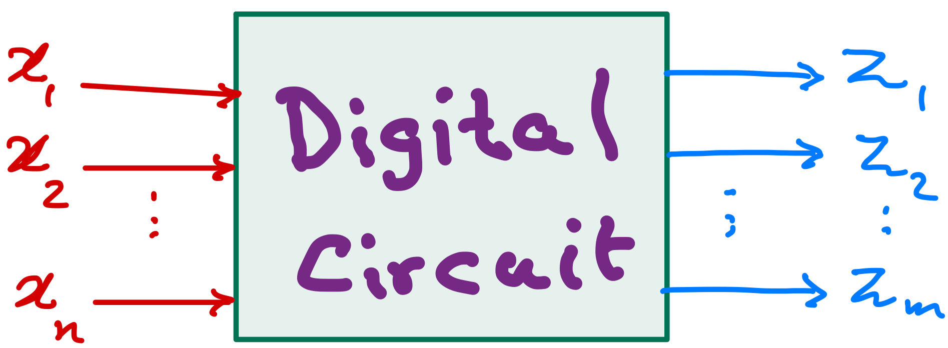
# Boolean Algebra

* Objective → Study design of digital circuits
* Digital circuits use binary system
* Signals can be either Logic '0' or '1' corresponding to High or Low voltages, resp.
* Physical value → High/Low voltage
* Logic value → 1 / 0
* Digital circuits takes binary signals and produce binary signals:




* $x_i$ is input # $i$ to the circuit, $z_i$ output # $i$
* Generally, $z_i = F(x_1, x_2, \ldots x_n)$ → binary fun. → Boolean fun.

  Requires binary logic algebra

* Binary logic algebra is commonly referred to as Boolean algebra, after G. Bool

* Boolean functions take Boolean variables and produce Boolean output

* To design digital circuits, we must know how to derive Boolean functions

* Digital circuits implementing Boolean function is commonly referred to as logic circuit

* Plan for the rest of the lesson:
  - Def of Boolean Algebra
  - Logic gates and operations
  - Basic identities and duality priciple
  - Algebraic manipulations

# Definition of Boolean Algebra

- Algebraic structure with $B = \{0, 1\}$, two binary operators $(\cdot, +)$, and one unary operator ($\bar{\ }$ or $'$), satisfying :

  * $\exists\ x, y \in B$ s.t. $x \neq y$
  * $\forall\ x \in B$ , $\bar{x} \in B$
  * closure under $\cdot$ and $+$
  * $0$ is identity for $+$,
    $1$ is identity for $\cdot$
  * Commutative under $+$ and $\cdot$
  * $\cdot$ is distributive over $+$
  * $+$ is distributive over $\cdot$

Usually + operator is OR

and · operator is AND

the ‾ unary operator is NOT

Ex1

operators

$$F = A \cdot B + 1$$

Variables Constant

Operator precedence:

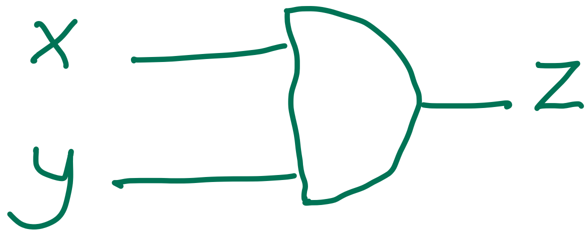
1. Parethesis (if any) {Highest priority}
2. Not operator
3. AND operator
4. OR operator {Lowest priority}

# Logic Gates & Operations

* In addition to representing logic circuits using Boolean algebra, we can represent them graphically

$\boxed{1}$ AND gate

$$Z = x \cdot y$$
$$= xy$$




Definion of AND using truth table

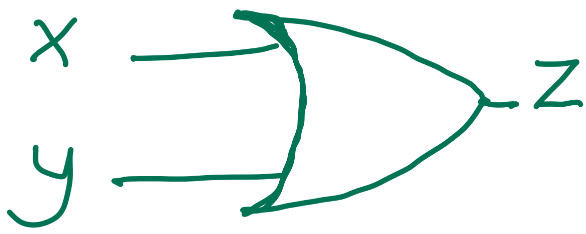
| x | y | xy |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | 0 |
| 1 | 0 | 0 |
| 1 | 1 | 1 |

Mathematically

$$xy = \begin{cases} 1 & \text{iff } x=1 \text{ and } y=1 \\ 0 & \text{otherwise} \end{cases}$$

## [2] OR gate

$$Z = x + y$$




Definion of OR using truth table

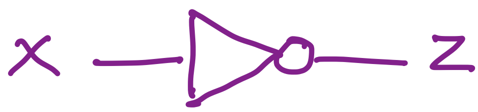
| $x$ | $y$ | $x+y$ |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | 1 |
| 1 | 0 | 1 |
| 1 | 1 | 1 |

Mathematically

$$x+y = \begin{cases} 1 & \text{iff } x=1 \text{ or } y=1 \\ 0 & \text{otherwise} \end{cases}$$

## [3] NOT gate

$$Z = \overline{X}$$




Definition of NOT gate using truth table

| $X$ | $\overline{X}$ |
|---|---|
| 0 | 1 |
| 1 | 0 |

Mathematically,

$$\overline{X} = \begin{cases} 0 & \text{iff } X=1 \\ 1 & \text{otherwise} \end{cases}$$

# Side Note

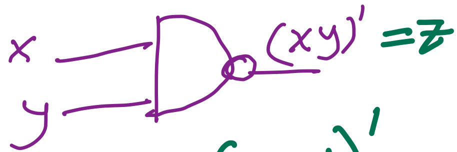
In addition to AND, OR, and NOT gates, there are other composite gates

details later

1. NAND gate

AND followed by NOT

| $x$ | $y$ | $xy$ | $(xy)'$ |
|---|---|---|---|
| 0 | 0 | 0 | 1 |
| 0 | 1 | 0 | 1 |
| 1 | 0 | 0 | 1 |
| 1 | 1 | 1 | 0 |

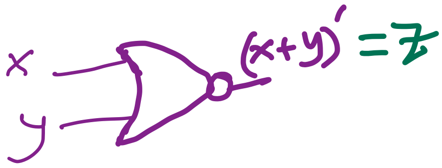



$z = (xy)'$

2. NOR gate

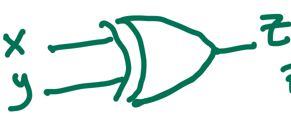
OR followed by NOT

| $x$ | $y$ | $x+y$ | $(x+y)'$ |
|---|---|---|---|
| 0 | 0 | 0 | 1 |
| 0 | 1 | 1 | 0 |
| 1 | 0 | 1 | 0 |
| 1 | 1 | 1 | 0 |




$z = (x+y)'$

3. XOR gate




$z = x \oplus y = \begin{cases} 1 & \text{iff } x \neq y \\ 0 & \text{otherwise} \end{cases}$

4. XNOR gate

$(x \oplus y)' = \begin{cases} 1 & \text{iff } x = y \\ 0 & \text{otherwise} \end{cases}$

Ex2 $F = a\bar{b} + b$

(a) Sketch the logic circuit of F.

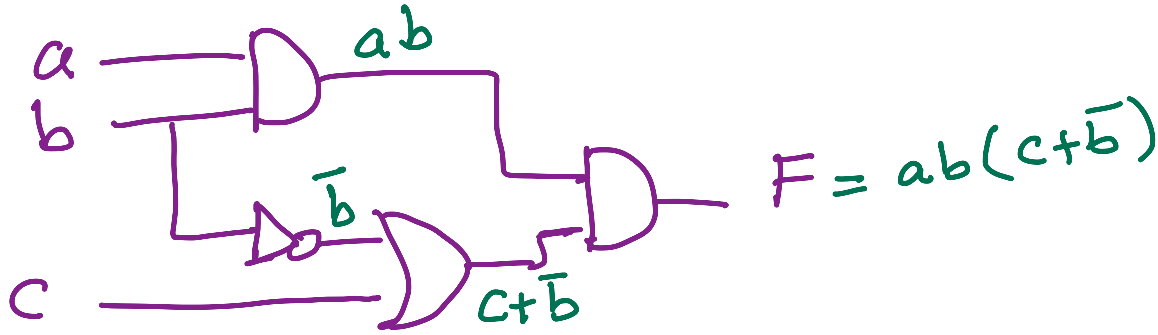

(b) Derive the truth table of F

| a | b | $\bar{b}$ | $a\bar{b}$ | F | a+b |
|---|---|---|---|---|---|
| 0 | 0 | 1 | 0 | 0 | 0 |
| 0 | 1 | 0 | 0 | 1 | 1 |
| 1 | 0 | 1 | 1 | 1 | 1 |
| 1 | 1 | 0 | 0 | 1 | 1 |

(c) What can you say about F?

$$F = a\bar{b} + b = a + b$$

Ex3




(a) Derive the Boolean expression of F.

(b) Derive the truth table of $F = ab(c+\bar{b})$

| a | b | c | $\bar{b}$ | ab | $c+\bar{b}$ | F | abc |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 0 | 1 | 0 | 0 |
| 0 | 0 | 1 | 1 | 0 | 1 | 0 | 0 |
| 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 1 | 1 | 0 | 0 | 1 | 0 | 0 |
| 1 | 0 | 0 | 1 | 0 | 1 | 0 | 0 |
| 1 | 0 | 1 | 1 | 0 | 1 | 0 | 0 |
| 1 | 1 | 0 | 0 | 1 | 0 | 0 | 0 |
| 1 | 1 | 1 | 0 | 1 | 1 | 1 | 1 |

$$\Rightarrow F = ab(c+\bar{b}) = abc$$

# Basic Identities and Duality Principle

* To obtain the dual expression of a given expression → change $\cdot$ to $+$
  change $+$ to $\cdot$
  change $1$ to $0$
  change $0$ to $1$

| Prop. Name | Expression | Dual |
|---|---|---|
| Identity | $x+0=x$ | $x\cdot 1=x$ |
| Null | $x+1=1$ | $x\cdot 0=0$ |
| Idempotence | $x+x=x$ | $x\cdot x=x$ |
| Complement | $x+\bar{x}=1$ | $x\cdot\bar{x}=0$ |
| Commutative | $x+y=y+x$ | $xy=yx$ |
| Associative | $(x+y)+z=x+(y+z)$ | $(xy)z=x(yz)$ |
| Distributive | $x(y+z)=xy+xz$ | $x+yz=(x+y)(x+z)$ |
| Absorption | $x+xy=x$ | $x\cdot(x+y)=x$ |
| Minimization | $xy+\bar{x}y=y$ | $(x+y)(\bar{x}+y)=y$ |
| Simplification | $x+\bar{x}y=x+y$ | $x\cdot(\bar{x}+y)=xy$ |
| Consensus | $xy+\bar{x}z+yz=xy+\bar{x}z$ | $(x+y)(\bar{x}+z)(y+z)=(x+y)(\bar{x}+z)$ |
| DeMorgan's | $(x+y)'=\bar{x}\cdot\bar{y}$ | $(xy)'=\bar{x}+\bar{y}$ |

# Algebraic Manipulations

* Use properties of Boolean algebra to minimize Boolean functions
* Minimization implies less hardware and less power consumption

Ex 4

Prove the Simplification Rule.

$$x + \bar{x}y = x + y$$

solution

$LHS = x + \bar{x}y$

$= x \cdot 1 + \bar{x}y$ {identity}

$= x(y+1) + \bar{x}y$ {Null}

$= xy + x \cdot 1 + \bar{x}y$ {dist.}

$= y(x+\bar{x}) + x$ {dist. + ident}

$= y \cdot 1 + x$ {Null}

$= x+y$ {comm.}

$= RHS$

## Ex5

Prove consensus theorem

$$xy + \bar{x}z + yz = xy + \bar{x}z$$

$$\begin{aligned}
LHS &= xy + \bar{x}z + yz \\
&= xy + \bar{x}z + (x+\bar{x})yz && \{\text{ident.} + \text{complem.}\} \\
&= \underline{xy} + \underline{\bar{x}z} + \underline{xyz} + \underline{\bar{x}yz} && \{\text{dist. law}\} \\
&= xy(1+z) + \bar{x}z(1+y) && \{\text{dist. law}\} \\
&= xy + \bar{x}z && \{\text{Null, ident.}\} \\
&= RHS
\end{aligned}$$

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## Ex6

$$F = a'b(c' + d'c) + b(a + a'dc)$$

Simplify F to a min. number of literals. Justify every step.

$$\begin{aligned}
F &= a'b(c' + d'c) + b(a + a'dc) \\
&= a'b(c' + d') + b(a + dc) && \{\text{simplif.}\} \\
&= b(a'(cd)' + a + cd) && \{\text{comm.} + \text{dist.} + \text{deMorg.}\} \\
&= b((cd)' + a + cd) && \{\text{simplification}\} \\
&= b && \{\text{comp.} + \text{Null}\}
\end{aligned}$$

1 literal

Ex7 $G = A\bar{B} + \bar{B}C + AB\bar{C} + AB$

Simplify to minimum number of literals.

$$G = A\bar{B} + \bar{B}C + AB\bar{C} + AB$$

$$= A(\bar{B}+B) + \bar{B}C + AB\bar{C}$$ {dist. law}

$$= A + \bar{B}C + AB\bar{C}$$ {comp. + AND identity}

$$= A(1+B\bar{C}) + \bar{B}C$$ {dist. law}

$$= A + \bar{B}C$$ {AND/OR identities}

3 literals

Ex8

find and simplify the complement of

$$F = [(A\bar{B})' + C]\bar{D} + A\bar{C}$$

to minimum number of literals.

$$\bar{F} = \{[(A\bar{B})' + C]\bar{D} + A\bar{C}\}'$$

$$= \{[(A\bar{B})' + C]\bar{D}\}' \cdot \{A\bar{C}\}'$$ {DeMorgan's}

$$= ([(A\bar{B})' + C]' + D) \cdot (\bar{A} + C)$$ ″

$$= (A\bar{B}\bar{C} + D)(\bar{A} + C)$$ ″

$$= \bar{A}(A\bar{B}\bar{C} + D) + C(A\bar{B}\bar{C} + D)$$ {dist. law}

$$= \bar{A}D + CD$$ {dist law + AND/OR id}

$$= (\bar{A} + C)D$$ {dist. law}

3 literals

Ex 9 Given $F = (A\bar{B}+C)\bar{D} + E$

without simplification, find

(a) $\bar{F}$

$$\bar{F} = [(A\bar{B}+C)\bar{D} + E]'$$

$$= [(A\bar{B}+C)\bar{D}]'\ \bar{E} \qquad \{DeMorgan's\}$$

$$= [(A\bar{B}+C)' + D]\ \bar{E}$$

$$= ((\bar{A}+B)\bar{C} + D)\ \bar{E}$$

(b) the dual of F

$$F_{dual} = ((A+\bar{B})C + \bar{D})\ E$$

Note that the dual is similar to $\bar{F}$ except for the complements.

## Ex 10

Simplify to min. literals

$$F = (A+B)(A+C)(\bar{A}+\bar{B})(A+\bar{C})$$

$$F = (A+B)(A+C)(\bar{A}+\bar{B})(A+\bar{C})$$

$$= (A + BC\bar{C})(\bar{A}+\bar{B})$$ {dist. law}

$$= A(\bar{A}+\bar{B})$$ {AND/OR identities}

$$= A\bar{B}$$ {AND/OR id. + dist. law}

2 literals

## Ex 11

Min. literals of $F = A + \bar{A}C + (A+C)(\bar{A}+\bar{C})$

$$F = A + \bar{A}C + (A+C)(\bar{A}+\bar{C})$$

$$= A + C + (A+C)(\bar{A}+\bar{C})$$ {Simp.}

$$= A + C$$ {Absorbtion}

2 literal's

## Ex 12

simplify $\quad F = AB + \bar{A}C + \bar{B}C + A\bar{C}$

$$
\begin{aligned}
F &= AB + \bar{A}C + \bar{B}C + A\bar{C} \\
&= AB + \bar{A}C + BC + \bar{B}C + A\bar{C} + A\bar{B} && \{\text{cons.}\} \\
&= C(B + \bar{B}) + A(B + \bar{B}) + \bar{A}C + A\bar{C} && \{\text{dist.}\} \\
&= C + A + \bar{A}C + A\bar{C} && \{\text{AND/OR id}\} \\
&= A + C + C + A && \{\text{comm., simp.}\} \\
&= A + C && \{\text{Idemp.}\}
\end{aligned}
$$