Software for multivariade regression of Distributional SD

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Regression models

This is an R Markdown presentation showing an application of the two-components regression model (A. Irpino and Verde 2015) on the `OzoneFull` dataset which is available in the `HistDAWass` package. We used the `HistDAWass` package and the therein build functions for the regression analysis.

Install and load the package

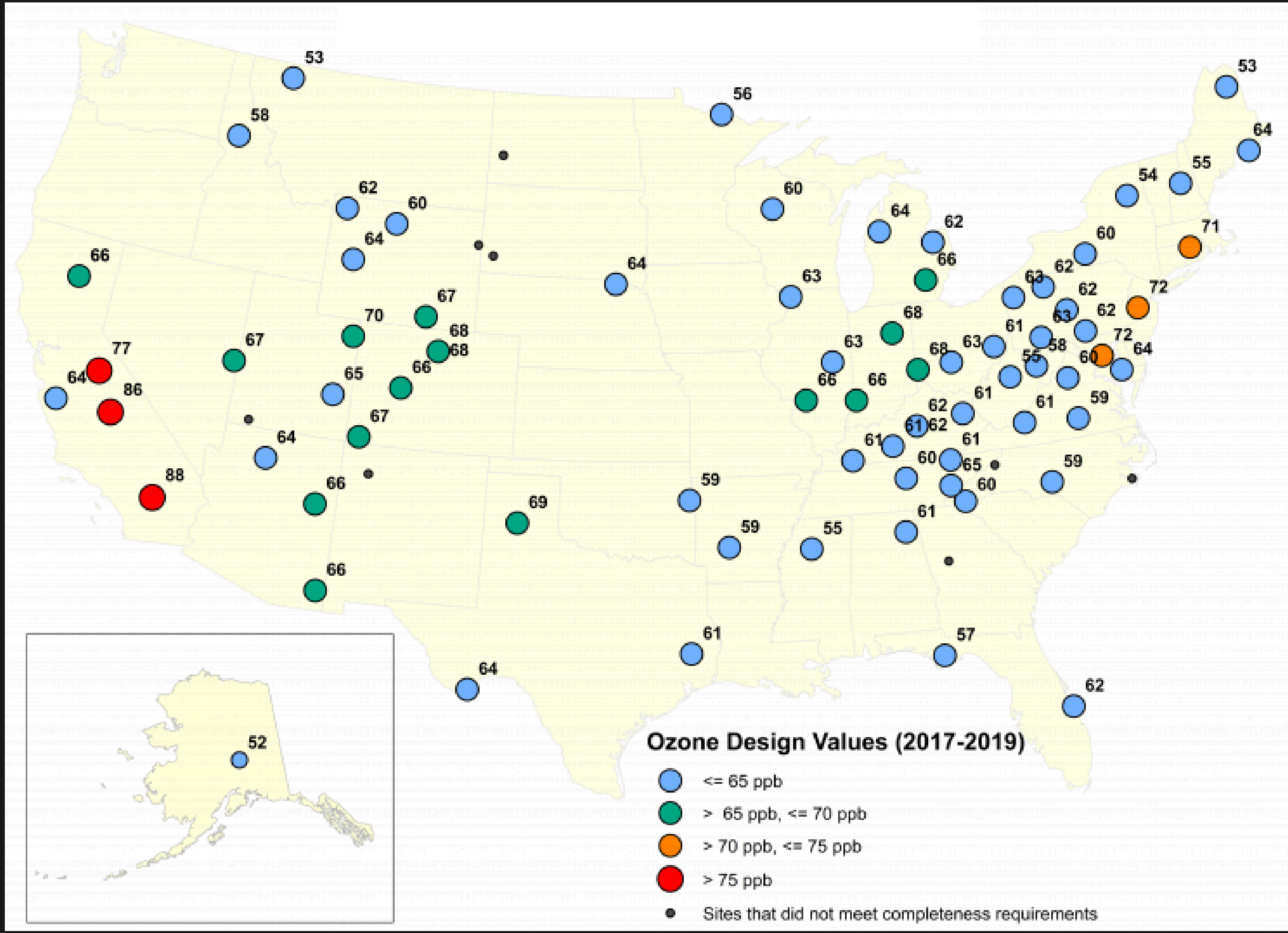
For installing and loading the package in your environment launch the following code:

```r
## if not installed in your environment
#  install.packages("HistDAWass")
library(HistDAWass) #load the package
#other useful packages
library(tidyverse)
library(plotly)
library(patchwork)
```

Data description

The `OzoneFull` dataset is a `MatH` object, namely, a table of histogram-valued data, representing aggregate raw data downloaded from the Clean Air Status and Trends Network (CASTNET) (<http://java.epa.gov/castnet/>), an air-quality monitoring network of the United States, designed to provide data to assess trends in air quality, atmospheric deposition and ecological effects due to changes in air pollutant emissions.

We selected data on the ozone concentration in 78 USA sites among those depicted in Fig for which the monitored data were complete (i.e., without missing values for each of the selected characteristics).



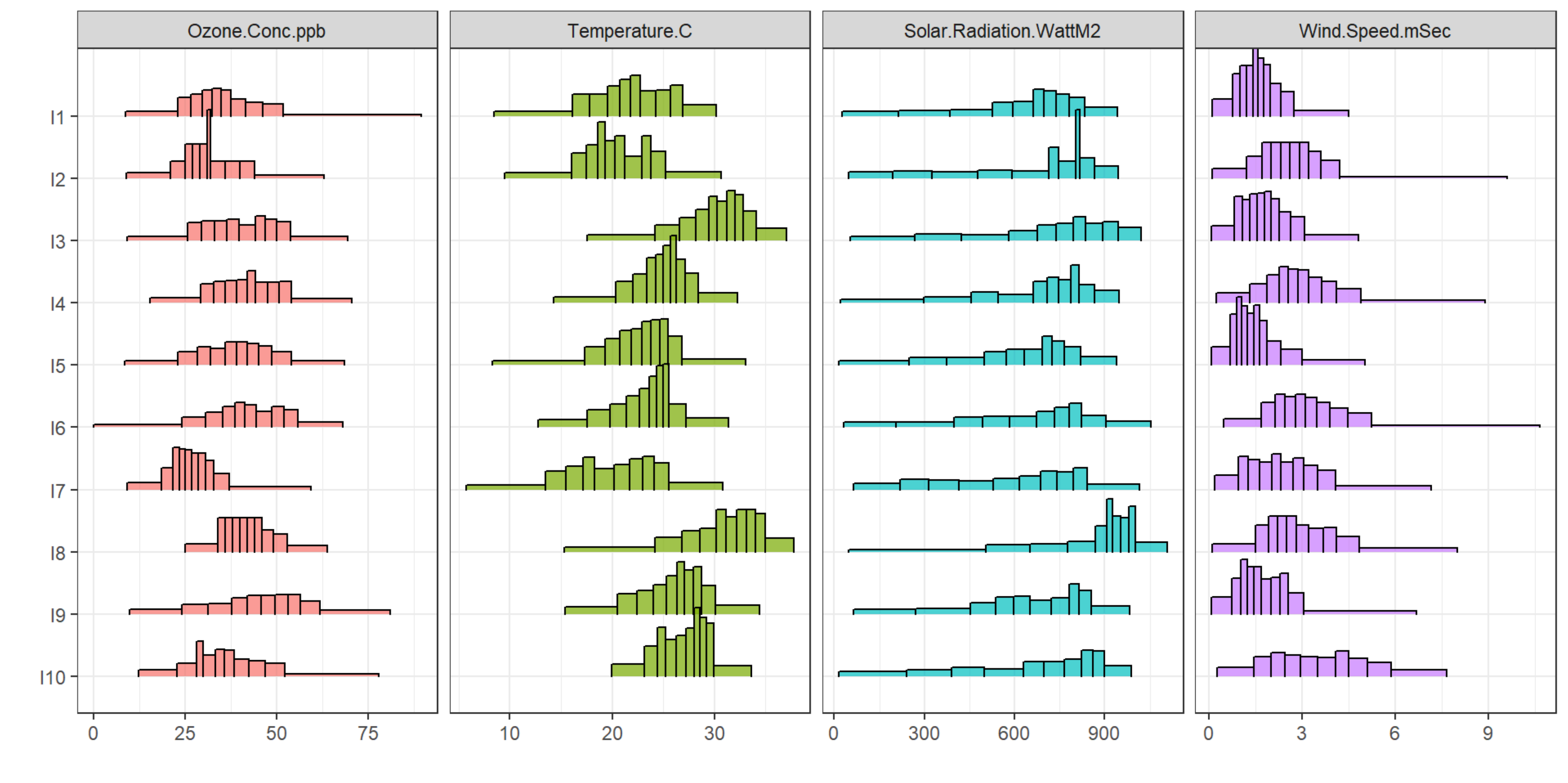
CASTNET network sites map

Ozone is a gas that can cause respiratory diseases. In the literature, several studies reported evidence of the relation between the ozone concentration level and temperature, wind speed and the solar radiation (see, for example,(Dueñas et al. 2002)).

Given the distribution of *temperature* (X_1) (degrees Celsius), the distribution of *solar radiation* (X_2) (Watts per square meter) and the distribution of *wind speed* (X_3) (meters per second), the main objective is to predict the distribution of *ozone concentration* (Y) (Particles per billion) using a linear model. CASTNET collects hourly data and, as the period of observation, we chose the summer season of 2010 and the central hours of the days (10 a.m.–5 p.m.).

We collected the histograms of the values of each site observed for the four variables. The histograms were constructed using 100 equi-frequent bins, namely, we have bins of different widths but of constant frequency. The histogram representation of varying bin-width histograms in not always pleasant, we plot the data using only ten equi-frequent bins. We show the first 5 of 78 sites.

```
plot(New_OZ[1:10,])+theme_bw()+xlab("")+ylab("")+theme(legend.position = "none")
```



The data table

European Commission

Each cell of the data table contains a histogram. We see the first three rows of the matrix

ST_ID	Bin	p	Bin1	p1	Bin2	p2	Bin3	p3
I1	8.77-16.62	0.01	8.45-11.65	0.01	25.29- 75.88	0.01	0.10-0.35	0.01
I1	16.62-17.54	0.01	11.65-13.06	0.01	75.88-108.27	0.01	0.35-0.41	0.01
I1	17.54-18.42	0.01	13.06-13.83	0.01	108.27-111.43	0.01	0.41-0.50	0.01
I1	18.42-18.90	0.01	13.83-14.12	0.01	111.43-114.09	0.01	0.50-0.55	0.01
I1	…-…	0.01	…-…	0.01	…-…	0.01	…-…	0.01
I1	65.68-67.78	0.01	28.87-29.23	0.01	914.12-933.30	0.01	3.52-3.79	0.01
I1	67.78-89.60	0.01	29.23-30.18	0.01	933.30-942.00	0.01	3.79-4.48	0.01
I2	9.00-15.00	0.01	9.50- 9.75	0.01	49.00- 56.16	0.01	0.10-0.55	0.01
I2	15.00-17.00	0.01	9.75-10.38	0.01	56.16- 71.50	0.01	0.55-0.80	0.01
I2	17.00-18.00	0.01	10.38-10.60	0.01	71.50-102.84	0.01	0.80-0.80	0.01
I2	18.00-19.00	0.01	10.60-11.24	0.01	102.84-133.40	0.01	0.80-0.90	0.01
I2	…-…	0.01	…-…	0.01	…-…	0.01	…-…	0.01
I2	54.24-58.00	0.01	29.02-29.60	0.01	910.00-916.84	0.01	7.52-8.37	0.01
I2	58.00-63.00	0.01	29.60-30.70	0.01	916.84-944.00	0.01	8.37-9.60	0.01
I3	9.25-17.99	0.01	17.57-20.13	0.01	52.57- 78.67	0.01	0.08-0.26	0.01
I3	17.99-20.31	0.01	20.13-20.63	0.01	78.67-105.48	0.01	0.26-0.38	0.01
I3	20.31-21.41	0.01	20.63-21.13	0.01	105.48-116.96	0.01	0.38-0.41	0.01
I3	21.41-22.20	0.01	21.13-21.61	0.01	116.96-140.19	0.01	0.41-0.45	0.01
I3	…-…	0.01	…-…	0.01	…-…	0.01	…-…	0.01

ST_ID	Bin	p	Bin1	p1	Bin2	p2	Bin3	p3
I3	62.38-64.11	0.01	36.10-36.42	0.01	979.18- 990.02	0.01	3.77-4.07	0.01
I3	64.11-69.45	0.01	36.42-37.07	0.01	990.02-1020.00	0.01	4.07-4.81	0.01

Basic Wasserstein-based statistics

The Frechét mean distributions of the the four variables

We start computing the Frechét mean of each distributional variable using the L_2 Wassertein distance as in (R. Irpino A.and Verde 2015)

Ozone

```
Output shows the first five and the last five bins due to eccesive length
                  X                  p
Bin_1   [14.789-19.479)             0.01
Bin_2   [19.479-21.567)             0.01
Bin_3   [21.567-22.872)             0.01
Bin_4   [22.872-23.912)             0.01
Bin_5   [23.912-24.894)             0.01
...               ...              ...
Bin_96 [57.865 ; 58.948)             0.01
Bin_97 [58.948 ; 60.278)             0.01
Bin_98 [60.278 ; 61.979)             0.01
Bin_99 [61.979 ; 64.515)             0.01
Bin_100 [64.515 ; 71.095)             0.01

 mean =  41.2147347282052   std  =  9.96802979889176
```

Temperature

```
Output shows the first five and the last five bins due to eccesive length
                 X                 p
Bin_1   [10.355-12.383)              0.01
Bin_2    [12.383-13.91)              0.01
Bin_3      [13.91-15.1)              0.01
Bin_4     [15.1-15.853)              0.01
Bin_5   [15.853-16.473)              0.01
...               ...              ...
Bin_96 [28.795 ; 29.106)              0.01
Bin_97 [29.106 ; 29.447)              0.01
Bin_98 [29.447 ; 29.888)              0.01
Bin_99 [29.888 ; 30.569)              0.01
Bin_100 [30.569 ; 31.602)              0.01

 mean =  23.2805152974359   std  =  3.76407404885491
```

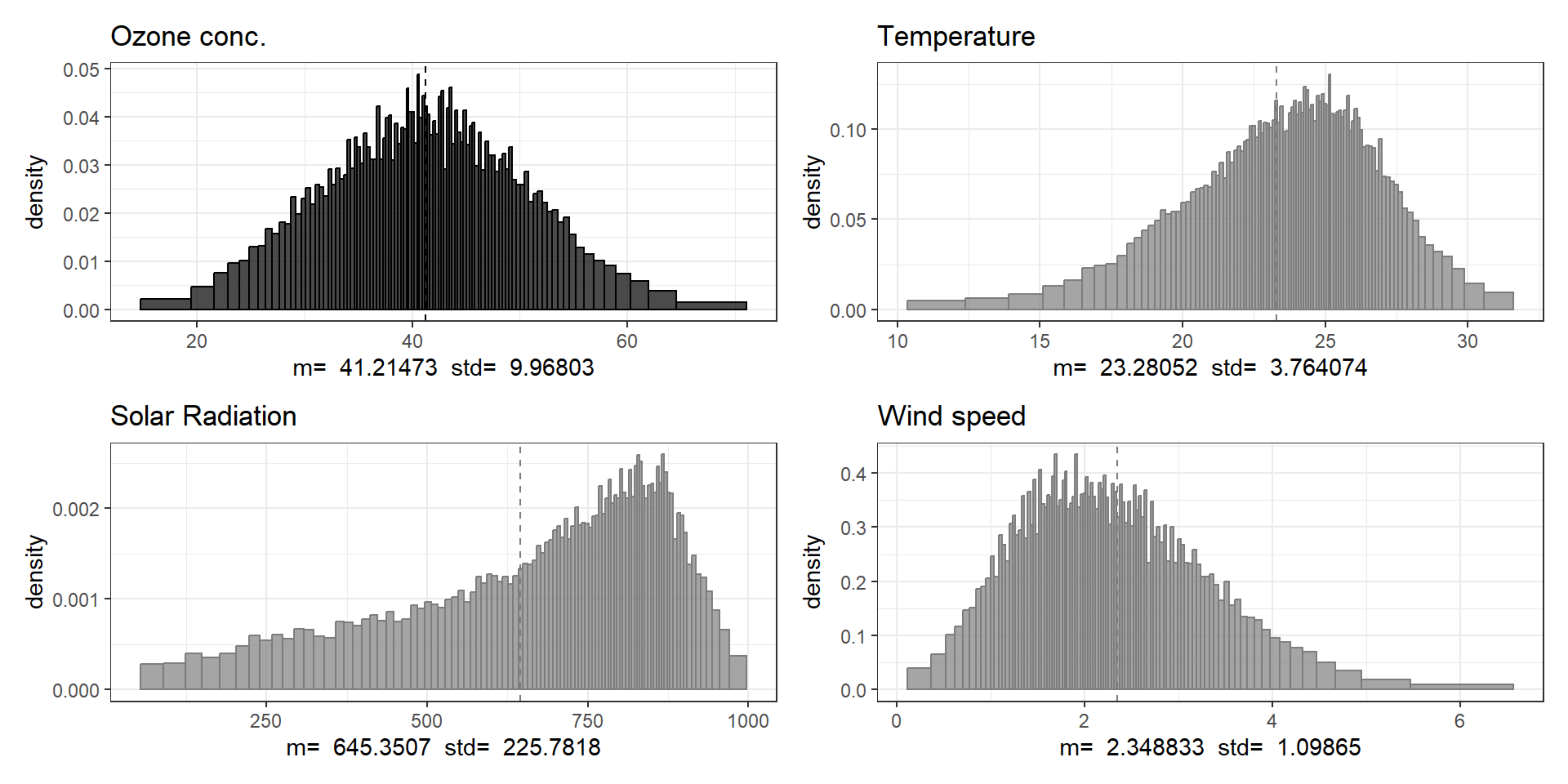
Solar Radiation

```
Output shows the first five and the last five bins due to eccesive length
                 X                  p
Bin_1   [54.186-89.738)               0.01
Bin_2   [89.738-123.48)               0.01
Bin_3   [123.48-148.68)               0.01
Bin_4    [148.68-177.1)               0.01
Bin_5    [177.1-202.25)               0.01
...               ...               ...
Bin_96 [926.63 ; 934.72)               0.01
Bin_97 [934.72 ; 943.94)               0.01
Bin_98 [943.94 ; 955.36)               0.01
Bin_99 [955.36 ; 970.49)               0.01
Bin_100 [970.49 ; 997.58)               0.01

 mean =  645.350728000001   std  =  225.781773829828
```

Wind Speed

```
Output shows the first five and the last five bins due to eccesive length
                 X                 p
Bin_1 [0.11386-0.36784)              0.01
Bin_2 [0.36784-0.52206)              0.01
Bin_3 [0.52206-0.62055)              0.01
Bin_4 [0.62055-0.70678)              0.01
Bin_5 [0.70678-0.77491)              0.01
...                 ...             ...
Bin_96 [4.3248 ; 4.4692)             0.01
Bin_97 [4.4692 ; 4.6702)             0.01
Bin_98 [4.6702 ; 4.9502)             0.01
Bin_99 [4.9502 ; 5.4696)             0.01
Bin_100 [5.4696 ; 6.5707)             0.01

 mean =  2.34883345512821   std  =  1.09865034695591
```

The plot of the four barycenters



The mean distributions for each variable

In Table [TAB:OZO_summarystat], we report the main summary statistics for the four histogram variables, while in Fig. [Fig: OZO_barycenters], we provide the four barycenters of the 78 sites for each variable. We note, for example, the different skewness of the barycenters. In general, when the barycenter is skewed, the observed distributions are in general skewed in the same direction. This is not in general true for symmetric barycenters, which can be generated both from left- and right-skewed distributions.

Basic statistics of barycenters

Variable	Mean	Std	First_Q	Median	Third_Q	Skewness	Kurtosis
Ozone	41.21	9.97	34.27	41.15	48.05	0.078	2.778
Temperature	23.28	3.76	21.04	23.71	25.93	-0.596	3.393
Solar Radiation	645.35	225.78	496.05	701.63	826.56	-0.715	2.580
Wind Speed	2.35	1.10	1.54	2.22	3.03	0.649	3.456

We show the covariances and correlations of the variables in the next two tables, and the standard deviations of the variables computed according to what is proposed in (R. Irpino A.and Verde 2015) :

```
# Covariance matrix
Cov_M<-WH.var.covar(OzoneFull)
knitr::kable(round(Cov_M,2), caption="Covariance matrix")
```

Covariance matrix

	Ozone.Conc.ppb	Temperature.C	Solar.Radiation.WattM2	Wind.Speed.mSec
Ozone.Conc.ppb	90.81	9.05	690.92	5.03
Temperature.C	9.05	14.76	197.71	0.72
Solar.Radiation.WattM2	690.92	197.71	12866.55	65.47
Wind.Speed.mSec	5.03	0.72	65.47	1.73

Standard deviations

	x
Ozone.Conc.ppb	9.53
Temperature.C	3.84
Solar.Radiation.WattM2	113.43
Wind.Speed.mSec	1.31

Correlation matrix

	Ozone.Conc.ppb	Temperature.C	Solar.Radiation.WattM2	Wind.Speed.mSec
Ozone.Conc.ppb	1.000	0.247	0.639	0.402
Temperature.C	0.247	1.000	0.454	0.143
Solar.Radiation.WattM2	0.639	0.454	1.000	0.439
Wind.Speed.mSec	0.402	0.143	0.439	1.000

Two-components regression model

Now we estimate the model

$$Y = \beta_0 + \beta_1 \mu_{X_1} + \beta_2 \mu_{X_2} + \beta_3 \mu_{X_3} + \gamma_1 X_1^c + \gamma_2 X_2^c + \gamma_3 X_3^c + \varepsilon$$

```
results<-WH.regression.two.components(OzoneFull,1,c(2:4))
knitr::kable(data.frame(Coeff.est.=round(results,3)),
             caption="regression parameters")
```

regression parameters

	Coeff.est.
(AV_Intercept)	2.927
AV_Temperature.C	-0.346
AV_Solar.Radiation.WattM2	0.070
AV_Wind.Speed.mSec	0.395
CEN_Temperature.C	0.915
CEN_Solar.Radiation.WattM2	0.018
CEN_Wind.Speed.mSec	1.887

We compute the goodness of fit statistics

```r
# we compute the expected distributions
expected<-WH.regression.two.components.predict(OzoneFull[,2:4],results)

# we compute the GOF measures

GOFS<- WH.regression.GOF(OzoneFull[,1],expected)
GOFS
```

```
$RMSE_W
[1] 7.000022

$OMEGA
[1] 0.7423388

$PSEUDOR2
$PSEUDOR2$index
[1] 0.4604191

$PSEUDOR2$details
      TotSSQ         SSQ.R         SSQ.E          Bias     SSQ.R.rel     SSQ.E.rel
7083.3190671 4070.7089580 3822.0239212 -809.4138121     0.5746895     0.5395809
SSQ.bias.rel
  -0.1142704
```

Bootstrap confidence intervals for the model parameters

We perform a bootstrap estimates of the confidence intervals of the parameters using 1,000 replications

Bootstrap results

	model.est.	boot.mean.est.	bias	q.2.5	q.97.5
(AV_Intercept)	2.927	3.209	-0.282	-11.587	15.136
AV_Temperature.C	-0.346	-0.352	0.006	-0.813	0.179
AV_Solar.Radiation.WattM2	0.070	0.070	0.000	0.051	0.090
AV_Wind.Speed.mSec	0.395	0.378	0.017	-1.301	1.942
CEN_Temperature.C	0.915	0.911	0.004	0.474	1.371
CEN_Solar.Radiation.WattM2	0.018	0.018	0.000	0.012	0.024
CEN_Wind.Speed.mSec	1.887	1.973	-0.087	1.044	3.118

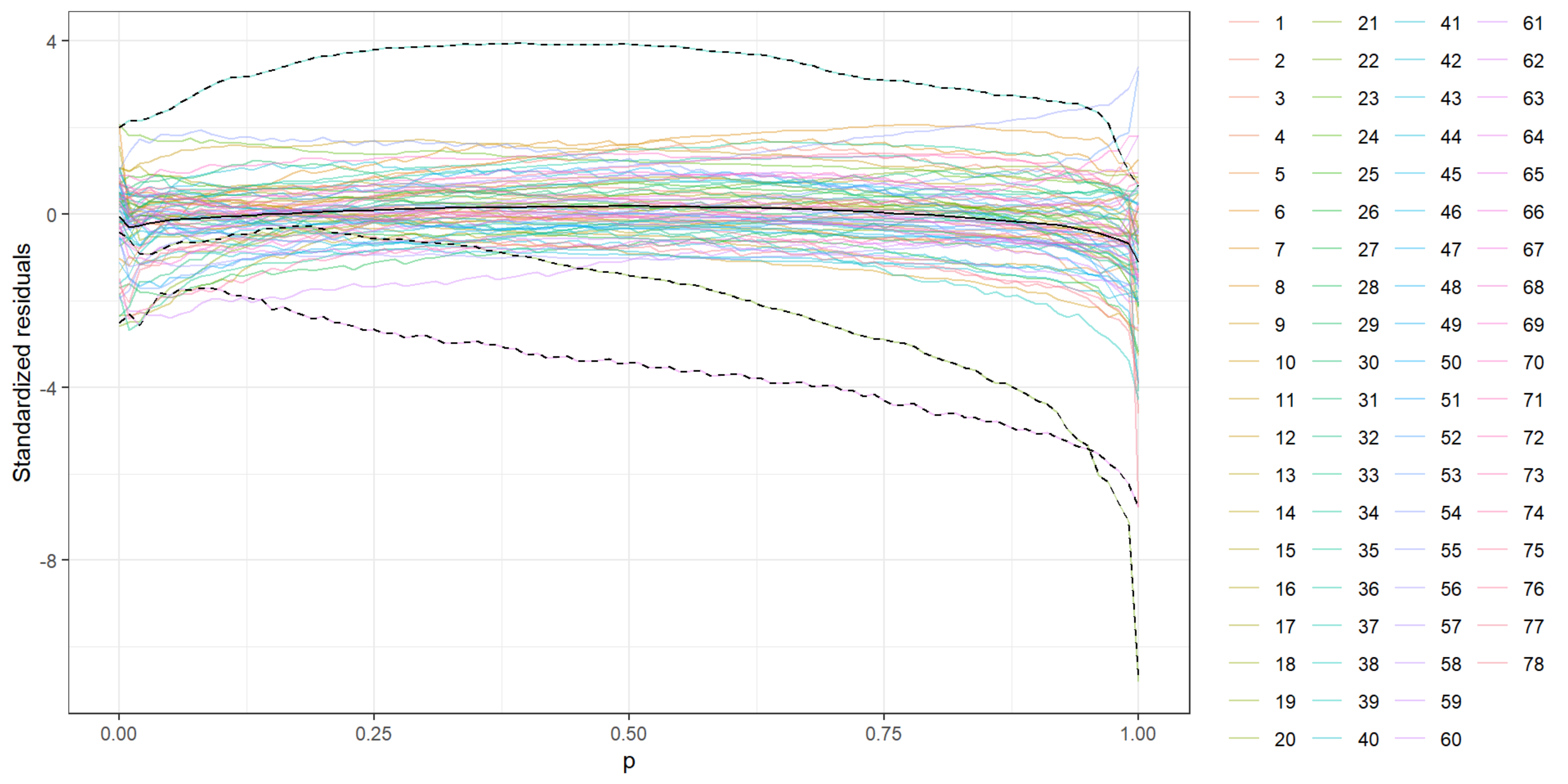
Reading the bootstrap results, we may assert that the ozone concentration distribution of a site depends on the mean *solar radiation*, where for each $\Delta Watt/m^2$ a $0.070\ (ppb)$ variation of the *ozone concentration* mean level is expected, while in general we cannot say that the mean levels of *temperature* and *wind speed* induce a significant variation of the *ozone concentration level*(95% bootstrap confidence intervals include the zero). Furthermore, the variability of the *ozone concentration* is almost the same of the *temperature*(0.928); a unitary variation in the variability of the *solar radiation* induces a variation of $0.018\ (ppb)$ and a variation in the variability of the *wind speed* causes an increase in the variability of $1.958\ (ppb)$.

Residual analysis

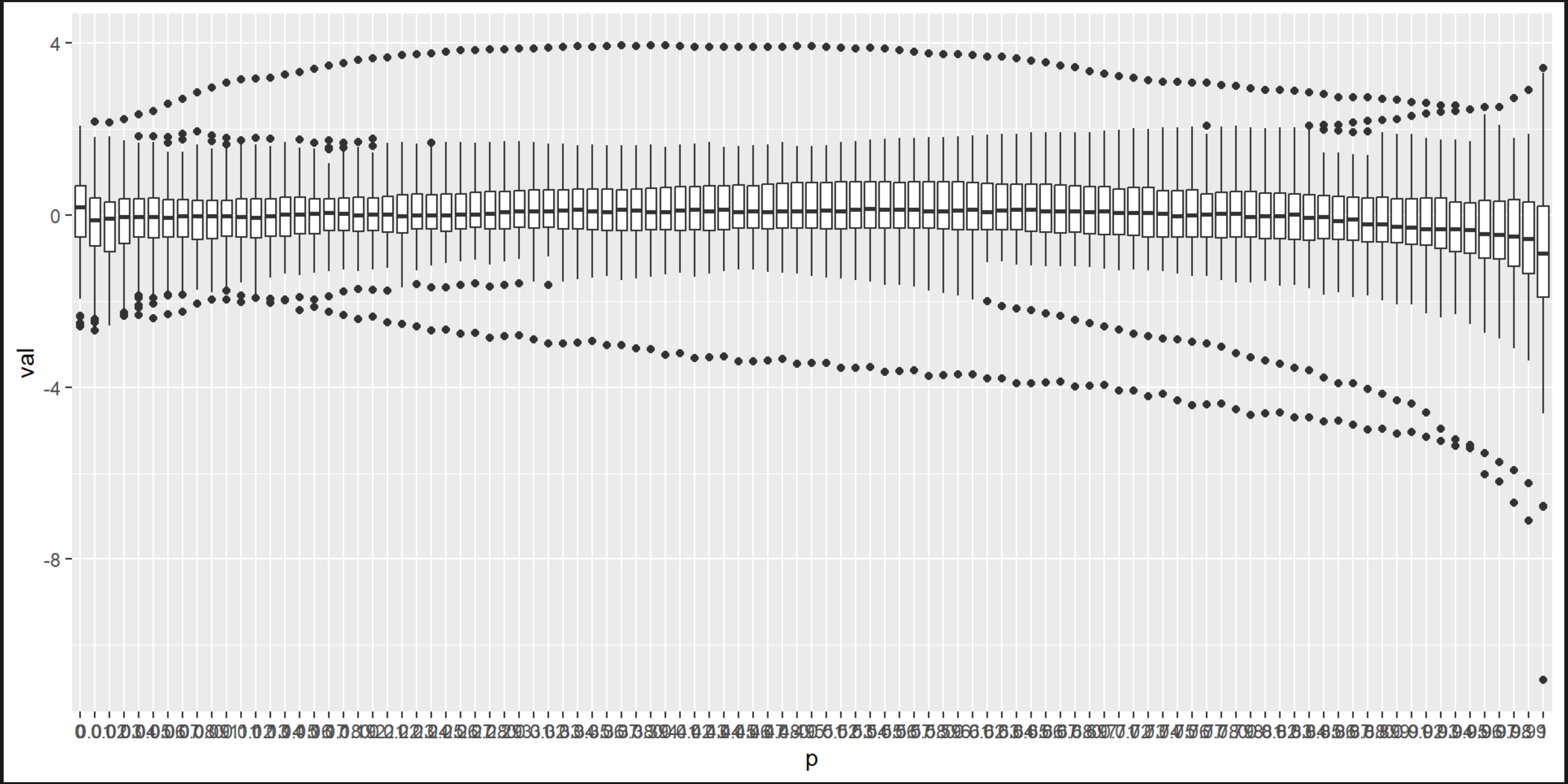
```r
Res<-matrix(0,n,length(expected@M[1,1][[1]]@x))
C<-matrix(0,n,length(expected@M[1,1][[1]]@x)-1)
R<-matrix(0,n,length(expected@M[1,1][[1]]@x)-1)
for(i in 1:n){
  Res[i,]<-expected@M[i,1][[1]]@x-OzoneFull@M[i,1][[1]]@x
  R[i,]<-diff(expected@M[i,1][[1]]@x-OzoneFull@M[i,1][[1]]@x)/2
  C[i,]<-(Res[i,1:(length(Res[i,])-1)]+
             Res[i,2:(length(Res[i,]))])/2
}

SDRES<-sqrt((0.01*(sum(C^2)-sum(colMeans(C)^2)+
                    1/3*(sum(R^2)-sum(colMeans(R)^2))))/n)

Res_st<-data.frame(ID=c(1:n),Res/SDRES)
Res<-data.frame(ID=c(1:n),Res)
```

```r
MyL<-Res %>% pivot_longer(-ID,names_to = "var",values_to = "val")%>% mutate(Q=rep(0:100,78)/100)
MyL_st<-Res_st %>% pivot_longer(-ID,names_to = "var",values_to = "val")%>% mutate(Q=rep(0:100,78)/100)
```

Plot of standardized residual functions



Boxplot of residual functions



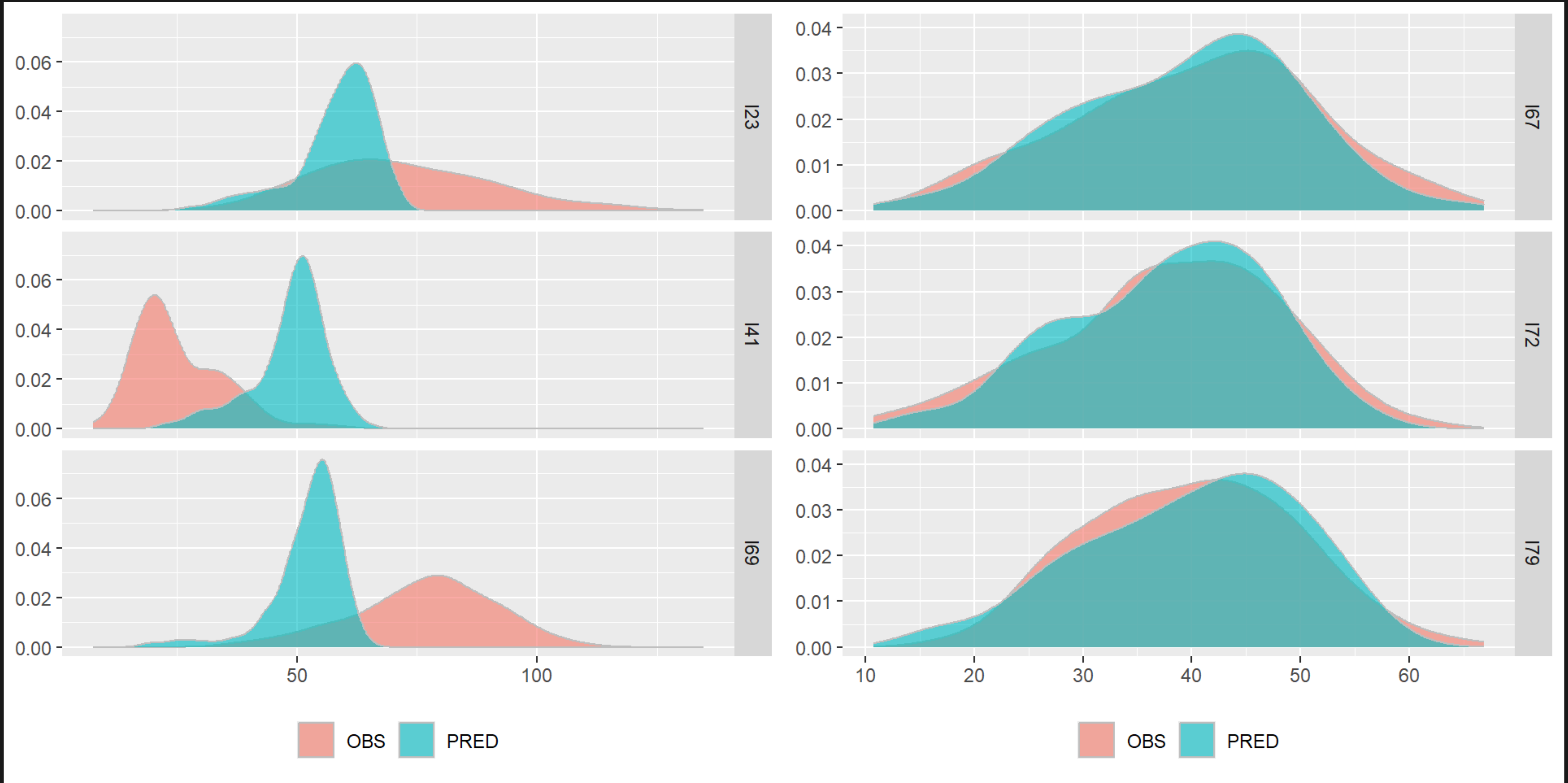
Observe that:

In general the standardized error functions are inside the $\pm 2\sigma$ band. Looking at the point-wise boxplots, we see that in general the average error function is close to the zero line, suggesting that the process generating the error functions has zero mean. Also the boxplots seem to suggest that the process have constant variance in $[0;1]$.

We note that stations $I23$, $I41$, and $I69$ present error functions which have the most part of the values far more than $\pm 2\sigma$.

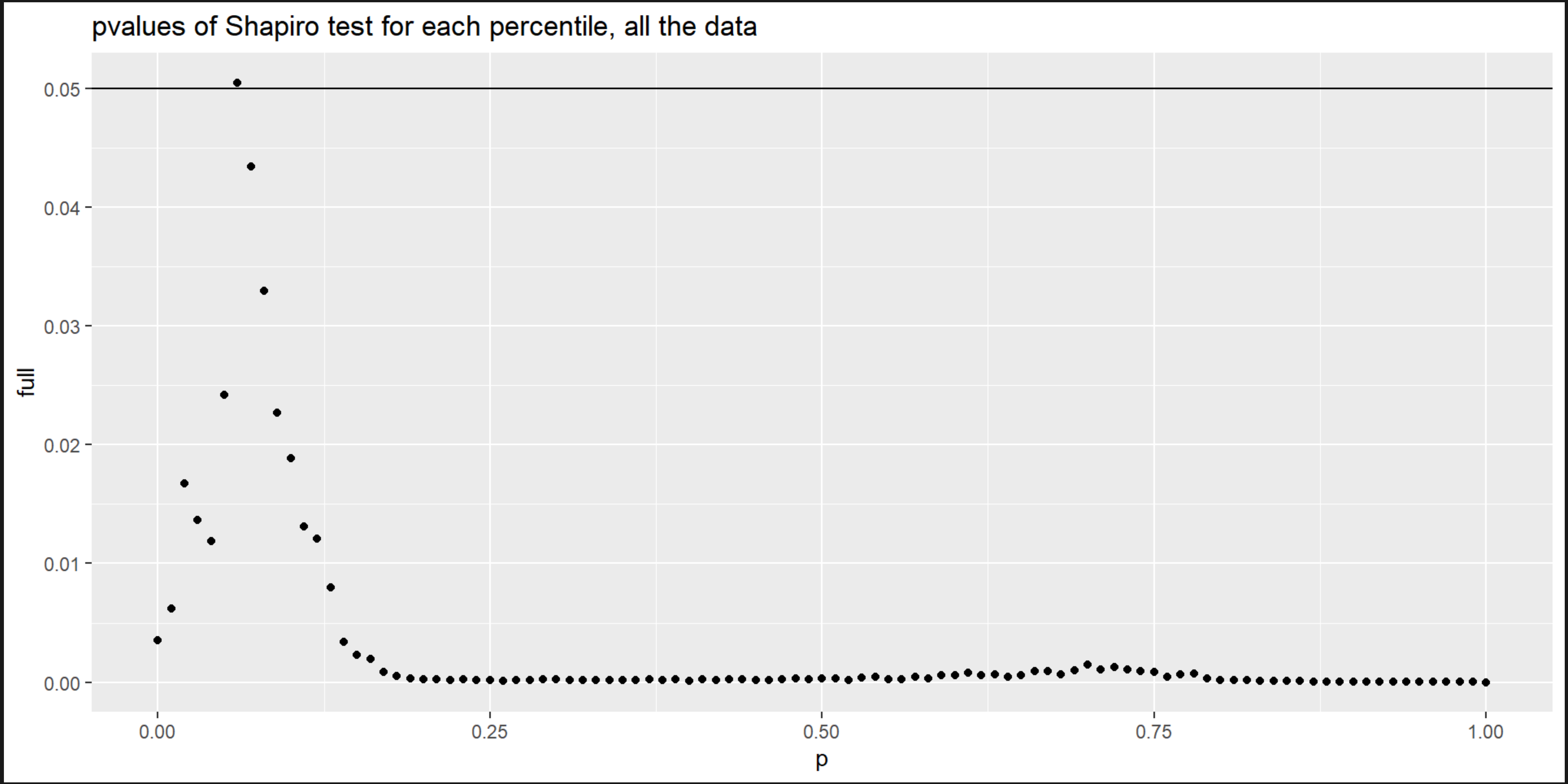
Worst and best predictions

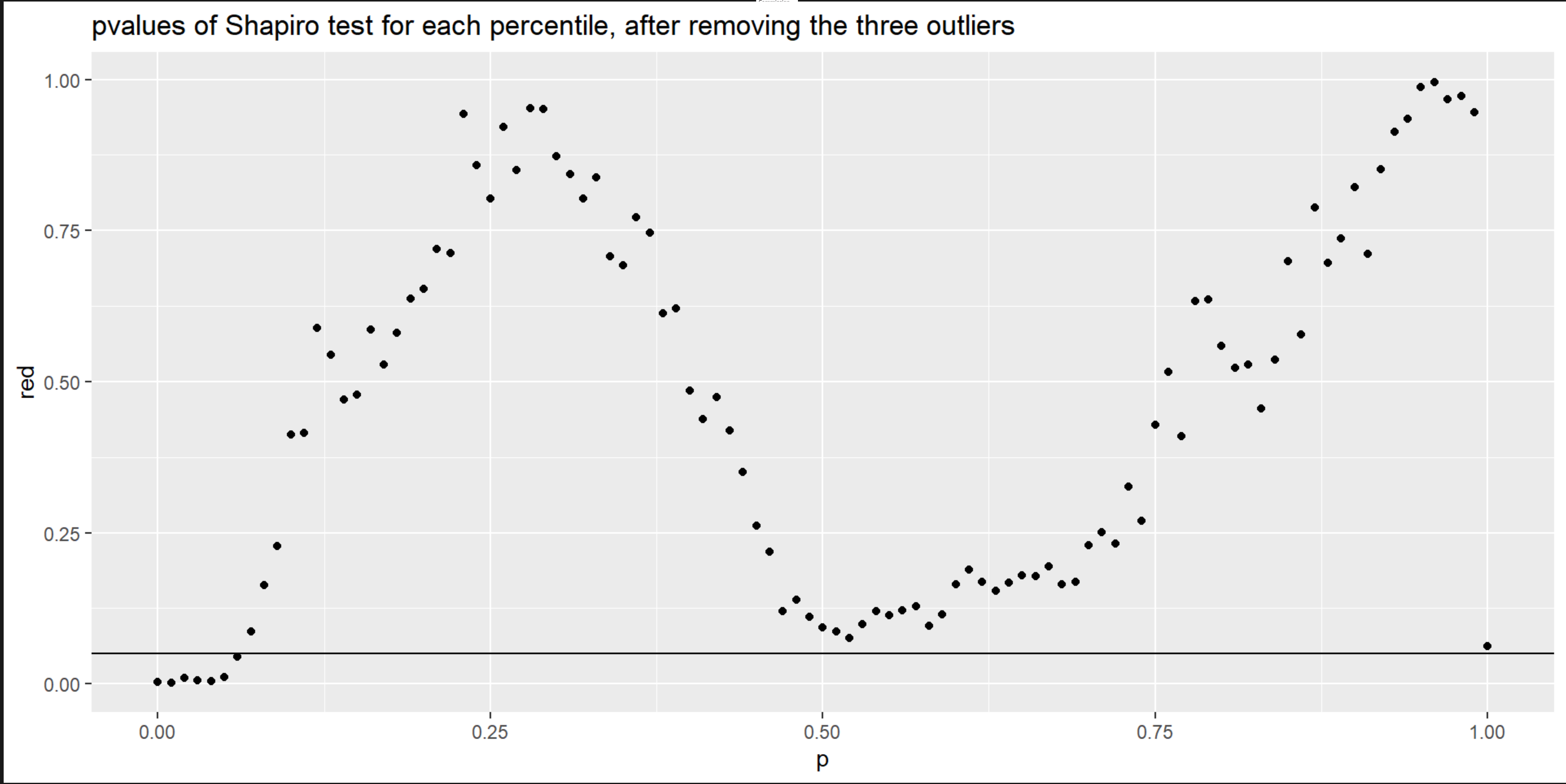
```r
HistDAWass::plotPredVsObs(expected[c(22,61,37,66,63,73),],OzoneFull[c(22,61,37,66,63,73),1],type="DENS")
```



Pointwise testing for normality

We compute the Shapiro test for each percentile with and without outliers.





It seems that removing the outliers the process is almost Gaussian after $p = 0.07$.

Conclusions

- The two-components regression for distributional data shows interesting patterns related to the position and the internal variability of distributional data.
- If data are intervals, we can consider a interval as a uniform distributions or a one-bin histograms.
- You can use histograms with different binning thanks to Wasserstein.
- If the domain of distribution is discrete, it is easy to generalize the method (Actually we are developing that).
- The use of scalar parameters helps interpretation, but the final fitting can be biased. We are developing some *correction* tools for that.

References

Dueñas, C., M. C. Fernández, S. Cañete, J. Carretero, and E. Liger. 2002. "Assessment of Ozone Variations and Meteorological Effects in an Urban Area in the Mediterranean Coast." *Science of The Total Environment* 299 (1-3): 97–113. https://doi.org/10.1016/s0048-9697(02)00251-6.

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