Central forces

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As announced at the beginning of the course, we finish by studying central forces. This, we must acknowledge, is somehow anticlimactic—having developed very general, abstract, and profound theories about mechanics and, more broadly, the physical world (dealing with variational principles, symmetry, conservation...), we end up solving one specific problem in mechanics. However, this is not just any problem—describing the motion of bodies subject to central fields, and especially of celestial bodies and objects on Earth subject to gravity, is one of the major achievements in the history of physics. Additionally, an important advantage of having covered Lagrangian and Hamiltonian mechanics before is that some arguments can be made much shorter than they would be if we had to use Newton's second law. Let's do it!

One body under a general central field

A **central force** is a force that **points radially towards the origin**[1] and whose magnitude **depends only on the distance** from the origin. Central forces are ubiquitous in nature, and thus the importance of learning to deal with them. Fortunately, they have a number of properties that make them uncharacteristically easy to deal with. Here, we use some of these properties to show that their description can be cast in a single dimension. We start by considering a single body in a central field created by a source that is fixed at the origin, without worrying about the source itself.

[1] Therefore, the force $\mathbf{F}$ is parallel to the the position of the body $\mathbf{r}$.

Reduction to one dimension and effective potential

First, we note that **motion under a central force always happens on a plane**. Indeed, consider the plane P defined by the initial position of the body $\mathbf{r}_0$ and the initial velocity $\mathbf{v}_0$. Since the force is parallel to $\mathbf{r}$, it cannot push the body out of the plane. Indeed, this is true at any point in the trajectory, so the trajectory never leaves the initial plane P.

Thus, we can reduce the initial 3D problem to 2D, and write the Lagrangian in polar coordinates. This is something that we have done in several occasions already; the result is

$$\mathcal{L} = \frac{m}{2}\left(r'^2 + r^2\theta'^2\right) - V(r),$$

where the central potential $V(r)$ only depends on the distance to the origin r.

Now, since θ is cyclic, the Euler-Lagrange equations give us conservation of angular momentum

$$p_\theta \equiv mr^2\theta' = \text{constant} \implies \theta' = \frac{p_\theta}{mr^2}\,.$$

Therefore, we see that, in practice, because of the immediate integration resulting from the rotational symmetry, **the problem is reduced to one dimension** r.

It is very useful to translate the problem to the Hamiltonian formalism. Indeed, since the Lagrangian is quadratic on the velocities, we know immediately that the Hamiltonian of the system is $H = T + V$, so[2]

$$H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + V(r) \equiv \frac{p_r^2}{2m} + V_{\text{eff}}(r)\,,$$

where the momentum is $p_r = mr'$ and the **effective potential energy** is

$$V_{\text{eff}}(r) \equiv \frac{p_\theta^2}{2mr^2} + V(r)\,.$$

Hamilton's canonical equations applied to this Hamiltonian (or, if you prefer, the Euler-Lagrange equation applied to the Lagrangian above) lead[3] to the following equation of motion for r

$$mr'' = \frac{p_\theta^2}{mr^3} - \frac{dV}{dr}\,,$$

with $p_\theta = \text{constant}$.

[2] Exercise: Get the Hamiltonian as the Legendre transform of the Lagrangian and verify that you get the same result.

[3] Exercise: Do it!

Formal solution of the 1D problem

The equation of motion for r above is a second-order differential equation. However, just like conservation of angular momentum allowed us to automatically integrate the equation for θ (so that we have a first-order differential equation, instead of second-order), we can immediately carry out an integral for r by noting that the total energy $E = H = T + V$ is conserved because $\mathcal{L}$ does not depend explicitly on time. Then, calling the angular momentum $p_\theta = L$ to highlight that it is a constant, we have

$$\frac{m}{2}r'^2 + \frac{L^2}{2mr^2} + V(r) = E\,,$$

which is a first-order differential equation for r.

In theory, this equation could be integrated to get $r(t)$

$$\int_{r_0}^{r} \frac{dr}{\sqrt{E - \frac{L^2}{2mr^2} - V(r)}} = \pm\int_{t_0}^{t} dt\sqrt{\frac{2}{m}} = \sqrt{\frac{2}{m}}(t - t_0)\,.$$

From $r(t)$, in theory one may then obtain $\theta(t)$ by integrating

$$\theta - \theta_0 = \int_{t_0}^{t} dt \frac{L}{m\,(r(t))^2}\,.$$

Unfortunately, these integrals cannot be carried out in closed form form most potentials $V(r)$. And even for those for which they can be carried out (which, luckily, turn out to be the interesting ones), this formal solution is not particularly helpful.

The Kepler problem

We now focus on the gravitational (attractive) potential $V(r) = -\frac{k}{r}$, that is, one specific potential that decays as the inverse of the distance r.[4]

[4] Until the next section, where we clarify the relationship to the two-body problem, k is just some arbitrary constant.

V_{eff} in the Kepler problem

The effective potential in the Kepler problem is

$$V_{\text{eff}}(r) \equiv \frac{L^2}{2mr^2} - \frac{k}{r}\,.$$

Figure 1 shows qualitatively the shape of this potential. At small values of r, V_{eff} is dominated by the positive $1/r^2$ term, thus creating an effective barrier that stops the body in the central field from collapsing towards $r = 0$. At large values of r, the negative $1/r$ term dominates and the potential approaches $V_{\text{eff}} = 0$ from the negative side.

By inspection of the potential, we identify different types of orbits depending on the value of the total energy E (Fig. 1, bottom):

- $E = E_1$: the body orbits at a fixed r_0 such that $dV_{\text{eff}}/dr|_{r=r_0} = 0$; that is, it follows a circular orbit around the origin.
- $E = E_2$: the body oscillates between two radii r_1 and r_2 such that $V_{\text{eff}}(r_1) = V_{\text{eff}}(r_2) = E$.
- $E \geq E_3$: the body has an unbounded orbit, approaching the origin until it reaches a minimum distance r_4 such that $V_{\text{eff}}(r_4) = E$, and then turning around towards infinite.

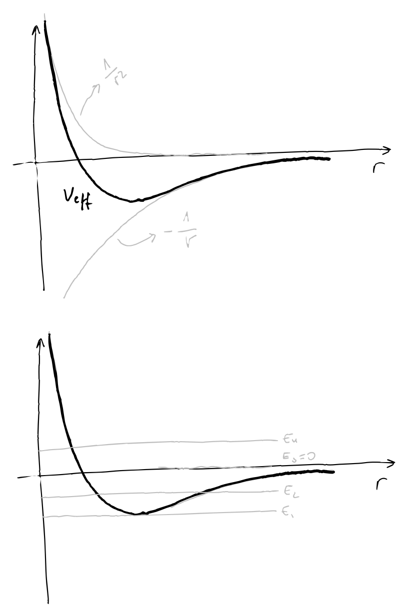

Figure 1: The effective potential in the Kepler problem and types orbit (bottom).

Solution of $r(\theta)$

We are now in a position to obtain a solution to the Kepler problem. Rather than following the path outlined above to obtain $r(t)$, and then $\theta(t)$, we will use a bunch of nifty tricks to get a solution of the trajectory of the form $r(\theta)$.

We start by considering the change of parametrization of the trajectory from $r(t)$ to $r(\theta(t))$

$$r' = \frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \frac{L}{mr^2}\frac{dr}{d\theta}\,.$$

Then, we rewrite the equation for the conservation of energy in terms of the new "velocity" $dr/d\theta$

$$\frac{m}{2}\left(\frac{L}{mr^2}\frac{dr}{d\theta}\right)^2 + \frac{L^2}{2mr^2} - \frac{k}{r} = E \implies \frac{L^2}{2m}\left[\left(\frac{1}{r^2}\frac{dr}{d\theta}\right)^2 + \frac{1}{r^2}\right] - \frac{k}{r} = E$$

Finally, one may realize that the equation is full of $1/r$ terms and, with a leap of faith, attempt one last change of variables $u = 1/r$ to get[5]

$$\frac{L^2}{2m}\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right] - ku = E\,, \qquad (1)$$

which, upon differentiation with respect to θ becomes[6]

$$\boxed{\frac{d^2u}{d\theta^2} + u = \frac{mk}{L^2}}\,.$$

[5] We use $du/d\theta = du/dr \times dr/d\theta = -1/r^2 \times dr/d\theta$.

[6] Note the term $du/d\theta$ throughout that cancels.

This is the equation of a harmonic oscillator with $\omega = 1$ and a constant driving force. By inspection, we can get the solution

$$u(\theta) = A\cos\theta + \frac{mk}{L^2}$$

where we have chosen an arbitrary origin for the angle $\theta_0 = 0$, and A is an integration constant whose value we can determine by substitution into Eq. (1) to get

$$A = \frac{mk}{L^2}\sqrt{1 + \frac{2EL^2}{mk^2}}\,.$$

Finally, the equation of the trajectory can be written as

$$\boxed{u(\theta) = \frac{1}{r(\theta)} = \frac{mk}{L^2}(1 + \epsilon\cos\theta) \quad \text{with} \quad \epsilon = \sqrt{1 + \frac{2EL^2}{mk^2}}}\,. \qquad (2)$$

The **eccentricity** ϵ turns out to be the key parameter in determining the actual shape of the orbit.

Conic orbits

We now show that trajectories under gravitational central fields are conic sections (Fig. 2), depending on the value of the eccentricity ϵ. To do that, it is convenient to write the equation of the trajectory (Eq. (2)) in Cartesian coordinates—defining $C \equiv L^2/mk$ we get that the trajectory satisfies[7]

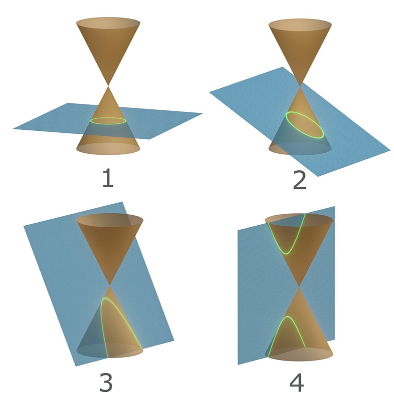

Figure 2: Conic sections. 1) Circle; 2) Ellipse; 3) Parabola; 4) Hyperbola.

[7] Do IT! Note that $\cos\theta = x/r$.

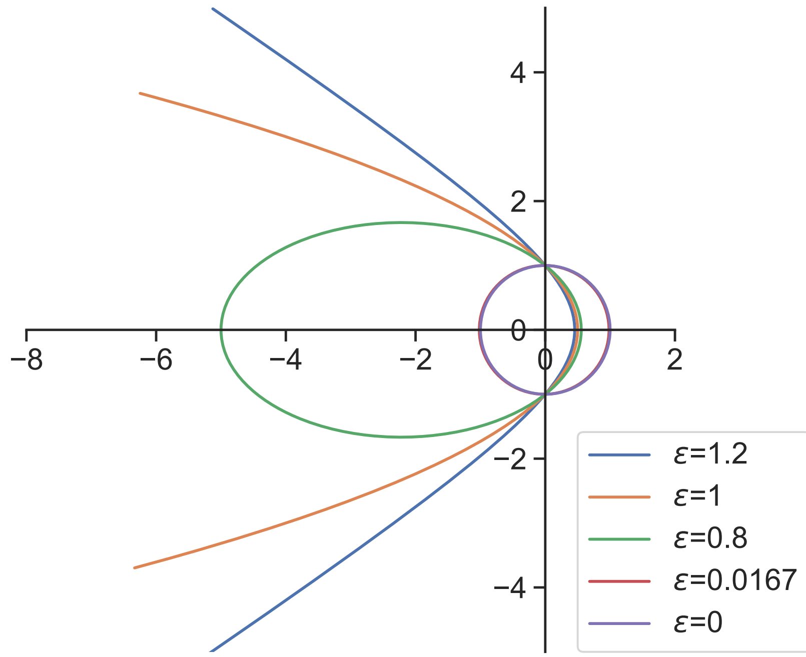

Figure 3: Different orbits as a function of the eccentricity ϵ. In all cases, the field is centered at the origin. The orbit of the Earth around the Sun has eccentricity $\epsilon = 0.0167$, so it is nearly circular.

$$x^2(1-\epsilon^2) + y^2 + 2C\epsilon x = C^2 . \tag{3}$$

We can just plot these trajectories to see their shape (Fig. 3).

Circular orbits for $\epsilon = 0$. For $\epsilon = 0$, the equation of the trajectory becomes

$$x^2 + y^2 = C^2 ,$$

which is the equation of a circle with the center at the origin and radius $C = L^2/mk$.

Elliptical orbits for $0 < \epsilon < 1$. For $0 < \epsilon < 1$, Eq. (3) can be rewritten as

$$(1-\epsilon^2)\left(x + \frac{C\epsilon}{1-\epsilon^2}\right)^2 + y^2 = \frac{C^2}{1-\epsilon^2} .$$

This has the form

$$\frac{(x-x_c)^2}{a^2} + \frac{y^2}{b^2} = 1 ,$$

which is the equation of an ellipse centered at x_c.

Parabolic orbits for $\epsilon = 1$. For $\epsilon = 1$, (which corresponds to $E = 0$), we have

$$y^2 = C^2 - 2C\epsilon x$$

which is the equation of a parabola with the focus located at the origin.

HYPERBOLIC ORBITS FOR $\epsilon > 1$. For $\epsilon > 1$, Eq. (3) can be rewritten as

$$(\epsilon^2 - 1)\left(x - \frac{C\epsilon}{\epsilon^2 - 1}\right)^2 + y^2 = \frac{C^2}{\epsilon^2 - 1}\,.$$

This has the form

$$\frac{(x - x_c)^2}{a^2} - \frac{y^2}{b^2} = 1\,,$$

which is the equation of a hyperbola.

Two-body problem and reduced mass

So far, we have not paid any attention to the source of the central field. Rather, we have considered that we have a single mass and a field that is created by an undetermined source at the origin. But, is this relevant in practice, for the realistic case in which we have two masses that attract each other? Here, we prove that, thankfully, everything we have seen is correct, except for some correction that we discuss next.

An elegant way to see this is by using Noether's theorem—let's do it. First, consider the two masses m_1 and m_2 and their respective positions $\mathbf{r}_1$ and $\mathbf{r}_2$. By definition, the central potential that describes their interaction depends only on the distance $r \equiv |\mathbf{r}_1 - \mathbf{r}_2|$ between them. Therefore, the Lagrangian of the two-body system can be written as

$$\mathcal{L} = \frac{m_1}{2}\mathbf{r}_1'^2 + \frac{m_2}{2}\mathbf{r}_2'^2 - V(r)\,.$$

From this, it is obvious that the Lagrangian is translationally invariant; in particular, it is invariant under the infinitesimal translation

$$\begin{aligned} \mathbf{r}_1 &\to \mathbf{r}_1 + \boldsymbol{\epsilon} \\ \mathbf{r}_2 &\to \mathbf{r}_2 + \boldsymbol{\epsilon} \end{aligned}$$

which means that, by Noether's theorem, the total momentum $\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2 = m_1\mathbf{r}_1' + m_2\mathbf{r}_2' =$ constant. In turn, that means that the "coordinate" $m_1\mathbf{r}_1' + m_2\mathbf{r}_2'$ moves like a free particle, and so does the center of mass $\mathbf{R}_{\text{CM}} = (m_1\mathbf{r}_1' + m_2\mathbf{r}_2')/M$, with $M = m_1 + m_2$ being the total mass.

Given these observations, let's rewrite the Lagrangian in terms of $\mathbf{R}_{\text{CM}}$ and $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, instead of $\mathbf{r}_1$ and $\mathbf{r}_2$—we get[8]

$$\mathcal{L} = \frac{M}{2}\mathbf{R}_{\text{CM}}'^2 + \frac{\mu}{2}\mathbf{r}'^2 - V(r) \quad \text{with} \quad \frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}$$

[8] EXERCISE: Do it!

where μ is the so-called **reduced mass**. From this, it is trivial to see that, if we choose the center of mass frame of reference, the problem is exactly as the one-body problem that we have been dealing with, except that we need to work with the reduced mass μ instead of the actual mass, and that the origin is at the center of mass of the two-body system.