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Design Optimization HW 2

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Problem 1

$$f = 2x_1^2 - 4x_1x_2 + 1.5x_2^2 + x_2$$

$$g = \begin{bmatrix} 4x_1 - 4x_2 \\ -4x_1 + 3x_2 + 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\cancel{4}x_1 - \cancel{4}x_2 = 0 \qquad x_1 = x_2$$

$$-4x_1 + 3x_2 = -1 \qquad -4x_1 + 3x_1 = -1$$

$$-1x_1 = -1$$

$$\boxed{x = 1 = x_2}$$

The stationary point is at $f(1,1)$

$$H = \begin{bmatrix} 4 & -4 \\ -4 & 3 \end{bmatrix} \Rightarrow \det = 12 - 16 = \boxed{-4 < 0}$$

Indefinite

$$f(x_1^*, x_2^*) = 2(1)^2 - 4(1)(1) + 1.5(1)^2 + 1 = 0.5$$

$$f(x) = f^* + g^T(x - x_0) + \frac{1}{2}(x - x_0)^T H (x - x_0)$$

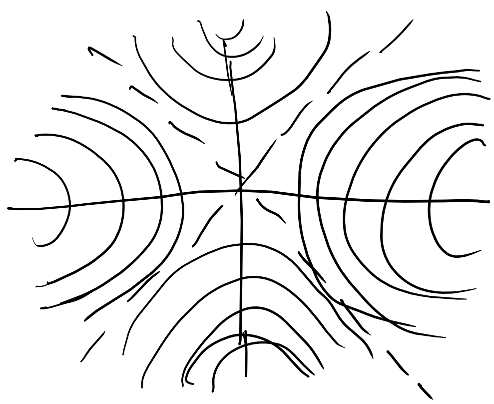
$$= 0.5 + [0, 0](x - x_0) + \frac{1}{2}(x - x_0)^T \begin{bmatrix} 4 & -4 \\ -4 & 3 \end{bmatrix}(x - x_0)$$

$$= 0.5 + \cancel{(0)(x_1 - 1)} + \cancel{(0)(x_2 - 1)} + \frac{1}{2}(x_1 - 1)^2(4) + (x_1 - 1)(x_2 - 1)(-4) + \frac{1}{2}(x_2 - 1)^2(3)$$

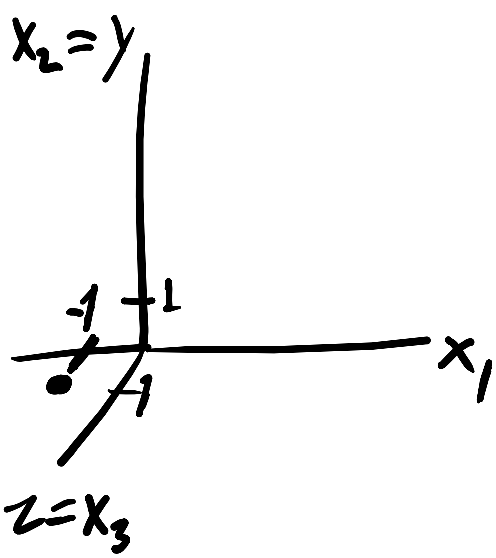
$$f(x) - 0.5 = 2\left(x_1^2 - 2x_1 + 1\right) - 4\left(x_1 x_2 - x_1 - x_2 + 1\right) + \frac{3}{2}\left(x_2^2 - 2x_2 + 1\right) \leq 0$$

$$\boxed{0 \geq 2x_1^2 + \frac{3}{2}x_2^2 + x_2 - 4x_1x_2 + 3}$$

This inequality defines both downslopes



Problem 2



$$obj = \sqrt{(x_1+1)^2 + (x_2 - 0)^2 + (x_3 - 1)^2}$$

$$\boxed{obj = (x_1+1)^2 + x_2^2 + (x_3-1)^2}$$

$$g = \begin{bmatrix} 2(x_1+1) \\ 2x_2 \\ 2(x_3-1) \end{bmatrix}$$

Problem 2

```
In [25]:  from scipy.optimize import minimize

          obj = lambda x: (x[0]+1)**2+x[1]+(x[2]-1)**2
          con = ({'type': 'eq', 'obj': lambda x: x[0]+2*x[1]+3*x[2]-1})
          bnds = ((None, None), (None, None), (None, None))
          res = minimize(obj, (10,10,10), method = 'SLSQP', constraints = con)
```

```
----> 6 res = minimize(obj, (10,10,10), method = 'SLSQP', constraints = con)

~\anaconda3\lib\site-packages\scipy\optimize\_minimize.py in minimize(fun, x0, args, method, jac, hess, hessp, bounds, co
nstraints, tol, callback, options)
    625         return _minimize_cobyla(fun, x0, args, constraints, **options)
    626     elif meth == 'slsqp':
--> 627         return _minimize_slsqp(fun, x0, args, jac, bounds,
    628                                constraints, callback=callback, **options)
    629     elif meth == 'trust-constr':

~\anaconda3\lib\site-packages\scipy\optimize\slsqp.py in _minimize_slsqp(func, x0, args, jac, bounds, constraints, maxite
r, ftol, iprint, disp, eps, callback, finite_diff_rel_step, **unknown_options)
    284         # check function
    285         if 'fun' not in con:
--> 286             raise ValueError('Constraint %d has no function defined.' % ic)
    287
    288         # check Jacobian

ValueError: Constraint 0 has no function defined.
```

Problem 3

Given $f(x)$and $g(x)$ are two convex functions defined on the convex set X

1) Prove that $af(x) + bg(x)$ is a convex set for $a > 0$ and $b > 0$

Let $a = 1, b = 1$, and $H(x) = f(x) + g(x)$

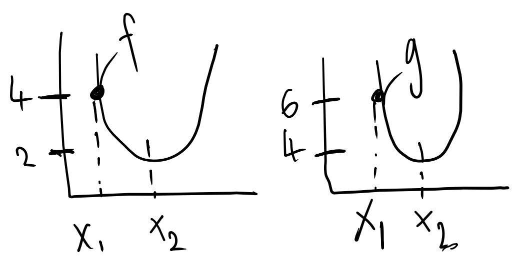
$\forall x_1, x_1 \in X, \lambda \in [0,1]$

$$H(\lambda x_1 + (1-\lambda)x_2) \leq \lambda H(x_1) + (1-\lambda)H(x_2)$$

Since $f(x)$and $g(x)$ are convex sets,

$$H(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda)f(x_2) + \lambda g(x_1) + (1-\lambda)g(x_2)$$

For every point in f, there is a corresponding point in g, when they are added together, there is an overall function that is convex as well. For example:



$f(x_1) + g(x_1) = 4 + 6 = 10$

$f(x_2) + g(x_2) = 2 + 4 = 6$

The convex inequality will hold true because the function of the sum of $f(x)$and $g(x)$ follows the same pattern as both functions

2) **In what conditions will $f(g(x))$ be convex**

If $f(x) * g(x)$ is convex, then $f(g(x))$ is convex. And the nature of $f(x) * g(x)$ is similar to $f(x) + g(x)$.

$$f(x_1) + g(x_1) = 24,$$

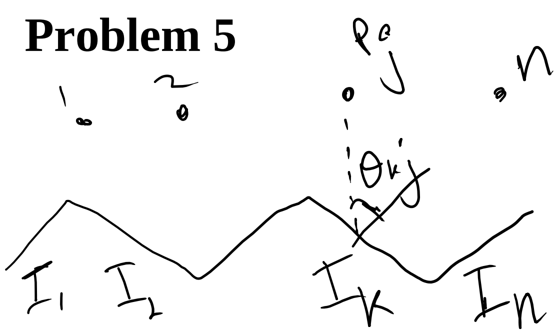
$$f(x_2) + g(x_2) = 8$$

It follows the pattern of both f and g but $f * g$ gives a function with larger values compared to f and g.

Problem 4

Show $f(x_1) \geq f(x_0) + g_{x_0}{}^T(x_1 - x_0)$ for a convex function $f(x)\colon X \to \mathbb{R}$ and for $x_0, x_1 \in X$

Problem 5



Target reflection intensity: I_t

actual reflection intensity: $a_k^T p$

$$\min_{\{P_j\}_{j=1}^{m}} \sum_{k=1}^{n} \left(I_t - a_{kj}^T P_j\right)^2 \leftarrow \text{difference is positive}$$

$$\text{s.t} \quad 0 \le P_j \le P_{max} \quad \forall j = 1:m$$

1) **Formulate this problem as an optimization problem.**

$$\min_{P_{j_{j=1}}^{m}} \sum_{k=1}^{n} \left(I_t - a_{kj}^T P_j\right)^2$$

$$\text{s.t. } 0 \leq P_j \leq P_{max} \ \forall\, j = 1{:}\, m$$

2) **Is your problem convex?**

Taking the gradient of the general objective function:

$$g(P) = \sum_{k=1}^{n} 2\left(I_t - a_k^T P\right) \cdot a_k$$

$$g(P) = \sum_{\mathrm{k=1}}^{\mathrm{n}} 2\mathrm{a}_{\mathrm{k}}^{\mathrm{T}} P \cdot a_k - 2I_t a_k$$

Taking the Hessian of the general objective function:

$$H(P) = \sum_{\mathrm{k=1}}^{\mathrm{n}} 2\mathrm{a}_{\mathrm{k}}^{\mathrm{T}} \cdot a_k$$

According to Lemma: if $d^T Hd \geq 0\ \forall\ d \neq 0$, then H is p.s.d.

$$d^T Hd = \sum_{\mathrm{k=1}}^{\mathrm{n}} 2d^T \left(a_k \mathrm{a}_{\mathrm{k}}^{\mathrm{T}}\right) d$$

The H is p.s.d when d is orthogonal to a_k because then $d^T a_k$ would be zero.

The problem is not convex because H is either p.s.d or indefinite.

3) **If we require the overall power output of any of the n lamps to be less than p^*, will the problem have a unique solution?**

If the function is convex, then the solution will inherently be unique. If the overall power out of the lamps in less than p^*, which is a constant across all planes making it a constant in p_j's space. Since it is a constant everywhere, p^* is a convex function. And p_j is a convex function too as it spans m number of dimensions. An intersection of 2 convex spaces is a convex function itself.

4) **If we require no more than half of the lamps to be switched on, will the problem have a unique solution?**

If there are only n/2 lamps that are switches on, there can be multiple combinations of lamps which can give an equivalent optimal power output. For example, given there are n=2 number of lamps that are switched on, the situation where 1 one of them being switched on would give the same solution as the other being switched on. This is because of the symmetry of the set-up and equivalent reflection intensity of the lamps. So, there are no unique solutions, but multiple optimal solutions.