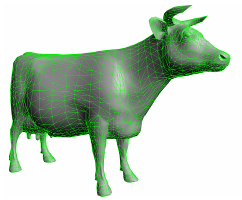
Entropy Coding

Connectivity coding



Connectivity data

Add 7, Add 6, Add 7, Add 5,...

TG output

CRRRLSLECRRE

Edgebreaker output

0010101010...

Entropy coder output

Entropy coding

- Lossles coder
- Input: a set of symbols
- Output: bitstream
- Idea
  - Assign each symbol a series of bits
  - Use less bits for common symbols

Definitions

- **Alphabet**
  Finite set containing at least one element
- **Symbol**
  Element in the alphabet
- **A string over the alphabet**
  Sequence of symbols from alphabet
- **Codeword**
  Bits representing coded symbol or string
- p_i
  Occurrence probability of s_i in input string
- L_i
  Length of codeword of s_i in bits

- **A** = {a, b, c, d, e}
- $s_i \in$ **A**
- S = ccdabdcaad
- 1101010011010100
- $p_i = P(s_i \in S), \quad \sum_{\forall i \in A} p_i = 1$

Entropy

- ***Entropy*** of the set $\{e_1,\ldots,e_n\}$ with probabilities $\{p_1,\ldots,p_n\}$

$$H\left(p_1,\ldots,p_n\right) \equiv -\sum_{i=1}^{n} p_i \log_2 p_i$$

- $-\log_2 p_i$ = **uncertainty** in symbol e_i
  - The "surprise" when we see this symbol
  - Entropy – average "surprise" on all symbols

- In our context
  - Minimal number of bits **on the average**, needed to represent a symbol
  - Average on all symbols code lengths
  - Assuming no dependencies between symbols' appearances

Entropy example 1

Entropy calculation for a two symbol alphabet.

Example 1:	A	p_A=0.5
	B	p_B=0.5

$$H(A,B) = -p_A \log_2 p_A - p_B \log_2 p_B =$$
$$= -0.5\log_2 0.5 - 0.5\log_2 0.5 = 1$$

We need 1 bit per symbol on average to represent the data.

Entropy example 2

Entropy calculation for a two symbol alphabet.

Example 1:	A	p_A=0.8
	B	p_B=0.2

$$H(A,B) = -p_A \log_2 p_A - p_B \log_2 p_B =$$
$$= -0.8\log_2 0.8 - 0.2\log_2 0.2 \cong 0.7219$$

We need LESS than 1 bit per symbol on average

Entropy examples

- Entropy of $\{e_1,\ldots e_n\}$ is **maximized** when

  $p_1=p_2=\ldots=p_n=1/n \rightarrow H(e_1,\ldots,e_n)=\log_2 n$

  - No symbol is "better" than the other or contains more information
  - 2^k symbols must be represented by k bits

- Entropy of $\{e_1,\ldots e_n\}$ is **minimized** when

  $p_1=1,\ p_2=\ldots=p_n=0 \rightarrow H(e_1,\ldots,e_n)=0$

Entropy coding

- Entropy
  - ***Lower bound*** on average number of bits needed for alphabet
  - Data compression limit

- Coding efficiency = *Bits Per Symbol*

$$BPS = \frac{\text{length(encoded message)}}{\text{length(original message)}}$$

- Entropy coding methods
  - Try to achieve entropy of alphabet: BPS → Entropy
  - If BPS = Entropy, code is optimal

Code types

- **Fixed-length codes**
  - All codewords have same length (number of bits)

A – 000, B – 001, C – 010, D – 011, E – 100, F – 101

- **Variable-length codes**
  - Codewords can have different lengths

A – 0, B – 00, C – 110, D – 111, E – 1000, F - 1011

Code types

- **Prefix code**
  - No codeword is a prefix of any other codeword

A = 0, B = 10, C = 110, D = 111

- **Uniquely decodable code**
  - Has only one possible source string producing it
  - Unambigously decoded
  - Examples:
    - Prefix code - end of codeword recognized without ambiguity

010011001110 → 0 | 10 | 0 | 110 | 0 | 111 | 0

    - Fixed-length code

Huffman code

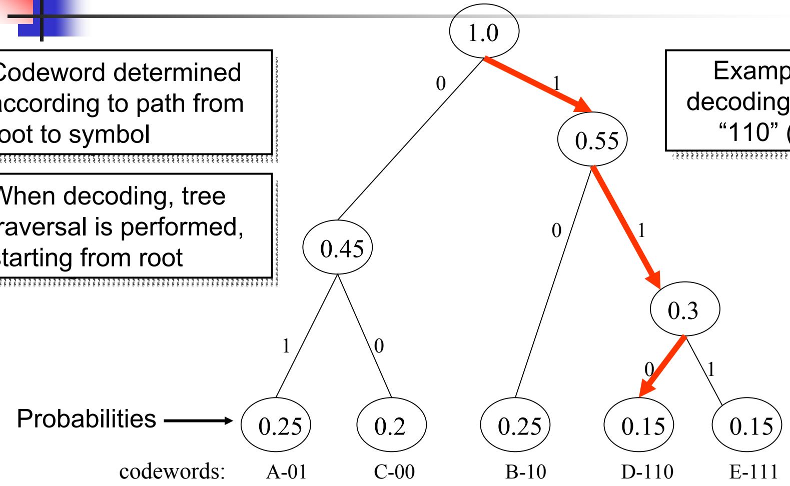
- A variable-length prefix code
- Codeword chosen by probability of appearance
  - High probability → short codeword
- Integral number of bits per codeword
- **Optimal** variable-length prefix code for known probablilities
- Encoding/decoding done using Huffman tree

Huffman tree example

Codeword determined according to path from root to symbol

When decoding, tree traversal is performed, starting from root

Example: decoding input "110" (D)



Huffman encoding example

Use previous codewords to encode "BCAE":

String:	B	C	A	E
Encoded:	10	00	01	111

Number of bits used: 9

The BPS is (9 bits/4 symbols) = **2.25**

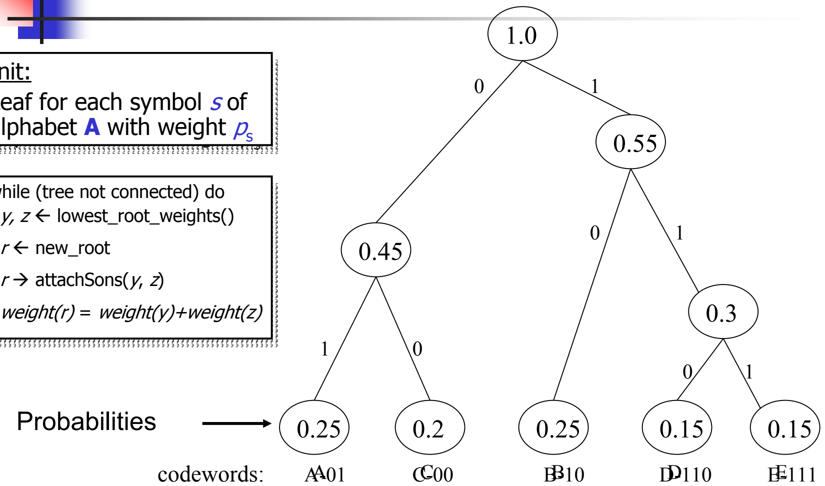
Entropy: - 0.25log0.25 - 0.25log0.25 - 0.2log0.2 - 0.15log0.15 - 0.15log0.15 = **2.2854**

BPS lower than entropy. **WHY ?**

Huffman tree construction

Init:
Leaf for each symbol *s* of alphabet **A** with weight p_s

while (tree not connected) do
  y, *z* ← lowest_root_weights()
  r ← new_root
  r → attachSons(*y*, *z*)
  weight(r) = *weight(y)+weight(z)*



Huffman tree construction

- Initialization
  - Leaf for each symbol *s* of alphabet **A** with weight p_s
  - Can work instead with integer weights - number of occurrences
- while (tree not connected) do
  - *Y*, *Z* ← lowest_root_weights_tree()
  - r ← new_root
  - r->attachSons(Y, Z)
    - attach one via a 0, the other via a 1, order not significant
  - *weight(r)* = *weight(Y)+weight(Z)*

Huffman encoding

- Build a table of per-symbol encodings - generated from Huffman tree
  - Globally known to both encoder and decoder
  - Sent by encoder, read by decoder
- Encode one symbol after the other, using encoding table.
- Encode the pseudo-eof symbol.

Huffman decoding

- Construct decoding tree based on encoding table
- Read coded message bit-by-bit
  - Traverse the tree top to bottom accordingly
  - When a leaf is reached, a codeword was found → corresponding symbol is decoded
- Repeat until the pseudo-eof symbol is reached
- No ambiguities - prefix code

Symbol probabilities

- How are the probabilities known?
  - Counting symbols in input string
    - Data must be given in advance
    - Requires an extra pass on the input string
  - Data source's distribution is known
    - Data not known in advance, but distribution is known

Example
"Global" English frequencies table

Letter	Prob.	Letter	Prob.
A	0.0721	N	0.0638
B	0.0240	O	0.0681
C	0.0390	P	0.0290
D	0.0372	Q	0.0023
E	0.1224	R	0.0638
F	0.0272	S	0.0728
G	0.0178	T	0.0908
H	0.0449	U	0.0235
I	0.0779	V	0.0094
J	0.0013	W	0.0130
K	0.0054	X	0.0077
L	0.0426	Y	0.0126
M	0.0282	Z	0.0026
Total: 1.0000			

Huffman entropy analysis

- Best results - entropy wise
  - **Only** when occurrence probabilities are negative powers of 2 (i.e. ½, ¼, …). Otherwise, BPS > entropy bound.
- Example

Symbol	Probability	Codeword
A	0.5	1
B	0.25	01
C	0.125	001
D	0.125	000

- Entropy = **1.75**
- An input stream which represens the probabilities
  - **AAAABBCD** Code: **11110101001000**
- BPS = (14 bits/8 symbols) = **1.75**

Huffman tree
Construction complexity

- Simple implementation - O(n^2).
- Using a Priority Queue - O($n \cdot \log(n)$):
  - Inserting a new node – O(log(n))
  - n nodes insertions - O($n \cdot \log(n)$)
  - Retrieving 2 smallest node weights – o(log(n))

Huffman summary

- Achieves entropy when occurrence probabilities are negative powers of 2
- Alphabet and distribution must be known in advance
- Given Huffman tree, very easy (and fast) to encode and decode
- Huffman code not unique (arbitrary decisions in tree construction)

Better than Huffman?

- Huffman optimal, so how can improve?
- Use fractional number of bits per codeword
  - Arithmetic coding
- Learn probabilities from bit stream
  - Lempel-Ziv coding
    - Unknown alphabet
    - Unknown probabilities
    - Handles dependencies between symbols