Graphs - Introduction

November 30, 2023

Graphs - Basic Concepts

- Basic definitions: vertices and edges
- More definitions: paths, simple paths, cycles, loops
- Connected and disconnected graphs
- Spanning trees
- Complete graphs
- Weighted graphs and networks
- Graph representations
  - Adjacency matrix
  - Adjacency lists

Graph

A graph is a mathematical object that can be used to model many problems:

- Flowchart of a program
- Road map
- Electric circuits
- Course curriculum
- Communication Network
- Social Networks (Twitter, Facebook ...)
- Internet

Graphs are important because they can be used to represent essentially any relationship.

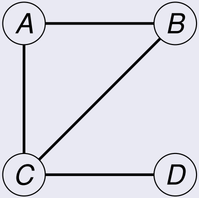
Vertices and Edges

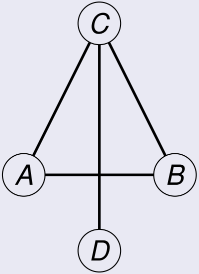
Definition: A graph ($\mathcal{G}$) is a collection (nonempty set) of vertices and edges

- Vertices ($\mathcal{V}$):
  - Can have names and properties
- Edges ($\mathcal{E}$):
  - Connect two vertices (each edge is a pair (v, w) where $v, w \in \mathcal{V}$)
  - Can have weights or cost
  - Can be directed
  - Edge set
    $$\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$$
  - Adjacent vertices: there is an edge between them
- We write: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$

Example

- Vertex set : $\mathcal{V} = \{A, B, C, D\}$
- Edge set : $\mathcal{E} = \{(A, B), (A, C), (B, C), (C, D)\}$

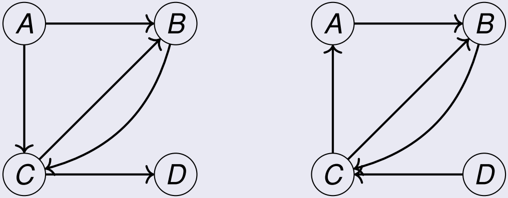




Two ways to draw the same graph

Directed versus Undirected

Directed Graph: each edge is associated with an ordered pair of vertices. A graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is undirected if edge $x, y \in \mathcal{E}$ implies that (y, x) is also in $\mathcal{E}$. If not, we say that the graph is directed.

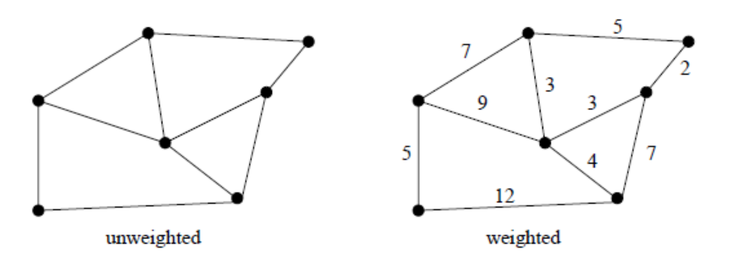


Different Graphs

E.g: Road networks between cities are typically undirected, while street networks within cities are almost always directed.

Weighted versus Unweighted

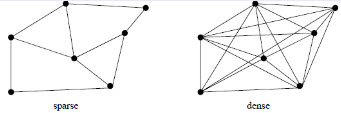
Each edge (or vertex) in a **weighted** graph $\mathcal{G}$ is assigned a numerical value, or weight.
E.g: The edges of a road network graph might be weighted with their length, drive-time, or speed limit



A complete undirected unweighted graph

is one where there is an edge connecting all possible pairs of vertices in a graph.
The complete graph with n vertices is denoted as K_n.

- Dense graphs: relatively few of the possible edges are missing
- Sparse graphs: relatively few of the possible edges are present



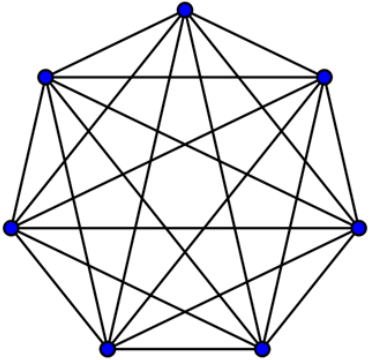
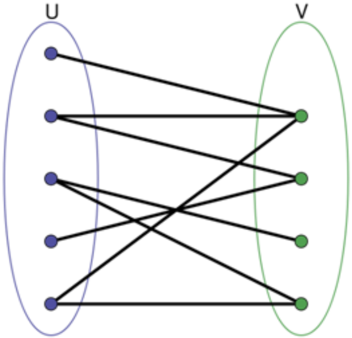


Figure: A complete graph

Bipartite Graph

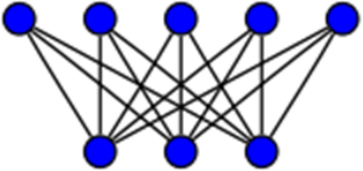
A graph is bipartite

- If there exists a way to partition the set of vertices V, in the graph into two sets V_1 and V_2
- Where $V_1 \cup V_2 = V$ and $V_1 \cap V_2 = \emptyset$, such that each edge in E contains one vertex from V_1 and the other vertex from V_2.
- Simply, it can be colored without conflicts while using only two colors.



Complete bipartite graph

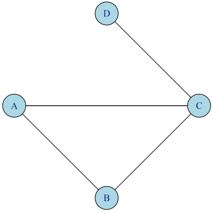
A complete bipartite graph on m and n vertices is denoted by $K_{m,n}$ and consists of $m + n$ vertices, with each of the first m vertices connected to all of the other n vertices, and no other vertices.



More definitions : Path

- A path of length n from vertex v_0 to vertex v_n is an alternating sequence of $n+1$ vertices and n edges beginning with vertex v_0 and ending with vertex v_n in which edge e_i is incident upon vertices v_{i-1} and v_i.
  - (The order in which these are connected matters for a path in a directed graph in the natural way.)
- A simple path is a path such that all vertices are distinct, except that the first and the last could be the same.

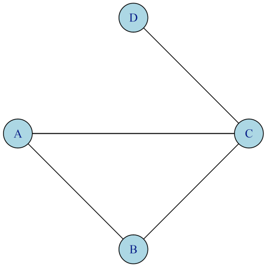
A B C
B A C D
A B C A B C A B C D
B A B A C



More definitions : Cycle

Simple path with distinct edges, except that the first vertex is equal to the last

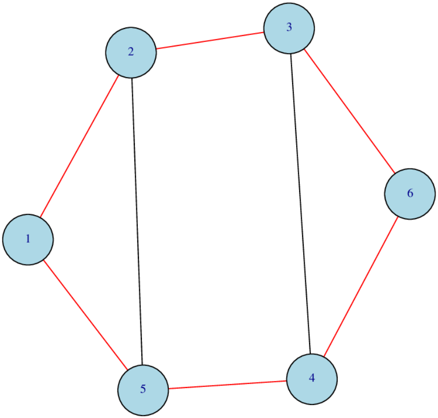
A B C A
B A C B
C B A C



A graph without cycles is called acyclic graph.

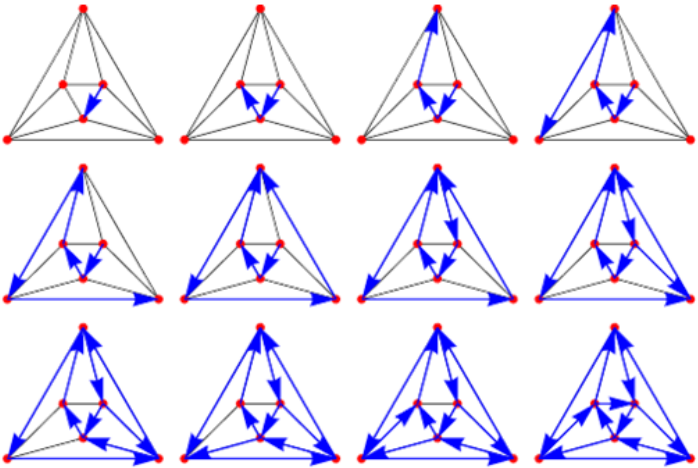
More definitions : Hamiltonian Cycle

- A Hamiltonian cycle is a simple cycle that contains all the vertices in the graph
- A cycle that travels exactly once over each vertex in a graph is called Hamiltonian.



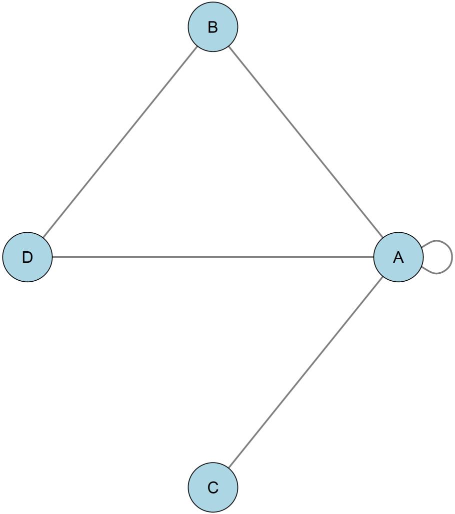
More definitions : Euler Cycle

- An Euler cycle is a cycle that contains every edge in the graph exactly once.
  - A cycle that travels exactly once over each edge in a graph is called Eulerian.
  - Note that a vertex may be contained in an Euler cycle more than once. Typically, these are known as Euler circuits, because a circuit has no repeated edges



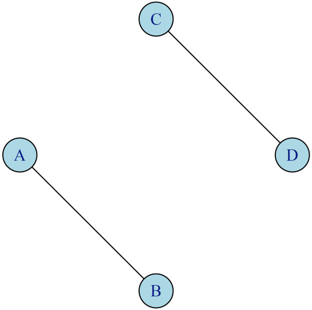
More definitions : Loop

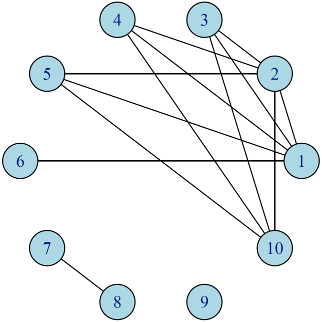
- Loop: An edge that connects the vertex with itself



Connected and Disconnected graphs

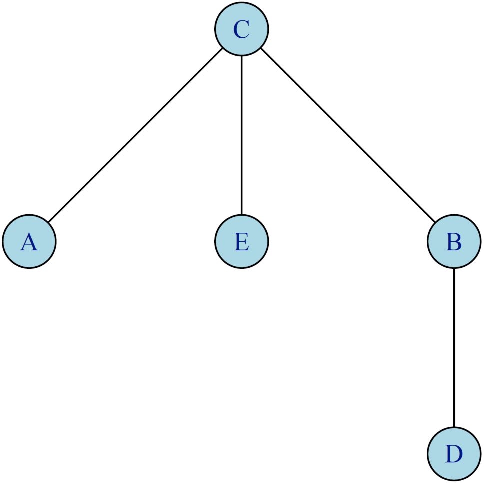
- Connected graph: There is a path between each two vertices
- Disconnected graph : There are at least two vertices not connected by a path.
- Examples of disconnected graphs:





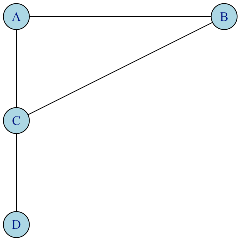
Graphs and Trees

- Tree: an undirected graph with no cycles, and a node chosen to be the root

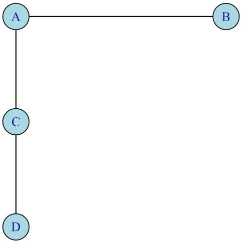


A spanning tree of an undirected graph

- Spanning tree: A sub-graph that contains all the vertices, and no cycles.
- a subset of edges from $\mathcal{E}$ forming a tree connecting all vertices of $\mathcal{V}$.
- If we add any edge to the spanning tree, it forms a cycle, and the tree becomes a graph

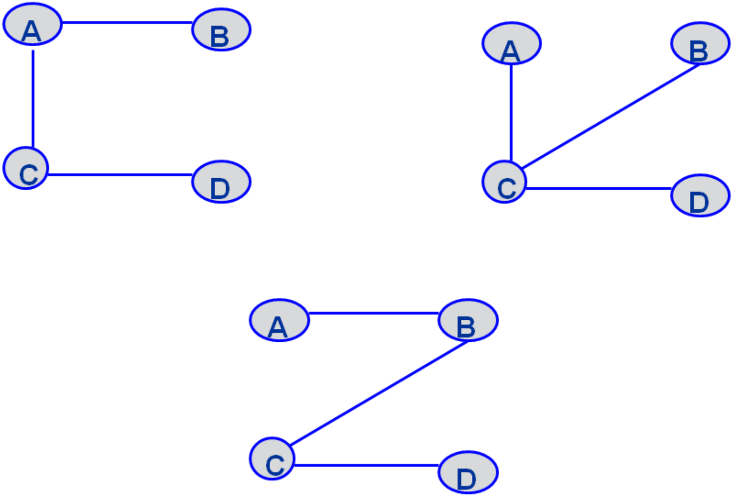


Graph



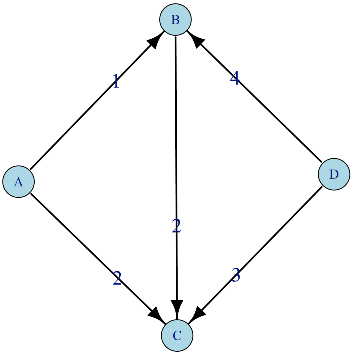
Spanning Tree

All spanning trees of the graph on the previous slide



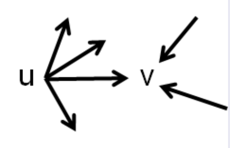
Weighted graphs and Networks

- Weighted graphs – weights are assigned to each edge (e.g. road map)
- Networks: directed weighted graphs (some theories allow networks to be undirected)



Some Graph Terminology

- u is the source , v is the sink of (u,v)
  $\mathbf{u} \rightarrow \mathbf{v}$
- u, v, b, c are the endpoints of (u,v) and (b, c)
- u, v are adjacent nodes. b, c are adjacent nodes



- outdegree of u in directed graph: number of edges for which u is source
- indegree of v in directed graph: number of edges for which v is sink
- degree of vertex w in undirected graph: number of edges of which w is an endpoint

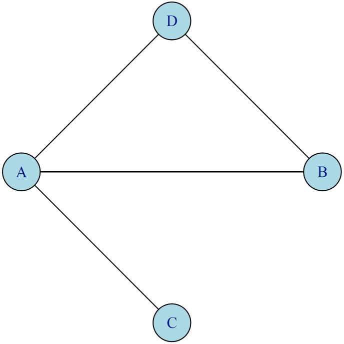
Graph Representation

- Adjacency matrix
- Adjacency lists

Adjacency matrix - undirected graphs

Vertices: A,B,C,D
Edges: AC, AB, AD, BD

	A	B	C	D
A	0	1	1	1
B	1	0	0	1
C	1	0	0	0
D	1	1	0	0



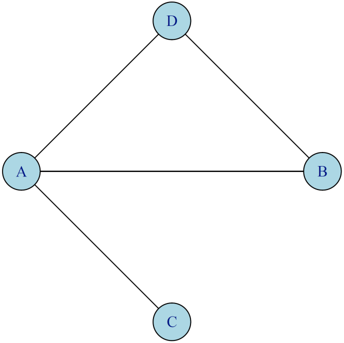
Adjacency matrix - undirected graphs

Vertices: A,B,C,D

Edges: AC, AB, AD, BD

The matrix is symmetrical

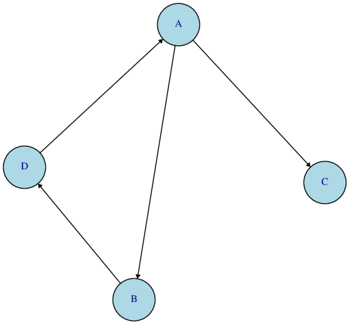
	A	B	C	D
A	0	1	1	1
B	1	0	0	1
C	1	0	0	0
D	1	1	0	0



Adjacency matrix - directed graphs

Vertices: A,B,C,D
Edges: AC, AB, BD, DA

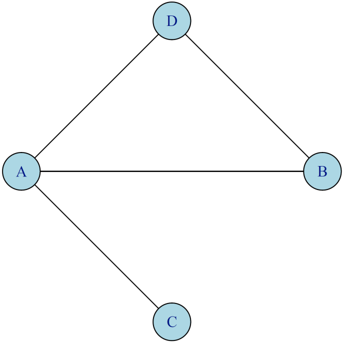
	A	B	C	D
A	0	1	1	0
B	0	0	0	1
C	0	0	0	0
D	1	0	0	0



Adjacency lists - undirected graphs

Vertices: A,B,C,D
Edges: AC, AB, AD, BD

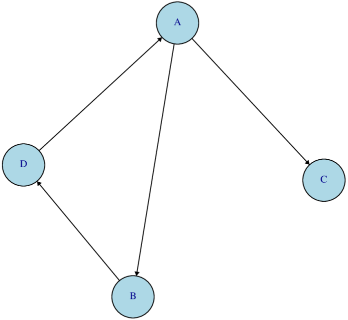
Heads	**Lists**
A	B C D
B	A D
C	A
D	A B



Adjacency lists – directed graphs

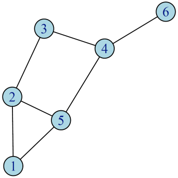
Vertices: A,B,C,D
Edges: AC, AB, BD, DA

Heads	**lists**
A	**B C**
B	**D**
C	**-**
D	**A**

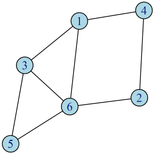


The complement of a graph

- The complement of a graph G is a graph G’ which contains all the vertices of G, but for each edge that exists in G, it is NOT in G’, and for each possible edge NOT in G, it IS in G’.



	1	2	3	4	5	6
1	0	1	0	0	1	0
2	1	0	1	0	1	0
3	0	1	0	1	0	0
4	0	0	1	0	1	1
5	1	1	0	1	0	0
6	0	0	0	1	0	0



	1	2	3	4	5	6
1	1	0	1	1	0	1
2	0	1	0	1	0	1
3	1	0	1	0	1	1
4	1	1	0	1	0	0
5	0	0	1	0	1	1
6	1	1	1	0	1	1