Graphs-Spanning Trees

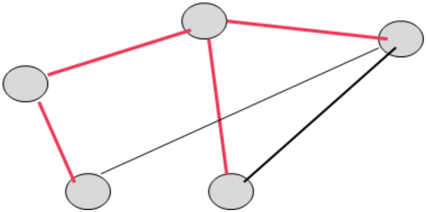
December 2, 2021

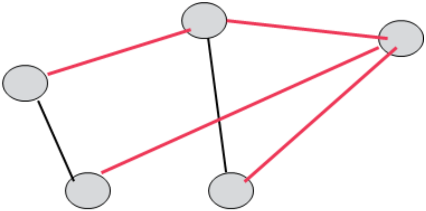
Topics

- Spanning Trees in Unweighted Graphs
- Minimum Spanning Trees in Weighted Graphs
    - Prim's Algorithm
    - Kruskal's Algorithm

Definitions

- Spanning tree: a tree that contains all vertices in the graph.
  - Sub graph of the original graph
  - Tree
  - Number of nodes: |V|
  - Number of edges: |V|-1





Spanning trees for unweighted graphs - data structures

- A table (an array) T
  - *size* = number of vertices,
  - T_v= parent of vertex v
- Adjacency lists
- A queue of vertices to be processed

Algorithm - initialization

1. Choose a vertex S and store it in a queue, set
   a *counter* $= 0$ (counts the processed nodes), and
   $T_s = 0$ (to indicate the root),
   $T_i = -1$,$i \neq s$ (to indicate vertex not processed)

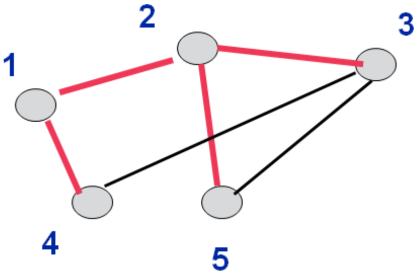
Algorithm - basic loop

2. While queue not empty and *counter* $< |V| - 1$:
   Read a vertex V from the queue
   For all adjacent vertices U :
   If $T_u = -1$ (not processed)
   $T_u \leftarrow V$
   $counter \leftarrow counter + 1$
   store U in the queue

Algorithm – results and complexity

- Result:
  - Root: S, such that Ts = 0
  - Edges in the tree: (Tv , V)
- Complexity: $O(|E| + |V|)$ - we process all edges and all nodes

Example



Table

0	1 is the root
1	edge (1,2)
2	edge (2,3)
1	edge (1,4)
2	edge (2,5)

Edge format: (Tv,V)

Definitions

- Minimum Spanning tree: a tree
  - that contains all vertices in the graph,
  - is connected,
  - is acyclic, and
  - has minimum total edge weight.

Applications of Minimum Spanning Trees

- Communication networks
- VLSI design
- Transportation systems

Minimum Spanning Tree - Prim's algorithm

- Given: Weighted graph.
  - Find a spanning tree with the minimal sum of the weights.
  - Similar to shortest paths in weighted graphs.
  - Difference: we record the weight of the current edge, not the length of the path .

Data Structures

- Three arrays:
  - cost[v] = the weight of the shortest edge connecting v to a known vertex
  - path[v] = the last vertex to cause a change in cost[v]
  - known[v] = True, if vertex v is fixed in the tree, False otherwise

Data Structures (cont.)

- Adjacency lists
- A priority queue of vertices to be processed.
  - Priority of each vertex is determined by the weight of edge that links the vertex to its parent.
  - The priority may change if we change the parent, provided the vertex is not yet fixed in the tree.

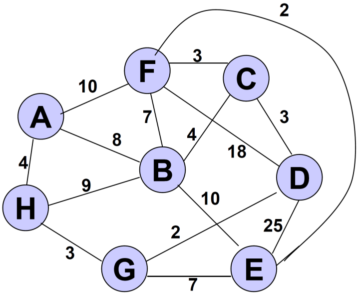
Prim's Algorithm

For all v
    $cost[v] = \infty$; $known[v] = false$; $path[v] = -$
$cost[s] = 0$
Insert s onto a priority queue that percolates vertices
    with lowest $cost[]$ to the top.
While priority queue not empty
    DeleteMin v from priority queue
    If not $known[v]$
      $known[v] = true$
      For all unknown successors w of v
        If $weight[v, w] < cost[w]$
          $cost[w] = weight[v, w]$
          $path[w] = v$
          insert w onto priority queue

Results and Complexity

- At the end of the algorithm, the tree would be represented with its edges
  - $\{(v, path[v])|v = 1, 2, \ldots, |V|\}$
- Complexity: $O(|E|log(|V|))$

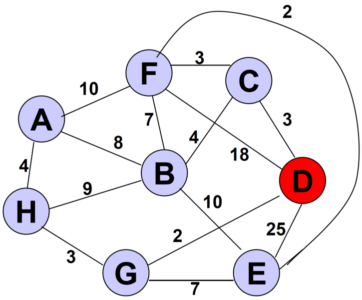
Example



Initialize array

	K	d_v	p_v
A	F	∞	–
B	F	∞	–
C	F	∞	–
D	F	∞	–
E	F	∞	–
F	F	∞	–
G	F	∞	–
H	F	∞	–

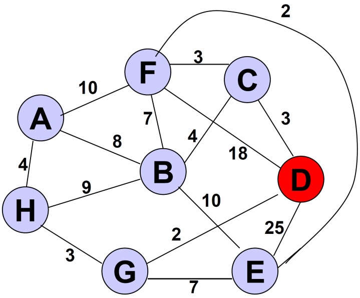
Example



Start with any node, say D

	K	d_v	p_v
A			
B			
C			
D	T	0	–
E			
F			
G			
H			

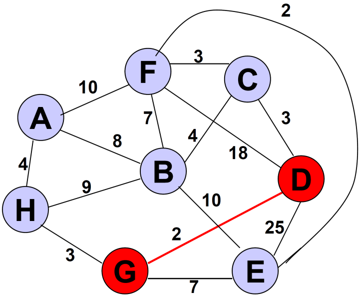
Example



Update distances of adjacent, unselected nodes

	K	d_v	p_v
A			
B			
C		3	D
D	T	0	–
E		25	D
F		18	D
G		2	D
H			

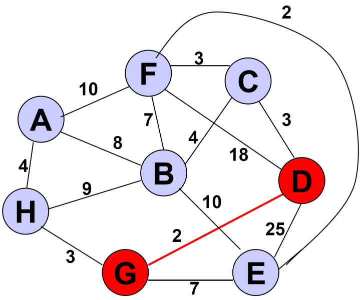
Example



Select node with minimum distance

	K	d_v	p_v
A			
B			
C		3	D
D	T	0	–
E		25	D
F		18	D
G	T	2	D
H			

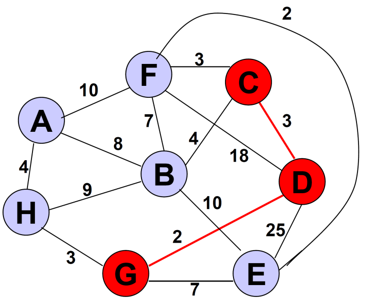
Example



Update distances of adjacent, unselected nodes

	K	d_v	p_v
A			
B			
C		3	D
D	T	0	–
E		7	G
F		18	D
G	T	2	D
H		3	G

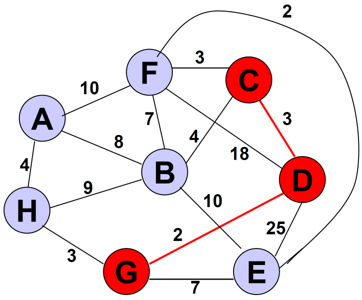
Example



Select node with minimum distance

	K	d_v	p_v
A			
B			
C	T	3	D
D	T	0	–
E		7	G
F		18	D
G	T	2	D
H		3	G

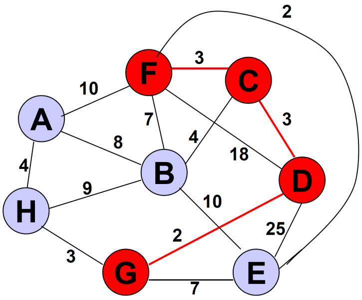
Example



Update distances of adjacent, unselected nodes

	K	d_v	p_v
A			
B		4	C
C	T	3	D
D	T	0	–
E		7	G
F		3	C
G	T	2	D
H		3	G

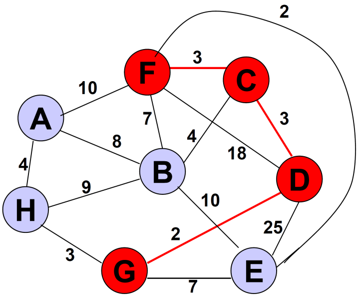
Example



Select node with minimum distance

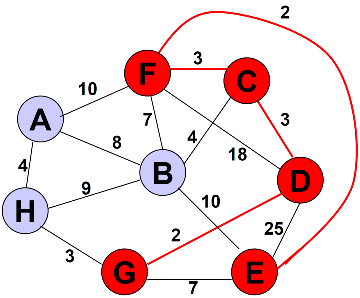
	K	d_v	p_v
A			
B		4	C
C	T	3	D
D	T	0	–
E		7	G
F	T	3	C
G	T	2	D
H		3	G

Example



Update distances of adjacent, unselected nodes

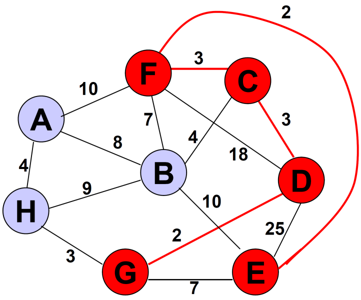
	K	d_v	p_v
A		10	F
B		4	C
C	T	3	D
D	T	0	–
E		2	F
F	T	3	C
G	T	2	D
H		3	G



Select node with minimum distance

	K	d_v	p_v
A		10	F
B		4	C
C	T	3	D
D	T	0	–
E	T	2	F
F	T	3	C
G	T	2	D
H		3	G

Example

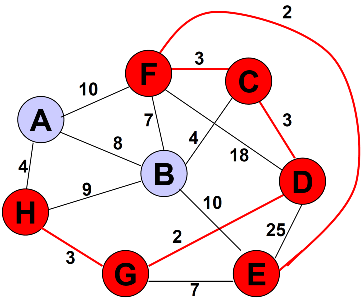


Update distances of adjacent, unselected nodes

	K	d_v	p_v
A		10	F
B		4	C
C	T	3	D
D	T	0	–
E	T	2	F
F	T	3	C
G	T	2	D
H		3	G

Table entries unchanged

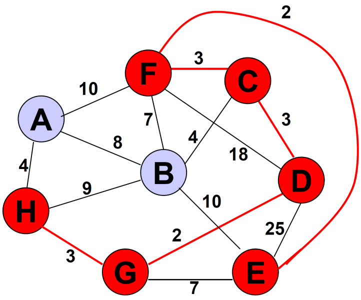
Example



Select node with minimum distance

	K	d_v	p_v
A		10	F
B		4	C
C	T	3	D
D	T	0	–
E	T	2	F
F	T	3	C
G	T	2	D
H	T	3	G

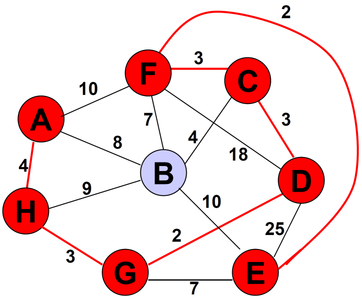
Example



Update distances of adjacent, unselected nodes

	K	d_v	p_v
A		4	H
B		4	C
C	T	3	D
D	T	0	–
E	T	2	F
F	T	3	C
G	T	2	D
H	T	3	G

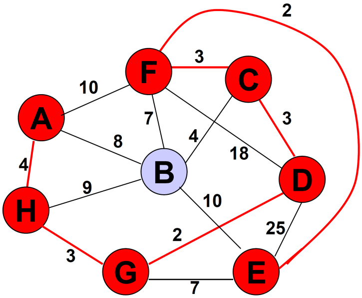
Example



Select node with minimum distance

	K	d_v	p_v
A	T	4	H
B		4	C
C	T	3	D
D	T	0	–
E	T	2	F
F	T	3	C
G	T	2	D
H	T	3	G

Example

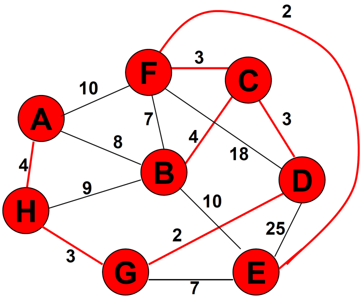


Update distances of adjacent, unselected nodes

	K	d_v	p_v
A	T	4	H
B		4	C
C	T	3	D
D	T	0	–
E	T	2	F
F	T	3	C
G	T	2	D
H	T	3	G

Table entries unchanged

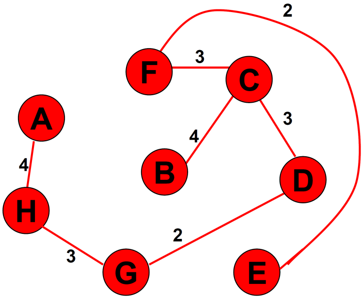
Example



Select node with minimum distance

	K	d_v	p_v
A	T	4	H
B	T	4	C
C	T	3	D
D	T	0	–
E	T	2	F
F	T	3	C
G	T	2	D
H	T	3	G

Example



Cost of Minimum
Spanning Tree = Σ d_v = **21**

	K	d_v	p_v
A	T	4	H
B	T	4	C
C	T	3	D
D	T	0	–
E	T	2	F
F	T	3	C
G	T	2	D
H	T	3	G

Done

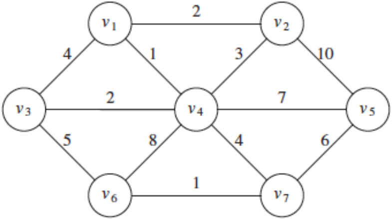
Kruskal's Algorithm

- The algorithm works with :
  - set of edges,
  - tree forests,
  - each vertex belongs to only one tree in the forest.

The Algorithm

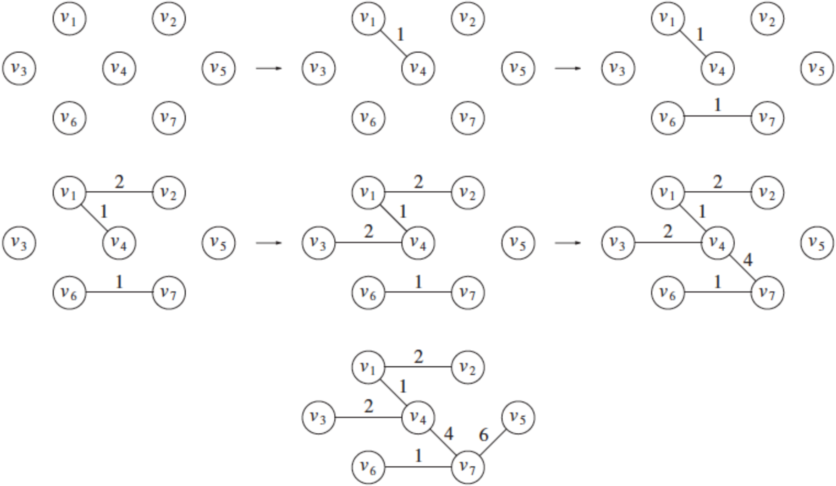
1. Build |V| trees of one vertex only - each vertex is a tree of its own. Store edges in priority queue
2. Choose an edge (u,v) with minimum weight
   if u and v belong to one and the same tree,
     do nothing
   if u and v belong to different trees,
     link the trees by the edge (u,v)
3. Perform step 2 until all trees are combined into one tree only

Example



Edge	Weight	Action
(v_1, v_4)	1	Accepted
(v_6, v_7)	1	Accepted
(v_1, v_2)	2	Accepted
(v_3, v_4)	2	Accepted
(v_2, v_4)	3	Rejected
(v_1, v_3)	4	Rejected
(v_4, v_7)	4	Accepted
(v_3, v_6)	5	Rejected
(v_5, v_7)	6	Accepted

Example



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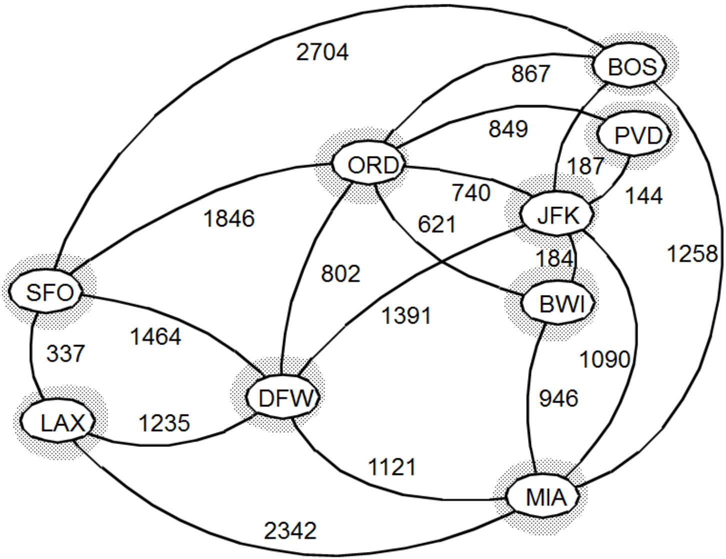

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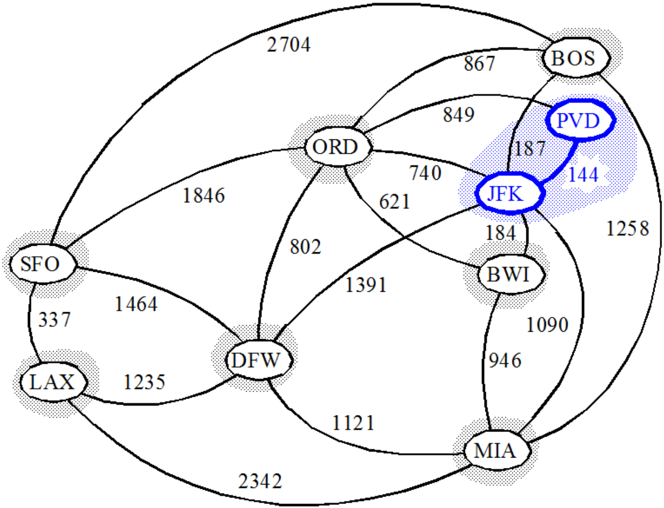

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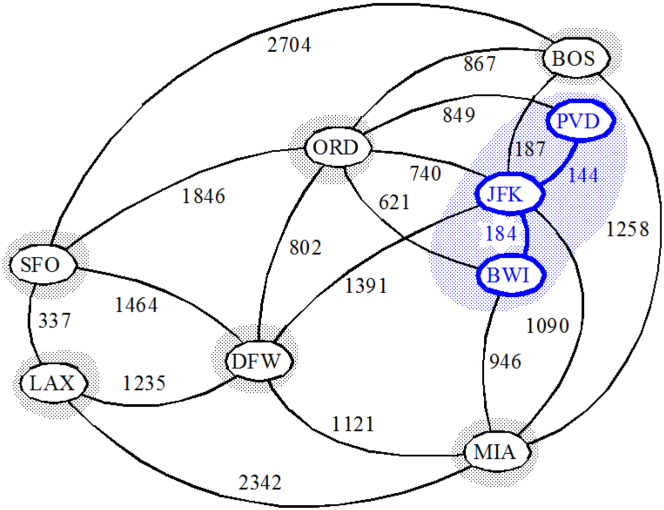

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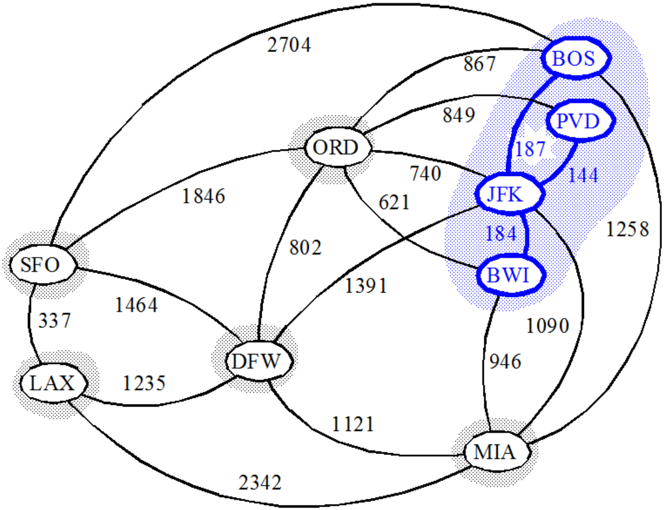

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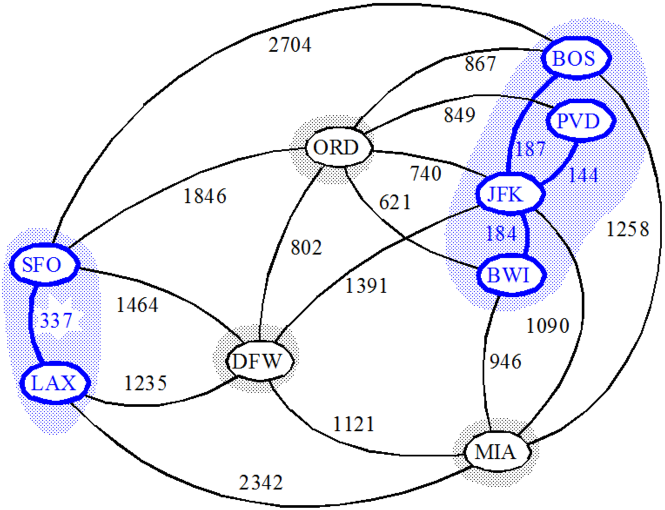
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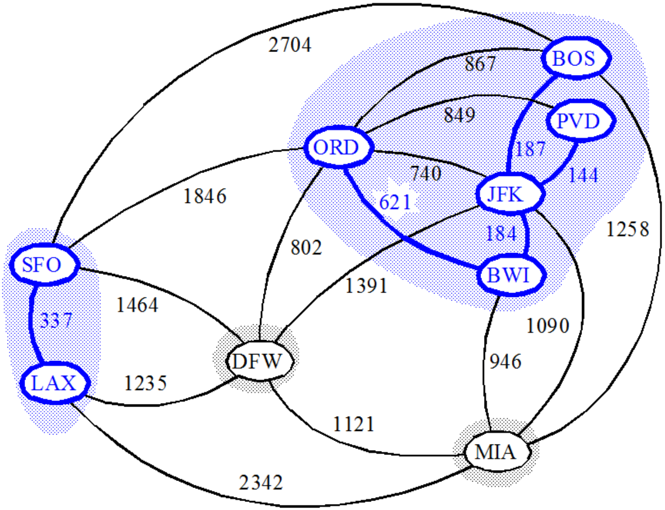

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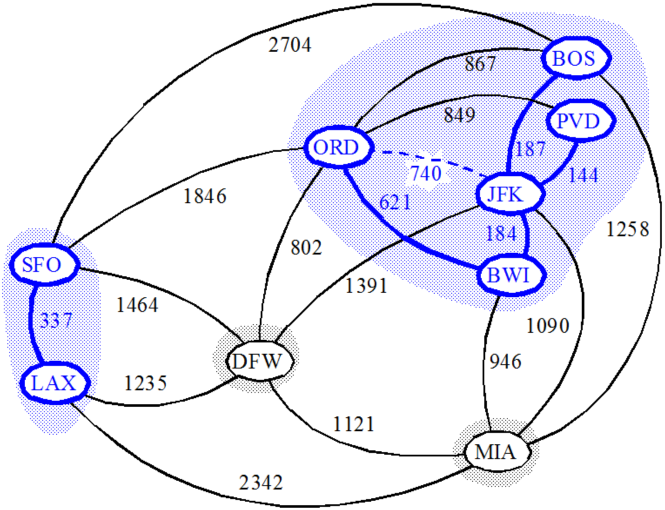

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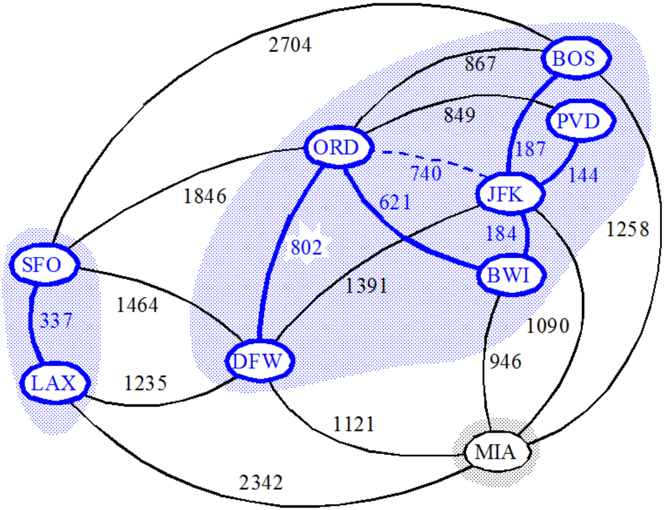
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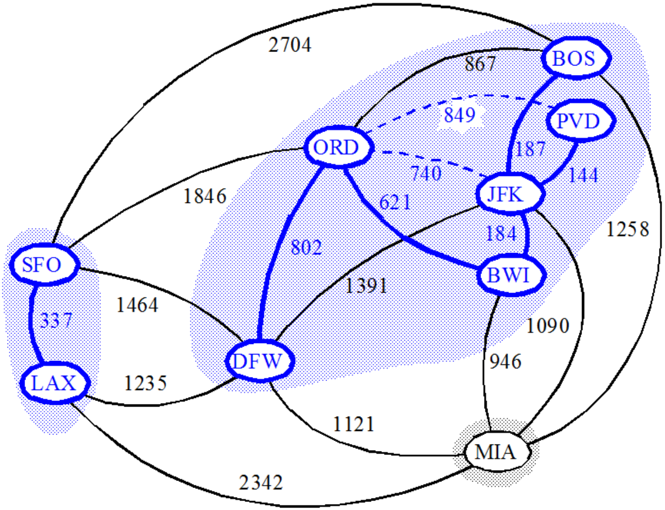

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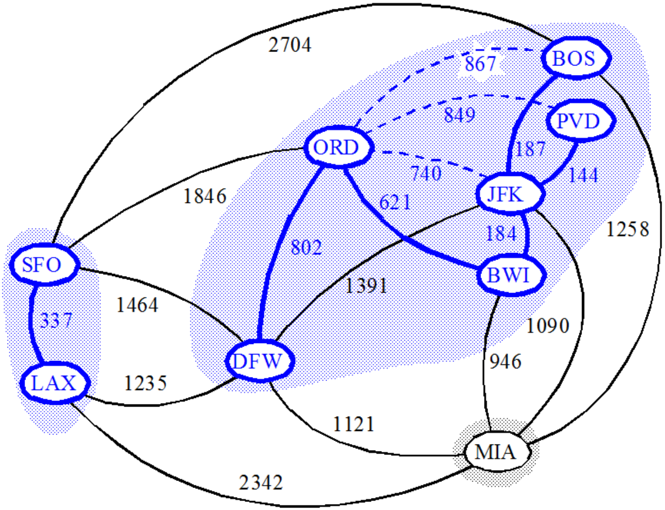

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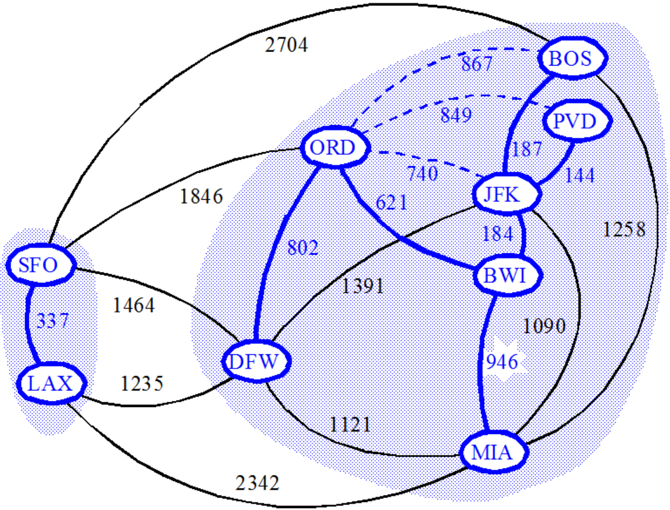

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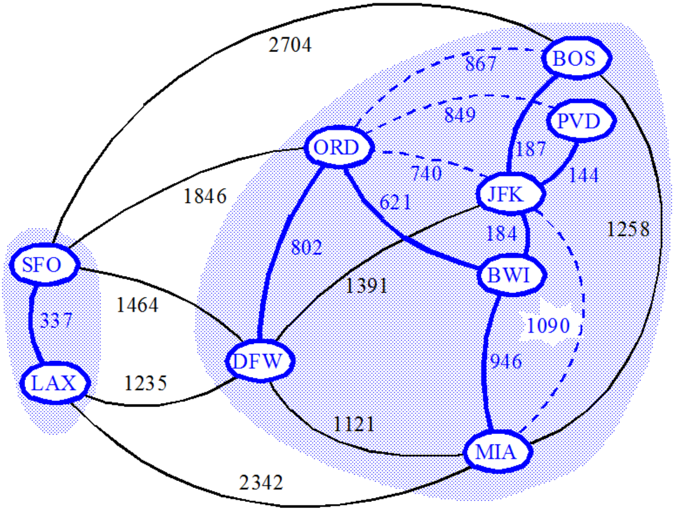

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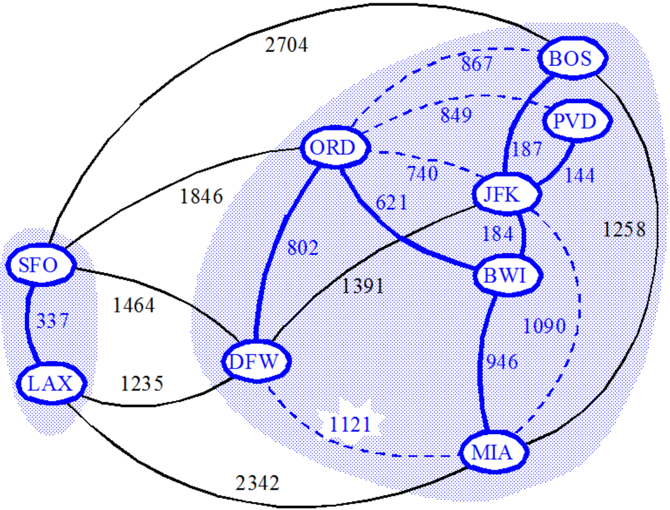

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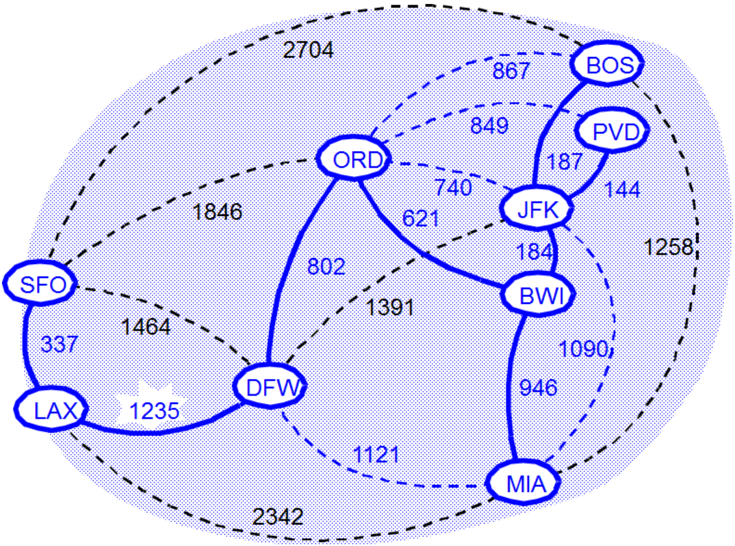

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Operations on Disjoint Sets

- Comparison: Are the vertices in the same tree?
- Union: Combine two disjoint sets to form one set - the new tree
- Implementation
  - Union/find operations: the unions are represented as trees.
  - The set of trees is implemented by an array.

Complexity of Kruskal's Algorithm

- The complexity is determined by the heap operations and the union/find operations
- Union/find operations on disjoint sets represented as trees: tree search operations, with complexity $O(log|V|)$
- DeleteMin in Binary heaps with N elements is $O(logN)$,
- When performed for N elements, it is $O(NlogN)$.

Complexity of Kruskal's Algorithm (Cont.)

- Kruskal's algorithm works with a binary heap that contains all edges: $O(|E|log(|E|))$
- However, a complete graph has
$$|E| = |V| \times (|V| - 1)/2, \text{ i.e. } |E| = O(|V|^2)$$
- Thus for the binary heap we have $O(|E|log(|E|)) = O(|E|log(|V|^2) = O(2|E|log(|V|)$ $= O(|E|log(|V|))$
- Since for each edge we perform DeleteMin operation and union/find operation, the overall complexity is:
$$O(|E|(log(|V|) + log(|V|)) = O(|E|log(|V|))$$

Discussion

- Sparse trees - Kruskal's algorithm :
  - Guided by edges
- Dense trees - Prim's algorithm :
  - The process is limited by the number of the processed vertices