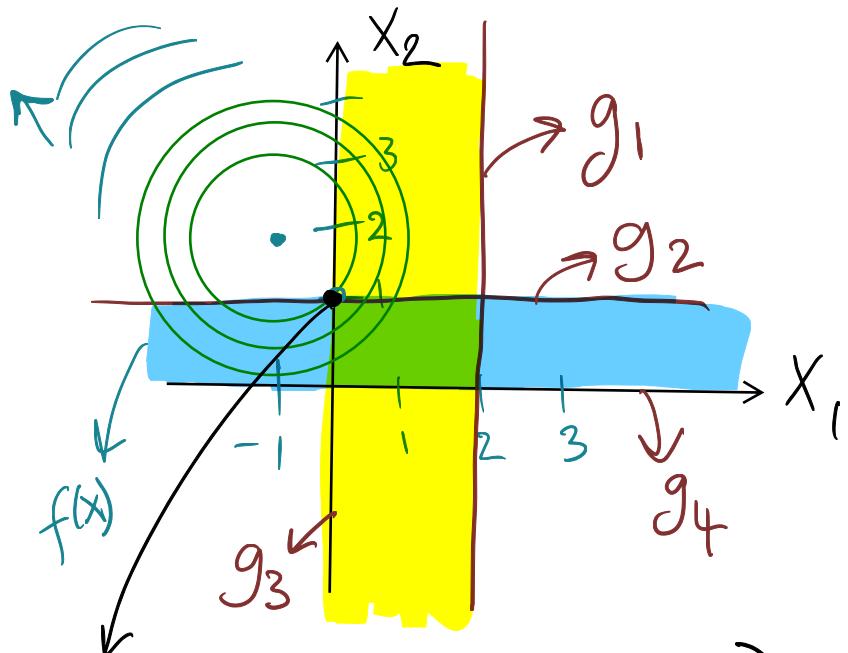
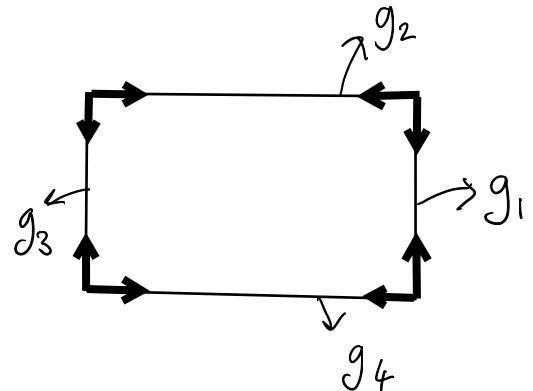


$$1) \quad \min_{x_1, x_2} f(x) = (x_1 + 1)^2 + (x_2 - 2)^2$$

s.t. $\begin{aligned} g_1 &= x_1 - 2 \leq 0, & g_3 &= -x_1 \leq 0 \\ g_2 &= x_2 - 1 \leq 0, & g_4 &= -x_2 \leq 0 \end{aligned}$



Feasible descent:



minimum point $(x_1, x_2) = (0, 1)$

Root point: $x = -1, y = 2$

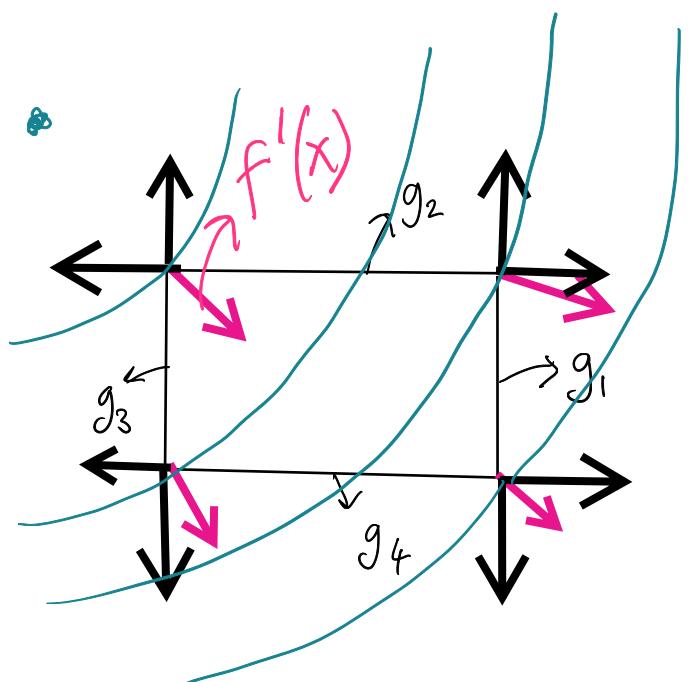
$$g_1 = x_1 - 2 \leq 0 \Rightarrow x_1 \leq 2$$

$$g_2 = x_2 - 1 \leq 0 \Rightarrow x_2 \leq 1$$

$$g_3 = -x_1 \leq 0 \Rightarrow x_1 \geq 0$$

$$g_4 = -x_2 \leq 0 \Rightarrow x_2 \geq 0$$

Gradient Directions:



$$L = f + \lambda^T h + \hat{\mu}^T \hat{g}$$

$$L = (x_1+1)^2 + (x_2-2)^2 + \mu_1(x_1-2) + \mu_2(x_2-1) + \mu_3(-x_1) + \mu_4(-x_2)$$

$$\frac{\partial L}{\partial x} = \begin{bmatrix} 2(x_1+1) + \mu_1 - \mu_3 \\ 2(x_2-2) + \mu_2 - \mu_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 2(0+1) + \cancel{\mu_1} - \mu_3 \\ 2(1-2) + \cancel{\mu_2} - \cancel{\mu_4} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

μ_1 and μ_4 are inactive

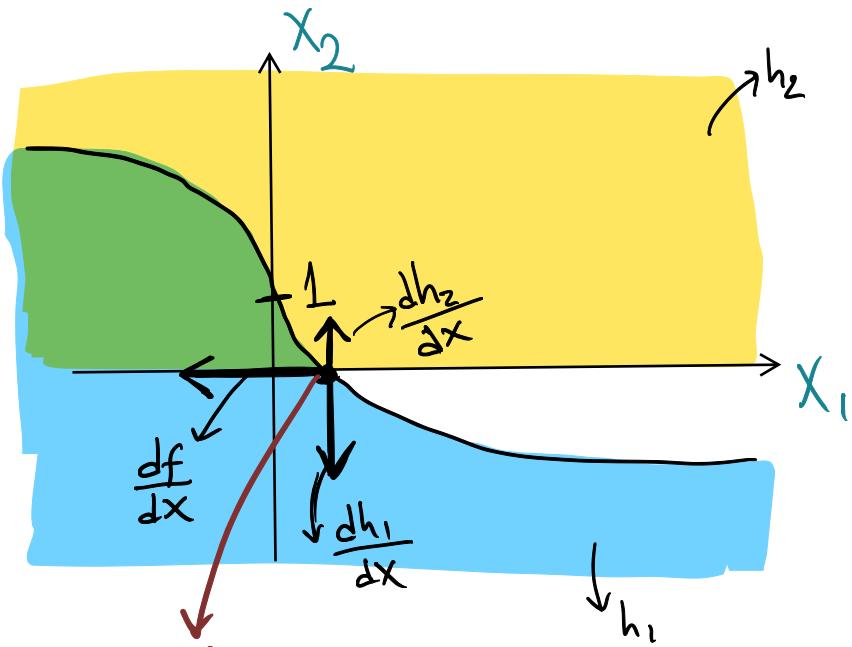
$$2 - \mu_3 = 0 \Rightarrow \boxed{\mu_3 = 2 > 0}$$

$$-2 + \mu_2 = 0 \Rightarrow \boxed{\mu_2 = 2 > 0}$$

Force balance occurs. This is the minimum point.

$$2) \min_{x_1, x_2} f = -x_1 + 0x_2 \rightarrow \max_{x_1, x_2} f = x_1 - 0x_2$$

s.t. $g_1 = x_2 - (1-x_1)^3 \leq 0$ and $x_2 \geq 0$



*Solution
since x_1 is at a maximum*

$$\frac{df}{dx} = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$\frac{dh_1}{dx} = \begin{bmatrix} \frac{\partial h_1}{\partial x_1} \\ \frac{\partial h_1}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 3(1-x_1)^2 \\ 1 \end{bmatrix}$$

$$\frac{dh_2}{dx} = \begin{bmatrix} \frac{\partial h_2}{\partial x_1} \\ \frac{\partial h_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

KKT conditions cannot be applied since the derivatives of the constraints are orthogonal to the derivative of the objective function. In other words, forces from the constraints do not cancel out the force from the function.

$$3) \begin{array}{ll} \max_{x_1, x_2, x_3} & f = x_1 x_2 + x_2 x_3 + x_1 x_3 \\ \text{s.t.} & h = x_1 + x_2 + x_3 - 3 = 0 \end{array}$$

Reduced Gradient:

$$m=1, n=3, s: x_1, d: x_2, x_3, \text{def} = n-m=2$$

$$\bar{x} = \begin{bmatrix} \bar{d} \\ \bar{s} \end{bmatrix} = \bar{x} = \begin{bmatrix} x_2 \\ x_3 \\ x_1 \end{bmatrix}$$

$$\boxed{\frac{df}{dd} = \frac{\partial f}{\partial d} - \left(\frac{\partial f}{\partial s} \right) \left(\frac{\partial h}{\partial s} \right)^{-1} \frac{\partial h}{\partial d}} \rightarrow \text{Reduced gradient}$$

$$\frac{\partial f}{\partial d} = \begin{bmatrix} \frac{\partial f}{\partial x_2} \\ \frac{\partial f}{\partial x_3} \end{bmatrix} = \begin{bmatrix} x_1 + x_3 \\ x_2 + x_1 \end{bmatrix}, \quad \frac{\partial f}{\partial s} = \frac{\partial f}{\partial x_1} = x_2 + x_3$$

$$\frac{\partial h}{\partial s} = \frac{\partial h}{\partial x_1} = 1, \quad \frac{\partial h}{\partial d} = \begin{bmatrix} \frac{\partial h}{\partial x_2} \\ \frac{\partial h}{\partial x_3} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\frac{df}{dd} = \begin{bmatrix} x_1 + x_3 \\ x_2 + x_1 \end{bmatrix} - \begin{bmatrix} x_2 + x_3 \\ x_1 + x_3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 + x_3 \\ x_2 + x_1 \end{bmatrix} - \begin{bmatrix} x_2 + x_3 \\ x_2 + x_3 \end{bmatrix}$$

$$\begin{aligned} \hookrightarrow x_1 + x_3 - (x_2 + x_3) &= 0 \Rightarrow \begin{cases} x_1 = x_2 \\ x_1 = x_3 \end{cases} \rightarrow x_1 = x_2 = x_3 = x \\ \hookrightarrow x_2 + x_1 - (x_2 + x_3) &= 0 \Rightarrow \end{aligned}$$

$$\begin{aligned} x_1 + x_2 + x_3 - 3 &= 0 \rightarrow x + x + x - 3 = 0 \\ 3x = 3 &\Rightarrow \boxed{x_1 = x_2 = x_3 = 1} \end{aligned}$$

Lagrangian

$$\mathcal{L}(x, \lambda) = f + \lambda^T h$$

$$L(x_1, x_2, x_3, \lambda) = x_1 x_2 + x_2 x_3 + x_1 x_3 + \lambda(x_1 + x_2 + x_3 - 3)$$

$$\frac{\partial L}{\partial x} = \begin{bmatrix} x_2 + x_3 + \lambda \\ x_1 + x_3 + \lambda \\ x_2 + x_1 + \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{l} x_2 + x_3 = -\lambda \quad \textcircled{1} \\ x_1 + x_3 = -\lambda \quad \textcircled{2} \\ x_2 + x_1 = -\lambda \quad \textcircled{3} \end{array}$$

$$\begin{array}{l} \textcircled{1} = \textcircled{2} = x_2 + x_3 = x_1 + x_3 \Rightarrow x_1 = x_2 \\ \textcircled{2} = \textcircled{3} = x_1 + x_3 = x_2 + x_1 \Rightarrow x_3 = x_2 \end{array} \Rightarrow \boxed{x_1 = x_2 = x_3} = x$$

$$x_1 + x_2 + x_3 - 3 = 0 \rightarrow x + x + x - 3 = 0 \\ 3x = 3 \Rightarrow \boxed{x_1 = x_2 = x_3 = 1}$$

$$\textcircled{1} \Rightarrow x_2 + x_3 = -\lambda$$

$$1 + 1 = -\lambda$$

$$\boxed{\lambda = -2}$$

* Note: Lagrangian is easier/quicker
when dealing w/ multi-variate problems

4) $\max_{x_1, x_2} f = 2x_1 + bx_2$

$$g_1 = x_1^2 + x_2^2 - 5 \leq 0$$

$$g_2 = x_1 - x_2 - 2 \leq 0$$

Given $x_1 = 1, x_2 = 2$

active $1^2 + 2^2 \leq 5 \Rightarrow 5 = 5$

inactive $1 - 2 \leq 2 \Rightarrow -2 < 2$

Reduced Gradient:

$$m=1, n=2, S: x_1, d: x_2, \text{ def } = n-m = 1$$

$$X = \begin{bmatrix} d \\ S \end{bmatrix} = X = \begin{bmatrix} x_2 \\ x_1 \end{bmatrix}$$

$$\frac{df}{dd} = \frac{\partial f}{\partial d} - \left(\frac{\partial f}{\partial S} \right) \left(\frac{\partial g}{\partial S} \right)^{-1} \frac{\partial g}{\partial d}$$

$$\frac{\partial f}{\partial d} = \frac{\partial f}{\partial x_2} = b, \quad \frac{\partial f}{\partial S} = \frac{\partial f}{\partial x_1} = 2$$

$$\frac{\partial g}{\partial S} = \frac{\partial g_1}{\partial x_1} = 2x_1, \quad \frac{\partial g}{\partial d} = \frac{\partial g_1}{\partial x_2} = 2x_2$$

$$\frac{df}{dd} = b - 2 \left(\frac{1}{2x_1} \right) (2x_2) = 0 \Rightarrow b - 2 \frac{x_2}{x_1} = 0 \Rightarrow x_1 b - 2x_2 = 0 \Rightarrow x_2 = \frac{x_1 b}{2}$$

$$x_1 b - 2x_2 = 0$$

$$(1) b - 2(2) = 0$$

$$b = 4$$

$$g_1 = x_1^2 + x_2^2 - 5 \leq 0$$

$$x_1^2 + \left(\frac{x_1 b}{2} \right)^2 - 5 \leq 0$$

$$1 + \left(\frac{1 \cdot b}{2} \right)^2 - 5 \leq 0$$

$$\frac{b^2}{4} - 4 \leq 0$$

$$b^2 \leq 16$$

$$b \leq 4$$

$$b \geq -4$$

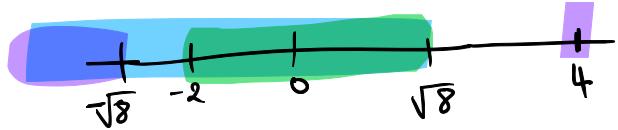
$$g_2 = x_1 - x_2 - 2 \leq 0$$

$$x_1 - \frac{x_1 b}{2} \leq 2$$

$$1 - \frac{b}{2} \leq 2$$

$$-\frac{b}{2} \leq 1$$

$$b \geq -2$$





I was stressed.

please give me
some extra credit Dr. Ren!

Just kidding. Hope you
enjoy something out of the
ordinary while grading, Dr. Ren