Combinatorics and Probability
Part I: Basic Combinatorics

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Contents

1. Basic principles of combinatorics
2. Samples
3. Principle of inclusion-exclusion

Sum and product

Proposition

Let A and B be two **disjoint** sets. Then,

$$|A \cup B| = |A| + |B|$$

Proposition

Product principle:

$$|A \times B| = |A||B|$$

Pigeonhole principle

Also known as *Dirichlet's box principle*.

Proposition

If we distribute n objects among k boxes, and n is larger than k, then one of the boxes must get more than one object.

More general version:

Proposition

Given natural numbers k and m, if $n = km + 1$ objects are distributed among k boxes, then at least one of the boxes will get at least $m + 1$ objects.

Applications of the pigeonhole principle (I)

Proposition

Let G be a finite simple graph, with $n \geq 2$ vertices. Then G must have at least two vertices of the same degree.

Proof: There are n vertices and the degree of each vertex is a number in the set $\{0, 1, \ldots, n-1\}$ (which has cardinality n). Now, if there is a vertex with degree 0, then no vertex can have degree $n-1$. Therefore the set of possible degrees would have cardinality $n-1$. By the pigeonhole principle, there must be two vertices with the same degree.

Classification of the samples

	Without repetition	With repetition
Ordered	Permutations without repetition	Permutations with repetition (words)
Unordered	Combinations without repetition	Combinations with repetition

Permutations without repetition

Proposition

The number of permutations of length k of n distinct elements, without repetition, is

$$P(n,k) = \underbrace{n \cdot (n-1) \cdot (n-2) \cdots (n-k+1)}_{k \text{ factors}} = \frac{n!}{(n-k)!}$$

Examples of permutations without repetitions (I)

In the ancient kingdom of Atlantis there was a round table with n seats, one of which was a special seat reserved for the king. The noblemen of the court sat around the king clockwise in any order.

Question:

In how many ways can we accommodate the remaining $n-1$ noblemen at the table?

Answer: $(n-1)!$

Example of permutation without repetitions (II)

In the ancient kingdom of Camelot there was also a round table with n seats, but all of them were the same, i.e. there was no distinguished seat, so that all the knights were considered equal.

Question:

In how many ways can we accommodate n knights at the table?

Example of permutation without repetitions (III)

Answer:

We are only interested in the relative position of each person in relation to the others. In other words, if two arrangements only differ in a rotation of the table, then we will consider that they are essentially the same arrangement.

Each arrangement is a permutation, so in principle there would be $n!$ arrangements. However, since we are only interested in the relative position of each person in relation to the others, we have to divide $n!$ by the number of different arrangements that can be obtained by rotating the table, which is n. Thus, the number of distinct arrangements would be

$$\frac{n!}{n} = (n-1)!$$

Permutations with repetition (words)

Proposition

The number of length-k permutations of n distinct elements, with repetition, is

$$\underbrace{n \cdot n \cdot n \cdots n}_{k \text{ factors}} = n^k$$

These permuations are also called *words of length k from an alphabet of n letters*.

Example of permutations with repetition (I)

Suppose that a password is required to have 8 characters, it has to start with a letter from the English alphabet, and the remaining 7 characters can be any combination of letters and digits. We do not distinguish between upper case and lower case letters. How many different passwords can we construct?

For the first letter we have 26 possibilities, and the remainder is a permutation with repetition of length 7, taken from an alphabet of 36 symbols. Then, the total number of distinct passwords is

$$26 \cdot 36^7 = 2\,037\,468\,266\,496$$

Combinations without repetition

Proposition

$$C(n,k) = \frac{P(n,k)}{P(k,k)} = \frac{n!}{k!(n-k)!}$$

The numbers $C(n,k)$ are also known as *binomial coefficients*, and they are usually denoted $\binom{n}{k}$.

Binomial coefficients

Definition in terms of the factorial:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Recursive formula:

$$\binom{n}{k} = \begin{cases} 1 & \text{if} \quad k = 0 \quad \text{or} \quad k = n, \\ \binom{n-1}{k-1} + \binom{n-1}{k} & \text{otherwise.} \end{cases}$$

Pascal triangle

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}
$$

Example of combination without repetitions (I)

The handshaking lemma:

A group of *n* friends gather at a party. Each one of them shakes hands with the rest. How many handshakes have taken place?

Answer:

There is a handshake for every distinct pair of friends. Since the pairs are unordered, the total number of pairs is $\binom{n}{2} = \frac{n(n-1)}{2}$.

By the way, this is also the maximum number of edges that a simple graph with n vertices can have.

Newton's binomial theorem

It was known long before Newton.

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

$$(a-b)^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} a^{n-k} b^k$$

Proof of Newton's binomial formula

Purely combinatorial proof:

We have

$$(a+b)^n = \underbrace{(a+b)\,(a+b)\cdots(a+b)}_{n \text{ factors}} = \sum_{k=0}^{n} C_k \; a^{n-k} b^k$$

The coefficient C_k is the number of ways in which we can pick k instances of the letter b from the n factors, and $n-k$ instances of the letter a, and this number is precisely $\binom{n}{k}$.

Proof of Newton's binomial formula

Proof by induction:

Base case: For $n = 1$ the property holds trivially.

Induction hypothesis: Assume that the property holds for $n \geq 1$.

Induction step: Prove that it also holds for $n + 1$.

By the induction hypothesis we have

$$(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

Now,

$$(a + b)^{n+1} = (a + b)(a + b)^n = a(a + b)^n + b(a + b)^n$$

Proof of Newton's binomial formula

Proof by induction (cont.):

Let $p(a,b)$ be a polynomial in the indeterminates a and b, and let's denote by $[a^i b^j]p(a,b)$ the coefficient of the term $a^i b^j$ in p. Then,

$$[a^{n-k}b^k]\,(a+b)^n = \binom{n}{k}.$$

We need to look into the coefficient $[a^{n+1-k}b^k]\,(a+b)^{n+1}$:

$$\begin{aligned}
[a^{n+1-k}b^k]\,(a+b)^{n+1} &= [a^{n+1-k}b^k]a\,(a+b)^n + [a^{n+1-k}b^k]b\,(a+b)^n \\
&= [a^{n-k}b^k]\,(a+b)^n + [a^{n+1-k}b^{k-1}]\,(a+b)^n \\
&= [a^{n-k}b^k]\,(a+b)^n + [a^{n-(k-1)}b^{k-1}]\,(a+b)^n \\
&= \binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}
\end{aligned}$$

Proof of Newton's binomial formula

Proof by induction (Cont):

Hence

$$(a+b)^{n+1} = \sum_{k=0}^{n+1} \binom{n+1}{k} a^{n+1-k} b^k$$

which is precisely what we wanted to prove in our induction step. Therefore

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

for all $n \in \mathbb{N}$.

Further properties of binomial coefficients

Symmetry:

$$\binom{n}{k} = \binom{n}{n-k}, \quad n \geq k \geq 0$$

Absorption:

$$\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}, \quad n \geq k \geq 1$$

Trinomial revision:

$$\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}, \quad n \geq m \geq k \geq 0$$

Sums of binomial coefficients

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Chu-Vandermonde:

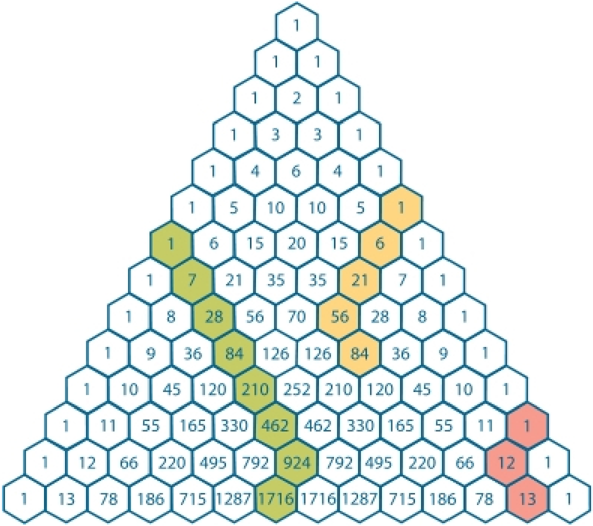
$$\sum_{j=0}^{k} \binom{m}{j} \binom{n-m}{k-j} = \binom{n}{k}$$

$$\sum_{j=0}^{m} \binom{m}{j}^2 = \binom{2m}{m}$$

Hockey-stick identity:

$$\sum_{m=k}^{n} \binom{m}{k} = \binom{n+1}{k+1}$$

Hockey-stick identity



Combinations with repetition

Proposition

The number of k-combinations with repetition taken from a set of cardinality n is

$$\binom{n-1+k}{k} = \binom{n-1+k}{n-1}$$

Proof: The k-sample can be associated with a scheme consisting of k star symbols $\bigstar$ and $n-1$ vertical bars $|$ that act as separators. The bars determine n blocks, and each block corresponds to one of the elements of the n-set. In the end we have a string of length $n-1+k$ in the alphabet $\{\bigstar, |\}$, and we are asking for the number of ways to place the k stars in the string, which is $\binom{n-1+k}{k}$, or the number of ways to place the $n-1$ bars in the string, which is $\binom{n-1+k}{n-1}$.

Example of combinations with repetition I

(Example taken from *Matemática Discreta*, by Juan A. Rodríguez Velázquez).

Suppose there are four types of sandwiches at the bar, and Peter is so hungry that he thinks he could eat three sandwiches. How many different combinations of sandwiches can he eat?

In this case there is nothing that suggests that the order of eating the sandwiches matters. Therefore, what we want to find is the number of 3-combinations with repetitions from a 4-set, i.e.

$$\binom{4-1+3}{3} = \binom{6}{3} = 20$$

Example of combinations with repetition II

Proposition

The number of nonnegative integer solutions of the equation

$$x_1 + x_2 + \cdots + x_n = k$$

is $\binom{n-1+k}{k}$

Proof: There is a bijection between the nonnegative integer solutions of the equation and k-combinations with repetition of the set $\{1, 2, \ldots, n\}$.

Example of combinations with repetition III

Corollary

The number of positive integer solutions of the equation

$$x_1 + x_2 + \cdots + x_n = k,$$

where $k \geq n$, is $\binom{k-1}{k-n} = \binom{k-1}{n-1}$

Proof: The number of positive integer solutions of the above equation agrees with the number of nonnegative integer solutions of the equation

$$(x_1 - 1) + (x_2 - 1) + \cdots + (x_n - 1) = k - n.$$

Multinomial coefficients

Definition in terms of the factorial:

$$\binom{n}{k_1, k_2, \ldots, k_m} = \frac{n!}{k_1!\, k_2! \cdots k_m!}$$

where $k_1 + k_2 + \cdots + k_m = n$.

Combinatorially, the multinomial coefficient $\binom{n}{k_1, k_2, \ldots, k_m}$ counts the number of different ways to partition an n-element set into disjoint subsets of sizes $k_1, k_2, \ldots, k_m$.

Note that the multinomial coefficient $\binom{n}{k, n-k}$ is simply the binomial coefficient $\binom{n}{k}$.

Example of multinomial coefficients I

(Adapted from *Matemática Discreta*, by Juan A. Rodríguez Velázquez).

In how many ways can we arrange ten kitchen knives in a row, given that there four identical carver knives, three identical chef knives, and three identical slicers?

Answer:

$$\binom{10}{4,3,3} = \frac{10!}{4!\,3!\,3!} = 4200$$

Example of multinomial coefficients II

(Adapted from *Matemática Discreta*, by Juan A. Rodríguez Velázquez).

In how many ways can we distribute five distinct objects into three distinct bins, so that the first and the second bin contain two objects each, and the third bin contains one object?

Answer:

$$\binom{5}{2,2,1} = \frac{5!}{2!\,2!\,1!} = 30$$

Example of multinomial coefficients III

(Adapted from Wikipedia: `https://en.wikipedia.org/wiki/Multinomial_theorem`).

The multinomial coefficient $\binom{n}{k_1, k_2, \ldots, k_m}$ also counts the number of distinct permutations of the letters of a word of length n, having k_1 occurrences of the first letter, k_2 occurrences of the second letter, and so on. Take for instance the word MISSISSIPPI. The number of distinct permutations is

$$\underbrace{\binom{11}{1,10}}_{\text{M}}\underbrace{\binom{10}{4,6}}_{\text{I}}\underbrace{\binom{6}{4,2}}_{\text{S}}\underbrace{\binom{2}{2,0}}_{\text{P}} = \binom{11}{1,4,4,2} = \frac{11!}{1!\,4!\,4!\,2!} = 34650$$

Properties of multinomial coefficients

Given $n = k_1 + k_2 + \cdots + k_m$,

$$\binom{n}{k_1, k_2, \ldots, k_m} = \binom{k_1}{k_1}\binom{k_1+k_2}{k_2}\cdots\binom{k_1+k_2+\cdots+k_m}{k_m},$$

or also

$$\begin{aligned}\binom{n}{k_1, k_2, \ldots, k_m} &= \binom{n}{k_1}\binom{n-k_1}{k_2}\cdots\binom{n-k_1-\cdots-k_{m-1}}{k_m} \\ &= \binom{n}{k_1}\binom{n-k_1}{k_2}\binom{n-k_1-k_2}{k_3}\cdots\binom{k_m}{k_m}.\end{aligned}$$

Prove !!

The multinomial theorem

The multinomial theorem is a generalization of Newton's binomial formula:

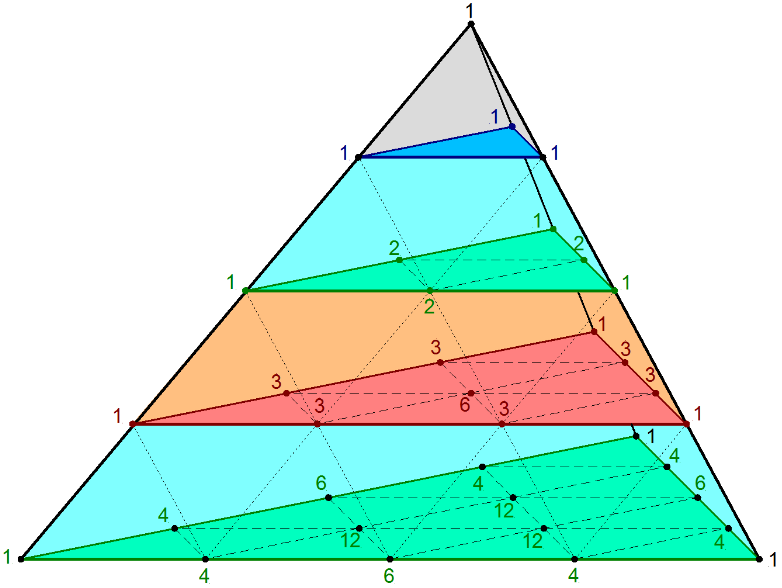
$$(x_1 + x_2 + \cdots + x_m)^n = \sum_{k_1+k_2+\cdots+k_m=n} \binom{n}{k_1, k_2, \ldots, k_m} \prod_{t=1}^{m} x_t^{k_t}$$

Example:

$$(a+b+c)^3 = a^3+b^3+c^3+3a^2b+3a^2c+3b^2a+3b^2c+3c^2a+3c^2b+6abc$$

This corresponds to the red level in the pyramid of the next slide.

Pascal's pyramid



More properties of multinomial coefficients

Setting $x_1 = x_2 = \cdots = x_m = 1$ in the multinomial theorem we get:

$$\sum_{k_1+k_2+\cdots+k_m=n} \binom{n}{k_1, k_2, \ldots, k_m} = (\underbrace{1+1+\cdots+1}_{m \text{ times}})^n = m^n$$

The number of terms in the sum above is $\binom{n+m-1}{m-1} \longleftarrow$ Prove !!

Principle of inclusion-exclusion for two sets

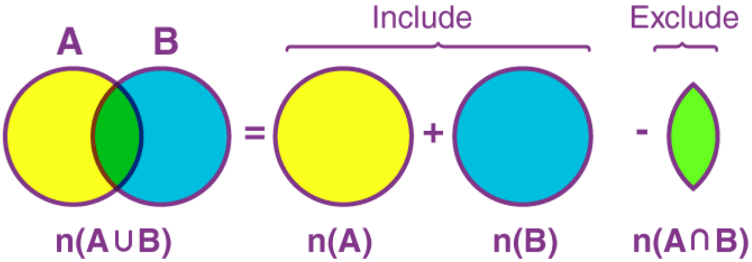
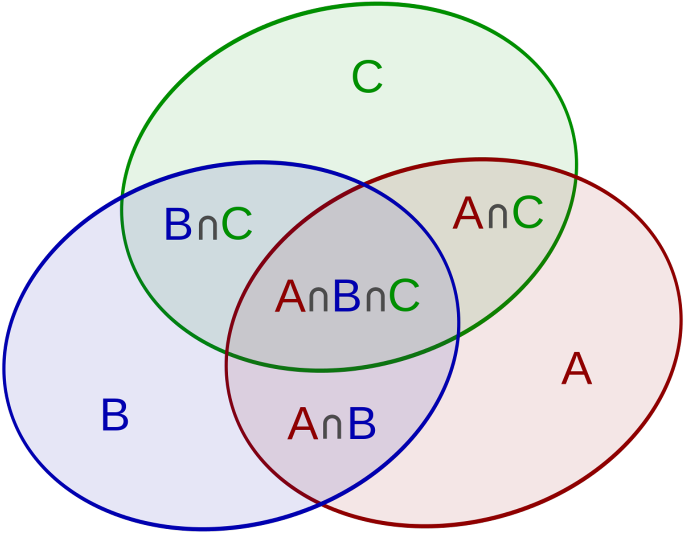


Figure: Two sets with non-empty intersection

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Principle of inclusion-exclusion for three sets



Inclusion-exclusion formula for three sets

Proposition

Let A, B and C three sets. Then

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

Proof: Left to the student

Example inclusion-exclusion I

How many integers in the set $S = \{1, \ldots, 100\}$ are not divisible by 2, 3 or 5? (Adapted from Wikipedia)

Answer: Let S_2, S_3 and S_5 denote the subsets of S consisting of those numbers that can be divided by 2, 3 and 5, respectively. We have

$$|S_2| = \left\lfloor \frac{100}{2} \right\rfloor = 50, \quad |S_3| = \left\lfloor \frac{100}{3} \right\rfloor = 33, \quad |S_5| = \left\lfloor \frac{100}{5} \right\rfloor = 20$$

$$|S_2 \cap S_3| = 16, \quad |S_2 \cap S_5| = 10, \quad |S_3 \cap S_5| = 6$$

$$|S_2 \cap S_3 \cap S_5| = 3$$

The number we are looking for is

$$|S| - |S_2 \cup S_3 \cup S_5| = 100 - (50 + 33 + 20) + (16 + 10 + 6) - 3 = 26$$

Inclusion-exclusion formula for n sets

Theorem

Let $A_1, A_2, \ldots A_n$ be sets. Then

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i| - \sum_{1 \leqslant i < j \leqslant n} |A_i \cap A_j| + \sum_{1 \leqslant i < j < k \leqslant n} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n-1} |A_1 \cap \cdots \cap A_n|$$

In compact form:

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{k=1}^{n} (-1)^{k+1} \left(\sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} |A_{i_1} \cap \cdots \cap A_{i_k}| \right)$$

Derangements

Definition

A **derangement** of the set is a permutation of the elements of a set, such that no element appears in its original position. In other words, a derangement is a permutation that has no fixed points.

Theorem

The number of derangements of an n-set is

$$!n = \sum_{i=2}^{n} (-1)^i \binom{n}{i} (n-i)!$$

The number $!n$ is called the *subfactorial*.

Proof of the formula for derangements (I)

Without loss of generality we can take $S = \{1, \ldots, n\}$. Denote by S_i the subset of the permutations of S that fix the element i. The number of derangements of S is

$$!n = n! - |S_1 \cup \cdots \cup S_n|.$$

Now, by the principle of inclusion-exclusion

$$|S_1 \cup \cdots \cup S_n| = \sum_i |S_i| - \sum_{i<j} \left|S_i \cap S_j\right| + \sum_{i<j<k} \left|S_i \cap S_j \cap S_k\right| + \\ + \cdots + (-1)^{n+1} \left|S_1 \cap \cdots \cap S_n\right|$$

Now,

$$\sum_{i_1<i_2<\cdots<i_r} \left|S_{i_1} \cap S_{i_2} \cap \cdots \cap S_{i_r}\right| = \binom{n}{r}(n-r)!,$$

hence

Proof of the formula for derangements (II)

$$
\begin{aligned}
|S_1 \cup \cdots \cup S_n| &= \binom{n}{1}(n-1)! - \binom{n}{2}(n-2)! + \cdots + (-1)^{n+1}\binom{n}{n}0! \\
&= n! - \binom{n}{2}(n-2)! + \cdots + (-1)^{n+1}\binom{n}{n}0! \\
&= \sum_{i=1}^{n}(-1)^{i+1}\binom{n}{i}(n-i)! = n! \sum_{i=1}^{n}\frac{(-1)^{i+1}}{i!},
\end{aligned}
$$

hence

$$
\begin{aligned}
!n = n! - \left[n! - \binom{n}{2}(n-2)! + \cdots + (-1)^{n+1}\binom{n}{n}0!\right] \\
= \binom{n}{2}(n-2)! - \cdots + (-1)^{n+1}\binom{n}{n}0! \\
= \sum_{i=2}^{n}(-1)^{i}\binom{n}{i}(n-i)!
\end{aligned}
$$

Subfactorial vs. factorial

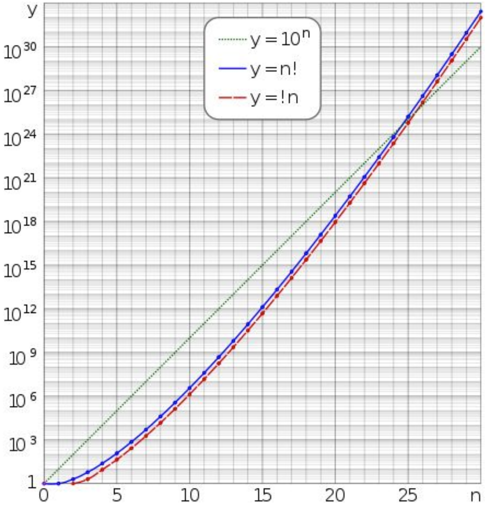


Figure: Subfactorial vs. factorial

Warm-up exercises

1. Find the number of
   (a) two-digit even numbers
   (b) two-digit odd numbers
   (c) two-digit odd numbers with distinct digits
   (d) two-digit even numbers with distinct digits
   (e) positive integers with distinct digits
2. Consider a tournament with n players, where every player plays against every other player. Suppose that each player wins at least one game. Show that there are at least two players having the same number of wins.
3. Let T be an equilateral triangle with sides of length 2. Show that if five points are chosen at random inside T, then at least a pair of points will have a separation of less than 1 unit.
4. Prove the identity of trinomial revision involving binomial coefficients.

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