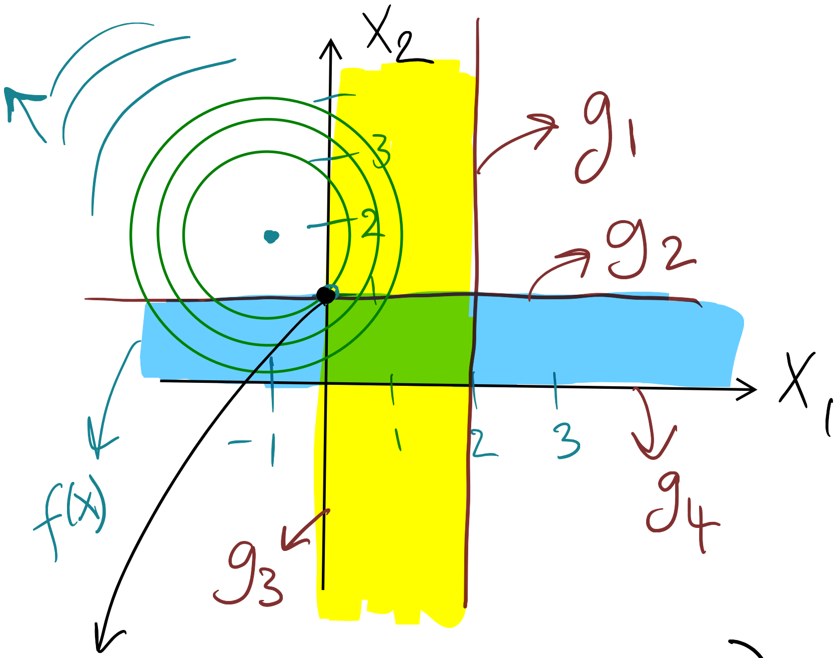
1) $\min_{x_1, x_2} \; f(x) = (x_1+1)^2 + (x_2-2)^2$

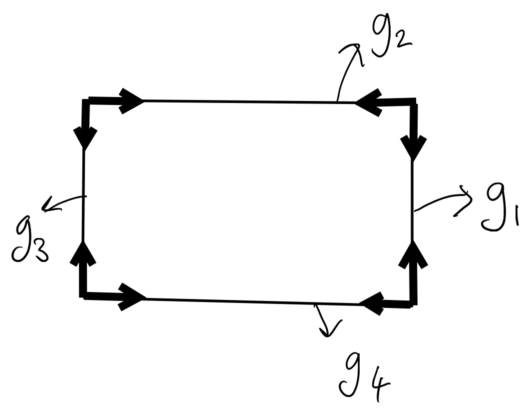
s.t. $g_1 = x_1 - 2 \le 0, \quad g_3 = -x_1 \le 0$

$g_2 = x_2 - 1 \le 0, \quad g_4 = -x_2 \le 0$




minimum point $(x_1, x_2) = (0, 1)$

## Feasible descent:




Root point: $x = -1, \; y = 2$

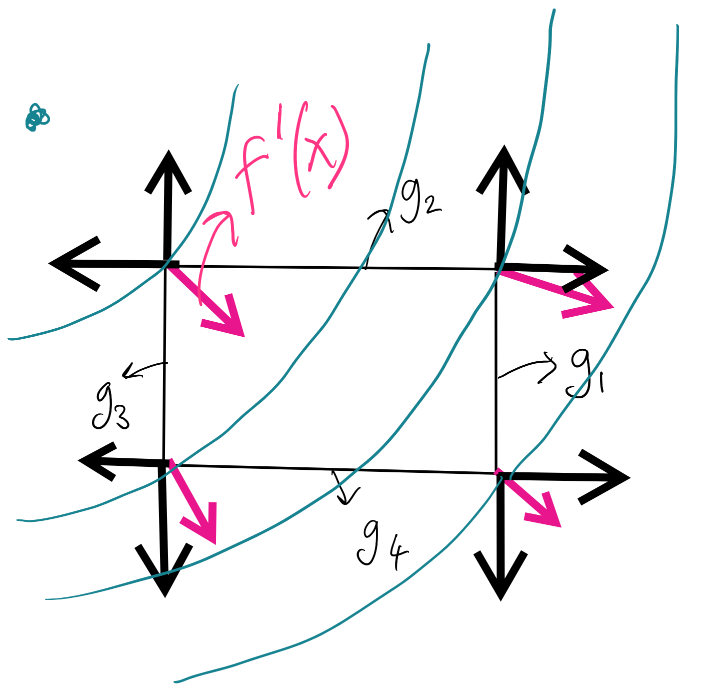
$g_1 = x_1 - 2 \le 0 \Rightarrow x_1 \le 2$

$g_2 = x_2 - 1 \le 0 \Rightarrow x_2 \le 1$

$g_3 = -x_1 \le 0 \Rightarrow x_1 \ge 0$

$g_4 = -x_2 \le 0 \Rightarrow x_2 \ge 0$

## Gradient Directions:



$$L = f + \lambda^T h + \hat{\mu}^T \hat{g}$$

$$\mathcal{L} = (x_1+1)^2 + (x_2-2)^2 + \mu_1(x_1-2) + \mu_2(x_2-1) + \mu_3(-x_1) + \mu_4(-x_2)$$

$$\frac{\partial L}{\partial x} = \begin{bmatrix} 2(x_1+1)+\mu_1-\mu_3 \\ 2(x_2-2)+\mu_2-\mu_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 2(0+1)+\cancel{\mu_1}-\mu_3 \\ 2(1-2)+\mu_2-\cancel{\mu_4} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\mu_1$ and $\mu_4$ are inactive

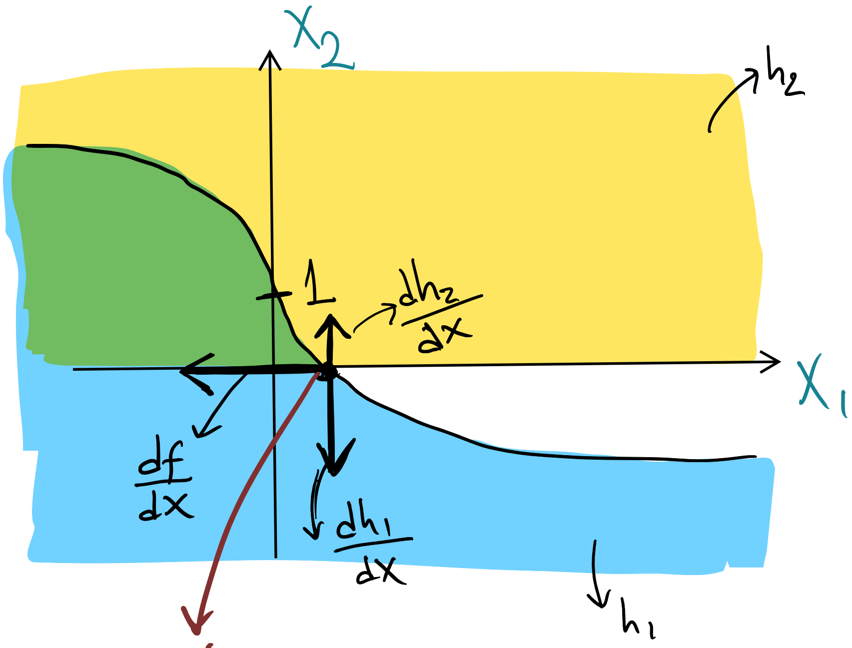
$$2-\mu_3 = 0 \Rightarrow \boxed{\mu_3 = 2 > 0}$$

$$-2+\mu_2 = 0 \Rightarrow \boxed{\mu_2 = 2 > 0}$$

Force balance occurs. This is the minimum point.

2) $\min_{x_1, x_2} \quad f = -x_1 + 0x_2 \rightarrow \max_{x_1, x_2} \quad f = x_1 - 0x_2$

s.t. $\quad \underbrace{g_1 = x_2 - (1-x_1)^3 \leq 0}_{h_1}$ and $\underbrace{x_2 \geq 0}_{h_2}$




$$\frac{df}{dx} = \begin{bmatrix} \partial f/\partial x_1 \\ \partial f/\partial x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$\frac{dh_1}{dx} = \begin{bmatrix} \partial h_1/\partial x_1 \\ \partial h_1/\partial x_2 \end{bmatrix} = \begin{bmatrix} 3(1-x_1)^2 \\ 1 \end{bmatrix}$$

$$\frac{dh_2}{dx} = \begin{bmatrix} \partial h_2/\partial x_1 \\ \partial h_2/\partial x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Solution since $x_1$ is at a maximum

KKT conditions cannot be applied since the derivatives of the constraints are orthogonal to the derivative of the objective function. In other words, forces from the constraints do not cancel out the force from the function.

3) $$\max_{x_1, x_2, x_3} \quad f = x_1 x_2 + x_2 x_3 + x_1 x_3$$

$$\text{s.t.} \quad h = x_1 + x_2 + x_3 - 3 = 0$$

Reduced Gradient:

$m = 1$, $n = 3$, $s: x_1$, $d: x_2, x_3$, $def = n - m = 2$

$$\bar{x} = \begin{bmatrix} \bar{d} \\ s \end{bmatrix} = \bar{x} = \begin{bmatrix} x_2 \\ x_3 \\ x_1 \end{bmatrix}$$

$$\boxed{\frac{df}{dd} = \frac{\partial f}{\partial d} - \left(\frac{\partial f}{\partial s}\right)\left(\frac{\partial h}{\partial s}\right)^{-1} \frac{\partial h}{\partial d}} \rightarrow \text{Reduced gradient}$$

$$\frac{\partial f}{\partial d} = \begin{bmatrix} \frac{\partial f}{\partial x_2} \\ \frac{\partial f}{\partial x_3} \end{bmatrix} = \begin{bmatrix} x_1 + x_3 \\ x_2 + x_1 \end{bmatrix}, \quad \frac{df}{\partial s} = \frac{\partial f}{\partial x_1} = x_2 + x_3$$

$$\frac{\partial h}{\partial s} = \frac{\partial h}{\partial x_1} = 1, \quad \frac{\partial h}{\partial d} = \begin{bmatrix} \frac{\partial h}{\partial x_2} \\ \frac{\partial h}{\partial x_3} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\frac{df}{dd} = \begin{bmatrix} x_1 + x_3 \\ x_2 + x_1 \end{bmatrix} - \begin{bmatrix} x_2 + x_3 \end{bmatrix} \begin{bmatrix} \frac{1}{1} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 + x_3 \\ x_2 + x_1 \end{bmatrix} - \begin{bmatrix} x_2 + x_3 \end{bmatrix}$$

$\hookrightarrow x_1 + \cancel{x_3} - (x_2 + \cancel{x_3}) = 0 \Rightarrow \boxed{x_1 = x_2}$

$\hookrightarrow \cancel{x_2} + x_1 - (\cancel{x_2} + x_3) = 0 \Rightarrow \boxed{x_1 = x_3}$

$\rightarrow x_1 = x_2 = x_3 = x$

$x_1 + x_2 + x_3 - 3 = 0 \rightarrow x + x + x - 3 = 0$

$3x = 3 \Rightarrow \boxed{x_1 = x_2 = x_3 = 1}$

# Lagrangian

$$\mathcal{L}(x,\lambda) = f + \lambda^T h$$

$$L(x_1, x_2, x_3, \lambda) = x_1x_2 + x_2x_3 + x_1x_3 + \lambda(x_1 + x_2 + x_3 - 3)$$

$$\frac{\partial L}{\partial x} = \begin{bmatrix} x_2 + x_3 + \lambda \\ x_1 + x_3 + \lambda \\ x_2 + x_1 + \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} x_2 + x_3 = -\lambda & (1) \\ x_1 + x_3 = -\lambda & (2) \\ x_2 + x_1 = -\lambda & (3) \end{matrix}$$

$$(1) = (2) = x_2 + \cancel{x_3} = x_1 + \cancel{x_3} \Rightarrow x_1 = x_2$$

$$(2) = (3) = \cancel{x_1} + x_3 = x_2 + \cancel{x_1} \Rightarrow x_3 = x_2$$

$$\boxed{x_1 = x_2 = x_3} = x$$

$$x_1 + x_2 + x_3 - 3 = 0 \longrightarrow x + x + x - 3 = 0$$

$$3x = 3 \Rightarrow \boxed{x_1 = x_2 = x_3 = 1}$$

$$(1) \Rightarrow x_2 + x_3 = -\lambda$$

$$1 + 1 = -\lambda$$

$$\boxed{\lambda = -2}$$

* Note: Lagrangian is easier/quicker when dealing w/ multi-variate problems

4) $\max\limits_{x_1, x_2} \quad f = 2x_1 + bx_2$

$$g_1 = x_1^2 + x_2^2 - 5 \le 0$$
$$g_2 = x_1 - x_2 - 2 \le 0$$

Given $x_1 = 1,\ x_2 = 2$

$\xrightarrow{\text{active}}$ $1^2 + 2^2 \le 5 \Rightarrow 5 = 5$

$\xrightarrow{\text{inactive}}$ $1 - 2 \le 2 \Rightarrow -2 < 2$

## Reduced Gradient:

$m = 1,\ n = 2,\ s: x_1,\ d: x_2,\ \text{def} = n - m = 1$

$$X = \begin{bmatrix} d \\ s \end{bmatrix} = X = \begin{bmatrix} x_2 \\ x_1 \end{bmatrix}$$

$$\frac{df}{dd} = \frac{\partial f}{\partial d} - \left(\frac{\partial f}{\partial s}\right)\left(\frac{\partial g}{\partial s}\right)^{-1} \frac{\partial g}{\partial d}$$

$$\frac{\partial f}{\partial d} = \frac{\partial f}{\partial x_2} = b, \quad \frac{\partial f}{\partial s} = \frac{\partial f}{\partial x_1} = 2$$

$$\frac{\partial g}{\partial s} = \frac{\partial g_1}{\partial x_1} = 2x_1, \quad \frac{\partial g}{d} = \frac{\partial g_1}{\partial x_2} = 2x_2$$

$$\frac{df}{dd} = b - 2\left(\frac{1}{2x_1}\right)(2x_2) = 0 \Rightarrow b - 2\frac{x_2}{x_1} = 0 \Rightarrow x_1 b - 2x_2 = 0 \Rightarrow \boxed{x_2 = \frac{x_1 b}{2}}$$

$x_1 b - 2x_2 = 0$

$(1)b - 2(2) = 0$

$\boxed{b = 4}$

$g_1 = x_1^2 + x_2^2 - 5 \le 0$

$x_1^2 + \left(\frac{x_1 b}{2}\right)^2 - 5 \le 0$

$1 + \left(\frac{1 \cdot b}{2}\right)^2 - 5 \le 0$

$\frac{b^2}{4} - 4 \le 0$

$b^2 \le 16$

$\boxed{\begin{matrix} b \le 4 \\ b \ge -4 \end{matrix}}$

$g_2 = x_1 - x_2 - 2 \le 0$

$x_1 - \frac{x_1 b}{2} \le 2$

$1 - \frac{b}{2} \le 2$

$-\frac{b}{2} \le 1$

$\boxed{b \ge -2}$

5) $\min_{x_1, x_2, x_3} \quad f = x_1^2 + x_2^2 + x_3^2$

s.t

$$h_1 = \frac{x_1^2}{4} + \frac{x_2^2}{5} + \frac{x_3^2}{25} - 1 = 0$$

$$h_2 = x_1 + x_2 - x_3 = 0$$

① $n = 3$, $m = 2$ $>$ des $= n - m = 1$

$s = [x_1 \quad x_2]$, $d = [x_3]$

② $x_0 = [s_0, d_0] = [x1_0 \quad x2_0 \quad x3_0]$

$$h(x_0) = \begin{bmatrix} \frac{x_1^2}{4} + \frac{x_2^2}{5} + \frac{x_3^2}{25} \\ x_1 + x_2 - x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \text{Assume } x_1 = x_2$$

$$2x_1 = x_3 \Rightarrow \frac{x_1^2}{4} + \frac{x_1^2}{5} + \frac{(2x_1)^2}{25} = 1 \Rightarrow 0.49x_1^2 = 1$$

$$\Rightarrow \boxed{x_1 = 1.3}, \boxed{x_2 = 1.3}, \boxed{x_3 = 2.6} \text{ guesses}$$

$x_0 = [1.3 \quad 1.3 \quad 2.6]$

③ $$\underset{1\times 1}{\frac{df}{dd}} = \frac{\partial f}{\partial d} - \frac{\partial f}{\partial s}\left(\frac{\partial h}{\partial s}\right)^{-1}\frac{\partial h}{\partial d}$$

$$\underset{1\times 1}{\frac{\partial f}{\partial d} = 2x_3}, \quad \underset{1\times m}{\frac{\partial f}{\partial s} = [2x_1 \quad 2x_2]}, \quad \underset{m\times m}{\frac{\partial h}{\partial s} = \begin{bmatrix} \frac{2}{4}x_1 & \frac{2}{5}x_2 \\ 1 & 1 \end{bmatrix}}, \quad \underset{2\times 1}{\frac{\partial h}{\partial d} = \begin{bmatrix} \frac{2x_3}{25} \\ -1 \end{bmatrix}}$$

④ $\varepsilon < \frac{df_0}{dd_0}$

While $\left\|\frac{df}{dd}\right\|^2 > \varepsilon$

(4.1) Line search

① $\alpha = 1$, $b = 0.5$, $t = 0.2$

② $$f(\alpha) = f\left(d_k - \alpha\frac{df}{dd}, \; s_k + \alpha\left(\left(\frac{\partial h}{\partial s}\right)^{-1}\left(\frac{\partial h}{\partial d}\right)\left(\frac{df}{dd}\right)^T\right)^T\right)$$

$$= f\left(x_3 - \alpha\frac{df}{dd}, \; x_1 + \alpha\right.$$

I was stressed.

please give me some extra credit Dr. Ren!

(Just kidding. Hope you enjoy something out of the ordinary while grading, Dr. Ren)