Chapter 2: Differentiation

2.3 Differentiation

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Key Points in this Section.

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1. Given $f : U \subset \mathbb{R}^3 \to \mathbb{R}$, where U is open, the ***partial derivative with respect to*** x is defined by

$$f_x(x, y, z) = \frac{\partial f}{\partial x}(x, y, z) = \lim_{h \to 0} \frac{f(x + h, y, z) - f(x, y, z)}{h}$$

   if it exists. The partial derivatives $\partial f/\partial y$ and $\partial f/\partial z$ are defined similarly and the extension to function of n variables is analogous.

2. The ***linear approximation*** to $f(x, y)$ at (x_0, y_0) is

$$\ell_{(x_0,y_0)}(x, y) = f(x_0, y_0) + \left[\frac{\partial f}{\partial x}(x_0, y_0)\right](x - x_0) + \left[\frac{\partial f}{\partial y}(x_0, y_0)\right](y - y_0)$$

3. The function $f(x, y)$ is ***differentiable*** at (x_0, y_0) if the partials exist at (x_0, y_0) *and* if

$$\lim_{(x,y)\to(x_0,y_0)} \frac{f(x, y) - \ell_{(x_0,y_0)}(x, y)}{\| (x, y) - (x_0, y_0) \|} = 0$$

4. If f is differentiable at (x_0, y_0), the ***tangent plane*** to the graph of f at (x_0, y_0, z_0), where $z_0 = f(x_0, y_0)$ is

$$z = \ell_{(x_0,y_0)}(x, y).$$

5. The definition of differentiability is motivated by the idea that the tangent plane should give a good approximation to the function.

6. If $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$ has partial derivatives at $\mathbf{x}_0 \in U$, the ***derivative matrix*** is the $m \times n$ matrix $\mathbf{D}f(\mathbf{x}_0)$ given by

$$\mathbf{D}f(\mathbf{x}_0) = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} & \cdots & \dfrac{\partial f_1}{\partial x_n} \\ \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2} & \cdots & \dfrac{\partial f_2}{\partial x_n} \\ \vdots & & & \vdots \\ \dfrac{\partial f_m}{\partial x_1} & \dfrac{\partial f_m}{\partial x_2} & \cdots & \dfrac{\partial f_m}{\partial x_n} \end{bmatrix}$$

where the partials are all evaluated at $\mathbf{x}_0$.

7. We say $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$ is ***differentiable*** at $\mathbf{x}_0$ provided the partials exist and

$$\lim_{\mathbf{x}\to\mathbf{x}_0} \frac{\|f(\mathbf{x}) - f(\mathbf{x}_0) - \mathbf{D}f(\mathbf{x}_0)\cdot(\mathbf{x} - \mathbf{x_0})\|}{\|\mathbf{x} - \mathbf{x}_0\|} = 0.$$

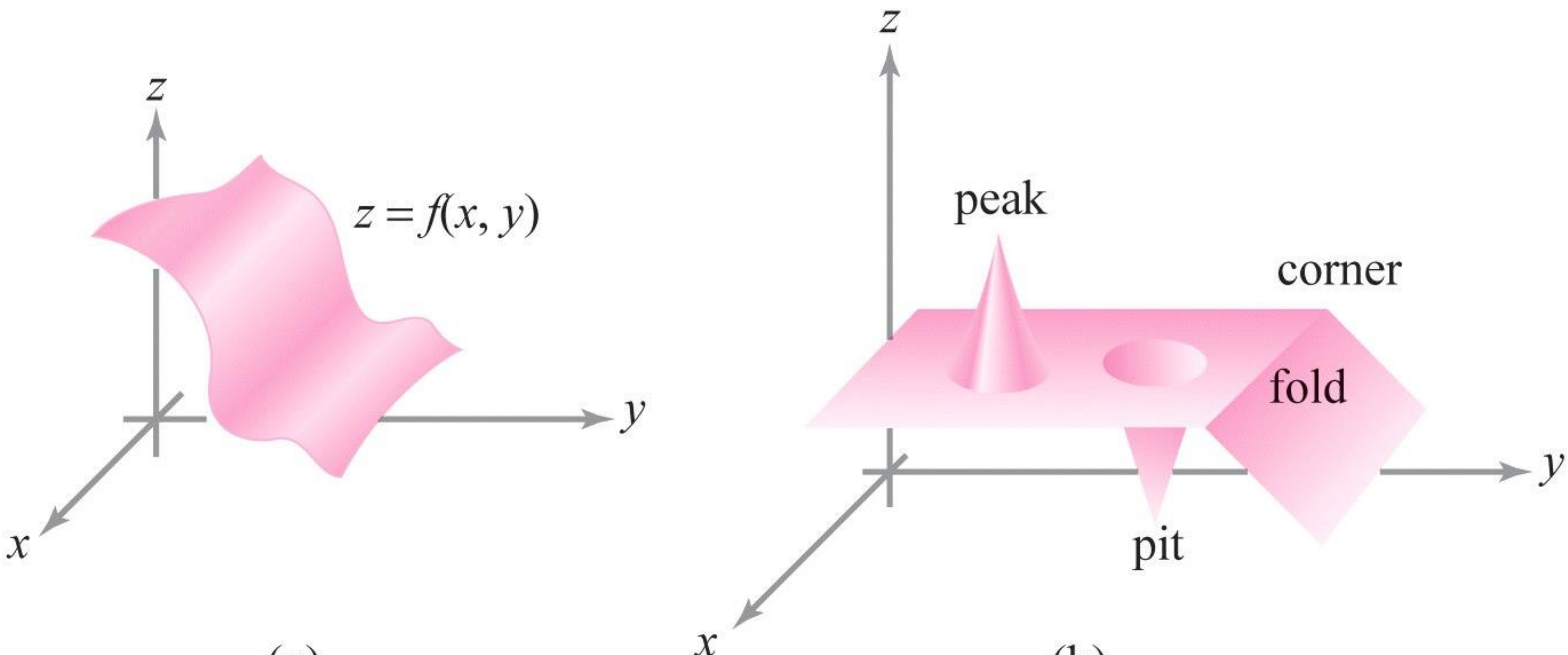
8. For $f : U \subset \mathbb{R}^3 \to \mathbb{R}$, its ***gradient*** is

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{i} + \frac{\partial f}{\partial z}\mathbf{k}.$$

   Similarly, for $f : U \subset \mathbb{R}^n \to \mathbb{R}$, ∇f is the vector with components

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right).$$

9. If f is differentiable at $\mathbf{x}_0$, then it is continuous at $\mathbf{x}_0$. If the partials exist and are continuous in a neighborhood of $\mathbf{x}_0$ (that is, f is C^1), then f is differentiable at $\mathbf{x}_0$.

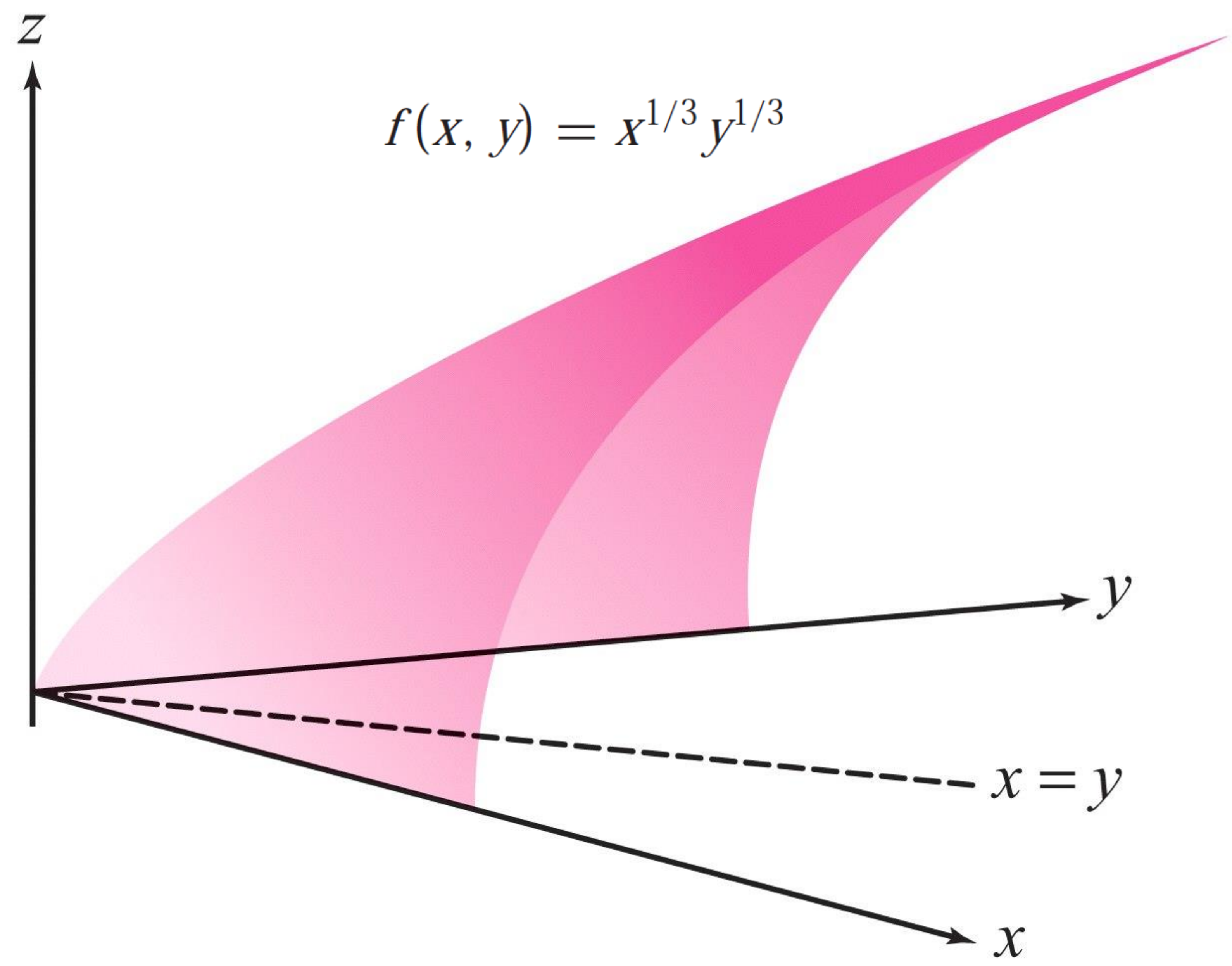


(a) (b)

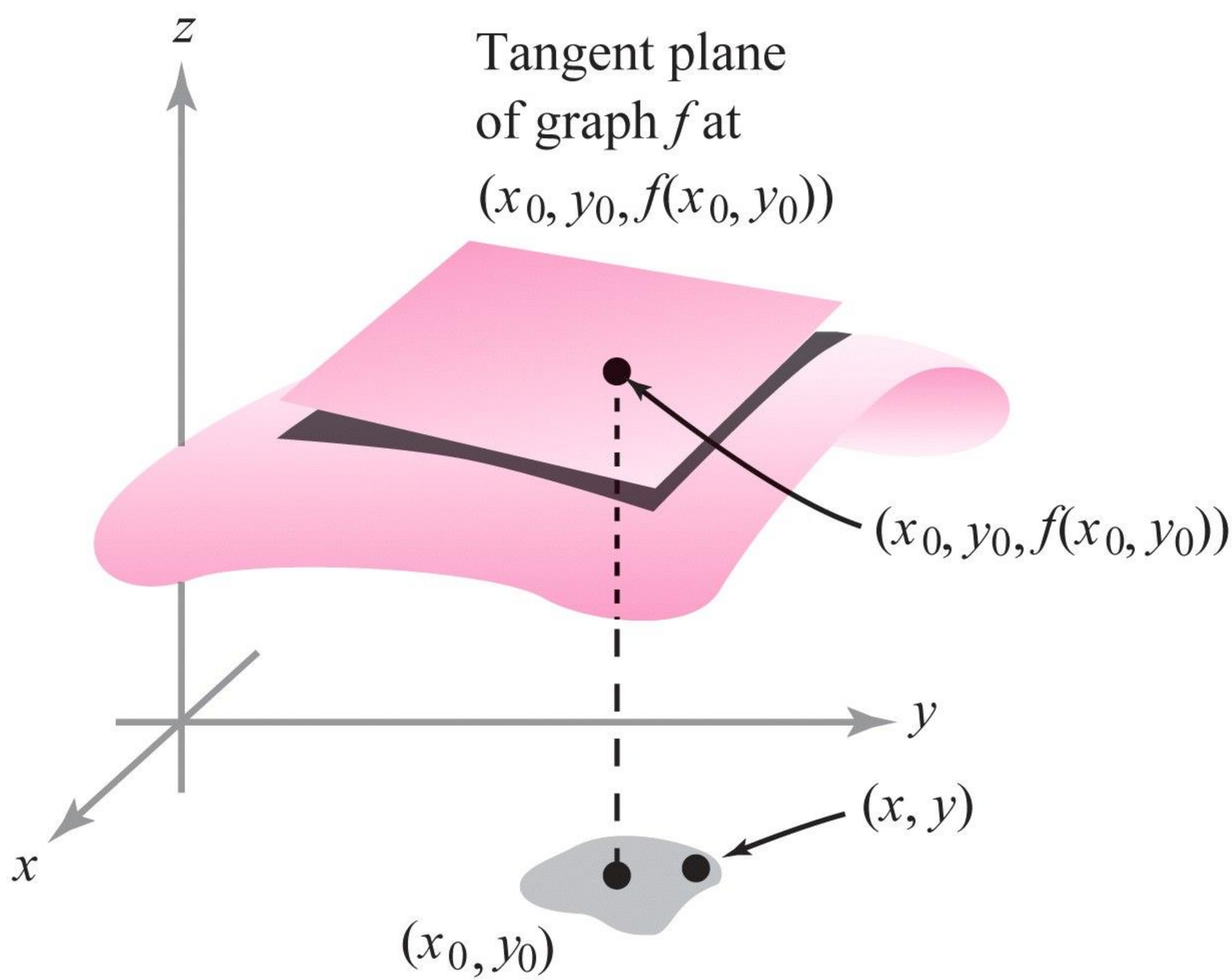
DEFINITION: Partial Derivatives Let $U \subset \mathbb{R}^n$ be an open set and suppose $f\colon U \subset \mathbb{R}^n \to \mathbb{R}$ is a real-valued function. Then $\partial f/\partial x_1, \ldots, \partial f/\partial x_n$, the ***partial derivatives*** of f with respect to the first, second, ..., nth variable, are the real-valued functions of n variables, which, at the point $(x_1, \ldots, x_n) = \mathbf{x}$, are defined by

$$\begin{aligned}\frac{\partial f}{\partial x_j}(x_1, \ldots, x_n) &= \lim_{h\to 0} \frac{f(x_1, x_2, \ldots, x_j + h, \ldots, x_n) - f(x_1, \ldots, x_n)}{h} \\ &= \lim_{h\to 0} \frac{f(\mathbf{x} + h\mathbf{e}_j) - f(\mathbf{x})}{h}\end{aligned}$$

if the limits exist, where $1 \le j \le n$ and $\mathbf{e}_j$ is the jth standard basis vector defined by $\mathbf{e}_j = (0, \ldots, 1, \ldots, 0)$, with 1 in the jth slot (see Section 1.5). The domain of the function $\partial f/\partial x_j$ is the set of $\mathbf{x} \in \mathbb{R}^n$ for which the limit exists.



z
$f(x, y) = x^{1/3}y^{1/3}$
y
x = y
x



z
Tangent plane
of graph f at
$(x_0, y_0, f(x_0, y_0))$
$(x_0, y_0, f(x_0, y_0))$
y
(x, y)
x
(x_0, y_0)

DEFINITION: Differentiable: Two Variables Let $f\colon \mathbb{R}^2 \to \mathbb{R}$. We say f is ***differentiable*** at (x_0, y_0), if $\partial f/\partial x$ and $\partial f/\partial y$ exist at (x_0, y_0) and if

$$\frac{f(x, y) - f(x_0, y_0) - \left[\dfrac{\partial f}{\partial x}(x_0, y_0)\right](x - x_0) - \left[\dfrac{\partial f}{\partial y}(x_0, y_0)\right](y - y_0)}{\|(x, y) - (x_0, y_0)\|} \to 0 \tag{2}$$

as $(x, y) \to (x_0, y_0)$. This equation expresses what we mean by saying that

$$f(x_0, y_0) + \left[\frac{\partial f}{\partial x}(x_0, y_0)\right](x - x_0) + \left[\frac{\partial f}{\partial y}(x_0, y_0)\right](y - y_0)$$

is a ***good approximation*** to the function f.

DEFINITION: Tangent Plane Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be differentiable at $\mathbf{x}_0 = (x_0, y_0)$. The plane in $\mathbb{R}^3$ defined by the equation

$$z = f(x_0, y_0) + \left[\frac{\partial f}{\partial x}(x_0, y_0)\right](x - x_0) + \left[\frac{\partial f}{\partial y}(x_0, y_0)\right](y - y_0),$$

is called the ***tangent plane*** of the graph of f at the point (x_0, y_0).

DEFINITION: Differentiable, n Variables, m Functions Let U be an open set in $\mathbb{R}^n$ and let $f\colon U \subset \mathbb{R}^n \to \mathbb{R}^m$ be a given function. We say that f is ***differentiable*** at $\mathbf{x}_0 \in U$ if the partial derivatives of f exist at $\mathbf{x}_0$ and if

$$\lim_{\mathbf{x}\to\mathbf{x}_0} \frac{\|f(\mathbf{x}) - f(\mathbf{x}_0) - \mathbf{T}(\mathbf{x} - \mathbf{x}_0)\|}{\|\mathbf{x} - \mathbf{x}_0\|} = 0, \tag{4}$$

where $\mathbf{T} = \mathbf{D}f(\mathbf{x}_0)$ is the $m \times n$ matrix with matrix elements $\partial f_i/\partial x_j$ evaluated at $\mathbf{x}_0$ and $\mathbf{T}(\mathbf{x} - \mathbf{x}_0)$ means the product of $\mathbf{T}$ with $\mathbf{x} - \mathbf{x}_0$ (regarded as a column matrix). We call $\mathbf{T}$ the ***derivative*** of f at $\mathbf{x}_0$.

$\mathbf{D}f(\boldsymbol{x})$: ***derivative***, ***matrix of partial derivatives***, or ***Jacobian matrix***

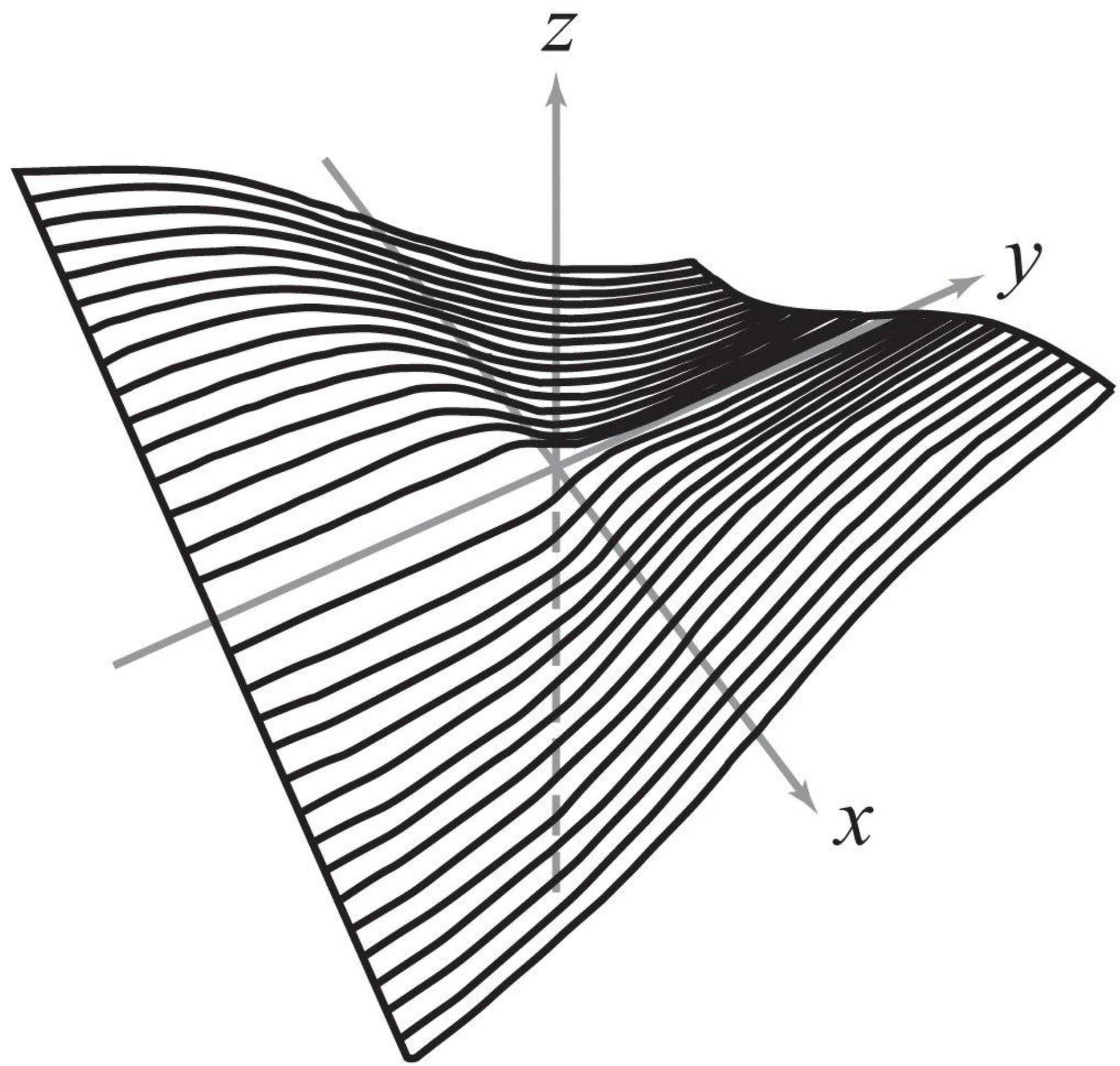
DEFINITION: Gradient Consider the special case $f : U \subset \mathbb{R}^n \to \mathbb{R}$. Here $\mathbf{D}f(\mathbf{x})$ is a $1 \times n$ matrix:

$$\mathbf{D}f(\mathbf{x}) = \begin{bmatrix} \dfrac{\partial f}{\partial x_1} & \cdots & \dfrac{\partial f}{\partial x_n} \end{bmatrix}.$$

We can form the corresponding vector $(\partial f/\partial x_1, \ldots, \partial f/\partial x_n)$, called the ***gradient*** of f and denoted by ∇f or grad f.

THEOREM 8 Let $f: U \subset \mathbb{R}^n \to \mathbb{R}^m$ be differentiable at $\mathbf{x}_0 \in U$. Then f is continuous at $\mathbf{x}_0$.

THEOREM 9 Let $f: U \subset \mathbb{R}^n \to \mathbb{R}^m$. Suppose the partial derivatives $\partial f_i/\partial x_j$ of f all exist and are continuous in a neighborhood of a point $\mathbf{x} \in U$. Then f is differentiable at $\mathbf{x}$.



$f(x, y) = xy/\sqrt{x^2 + y^2}$ [with $f(0, 0) = 0$]

1. Find $\partial f/\partial x, \partial f/\partial y$ if

(a) $f(x, y) = xy$

(b) $f(x, y) = e^{xy}$

(c) $f(x, y) = x \cos x \cos y$

(d) $f(x, y) = (x^2 + y^2) \log (x^2 + y^2)$

2. Evaluate the partial derivatives $\partial z/\partial x, \partial z/\partial y$ for the given function at the indicated points.

(a) $z = \sqrt{a^2 - x^2 - y^2}$; $(0, 0), (a/2, a/2)$

(b) $z = \log \sqrt{1 + xy}$; $(1, 2), (0, 0)$

(c) $z = e^{ax} \cos (bx + y)$; $(2\pi/b, 0)$

3. In each case following, find the partial derivatives $\partial w/\partial x, \partial w/\partial y$.

(a) $w = xe^{x^2+y^2}$

(b) $w = \dfrac{x^2 + y^2}{x^2 - y^2}$

(c) $w = e^{xy} \log (x^2 + y^2)$

(d) $w = x/y$

(e) $w = \cos (ye^{xy}) \sin x$