Unit 7
Maximum-Margin Classification and the Support Vector Machine

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ECE 4300: Introduction to Machine Learning, Sp20

Learning objectives

- Gain experience with linear classification of images
  - representing images, displaying images, classifying images
- Understand the geometry of the linear classification boundary:
  - orthogonality to $\boldsymbol{w}$, offset b, margin $1/\|\boldsymbol{w}\|$
- Understand the margin-maximizing classifier
- Understand the support vector classifier (SVC)
- Understand the support vector machine (SVM)
- Understand how to implement the SVC and SVM with sklearn

Outline

- Motivating Example: Recognizing Handwritten Digits
- Maximum-Margin Classification
- The Support Vector Classifier
- The Support Vector Machine

MNIST digit classification

- Goal: Handwriting recognition
- Dataset: MNIST
  - Only digits 0–9
  - 28x28 images (grayscale)
  - 70,000 examples
  - Collected by American Census Bureau in 1980s
- Why use it?
  - Simple
  - Widely accessible
  - Many benchmarks

```
from sklearn.datasets import fetch_openml
mnist = fetch_openml('mnist_784', version=1, cache=True)
mnist.data.shape
```

```
(70000, 784)
```

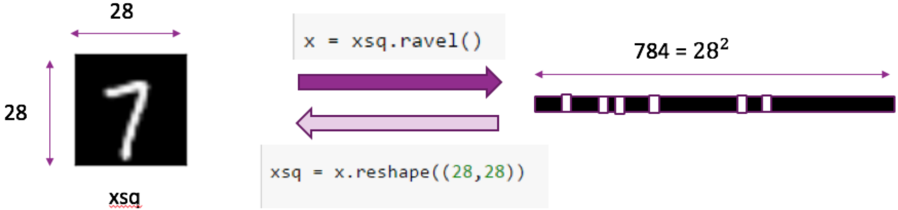
MNIST benchmarks

- In this unit, we focus on the support vector machine (SVM).
- Not state-of-the-art, but quite good, and widely used in machine learning

Type	Classifier	Distortion	Preprocessing	Error rate (%)
Linear classifier	Pairwise linear classifier	None	Deskewing	7.6[9]
Non-linear classifier	40 PCA + quadratic classifier	None	None	3.3[9]
Deep Neural network	2-layer 784-800-10	None	None	1.6[21]
Boosted Stumps	Product of stumps on Haar features	None	Haar features	0.87[19]
Deep Neural network	2-layer 784-800-10	elastic distortions	None	0.7[21]
Support vector machine	Virtual SVM, deg-9 poly, 2-pixel jittered	None	Deskewing	0.56[20]
K-Nearest Neighbors	K-NN with non-linear deformation (P2DHMDM)	None	Shiftable edges	0.52[18]
Deep neural network	6-layer 784-2500-2000-1500-1000-500-10	elastic distortions	None	0.35[22]
Convolutional neural network	6-layer 784-40-80-500-1000-2000-10	None	Expansion of the training data	0.31[15]
Convolutional neural network	6-layer 784-50-100-500-1000-10-10	None	Expansion of the training data	0.27[23]
Convolutional neural network	Committee of 35 CNNs, 1-20-P-40-P-150-10	elastic distortions	Width normalizations	0.23[8]
Convolutional neural network	Committee of 5 CNNs, 6-layer 784-50-100-500-1000-10-10	None	Expansion of the training data	0.21[17]

Representing images in Numpy

- Each pixel is a value between 0 (black) and 255 (white)
- Images can be represented as matrices or vectors



- In Numpy, vectorization is done by stacking rows, but in most papers/books it is done by stacking columns:

$$\boldsymbol{X} = \begin{bmatrix} x_1 & x_{29} & \cdots & x_{757} \\ x_2 & x_{30} & \cdots & x_{758} \\ \vdots & \vdots & & \vdots \\ x_{28} & x_{56} & \cdots & x_{784} \end{bmatrix} = \mathrm{mat}(\boldsymbol{x}), \qquad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{784} \end{bmatrix} = \mathrm{vec}(\boldsymbol{X})$$

  - Basically, Numpy works with $\boldsymbol{X}^\mathsf{T}$ and $\boldsymbol{x}^\mathsf{T}$ for the $\boldsymbol{X}$ and $\boldsymbol{x}$ defined above

Displaying images with matplotlib

- plt.imshow is the main plot command
- We randomly permute the sample indices using `Iperm`
  - Else training set might include different digits than testing set
  - Do this for any dataset!
- A random sample of 4 MNIST digits:

  - A human could easily classify these

```
def plt_digit(x):
    nrow = 28
    ncol = 28
    xsq = x.reshape((nrow,ncol))
    plt.imshow(xsq,  cmap='Greys_r')
    plt.xticks([])
    plt.yticks([])

# Convert data to a matrix
X = mnist.data
y = mnist.target.astype(np.int8) # fetch_ope

# Select random digits
nplt = 4
nsamp = X.shape[0]
Iperm = np.random.permutation(nsamp)

# Plot the images using the subplot command
for i in range(nplt):
    ind = Iperm[i]
    plt.subplot(1,nplt,i+1)
    plt_digit(X[i,:])
```

Try multinomial logistic regression (MLR)

- Train and test on 10000 samples
  - More training leads to better results
  - But can be very slow
- Select a fast solver (lbfgs)
  - Even this can take minutes

```
ntr = 10000
nts = 10000
Xs = X/255.0*2 - 1 # convert to range [-1,1]
Xtr = Xs[Iperm[:ntr],:]
ytr = y[Iperm[:ntr]]
Xts = Xs[Iperm[ntr:ntr+nts],:]
yts = y[Iperm[ntr:ntr+nts]]
```

```
logreg = linear_model.LogisticRegression(verbose=1, multi_class='multinomial',
                                         solver='lbfgs', max_iter=500)
logreg.fit(Xtr,ytr)
```

```
[Parallel(n_jobs=1)]: Using backend SequentialBackend with 1 concurrent workers
/Users/schniter.1/anaconda3/lib/python3.6/site-packages/sklearn/linear_model/lo
ConvergenceWarning: lbfgs failed to converge. Increase the number of iterations
  "of iterations.", ConvergenceWarning)
[Parallel(n_jobs=1)]: Done   1 out of   1 | elapsed:    7.2s finished
```

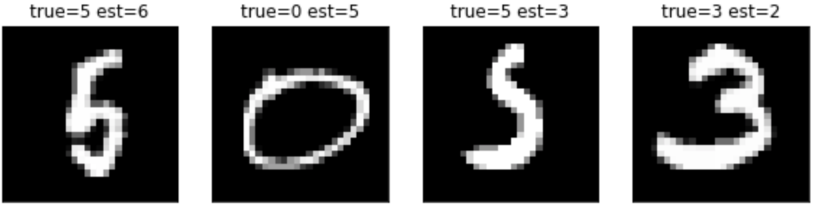
```
LogisticRegression(C=1.0, class_weight=None, dual=False, fit_intercept=True,
          intercept_scaling=1, max_iter=500, multi_class='multinomial',
          n_jobs=None, penalty='l2', random_state=None, solver='lbfgs',
          tol=0.0001, verbose=1, warm_start=False)
```

Performance of MLR

- Accuracy = 89% (not very good)
- Most of the errors MLR made would be obvious to a human, for example:

```
yhat = logreg.predict(Xts)
acc = np.mean(yhat == yts)
print('Accuracy = {0:f}'.format(acc))
```

```
Accuracy = 0.892900
```

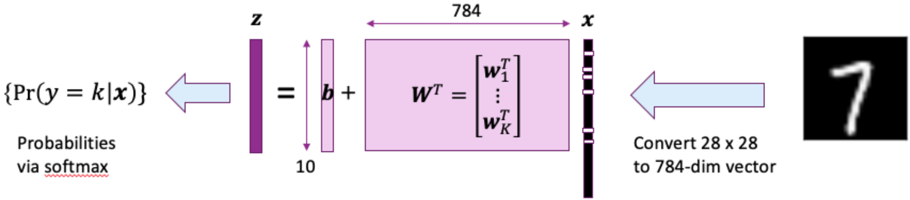


- What went wrong?
- Can we do better?

Review of MLR

- To classify a test vector $\boldsymbol{x}$ (e.g., an MNIST image), we do the following:
- For each class hypothesis $k = 1, \dots, K$,
  - design weights b_k and $\boldsymbol{w}_k = [w_{k1}, \dots, w_{kd}]^\mathsf{T}$ that classify $\boldsymbol{x}$ as in-k or not-in-k
  - compute a linear score $z_k = b_k + \boldsymbol{x}^\mathsf{T}\boldsymbol{w}_k$

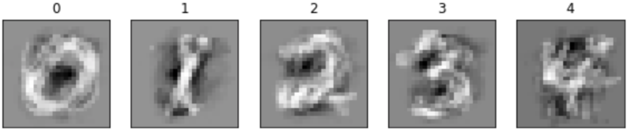
then classify as $\widehat{y} = \arg\max_k \underbrace{\Pr\{y=k \mid \boldsymbol{x}\}}_{\substack{\text{probability of}\\ \boldsymbol{x} \text{ in class } k}} = \arg\max_k \underbrace{\frac{e^{z_k}}{\sum_{l=1}^{K} e^{z_l}}}_{\text{softmax}} = \arg\max_k z_k$



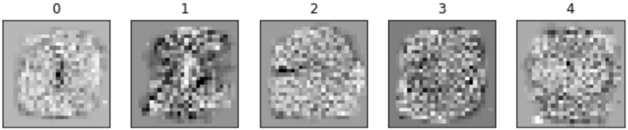
What weights does MLR learn?

- Intuitively, we want $z_k = b_k + \boldsymbol{x}^\mathsf{T}\boldsymbol{w}_k$ to be large when $\boldsymbol{x}$ is from class k, and small otherwise.
- Let's visualize the learned $\boldsymbol{w}_k$ for $k = 0, \ldots, 4$ by plotting them as images:

Learned from 100 training samples:



Learned from 10000 training samples:



- MLR struggles to match *all* the training examples from each class due to variations across that class, especially when number of training samples is large!

Outline

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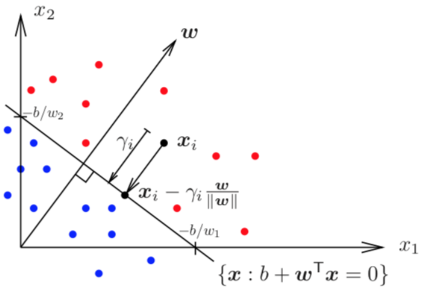
Hyperplane geometry

- In *binary* linear classification, the decision boundary is $z = 0$ or
$$\{\boldsymbol{x} : b + \boldsymbol{w}^\mathsf{T}\boldsymbol{x} = 0\}$$
which is a hyperplane
- This hyperplane is orthogonal to $\boldsymbol{w}$:
  - For any $\{\boldsymbol{x}_1, \boldsymbol{x}_2\}$ on the hyperplane, $b + \boldsymbol{w}^\mathsf{T}\boldsymbol{x}_i = 0$ for $i = 1, 2$.
  - Thus $\boldsymbol{w}^\mathsf{T}(\boldsymbol{x}_1 - \boldsymbol{x}_2) = 0$



- From the figure, the (signed) distance from a point $\boldsymbol{x}_i$ to the hyperplane is γ_i:

$$0 = b + \left(\boldsymbol{x}_i - \gamma_i \frac{\boldsymbol{w}}{\|\boldsymbol{w}\|}\right)^\mathsf{T}\boldsymbol{w} = b + \boldsymbol{x}_i^\mathsf{T}\boldsymbol{w} - \gamma_i\|\boldsymbol{w}\| \quad \Rightarrow \quad \boxed{\gamma_i = \frac{b + \boldsymbol{w}^\mathsf{T}\boldsymbol{x}_i}{\|\boldsymbol{w}\|}}$$

- The classifier $(\alpha b, \alpha\boldsymbol{w})$ is equivalent to the classifier $(b, \boldsymbol{w})$ for all $\alpha > 0$

Review of linear separability

- Assume binary training data $\{(\boldsymbol{x}_i, y_i)\}_{i=1}^n$
- The data is "linearly separable" if there exists $(b, \boldsymbol{w})$ such that

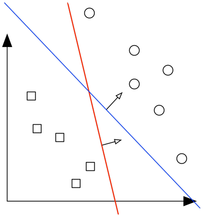
$$b + \boldsymbol{x}_i^{\mathsf{T}} \boldsymbol{w} > 1 \qquad \text{whenever } y_i = 1$$
$$b + \boldsymbol{x}_i^{\mathsf{T}} \boldsymbol{w} < -1 \qquad \text{whenever } y_i = -1$$

  or, in a more compact form,

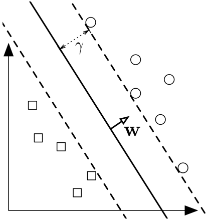
$$y_i(b + \boldsymbol{x}_i^{\mathsf{T}} \boldsymbol{w}) > 1 \quad \forall i$$

  - Note: $(b, \boldsymbol{w})$ is not unique!
  - "1" chosen for convenience; could use any value
- If separable, the distance γ_i from the boundary to datapoint $\boldsymbol{x}_i$ obeys $\boxed{y_i \gamma_i \geq 1/\|\boldsymbol{w}\| \triangleq \gamma}$, where γ is known as the "margin"

linear separability:



margin maximizing $\boldsymbol{w}$:

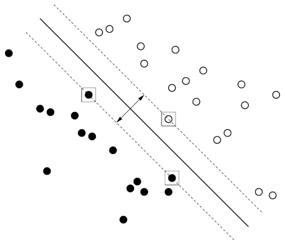


Maximum-margin classification

- To make the classifier robust, we should *maximize the margin* $1/\|\boldsymbol{w}\|$!
  - Equivalently, we should minimize $\|\boldsymbol{w}\|$
- The hard-margin classifier is defined as the $(b, \boldsymbol{w})$ that solves

$$\min \frac{1}{2}\|\boldsymbol{w}\|^2 \;\text{ s.t. }\; y_i(b + \boldsymbol{x}_i^\mathsf{T}\boldsymbol{w}) \geq 1 \;\; \forall i = 1, \dots, n$$

- The orientation of its boundary depends only on a few samples $\boldsymbol{x}_i$ near the boundary:
  - Called the support vectors
  - More on this later ...



- Problem: This classifier exists only if the data is linear separable!

Outline

The support vector classifier

- To handle data that is not linearly separable, we replace the hard constraint

$$y_i(b + \boldsymbol{x}_i^\mathsf{T}\boldsymbol{w}) \geq 1\ \forall i \quad \Leftrightarrow \quad 0 \geq 1 - y_i(b + \boldsymbol{x}_i^\mathsf{T}\boldsymbol{w})\ \forall i$$

  with the cost function $\underbrace{\sum_{i=1}^{n} \max\left\{0, 1 - y_i(b + \boldsymbol{x}_i^\mathsf{T}\boldsymbol{w})\right\}}_{=\,0 \text{ iff linearly separable}}$

- This results in the soft-margin classifier or support-vector classifier (SVC):

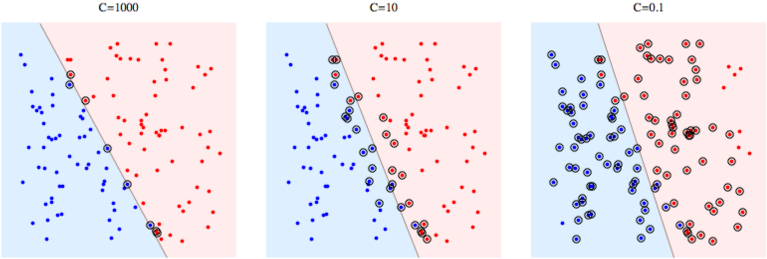
$$\min_{b,\boldsymbol{w}} \left\{ C \sum_{i=1}^{n} \max\left\{0, 1 - y_i(b + \boldsymbol{x}_i^\mathsf{T}\boldsymbol{w})\right\} + \frac{1}{2}\|\boldsymbol{w}\|^2 \right\} \text{ with tunable parameter } C > 0$$

- C trades off between margin size $1/\|\boldsymbol{w}\|$ and number of margin violations
  - Larger C penalizes violations more and thus yields a smaller margin $1/\|\boldsymbol{w}\|$
  - For linearly separable data and sufficiently large C, soft-margin $\Leftrightarrow$ hard-margin

Effect of SVC penalty C

C controls the margin $\frac{1}{\|\boldsymbol{w}\|}$, and thus the number of support vectors

- Larger C: smaller margin, fewer support vectors
  - More sensitive to datapoints close to hyperplane: Lower bias, higher variance
- Smaller C: larger margin, more support vectors
  - Less sensitive to data: Higher bias, lower variance



Support vectors are circled

The support vectors are the $\boldsymbol{x}_i$ with signed distance $< \frac{1}{\|\boldsymbol{w}\|}$ from hyperplane

The SVC and hinge loss

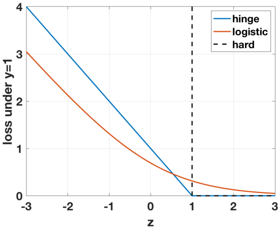
- Recalling L2-regularized logistic regression:

$$\min_{\boldsymbol{w},b}\left\{\sum_{i=1}^{n}\underbrace{\left(\ln[1+e^{z_i}]-y_i z_i\right)}_{\text{“logistic loss”}}+\frac{1}{2C}\|\boldsymbol{w}\|^2\right\} \quad \text{for} \quad z_i=b+\boldsymbol{x}_i^\mathsf{T}\boldsymbol{w}$$

we see that the SVC looks similar, but with a different loss:

$$\min_{\boldsymbol{w},b}\left\{\sum_{i=1}^{n}\underbrace{\max\left\{0,1-y_i z_i\right\}}_{\text{“hinge loss”}}+\frac{1}{2C}\|\boldsymbol{w}\|^2\right\} \quad \text{for} \quad z_i=b+\boldsymbol{x}_i^\mathsf{T}\boldsymbol{w}$$

- Comparison of loss functions:
  - hinge loss places *no penalty* on samples lying on good side of margin (i.e., $y_i z_i > 1$)
  - thus, the SVC uses only the samples $\boldsymbol{x}_i$ that cause $y_i z_i \leq 1$, called "support vectors"
  - The SVC ignores the other samples!



A closer look at the SVC

- We just saw that the SVC solution $(\boldsymbol{w}_*, b_*)$ minimizes the cost

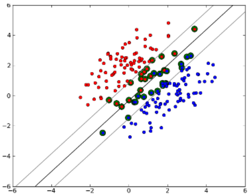
$$J_{\text{svc}}(\boldsymbol{w}, b) \triangleq C \sum_{i=1}^{n} \max\{0, 1 - y_i \underbrace{(b + \boldsymbol{w}^\mathsf{T}\boldsymbol{x}_i)}_{z_i}\} + \frac{1}{2}\|\boldsymbol{w}\|^2$$

- The gradient must equal zero at $(\boldsymbol{w}_*, b_*)$:

$$0 = \frac{\partial}{\partial b} J(\boldsymbol{w}_*, b_*) = -C \sum_{i=1}^{n} \alpha_i y_i \quad \text{for} \quad \alpha_i = \begin{cases} 1 & y_i(b_* + \boldsymbol{w}_*^\mathsf{T}\boldsymbol{x}_i) < 1 \\ ? & y_i(b_* + \boldsymbol{w}_*^\mathsf{T}\boldsymbol{x}_i) = 1 \\ 0 & y_i(b_* + \boldsymbol{w}_*^\mathsf{T}\boldsymbol{x}_i) > 1 \end{cases}$$

$$\mathbf{0} = \nabla_{\boldsymbol{w}} J(\boldsymbol{w}_*, b_*) = -C \sum_{i=1}^{n} \alpha_i y_i \boldsymbol{x}_i + \boldsymbol{w}_*$$

which implies $\boldsymbol{w}_* = C \sum_{i=1}^{n} \alpha_i y_i \boldsymbol{x}_i$ and $\sum_{i=1}^{n} \alpha_i y_i = 0$



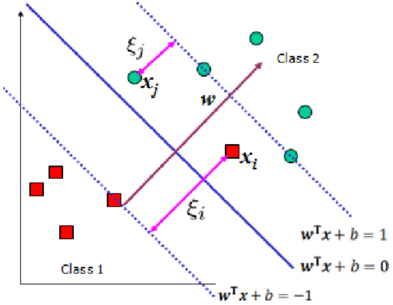
- The equation for $\boldsymbol{w}_*$ shows that it depends only on the support vectors $\{\boldsymbol{x}_i : y_i(b_* + \boldsymbol{x}_i^\mathsf{T}\boldsymbol{w}_*) \leq 1\}$

Common form of the SVC

- The SVC is often written using "slack variables" $\xi_i \triangleq \max\{0, 1 - y_i z_i\}$:

$$\min_{b,\boldsymbol{w}} \left\{ C \sum_{i=1}^{n} \xi_i + \frac{1}{2}\|\boldsymbol{w}\|^2 \right\} \text{ s.t. } y_i(b + \boldsymbol{x}_i^\mathsf{T}\boldsymbol{w}) \geq 1 - \xi_i \text{ and } \xi_i \geq 0 \text{ for all } i$$

- ξ_i measures violation of the margin
  - $\xi_i = 0$: sample outside "street" (on correct side of hyperplane)
  - $\xi_i \in (0, 2)$: sample in "street"
  - $\xi_i > 1$: sample misclassified (wrong side of hyperplane)
- Any $\xi_i > 0$ is considered a violation



Applying the SVC to test data

- Given a test vector $\boldsymbol{x}$, the SVC computes the score

$$z = b_* + \boldsymbol{w}_*^{\mathsf{T}}\boldsymbol{x} \quad \text{with} \quad \boldsymbol{w}_* = C\sum_{i=1}^{n}\alpha_i y_i \boldsymbol{x}_i \ \text{ and } \ \alpha_i = 0 \text{ for non-support } \boldsymbol{x}_i$$

- The resulting score

$$z = b_* + C\sum_{i=1}^{n}\alpha_i y_i \boldsymbol{x}_i^{\mathsf{T}}\boldsymbol{x} \quad \text{used for} \quad \widehat{y} = \operatorname{sgn}(z)$$

  is a weighted linear combination of the "correlations" $\boldsymbol{x}_i^{\mathsf{T}}\boldsymbol{x}$ between $\boldsymbol{x}$ and each support vector $\boldsymbol{x}_i$.
  - The SVC uses only $\{\boldsymbol{x}_i\}$ that are support vectors, whereas logistic regression uses *all* the training samples $\{\boldsymbol{x}_i\}$
  - The SVC tries to focus on what is most important when making a decision

Multiclass SVC

- In multiclass linear classification with K classes, we compute the score *vector*

$$\boldsymbol{z}_i = \begin{bmatrix} b_1 \\ \vdots \\ b_K \end{bmatrix} + \begin{bmatrix} \boldsymbol{w}_1^\mathsf{T} \\ \vdots \\ \boldsymbol{w}_K^\mathsf{T} \end{bmatrix} \boldsymbol{x}_i = \boldsymbol{b} + \boldsymbol{W}^\mathsf{T} \boldsymbol{x}_i \in \mathbb{R}^K$$

- One option is to *separately* design one-vs-all SVCs $\{(b_k, \boldsymbol{w}_k)\}_{k=1}^K$

- Another option is to *jointly* design the multiclass SVC $(\boldsymbol{b}, \boldsymbol{W})$ via

$$\min_{\boldsymbol{b},\boldsymbol{W}} \left\{ \frac{1}{n} \sum_{i=1}^n \sum_{j \neq y_i} \max\left\{0, 1 + z_{ij} - z_{i,y_i}\right\} + \frac{1}{2C} \|\boldsymbol{W}\|_F^2 \right\}$$

  - Note that, when $K = 2$, we get (using $j, y_i \in \{-1, 1\}$)

$$\begin{aligned} \sum_{j \neq y_i} \max\left\{0, 1 + z_{ij} - z_{i,y_i}\right\} &= \max\left\{0, 1 + z_{i,-y_i} - z_{i,y_i}\right\} \\ &= \max\left\{0, 1 - y_i(z_{i,1} - z_{i,-1})\right\} \\ &= \max\left\{0, 1 - y_i(b + \boldsymbol{w}^\mathsf{T} \boldsymbol{x}_i)\right\} \end{aligned}$$

  for $b \triangleq b_1 - b_{-1}$ and $\boldsymbol{w} \triangleq \boldsymbol{w}_1 - \boldsymbol{w}_{-1}$, which matches our earlier binary SVC

Implementing the SVC in sklearn

- The SVC is easy to implement in sklearn using svm.SVC
  - The parameter C penalizes the slack variables
  - As we discussed, larger C means fewer support vectors

```
from sklearn import svm
svc = svm.SVC(probability=False, kernel="linear", verbose=1)
svc.fit(Xtr,ytr)
yhat_ts = svc.predict(Xts)
acc = np.mean(yhat_ts == yts)
print('Accuracy = {0:f}'.format(acc))
```

```
Accuracy = 0.907100
```

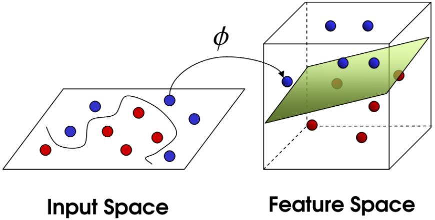
- The SVC outperforms logistic regression by about 2% on this dataset
  - good, but not a huge improvement
- For large datasets, it is recommended to use svm.LinearSVC instead

Outline

Support vector machine

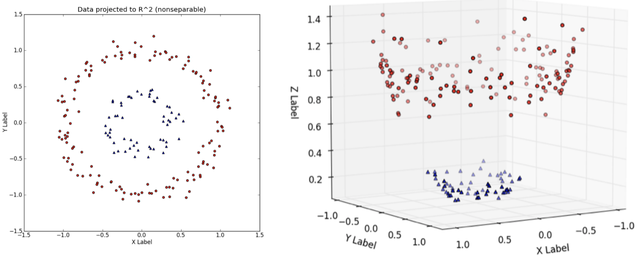
- Support vector classifier (SVC)
  - works very well when the features are roughly linearly separable
- Support vector machine (SVM)
  - SVC plus ...

1) a feature transformation $\boldsymbol{x} \mapsto \boldsymbol{\phi}(\boldsymbol{x})$ to enhance linear separability
2) the "kernel trick" for fast implementation



Feature transformations

- We want a feature transformation $\boldsymbol{x} \mapsto \boldsymbol{\phi}(\boldsymbol{x})$ that leads to linear separability
- Usually we do this by mapping to a higher-dimensional space
- Example: For the $\boldsymbol{x} = [x_1, x_2]^{\mathsf{T}}$ data below, which is not linearly separable, we can use $\boldsymbol{\phi}(\boldsymbol{x}) = [x_1, x_2, x_1^2 + x_2^2]^{\mathsf{T}}$ to get linear separability!



The problem of dimensionality

- In principle, given transformed features $\boldsymbol{\phi}(\boldsymbol{x})$, we could train via

$$(\boldsymbol{w}_*, b_*) = \min_{\boldsymbol{w},b} \left\{ C \sum_{i=1}^{n} \max\{0, 1 - y_i z_i\} + \frac{1}{2}\|\boldsymbol{w}\|^2 \right\} \text{ for } z_i = b + \boldsymbol{w}^\mathsf{T}\boldsymbol{\phi}(\boldsymbol{x}_i)$$

  and classify a test vector $\boldsymbol{x}$ via

$$\widehat{y} = \operatorname{sgn}(z) = \operatorname{sgn}\left[b_* + C \sum_{i=1}^{n} \alpha_i y_i \boldsymbol{\phi}(\boldsymbol{x}_i)^\mathsf{T}\boldsymbol{\phi}(\boldsymbol{x}) \right]$$

- But what if the dimension of $\boldsymbol{\phi}(\boldsymbol{x})$ is very high?
  - The dimensionality of $\boldsymbol{w}$ will be equally high
  - The optimization problem will be computationally expensive!
- And what if $\boldsymbol{\phi}(\boldsymbol{x})$ is infinite-dimensional?
  - The optimization problem can't be solved using the above formulation
  - This might sound crazy, but it's not unusual (more details soon . . .)

The Lagrangian dual formulation of the SVC

- Can avoid these problems by using the "Lagrangian dual" form of the SVC:

$$\boldsymbol{w}_* = \sum_{i=1}^{n} \lambda_i y_i \boldsymbol{x}_i \quad \text{and} \quad b_* = \boldsymbol{w}_*^\mathsf{T} \boldsymbol{x}_{i_*} - y_{i_*} \quad \text{for any } i_* \text{ s.t. } \lambda_{i_*} > 0$$

  where the Lagrange multipliers $\{\lambda_i\}$ solve the quadratic programming problem

$$\min_{\boldsymbol{\lambda}} \left\{ \sum_{i=1}^{n} \lambda_i - \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_i \lambda_j y_i y_j \boldsymbol{x}_i^\mathsf{T} \boldsymbol{x}_j \right\} \text{ s.t. } \lambda_i \in [0, C] \text{ and } \sum_{i=1}^{n} \lambda_i y_i = 0$$

  - The derivation is outside the scope of this course

    See Sec. 12.2 of Hastie, Tibshirani, Friedman, *The Elements of Statistical Learning, 2nd Ed.*, 2009

  - But some terms may look familiar from our gradient analysis of the SVC cost

- Note that we can avoid computing $\boldsymbol{w}_*$ by directly computing the decision

$$\widehat{y} = \operatorname{sgn}\left(b_* + \sum_{i=1}^{n} \lambda_i y_i \boldsymbol{x}_i^\mathsf{T} \boldsymbol{x} \right) \text{ where } b_* = \sum_{i=1}^{n} \lambda_i y_i \boldsymbol{x}_i^\mathsf{T} \boldsymbol{x}_{i_*} - y_{i_*}$$

The kernel trick

- Now consider the dual-form SVC with $\boldsymbol{x}$ replaced by the transformation $\boldsymbol{\phi}(\boldsymbol{x})$:

$$\widehat{y} = \operatorname{sgn}\left(b_* + \sum_{i=1}^n \lambda_i y_i \boldsymbol{\phi}(\boldsymbol{x}_i)^\mathsf{T} \boldsymbol{\phi}(\boldsymbol{x})\right) \text{ where } b_* = \sum_{i=1}^n \lambda_i y_i \boldsymbol{\phi}(\boldsymbol{x}_i)^\mathsf{T} \boldsymbol{\phi}(\boldsymbol{x}_{i_*}) - y_{i_*}$$

  where i_* is any support index, and where $\{\lambda_i\}$ solve

$$\min_{\boldsymbol{\lambda}} \left\{ \sum_{i=1}^n \lambda_i - \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \lambda_i \lambda_j y_i y_j \boldsymbol{\phi}(\boldsymbol{x}_i)^\mathsf{T} \boldsymbol{\phi}(\boldsymbol{x}_j) \right\} \text{ s.t. } \begin{cases} \lambda_i \in [0, C] \\ \sum_{i=1}^n \lambda_i y_i = 0 \end{cases}$$

- The transformed features appear only in the form of a "kernel" function

$$\kappa(\boldsymbol{x}, \boldsymbol{x}') \triangleq \boldsymbol{\phi}(\boldsymbol{x})^\mathsf{T} \boldsymbol{\phi}(\boldsymbol{x}')$$

  - Often the kernel is easier to compute than $\boldsymbol{\phi}(\boldsymbol{x})$
  - For both learning and inference, we only need to compute the kernel (not $\boldsymbol{\phi}(\boldsymbol{x})$)!
  - Called the "kernel trick"

Popular kernels

- Popular kernels include:
  - radial basis function (RBF): $\kappa(\boldsymbol{x}, \boldsymbol{x}') = \exp(-\gamma \|\boldsymbol{x} - \boldsymbol{x}'\|^2)$ for "width" $\gamma > 0$
  - polynomial: $\kappa(\boldsymbol{x}, \boldsymbol{x}') = (\gamma \boldsymbol{x}^\mathsf{T} \boldsymbol{x}' + r)^d$ where $\gamma > 0$, $r \geq 0$, and often $d = 2$
  - sigmoidal: $\kappa(\boldsymbol{x}, \boldsymbol{x}') = \tanh(\gamma \boldsymbol{x}^\mathsf{T} \boldsymbol{x}' + r)$ for some $\gamma > 0$ and $r > 0$
  - linear: $\kappa(\boldsymbol{x}, \boldsymbol{x}') = \boldsymbol{x}^\mathsf{T} \boldsymbol{x}'$ . . . corresponds to trivial transformation $\boldsymbol{\phi}(\boldsymbol{x}) = \boldsymbol{x}$
- Each kernel measures a "similarity" between $\boldsymbol{x}$ and $\boldsymbol{x}'$
- Note: The feature space of the RBF kernel is infinite dimensional!

$$\exp(-\tfrac{1}{2}\|\boldsymbol{x} - \boldsymbol{x}'\|^2) = \exp(-\tfrac{1}{2}\|\boldsymbol{x}\|^2) \exp(-\tfrac{1}{2}\|\boldsymbol{x}'\|^2) \sum_{j=0}^{\infty} \frac{(\boldsymbol{x}^\mathsf{T} \boldsymbol{x}')^j}{j!}$$

  - Can't implement the RBF kernel via $\boldsymbol{\phi}(\boldsymbol{x})$
  - But can implement RBF efficiently using the kernel trick!

Implementing the SVM in sklearn

- The SVM is easy to implement in sklearn using svm.SVC
  - Choose from several kernels: rbf (default), poly, sigmoid, linear, or custom.
  - Specify the kernel width $\gamma > 0$
  - Control number of support vectors via $C > 0$
  - Ideally optimize all parameters via cross-validation
    - See http://peekaboo-vision.blogspot.com/2010/09/mnist-for-ever.html
    - Below we use settings from https://martin-thoma.com/svm-with-sklearn/

```
from sklearn import svm
svcrbf = svm.SVC(probability=False, kernel="rbf", C=2.8, gamma=.0073, verbose=1)
svcrbf.fit(Xtr,ytr)
yhat_ts = svcrbf.predict(Xts)
acc = np.mean(yhat_ts == yts)
print('Accuracy = {0:f}'.format(acc))
```

```
Accuracy = 0.970500
```

- The SVM significantly outperforms both logistic regression and SVC!
  - Could do even better with more training samples and tweaked parameters

Learning objectives

- Gain experience with linear classification of images
  - representing images, displaying images, classifying images
- Understand the geometry of the linear classification boundary:
  - orthogonality to $\boldsymbol{w}$, offset b, margin $1/\|\boldsymbol{w}\|$
- Understand the margin-maximizing classifier
- Understand the support vector classifier (SVC)
- Understand the support vector machine (SVM)
- Understand how to implement the SVC and SVM with sklearn