Numerical Methods

Interpolation

- Classical Interpolating Polynomial

2022-2023

[DΣIM]

Interpolation

- Interpolation comes from the Latin words ***inter*** (between) and ***polare*** (clean up) and it refers to calculation of new values in between known values
- Suppose we have a function f which is known at a set of points known as ***nodes***. The object of interpolation is to ***estimate*** values of the function at other points and ***bound the error*** between the estimated and true values.

Interpolation

- Our interpolation formula will have the form

$$f(x)=\sum_{i=1}^{n} L_i(x) f(a_i)+E(x)=y(x)+E(x)$$

- Or if we use derivatives

$$f(x)=\sum_{i=1}^{n}\sum_{j=0}^{m_i} A_{ij}(x) f^{(j)}(a_i)+E(x)$$

- Where $\left(a_i, f(a_i)\right), i=0, \ldots, n$ are the nodal values and the functions $L_i(x)$ will be the basis functions of the interpolation.

Interpolation

- Our object is to determine the basis functions $L_i(x)$ and $A_{ij}(x)$ so that:

$$E(a_i) = 0 \qquad i = 1, \ldots, n$$

- Not depending on the function $f(x)$ studied.
- In general, however, for non nodal points we will have $E(x) \neq 0$, So we need a representation of $E(x)$ which will enable us to estimate or at least bound the error for values of $x \neq a_i, i = 1, \ldots, n$

Polynomial Interpolation

- We will state our classical problem as follows. Given any table with n+1 values

x	x_0	x_1	.....	x_n
y	y_0	y_1	.....	y_n

- Is this possible to find any polynomial which satisfies each single point of this table?

Polynomial Interpolation

- The polynomial $P(x)$ which satisfies the ***interpolation conditions***:

$$P(x_i) = y_i \quad \text{when} \quad 1 \leq i \leq n$$

- Will be called the ***interpolating polynomial,*** while the points x_i will be called the ***interpolating nodes***
- This polynomial will depend on the given table and its degree will never be greater that n

Polynomial Interpolation

- ***Theorem. Existence and Uniqueness***: Given n+1 interpolating points, there exists a ***unique*** polynomial $P_n(x)$ of degree equal or smaller than n for which

$$P(x_i) = y_i \quad \text{when} \quad 1 \le i \le n$$

- This polynomial must be of the form

$$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

Interpolation

- In order to generate this polynomial, we must know the n+1 constant coefficients a_i.
- This polynomial must satisfy the n+1 interpolating points, so we have

$$
\begin{aligned}
&a_n x_0^n + \cdots + a_1 x_0 + a_0 = y_0 \\
&a_n x_1^n + \cdots + a_1 x_1 + a_1 = y_1 \\
&\vdots \\
&a_n x_n^n + \cdots + a_1 x_n + a_n = y_n
\end{aligned}
$$

Interpolation

- This is a linear system with $n+1$ equations, where the $n+1$ unknowns are the a_i values. The matrix of the system is no non-singular as determinant of the system is the known Vandermonde determinant:

$$\begin{vmatrix} 1 & x_0 & \cdots & x_0^n \\ 1 & x_1 & \cdots & x_1^n \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & \cdots & x_n^n \end{vmatrix} = \prod_{k>i} (x_k - x_i) \neq 0$$

- And this linear system has thus a ***unique solution***

Lagrange Interpolation

- ***Lagrange polynomial***. Suppose that $n \geq 1$. There are polynomials $L_k \in P_n(\mathbb{R}), k = 0,1,\dots,n$ such that:

$$L_k(x_i) = \delta_{ki} = \begin{cases} 1, & i = k \\ 0, & i \neq k \end{cases}$$

- For $i, k = 0,1,\dots,n$. Moreover, the polynomial

$$p_n(x) = \sum_{k=0}^{n} L_k(x) y_k = \sum_{k=0}^{n} L_k(x) f(x_k)$$

- Satisfies the interpolation conditions.

Lagrange Interpolation

- For each fixed $k, 0 \leq k \leq n$, L_k is required to have n zeros at $x_i, i = 0,1,\dots,n, i \neq k$. $L_k(x)$ must be of the form

$$L_k(x) = c_k \prod_{\substack{i=0 \\ i \neq k}}^{n} (x - x_i)$$

- Where $c_k \in \mathbb{R}$, is a constant to be determined. As we want that $L_k(x_k) = 1$, we need:

$$c_k = \prod_{\substack{i=0 \\ i \neq k}}^{n} \frac{1}{x_k - x_i}.$$

Lagrange Interpolation

- We obtain

$$\boxed{L_k(x) = \prod_{\substack{i=0 \\ i\neq k}}^{n} \frac{x - x_i}{x_k - x_i}.}$$

- As the function p_n is a linear combination of the polynomials $L_k \in P_n(\mathbb{R}), k = 0,1,\ldots,n$ we have also $p_n \in P_n(\mathbb{R})$. Finally, we have from the construction that $p_n(x_i) = y_i$.

Lagrange Interpolation

- Consider the case n=0. Then, the polynomial is simple the horizontal line:

$$P_0(x) = y_0$$

- The following case, n=1, corresponds to the straight line through the two nodal points: (x_1,y_1) and (x_2,y_2)

$$P_1(x) = \frac{x - x_2}{x_1 - x_2} y_1 + \frac{x - x_1}{x_2 - x_1} y_2 = y_1 + \left(\frac{y_2 - y_1}{x_2 - x_1} \right) (x - x_1)$$

Linear Interpolation

- We could think that straight lines or plane polygons pieces are not adequate to represent smooth curves or smooth surfaces
- This is not always true. Any curve or surface can be approximated piecewise by many (many!) small straight lines or small polygons. In fact, this is how fast rendering graphics work.

Lagrange Interpolation

- Note that all the functions $L_k(x)$ are implicitly, polynomials of degree n.
- This expression for the interpolating polynomial is ***difficult to evaluate***. This is especially true if we are dealing with a great number of data points
- The polynomial interpolation is thus limited to polynomials of degree n=2 or n=3. This is reasonable, as high degree polynomials have many roots and are oscillatory functions.

Lagrange Interpolation

- Two main reasons make the Lagrange's polynomial unsuitable:
  - Polynomials may have n real roots and so the must cut the x axis n times. If n is big the polynomial curve must oscillate often
  - Even if this polynomial is unique, this may not be the best suited curve for our $n+1$ data points

Newton's Interpolation

- In the Lagrange representation, if we add a new node, the new higher degree interpolation polynomial must be completely recalculated, as we change all the basis functions. This contrasts with Taylor's series expansion or to least squares expansion in orthogonal functions.
- We seek a representation of the interpolation polynomial which has the property that the next higher degree interpolation polynomial is found by simply adding a new term

Newton's Interpolation

- Let $Q_k(x)$ be the interpolation polynomial for $f(x)$, of degree at most k, with respect to the $k+1$ distinct points $x_0, x_1, \ldots, x_k$. We seek the successive interpolation polynomials, $\{Q_k(x)\}$, of degree at most k in the form $Q_0(x) \equiv f(x_0)$ and

$$Q_k(x) = Q_{k-1}(x) + q_k(x), \quad \text{for } k = 1, 2, \ldots, n$$

- Where $q_k(x)$ has at most degree k.

Newton's Interpolation

- Since we require

$$Q_k(x_j) = f(x_j) = Q_{k-1}(x_j), \quad j = 0,1,\ldots,k-1$$

- It follows that $q_k(x_j) = 0$ at these k points. Thus, we may write

$$q_k(x) = a_k \prod_{j=0}^{k-1} (x - x_j)$$

- Which represents the most general polynomial of degree at most k that vanishes at the indicated k points.

Newton's Interpolation

- The constant a_k remains to be determined. But, in order that $Q_k(x_k) = f(x_k)$, it follows that:

$$a_k = \frac{f(x_k) - Q_{k-1}(x_k)}{\prod_{j=0}^{k-1}(x_k - x_j)}, \qquad \text{for } k = 1, 2, \ldots, n$$

- The zero-degree interpolation polynomial for the initial point x_0 is $Q_0(x) \equiv f(x_0)$. Thus, with $a_0 = f(x_0)$

Newton’s Interpolation

- Using recursively these expressions we can write the interpolating polynomial of degree at most n in the form:

$$Q_n(x) = a_0 + (x - x_0)a_1 + \cdots + (x - x_0)\cdots(x - x_{n-1})a_n$$

- The kth coefficient is called the kth order divided difference and is usually expressed in the notation

$$a_0 = f[x_0]$$
$$a_k = f[x_0, x_1, \ldots, x_k], \quad k = 1, 2, \ldots$$

Newton's Interpolation

- Since $Q_n(x)$ is the unique interpolation polynomial of degree n, we may equate the leading coefficient, a_n, in this form with that obtained by using the Lagrange form. Then from

$$Q_n(x) = \sum_{j=0}^{n} f(x_j) \prod_{\substack{k=0 \\ k \neq j}}^{n} \frac{x - x_k}{x_j - x_k}$$

- We obtain

$$a_n = f[x_0, x_1, \ldots, x_n] = \sum_{j=0}^{n} \frac{f(x_j)}{\prod_{\substack{k=0 \\ k \neq j}}^{n} (x_j - x_k)}$$

Newton's Interpolation

- From this representation it follows that the divided differences are symmetric functions of their arguments. That is, if we adopt the additional notation

$$f_{i,j,k,\ldots} \equiv f[x_i, x_j, x_k, \ldots]$$

- Then this symmetry is expressed by

$$f_{0,1,\ldots,n} = f_{j_0, j_1, \ldots, j_n}$$

- Where $(j_0, j_1, \ldots, j_n)$ is any permutation of the integers $(0,1,\ldots,n)$.

Newton's Interpolation

- We can derive a more convenient form for computing the divided differences using this fact and the uniqueness of the interpolation polynomial
- We may construct the polynomial $Q_n(x)$ by matching the values of $f(x_j)$ in the reverse order $j = n, n-1, \ldots, 1,0$. In this way we would obtain, say

$$Q_n(x) \equiv b_0 + (x - x_n)b_1 + \cdots + (x - x_n)\cdots(x - x_1)b_n$$

Newton's Interpolation

- Where:

$$b_k = f[x_n, x_{n-1}, \ldots, x_{n-k}] \quad \text{and} \quad b_0 = f[x_n] = f(x_n).$$

- But $a_n = b_n$ since they are the coefficients of x^n in the unique polynomial $Q_n(x)$. Subtracting the two representations, we obtain

$$0 \equiv [(x - x_0) - (x - x_n)](x - x_1) \cdots (x - x_{n-1}) a_n$$
$$+ (a_{n-1} - b_{n-1}) x^{n-1} + p_{n-2}(x)$$

- Where $p_{n-2}(x)$ is a polynomial of degree at most $n - 2$.

Newton's Interpolation

- Since this expression vanishes identically all the coefficients must be zero. Hence for the coefficient of x^{n-1} we have

$$0 = [x_0 - x_n]a_n + (b_{n-1} - a_{n-1})$$

- The symmetry of the divided differences, implies that

$$b_{n-1} = f[x_n, x_{n-1}, \ldots, x_1] = f[x_1, x_2, \ldots, x_n]$$

- And we have also:

$$a_{n-1} = f[x_0, x_1, \ldots, x_{n-1}]$$

Newton's Interpolation

- From $a_n = (a_{n-1} - b_{n-1})/(x_n - x_0)$ we have:

$$f[x_0, x_1, \ldots, x_n] = \frac{f[x_1, x_2, \ldots, x_n] - f[x_0, x_1, \ldots, x_{n-1}]}{x_n - x_0}$$

- Defining for completeness:

$$f[x_0] = f(x_0)$$

- The polynomial can be written as the Newton's divided difference interpolation formula

$$Q_n(x) = f[x_0] + (x - x_0) f[x_0, x_1] + \cdots + (x - x_0) \cdots (x - x_{n-1}) f[x_0, x_1, \ldots, x_n]$$

Newton's Interpolation

- Applying the expression of the divided differences to $(x_j, x_{j+1}, \ldots, x_{j+n})$ we can construct a table of divided differences in a symmetric manner based on

$$f_{j,j+1,\ldots,j+n} = \frac{f_{j+1,j+2,\ldots,j+n} - f_{j,j+1,\ldots,j+n-1}}{x_{j+n} - x_j}$$

Newton's Interpolation

- Consider table with $n + 1$ data points:

x	x_0	x_1	x_2	…	x_n
$f(x)$	$f(x_0)$	$f(x_1)$	$f(x_2)$	…	$f(x_n)$

- The algorithm to determine the constants $a_0, a_1, \ldots, a_n$ as a function of the data points $x_0, x_1, \ldots, x_n$ and $y_0, y_1, \ldots, y_n$ will build recursively the table:

Newton's Interpolation

x	$f(x)$	$f[x, x]$	$f[x, x, x]$	$f[x, x, x, x]$	$\cdots$
x_0	f_0				
		$\frac{f_1 - f_0}{x_1 - x_0} \equiv f_{01}$			
x_1	f_1		$\frac{f_{12} - f_{01}}{x_2 - x_0} \equiv f_{012}$		
		$\frac{f_2 - f_1}{x_2 - x_1} \equiv f_{12}$		$\frac{f_{123} - f_{012}}{x_3 - x_0} \equiv f_{0123}$	
x_2	f_2		$\frac{f_{23} - f_{12}}{x_3 - x_1} \equiv f_{123}$		$\cdots$
		$\frac{f_3 - f_2}{x_3 - x_2} \equiv f_{23}$		$\frac{f_{234} - f_{123}}{x_4 - x_1} \equiv f_{1234}$	
x_3	f_3		$\frac{f_{34} - f_{23}}{x_4 - x_2} \equiv f_{234}$	$\vdots$	
		$\frac{f_4 - f_3}{x_4 - x_3} \equiv f_{34}$	$\vdots$		
x_4	f_4				
		$\vdots$			
$\vdots$	$\vdots$				

Numerical Methods

2022-2023

Newton's Interpolation

- The first divided differences are computed as:

$$f[x_i] = f(x_i)$$

$$f[x_i, x_{i+1}] = \frac{f[x_{i+1}] - f[x_i]}{x_{i+1} - x_i}$$

$$f[x_i, x_{i+1}, x_{i+2}] = \frac{f[x_{i+1}, x_{i+2}] - f[x_i, x_{i+1}]}{x_{i+2} - x_i}$$

Newton's Interpolation

- Example 1

Example 1 Compute a divided difference table for these function values:

x	3	1	5	6
$f(x)$	1	-3	2	4

(14)

Solution We arrange the given table vertically and compute divided differences by use of Formula (11), arriving at

3	1	2	$-3/8$	$7/40$
1	-3	$5/4$	$3/20$	
5	2	2		
6	4			

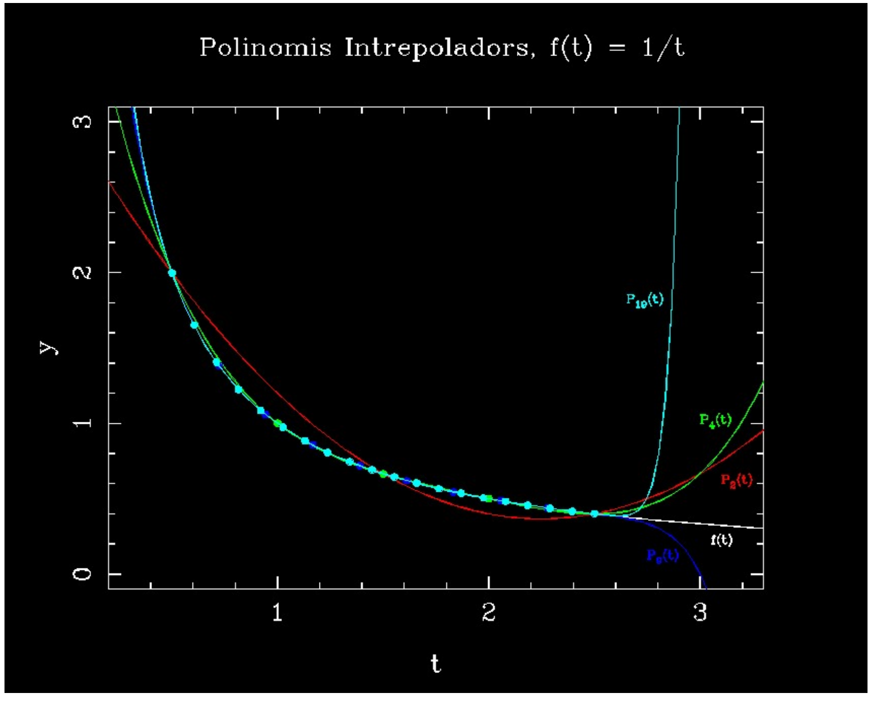
Newton's Interpolation

- Example 2

		First divided differences	Second divided differences	Third divided differences	Fourth divided differences
1.0	0.7651977				
		−0.4837057			
1.3	0.6200860		−0.1087339		
		−0.5489460		0.0658784	
1.6	0.4554022		−0.0494433		0.0018251
		−0.5786120		0.0680685	
1.9	0.2818186		0.0118183		
		−0.5715210			
2.2	0.1103623				

Polynomial Interpolation

- The interpolating polynomials are only useful within the interpolation interval



2022-2023

[DΣIM]

Interpolation Error

- Although the values of the function f an those of its Lagrange interpolation polynomial coincide at the interpolation points, $f(x)$ may be quite different from $p_n(x)$ when x is not an interpolation point.
- Thus, it is natural to ask just how large can be the difference $f(x) - p_n(x)$ when $x \neq x_i$, for $i = 0, \ldots, n$. Assuming that the function is sufficiently smooth, we need an estimate of the interpolation error.

Interpolation Error

- Suppose that $n \geq 0$, and that f is a smooth real valued function defined and continuous in the closed interval $[a, b]$, such that the derivative of order $n+1$ exists and is continuous in $[a, b]$. Then, given that $x \in [a, b]$, there exists $\xi = \xi(x)$ in (a, b) such that

$$\boxed{f(x) - p_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\pi_{n+1}(x),}$$

- Where:

$$\pi_{n+1}(x) = \prod_{i=0}^{n}(x - x_i) = (x - x_0)\cdots(x - x_n).$$

Interpolation Error

- Moreover

$$\left| f(x) - p_n(x) \right| \leq \frac{M_{n+1}}{(n+1)!} \left| \pi_{n+1}(x) \right|$$

- Where:

$$M_{n+1} = \max_{\xi \in (a,b)} \left| f^{(n+1)}(\xi) \right|$$

- When $x = x_i$ for some $i, i = 0,1,\ldots,n$, both sides are identically zero and the relation is identically satisfied.

Interpolation Error

- Suppose then that $x \in [a, b]$ and $x \neq x_i$ for any of the nodes. Consider the auxiliary real valued function $t \to \varphi(t)$ defined on $[a, b]$:

$$\varphi(t) = f(t) - p_n(t) - \frac{f(x) - p_n(x)}{\pi_{n+1}(x)}\pi_{n+1}(t).$$

- Clearly $\varphi(x_i) = 0, i = 0,1, \dots, n$ and $\varphi(x) = 0$. Thus, φ vanishes at $n + 2$ points which are all distinct in $[a, b]$

Interpolation Error

- By Rolle's theorem, $\varphi'(t)$ must vanish at $n+1$ points in (a,b), one between each pair of consecutive points at which φ vanishes.
- If $n = 0$, we deduce the existence of $\xi = \xi(x)$ in the interval (a,b), such that $\varphi'(\xi) = 0$. Since $p_0(x) \equiv f(x_0)$ and $\pi_1(t) = t - x_0$, it follows that

$$0 = \varphi'(\xi) = f'(\xi) - \frac{f(x) - p_0(x)}{\pi_1(x)}$$

- Which confirms the case $n = 0$.

Interpolation Error

- Now suppose that $n \geq 1$. As $\varphi'(t)$ vanishes at $n+1$ points in (a,b), one between each pair of consecutive points at which φ vanishes, applying Rolle's theorem again, we see that φ'' vanishes at n distinct points.
- Then $\varphi^{(n+1)}$ vanishes at some point $\xi \in (a,b)$, the exact value of ξ being dependent on the value of x.

Numerical Methods

Interpolation Error

- Differentiating $n+1$ times the function φ with respect to t, and noting that p_n is a polynomial of degree n or less, it follows:

$$0 = \varphi^{(n+1)}(\xi) = f^{(n+1)}(\xi) - \frac{f(x) - p_n(x)}{\pi_{n+1}(x)}(n+1)!$$

- Hence

$$f(x) - p_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\pi_{n+1}(x).$$

- Finally, if $f^{(n+1)}$ is continuous, the same is true for $\left|f^{(n+1)}\right|$. The function is bounded in $[a, b]$ and achieves the maximum there.

Interpolation Error

- It is worth noting that the location of ξ in the interval $[a, b]$ is unknown and we can not know exactly the value of the error in the interpolation.
- On the other hand, given the function f, an upper bound on the maximum value of $f^{(n+1)}$ over $[a, b]$ is, at least in principle, possible to obtain, and thereby we can provide an upper bound on the size of the interpolation error.

Convergence

- An important theoretical question if whether a sequence (p_n) of interpolation polynomials for a continuous function f converges to f as $n \to \infty$.
- The question needs to be made more specific as p_n depends on the interpolation points. Suppose that we agree to choose equally spaced points:

$$x_j = a + j\frac{(b-a)}{n}, \qquad j = 0,1,\ldots,n$$

Convergence

- The question of convergence then clearly depends on the behavior of M_{n+1} as n increases. If

$$\lim_{n\to\infty} \frac{M_{n+1}}{(n+1)!} \max_{x\in[a,b]} \left|\pi_{n+1}(x)\right| = 0$$

- Then

$$\lim_{n\to\infty} \max_{x\in[a,b]} \left|f(x) - p_n(x)\right| = 0$$

- An we could say that the sequence of interpolation polynomials (p_n), converges uniformly to f as $n \to \infty$ in $[a, b]$.

Convergence

- We may think that if all derivatives of f exist and are continuous on $[a, b]$ this will hold. Unfortunately, this is not so, since the sequence

$$\left(M_{n+1}\right) \max_{x\in[a,b]} \left|\pi_{n+1}(x)\right|$$

- May tend to ∞, as $n \rightarrow \infty$ faster than the sequence $1/(n+1)!$ tends to zero.

Errors of the interpolating polynomial

- In general, it is not true that

$$\lim_{n\to\infty}\max_{x_0\le x\le x_n}\left|f(x)-P_n(x)\right|=0$$

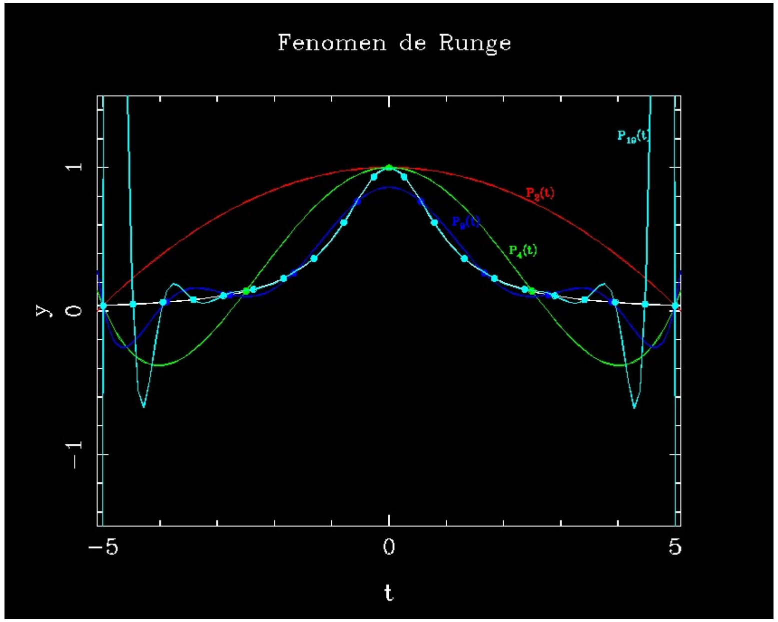
- The limit can be infinite in some points. An example of this situation is found in Runge phenomenon. Consider the function

$$f(x)=\frac{1}{1+x^2}$$

Numerical Methods

Runge's phenomenon

- The polynomials of higher degree are not always better suited



[DΣIM]

Chebyshev Nodes

- The error in the interpolating polynomial can be optimized if we are free to choose the location of the nodes.
- This procedure leads to the system of polynomials called ***Chebyshev polynomials***. There are recursively defined as follows:

$$T_0(x) = 1 \qquad T_1(x) = x$$

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) \quad (n \geq 1)$$

Chebyshev Nodes

- If $x \in [-1,1]$ the Chebyshev polynomials can be expressed in closed form as:

$$T_n(x) = \cos(n \arcsin(x)) \quad (n \geq 0)$$

- To prove this relation, we start with the trigonometric expression:

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

Chebyshev Nodes

- Then we have:

$$\cos(n+1)\theta = \cos\theta\cos n\theta - \sin\theta\sin n\theta$$
$$\cos(n-1)\theta = \cos\theta\cos n\theta + \sin\theta\sin n\theta$$

- Adding these relations and rearranging, we obtain

$$\cos(n+1)\theta = 2\cos\theta\cos n\theta - \cos(n-1)\theta$$

- If we write $\theta = arcos(x)$ and $x = \cos(x)$, this relation is the Chebyshev recursive relation for the functions

$$f_n(x) = \cos(n\ \cos^{-1} x)$$

Chebyshev Nodes

- A ***monic polynomial*** is one in which the term of highest degree has coefficient unity. From the recursive definition of Chebyshev polynomials, we see that in $T_n(x)$ the term of highest degree is $2^{n-1}x^n$. Therefore, $2^{1-n}T_n$ is a monic polynomial.
- From the closed from obtain also

$$|T_n(x)| \leq 1 \qquad (-1 \leq x \leq 1)$$

$$T_n(\cos(i\pi / n)) = (-1)^i \quad (0 \leq i \leq n)$$

Chebyshev Nodes

- If p is a monic polynomial of degree n, we have:

$$\|p\|_\infty = \max_{-1\le x\le 1} |p(x)| \ge 2^{1-n}$$

- To prove this, we proceed by contradiction. Suppose that $|p(x)| < 2^{1-n}$ when $|x| \le 1$. Let $q = 2^{1-n} T_n$ and take $x_i = \cos((i\pi)/n)$. As q is a monic polynomial of degree n

$$(-1)^i p(x_i) \le |p(x_i)| < 2^{1-n} = (-1)^i q(x_i)$$

Chebyshev Nodes

- Consequently:

$$(-1)^i \left[q(x_i) - p(x_i)\right] > 0 \quad (0 \leq i \leq n).$$

- And the polynomial $q - p$ oscillates in sign $n + 1$ times on the interval $[-1,1]$. This is not possible as $q - p$ has degree at most $n - 1$. Hence, we have a contradiction. Thus, for a monic polynomial p:

$$\|p\|_\infty = \max_{-1 \leq x \leq 1} |p(x)| \geq 2^{1-n}$$

Chebyshev Nodes

- Assume now that the interpolation nodes are in the interval $[-1,1]$. If x is in this interval we have for the interpolation error:

$$\max_{|x|\le 1}\left|f(x)-p(x)\right| \le \frac{1}{(n+1)!}\max_{|x|\le 1}\left|f^{(n+1)}(x)\right|\max_{|x|\le 1}\left|\prod_{i=0}^{n}(x-x_i)\right|$$

- But

$$\max_{|x|\le 1}\left|\prod_{i=0}^{n}(x-x_i)\right| \ge 2^{-n}$$

- As this is a monic polynomial.

Chebyshev Nodes

- The minimum will be attained if

$$\prod_{i=0}^{n}(x-x_i)=2^{-n}T_{n+1}(x)$$

- Is the monic multiple of T_{n+1}.
- The nodes x_i are then the roots of T_{n+1}. These are

$$x_i=\cos\left(\frac{2i+1}{2n+2}\pi\right) \quad (0\le i\le n)$$

Chebyshev Nodes

- If the nodes x_i are the roots of the Chebyshev polynomial T_{n+1}, then the error formula yields for $|x| \leq 1$

$$\left|f(x)-p(x)\right| \leq \frac{1}{2^n(n+1)!}\max_{|t|\leq 1}\left|f^{(n+1)}(t)\right|$$

- For a general interval $[a, b]$ we need to transform the interval $[-1,1]$ into $[a, b]$, then

$$x_i = \frac{b-a}{2}\cos\left[\frac{2i+1}{2n+2}\pi\right]+\frac{a+b}{2}.$$

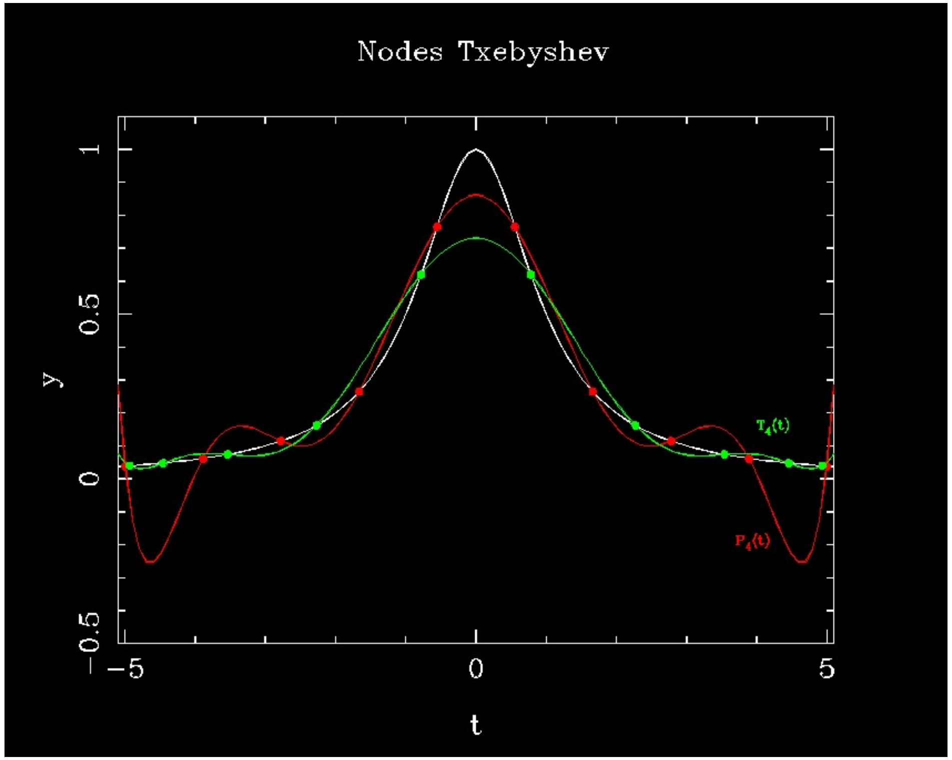
Chebyshev's Nodes

- If we have the freedom to select the interpolating nodes, we could choose these nodes in order minimize the error of the interpolating polynomial
- These nodes are called the ***Chebyshev nodes***

$$x_i = \frac{a+b}{2} + \frac{b-a}{2}\cos\left(\frac{i}{n-1}\pi\right) \quad 0 \le i \le n$$

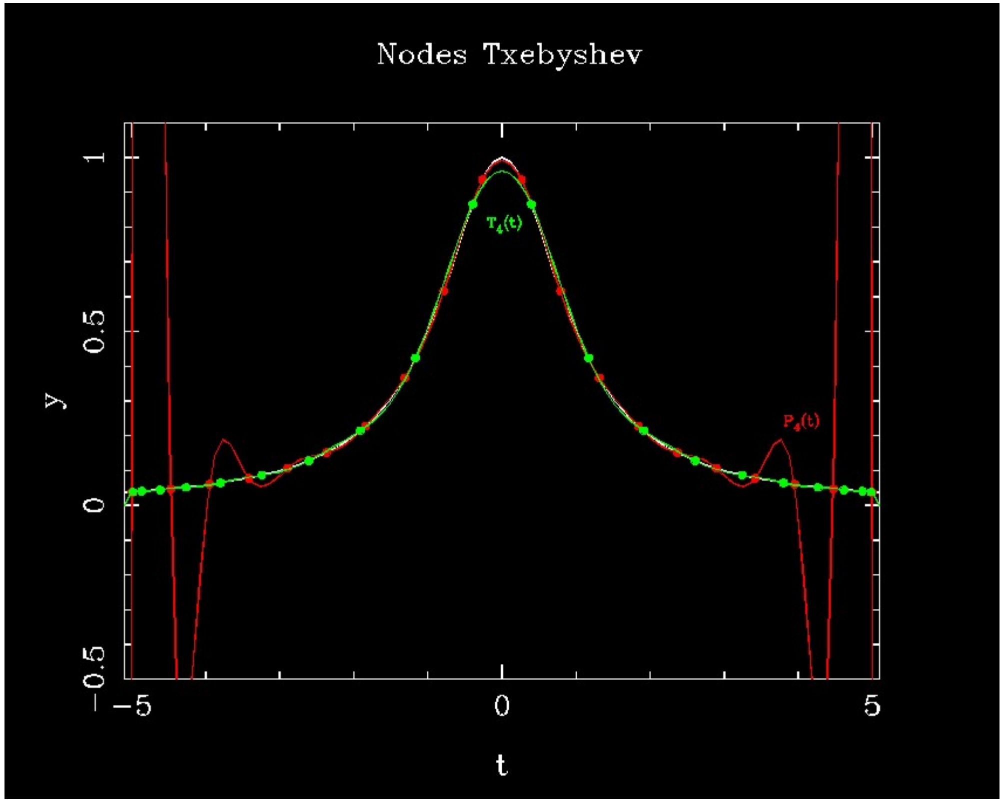
Chevyshev's Nodes

- Ten node interpolation



Nodes de Txebyshev

- 20 node interpolation



Hermite Interpolation

- Lagrange interpolation can be generalized to include values of the derivative of the function. The interpolating polynomial must satisfy not only the values of the function at the nodes, but also the values of the derivatives until certain order.
- Given the nodes $x_i, i = 0,1,\dots,n$ and two sets of real values $y_i, z_i\ , i = 0,1,\dots,n$ we need a polynomial $p_{2n+1} \in P_{2n+1}(\mathbb{R})$ such that

$$p_{2n+1}(x_i) = y_i, \quad p'_{2n+1}(x_i) = z_i, \quad i = 0,1,\dots,n$$

Hermite Interpolation

- Let $n \geq 0$, and suppose that $x_i, i = 0,1, \dots, n$ are distinct real numbers. Given two sets of real values y_i, z_i , $i = 0,1, \dots, n$ there is a unique polynomial $p_{2n+1} \in P_{2n+1}(\mathbb{R})$ such that:

$$p_{2n+1}(x_i) = y_i, \quad p'_{2n+1}(x_i) = z_i, \quad i = 0,1,\dots,n$$

- We will need to sets of auxiliary polynomials H_k and K_k, with $k = 0,1, \dots, n$ defined by

$$H_k(x) = \left[L_k(x)\right]^2 (1 - 2L'_k(x)(x - x_k))$$
$$K_k(x) = \left[L_k(x)\right]^2 (x - x_k)$$

Hermite Interpolation

- Here $L_k(x)$ are the Lagrange polynomials

$$L_k(x) = \prod_{\substack{i=0 \\ i\neq k}}^{n} \frac{x - x_i}{x_k - x_i}.$$

- H_k and K_k are polynomials of degree $2n+1,$ verifying that:

$$H_k(x_i) = \begin{cases} 1, & i = k \\ 0, & i \neq k \end{cases} \qquad H'_k(x_i) = 0, \qquad i,k = 0,1,\dots,n$$

$$K'_k(x_i) = 0, \qquad K_k(x_i) = \begin{cases} 1, & i = k \\ 0, & i \neq k \end{cases} \qquad i,k = 0,1,\dots,n$$

Hermite Interpolation

- The ***Hermite polynomial*** of degree $2n + 1$:

$$p_{2n+1}(x) = \sum_{k=0}^{n} \left[H_k(x) y_k + K_k(x) z_k \right]$$

- Satisfies these conditions. It certainly verifies that the polynomial $p_{2n+1} \in P_{2n+1}(\mathbb{R})$. To see that this is the unique polynomial, suppose that there exists a polynomial $q_{2n+1} \in P_{2n+1}(\mathbb{R})$, such that

$$q_{2n+1}(x_i) = y_i, \quad q'_{2n+1}(x_i) = z_i, \quad i = 0,1,\ldots,n$$

Hermite Interpolation

- The polynomial $p_{2n+1} - q_{2n+1}$ will have $n + 1$ distinct zeros; therefore, Rolle's theorem implies that $p'_{2n+1} - q'_{2n+1}$ will vanish at additional n points which interlace the x_i. Then this polynomial of degree $2n$ will have $2n + 1$ zeros and must be identically zero. This implies that $p_{2n+1} - q_{2n+1}$ is a constant function.
- However $(p_{2n+1} - q_{2n+1})(x_i) = 0$ contradicting the hypothesis that p_{2n+1} and q_{2n+1} are distinct.

Hermite Interpolation

- When $n = 0$, we define $H_0(x) \equiv 1$ and $K_0(x) \equiv x - x_0$. This corresponds to taking $L_0(x) \equiv 1$.
- Then p_1 results in:

$$p_1(x) = H_0(x)y_0 + K_0(x)z_0 = y_0 + (x - x_0)z_0$$

- Which is the unique polynomial such that $p_1(x_0) = y_0$ and $p'_1(x_0) = z_0$.

Hermite Interpolation

- Suppose that $n \geq 0$ and let f be a real-valued function, defined, continuous and $2n+2$ times differentiable on the interval $[a, b]$, such that $f^{(2n+2)}$ is continuous on $[a, b]$. Further, let p_{2n+1} denote the Hermite interpolation polynomial of f. Then, for each $x \in [a, b]$ there exists $\xi = \xi(x)$ in (a, b) such that

$$f(x) - p_{2n+1}(x) = \frac{f^{(2n+2)}(\xi)}{(2n+2)!}\left[\pi_{n+1}(x)\right]^2$$

Hermite Interpolation

- Here π_{n+1} is defined as usual:

$$\pi_{n+1}(x) = \prod_{i=0}^{n}(x - x_i)$$

- Moreover, we have

$$\left|f(x) - p_{2n+1}(x)\right| \leq \frac{M_{2n+2}}{(2n+2)!}\left[\pi_{n+1}(x)\right]^2.$$

- Where:

$$M_{2n+2} = \max_{\zeta \in [a,b]}\left|f^{(2n+2)}(\zeta)\right|$$

Hermite Interpolation

- This relation is true for $x = x_i$, $i = 0,1,\dots,n$. For $x \in [a,b]$, $x \neq x_i$, we define the function

$$\psi(t) = f(t) - p_{2n+1}(t) - \frac{f(x) - p_{2n+1}(x)}{\left[\pi_{n+1}(x)\right]^2}\left[\pi_{n+1}(t)\right]^2$$

- Then $\psi(x_i) = 0$, $i = 0,1,\dots,n$ and $\psi(x) = 0$. $\psi'(t)$ must vanish at $n+1$ points which lie strictly between each pair of consecutive points from the set $\{x_0, x_1, \dots, x_n, x\}$.

Hermite Interpolation

- Also, $\psi'(x_i) = 0, i = 0,1, \dots, n$; hence ψ' vanishes at a total of $2n + 2$ distinct points in $[a, b]$. Applying Rolle's theorem repeatedly, we find eventually that $\psi^{(2n+2)}$ vanishes at some point $\xi \in (a, b)$,
- For this point we have $\psi^{(2n+2)}(\xi) = 0$ and $p_{2n+1}^{(2n+2)}(\xi) = 0$ and we obtain the desired result.

Hermite Interpolation

- To obtain a more computable expression for the Hermite interpolation polynomial we need to introduce a generalization of divided differences when using repeated nodes. As:

$$\lim_{x \to x_0} f[x_0, x] = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0)$$

- We define

$$f[x_i, x_i] = f'(x_i) \quad i = 0, 1, \ldots, n$$

Hermite Interpolation

- Using this definition, we build now the following table of divided differences:

$$
\begin{array}{lllll}
x_0 & f(x_0) & f'(x_0) & f[x_0,x_0,x_1] & f[x_0,x_0,x_1,x_1] \\
x_0 & f(x_0) & f[x_0,x_1] & f[x_0,x_1,x_1] & \\
x_1 & f(x_1) & f'(x_1) & & \\
x_1 & f(x_1) & & & \\
\vdots & \vdots & & &
\end{array}
$$

Hermite Interpolation

- From this table we build the polynomial:

$$
\begin{aligned}
p_{2n+1}(x) = f(x_0) + (x-x_0) f[x_0,x_0] + (x-x_0)^2 f[x_0,x_0,x_1] \\
+ (x-x_0)^2 (x-x_1) f[x_0,x_0,x_1,x_1] + \cdots \\
+ (x-x_0)^2 \cdots (x-x_{n-1})^2 (x-x_n) f[x_0,x_0,\ldots,x_n,x_n]
\end{aligned}
$$

- This polynomial is of degree $2n+1$. We will prove that this is the Hermite interpolation polynomial of the nodes $\{x_0, x_1, \ldots, x_n\}$.

Hermite Interpolation

- The error in the Hermite polynomial can be written as:

$$f(x) - p_{2n+1}(x) = (x - x_0)^2 \cdots (x - x_n)^2 f[x_0, x_1, \ldots, x_n, x]$$

- If we construct the interpolating polynomial at the node points $x_0, x_1, \ldots, x_n, x$ we have:

$$\begin{aligned} p_{n+1}(x) &= f(x_0) + (x - x_0) f[x_0, x_1] + \cdots + (x - x_0) \cdots (x - x_{n-1}) f[x_0, \ldots, x_n] + \\ &\quad + (x - x_0) \cdots (x - x_{n-1})(x - x_n) f[x_0, \ldots, x_n, x] \\ &= p_n(x) + (x - x_0) \cdots (x - x_{n-1})(x - x_n) f[x_0, \ldots, x_n, x] \end{aligned}$$

Hermite Interpolation

- Thus, we can write:

$$f(x) - p_n(x) = (x - x_0)\cdots(x - x_n) f[x_0, \ldots, x_n, x]$$

- Which gives an alternative expression for the interpolation error, with

$$f[x_0, \ldots, x_n, x] = \frac{f^{(n+1)}(\xi_x)}{(n+1)!}, \quad \xi_x \in (x_0, x_n)$$

- And hence, we can write, for the Hermite polynomial

$$f(x) - p_{2n+1}(x) = (x - x_0)^2 \cdots (x - x_n)^2 f[x_0, x_1, \ldots, x_n, x]$$

Hermite Interpolation

- Then, we note that:

$$f(x_i) - p_{2n+1}(x_i) = 0, \quad i = 0,1,\ldots,n$$

- We have also that,

$$f'(x) - p'_{2n+1}(x) = (x-x_0)^2 \cdots (x-x_n)^2 \frac{d}{dx} f[x_0, x_0, \ldots, x_n, x_n, x]$$

$$+2 f[x_0, x_0, \ldots, x_n, x_n, x] \sum_{i=0}^{n} (x-x_i) \prod_{\substack{j=0 \\ j\neq i}}^{n} (x-x_j)^2$$

- Hence

$$f'(x_i) - p'_{2n+1}(x_i) = 0, \quad i = 0,1,\ldots,n$$

- Thus, thi interpolating polynomial satisfies the Hermite conditions and must be the unique Hermite interpolating polynomial.