CHAPTER

SEVEN

Generating Functions

A standard topic of study in first-year calculus is the representation of functions as infinite sums called power series; such a representation has the form $F(x) = \sum_{n=0}^{\infty} a_n x^n$. Perhaps surprisingly these power series can also serve as very powerful enumerative tools. In a combinatorial setting, we consider such power series of this type as another way of encoding the values of a sequence $\{a_n : n \geq 0\}$ indexed by the non-negative integers. The strength of power series as an enumerative technique is that they can be manipulated just like ordinary functions, i.e., they can be added, subtracted and multiplied, and for our purposes, we generally will not care if the power series converges, which anyone who might have found all of the convergence tests studied in calculus daunting will likely find reassuring. However, when we find it convenient to do so, we will use the familiar techniques from calculus and differentiate or integrate them term by term, and for those familiar series that do converge, we will use their representations as functions to facilitate manipulation of the series.

7.1 Basic Notation and Terminology

With a sequence $\sigma = \{a_n : n \geq 0\}$ of real numbers, we associate a "function" $F(x)$ defined by

$$F(x) = \sum_{n=0}^{\infty} a_n x^n.$$

The word "function" is put in quotes as we do not necessarily care about substituting a value of x and obtaining a specific value for $F(x)$. In other words, we consider $F(x)$ as a formal power series and frequently ignore issues of convergence.

It is customary to refer to $F(x)$ as the *generating function* of the sequence σ. As we have already remarked, we are not necessarily interested in calculating $F(x)$ for specific values of x. However, by convention, we take $F(0) = a_0$.

Example 7.1. Consider the constant sequence $\sigma = \{a_n : n \geq 0\}$ with $a_n = 1$ for every $n \geq 0$. Then the generating function $F(x)$ of σ is given by

$$F(x) = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + \cdots .$$

You probably remember that this last expression is the Maclaurin series for the function $F(x) = 1/(1-x)$ and that the series converges when $|x| < 1$. Since we want to think in terms of formal power series, let's see that we can justify the expression

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + \cdots = \sum_{n=0}^{\infty} x^n$$

without any calculus techniques. Consider the product

$$(1-x)(1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + \cdots)$$

and notice that, since we multiply formal power series just like we multiply polynomials (power series are pretty much polynomials that go on forever), we have that this product is

$$(1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + \cdots) - x(1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + \cdots) = 1.$$

Now we have that

$$(1-x)(1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + \cdots) = 1,$$

or, more usefully, after dividing through by $1-x$,

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.$$

Example 7.2. Just like you learned in calculus for Maclaurin series, formal power series can be differentiated and integrated term by term. The rigorous mathematical framework that underlies such operations is not our focus here, so take us at our word that this can be done for formal power series without concern about issues of convergence.

To see this in action, consider differentiating the power series of the previous example. This gives

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + 7x^6 + \cdots = \sum_{n=1}^{\infty} nx^{n-1}.$$

Integration of the series represented by $1/(1+x) = 1/(1-(-x))$ yields (after a bit of algebraic manipulation)

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \cdots = \sum_{n=1}^{\infty} \frac{x^n}{n}$$

Before you become convinced that we're only going to concern ourselves with generating functions that actually converge, let's see that we can talk about the formal power series

$$F(x) = \sum_{n=0}^{\infty} n! x^n,$$

even though it has radius of convergence 0, i.e., the series $F(x)$ converges only for $x = 0$, so that $F(0) = 1$. Nevertheless, it makes sense to speak of the formal power series $F(x)$ as the generating function for the sequence $\{a_n : n \geq 0\}$, $a_0 = 1$ and a_n is the number of permutations of $\{1, 2, \ldots, n\}$ when $n \geq 1$.

For reference, we state the following elementary result, which emphasizes the form of a product of two power series.

Proposition 7.3. *Let $A(x) = \sum_{n=0}^{\infty} a_n x^n$ and $B(x) = \sum_{n=0}^{\infty} b_n x^n$ be generating functions. Then $A(x)B(x)$ is the generating function of the sequence whose n^{th} term is given by*

$$a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_n b_0 = \sum_{k=0}^{n} a_k b_{n-k}.$$

7.2 Making distributions with restrictions

A recurring problem so far in this book as been to consider problems that ask about distributing indistinguishable objects (say apples) to distinct entities (say children). We started in chapter 2 by asking how many ways there were to distribute 40 apples to 5 children so that each child is guaranteed to get at least one apple and saw that the answer was $C(39, 4)$. We even saw how to restrict the situation so that one of the children was limited and could receive at most 10 apples. In chapter 6, we learned how to extend the restrictions so that more than one child had restrictions on the number of apples allowed by taking advantage of the Principle of Inclusion-Exclusion. Before moving on to see how generating functions can allow us to get even more creative with our restrictions, let's take a moment to see how generating functions would allow us to solve the most basic problem at hand.

Example 7.4. We already know that the number of ways to distribute n apples to 5 children so that each child gets at least one apple is $C(n-1, 4)$, but it will be instructive to see how we can derive this result using generating functions. Let's start with an even simpler problem: how many ways are there to distribute n apples to *one* child so that each child receives at least one apple? Well, this isn't too hard, there's only one way to do it—give all the apples to the lucky kid! Thus the *sequence* that enumerates the number of ways to do this is $\{a_n : n \geq 1\}$ with $a_n = 1$ for all $n \geq 1$. Then the generating function for this sequence is

$$x + x^2 + x^3 + \cdots = x(1 + x + x^2 + x^3 + \cdots) = \frac{x}{1-x}.$$

How can we get from this fact to the question of five children? Notice what happens when we multiply

$$(x+x^2+x^3+\cdots)(x+x^2+x^3+\cdots)(x+x^2+x^3+\cdots)(x+x^2+x^3+\cdots)(x+x^2+x^3+\cdots).$$

To see what this product represents, first consider how many ways can we get an x^6? We could use the x^2 from the first factor and x from each of the other four, or x^2 from the second factor and x from each of the other four, etc., meaning that the coefficient on x^6 is $5 = C(5,4)$. More generally, what's the coefficient on x^n in the product? In the expansion, we get an x^n for every product of the form $x^{k_1}x^{k_2}x^{k_3}x^{k_4}x^{k_5}$ where $k_1 + k_2 + k_3 + k_4 + k_5 = n$. Returning to the general question here, we're really dealing with distributing n apples to 5 children, and since $k_i > 0$ for $i = 1, 2, \ldots, 5$, we also have the guarantee that each child receives at least one apple, so the product of the generating function for *one* child gives the generating function for *five* children.

Let's pretend for a minute that we didn't know that the coefficients must be $C(n-1,4)$. How could we figure out the coefficients just from the generating function? The generating function we're interested in is $x^5/(1-x)^5$, which you should be able to pretty quickly see satisfies

$$\begin{aligned}\frac{x^5}{(1-x)^5} &= \frac{x^5}{4!}\frac{d^4}{dx^4}\left(\frac{1}{1-x}\right) = \frac{x^5}{4!}\sum_{n=0}^{\infty} n(n-1)(n-2)(n-3)x^{n-4}\\ &= \sum_{n=0}^{\infty}\frac{n(n-1)(n-2)(n-3)}{4!}x^{n+1} = \sum_{n=0}^{\infty}\binom{n}{4}x^{n+1}.\end{aligned}$$

The coefficient on x^n in this series $C(n-1,4)$, just as we expected.

We could revisit an example from chapter 6 to see that if we wanted to limit a child to receive at most 4 apples, we would use $(x + x^2 + x^3 + x^4)$ as its generating function instead of $x/(1-x)$, but rather than belabor that here, let's try something a bit more exotic.

Example 7.5. A grocery store is preparing holiday fruit baskets for sale. Each fruit basket will have 20 pieces of fruit in it, chosen from apples, pears, oranges, and grapefruit. How many different ways can such a basket be prepared if there must be at least one apple in a basket, a basket cannot contain more than three pears, and the number of oranges must be a multiple of four?

In order to get at the number of baskets consisting of 20 pieces of fruit, let's solve the more general problem where each basket has n pieces of fruit. Our method is simple: find the generating function for how to do this with each type of fruit individually and then multiply them. As in the previous example, the product will contain the term x^n for every way of assembling a basket of n pieces of fruit subject to our restrictions. The apple generating function is $x/(1-x)$, since we only want positive powers of x (corresponding to ensuring at least one apple). The generating function for pears is $(1+x+x^2+x^3)$, since we can have only zero, one, two, or three pears in basket. For oranges we have $1/(1-x^4) = 1+x^4+x^8+\cdots$, and the unrestricted grapefruit give us a factor of $1/(1-x)$. Multiplying, we have

$$\frac{x}{1-x}(1+x+x^2+x^3)\frac{1}{1-x^4}\frac{1}{1-x} = \frac{x}{(1-x)^2(1-x^4)}(1+x+x^2+x^3).$$

Now we want to make use of the fact that $(1+x+x^2+x^3) = (1-x^4)/(1-x)$ to see that our generating function is

$$\frac{x}{(1-x)^3} = \frac{x}{2}\sum_{n=0}^{\infty} n(n-1)x^{n-2} = \sum_{n=0}^{\infty}\frac{n(n-1)}{2}x^{n-1} = \sum_{n=0}^{\infty}\binom{n}{2}x^{n-1} = \sum_{n=0}^{\infty}\binom{n+1}{2}x^n.$$

Thus, there are $C(n+1,2)$ possible fruit baskets containing n pieces of fruit, meaning that the answer to the question we originally asked is $C(21,2) = 210$.

Example 7.6. Find the number of integer solutions to the equation

$$x_1 + x_2 + x_3 = n$$

($n \geq 0$ an integer) with $x_1 \geq 0$ even, $x_2 \geq 0$, and $0 \leq x_3 \leq 2$.

Again, we want to look at the generating function we would have if each variable existed individually and take their product. For x_1, we get a factor of $1/(1-x^2)$; for x_2, we have $1/(1-x)$; and for x_3 our factor is $(1+x+x^2)$. Therefore, the generating function for the number of solutions to the equation above is

$$\frac{1+x+x^2}{(1-x)(1-x^2)} = \frac{1+x+x^2}{(1+x)(1-x)^2}.$$

In calculus, when we wanted to integrate a rational function of this form, we would use the method of partial fractions to write it as a sum of "simpler" rational functions whose antiderivatives we recognized. Here, our technique is the same, as we can readily recognize the formal power series for many rational functions. Our goal is to write

$$\frac{1+x+x^2}{(1+x)(1-x)^2} = \frac{A}{1+x} + \frac{B}{1-x} + \frac{C}{(1-x)^2}$$

for appropriate constants, A, B, and C. To find the constants, we clear the denominators, giving

$$1 + x + x^2 = A(1-x)^2 + B(1-x^2) + C(1+x).$$

Equating coefficients on terms of equal degree, we have:

$$\begin{aligned} 1 &= A + B + C \\ 1 &= -2A + C \\ 1 &= A - B \end{aligned}$$

Solving the system, we find $A = 1/4$, $B = -3/4$, and $C = 3/2$. Therefore, our generating function is

$$\frac{1}{4}\frac{1}{1+x} - \frac{3}{4}\frac{1}{1-x} + \frac{3}{2}\frac{1}{(1-x)^2} = \frac{1}{4}\sum_{n=0}^{\infty}(-1)^n x^n - \frac{3}{4}\sum_{n=0}^{\infty} x^n + \frac{3}{2}\sum_{n=0}^{\infty} n x^{n-1}.$$

The solution to our question is thus the coefficient on x^n in the above generating function, which is

$$\frac{(-1)^n}{4} - \frac{3}{4} + \frac{3(n+1)}{2},$$

a surprising answer that would not be too easy to come up with via other methods!

7.3 Newton's Binomial Theorem

In chapter 2, we discussed the binomial theorem and saw that the following formula holds for all integers $p \geq 1$:

$$(1+x)^p = \sum_{k=0}^{p} \binom{p}{n} x^n.$$

You should quickly realize that this formula implies that the generating function for the number of n-element subsets of a p-element set is $(1+x)^p$. The topic of generating functions is what leads us to consider what happens if we encounter $(1+x)^p$ as a generating function with p not a positive integer. It turns out that, by suitably extending the definition of the binomial coefficients to real numbers, we can also extend the binomial theorem in a manner originally discovered by Sir Isaac Newton.

We've seen several expressions that can be used to calculate the binomial coefficients, but in order to extend $C(p,k)$ to real values of p, we will utilize the form

$$\binom{p}{k} = \frac{P(p,k)}{k!},$$

recalling that we've defined $P(p,k)$ recursively as $P(p,0) = 1$ for all integers $p \geq 0$ and $P(p,k) = pP(p-1,k-1)$ when $p \geq k > 0$ (k an integer). Notice here, however, that the expression for $P(p,k)$ really makes sense for any real number p, so long as k is any positive integer and we've defined $P(p,0)=1$ for all real numbers p. Therefore we will make this definition formal.

Definition 7.7. For all real numbers p and nonnegative integers k, the number $P(p,k)$ is defined by

1. $P(p,0)=1$ for all real numbers p and
2. $P(p,k)=pP(p-1,k-1)$ for all real numbers p and integers $k>0$.

(Notice that this definition does not require $p \geq k$ as we did with integers.)

We are now prepared to extend the definition of binomial coefficient so that $C(p,k)$ is defined for all real p and nonnegative integer values of k. We do this as follows.

Definition 7.8. For all real numbers p and nonnegative integers k,

$$\binom{p}{k} = \frac{P(p,k)}{k!}.$$

Note that $P(p,k)=C(p,k)=0$ when p and k are integers with $0 \leq p < k$. On the other hand, we have some interesting new concepts such as $P(-5,4)=(-5)(-6)(-7)(-8)$ and

$$\binom{-7/2}{5} = \frac{(-7/2)(-9/2)(-11/2)(-13/2)(-15/2)}{5!}.$$

With this more general definition of binomial coefficients in hand, we're ready to state Newton's Binomial Theorem for all non-zero real numbers. The proof of this theorem can be found in most advanced calculus books.

Theorem 7.9. *For all real p with $p \neq 0$,*

$$(1+x)^p = \sum_{n=0}^{\infty} \binom{p}{n} x^n.$$

Note that the general form reduces to the original version of the binomial theorem when p is a positive integer.

7.4 Using the Binomial Theorem to Count Trees

In this section, we will use generating functions to enumerate the a certain type of trees and make a connection to a counting sequence we encountered earlier in the text. In order to do so, we must introduce a bit of terminology. A tree is *rooted* if we have designated a special vertex called its *root*. We will always draw our trees with the root at the top and all other vertices below it. An *unlabeled* tree is one in which we do not make distinctions based upon names given to the vertices. For our purposes, a *binary* tree is one in which each vertex has 0 or 2 children, and an *ordered* tree is one in which the children of a vertex have some ordering (first, second, third, etc.). Since we will be focusing on rooted, unlabeled, binary, ordered trees (RUBOTs for short), we will call the two children of vertices that have children the *left* and *right* children.

In Figure 7.1, we show the rooted, unlabeled, binary, ordered trees with n leaves for $n \leq 4$.

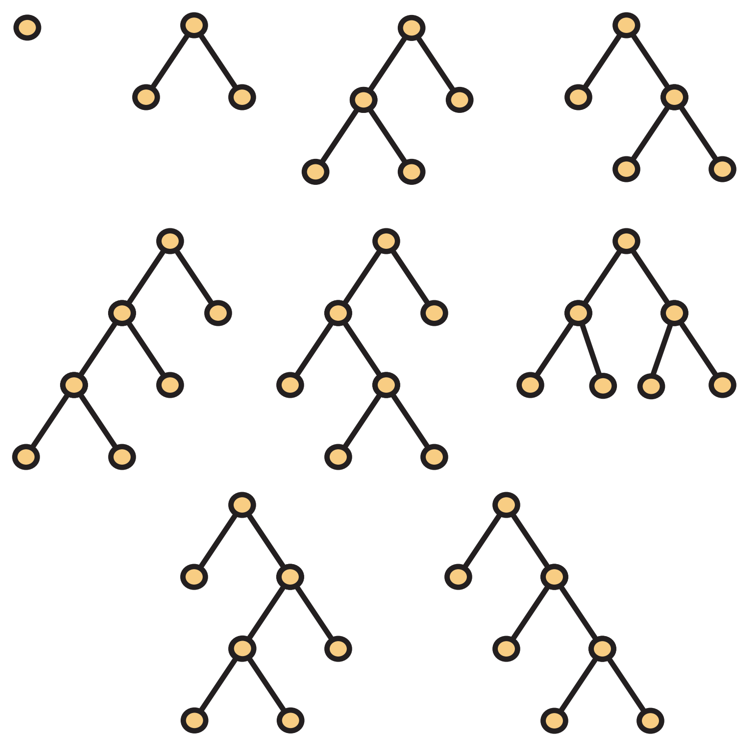

FIGURE 7.1: THE RUBOTS WITH n LEAVES FOR $n \leq 4$

Let $C(x) = \sum_{n=0}^{\infty} c_n x^n$ be the generating function for the sequence $\{c_n : n \geq 0\}$ where c_n is the number of RUBOTs with n leaves. (We take $c_0 = 0$ for convenience.) Then we can see from Figure 7.1 that $C(x) = x + x^2 + 2x^3 + 5x^4 + \cdots$. But what are the remaining coefficients? Let's see how we can break a RUBOT with n leaves down into a combination of two smaller RUBOTs to see if we can express c_n in terms of some c_k for $k < n$. When we look at a RUBOT with $n \geq 2$ leaves, we notice that the root vertex must have two children, and those children are really just root nodes of smaller RUBOTs, say the left child roots a RUBOT with k leaves, meaning that the right child roots a RUBOT with $n-k$ leaves. Since there are c_k possible sub-RUBOTs for the left child and c_{n-k} sub-RUBOTs for the right child, there are a total of $c_k c_{n-k}$ RUBOTs in which the root's left child has k leaves on its sub-RUBOT. We can do this for any $k = 1, 2, \ldots, n-1$, giving us that

$$c_n = \sum_{k=1}^{n-1} c_k c_{n-k}.$$

Since $c_0 = 0$, we can actually write this as

$$c_n = \sum_{k=0}^{n} c_k c_{n-k}.$$

Let's look at the square of the generating function $C(x)$. We have

$$\begin{aligned} C^2(x) &= c_0^2 + (c_0c_1 + c_1c_0)x + (c_0c_2 + c_1c_1 + c_2c_0)x^2 + \cdots \\ &= 0 + 0 + (c_0c_2 + c_1c_1 + c_2c_0)x^2 + (c_0c_3 + c_1c_2 + c_2c_1 + c_3c_0)x^3 + \cdots . \end{aligned}$$

But now we see from our recursion above that the coefficient on x^n in $C^2(x)$ is nothing but c_n for $n \geq 2$. All we're missing is the x term, so adding it in gives us that

$$C(x) = x + C^2(x).$$

Now this is a quadratic equation in $C(x)$, so we can solve for $C(x)$ and have

$$C(x) = \frac{1 \pm \sqrt{1-4x}}{2} = \frac{1 \pm (1-4x)^{1/2}}{2}.$$

Hence, we can use Newton's binomial theorem to expand $C(x)$. To do so, we use the following lemmas.

Lemma 7.10. *For each $k \geq 0$, $P(p, k+1) = P(p, k)(p-k)$.*

Proof. When $k = 0$, both sides evaluate to p. Now assume validity when $k = m$ for some non-negative integer m. Then

$$\begin{aligned} P(p, m+2) &= pP(p-1, m+1) \\ &= p[P(p-1, m)(p-1-m)] \\ &= [pP(p-1, m)](p-1-m) \\ &= P(p, m+1)[p-(m+1)]. \end{aligned}$$

□

Lemma 7.11. *For each $k \geq 1$,*

$$\binom{1/2}{k} = \frac{(-1)^{k-1}}{k} \frac{\binom{2k-2}{k-1}}{2^{2k-1}}.$$

Proof. We proceed by induction on k. Both sides reduce to $1/2$ when $k = 1$. Now assume validity when $k = m$ for some non-negative integer m. Then

$$\begin{aligned}
\binom{-1/2}{m+1} &= \frac{P(1/2, m+1)}{(m+1)!} \\
&= \frac{P(1/2, m)(1/2 - m)}{(m+1)m!} \\
&= \frac{1/2 - m}{m+1} \binom{1/2}{m} \\
&= (-1)\frac{2m-1}{2(m+1)} \frac{(-1)^{m-1}}{m} \frac{\binom{2m-2}{m-1}}{2^{2m-1}} \\
&= \frac{(-1)^m}{m+1} \frac{\binom{2m}{m}}{2^{2m+1}}.
\end{aligned}$$

□

Now we see that

$$\begin{aligned}
C(x) &= \frac{1}{2} \pm \frac{1}{2} \sum_{n=0}^{\infty} \binom{1/2}{n} (-4)^n x^n = \frac{1}{2} \pm \frac{1}{2} \left(1 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \frac{\binom{2n-2}{n-1}}{2^{2n-1}} (-4)^n x^n \right) \\
&= \frac{1}{2} \pm \frac{1}{2} \mp \sum_{n=1}^{\infty} \frac{\binom{2n-2}{n-1}}{n} x^n.
\end{aligned}$$

Since we need $c_n \geq 0$, we take the "minus" option from the "plus-or-minus" in the quadratic formula and thus have the following theorem.

Theorem 7.12. *The generating function for the number c_n of rooted, unlabeled, binary, ordered trees with n leaves is*

$$C(x) = \frac{1 - \sqrt{1-4x}}{2} = \sum_{n=1}^{\infty} \frac{1}{n} \binom{2n-2}{n-1} x^n.$$

Notice that c_n is a Catalan number, which we first encountered in chapter 2.

As a consequence of Theorem 7.12, we have the following corollary.

Corollary 7.13. *The generating functon of the sequence $\{\binom{2n}{n} : n \geq 0\}$ is $f(x) = (1-4x)^{-1/2}$.*

Proof. Differentiating $C(x)$ first as a function of x and then as a formal power series gives the desired result. □

Since the generating function $f(x)$ of Corollary 7.13 squares to $1/(1-4x)$, we have the following corollary as a nice identity regarding binomial coefficients.

Corollary 7.14. *For all $n \geq 0$,*

$$2^{2n} = \sum_{k=0}^{n} \binom{2k}{k}\binom{2n-2k}{k}.$$

7.5 Partitions of an Integer

A recurring theme in this course has been to count the number of integer solutions to an equation of the form $x_1 + x_2 + \cdots + x_k = n$. What if we wanted to count the number of such solutions but didn't care what k was? How about if we took this new question and required that the x_i be *distinct* (i.e., $x_i \neq x_j$ for $i \neq j$)? What about if we required that each x_i be odd? These certainly don't seem like easy questions to answer at first, but generating functions will allow us to say something very interesting about the answers to the last two questions.

By a *partition* P of an integer, we mean a collection of (not necessarily distinct) positive integers such that $\sum_{i\in P} i = n$. (By convention, we will write the elements of P from largest to smallest.) For example, $2+2+1$ is a partition of 5. For each $n \geq 0$, let p_n denote the number of partitions of the integer n (with $p_0 = 1$ by convention). Note that $p_8 = 22$ as evidenced by the list in Table 7.1. Note that there are 6 partitions of 8 into *distinct* parts. Also there are 6 partitions of 8 into *odd* parts. While it might seem that this is a coincidence, it in fact is always the case as the following theorem states.

Theorem 7.15. *For each $n \geq 1$, the number of partitions of n into distinct parts is equal to the number of partitions of n into odd parts.*

8 distinct parts	7+1 distinct parts, odd parts	6+2 distinct parts
6+1+1	5+3 distinct parts, odd parts	5+2+1 distinct parts
5+1+1+1 odd parts	4+4	4+3+1 distinct parts
4+2+2	4+2+1+1	4+1+1+1+1
3+3+2	3+3+1+1 odd parts	3+2+2+1
3+2+1+1+1	3+1+1+1+1+1 odd parts	2+2+2+2
2+2+2+1+1	2+2+1+1+1+1	2+1+1+1+1+1+1
	1+1+1+1+1+1+1+1 odd parts	

TABLE 7.1: THE PARTITIONS OF 8, NOTING THOSE INTO DISTINCT PARTS AND THOSE INTO ODD PARTS.

Proof. Our proof begins by considering the generating function $P(x)$ for p_n, the number of partitions of n. This is

$$P(x) = \prod_{n=1}^{\infty} \frac{1}{1-x^n} = \left(\sum_{n=0}^{\infty} x^n\right)\left(\sum_{n=0}^{\infty} x^{2n}\right)\cdots\left(\sum_{n=0}^{\infty} x^{kn}\right)\cdots,$$

since an x^n term in the product arises for eqch partition by picking the $(x^k)^j$ term from the k^{th} factor in the product, where j is the number of k's appearing in the partition in question.For distinct parts, we're no longer allowed to use an integer more than one time in a partition, so we have the generating function

$$D(x) = \prod_{n=1}^{\infty}(1+x^n)$$

for the number of partitions of an integer into distinct part. Finally, the generating function $O(x)$ for the number of partitions of n into odd parts is

$$O(x) = \prod_{n=1}^{\infty} \frac{1}{1-x^{2n-1}}$$

by the same argument as used for $P(x)$. To see that $D(x) = O(x)$, we note that $1-x^{2n} = (1-x^n)(1+x^n)$ for all $n \geq 1$. Therefore,

$$\begin{aligned} D(x) &= \prod_{n=1}^{\infty}(1+x^n) = \prod_{n=1}^{\infty}\frac{1-x^{2n}}{1-x^n} = \frac{\prod_{n=1}^{\infty}(1-x^{2n})}{\prod_{n=1}^{\infty}(1-x^n)} \\ &= \frac{\prod_{n=1}^{\infty}(1-x^{2n})}{\prod_{n=1}^{\infty}(1-x^{2n-1})\prod_{n=1}^{\infty}(1-x^{2n})} = \prod_{n=1}^{\infty}\frac{1}{1-x^{2n-1}} \\ &= O(x), \end{aligned}$$

and thus the number of partitions of an integer n into distinct parts equals the number of partitions into odd parts. □

7.6 Exercises

1. What is the generating function for the number of ways to select a group of n students from a class of p students?

2. What is the generating function for the number of partitions of an integer into even parts?

3. How many rooted, unlabeled, binary, ordered, trees (RUBOTs) with 6 leaves are there? Draw 6 distinct RUBOTs with 6 leaves.

4. What is the generating function for the number of ways to distribute folders to four office workers, Alberto, Bonnie, Clyde, and Denise, such that Alberto gets at least two folders, Bonnie gets at most three folders, the number of folders Clyde receives is a multiple of three, and Denise gets at least one folder but no more than six folders? (You do not need to find the coefficient on x^n in the generating function, just provide the generating function.)