Number Theory 1

Veteran Track

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Guess the Next Term!

$4096, 8192, 16384, \_$

Guess the Next Term!

$4096, 8192, 16384, \mathbf{32768}$

- Powers of two!
- The next term is double the last ^^

Guess the Next Term!

$1, 1, 2, 3, 5, 8, 13, \_$

Guess the Next Term!

$1, 1, 2, 3, 5, 8, 13, \mathbf{21}$

- Fibonacci Numbers
- The next term is the sum of the past two terms

Guess the Next Term!

$1, 1, 1, 1, 2, 1, 1, 3, 3, 1, 1, 4, \_$

Guess the Next Term!

$1, 1, 1, 1, 2, 1, 1, 3, 3, 1, 1, 4, \mathbf{6}$

- It's Pascal's Triangle, but the entries have been flattened out
- Again,

Guess the Next Term!

$1, 1, 2, 5, 14, \_$

Guess the Next Term!

$1, 1, 2, 5, 14, \mathbf{42}$

- It's the Catalan numbers!

You've Seen these Sequences Before!

Here are some new ones that you may or may not know ^^

Guess the Next Term!

$2, 3, 5, 7, 11, 13, 17, 19, \_$

Guess the Next Term!

$2, 3, 5, 7, 11, 13, 17, 19, \mathbf{23}$

- Yep! It's just the prime numbers! I'm sure you got this right ^^

Guess the Next Term!

$1, 1, 2, 2, 4, 2, 6, 4, 6, 4, \_$

Guess the Next Term!

$1, 1, 2, 2, 4, 2, 6, 4, 6, 4, \mathbf{10}$

- This one might be hard. These are the first few terms of **Euler's Totient Function**.
- The nth term (denoted $\varphi\left(n\right)$) is the number of numbers in the range $\left[1, n\right]$ that are coprime with n.
- For instance, the first term is $\varphi\left(1\right) = 1$ since 1 is coprime with 1.
- $\varphi(6) = 2$ since two numbers, namely 1 and 5, are coprime with 6, but $2, 3, 4,$ and 6 are not.
- $\varphi(11) = 10$ since $1, 2, 3, 4, 5, 6, 7, 8, 9,$ and 10 are coprime with 11, so the next term is 10.

Guess the Next Term!

$1, -1, -1, 0, -1, 1, -1, \_$

- This one is difficult 👀

Guess the Next Term!

$1, -1, -1, 0, -1, 1, -1, \mathbf{0}$

- These are the first few terms of the **mobius function**.
- The nth term of the **mobius function**, denoted as $\mu\left(n\right)$, is defined as follows:
- Consider the **prime factorization of** n. If the number has a factor that is a perfect square, $\mu(n) = 0$. Otherwise, if n has an odd number of prime factors, $\mu(n) = -1$. Finally, if n has an even number of prime factors, $\mu(n) = 1$.
- For example, take $n = 30 = 2^1 \cdot 3^1 \cdot 5^1$. Since there are no primes with an exponent greater than one, and there are 3 prime factors (2, 3, and 5), $\mu(30) = -1$.

Guess the Next Term!

$1, -1, -1, 0, -1, 1, -1, \mathbf{0}$

- Now, consider $n = 24 = 2^3 \cdot 3$. Since there are no primes with an exponent greater than one, and there are 2 prime factors (2 and 3), $\mu(24) = 1$.
- Finally, consider $n = 8 = 2^3$. Notice that the exponent of 2 is greater than 1. This means that there's a square that divides the number 8. In this case, it's $2^2 = 4$. Therefore, $\mu(8) = 0$.

Recap

- To recap, we learned three new sequences/functions:
    1. The prime numbers
    2. Euler's totient function, φ
    3. The Mobius function, μ

Prime Numbers and Prime Factorization

- A positive integer p is prime if $p \geq 2$ and its only factors are 1 and itself
- A number n can be uniquely expressed as a product of primes (up to reordering the prime factors)
- For instance, $n = 360 = 2^3 \cdot 3^2 \cdot 5$. $2^3 \cdot 3^2 \cdot 5$ is known as the **prime factorization** of 360
- The result that all numbers have a unique prime factorization is known as the **fundamental theorem of arithmetic**

Primality Checking

- How can we check if a number is prime?
- Well, all we need to do is to check whether its only factors are 1 and itself, so something like this should work:

```
bool is_prime(int n) {
    for(int i = 2; i < n; i++) {
        if((n % i) == 0) return false;
    }
    return true;
}
```

- This works, and it gives a time complexity of $O(n)$. However, what if n is big? Could we do better?

Primality Checking

- It turns out that we can! Let's say that n has a factor i that's not either 1 or itself.
- Then, notice that $\frac{n}{i}$ is also another factor of n that's not one or itself. Let $j = \frac{n}{i}$. Then, $ij = n$. In other words, i and j form a pair of factors.
- One thing you may know about factor pairs is that one of the numbers in the pair is always less than or equal to $\sqrt{n}$. An easy way to see this is that if $i, j > \sqrt{n}$, then $ij > n$, which is a contradiction.
- Therefore, it suffices to check the range $[1, \sqrt{n}]$ for factors, which gives us a $O(\sqrt{n})$ algorithm.

Primality Checking

- Here's the implementation of the $O(\sqrt{n})$ primality checker:

```
bool is_prime(int n) {
    for(int i = 2; i * i <= n; i++) {
        if((n % i) == 0) return false;
    }
    return true;
}
```

Primality Checking

- $\sqrt{N}$ is definitely fast, but what if we want to check whether a really large number (say around the size of 10^{18}) is prime or not?

Primality Checking

- $\sqrt{N}$ is definitely fast, but what if we want to check whether a really large number (say around the size of 10^{18}) is prime or not?
- This is where we can use randomized algorithms such as the **Fermat Primality Test**. Such algorithms can determine whether a number is prime or not really quickly (in $O(\log n)$ time), but come with the disadvantage that the algorithm is *probabilistic*, so there's a chance that it fails.
- Also, in CompProg, the Fermat Primality Test is often not used. I haven't ever had to use it yet in CompProg, but it's good to know it exists in case you need it.
- If you want to know more about the Fermat Primality Test, check this article: https://cp-algorithms.com/algebra/primality_tests.html#fermat-primality-test

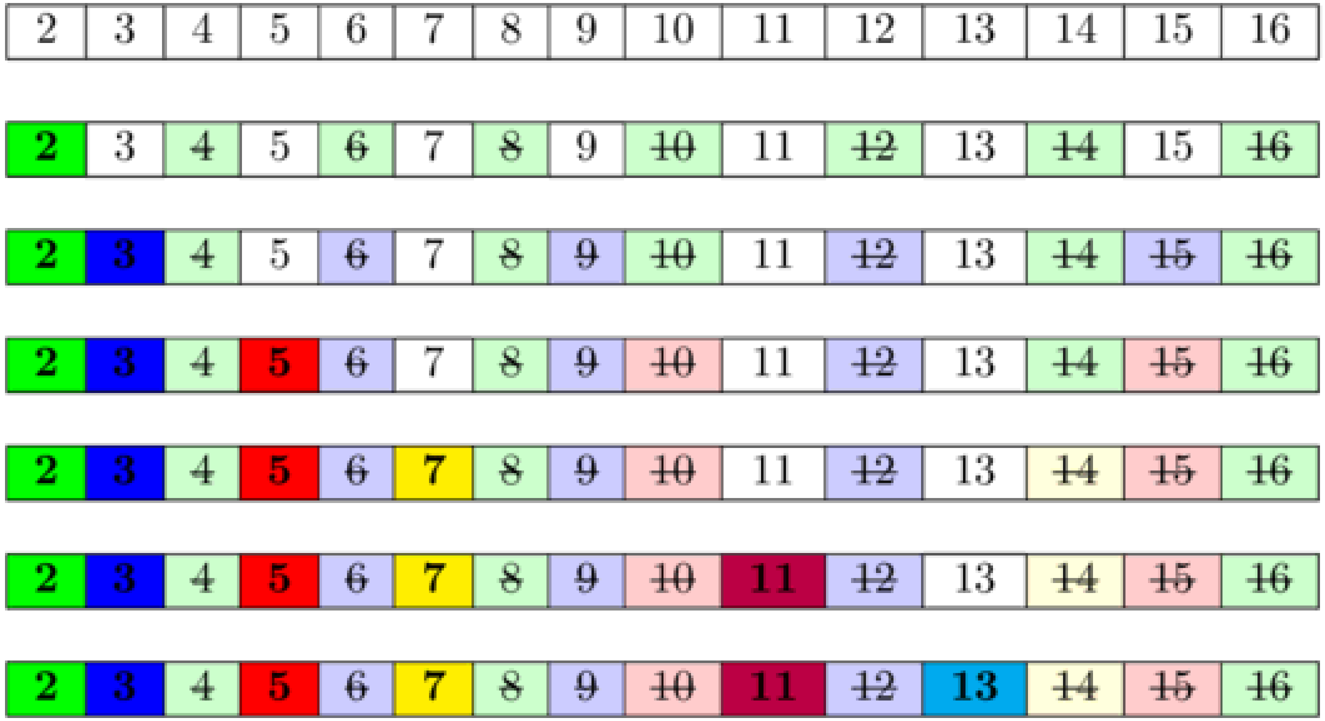
Finding Primes in a Range

- Sometimes, we may need to find all prime numbers in the range $[1, N]$.
- We can use the previous primality checker to go through each number in the range and determine whether it is prime or not in $O(\sqrt{N})$ time. This gives an $O(N\sqrt{N})$ algorithm. Could we do better?

Finding Primes in a Range

- It turns out that we can do better!
- The idea is to **progressively filter out composite numbers in the range** $[1, N]$. Maintain an array of booleans isPrime. $\text{isPrime}[i]$ is true if and only if i is prime.
- Initially, isPrime is all true. Then, we will iterate along the range $i \in [2, N]$.
  - If $\text{isPrime}[i] = \text{true}$, then we will mark all multiples of i (i.e., $2i$, $3i$, $4i$, etc... until $ki \leq N$) as *not* prime.

Finding Primes in a Range



Source: https://cp-algorithms.com/algebra/sieve-of-eratosthenes.html

Finding Primes in a Range

- This method of filtering out the composite numbers is similar to *sieving* out the composite numbers, leaving only the prime numbers behind. This is why this method is known as the **Sieve of Eratosthenes**.
- What is the time complexity of this algorithm? In the worst case, it looks like we'll have to mark $O(N)$ numbers as not prime in each step, so naively, it seems like this algorithm takes $O(N^2)$ time! That seems slow! What now?

Finding Primes in a Range

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- The idea is that $O(N^2)$ is not a *tight* bound of the time complexity. Let's try to compute it more accurately!

Finding Primes in a Range

- Again, the Sieve of Eratosthenes will iterate from $i \in [2, N]$. For each i, it will mark $O(\frac{N}{i})$ numbers as not prime.
- Summing this value over $i \in [2, N]$, we get $O(N\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{N}\right)) = O(N\sum_{i=2}^{N}\frac{1}{i})$. Note that $\sum_{i=2}^{N}\frac{1}{i} < \sum_{i=1}^{N}\frac{1}{i} = H_N$, where H_N is the Nth harmonic number.
- A key fact about harmonic numbers is that $H_N = \sum_{i=1}^{i}\frac{1}{N} = O(\log N)$. Therefore, the entire algorithm runs in $O(N\log N)$!

Finding Primes in a Range

- In fact, we can find an even tighter bound by considering that we only mark the multiples of i as not prime if and only if i itself is prime. Therefore, the time complexity of the algorithm is $O(N \sum_{p \leq N, p \text{ is prime}} \frac{1}{p})$. By exploiting the distribution of the primes, we can show that the sieve of Eratosthenes runs in $O(N \log \log N)$ 🤯

Sieve of Eratosthenes: Implementation

```
int n;
cin >> n;
vector<bool> is_prime(n + 1, true);
is_prime[0] = is_prime[1] = false;
for(int i = 2; i <= n; i++) {
    if(is_prime[i]) {
        for(int j = 2; j * i <= n; j++) is_prime[j * i] = false;
    }
}
```

A Different Sieve

- Challenge: the code below has been slightly modified. What do you think it does?

```
int n;
cin >> n;
vector<bool> is_something(n + 1, false);
is_something[0] = true;
for(int i = 1; i <= n; i++) {
    for(int j = 1; j * i <= n; j++) is_something[j * i] = !is_something[j * i];
}
```

A Different Sieve

- Challenge: the code below has been slightly modified. What do you think it does?

```
int n;
cin >> n;
vector<bool> is_something(n + 1, false);
is_something[0] = true;
for(int i = 1; i <= n; i++) {
    for(int j = 1; j * i <= n; j++) is_something[j * i] = !is_something[j * i];
}
```

- Yep! It **determines whether all numbers in the range** $[1, N]$ are squares or not!
- In the code above, $\text{isSomething}\,[i]$ is true if and only if it has been flipped an odd number of times. This happens when i has an odd number of factors.
- i has an odd number of factors if and only if it's a perfect square!

Counting Divisors

- You can use sieves yet again to compute the number of divisors $\tau(i)$ of all numbers in the range $[1, N]$ in $O(N \log N)$.
- Start with $\tau[i] = 0$, then, for each number $i \in [1, N]$, increase $\tau[ki]$ by 1.

```
int n;
cin >> n;
vector<int> num_div(n + 1, 0);
for(int i = 1; i <= n; i++) {
    for(int j = 1; j * i <= n; j++) num_div[j * i]++;
}
```

Euler's Totient Function

- Now, we will delve into computing Euler's Totient Function.
- Recall that $\varphi\left(n\right)$ is the number of numbers in the range $[1, n]$ that are coprime with n.
- We can use **sieves** to compute φ over $[1, N]$ in $O(N \log\log N)$.
- To do this, we will exploit the following property. Given the prime factorization of n, $n = \prod_i {p_i}^{q_i}$, where p_i is the ith prime, we can compute $\varphi(n)$ as follows:

$$\phi\left(n\right) = \phi\left(\prod_i {p_i}^{q_i}\right) = \prod_i {p_i}^{q_i - 1} \cdot \varphi(p_i) = \prod_i {p_i}^{q_i - 1} \cdot (p_i - 1)$$

Euler's Totient Function

$$\phi(n) = \prod_i {p_i}^{q_i - 1} \cdot (p_i - 1) = \prod_i {p_i}^{q_i} \cdot \left(1 - \frac{1}{p_i}\right) = \prod_i {p_i}^{q_i} \cdot \prod_i \left(1 - \frac{1}{p_i}\right) = n \prod_i \left(1 - \frac{1}{p_i}\right)$$

- In other words, we can obtain $\varphi(n)$ by adjusting n by a "correction factor" for each of its prime divisors. The "correction factor" involves multiplying n by $1 - \frac{1}{p}$ for each prime divisor.
- Equivalently, we can set $\varphi[n] \leftarrow \varphi[n] - \frac{\varphi[n]}{p}$ for each of its prime divisors. Thus, doing something like `phi[n] -= phi[n]/p` for each prime p dividing n works.
- We can set $\varphi[i] = i$ initially. Then, to detect when a number is prime, it suffices to check whether $\varphi[i] = i$ the moment we iterate over it (since this implies that no other prime factors have adjusted its value, implying that it has *no other prime factors* and thus is prime).

Euler's Totient Function: Implementation

- The time complexity of this implementation is $O(n \log \log n)$. It is based on the implementation in https://cp-algorithms.com/algebra/phi-function.html#etf_1_to_n

```
ll n;
cin >> n;
vector<ll> phi(n + 1, 0ll);
for(ll i = 0; i <= n; i++) {
    phi[i] = i;
}
for(ll i = 2ll; i <= n; i++) {
    if(phi[i] == i) {
        for(ll j = 1ll; i * j <= n; j++) {
            phi[i * j] -= phi[i * j] / i;
        }
    }
}
```

Euler's Totient Function: Applications

- Euler's Totient Function can be used to **compute power towers**. In particular, we have the following generalization of **Fermat's Little Theorem**:

$$a^n \equiv a^{n \mod \varphi(m)} \mod m$$

- One can use this to compute power towers quickly! For instance, try finding:

$$2023^{2022^{2021^{2020^{\cdots}}}} \mod 24$$

Mobius Function

- Finally, we get to the **Mobius Function**. The mobius function (denoted as μ), is an important number-theoretic function, as we will see in a while.
- It is defined as:

$$\mu(n) = \begin{cases} 0 \text{ if } n \text{ has a perfect square as a factor} \\ 1 \text{ if } n \text{ has no perfect square factors and has an even number of prime factors} \\ -1 \text{ if } n \text{ has no perfect square factors and has an odd number of prime factors} \end{cases}$$

- Though this function may seem arbitrary, it has a lot of applications in number theory.

Mobius Inversion

- Note the following identity related to Euler's Totient Function:

$$n = \sum_{d|n} \varphi(d)$$

- It turns out that there's a corresponding "shadow" identity related to the one above,

$$\varphi(n) = \sum_{d|n} \mu(d) \cdot \frac{n}{d}$$

- In a sense, the new identity is somehow an "inverse" of the previous one, since we've taken out the totient function from the summation

Mobius Inversion

- In fact, suppose f and g are functions over the positive integers. Then, if f and g are related as follows,

$$f(n) = \sum_{d|n} g\left(d\right)$$

- We have a corresponding inverse relationship between f and g,

$$g(n) = \sum_{d|n} \mu(d) f\left(\frac{n}{d}\right)$$

- This is known as the **Mobius Inversion Formula**. It is one of the most powerful applications of the Mobius function.

Calculating the Mobius Function

$$\mu\left(n\right)=\begin{cases}0\text{ if }n\text{ has a perfect square as a factor}\\1\text{ if }n\text{ has no perfect square factors and has an even number of prime factors}\\-1\text{ if }n\text{ has no perfect square factors and has an odd number of prime factors}\end{cases}$$

- Calculating the nth term of the Mobius function can be done with a sieve as well; however, it is not as simple as the other applications of the sieve technique.
- Writing a program to compute the Mobius function will be left to you as an exercise! ^^
- To help guide you, here are some things that will help you compute μ:
    i. You need to know how many prime factors a number n has
    ii. Then, you must determine whether or not a number is squarefree (i.e., has no perfect squre factors)

Takeaway

Sieve Methods are POWERFUL

Homework

- Check the Reboot Website for the homework this week. The homework problems for this week may be challenging, so feel free to **collaborate and discuss with your fellow trainees**. You may also **ask for help from the trainers** and even **read the editorial (but only when you're really stuck)** 😄

References

1. *Euler's totient function*. (2024, January 27). CP-Algorithms. https://cp-algorithms.com/algebra/phi-function.html

Appendix

Euler's Totient Function: Additional Properties

- Here are some properties of Euler's Totient Function that may help. These are mostly useful in more mathematical settings, but it's good to know these properties exist ^^

$$\varphi(mn) = \varphi(m) \cdot \varphi(n), \text{ where } m \text{ and } n \text{ are coprime}$$

- A more general version of the multiplicative rule:

$$\varphi(mn) = \varphi(m) \cdot \varphi(n) \cdot \frac{\gcd(m, n)}{\varphi(\gcd(m, n))}$$

$$\varphi(n^k) = \varphi(n) \cdot n^{k-1}$$