NOI.PH Training: Extras

Circles and Lines

Kevin Charles Atienza

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1 Circle-circle intersection

How do you find the intersection of two circles on the 2D plane?

The circle with center (x_0, y_0) and radius r_0 is the set of points (x, y) such that

$$(x - x_0)^2 + (y - y_0)^2 = r_0^2.$$

So we have two such equations and we want to find the set of common solutions. This is not a linear system, so there may be more than one solution.

Why discuss this problem? One reason I have is that I've seen so many students overcomplicate the solution by using trigonometry and stuff. While such solutions are usually perfectly fine, it gives a wrong impression that the problem is harder than it is—in fact, basic algebra is enough!

I'll present two solutions below.

1.1 Solution 1: Transformation and Simplification

We want to find the solutions to the following system:

$$\begin{aligned}(x - x_1)^2 + (y - y_1)^2 &= r_1^2 \\ (x - x_2)^2 + (y - y_2)^2 &= r_2^2.\end{aligned}$$

Actually, I'm going to write it as

$$\begin{aligned}(x - \square)^2 + (y - \square)^2 &= \square \\ (x - \square)^2 + (y - \square)^2 &= \square\end{aligned}$$

where $\square$ stands for "something we know, but I don't want to write out explicitly". This is because the particular expressions don't really matter as much as the derivation itself. You can always reconstruct the $\square$s explicitly yourself if you need to.

Anyway, the first step is to notice that we can reduce it to the special case where one of the centers is the origin—we can simply translate the system so that one center goes to the origin, and then un-translate back at the end, after getting the answer. Algebraically, this simplifies our system to:

$$\begin{aligned}x^2 + y^2 &= \square \\ (x - \square)^2 + (y - \square)^2 &= \square.\end{aligned}$$

The first equation is now simpler, which is nice. The second equation is still a bit messy though. However, notice that we have another freedom besides translation: rotation. Let's take advantage of that. For example, we could rotate until the x-coordinate of the other center becomes zero. The equations now become:

$$\begin{aligned}x^2 + y^2 &= \square \\ x^2 + (y - \square)^2 &= \square.\end{aligned}$$

Nice! The second equation is a bit simpler now, and can probably expand it now:

$$\begin{aligned}x^2 + (y - \square)^2 &= \square \\ x^2 + y^2 - 2\square y + \square^2 &= \square \\ x^2 + y^2 + \square y + \square &= \square \\ x^2 + y^2 + \square y &= \square.\end{aligned}$$

Now, we know the value of $x^2 + y^2$ from the first equation, so we can substitute:

$$\begin{aligned} x^2 + y^2 + \square y &= \square \\ \square + \square y &= \square \\ \square y &= \square \\ y &= \frac{\square}{\square} = \square. \end{aligned}$$

Assuming the coefficient of y is nonzero, we've now obtained the y-coordinate (of all solutions)! We can now substitute this into the first equation to get:

$$\begin{aligned} x^2 + y^2 &= \square \\ x^2 + \square^2 &= \square \\ x^2 &= \square. \\ x &= \pm\sqrt{\square}. \end{aligned}$$

And now, this gives us the x-coordinates of all solutions!

Note that we assumed that the coefficient of y is nonzero so that we can divide. If you actually do it explicitly, you'll see that the coefficient is nonzero if and only if the two centers are not equal. If the two centers are equal, then the system is degenerate, and there may be zero or infinitely many solutions.

Also, the value inside the square root can be positive, zero, or negative, and this corresponds to having 2, 1 or 0 solutions. This matches what happens geometrically as well, since circles can intersect in 2, 1 or 0 points (except for the degenerate case).

1.2 Solution 2: Vectors

Let's rewrite the system in terms of vectors.[1] Let $\mathbf{c}_1 = \langle x_1, y_1 \rangle$ and $\mathbf{c}_2 = \langle x_2, y_2 \rangle$ represent the centers. The distance between two points $\mathbf{p}$ and $\mathbf{q}$ can be expressed in vector language: it is the length of the vector $\mathbf{p} - \mathbf{q}$. Therefore, we can express our two equations as follows, denoting our unknown point as $\mathbf{p}$ and the length of a vector $\mathbf{v}$ as $|\mathbf{v}|$:

$$\begin{aligned} |\mathbf{p} - \mathbf{c}_1|^2 &= r_1^2 \\ |\mathbf{p} - \mathbf{c}_2|^2 &= r_2^2. \end{aligned}$$

Again, I'll just write this as

$$\begin{aligned} |\mathbf{p} - \boldsymbol{\square}|^2 &= \square \\ |\mathbf{p} - \boldsymbol{\square}|^2 &= \square \end{aligned}$$

where "$\boldsymbol{\square}$" is like $\square$ except it's a vector.

Furthermore, $|\mathbf{v}|^2$ is just $\mathbf{v} \cdot \mathbf{v}$ (the dot product), and the dot product is distributive, so we can simplify each equation above as

$$\begin{aligned} |\mathbf{p} - \boldsymbol{\square}|^2 &= \square \\ (\mathbf{p} - \boldsymbol{\square}) \cdot (\mathbf{p} - \boldsymbol{\square}) &= \square \\ \mathbf{p} \cdot \mathbf{p} - 2\mathbf{p} \cdot \boldsymbol{\square} + \boldsymbol{\square} \cdot \boldsymbol{\square} &= \square \\ \mathbf{p} \cdot \mathbf{p} + \mathbf{p} \cdot \boldsymbol{\square} + \square &= \square \\ \mathbf{p} \cdot \mathbf{p} + \mathbf{p} \cdot \boldsymbol{\square} &= \square. \end{aligned}$$

[1] We will treat "points" and "vectors" as mostly the same. See https://math.ucr.edu/home/baez/torsors.html for a "proper" way to treat these.

Our system now looks like:

$$\begin{aligned}
\mathbf{p} \cdot \mathbf{p} + \mathbf{p} \cdot \text{❑} &= \square \\
\mathbf{p} \cdot \mathbf{p} + \mathbf{p} \cdot \text{❑} &= \square.
\end{aligned}$$

The first term is a bit annoying since it's "quadratic". However, we can remove it by subtracting the first equation from the second, giving us

$$\begin{aligned}
\mathbf{p} \cdot \text{❑} - \mathbf{p} \cdot \text{❑} &= \square - \square \\
\mathbf{p} \cdot (\text{❑} - \text{❑}) &= \square \\
\mathbf{p} \cdot \text{❑} &= \square.
\end{aligned}$$

At this point, you've probably noticed the similarity of this solution so far with the previous one, so maybe the next step is the same. However, unlike before, we seem to be stuck here since we can't simply "divide" from a dot product. We need to find a substitute for the "divide" step.

Now, notice that, geometrically, the equation "$\mathbf{p} \cdot \text{❑} = \square$" is just a line, and so we expect the solution set to be of the form $\{\text{❑} + t\text{❑} \mid t \in \mathbb{R}\}$. Let's assume we can express our equation this way. Using this, our equation simply becomes

$$\mathbf{p} = \text{❑} + t\text{❑}.$$

This is now of the form that we can substitute! We have effectively "divided" somehow. So, substituting to the remaining equation, we get:

$$\begin{aligned}
\mathbf{p} \cdot \mathbf{p} + \mathbf{p} \cdot \text{❑} &= \square \\
(\text{❑} + t\text{❑}) \cdot (\text{❑} + t\text{❑}) + (\text{❑} + t\text{❑}) \cdot \text{❑} &= \square \\
\text{❑} \cdot \text{❑} + 2t\text{❑} \cdot \text{❑} + t^2\text{❑} \cdot \text{❑} + \text{❑} \cdot \text{❑} + t\text{❑} \cdot \text{❑} &= \square \\
\square + 2t\square + t^2\square + \square + t\square &= \square \\
\square t^2 + \square t + \square &= 0.
\end{aligned}$$

Finally, we get a quadratic equation which we can solve to find all solutions to the original system! And again, notice that there may be 2, 1 or 0 solutions.

The similarity of this solution to the previous one is pretty cool. Both solutions essentially go as "circle-circle system" $\to$ "circle-line system" $\to$ "substitute the line equation to the circle equation" $\to$ "solve a quadratic".

2 Lines on a plane

However, there remains the question of how to convert $\mathbf{p} \cdot$ ❑ $=$ □ into the form $\mathbf{p} =$ ❑ $+\, t$❑. That is an interesting subproblem that deserves its own section.

Let's make the "□" explicit and use variables, so our equation is $\mathbf{p} \cdot \mathbf{v} = c$ where $\mathbf{p}$ is the only unknown.

Geometrically, this equation is just a line, so we know there should be infinitely many solutions. But how do we find even one? Well, notice that a line intersects any other line (except if it is parallel), so maybe we can just introduce a "random" line and intersect it with the given line. Let's use a random line through $\mathbf{q}$ pointing at direction $\mathbf{w}$, so the set of points on it is simply $\{\mathbf{q} + t\mathbf{w} \mid t \in \mathbb{R}\}$. Thus, we want to solve the system:

$$\begin{aligned} \mathbf{p} \cdot \mathbf{v} &= c \\ \mathbf{p} &= \mathbf{q} + t\mathbf{w} \end{aligned}$$

and our unknowns are $\mathbf{p}$ and t. Substituting the second into the first, we get:

$$\begin{aligned} (\mathbf{q} + t\mathbf{w}) \cdot \mathbf{v} &= c \\ \mathbf{q} \cdot \mathbf{v} + t\mathbf{w} \cdot \mathbf{v} &= c \\ t\mathbf{w} \cdot \mathbf{v} &= c - \mathbf{q} \cdot \mathbf{v} \\ t &= \frac{c - \mathbf{q} \cdot \mathbf{v}}{\mathbf{w} \cdot \mathbf{v}}. \end{aligned}$$

We now have a t, which means we can solve for $\mathbf{p} = \mathbf{q} + t\mathbf{w}$, which is the point of intersection of the two lines. What's more, we have complete freedom to choose $\mathbf{q}$ and $\mathbf{w}$, and that will always give us a new point on the line! Well, almost—this only works if "we can divide", that is, $\mathbf{w} \cdot \mathbf{v}$ is not zero, that is, $\mathbf{w}$ is not perpendicular to $\mathbf{v}$. So we should only choose such $\mathbf{w}$, but that's okay since there's a lot of them. Anyway, the point is that we now have lots of solutions.

Using the choice $\mathbf{q} = \mathbf{0}$ (the origin) and $\mathbf{w} = \mathbf{v}$ (which is clearly not perpendicular to $\mathbf{v}$ unless $\mathbf{v} = \mathbf{0}$), we get the solution

$$\begin{aligned} t &= \frac{c - \mathbf{0} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} = \frac{c}{\mathbf{v} \cdot \mathbf{v}} \\ \mathbf{p} &= \mathbf{0} + t\mathbf{v} = \frac{c\mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}. \end{aligned}$$

We now have a point on the line, but what about the others? Letting $\mathbf{p}'$ be another solution, we now have two equations:

$$\begin{aligned} \mathbf{p} \cdot \mathbf{v} &= c \\ \mathbf{p}' \cdot \mathbf{v} &= c. \end{aligned}$$

Subtracting the first from the second, we get

$$\begin{aligned} \mathbf{p}' \cdot \mathbf{v} - \mathbf{p} \cdot \mathbf{v} &= c - c \\ (\mathbf{p}' - \mathbf{p}) \cdot \mathbf{v} &= 0 \end{aligned}$$

which says that $\mathbf{p}' - \mathbf{p}$ must be a direction perpendicular to $\mathbf{v}$. However, since we're on a plane, there's only one such direction, namely its 90-degree rotation. Denoting that direction by $\mathbf{v}^\perp$, we must have $\mathbf{p}' - \mathbf{p} = t\mathbf{v}^\perp$ for some $t \in \mathbb{R}$. In other words, any other solution $\mathbf{p}'$ must be in the line $\mathbf{p} + t\mathbf{v}^\perp$. We have now found our solution set! Specifically, we've shown that the line determined by the equation $\mathbf{p} \cdot \mathbf{v} = c$ is equivalent to the line $\{\mathbf{p} + t\mathbf{v}^\perp \mid t \in \mathbb{R}\}$, where $\mathbf{p}$ is any fixed solution.[2] Furthermore, we've found an explicit such $\mathbf{p}$ above.

[2]Strictly speaking, we've only shown that the solution is a subset, but it's easy to check that if $\mathbf{p}$ is a solution to $\mathbf{p} \cdot \mathbf{v} = c$, then $\mathbf{p} + t\mathbf{v}^\perp$ is also a solution. The only thing you need is that $\mathbf{v}^\perp \cdot \mathbf{v} = 0$.

3 Line-line intersection

We now describe how to find the intersection of two lines on a plane. I know it's strange to describe line-line intersection after circle-circle intersection, but please bear with me :P

Of course, using high-school algebra, this just amounts to solving a system of two linear equations in two variables, and you probably already know how to do that. However, the point of this section is that we want to do it in "vector terms". Let's say the two lines that we have are $\{\mathbf{p}_1 + t\mathbf{v}_1 \mid t \in \mathbb{R}\}$ and $\{\mathbf{p}_2 + t'\mathbf{v}_2 \mid t' \in \mathbb{R}\}$, so we want to solve the system

$$\begin{aligned}\mathbf{p} &= \mathbf{p}_1 + t\mathbf{v}_1\\ \mathbf{p} &= \mathbf{p}_2 + t'\mathbf{v}_2.\end{aligned}$$

In other words, we want to find two real numbers t and t' such that

$$\mathbf{p}_1 + t\mathbf{v}_1 = \mathbf{p}_2 + t'\mathbf{v}_2.$$

However, we seem to be stuck; there doesn't seem to be anything we can do here algebraically.

But that can't be right, can it? It shouldn't be that hard to find the intersection of two lines, right? In fact, we just did it in the last section! The trick is to express one of the equations as $\mathbf{p} \cdot ❑ = \square$ (rather than $\mathbf{p} = ❑ + t❑$).

So let's do that. Let's try to convert $\mathbf{p} = \mathbf{p}_1 + t\mathbf{v}_1$ into the form $\mathbf{p} \cdot \mathbf{w}_1 = c_1$. From our previous solution, we know that $\mathbf{v}_1$ and $\mathbf{w}_1$ must be perpendicular, so we must have $\mathbf{w}_1 = \mathbf{v}_1^\perp$, so we just need to find c_1. To do so, we need to find at least one point $\mathbf{p}$ on the line, and we can get c_1 as just $\mathbf{p} \cdot \mathbf{w}_1$. But we know one such point, namely $\mathbf{p}_1$. Therefore, we must have $c_1 = \mathbf{p}_1 \cdot \mathbf{w}_1$, and our system now becomes

$$\begin{aligned}\mathbf{p} \cdot \mathbf{w}_1 &= c_1\\ \mathbf{p} &= \mathbf{p}_2 + t'\mathbf{v}_2.\end{aligned}$$

By substituting the second to the first, we get a linear equation in t', which means we solve the problem just like before. It's nice to know that we can still find the intersection of two lines. Whew!

What went wrong originally? Let's look at our original equation

$$\mathbf{p}_1 + t\mathbf{v}_1 = \mathbf{p}_2 + t'\mathbf{v}_2.$$

Thinking about it, there's something unusual about this equation. For example, it's one equation, but there are two unknowns! Now, of course, this "2D vector equation" is really two equations under the hood, so it doesn't seem strange at all to expect a unique solution. However, this does tell us that this equation is only solvable in 2D (in general). And in retrospect, this makes sense; after all, in more than two dimensions, two random lines don't generally intersect even if they're not "parallel". For example, they may not even be coplanar!

In contrast, consider the form that worked for us:

$$\begin{aligned}\mathbf{p} \cdot \mathbf{w}_1 &= c_1\\ \mathbf{p} &= \mathbf{p}_2 + t'\mathbf{v}_2.\end{aligned}$$

In our case, we determined $\mathbf{w}_1$ to be $\mathbf{v}_1^\perp$ by appealing to the fact that we're in 2D, but after that, the rest of our solution doesn't use the dimension at all. In other words, this system is solvable in any dimension! (Well, except in the degenerate case $\mathbf{w}_1 \cdot \mathbf{v}_2 = 0$.)

What's going on here is that in n dimensions, the equation $\mathbf{p} = \mathbf{p}_2 + t'\mathbf{v}_2$ determines a line, i.e., a subspace[3] of dimension 1, while the equation $\mathbf{p} \cdot \mathbf{w}_1 = c_1$ determines a hyperplane of $(n-1)$ dimensions, or in modern terms, a subspace of "codimension 1". (The codimension is just n minus the dimension.) For example, in 3D, the first equation is a line, and the second equation is a plane. And the fact that the system is solvable in any dimension is essentially due to the fact that $1 + (n-1) = n$.

[3]throughout this article, by "subspace", we'll usually mean an "affine subspace".

4 Subspaces and codimensions

Okay, we have these dimensionality facts, but so what?

Well, here's what. Let's expand the equation

$$\mathbf{p} \cdot \mathbf{w} = c$$

explicitly. Let $\mathbf{p} = \langle x_1, x_2, \dots, x_n \rangle$ and $\mathbf{w} = \langle a_1, a_2, \dots, a_n \rangle$. Then the equation becomes

$$a_1 x_1 + a_2 x_2 + \dots + a_n x_n = c,$$

which is just a linear equation. Geometrically, we know this determines an $(n-1)$-dimensional hyperplane, i.e., a subspace of codimension 1.

Now, consider two such equations, and suppose we want to find the solutions to this system. Geometrically, we imagine intersecting two $(n-1)$-dimensional hyperplanes. Let's look at a couple of examples:

- In 3D, the intersection of two 2D planes is usually a line, i.e., 1D.
- In 2D, the intersection of two 1D lines is usually a point, i.e., 0D.
- In 1D, the intersection of two 0D points is usually empty.

So, guessing from this (admittedly limited) data set, we expect that in n dimensions, the interesection of two $(n-1)$-dimensional subspaces is $(n-2)$-dimensional.

We don't know yet if this is true, but let's run with it for a moment, and now consider the case of three equations, i.e., three $(n-1)$-dimensional hyperplanes. The intersection of two of them is $(n-2)$-dimensional (or at least we guess so). So now, what's the interesection of *this* with the third hyperplane? In other words, what is the intersection of an $(n-2)$-dimensional subspace with an $(n-1)$-dimensional subspace? Let's look at small cases again.

- In 3D, the intersection of a 1D line and a 2D plane is usually a 0D point.
- In 2D, the intersection of a 0D point and a 1D line is usually empty.

So, guessing from this (even more limited) data set, maybe it's $(n-3)$-dimensional!? In fact, let's go ahead and make the most general guess:

Guess 4.1. The intersection of two generic subspaces of dimensions $(n - e_1)$ and $(n - e_2)$

- has dimension $(n - e_1 - e_2)$ if $e_1 + e_2 \leq n$.
- is empty if $e_1 + e_2 > n$.

This is a bit cleaner to state in terms of "codimension":

Guess 4.2. The intersection of two generic subspaces of codimensions e_1 and e_2

- has codimension $e_1 + e_2$ if $e_1 + e_2 \leq n$.
- is empty if $e_1 + e_2 > n$.

It turns out that this is true! This is what we expect anyway, since we're dealing with "linear" things and we already expect linear things to be "nice". We won't prove it here, although it will simply become intuitively clear as we proceed with our discussion. But intuitively, the main idea is that an equation such as $\mathbf{p} \cdot \mathbf{w} = c$ represents a "one-dimensional constraint", i.e., a decrease of one degree of freedom, so e such equations represents a decrease of e degrees of freedom. And since we had n degrees of freedom at the beginning (since we're at n dimensions), we are left with $n - e$ degrees, i.e., an $(n - e)$-dimensional subspace of possibilities, a.k.a., "codimension e".

Furthermore, a subspace of codimension e corresponds to the set of solutions of a system of e equations (of the form $\mathbf{p} \cdot \mathbf{w} = c$), and the intersection of subspaces with codimensions e_1 and e_2 corresponds to just collecting all corresponding $e_1 + e_2$ equations together as a single system, so it makes sense for the intersection to have codimension $e_1 + e_2$. So for now, you can intuitively think of "e equations" and "codimension e" as roughly the same.

Anyway, this tells us that codimensions are nice and "additive" with respect to intersection. But what about regular *dimensions*? In math, we expect things named "something" and "co-something" to be highly analogous, and so we expect that there must be also be something "additive" about dimensions. And indeed, there is.

Consider a point $\mathbf{q}$ and d direction vectors $\mathbf{v}_1, \dots, \mathbf{v}_d$ (which we assume are independent[4]). Then the set

$$\{\mathbf{q} + t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \dots + t_d\mathbf{v}_d \mid t_1, \dots, t_d \in \mathbb{R}\}$$

clearly represents a subspace of dimension d. Let's denote this subspace by[5]

$$\mathbf{q} + \langle \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_d \rangle.$$

Our analog of Guess 4.2 is simply the following easy theorem:

Theorem 4.1. The "combined space" of two generic subspaces of dimensions d_1 and d_2 containing a common point $\mathbf{q}$

- has dimension $d_1 + d_2$ and has trivial intersection if $d_1 + d_2 \le n$.
- is the whole space and has nontrivial intersection if $d_1 + d_2 > n$.

By "trivial intersection", we mean that it is just $\{\mathbf{q}\}$.

Let's denote the combined subspace of A and B by $A \vee B$. Here, by the "combined space", we don't mean their union, since that might not be a subspace. For example, the union of two lines is not a plane. Rather, we define $A \vee B$ to the *smallest* subspace containing both A and B. Explicitly, if

$$\begin{aligned} A &= \mathbf{q} + \langle \mathbf{v}_1, \dots, \mathbf{v}_{d_1} \rangle \\ B &= \mathbf{q} + \langle \mathbf{w}_1, \dots, \mathbf{w}_{d_2} \rangle, \end{aligned}$$

then the combined space of A and B is just

$$\mathbf{q} + \langle \mathbf{v}_1, \dots, \mathbf{v}_{d_1}, \mathbf{w}_1, \dots, \mathbf{w}_{d_2} \rangle.$$

Alternatively, the combined space can be described as $\{\mathbf{a} + \mathbf{b} - \mathbf{q} \mid \mathbf{a} \in A, \mathbf{b} \in B\}$.[6]

[4] throughout this article, by "independent" we'll usually mean "linearly independent."

[5] a.k.a. coset notation

[6] This can also be denoted as $A + B - \mathbf{q}$.

4.1 Dimension and codimension

Using this, we can now see why the fact that "$1 + (n - 1) = n$" allowed us to solve our previous system in any number of dimensions: we were intersecting two subspaces of dimensions 1 and $n - 1$, hence of codimensions $n - 1$ and 1, and so the intersection must have codimension $(n-1)+1 = n$, i.e., dimension 0, i.e., a point. Whereas when we were solving $\mathbf{p}_1+t\mathbf{v}_1 = \mathbf{p}_2+t'\mathbf{v}_2$, we were intersecting two spaces of dimension 1, hence of codimensions $n-1$, so the intersection has codimension $2n - 2$, or dimension $n - (2n - 2) = 2 - n$, which is ≥ 0 iff $n \leq 2$. In other words, we expect no solution in general for more than two dimensions.

Rephrasing the above, we see that the intersection of a subspace of dimension 1 and a subspace of codimension 1 has dimension 0, i.e., a unique point (generically). More generally, we expect the intersection of a subspace of dimension k and a subspace of codimension k to be unique as well. Let's take a look at the case $k = 2$. Our subspace of dimension 2 looks like

$$\mathbf{q} + \langle \mathbf{v}_1, \mathbf{v}_2 \rangle = \{\mathbf{q} + t_1\mathbf{v}_1 + t_2\mathbf{v}_2 \mid t_1, t_2 \in \mathbb{R}\}$$

and our subspace of codimension 2 looks like the intersection of two subspaces of codimension 1, say, given by 2 equations

$$\begin{aligned} \mathbf{p} \cdot \mathbf{w}_1 &= c_1 \\ \mathbf{p} \cdot \mathbf{w}_2 &= c_2. \end{aligned}$$

(Remember to think of "k equations" and "codimension k" as roughly the same.)

Now, let's solve this system. We want a $\mathbf{p}$ satisfying the two equations above and also

$$\mathbf{p} = \mathbf{q} + t_1\mathbf{v}_1 + t_2\mathbf{v}_2.$$

Substituting this to the first equation, we get

$$\begin{aligned} (\mathbf{q} + t_1\mathbf{v}_1 + t_2\mathbf{v}_2) \cdot \mathbf{w}_1 &= c_1 \\ \mathbf{q} \cdot \mathbf{w}_1 + t_1\mathbf{v}_1 \cdot \mathbf{w}_1 + t_2\mathbf{v}_2 \cdot \mathbf{w}_1 &= c_1 \\ \text{❑} \cdot \text{❑} + t_1\mathbf{v}_1 \cdot \mathbf{w}_1 + t_2\mathbf{v}_2 \cdot \mathbf{w}_1 &= \square \\ \square + t_1\mathbf{v}_1 \cdot \mathbf{w}_1 + t_2\mathbf{v}_2 \cdot \mathbf{w}_1 &= \square \\ t_1(\mathbf{v}_1 \cdot \mathbf{w}_1) + t_2(\mathbf{v}_2 \cdot \mathbf{w}_1) &= \square \end{aligned}$$

where I've become lazy again and used ❑ and $\square$. Similarly, from the second equation, we get

$$t_1(\mathbf{v}_1 \cdot \mathbf{w}_2) + t_2(\mathbf{v}_2 \cdot \mathbf{w}_2) = \square.$$

So now, we have a system of two linear equations. In general, we expect this system to be nondegenerate, which will give us a unique point of intersection, as expected!

Similarly, intersecting a subspace of dimension k and a subspace of codimension k leaves us with a system of k equations in k unknowns, which has a unique solution in general.

5 Intersection of subspaces

Let's now tackle the most general problem along these lines: given a bunch of subspaces, find their intersection. The intersection is not necessarily a point—it may be a subspace of positive dimension. Of course, in general, we expect our addition rule for codimension to hold: we expect the intersection to have a codimension equal to the sum of the codimensions of the given subspaces (except for degenerate cases).

But first, how are the subspaces represented? From the above, we actually have two different kinds of representations. First, we have the "dimension"-based version: given a point $\mathbf{q}$ and d vectors $\mathbf{v}_1, \ldots, \mathbf{v}_d$, we have the following subspace of dimension d (assuming the $\mathbf{v}_i$s are independent):

$$\{\mathbf{q} + t_1\mathbf{v}_1 + \ldots + t_d\mathbf{v}_d \mid t_1, \ldots, t_d \in \mathbb{R}\},$$

which we originally denoted by $\mathbf{q} + \langle \mathbf{v}_1, \ldots, \mathbf{v}_d \rangle$. But for this section, we'll denote it by a different notation: $P_{\mathbf{q},\{\mathbf{v}_1,\ldots,\mathbf{v}_d\}}$. Notice that if V_1 and V_2 are sets of vectors, then

$$P_{\mathbf{q},V_1} \vee P_{\mathbf{q},V_2} = P_{\mathbf{q},V_1 \cup V_2}.$$

Second, we have the "codimension"-based version: given e pairs $\{(\mathbf{w}_1, c_1), \ldots, (\mathbf{w}_e, c_e)\}$, we have a subspace of codimension e (assuming the $\mathbf{w}_i$s are independent) determined by the set of common solutions $\mathbf{p}$ to the system

$$\begin{aligned}
\mathbf{p} \cdot \mathbf{w}_1 &= c_1 \\
\mathbf{p} \cdot \mathbf{w}_2 &= c_2 \\
\ldots &= \ldots \\
\mathbf{p} \cdot \mathbf{w}_e &= c_e.
\end{aligned}$$

Let's denote this subspace by $N_{\{(\mathbf{w}_1,c_1),\ldots,(\mathbf{w}_e,c_e)\}}$. Notice that if W_1 and W_2 are sets of pairs as above, then

$$N_{W_1} \cap N_{W_2} = N_{W_1 \cup W_2}.$$

So which representation do our inputs come in? Well, since we're solving the "most general" problem, we assume that the inputs may be given in either form. This also means that we ought to be able to convert from one form to another, e.g., from P_* to N_* and vice versa. We'll discuss that shortly.

There's also a related question: which representation do we use for our output? Well, if you look at both forms, then you'll probably agree that the P_* form is "better", since it actually gives us a handle on the points in the set. For example, we know at least one point from the set $P_{\mathbf{q},\{\mathbf{v}_1,\ldots,\mathbf{v}_d\}}$, namely $\mathbf{q}$, and we can get everything else by independently choosing d real parameters $t_1, \ldots, t_d$ and computing $\mathbf{q} + t_1\mathbf{v}_1 + \ldots + t_d\mathbf{v}_d$. In other words, there's a simple bijection between this set and $\mathbb{R}^d$ (assuming the $\mathbf{v}_i$'s are independent). On the other hand, from a set like $N_{\{(\mathbf{w}_1,c_1),\ldots,(\mathbf{w}_e,c_e)\}}$, we can't even find a single point easily. Indeed, finding a point $\mathbf{p}$ corresponds to solving a system of e linear equations, and that involves at least a bit more linear algebra. Thus, we'll consider the P_* form to be the "solved form". (Of course, the N_* form has the advantage that it's straightforward to check whether a given point is in it, whereas it is hard for the P_* form, for which it requires linear algebra.)

We can now state the problem a bit more precisely:

Problem 5.1. Given sets $\{P_1, P_2, \ldots, P_k, N_1, N_2, \ldots, N_m\}$ where each P_i is in P_* form and each N_i is in N_* form, find their intersection in P_* form.

5.1 Converting between representations

Let's now discuss how to convert between the P_* and N_* forms. Recall that a P_* set of dimension d should be equivalent to an N_* set of codimension $n-d$.

First, it's actually easy to convert P_* to N_* (which is to be expected, since P_* is the "nice" form), so let's discuss that first. From a set $P_{\mathbf{q},\{\mathbf{v}_1,\ldots,\mathbf{v}_d\}}$ with the $\mathbf{v}_i$s *independent*, we want to find $n-d$ pairs $(\mathbf{w}_1, c_1), \ldots, (\mathbf{w}_{n-d}, c_{n-d})$ such that the system of equations $\mathbf{p} \cdot \mathbf{w}_i = c_i$, $i = 1, \ldots, n-d$ is satisfied by the points of P and nothing else. Of course, there are choices like $\mathbf{w}_i = \mathbf{0}$ and $c_i = 0$ which we don't want, so we want the $\mathbf{w}_i$s to be *independent* as well.

Anyway, the points of $P_{\mathbf{q},\{\mathbf{v}_1,\ldots,\mathbf{v}_d\}}$ must satisfy the equations, so in particular, $\mathbf{q}$ must satisfy them, so we must have $c_i = \mathbf{q} \cdot \mathbf{w}_i$. So once we determine the $\mathbf{w}_i$s, we get the c_is for free. However, all the other points are of the form $\mathbf{q} + t_1\mathbf{v}_1 + \ldots + t_d\mathbf{v}_d$, so by substituting it to the equation $\mathbf{p} \cdot \mathbf{w}_i = c_i$, we get

$$\begin{aligned}
(\mathbf{q} + t_1\mathbf{v}_1 + \ldots + t_d\mathbf{v}_d) \cdot \mathbf{w}_i &= c_i \\
\mathbf{q} \cdot \mathbf{w}_i + t_1\mathbf{v}_1 \cdot \mathbf{w}_i + \ldots + t_d\mathbf{v}_d \cdot \mathbf{w}_i &= c_i \\
c_i + t_1\mathbf{v}_1 \cdot \mathbf{w}_i + \ldots + t_d\mathbf{v}_d \cdot \mathbf{w}_i &= c_i \\
t_1\mathbf{v}_1 \cdot \mathbf{w}_i + \ldots + t_d\mathbf{v}_d \cdot \mathbf{w}_i &= 0.
\end{aligned}$$

Now, note that this equation must be true for **all** choices of the t_js. Therefore, each of the coefficients must be zero, for if, say, the coefficient of t_j is nonzero, then we can set $t_j = 1$ and all other t_k's to zero, and the equation won't be true anymore. This means that we must have $\mathbf{v}_j \cdot \mathbf{w}_i = 0$ for all i and j, i.e., all the $\mathbf{w}_i$s are in the orthogonal complement of $\langle \mathbf{v}_1, \ldots, \mathbf{v}_d \rangle$. But this also means that the whole collection $\{\mathbf{v}_1, \ldots, \mathbf{v}_d, \mathbf{w}_1, \ldots, \mathbf{w}_{n-d}\}$ must actually be a *basis* of the whole space, because the $\mathbf{w}_i$s are independent, the $\mathbf{v}_j$s are independent, and the intersection of a subspace and its orthogonal complement is just $\{\mathbf{0}\}$. (See exercise.) Therefore, we only need to extend $\mathbf{v}_1, \ldots, \mathbf{v}_d$ into a complete basis, with the additional vectors in the orthogonal complement. And there's a standard way to do this! (See Appendix A.) The resulting $n-d$ vectors will be our $\mathbf{w}_i$s, so we're now able to convert the set from P_*-form to N_*-form!

Exercise 5.1. Show that the only vector that is both in a subspace and its orthogonal complement is $\mathbf{0}$.

Exercise 5.2. Show that *any* selection of $\mathbf{w}_i$s that span the orthogonal complement of the $\mathbf{v}_j$s works.

Now, converting from N_* to P_* is much harder. First, the P_* form requires a point $\mathbf{q}$ that is in the given set N_*, i.e., that satisfies all e equations $\mathbf{p} \cdot \mathbf{w}_i = c_i$. But this is basically a system of linear equations, so we'll definitely need a bit more linear algebra just to find such a point $\mathbf{q}$. It turns out that it's much easier to just solve the general problem (Problem 5.1) anyway. Then converting a set N from the form N_* to the form P_* is just a special case of the problem on the input set $\{N\}$!

5.2 Solving the problem

Now, we tackle Problem 5.1. We have sets $\{P_1, P_2, \ldots, N_1, N_2, \ldots\}$ in P_* and N_* form, and we want to find their intersection.

Let's first look at the P_is. Actually, we don't have a good way (yet) to intersect two sets of the form P_*. So let's do the lazy thing and just convert all of them to N_* form using our algorithm above :P

Actually, since we want our output to be in P_* form, we should probably keep one of the P_is around, i.e., convert *all but one* of the P_is. Thus, we have simplified the problem into a form with *exactly* one set of the form P_*. (If we don't have a P_* set initially, we can use the set $P_{\mathbf{0},\{\mathbf{e}_1,\dots,\mathbf{e}_n\}}$, where $\mathbf{e}_1, \dots, \mathbf{e}_n$ is your favorite basis of the whole space. This set represents the whole space.)

Next, let's look at the N_is. Remember that $N_{W_1 \cup W_2} = N_{W_1} \cap N_{W_2}$, so we can further reduce to the case where each N_i is of the form $N_{\{(\mathbf{w},c)\}}$ for a *single* pair $(\mathbf{w}, c)$. Our input now looks like:

$$\{P_{\mathbf{q},\{\mathbf{v}_1,\dots,\mathbf{v}_d\}}, N_{\{(\mathbf{w}_1,c_1)\}}, N_{\{(\mathbf{w}_2,c_2)\}}, \dots, N_{\{(\mathbf{w}_m,c_m)\}}\}.$$

Now, it would be nice if we had an algorithm for the special case of $m = 1$:

Problem 5.2. Given two sets $P_{\mathbf{q},\{\mathbf{v}_1,\dots,\mathbf{v}_d\}}$ and $N_{\{(\mathbf{w},c)\}}$, compute their intersection

$$P_{\mathbf{q},\{\mathbf{v}_1,\dots,\mathbf{v}_d\}} \cap N_{\{(\mathbf{w},c)\}}$$

in P_* form.

If we can solve Problem 5.2, then we can solve Problem 5.1 by just repeatedly applying it until we run out of N_* sets.

So let's go ahead and solve it. First, the P_* form requires finding two things:

- at least one point in the intersection, say $\mathbf{p}$, and
- a subspace of $\langle \mathbf{v}_1, \dots, \mathbf{v}_d \rangle$ corresponding to the allowed directions in the intersection, say $\langle \mathbf{v}'_1, \mathbf{v}'_2 \dots, \rangle$. (Warning: $\mathbf{v}'_i$ is not necessarily one of the $\mathbf{v}_j$s!)

The final set will be $P_{\mathbf{p},\{\mathbf{v}'_1,\mathbf{v}'_2,\dots\}}$. It turns out that we've basically already done the important parts of the solution in some way!

So first of all, we want to find a solution $(t_1, \dots, t_d) \in \mathbb{R}^d$ to the following system:

$$\begin{aligned} \mathbf{p} &= \mathbf{q} + t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \dots + t_d\mathbf{v}_d \\ \mathbf{p} \cdot \mathbf{w} &= c. \end{aligned}$$

This is now all too familiar. Substituting the first into the second, we get

$$\begin{aligned} (\mathbf{q} + t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \dots + t_d\mathbf{v}_d) \cdot \mathbf{w} &= c \\ \mathbf{q} \cdot \mathbf{w} + t_1\mathbf{v}_1 \cdot \mathbf{w} + t_2\mathbf{v}_2 \cdot \mathbf{w} + \dots + t_d\mathbf{v}_d \cdot \mathbf{w} &= c \\ t_1\mathbf{v}_1 \cdot \mathbf{w} + t_2\mathbf{v}_2 \cdot \mathbf{w} + \dots + t_d\mathbf{v}_d \cdot \mathbf{w} &= \square. \end{aligned}$$

Now, if at least one of the coefficients $\mathbf{v}_i \cdot \mathbf{w}$ is nonzero, then we can solve this by simply setting $t_i = \frac{\square}{\mathbf{v}_i \cdot \mathbf{w}}$ and $t_j = 0$ for $j \neq i$. On the other hand, if all of them are 0, then this is a degenerate case—either there's no solution at all, or all choices of the t_is are solutions, depending on whether $\square \neq 0$ or not.

Geometrically, if all $\mathbf{v}_i \cdot \mathbf{w}$ are zero, then $\mathbf{w}$ is perpendicular to all of $\mathbf{v}_1, \dots, \mathbf{v}_d$. Also, if $\mathbf{p} \in N_{\{(\mathbf{w},c)\}}$, then any other point $\mathbf{p}'$ in it must satisfy

$$(\mathbf{p}' - \mathbf{p}) \cdot \mathbf{w} = \mathbf{p}' \cdot \mathbf{w} - \mathbf{p} \cdot \mathbf{w} = c - c = 0,$$

i.e., $N_{\{(\mathbf{w},c)\}}$ is the set of points that lie in a direction from $\mathbf{p}$ perpendicular to $\mathbf{w}$. And since $\mathbf{w}$ is perpendicular to both sets, and perpendicular + perpendicular = parallel,[7] this means that the subspace $P_{\mathbf{q},\{\mathbf{v}_1,\ldots,\mathbf{v}_d\}}$ is actually parallel to $N_{\{(\mathbf{w},c)\}}$, and so this is indeed a degenerate case.

We have now found our point $\mathbf{p}$. What about the rest? Well, any other point in $P_{\mathbf{q},\{\mathbf{v}_1,\ldots,\mathbf{v}_d\}}$ looks like $\mathbf{p}+t_1\mathbf{v}_1+\ldots+t_d\mathbf{v}_d$ (the base point can be anything in the set), and we want to find the set of all such points that also satisfy the equation for $(\mathbf{w},c)$, i.e.,

$$\begin{aligned}(\mathbf{p}+t_1\mathbf{v}_1+\ldots+t_d\mathbf{v}_d)\cdot\mathbf{w}&=c\\ \mathbf{p}\cdot\mathbf{w}+t_1\mathbf{v}_1\cdot\mathbf{w}+\ldots+t_d\mathbf{v}_d\cdot\mathbf{w}&=c\\ c+t_1\mathbf{v}_1\cdot\mathbf{w}+\ldots+t_d\mathbf{v}_d\cdot\mathbf{w}&=c\\ t_1\mathbf{v}_1\cdot\mathbf{w}+\ldots+t_d\mathbf{v}_d\cdot\mathbf{w}&=0.\end{aligned}$$

(Note that it was important to change base points from $\mathbf{q}$ to $\mathbf{p}$ for this to work.)

Thus, we want to restrict the choices of the t_is to only those satisfying this. Now, from the previous step, we assumed that we're in the nondegenerate case where there is at least one nonzero $\mathbf{v}_j\cdot\mathbf{w}$. Without loss of generality, assume $\mathbf{v}_d\cdot\mathbf{w}\neq 0$. With this coefficient nonzero, we can now solve for t_d, so t_d is determined by all the other t_is. Thus, we can write $t_d=\square t_1+\ldots+\square t_{d-1}$, and so

$$\begin{aligned}&t_1\mathbf{v}_1+\ldots+t_{d-1}\mathbf{v}_{d-1}+t_d\mathbf{v}_d\\ &=t_1\mathbf{v}_1+\ldots+t_{d-1}\mathbf{v}_{d-1}+(\square t_1+\square t_2+\ldots+\square t_{d-1})\mathbf{v}_d\\ &=t_1\blacksquare+t_2\blacksquare+\ldots+t_{d-1}\blacksquare.\end{aligned}$$

Where the $d-1$ vectors $\blacksquare$ are the generators of the subspace that we want, which is

$$\begin{aligned}&\{\mathbf{p}+t_1\mathbf{v}_1+\ldots+t_{d-1}\mathbf{v}_{d-1}+t_d\mathbf{v}_d\mid t_1,\ldots,t_d\in\mathbb{R},t_1\mathbf{v}_1\cdot\mathbf{w}+\ldots++t_d\mathbf{v}_d\cdot\mathbf{w}=0\}\\ &=\{\mathbf{p}+t_1\blacksquare+t_2\blacksquare+\ldots+t_{d-1}\blacksquare\mid t_1,\ldots,t_{d-1}\in\mathbb{R}\}\\ &=P_{\mathbf{p},\{\blacksquare,\ldots,\blacksquare\}}.\end{aligned}$$

So the latter set is the set $P_{\mathbf{p},\{\mathbf{v}_1',\mathbf{v}_2',\ldots\}}$ that we're after. With this, we've finally solved Problem 5.2, and thus Problem 5.1!

Under the hood, what we're doing is really just solving a system of linear equations. We're just phrasing it in terms of vectors. I hope that the above at least casts what you already know from linear algebra in a different light and maybe make you appreciate it more.

By the way, there's something worth mentioning about this algorithm. In one of the first steps, we converted most of the P_* sets into N_*. This can be quite inefficient since a subspace of dimension 1 has codimension $n-1$ so it needs $n-1$ equations to represent. This bloats the input (and hence the running time) by a factor of n. An improvement is described in Appendix B.

Problem 5.3. In 3D, find the intersection of three spheres.

Problem 5.4. In 3D, find the intersection of two spheres.

For bonus points, use vectors to solve these problems :D

[7] The "perpendicular + perpendicular = parallel" thing is actually a bit tricky if you think about it. For example, if $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ are basis vectors, then $\mathbf{e}_1$ and $\mathbf{e}_2$ are perpendicular, and $\mathbf{e}_2$ and $\mathbf{e}_3$ are perpendicular, but $\mathbf{e}_1$ and $\mathbf{e}_3$ are NOT parallel (in fact, they're perpendicular). Can you make the argument correct?

A Orthogonalization

At some point, we needed to solve the following subproblem:

Problem A.1. Given independent vectors

$$\mathbf{v}_1, \dots, \mathbf{v}_d,$$

extend it into a basis

$$\mathbf{v}_1, \dots, \mathbf{v}_d, \mathbf{w}_1, \dots, \mathbf{w}_{n-d}$$

such that each $\mathbf{w}_i$ is perpendicular to $\mathbf{v}_j$.

We mentioned that there's a standard way to solve it. Indeed, it can be solved because we can solve the following problem:

Problem A.2. Given a set $\{\mathbf{v}_1, \dots, \mathbf{v}_d\}$ of nonzero vectors perpendicular to each other, and a vector $\mathbf{w}$, find the orthogonal complement of $\langle \mathbf{v}_1, \dots, \mathbf{v}_d \rangle$ in $\langle \mathbf{v}_1, \dots, \mathbf{v}_d, \mathbf{w} \rangle$, i.e., the vector subspace of $\langle \mathbf{v}_1, \dots, \mathbf{v}_d, \mathbf{w} \rangle$ of all its vectors that are perpendicular to all the $\mathbf{v}_i$s.

Since $\langle \mathbf{v}_1, \dots, \mathbf{v}_d \rangle$ is d-dimensional and $\langle \mathbf{v}_1, \dots, \mathbf{v}_d, \mathbf{w} \rangle$ is at most $(d+1)$-dimensional, the required subspace in Problem A.2 is at most one-dimensional, i.e., it is the space of all scalar multiples of some vector (which may be $\mathbf{0}$), e.g., $\{t\mathbf{w}' \mid t \in \mathbb{R}\}$. Also, clearly $\langle \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{w} \rangle = \langle \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{w}' \rangle$ since a subspace and its orthogonal complement span the whole space. This will actually become pretty clear once we solve the problems above.

Anyway, we can use Problem A.2 to solve Problem A.1 as follows. First, we assume that the $\mathbf{v}_i$s are perpendicular to each other.

1. Let W be our current collection of $\mathbf{w}_j$ vectors. Initially, it is the empty set.
2. Let $\mathbf{e}_1, \dots, \mathbf{e}_n$ be your favorite basis of the whole space.
3. For each i from 1 to n:
   (a) Solve Problem A.2 on the set $\{\mathbf{v}_1, \dots, \mathbf{v}_d\} \cup W$ and $\mathbf{w} = \mathbf{e}_i$. If the resulting vector is nonzero, add it to W.
4. Return W.

It can be shown that this algorithm correctly solves Problem A.2, and that since there are d $\mathbf{v}_i$s and the $\mathbf{e}_i$s are a basis, the set W will have at least $n-d$ members in the end, and that if the $\mathbf{v}_i$s are nonzero and orthogonal to each other, then W will have exactly $n-d$ members.

Of course, the given $\mathbf{v}_i$s are not necessarily orthogonal to each other. But we can also easily arrange that using Problem A.2 again. Given a set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_d\}$, we will find a new set of vectors spanning the same subspace but are perpendicular to each other.

1. Let V be our current collection of vectors. Initially, it is the empty set.
2. For each i from 1 to n:
   (a) Solve Problem A.2 on the set V and $\mathbf{w} = \mathbf{v}_i$. If the resulting vector is nonzero, add it to V.
3. Return V.

Finally, we now only need to solve Problem A.2. We want to remove the "component" of $\mathbf{w}$ that's not perpendicular to the $\mathbf{v}_i$s. Let's try it for a single vector first, say $\mathbf{v}_1$. We want $\alpha\mathbf{w} + \beta\mathbf{v}_1$ to be perpendicular to $\mathbf{v}_1$, so

$$\begin{aligned}(\alpha\mathbf{w} + \beta\mathbf{v}_1) \cdot \mathbf{v}_1 &= 0\\ \alpha\mathbf{w} \cdot \mathbf{v}_1 + \beta\mathbf{v}_1 \cdot \mathbf{v}_1 &= 0\\ \beta &= -\alpha\frac{\mathbf{w} \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\end{aligned}$$

and so the combination $\alpha\mathbf{w} + \beta\mathbf{v}_1$ that we want is

$$\alpha\left(\mathbf{w} - \frac{\mathbf{w} \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1\right)$$

which you can easily check to be perpendicular to $\mathbf{v}_1$ for any α, i.e., dot product zero. What's more, we've proven that these are the only solutions!

Now, what about two vectors, $\mathbf{v}_1$ and $\mathbf{v}_2$? Actually, since $\mathbf{v}_1$ and $\mathbf{v}_2$ are perpendicular to each other, this means we can actually solve this for each one separately! Namely, we simply do the above computation for $\mathbf{v}_1$, and then for $\mathbf{v}_2$. What we'll end up with is the vector

$$\alpha\left(\mathbf{w} - \frac{\mathbf{w} \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{w} \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2\right)$$

which you can check to be perpendicular to both $\mathbf{v}_1$ and $\mathbf{v}_2$, i.e., dot product zero. (Remember that $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$!) Again, we can easily show that this is the only solution up to scaling.

This extends to any number of vectors, so this solves Problem A.2.

This process of making all $\mathbf{v}_i$s perpendicular to each other (and making them norm 1 if we want) is called the Gram-Schmidt orthonormalization process. ("Orthonormalization" means orthogonalization and making everything of norm 1.)

Exercise A.1. Show that our algorithm for Problem A.2 is correct. Show that the set W will have at least $n - d$ members in the end, and that if the $\mathbf{v}_i$s are nonzero and orthogonal to each other, then W will have exactly $n - d$ members.

Exercise A.2. Show that the orthogonalization process is correct, i.e., for each i, the subspace spanned by the first i input vectors is the same as the subspace spanned by V after the ith iteration, and that the resulting vectors are orthogonal to each other.

Exercise A.3. Can you explain why the orthonormalization process is equivalent to inverting a certain lower-triangular matrix via Gaussian elimination?

B Intersection of two P_* sets

As mentioned earlier, our algorithm works by converting most P_* sets to N_* sets, which can be slow. So we'd like to optimize our algorithm somewhat in that case. To be clear, we want an algorithm to find the intersection of several P_* sets that's faster than converting them first to N_*.

One thing we could do to improve our method a bit is to do the conversion *gradually*. Specifically, instead of converting a P_* set into an N_* set wholesale, we can convert it one dimension at a time: A P_* set of dimension d can be converted into $P \cap N$ where P is a P_* set of dimension $d+1$ and N is an N_* set of codimension 1, i.e., essentially the reverse of Problem 5.2. The way to do this is to partially do our P_*-to-N_*-conversion algorithm for only one $\mathbf{w}$. Once express our P_* set this way, we can intersect the new N with our current "running" P_* set.[8] The details are left to the reader. However, this may still be slow.

Of course, another way would be just to forgo all this vector stuff and just solve this linear algebra problem traditionally (say with Gaussian elimination), but we'll try to resist that temptation :P

Anyway, somehow, we'd like to avoid this conversion process altogether. In other words, we want to be able to intersect two P_* sets on their own. This is especially worth it if the sets have small dimension compared to n, since the codimension will be large.

However, this is the same difficulty we ran into earlier. Remember that we were trying to intersect two lines written in the form $\mathbf{p} = \mathbf{p}_1 + t\mathbf{v}_1$ and $\mathbf{p} = \mathbf{p}_2 + t'\mathbf{v}_2$. We noted that this seemed hard. But now we're going to try to attack it anyway.

First, it's relatively not that hard to intersect two P_* sets with the same common point, since

$$P_{\mathbf{q},V_1} \cap P_{\mathbf{q},V_2} = P_{\mathbf{q},V}$$

where V is any generating set of the intersection of subspaces $\langle V_1 \rangle \cap \langle V_2 \rangle$. We can compute V from V_1 and V_2 by (somewhat counterintuitively) taking generators of the *combined* subspace $\langle V_1 \rangle + \langle V_2 \rangle$ and "removing redundant generators", and it turns out, each generator we remove corresponds to a generator of $\langle V_1 \rangle \cap \langle V_2 \rangle$.

Assuming $V_1 = \{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ and $V_2 = \{\mathbf{w}_1, \dots, \mathbf{w}_m\}$ are each independent (though they may not be independent when combined), here's the usual greedy algorithm to combine them and remove redundant generators:

1. Let S be our current collection of independent generators, initially V_1.
2. For i from 1 to m (the size of V_2):
   (a) Try to express $\mathbf{w}_i$ in terms of the generators S if possible. If it is not possible, add $\mathbf{w}_i$ to S.
3. Return S.

It is a textbook exercise in linear algebra[9] to show that this works. But there's a neat side effect in our case: for every $\mathbf{w}_i$ we *don't* add to S, we have an expression of $\mathbf{w}_i$ in terms of S. In other words, for each such i, there are constants $\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_{i-1}$ such that

$$\mathbf{w}_i = \alpha_1\mathbf{v}_1 + \ldots + \alpha_k\mathbf{v}_k + \beta_1\mathbf{w}_1 + \ldots + \beta_{i-1}\mathbf{w}_{i-1},$$

[8] To speed things up a bit, we can use the P_* set with the smallest dimension as our starting P_* set, and intersect the other P_* sets with it one by one.

[9] also read about matroids.

or, after rearranging,

$$\alpha_1\mathbf{v}_1 + \ldots + \alpha_k\mathbf{v}_k = \mathbf{w}_i - \beta_1\mathbf{w}_1 - \ldots - \beta_{i-1}\mathbf{w}_{i-1}.$$

The left-hand side is in $\langle V_1\rangle$, and the right-hand side is in $\langle V_2\rangle$. Also, the right-hand side is nonzero (why?). In other words, we have just found a nonzero element in the intersection $\langle V_1\rangle\cap\langle V_2\rangle$! Also, these vectors are independent of each other (check!). Furthermore, the number of such equations is exactly the number of thrown-away $\mathbf{w}_i$s, which is

$$\dim(\langle V_1\rangle) + \dim(\langle V_1\rangle) - \dim(\langle V_1\rangle + \langle V_2\rangle),$$

since we're removing every *redundant* generator. Finally, by the dimension formula, this is equal to $\dim(\langle V_1\rangle \cap \langle V_2\rangle)$, so we actually have a basis of the intersection! Thus, we can take our set V to be these generators.

Exercise B.1. Verify that the vectors generated above are nonzero and independent of each other.

Thus, we can solve the problem if we can find a common point of our two sets $P_{\mathbf{p},V_1}$ and $P_{\mathbf{q},V_2}$ (where $\mathbf{p}$ and $\mathbf{q}$ are not necessarily the same), since we can just change base points. Any such base point will do, so our goal is to find one, or to determine if none exist.

Recalling the definition of P_*, we want to find real numbers $t_1, \ldots, t_k, t'_1, \ldots, t'_m$ such that

$$\mathbf{p} + t_1\mathbf{v}_1 + \ldots + t_k\mathbf{v}_k = \mathbf{q} + t'_1\mathbf{w}_1 + \ldots + t'_m\mathbf{w}_m.$$

Rearranging, we get

$$\mathbf{q} - \mathbf{p} = t_1\mathbf{v}_1 + \ldots + t_k\mathbf{v}_k - t'_1\mathbf{w}_1 - \ldots - t'_m\mathbf{w}_m.$$

In other words, we want to express a given vector $\mathbf{q} - \mathbf{p}$ in terms of a collection of other vectors $V_1 \cup V_2$. How does one do that? And if you think about it, in the greedy algorithm above, we also sneakily assumed that we can already do it, since we wanted to express $\mathbf{w}_i$ in terms of S.

So this is the final problem we need to solve: Given a vector $\mathbf{w}$ and a collection of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_d$, express $\mathbf{w}$ in terms of the $\mathbf{v}_i$s. But this is exactly what Problem A.2 is designed to do. So we can in fact solve the problem using it! Of course, it assumes that the given set of vectors are independent and perpendicular to each other, but we already know how to arrange for that!

I'll leave it to you to compute the complexity of our algorithms so far. There's also the analogous co-question of finding the "combined space" of subspaces: $P_1 \vee \ldots \vee P_k \vee N_1 \vee \ldots \vee N_m$. I'll let you have fun with that :D

B.1 Line-line intersection (again)

Finally, we can now go full circle (he he) and find out how to intersect two lines (again!), this time in P_* form:

$$\mathbf{p}_1 + t\mathbf{v}_1 = \mathbf{p}_2 + t'\mathbf{v}_2.$$

Unlike before, we don't assume that we're in 2D. This is just a special case of Appendix B, but we'll do it (semi-)explicitly for fun.

We assume that $\mathbf{v}_1$ and $\mathbf{v}_2$ are not parallel for simplicity. Rearranging the above, we get

$$\mathbf{p}_2 - \mathbf{p}_1 = t\mathbf{v}_1 - t'\mathbf{v}_2.$$

Now, we're about to use Problem A.2, and to do that, we must replace $\mathbf{v}_2$ with something perpendicular to $\mathbf{v}_1$. The orthogonalization process gives us one such vector, namely

$$\mathbf{v}_2 - \frac{\mathbf{v}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1.$$

Since we assume that $\mathbf{v}_1$ and $\mathbf{v}_2$ are not parallel, we are guaranteed that this vector is nonzero. Let's call this vector $\mathbf{w}$.

Now, we apply Problem A.2 again, this time with $\mathbf{q} = \mathbf{p}_2 - \mathbf{p}_1$. We want to remove its parts that are not perpendicular to $\mathbf{v}_1$ and $\mathbf{w}$. Again, the orthogonalization process gives us

$$\mathbf{q} - \frac{\mathbf{q} \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{q} \cdot \mathbf{w}}{\mathbf{w} \cdot \mathbf{w}}\mathbf{w}.$$

This is perpendicular to $\mathbf{v}_1$ and $\mathbf{w}$, and thus to $\mathbf{v}_2 = \square\mathbf{v}_1 + \mathbf{w}$, so the only way it can be expressed in terms of $\mathbf{v}_1$ and $\mathbf{v}_2$ is in the rare case that it is zero (why?). Therefore, if this is nonzero, then we know that the above vector, and thus $\mathbf{q}$, cannot be expressed in terms of $\mathbf{v}_1$ and $\mathbf{v}_2$, and the two lines have no intersection. But if it is zero, then

$$\begin{aligned}
\mathbf{q} - \frac{\mathbf{q} \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{q} \cdot \mathbf{w}}{\mathbf{w} \cdot \mathbf{w}}\mathbf{w} &= 0 \\
\mathbf{q} &= \frac{\mathbf{q} \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 + \frac{\mathbf{q} \cdot \mathbf{w}}{\mathbf{w} \cdot \mathbf{w}}\mathbf{w} \\
\mathbf{p}_2 - \mathbf{p}_1 &= \frac{\mathbf{q} \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 + \frac{\mathbf{q} \cdot \mathbf{w}}{\mathbf{w} \cdot \mathbf{w}}\mathbf{w} \\
\mathbf{p}_2 - \mathbf{p}_1 &= \square\mathbf{v}_1 + \square\mathbf{w} \\
\mathbf{p}_2 - \mathbf{p}_1 &= \square\mathbf{v}_1 + \square(\mathbf{v}_2 - \frac{\mathbf{v}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1) \\
\mathbf{p}_2 - \mathbf{p}_1 &= \square\mathbf{v}_1 + \square(\mathbf{v}_2 + \square\mathbf{v}_1) \\
\mathbf{p}_1 + \square\mathbf{v}_1 &= \mathbf{p}_2 + \square\mathbf{v}_2,
\end{aligned}$$

and we've finally found our intersection point. This works in any number of dimensions!