by listing the elements from least to greatest. For example, $L = [b, c, d, a, f, g, e]$ is a linear order on the ground set $\{a, b, c, d, e, f, g\}$. Note that $d < f$, $c < g$ and $e > b$ in L. Also, note that the set of real numbers comes equipped with a total order. For example, $1 < 7/5 < \sqrt{2} < \pi$ in the natural total order on real numbers. But in this chapter, we will be interested primarily with partial orders that are *not* linear orders.

Let $\mathbf{P} = (X, P)$ be a poset and let x and y be distinct points from X. We say x is *covered* by y in P[1] when $x < y$ in P, and there is no point $z \in X$ for which $x < z$ in P and $z < y$ in P. Given a poset $\mathbf{P} = (X, P)$, We can then define a *cover graph* $\mathbf{G}$ whose vertex set is X with xy an edge in $\mathbf{G}$ if and only if one of x and y covers the other in $\mathbf{P}$.

It is convenient to illustrate a poset with a suitably drawn diagram of the cover graph in the Euclidean plane. We choose a standard horizontal/vertical coordinate system in the plane and require that the vertical coordinate of the point corresponding to y be larger than the vertical coordinate of the point corresponding to x whenever y covers x in P. Each edge in the cover graph is represented by a straight line segment which contains no point corresponding to any element in the poset other than those associated with its two end points. Such diagrams are called *Hasse diagrams* (*poset diagrams*, *order diagrams*, or just *diagrams*).

In Figure 5.2, we illustrate a poset with ground set $X = [17] = \{1, 2, \ldots, 17\}$.

[1]Reflecting the vagaries of the English language, many mathematicians use the phrases: (1) x is covered by y in P; (2) y covers x in P; and (3) (x, y) is a cover in P interchangeably.

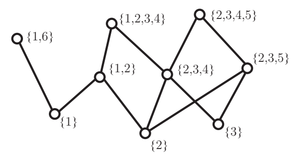


FIGURE 5.3: AN INCLUSION ORDER

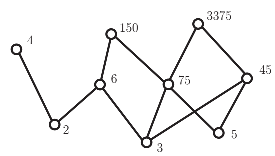


FIGURE 5.4: POSITIVE INTEGERS ORDERED BY DIVISION

There are several quite natural ways to construct posets. Here are four such examples.

1. A family $\mathcal{F}$ of sets is partially ordered by inclusion, i.e., set $A \leq B$ if and only if A is a subset of B.

2. A set X of positive integers is partially ordered by division—without remainder, i.e., set $m \leq n$ if and only if $n \equiv 0 \pmod{m}$.

3. A set X of t-tuples of real numbers is partially ordered by the rule: $(a_1, a_2, \ldots, a_t) \leq (b_1, b_2, \ldots, b_t)$ if and only if $a_i \leq b_i$ in $\mathbb{R}$ for $i = 1, 2, \ldots, t$.

4. When L_1, $L_2, \ldots, L_k$ are linear orders on the same set X, we can define a partial order P on X by setting $x \leq y$ in P if and only if $x \leq y$ in L_i for all $i = 1, 2, \ldots, k$. As is now clear, in the discussion at the very beginning of this chapter, Bob was drawing diagrams for the posets determined by the intersection of the linear orders given by Alice and the movie critic.

In fact, every finite poset arises from each of these four models. We illustrate this with the poset shown in Figures 5.3, 5.4 and 5.5 for the first three examples, and leave the last an exercise.

When $\mathbf{P} = (X, P)$ is a poset and $Y \subseteq X$, the binary relation $Q = P \cap (Y \times Y)$ is a partial order on Y, and we call the poset (Y, Q) a *subposet* of $\mathbf{P}$. Here are two important classes of subposets.

When $\mathbf{P} = (X, P)$ is a poset and C is a subset of X, we say that C is a *chain* if every distinct pair of points from C is comparable in P. When P is a linear order, the entire ground set X is a chain. Dually, if A is a subset of X, we say that A is an *antichain*

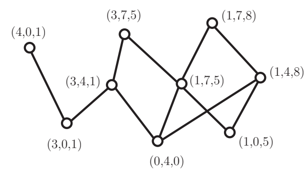


FIGURE 5.5: THE PRODUCT ORDER ON EUCLIDEAN SPACE

if every distinct pair of points from A is incomparable in P. Note that a one-element subset is both a chain and an antichain. Also, we consider the emptyset as both a chain and an antichain.

The *height* of a poset (X, P), denoted $\text{height}(P)$, is the largest h for which there exists a chain of h points in P. Dually, the *width* of a poset $\mathbf{P} = (X, P)$, denoted $\text{width}(P)$, is the largest w for which there exists an antichain of w points in P.

Question 5.1. Given a poset $\mathbf{P} = (X, P)$, how hard is to determine its height and width?

Bob says that it is very easy. For example, to find the width, just list all the subsets of X. Delete those which are not antichains. The answer is the size of the largest subset that remains.

Alice groans at Bob's naivety and suggests that he should read further in this chapter.

5.2 Additional Concepts for Posets

We say (X, P) and (Y, Q) are *isomorphic*, and write $(X, P) \cong (Y, Q)$ if there exists a bijection (1–1 and onto map) $f : X \to Y$ so that $x_1 \leq x_2$ in P if and only if $f(x_1) \leq f(x_2)$ in Q. In this definition, the map f is called an *isomorphism* from $\mathbf{P}$ to $\mathbf{Q}$. An isomorphism from $\mathbf{P}$ to $\mathbf{P}$ is called an *automorphism* of $\mathbf{P}$. An isomorphism from $\mathbf{P}$ to a subposet of $\mathbf{Q}$ is called an *embedding* of $\mathbf{P}$ in $\mathbf{Q}$. In most settings, we will not distinguish between isomorphic posets, and we will say that a poset $\mathbf{P} = (X, P)$ is *contained* in $\mathbf{Q} = (Y, Q)$ (also $\mathbf{Q}$ *contains* $\mathbf{P}$) when there is an embedding of $\mathbf{P}$ in $\mathbf{Q}$. Also, we will frequently say $\mathbf{P} = \mathbf{Q}$ when $\mathbf{P}$ and $\mathbf{Q}$ are isomorphic.

With the notion of isomorphism, we are lead naturally to the notion of an "unlabelled" posets, and in Figure 5.6, we show a diagram for such a poset.

Note that the poset shown in Figure 5.6 has the property that there is only one maximal point. Such a point is sometimes called a *one*, denoted not surprisingly as 1. Also, there is only one minimal point, and it is called a *zero*, denoted 0.

The *dual* of a partial order P on a set X is denoted by P^d and is defined by $P^d = \{(y, x) : (x, y) \in P\}$. The *dual* of the poset $\mathbf{P} = (X, P)$ is denoted by $\mathbf{P}^d$ and is defined by $\mathbf{P}^d = (X, P^d)$. A poset $\mathbf{P}$ is *self-dual* if $\mathbf{P} = \mathbf{P}^d$.

A poset $\mathbf{P} = (X, P)$ is *connected* if for every $x, y \in X$ with $x \neq y$, there is a finite sequence $x = x_0, x_1, \ldots, x_n = y$ of points from X so that x_i is comparable to x_{i+1} in P for $i = 0, 1, 2, \ldots, n-1$. A subposet $(Y, P(Y))$ of (X, P) is called a *component* of $\mathbf{P}$ if $(Y, P(Y))$ is connected and there is no subset $Z \subseteq X$ containing Y as a proper subset for which $(Z, P(Z))$ is connected. A one-point component is *trivial* (also, a *loose* point or *isolated* point); components of two or more points are *nontrivial*. Note that a loose point is both a minimal element and a maximal element. Returning to the poset shown in Figure 5.2, we see that it has two components.

With a poset $\mathbf{P} = (X, P)$, we associate a *comparability* graph $\mathbf{G}_1 = (X, E_1)$, an *incomparability* graph $\mathbf{G}_2 = (X, E_2)$, and a *cover* graph $\mathbf{G}_3 = (X, E_3)$. The edges in the comparability graph $\mathbf{G}_1$ consist of the comparable pairs and the edges in the incomparability graph are the incomparable pairs. The edges of the cover graph consist of those pairs xy for which $x <: y$ in P or $y <: x$ in P. Not every graph is a comparability graph. Also, not every graph is a cover graph. Diagrams of posets are drawings of cover graphs—with additional restrictions placed on the relative height of x and y when y covers x in the poset. On the other hand, there are no such restrictions when drawing comparability and incomparability graphs.

In Figure 5.7, we show a poset $\mathbf{P}$ and its associated comparability and incomparability graphs.

5.3 Dilworth's Chain Covering Theorem and its Dual

In this section, we prove the following theorem of R. P. Dilworth, which is truly one of the classic results of combinatorial mathematics.

Theorem 5.2 (Dilworth's Theorem)**.** *If $\mathbf{P} = (X, P)$ is a poset and $\mathrm{width}(X, P) = w$, then there exists a partition $X = C_1 \cup C_2 \cup \cdots \cup C_w$, where C_i is a chain for $i = 1, 2, \ldots, w$. Furthermore, there is no chain partition into fewer chains.*

Before proceeding with the proof of Dilworth's theorem, we pause to discuss the dual version for partitions into antichains, as it is even easier to prove.

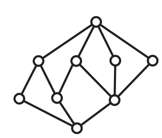


FIGURE 5.6: AN UNLABELLED PARTIALLY ORDERED SET

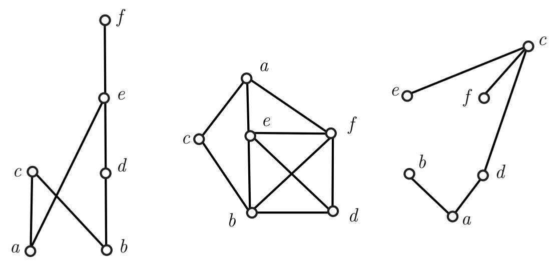


FIGURE 5.7: POSET WITH COMPARABILITY AND INCOMPARABILITY GRAPHS

Theorem 5.3. *If* $\mathbf{P} = (X, P)$ *is a poset and* $\mathrm{height}(P) = h$*, then there exists a partition* $X = A_1 \cup A_2 \cup \cdots \cup A_h$*, where* A_i *is an antichain for* $i = 1, 2, \ldots, h$*. Furthermore, there is no partition using fewer antichains.*

Proof. For each $x \in X$, let $\mathrm{height}(x)$ be the largest integer t for which there exists a chain

$$x_1 < x_2 < \cdots < x_t$$

with $x = x_t$. Evidently, $\mathrm{height}(x) \leq h$ for all $x \in X$. Then for each $i = 1, 2, \ldots, h$, let $A_i = \{x \in X : \mathrm{height}(x) = i\}$. It is easy to see that each A_i is an antichain, as if $x, y \in A_i$ are such that $x < y$, then there is a chain $x_1 < x_2 < \cdots < x_i = x < x_{i+i} = y$, so $\mathrm{height}(y) = i + 1$. Since $\mathrm{height}(P) = h$, there is a maximum chain $C = \{x_1, x_2, \ldots, x_h\}$. If it were possible to partition $\mathbf{P}$ into $t < h$ antichains, then by the pigeonhole principle, one of the antichains would contain two points from C, but this is not possible. $\square$

When $\mathbf{P} = (X, P)$ is a poset, a point $x \in X$ with $\mathrm{height}(x) = 1$ is called a *minimal* point of $\mathbf{P}$. We denote the set of all minimal points of a poset $\mathbf{P} = (X, P)$ by $\min(X, P)$ [2].

The argument given for the proof of Theorem 5.3 yields an efficient algorithm, one that is defined recursively. Set $\mathbf{P}_0 = \mathbf{P}$. If P_i has been defined and $P_i \neq \emptyset$, let $A_i = \min(\mathbf{P}_i)$ and then let $\mathbf{P}_{i+1}$ denote the subposet remaining when A_i is removed from $\mathbf{P}_i$.

In Figure 5.8, we illustrate the antichain partition provided by this algorithm. Note that the poset used here is the same poset appearing previously in Figure 5.2.

Remark 5.4. Alice claims that it is very easy to find the set of minimal elements of a poset. Do you agree?

Dually, we can speak of the set $\max(\mathbf{P})$ of *maximal* points of $\mathbf{P}$. We can also partition $\mathbf{P}$ into $\mathrm{height}(\mathbf{P})$ antichains by recursively removing the set of maximal points.

[2]Since we use the notation $\mathbf{P} = (X, P)$ for a poset, the set of minimal elements can be denoted by $\min(\mathbf{P})$ or $\min(X, P)$. This convention will be used for all set valued and integer valued functions of posets.

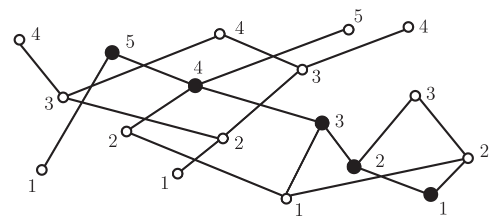


FIGURE 5.8: A POSET OF HEIGHT 5

We pause to remark that when $\mathbf{P} = (X, P)$ is a poset, the set of all chains of $\mathbf{P}$ is itself partially ordered by inclusion. So it is natural to say that a chain C is *maximal* when there is no chain C' containing C as a proper subset. Also, a chain C is *maximum* when there is no chain C' with $|C| < |C'|$. Of course, a maximum chain is maximal, but maximal chains need not be maximum.

Maximal antichains and maximum antichains are defined analogously.

With this terminology, the thrust of Theorem 5.3 is that it is easy to find the height h of a poset as well as a maximum chain C consisting of h points from $\mathbf{P}$.

5.3.1 Proof of Dilworth's Theorem

The argument for Dilworth's theorem is simplified by the following notation. When $\mathbf{P} = (X, P)$ is a poset and $x \in X$, we let $D(x) = \{y \in X : y < x \text{ in } P\}$; $D[x] = \{y \in X : y \leq x \text{ in } P\}$; $U(x) = \{y \in X : y > x \text{ in } P\}$; $U[x] = \{y \in X : y \geq x\}$; and $I(x) = \{y \in X - \{x\} : x \| y \text{ in } P\}$. When $S \subseteq X$, we let $D(S) = \{y \in X : y < x \text{ in } P, \text{ for some } x \in S\}$ and $D[S] = S \cup D(S)$. The subsets $U(S)$ and $U[S]$ are defined dually. Note that when A is a maximal antichain in $\mathbf{P}$, the ground set X is partitioned into pairwise disjoint sets: $X = A \cup D(A) \cup U(A)$.

We are now ready for the proof. Let $\mathbf{P} = (X, P)$ be a poset and let w denote the width of $\mathbf{P}$. As in Theorem 5.3, the pigeonhole principle implies that we require at least w chains in any chain partition of $\mathbf{P}$. To prove that w suffice, we proceed by induction on $|X|$, the result being trivial if $|X| = 1$. Assume validity for all posets with $|X| \leq k$ and suppose that $\mathbf{P} = (X, P)$ is a poset with $|X| = k + 1$. Without loss of generality, $w > 1$; else the trivial partition $X = C_1$ satisfies the conclusion of the theorem. Furthermore, we observe that if C is a (nonempty) chain in (X, P), then we may assume that the subposet $(X - C, P(X - C))$ also has width w. To see this, observe that the theorem holds for the subposet, so that if $\mathrm{width}(X - C, P(X - C)) = w' < w$, then we can partition $X - C$ as $X - C = C_1 \cup C_2 \cup \cdots \cup C_{w'}$, so that $X = C \cup C_1 \cup \cdots \cup C_{w'}$ is a partition into $w' + 1$ chains. Since $w' < w$, we know $w' + 1 \leq w$, so we have a partition of X into at most w chains. Since any partition of X into chains must use at least w chains, this is exactly the

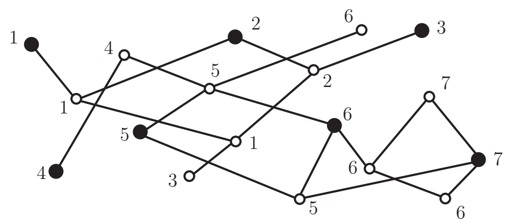


FIGURE 5.9: A POSET OF WIDTH 7

partition we seek.

Choose a maximal point x and a minimal point y with $y \leq x$ in P. Then let C be the chain containing x and y. Note that C contains either one or two elements depending on whether x and y are distinct.

Let $Y = X - C$ and $Q = P(Y)$ and let A be a w element antichain in the subposet (Y, Q). In the partition $X = A \cup D(A) \cup U(A)$, the fact that y is a minimal point while A is a maximal antichain imply that $y \in D(A)$. Similarly, $x \in U(A)$. In particular, this shows that x and y are distinct.

Label the elements of A as $\{a_1, a_2, \ldots, a_w\}$. Note that $U[A] \neq X$ since $y \notin U[A]$, and $D[A] \neq X$ since $x \notin D[A]$. Therefore, we may apply the inductive hypothesis to the suposets of $\mathbf{P}$ determined by $D[A]$ and $U[A]$, respectively, and partition each of these two subposets into w chains:

$$U[A] = C_1 \cup C_2 \cup \cdots \cup C_w \quad \text{and} \quad D[A] = D_1 \cup D_2 \cup \cdots \cup D_w$$

Without loss of generality, we may assume these chains have been labeled so that $a_i \in C_i \cap D_i$ for each $i = 1, 2, \ldots, w$. However, this implies that

$$X = (C_1 \cup D_1) \cup (C_2 \cup D_2) \cup \cdots \cup (C_w \cup D_w)$$

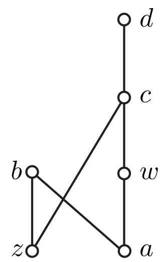
is the desired partition which in turn completes the proof.

In Figure 5.9, we illustrate Dilworth's chain covering theorem with a poset of width 7 that has been partitioned into 7 chains.

Remark 5.5. The ever alert Alice notes that the proof given above for Dilworth's theorem does not seem to provide an efficient algorithm for finding the width w of a poset, much less a partition of the poset into w chains. Bob has yet to figure out why listing all the subsets of X is a bad idea. We will return to this issue later in the course.

5.4 Linear Extensions of Partially Ordered Sets

Let $\mathbf{P} = (X, P)$ be a partially ordered set. A linear order L on X is called a *linear extension* (also, a *topological sort*) of P, if $x < y$ in L whenever $x < y$ in P. For example, the poset shown in Figure 5.10 and has 11 linear extensions.



L_1	L_2	L_3	L_4	L_5	L_6	L_7	L_8	L_9	L_{10}	L_{11}
d	d	d	b	d	d	d	b	d	d	b
c	c	b	d	c	c	b	d	c	b	d
w	b	c	c	w	b	c	c	b	c	c
b	w	w	w	b	w	w	w	z	z	z
a	a	a	a	z	z	z	z	w	w	w
z	z	z	z	a	a	a	a	a	a	a

FIGURE 5.10: A POSET AND ITS LINEAR EXTENSIONS

The classical sorting problem studied in all elementary computer science courses is to determine an unknown linear order L of a set X by asking a series of questions of the form: Is $x < y$ in L?

Here is an important special case: determine an unknown linear extension L of a poset $\mathbf{P}$ by asking a series of questions of the form: Is $x < y$ in L?

Question 5.6. Given a poset $\mathbf{P} = (X, P)$ and the problem of determining an unknown linear extension of P, how should you decide which question (of the form: Is $x < y$ in L?) to ask?

5.5 The Subset Lattice

When X is a finite set, the family of all subsets of X forms a *subset lattice*. We illustrate this in Figure 5.11 where we show the lattice of all subsets of $\{1, 2, 3, 4\}$. In this figure, note that we are representing sets by bit strings, and we have further abbreviated the notation by writing strings without commas and parentheses.

For a positive integer t, we let $\mathbf{2}^t$ denote the subset lattice consisting of all subsets of $\{1, 2, \ldots, t\}$ ordered by inclusion. Some elementary properties of this poset are:

1. The height is $t + 1$ and all maximal chains have exactly $t + 1$ points.

2. The size of the poset $\mathbf{2}^t$ is 2^t and the elements are partitioned into ranks (antichains) $A_0, A_1, \ldots, A_t$ with $|A_i| = \binom{t}{i}$ for each $i = 0, 1, \ldots, t$.

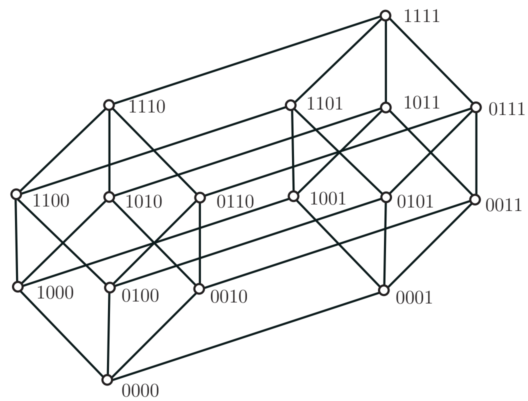


FIGURE 5.11: A SUBSET LATTICE

3. The maximum size of a rank in the subset lattice occurs in the middle, i.e. if $s = \lfloor t/2 \rfloor$, then the largest binomial coefficient in the sequence $\binom{t}{0}, \binom{t}{1}, \binom{t}{2}, \ldots, \binom{t}{t}$ is $\binom{t}{s}$. Note that when t is odd, there are two ranks of maximum size, but when t is even, there is only one.

5.5.1 Sperner's Theorem

For the width of the subset lattice, we have the following classic result due to Sperner.

Theorem 5.7 (Sperner)**.** *For each $t \geq 1$, the width of the subset lattice $\mathbf{2}^t$ is the maximum size of a rank, i.e.,*

$$\mathrm{width}(\mathbf{2^t}) = \binom{\mathbf{t}}{\lfloor \frac{\mathbf{t}}{\mathbf{2}} \rfloor}$$

Proof. The width of the poset $\mathbf{2}^t$ is at least $C(t, \lfloor \frac{t}{2} \rfloor)$ since the set of all $\lfloor \frac{t}{2} \rfloor$-element subsets of $\{1, 2, \ldots, t\}$ is an antichain. We now show that the width of $\mathbf{2}^t$ is at most $C(t, \lfloor \frac{t}{2} \rfloor)$.

Let w be the width of $\mathbf{2}^t$ and let $\{S_1, S_2, \ldots, S_w\}$ be an antichain of size w in this poset, i.e., each S_i is a subset of $\{1, 2, \ldots, t\}$ and if $1 \leq i < j \leq w$, then $S_i \not\subseteq S_j$ and $S_j \not\subseteq S_i$.

For each i, consider the set $\mathcal{S}_i$ of all maximal chains which pass through S_i. It is easy to see that if $|S_i| = k_i$, then $|\mathcal{S}_i| = k_i!(t - k_i)!$. This follows from the observation that to form such a maximum chain beginning with S_i as an intermediate point, you delete the elements of S_i one at a time to form the sets of the lower part of the chain. Also, to form the upper part of the chain, you add the elements not in S_i one at a time.

Note further that if $1 \le i < j \le w$, then $\mathcal{S}_i \cap \mathcal{S}_j = \emptyset$, for if there was a maximum chain belonging to both $\mathcal{S}_i$ and $\mathcal{S}_j$, then it would imply that one of S_i and S_j is a subset of the other.

Altogether, there are exactly $t!$ maximum chains in $\mathbf{2}^t$. This implies that

$$\sum_{i=1}^{i=w} k_i!(t-k_i)! \le t!.$$

This implies that

$$\sum_{i=1}^{i=w} \frac{k_i!(t-k_i)!}{t!} = \sum_{i=1}^{i=w} \frac{1}{\binom{t}{k_i}} \le 1.$$

It follows that

$$\sum_{i=1}^{i=w} \frac{1}{\binom{t}{\lceil \frac{t}{2} \rceil}} \le 1$$

Thus

$$w \le \binom{t}{\lceil \frac{t}{2} \rceil}. \qquad \square$$

5.6 An Alternative Proof of Sperner's Theorem

A poset $\mathbf{P} = (X, P)$ is said to be *ranked* if all maximal chains have the same cardinality. When a poset is ranked, then there is a partition $X = A_1 \cup A_2 \cup \ldots A_h$ so that every maximal chain consists of exactly one point from each A_i. We call this partition its *partition into ranks*.

A ranked poset is said to be *sperner* if the width of the poset is just the maximum cardinality of a rank. The preceding theorem can now be reinterpreted as saying that the subset lattice is a sperner poset. Let (X, P) be a ranked poset of height h and let $X = A_1 \cup A_2 \cup \ldots A_h$ be its partition into ranks. A chain $C = \{x_1, x_2, \ldots, x_k\}$ in (X, P) is called a *symmetric* chain if there exists an integer s so that C contains exactly one point from each rank $A_s, A_{s+1}, \ldots, A_{h+1-s}$. Intuitively, a symmetric chain is balanced about the middle of the poset (X, P) and dense in the sense that it is not possible to insert a point in between two consecutive points in C.

The following proposition is self evident.

Proposition 5.8. *If a ranked poset has a partition into symmetric chains, then it is a sperner poset. In fact, its width is just the size of the middle rank(s).*

Here is an alternative proof of Sperner's theorem.

Theorem 5.9. *For each $t \ge 1$, the subset lattice $\mathbf{2}^t$ has a symmetric chain partition.*

We actually prove an even stronger result. For posets $\mathbf{P} = (X, P)$ and $\mathbf{Q} = (Y, Q)$, we define the *cartesian product* $\mathbf{P} \times \mathbf{Q}$ as follows. The point set $X \times Y$ with $(x_1, y_1) \leq (x_2, y_2)$ in $\mathbf{P} \times \mathbf{Q}$ if and only if $x_1 \leq x_2$ in $\mathbf{P}$ and $y_1 \leq y_2$ in $\mathbf{Q}$. Note that the subset lattice $\mathbf{2}^t$ is just the cartesian product $\mathbf{2} \times \mathbf{2} \times \cdots \times \mathbf{2}$, with a total of t factors of $\mathbf{2}$.

Lemma 5.10. *Let m and n be positive integers. Then the cartesian product $\mathbf{m} \times \mathbf{n}$ has a symmetric chain partition.*

Proof. The point set of $\mathbf{m} \times \mathbf{n}$ is just $\{(i, j) : 0 \leq i < m, 0 \leq j < n\}$. Without loss of generality $m \leq n$, so that the width of $\mathbf{m} \times \mathbf{n}$ is m. Then for each $i = 0, 1, \ldots, m - 1$, let

$$C_i = \{(i, 0), (i, 1), \ldots, (i, n - 1 - i), (i + 1, n - 1 - i), \ldots, (m - 1, n - 1 - i)\}.$$

Then the family $\{C_1, C_2, \ldots, C_m\}$ is a symmetric chain partition of $\mathbf{m} \times \mathbf{n}$. $\square$

Theorem 5.11. *If $\mathbf{P}$ and $\mathbf{Q}$ are ranked posets and each has a symmetric chain partition, then $\mathbf{P} \times \mathbf{Q}$ is ranked and has a symmetric chain partition.*

Proof. It is easy to see that if $\mathbf{P}$ is ranked and has height h_1 and $\mathbf{Q}$ is ranked and has height h_2, then $\mathbf{P} \times \mathbf{Q}$ is ranked and has height $h_1 + h_2 - 1$. Now suppose that $\mathbf{P}$ and $\mathbf{Q}$ have symmetric chain partitions. Let C be a chain from the partition of $\mathbf{P}$ and let D be a chain from the partition of $\mathbf{Q}$. Then apply the preceding lemma to obtain a partition of the product $C \times D$. What results is a saturated partition of $\mathbf{P} \times \mathbf{Q}$. $\square$

5.7 Dedekind's Problem

The problem of counting the number $A(t)$ of antichains in the subset lattice $\mathbf{2}^t$ is a famous problem known in the literature as Dedekind's problem. Here is a table of the numbers which have been computed.

$$\begin{aligned}
A(1) &= 3\\
A(2) &= 6\\
A(3) &= 20\\
A(4) &= 168\\
A(5) &= 7781\\
A(6) &= 7828354\\
A(7) &= 2414682040998\\
A(8) &= 56130437228687557907788
\end{aligned}$$

Perhaps, the calculation of $A(10)$ is beyond reach.

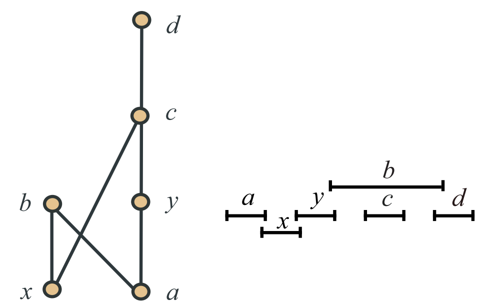


FIGURE 5.12: AN INTERVAL ORDER AND ITS REPRESENTATION

5.8 Interval Orders

When we discussed Dilworth's theorem, we commented that the algorithmic aspects would be deferred until later in the text. But there is one important class of orders for which the full solution is easy to obtain.

A poset $\mathbf{P} = (X, P)$ is called an *interval order* if there exists a function I assigning to each element $x \in X$ a closed interval $I(x) = [a_x, b_x]$ of the real line $\mathbb{R}$ so that for all x, $y \in X$, $x < y$ in P if and only if $b_x < a_y$ in L. We call I an *interval representation* of $\mathbf{P}$, or just a *representation* for short. For brevity, whenever we say that I is a representation of an interval order $\mathbf{P} = (X, P)$, we will use the alternate notation $[a_x, b_x]$ for the closed interval $I(x)$. Also, we let $|I(x)|$ denote the *length* of the interval, i.e., $|I(x)| = b_x - a_x$. The poset shown first in Figure 5.7 above is an interval order as the representation given in Figure 5.12 verifies.

Note that end points of intervals used in a representation need not be distinct. In fact, distinct points x and y from X may satisfy $I(x) = I(y)$. We even allow degenerate intervals. On the other hand, a representation is said to be *distinguishing* if all intervals are non-degenerate and all end points are distinct. It is easy to see that every interval order has a distinguishing representation. In fact, since we are concerned only with finite posets, we could have just as well required that all intervals used in the representation be open.

Theorem 5.12 (Fishburn)**.** *Let $\mathbf{P} = (X, P)$ be a poset. Then $\mathbf{P}$ is an interval order if and only if it does not contain $\mathbf{2} + \mathbf{2}$ as a subposet.*

Proof. Suppose first that $\{x, y, z, w\} \subseteq X$ and the subposet determined by these four points is isomorphic to $\mathbf{2} + \mathbf{2}$. We show that $\mathbf{P}$ is not an interval order. Suppose to the contrary that $\mathbf{P}$ is an interval order and let I be an interval representation of $\mathbf{P}$. Without loss of generality, we may assume that $x < y$ and $z < w$ in P. Thus $x \| w$ and $z \| y$ in P. Then $b_x < a_y$ and $b_z < a_w$ in $\mathbb{R}$ so that $a_w \leq b_x < a_y \leq b_z$, which is a contradiction.

Next, we assume that $\mathbf{P}$ does not contain $\mathbf{2}+\mathbf{2}$ as a subposet and show that $\mathbf{P}$ is an interval order. We proceed by induction on $n = |X|$. The result is trivially true when $n = 1$. Now assume that the result holds for all n with $n \leq k$, where k is some positive integer, and suppose that $|X| = k+1$. Then among the maximal elements, choose one which is greater than as many other elements of $\mathbf{P}$ as possible. Denote this element by x_0, and let $\mathbf{Q}$ denote the subposet obtained by removing x_0 from $\mathbf{P}$. Then $\mathbf{Q}$ has k points and does not contain a subposet isomorphic to $\mathbf{2}+\mathbf{2}$. Therefore $\mathbf{Q}$ is an interval order. Let I be an interval representation of $\mathbf{Q}$.

Choose a real number r with $r > b_y$ for every $y \neq x_0$. If x_0 is comparable to all other elements of X, then we may extend I to all of X by simply taking $I(x_0) = [r, r+1]$. Hence, we may assume that the set $S = \{y \in X : x_0 \| y \text{ in } P\}$ is non-empty.

Claim. *We have $S \subseteq \max(P)$.*

Proof. Since x_0 is maximal and $x_0 \| y$ for all $y \in S$, it is enough to prove that S is an antichain. Suppose to the contrary that $y_1, y_2 \in S$ and $y_1 < y_2$. Choose a maximal element $y_3 \in X$ with $y_2 \leq y_3$ in P. Evidently, $y_3 \neq x_0$, as $x_0 \| y_2$ but $y_2 \leq y_3$. Since x_0 is greater than as many elements of $\mathbf{P}$ as possible, there must be some $u \in X$ with $u < x_0$ in P and $u \| y_3$. Now we have that $x_0 \geq u$, $y_3 \geq y_1$, $x_0 \| y_1$, $x_0 \| y_3$, and $y_3 \| u$ immediately from our choices of these points. Furthermore, in order to preserve $x_0 \| y_1$ and $u \| y_3$, we cannot have $u \perp y_1$. Therefore, it follows that $\{u, x_0, y_1, y_3\}$ determine a subposet isomorphic to $\mathbf{2}+\mathbf{2}$. This completes the proof of the claim.

Now we modify the interval representation as follows:

$$\widehat{I}(u) = \begin{cases} I(u) & \text{if } u \notin S \text{ and } u \neq x_0; \\ [a_u, r] & \text{if } u \in S; \\ [r, r+1] & \text{if } u = x. \end{cases}$$

It is easy to see that $\widehat{I}$ is an interval representation of $\mathbf{P}$. □

5.9 Finding a Representation of an Interval Order

When $\mathbf{P} = (X, P)$ is an interval order and n is a positive integer, there may be many different ways to represent $\mathbf{P}$ using intervals with integer end points in $[n]$. But there is certainly a least n for which a representation can be found, and here the representation is unique. The discussion will again make use of the notation for down sets and up sets that we introduced prior to the proof of Dilworth's Theorem. As a reminder, we repeat it here. For a poset $\mathbf{P} = (X, P)$ and a subset $S \subset X$, let $D(S) = \{y \in X : \text{there exists some } x \in S \text{ with } y < x \text{ in } P\}$. Also, let $D[S] = D(S) \cup S$. When $|S| = 1$, say $S = \{x\}$, we write $D(x)$ and $D[x]$ rather than $D(\{x\})$ and $D[\{x\}]$. Dually, for a subset $S \subseteq X$, we define $U(S) = \{y \in X : \text{there exists some } x \in X \text{ with } y > x \text{ in } P\}$. As before, set $U[S] = U(S) \cup S$. And when $S = \{x\}$, we just write $U(x)$ for $\{y \in X : x < y \text{ in } P\}$.

Let $\mathbf{P} = (X, P)$ be a poset and let

$$\mathcal{D} = \{D(x) : x \in X\} \quad \text{and} \quad \mathcal{U} = \{U(x) : x \in X\}$$

Here are several elementary propositions which follow easily from Fishburn's theorem.

Proposition 5.13. *Let $\mathbf{P} = (X, P)$ be a poset. Then the following statements are equivalent.*

1. *$\mathbf{P}$ is an interval order.*
2. *Any two distinct sets in $\mathcal{D}$ are ordered by inclusion.*
3. *Any two distinct sets in $\mathcal{U}$ are ordered by inclusion.*

Proposition 5.14. *When $\mathbf{P}$ is an interval order, $|\mathcal{D}| = |\mathcal{U}|$.*

Let $\mathbf{P}$ be an interval order and let $d = |\mathcal{D}|$. Then label the sets in $\mathcal{D}$ and $\mathcal{U}$ respectively as $D_1, D_2, \ldots, D_d$ and $U_1, U_2, \ldots, U_d$ so that

$$\emptyset = D_1 \subset D_2 \subset D_3 \subset \cdots \subset D_d \quad \text{and}$$

$$U_1 \supset U_2 \cdots \supset U_{d-2} \supset U_{d-1} \supset \cdots \supset U_d = \emptyset.$$

Then we can form an interval representation I of $\mathbf{P}$ by the following rule: For each $x \in X$, set $I(x) = [i, j]$, where $D(x) = D_i$ and $U(x) = U_j$.

Proposition 5.15. *Let $\mathbf{P}$ be an interval order, and let $d = |\mathcal{D}|$ and I be defined as above. Then*

1. *For each $x \in X$, if $I(x) = [i, j]$, then $i \leq j$ in $\mathbb{R}$.*
2. *For each $x, y \in X$, if $I(x) = [i, j]$ and $I(y) = [k, l]$, then $x < y$ in P if and only if $j < k$ in $\mathbb{R}$.*
3. *The integer d is the least positive integer for which $\mathbf{P}$ has an interval representation using integer end points from $[d]$. This representation is unique.*

Consider the poset shown in Figure 5.13.

Then $d = 5$ with $D_1 = \emptyset$, $D_2 = \{3\}$, $D_3 = \{3, 6, 7\}$, $D_4 = \{3, 6, 7, 8\}$, and $D_5 = \{1, 3, 6, 7, 8, 10\}$. Also $U_1 = \{1, 2, 4, 5, 8, 9, 10\}$, $U_2 = \{1, 2, 5, 8, 9, 10\}$, $U_3 = \{2, 5, 9\}$,

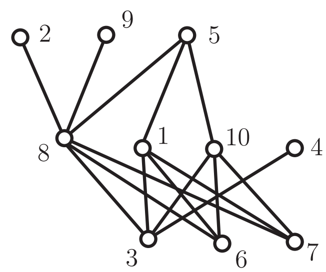


FIGURE 5.13: AN INTERVAL ORDER ON 10 POINTS

$U_4 = \{5\}$, and $U_5 = \emptyset$. So

$$\begin{aligned} I(1) &= [3,4] \\ I(2) &= [4,5] \\ I(3) &= [1,1] \\ I(4) &= [2,5] \\ I(5) &= [5,5] \\ I(6) &= [1,2] \\ I(7) &= [1,2] \\ I(8) &= [3,3] \\ I(9) &= [4,5] \\ I(10) &= [3,4] \end{aligned}$$

Also, this method yields an efficient algorithm for testing whether a poset is an interval order. You simply record the down sets in $\mathcal{D}$ and see if they are ordered by inclusion. When a poset is not an interval order, you will find distinct points x and y for which $D(x) \not\subseteq D(y)$ and $D(y) \not\subseteq D(x)$. This implies that there exist distinct points z and w with $z \in D(x) - D(y)$ and $w \in D(y) - D(x)$. The four points x, y, z and w are then seen to form a copy of $\mathbf{2} + \mathbf{2}$.

To illustrate this concept, erase the line joining points 3 and 10 in the figure above. You will then find that $D(10) = \{6, 7\}$ and $D(4) = \{3\}$. Therefore in 3, 4, 6 and 10 form a copy of $\mathbf{2} + \mathbf{2}$ in this modified poset.

5.10 Dilworth's Theorem for Interval Orders

As remarked previously, we do not yet have an efficient process for determining the width of a poset and a minimum partition into chains. For interval orders, there is indeed a simple way to find both. The explanation is just to establish a connection with coloring of interval graphs as discussed in Chapter 4.

Let $\mathbf{P} = (X, P)$ be an interval order and let $\{[a_x, b_x] : x \in X\}$ be intervals of the real line so that $x < y$ in $\mathbf{P}$ if and only $b_x < a_y$. Then let $\mathbf{G}$ be the interval graph determined by this family of intervals. Note that if x and y are distinct elements of X, then x and y are incomparable in $\mathbf{P}$ if and only if xy is an edge in $\mathbf{G}$. In other words, $\mathbf{G}$ is just the incomparability graph of $\mathbf{P}$.

Recall from Chapter 4 that interval graphs are perfect, i.e., $\chi(\mathbf{G}) = \omega(\mathbf{G})$ for every interval graph $\mathbf{G}$. Furthermore, you can find an optimal coloring of an interval graph by applying first fit to the vertices in a linear order that respects left end points. Such a coloring concurrently determines a partition of $\mathbf{P}$ into chains.

In fact, if you want to skip the part about interval representations, take any linear ordering of the elements as $x_1, x_2, \ldots, x_n$ so that $i < j$ whenever $D(x)$ is a proper subset of $D(y)$. Then apply First Fit with respect to chains. For example, using the 10 point interval order illustrated in Figure 5.13, here is such a labeling:

$$\begin{array}{ccccc} x_1 = 7 & x_2 = 6 & x_3 = 3 & x_4 = 4 & x_5 = 8 \\ x_6 = 1 & x_7 = 10 & x_8 = 2 & x_9 = 9 & x_{10} = 5 \end{array}$$

Now apply the **First Fit** algorithm to the points of $\mathbf{P}$, in this order, to assign them to chains $C_1, C_2, \ldots$. In other words, assign x_1 to chain C_1. Thereafter if you have assigned points $x_1, x_2, \ldots, x_i$ to chains, then assign x_{i+1} to chain C_j where j is the least positive integer for which x_{i+1} is comparable to x_k whenever $1 \leq k \leq i$ and x_k has already been assigned to C_j. For example, this rule results in the following chains for the interval order $\mathbf{P}$ shown in Figure 5.13.

$$\begin{aligned} C_1 &= \{7, 8, 2\} \\ C_2 &= \{6, 1, 5\} \\ C_3 &= \{3, 4\} \\ C_4 &= \{10\} \\ C_5 &= \{9\} \end{aligned}$$

In this case, it is easy to see that the chain partition is optimal since the width of $\mathbf{P}$ is 5 and $A = \{1, 2, 4, 9, 10\}$ is a 5-element antichain.

But students should be very careful in applying First Fit to find optimal chain partitions of posets—just as one must be leary of using First Fit to find optimal colorings of graphs. In general, there is always *some* linear ordering of the elements of the ground set for which First Fit will work. However, there is no known method for finding such a linear order, and it might well be a hopelessly difficult assignment.

5.11 Exercises

1. We say that a relation R on a set X is *symmetric* if $(x, y) \in R$ implies $(y, x) \in R$ for all $x, y \in X$. If $X = \{a, b, c, d, e, f\}$, how many symmetric relations are there on X? How many of these are reflexive?

2. A relation R on a set X is an *equivalence relation* if R is reflexive, symmetric, and transitive. Fix an integer $m \geq 2$. Show that the relation defined on the set $\mathbb{Z}$ of integers by aRb ($a, b \in \mathbb{Z}$) if and only if $a \equiv b \pmod{m}$ is an equivalence relation. (Recall that $a \equiv b \pmod{m}$ means that when dividing a by m and b by m you get the same remainder.)

3. Draw the diagram of the poset $\mathbf{P} = (X, P)$ where $X = \{1, 2, 3, 5, 6, 10, 15, 30\}$ and $x \leq y$ in P if and only if $x|y$.

4. A *linear extension* of a poset $\mathbf{P} = (X, P)$ is a total order L on X such that if $x \leq y$ in P, then $x \leq y$ in L. Give a linear extension of the poset $\mathbf{P}$ in Exercise 3. How many linear extensions of $\mathbf{P}$ are there? (Do **not** just list them all.)

5. Alice and Bob are considering posets $\mathbf{P}$ and $\mathbf{Q}$. They soon realize that $\mathbf{Q}$ is isomorphic to $\mathbf{P}^d$. After 10 minutes of work, they figure out that $\mathbf{P}$ has height 5 and width 3. Bob doesn't want do find the height and width of $\mathbf{Q}$, since he figures it will take (at least) another 10 minutes to answer these questions for $\mathbf{Q}$. Alice says Bob is crazy and that she already knows the height and width of $\mathbf{Q}$. Who's right and why?

6. Find the height and width of the poset in Figure 5.2.