CHAPTER

TWO

Strings, Sets, and Binomial Coefficients

Much of combinatorial mathematics can be reduced to the study of strings, as they form the basis of all written human communications. Also, strings are the way humans communicate with computers, as well as the way one computer communicates with another. As we shall see, sets and binomial coefficients are topics that fall under the string umbrella. So it makes sense to begin our in-depth study of combinatorics with strings.

2.1 Strings: A First Look

Let n be a positive integer. Throughout our course, we will use the shorthand notation $[n]$ to denote the n-element set $\{1, 2, \ldots, n\}$. Now let X be a set. Then a function $s\colon [n] \to X$ is also called an X-*string of length* n. In discussions of X-strings, it is customary to refer to the elements of X as *characters*, while the element $s(i)$ is the i^{th} character of s. Whenever possible, we prefer to denote a string s by writing $s =$"$x_1x_2x_3 \ldots x_n$", rather than the more cumbersome notation $s(1) = x_1$, $s(2) = x_2$, …, $s(n) = x_n$.

There are several alternatives for the notation and terminology associated with strings. First, the characters in a string s are frequently written using subscripts as $s_1, s_2, \ldots, s_n$, so the i^{th}-term of s can be denoted s_i rather than $s(i)$. Strings are also called *sequences*, especially when X is a set of numbers and the function s is defined by an algebraic rule. For example, the sequence of odd integers is defined by $s_i = 2i - 1$.

Alternatively, strings are called *words* and the set X is called the *alphabet*. For example, $aababbccabcbb$ is a 13-letter word on the 3-letter alphabet $\{a, b, c\}$.

In many computing languages, strings are called *arrays*. Also, when the character $s(i)$ is constrained to belong to a subset $X_i \subseteq X$, a string can be considered as an element of the cartesian product $X_1 \times X_2 \times \cdots \times X_n$.

Example 2.1. In the state of Georgia, license plates consist of four digits followed by a space followed by three capital letters. The first digit cannot be a 0. How many license plates are possible?

Let X consist of the digits $\{0, 1, 2, \ldots, 9\}$, let Y be the singleton set whose only element is a dash, and let Z denote the set of capital letters. A valid license plate is just a string from

$$(X - \{0\}) \times X \times X \times X \times Y \times Z \times Z \times Z$$

so the number of different license plates is $9 \times 10^3 \times 1 \times 26^3 = 158184000$, since the size of a product of sets is the product of the sets' sizes.

In the case that $X = \{0, 1\}$, an X-string is called a 0–1 string (also, a *bit string.*). When $X = \{0, 1, 2\}$, an X-string is also called a *ternary* string.

Example 2.2. A machine instruction in a 32-bit operating system is just a bit string of length 32. So the number of such strings is $2^{32} = 4294967296$. In general, the number of bit strings of length n is 2^n.

2.2 Permutations

Let X be a finite set and let n be a positive integer. An X-string $s = x_1x_2 \ldots x_n$ is called a *permutation* if all n characters used in s are distinct. Clearly, the existence of an X-permutation of length n requires that $|X| \geq n$.

When n is a positive integer, we define $n!$ (read "n *factorial*") by

$$n! = n \cdot n - 1 \cdot n - 2 \cdot \ldots \cdot 3 \cdot 2 \cdot 1.$$

By convention, we set $0! = 1$. As an example,

$$7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040.$$

Now for integers m, n with $m \geq n \geq 0$ define $P(m, n)$ by

$$P(m, n) = \frac{m!}{(m - n)!}$$

For example, $P(9, 3) = 9 \cdot 8 \cdot 7 = 504$ and $P(8, 4) = 8 \cdot 7 \cdot 6 \cdot 5 = 1680$. Also, Maple reports that

$$P(68, 23) = 20732231223375515741894286164203929600000.$$

Proposition 2.3. *If X is an m-element set and n is a positive integer with $m \geq n$, then the number of X-strings of length n that are permutations is $P(m, n)$.*

Proof. The proposition is true since when constructing a permutation $s = x_1x_2, \dots x_n$ from an m-element set, we see that there are m choices for x_1. After fixing x_1, we have that for x_2, there are $m-1$ choices, as we can use any element of $X - \{x_1\}$. For x_3, there are $m-2$ choices, since we can use any element in $X - \{x_1, x_2\}$. For x_n, there are $m-n+1$ choices, because we can use any element of X except $x_1, x_2, \dots x_{n-1}$. Noting that

$$P(m,n) = \frac{m!}{(m-n)!} = m(m-1)(m-2)\dots(m-n+1),$$

our proof is complete. □

Example 2.4. A four-person slate of officers, President, Vice-President, Secretary and Treasurer, is to be elected from a class of 80 students. How many different results are possible? Answer: $P(80,4) = 37957920$.

2.3 Combinations

Let X be a finite set and let k be an integer with $0 \leq k \leq |X|$. Then a k-element subset of X is also called a *combination* of size k. When $|X| = n$, the number of k-element subsets of X is denoted $\binom{n}{k}$. Numbers of the form $\binom{n}{k}$ are called *binomial coefficients*, and many combinatorists read $\binom{n}{k}$ as n *choose* k. When we need an in-line version, the preferred notation is $C(n,k)$. Also, the quantity $C(n,k)$ is referred to as the number of combinations of n things, taken k at a time.

Proposition 2.5. *If n and k are integers with $0 \leq k \leq n$, then*

$$\binom{n}{k} = \frac{P(n,k)}{k!} = \frac{n!}{(n-k)!}\frac{1}{k!} = \frac{n!}{k!(n-k)!}$$

Proof. Let X be an n-element set. The quantity $P(n,k)$ counts the number of X-permutations of length k. Since there are $k!$ permutations of a k-element set, each k-element subset of X is counted $k!$ times among the permutations, so dividing by $k!$ gives the number of k-element subsets. □

Our argument above illustrates a common combinatorial counting strategy. We counted one thing and determined that the objects we wanted to count were *overcounted* the same number of times each, so we divided by that number ($k!$ in this case). The following result is tantamount to saying that choosing elements to belong to a set is the same as choosing those elements which are to be denied membership.

Proposition 2.6. *For all integers n and k with $0 \leq k \leq n$,*

$$\binom{n}{k} = \binom{n}{n-k}.$$

Example 2.7. Let n be a positive integer and let X be an n-element set. Then there is a natural one-to-one correspondence between subsets of X and bit strings of length n. To be precise, let $X = \{x_1, x_2, \dots, x_n\}$. Then a subset $A \subseteq X$ corresponds to the string s where $s(i) = 1$ if and only if $i \in A$. For example, if $X = \{a, b, c, d, e, f, g, h\}$, then the subset $\{b, c, g\}$ corresponds to the bit string 01100010.

2.4 Combinatorial Proofs

Combinatorial arguments are among the most beautiful in all of mathematics. Oftentimes, statements that can be proved by other, messier methods (usually involving large amounts of tedious algebraic manipulations) have very short proofs once you can make a connection to counting. In this section, we introduce a new way of thinking about combinatorial problems with several examples. Our goal is to help you develop a "gut feeling" for combinatorial problems.

Example 2.8. Let n be a positive integer. Explain why

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

Consider an $(n+1) \times (n+1)$ array of dots as depicted in Figure 2.1. There are $(n+1)^2$ dots altogether, with exactly $n+1$ on the main diagonal. The off-diagonal entries split naturally into two equal size parts, those above and those below the diagonal.

Furthermore, each of those two parts has $S(n) = 1 + 2 + 3 + \cdots + n$ dots. It follows that

$$S(n) = \frac{(n+1)^2 - (n+1)}{2}$$

and this is obvious! Now a little algebra on the right hand side of this expression produces the formula given earlier.

Here is another way to get the same formula. Let n be a positive integer. Then the number of 2-element subsets of $[n+1] = \{1, 2, \dots, n+1\}$ is $\binom{n+1}{2}$. On the other hand, for each $i = 1, 2, \dots, n$, exactly $n+1-i$ of these 2-element sets have i as their least element. This shows that

$$\sum_{i=1}^{n} (n+1-i) = 1 + 2 + 3 + \cdots + n = \binom{n+1}{2} = \frac{n(n+1)}{2}.$$

Example 2.9. Let n be a positive integer. Explain why

$$1 + 3 + 5 + \cdots + 2n - 1 = n^2.$$

The left hand side is just the sum of the first n odd integers. But as suggested in Figure 2.2, this is clearly equal to n^2.

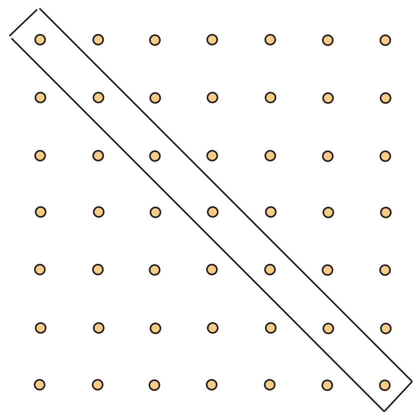

FIGURE 2.1: THE SUM OF THE FIRST n INTEGERS

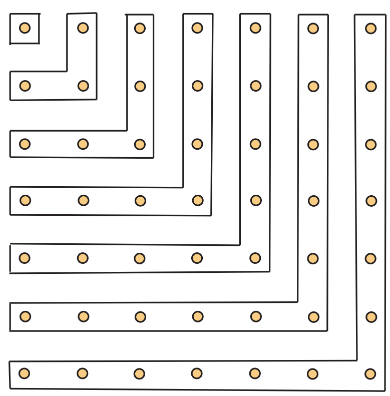

FIGURE 2.2: THE SUM OF THE FIRST n ODD INTEGERS

Example 2.10. Let n be a positive integer. Explain why

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n.$$

Both sides count the number of bit strings of length n, with the left side first grouping them according to the number of 0's.

Example 2.11. Let n and k be integers with $0 \leq k < n$. Then

$$\binom{n}{k+1} = \binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \cdots + \binom{n-1}{k}.$$

To prove this formula, we simply observe that both sides count the number of bit strings of length n that contain $k+1$ 1's with the right hand side first partitioning them according to the last occurence of a 1. (For example, if the last 1 occurs in position $k+5$, then the remaining k 1's must appear in the preceding $k+4$ positions, giving $C(k+4, k)$ strings of this type.)

Example 2.12. Explain the identity

$$3^n = \binom{n}{0}2^0 + \binom{n}{1}2^1 + \binom{n}{2}2^2 + \cdots + \binom{n}{n}2^n.$$

Both sides count the number of $\{0, 1, 2\}$-strings of length n, the right hand side first partitioning them according to positions in the string which are not 2. (For instance, if 6 of the positions are not 2, we must first choose those 6 positions in $C(n, 6)$ ways and then there are 2^6 ways to fill in those six positions by choosing either a 0 or a 1 for each position.)

Example 2.13. For each non-negative integer n,

$$\binom{2n}{n} = \binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2.$$

Both sides count the number of bit strings of length $2n$ with half the bits being 0's, with the right side first partitioning them according to the number of 1's occurring in the first n positions of the string. Note that we are also using the trivial identity $\binom{n}{k} = \binom{n}{n-k}$.

2.5 The Ubiquitous Nature of Binomial Coefficients

In this section, we present several combinatorial problems that can be solved by appeal to binomial coefficients, even though at first glance, they do not appear to have anything to do with sets.

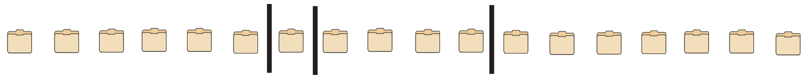

FIGURE 2.3: DISTRIBUTING IDENTICAL OBJECTS INTO DISTINCT CELLS

Example 2.14. The office assistant is distributing supplies. In how many ways can he distribute 18 identical folders among four office employees: Alice, Bob, Charlie and Dawn, with the additional restriction that each will receive at least one folder?

Imagine the folders placed in a row. Then there are 17 gaps between them. Of these gaps, choose three and place a divider in each. Then this choice divides the folders into four non-empty sets. The first goes to Alice, the second to Bob, etc. Thus the answer is $C(17,3)$. In Figure 2.3, we illustrate this scheme with Alice receiving 6 folders, Bob getting 1, Charlie 4 and Dawn 7.

Example 2.15. Suppose we redo the preceding problem but drop the restriction that each of the four employees gets at least one folder. Now how many ways can the distribution by made?

The solution involves a "trick" of sorts. First, we convert the problem to one that we already know how to solve. This is accomplished by *artificially* inflating everyone's allocation by one. In other words, if Bob will get 7 folders, we say that he will get 8. Also, artificially inflate the number of folders by 4, one for each of the four persons. So now imagine a row of $22 = 18 + 4$ folders. Again, choose 3 gaps. This determines a non-zero allocation for each person. The actual allocation is one less—and may be zero. So the answer is $C(21,3)$.

Example 2.16. Again we have the same problem as before, but now we want to count the number of distributions where only Alice and Charlie are guaranteed to get a folder. Bob and Dawn are allowed to get zero folders. Now the trick is to artificially inflate Bob and Dawn's allocation, but leave the numbers for Alice and Charlie as is. So the answer is $C(19,3)$.

Example 2.17. Here is a reformulation of the preceding discussion expressed in terms of integer solutions of inequalities.

We count the number of integer solutions to the inequality

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \leq 538$$

subject to various sets of restrictions on the values of $x_1, x_2, \ldots, x_6$. Some of these restrictions will require that the inequality actually be an equation.

The number of integer solutions is:

1. $C(537,5)$, when all $x_i > 0$ and equality holds.

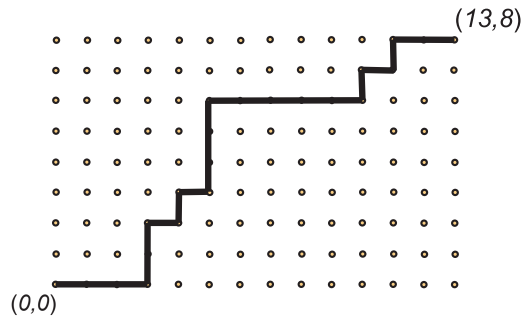

FIGURE 2.4: A LATTICE PATH

2. $C(543, 5)$, when all $x_i \geq 0$ and equality holds.

3. $C(291, 3)$, when $x_1, x_2, x_4, x_6 > 0$, $x_3 = 52$, $x_5 = 194$, and equality holds.

4. $C(537, 6)$, when all $x_i > 0$ and the inequality is strict. *Hint:* Imagine a new variable x_7 which is the balance. Note that x_7 must be positive.

5. $C(543, 6)$, when all $x_i \geq 0$ and the inequality is strict. *Hint:* Add a new variable x_7 as above. Now it is the only one which is required to be positive.

6. $C(544, 6)$, when all $x_i \geq 0$.

A classical enumeration problem (with connections to several problems) involves counting lattice paths. A *lattice path* in the plane is a sequence of ordered pairs of integers:

$$(m_1, n_1), (m_2, n_2), (m_3, n_3), \ldots, (m_t, n_t)$$

so that for all $i = 1, 2, \ldots, t-1$, either

1. $m_{i+1} = m_i + 1$ and $n_{i+1} = n_i$, or

2. $m_{i+1} = m_i$ and $n_{i+1} = n_i + 1$.

In Figure 2.4, we show a lattice path from $(0, 0)$ to $(13, 8)$.

Example 2.18. The number of lattice paths from (m, n) to (p, q) is $C((p-m)+(q-n), p-m)$.

To see why this formula is valid, note that a lattice path is just an X-string with $X = \{H, V\}$, where H stands for *horizontal* and V stands for *vertical*. In this case, there are exactly $(p-m)+(q-n)$ moves, of which $p-m$ are horizontal.

Example 2.19. Let n be a non-negative integer. Then the number of lattice paths from $(0,0)$ to (n,n) which never go above the diagonal line $y=x$ is the Catalan number

$$C(n) = \frac{1}{n+1}\binom{2n}{n}.$$

To see that this formula holds, consider the family $\mathcal{P}$ of all lattice paths from $(0,0)$ to (n,n). A lattice path from $(0,0)$ to (n,n) is just a $\{H,V\}$-string of length $2n$ with exactly n H's. So $|\mathcal{P}| = \binom{2n}{n}$. We classify the paths in $\mathcal{P}$ as *good* if they never go over the diagonal; otherwise, they are *bad*. A string $s \in \mathcal{P}$ is good if the number of V's in an initial segment of s never exceeds the number of H's. For example, the string "$HHVHVVHHHVHVVV$" is a good lattice path from $(0,0)$ to $(7,7)$, while the path "$HVHVHHVVVHVHHV$" is bad. In the second case, note that after 9 moves, we have 5 V's and 4 H's.

Let $\mathcal{G}$ and $\mathcal{B}$ denote the family of all good and bad paths, respectively. Of course, our goal is to determine $|\mathcal{G}|$.

Consider a path $s \in \mathcal{B}$. Then there is a least integer i so that s has more V's than H's in the first i positions. By the minimality of i, it is easy to see that i must be odd (otherwise, we can back up a step), and if we set $i = 2j+1$, then in the first $2j+1$ positions of s, there are exactly j H's and $j+1$ V's. The remaining $2n-2j-1$ positions (the "tail of s") have $n-j$ H's and $n-j-1$ V's. We now transform s to a new string s' by replacing the H's in the tail of s by V's and the V's in the tail of s by H's and leaving the initial $2j+1$ positions unchanged. For example, see Figure 2.5, where the path s is shown solid and s' agrees with s until it crosses the line $y=x$ and then is the dashed path. Then s' is a string of length $2n$ having $(n-j)+(j+1) = n+1$ V's and $(n-j-1)+j = n-1$ H's, so s' is a lattice path from $(0,0)$ to $(n-1,n+1)$. Note that there are $\binom{2n}{n-1}$ such lattice paths.

We can also observe that the transformation we've described is in fact a bijection between $\mathcal{B}$ and $\mathcal{P}'$, the set of lattice paths from $(0,0)$ to $(n-1,n+1)$. To see that this is true, note that every path s' in $\mathcal{P}'$ must cross the line $y=x$, so there is a first time it crosses it, say in position i. Again, i must be odd, so $i=2j+1$ and there are j H's and $j+1$ V's in the first i positions of s'. Therefore the tail of s' contains $n+1-(j+1) = n-j$ V's and $(n-1)-j$ H's, so interchanging H's and V's in the tail of s' creates a new string s that has n H's and n V's and thus represents a lattice path from $(0,0)$ to (n,n), but it's still a bad lattice path, as we did not adjust the first part of the path, which results in crossing the line $y=x$ in position i. Therefore, $|\mathcal{B}| = |\mathcal{P}'|$ and thus

$$C(n) = |\mathcal{G}| = |\mathcal{P}| - |\mathcal{B}| = |\mathcal{P}| - |\mathcal{P}'| = \binom{2n}{n} - \binom{2n}{n-1} = \frac{1}{n+1}\binom{2n}{n},$$

after a bit of algebra.

It is worth observing that in the preceding example, we made use of two common enumerative techniques: giving a bijection between two classes of objects, one of which

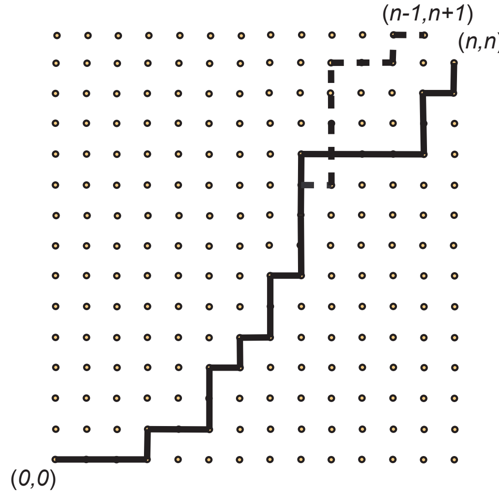

FIGURE 2.5: TRANSFORMING A LATTICE PATH

is "easier" to count than the other, and counting the objects we do *not* wish to enumerate and deducting their number from the total.

2.6 The Binomial Theorem

Here is a truly basic result from combinatorics kindergarten.

Theorem 2.20 (Binomial Theorem)**.** *Let x and y be real numbers with x, y and $x+y$ non-zero. Then for every $n \in \mathbb{N}_0$,*

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i.$$

Proof. View $(x+y)^n$ as a product

$$(x+y)^n = \underbrace{(x+y)(x+y)(x+y)(x+y)\dots(x+y)(x+y)}_{n \text{ factors}}.$$

Each term of the expansion of the product results from choosing either x or y from one of these factors. If x is chosen $n-i$ times and y is chosen i times, then the resulting product is $x^{n-i}y^i$. Clearly, the number of such terms is $C(n,i)$, i.e., out of the n factors, we choose the element y from i of them, while we take x in the remaining $n-i$. □

Example 2.21. There are times when we are interested not in the full expansion of a power of a binomial, but just the coefficient on one of the terms. The Binomial Theorem gives that the coefficient of x^5y^8 in $(2x-3y)^{13}$ is $\binom{13}{5}2^5(-3)^8$.

2.7 Multinomial Coefficients

Let X be a set of n elements. Suppose that we have two colors of paint, say red and blue and we are going to choose a subset of k elements to be painted red with the rest painted blue. Then the number of different ways this can be done is just the binomial coefficient $\binom{n}{k}$. Now suppose that we have three different colors, say red, blue, and green We will choose k_1 to be colored red, k_2 to be colored blue, with the remaining $k_3 = n-(k_1+k_2)$ colored green. We may compute the number of ways to do this by first choosing k_1 of the n elements to paint red, then from the remaining $n-k_1$ choosing k_2 to paint blue, and then painting the remaining k_3 green. It is easy to see that the number of ways to do this is

$$\binom{n}{k_1}\binom{n-k_1}{k_2} = \frac{n!}{k_1!(n-k_1)!}\frac{(n-k_1)!}{k_2!(n-(k_1+k_2))!} = \frac{n!}{k_1!k_2!k_3!}$$

Numbers of this form are called *multinomial coefficients*; they are an obvious generalization of the binomial coefficients. The general notation is:

$$\binom{n}{k_1,k_2,k_3,\ldots,k_r} = \frac{n!}{k_1!k_2!k_3!\ldots k_r!}.$$

For example,

$$\binom{8}{3,2,1,2} = \frac{8!}{3!2!1!2!} = \frac{40320}{6\cdot 2\cdot 1\cdot 2} = 1680.$$

Note that there is some "overkill" in this notation, since the value of k_r is determined by n and the values for $k_1, k_2, \ldots, k_{r-1}$. For example, with the ordinary binomial coeffients, we just write $\binom{8}{3}$ and not $\binom{8}{3,5}$.

Example 2.22. How many different rearrangements of the string:

PROFESSORTROTTERANDGTAKELLERAREGENIUSES!!

are possible if all letters and characters must be used?

To answer this question, we note that there are a total of 41 characters distributed as follows: 3 A's, 1 D, 7 E's, 1 F, 2 G's, 1 I, 1 K, 2 L's, 2 N's, 3 O's, 1 P, 6 R's, 4 S's, 4 T's, 1 U and 2 !'s. So the number of rearrangements is

$$\frac{41!}{3!1!7!1!2!1!1!2!2!3!1!6!4!4!1!2!}.$$

Just as with binomial coefficients and the Binomial Theorem, the multinomial coefficients arise in the expansion of powers of a multinomial:

Theorem 2.23 (Multinomial Theorem)**.** *Let $x_1, x_2, \ldots, x_r$ be nonzero real numbers with $\sum_{i=1}^{r} x_i \neq 0$. Then for every $n \in \mathbb{N}_0$,*

$$(x_1 + x_2 + \cdots + x_r)^n = \sum_{k_1+k_2+\cdots+k_r=n} \binom{n}{k_1, k_2, \ldots, k_r} x_1^{k_1} x_2^{k_2} \cdots x_r^{k_r}.$$

Example 2.24. What is the coefficient of $x^{99}y^{60}z^{14}$ in $(2x^3 + y - z^2)^{100}$? What about $x^{99}y^{61}z^{13}$?

By the Multinomial Theorem, the expansion of $(2x^3 + y - z^2)^{100}$ has terms of the form

$$\binom{100}{k_1, k_2, k_3}(2x^3)^{k_1} y^{k_2} (-z^2)^{k_3} = \binom{100}{k_1, k_2, k_3} 2^{k_1} x^{3k_1} y^{k_2} (-1)^{k_3} z^{2k_3}.$$

The $x^{99}y^{60}z^{14}$ arises when $k_1 = 33$, $k_2 = 60$, and $k_3 = 7$, so it must have coefficient

$$-\binom{100}{33, 60, 7} 2^{33}.$$

For $x^{99}y^{61}z^{13}$, the exponent on z is odd, which cannot arise in the expansion of $(2x^3 + y - z^2)^{100}$, so the coefficient is 0.

2.8 Exercises

1. Is 838200020310007224300 a Catalan number?

2. Suppose we are making license plates of the form $l_1l_2l_3 - d_1d_2d_3$ where l_1, l_2, l_3 are capital letters in the English alphabet and d_1, d_2, d_3 are decimal digits (i.e., in $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$) subject to the restriction that at least one digit is nonzero and at least one letter is K. How many license plates can we make?

3. How many strings of the form $l_1l_2d_1d_2d_3l_3l_4d_4l_5l_6$ are there where
   - for $1 \leq i \leq 6$, l_i is a capital letter in the English alphabet;
   - for $1 \leq i \leq 4$, d_i is a decimal digit;
   - l_2 is not a vowel (i.e., $l_2 \notin \{$A,E,I,O,U$\}$); and
   - the digits d_1, d_2, and d_3 are distinct (i.e., $d_1 \neq d_2 \neq d_3 \neq d_1$).

4. How many ternary strings of length $2n$ are there in which the zeroes appear only in odd-numbered positions?

5. Suppose that a teacher wishes to distribute 25 identical pencils to Ahmed, Barbara, Carlos, and Dieter such that Ahmed and Dieter receive at least one pencil each, Carlos receives no more than five pencils, and Barbara receives at least four pencils. In how many ways can such a distribution be made?

6. How many integer solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 = 132$$

provided that $x_1 > 0$, and $x_2, x_3, x_4 \geq 0$? What if we add the restriction that $x_4 < 17$?

7. How many integer solutions are there to the inequality

$$x_1 + x_2 + x_3 + x_4 + x_5 \leq 782$$

provided that $x_1, x_2 > 0$, $x_3 \geq 0$, and $x_4, x_5 \geq 10$?

8. Give a combinatorial argument to prove the identity

$$k\binom{n}{k} = n\binom{n-1}{k-1}.$$

Hint: Think of choosing a team with a captain.

9. How many lattice paths from $(0, 0)$ to $(17, 12)$ are there that pass through $(7, 6)$ and $(12, 9)$?

10. How many lattice paths from $(0, 0)$ to $(14, 73)$ are there that do *not* pass through $(6, 37)$?

11. Determine the coefficient on $x^{15}y^{120}z^{25}$ in $(2x + 3y^2 + z)^{100}$.

12. Determine the coefficient on $x^{12}y^{24}$ in $(x^3 + 2xy^2 + y + 3)^{18}$. (Be careful, as x and y now appear in multiple terms!)

13. For each word below, determine the number of rearrangements of the word in which all letters must be used.
    a) OVERNUMEROUSNESSES
    b) OPHTHALMOOTORHINOLARYNGOLOGY
    c) HONORIFICABILITUDINITATIBUS (the longest word in the English language consisting strictly of alternating consonants and vowels[1])

[1] http://www.rinkworks.com/words/oddities.shtml

14. How many ways are there to paint a set of 27 elements such that 7 are painted white, 6 are painted old gold, 2 are painted blue, 7 are painted yellow, 5 are painted green, and 0 of are painted red?

15. There are many useful sets that are enumerated by the Catalan numbers. (Volume two of R.P. Stanley's *Enumerative Combinatorics* contains a famous (or perhaps infamous) exercise in 66 parts asking readers to find bijections that will show that the number of various combinatorial structures is $C(n)$, and his web page boasts an additional list of at least 100 parts.) Give bijective arguments to show that each class of objects below is enumerated by $C(n)$. (All three were selected from the list in Stanley's book.)

    a) The number of ways to fully-parenthesize a product of $n+1$ factors as if the "multiplication" operation in question were not necessarily associative. For example, there is one way to parenthesize a product of two factors (a_1a_2), there are two ways to parenthesize a product of three factors $((a_1(a_2a_3))$ and $((a_1a_2)a_3))$, and there are five ways to parenthesize a product of four factors:

    $$(a_1(a_2(a_3a_4))), (a_1((a_2a_3)a_4)), ((a_1a_2)(a_3a_4)), ((a_1(a_2a_3))a_4), (((a_1a_2)a_3)a_4).$$

    b) Sequences of n 1's and n -1's in which the sum of the first i terms is nonnegative for all i.

    c) Sequences $1 \leq a_1 \leq \cdots \leq a_n$ of integers with $a_i \leq i$. For example, for $n = 3$, the sequences are

    $$111 \qquad 112 \qquad 113 \qquad 122 \qquad 123.$$

    Hint: Think about drawing lattice paths on paper with grid lines and (basically) the number of boxes below a lattice path in a particular column.