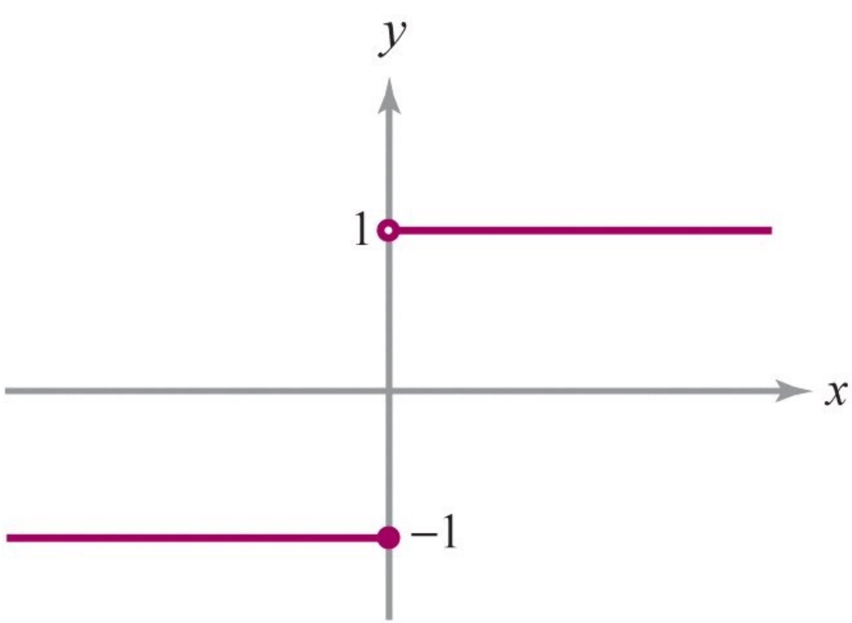
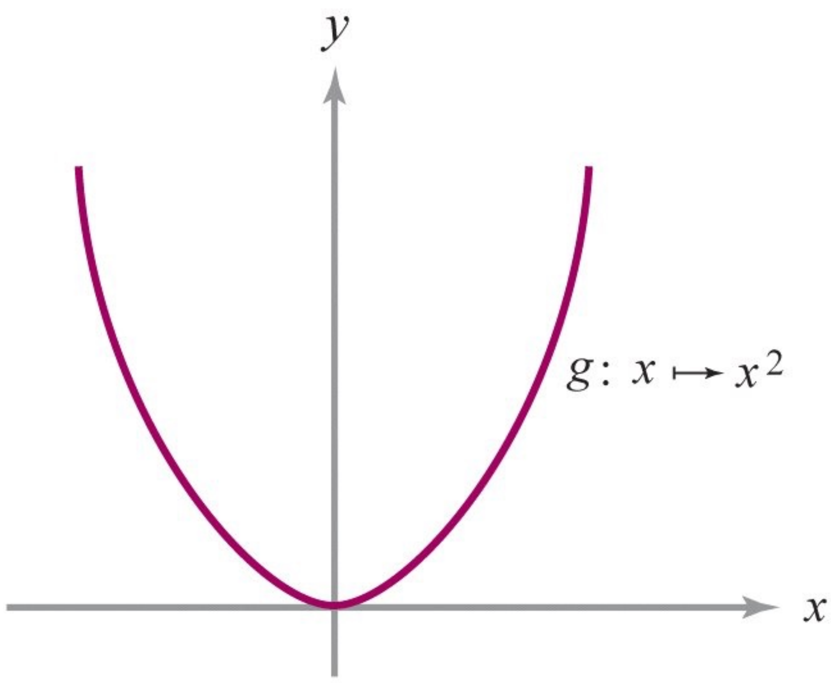
Chapter 2:

2.2 Limits and Continuity

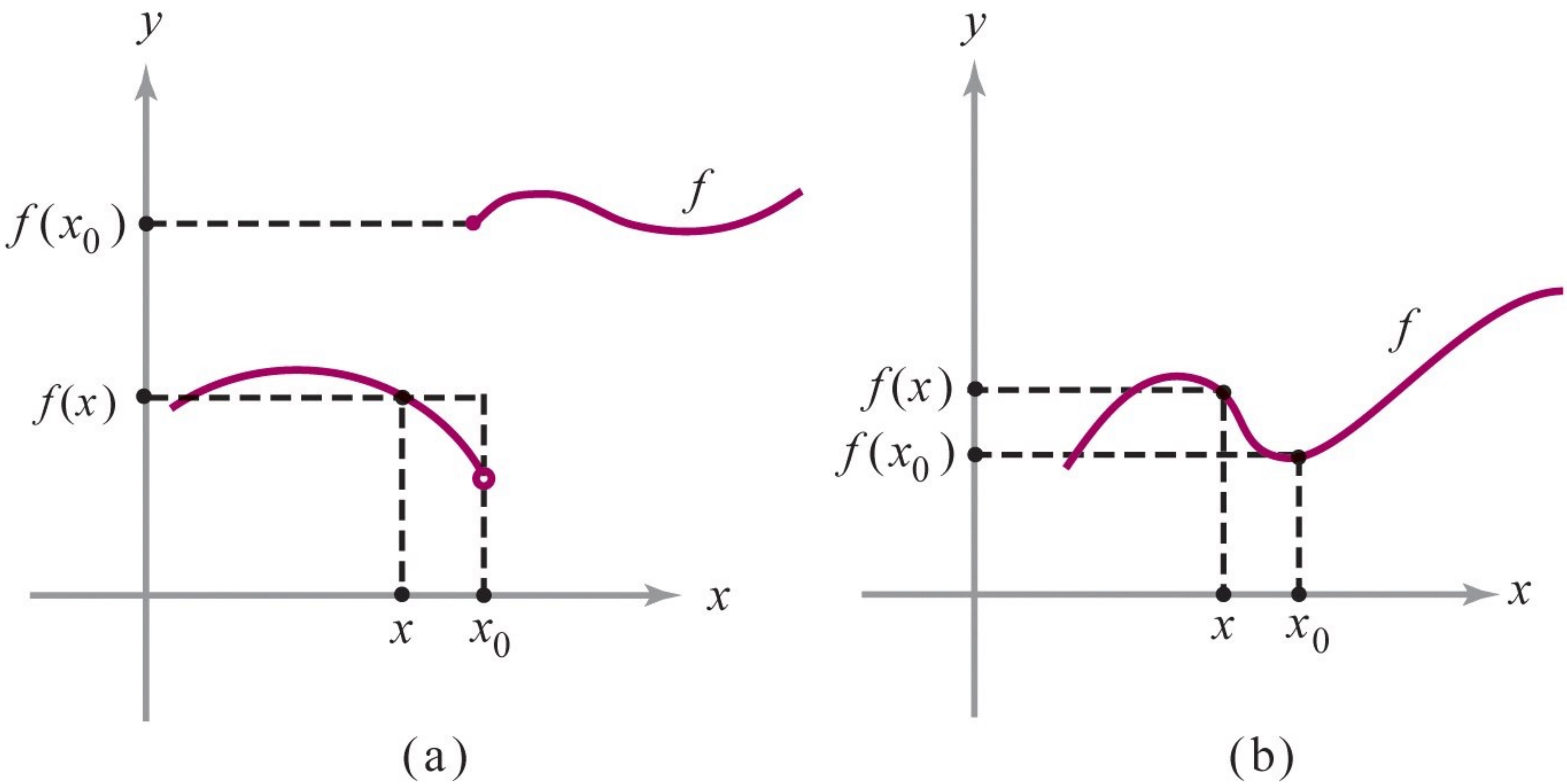
Continuity



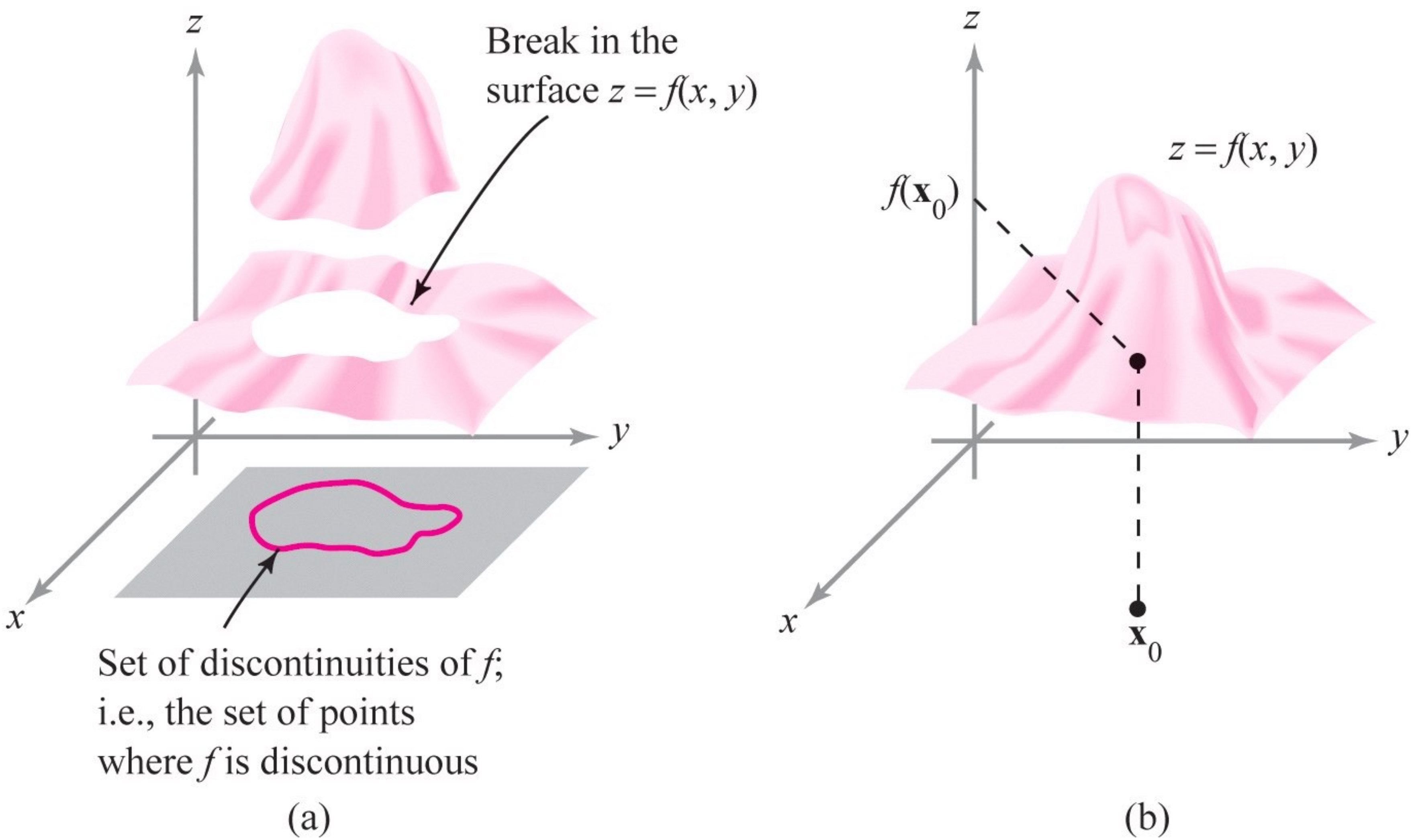
(a)



(b)



(a) Discontinuous function for which limit $x \rightarrow x_0$ f (x) does not exist. (b) Continuous function for which this limit exists and equals $f(x_0)$.



(a) A discontinuous function of two variables.
(b) (b) A continuous function

DEFINITION: Continuity Let $f\colon A \subset \mathbb{R}^n \to \mathbb{R}^m$ be a given function with domain A. Let $\mathbf{x}_0 \in A$. We say f is ***continuous*** at $\mathbf{x}_0$ if and only if

$$\operatorname*{limit}_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = f(\mathbf{x}_0).$$

If we just say that f is ***continuous***, we shall mean that f is continuous at each point $\mathbf{x}_0$ of A. If f is not continuous at $\mathbf{x}_0$, we say f is ***discontinuous*** at $\mathbf{x}_0$. If f is discontinuous at some point in its domain, we say f is ***discontinuous***.

THEOREM 4: Properties of Continuous Functions Suppose that $f\colon A \subset \mathbb{R}^n \to \mathbb{R}^m$, $g\colon A \subset \mathbb{R}^n \to \mathbb{R}^m$, and let c be a real number.

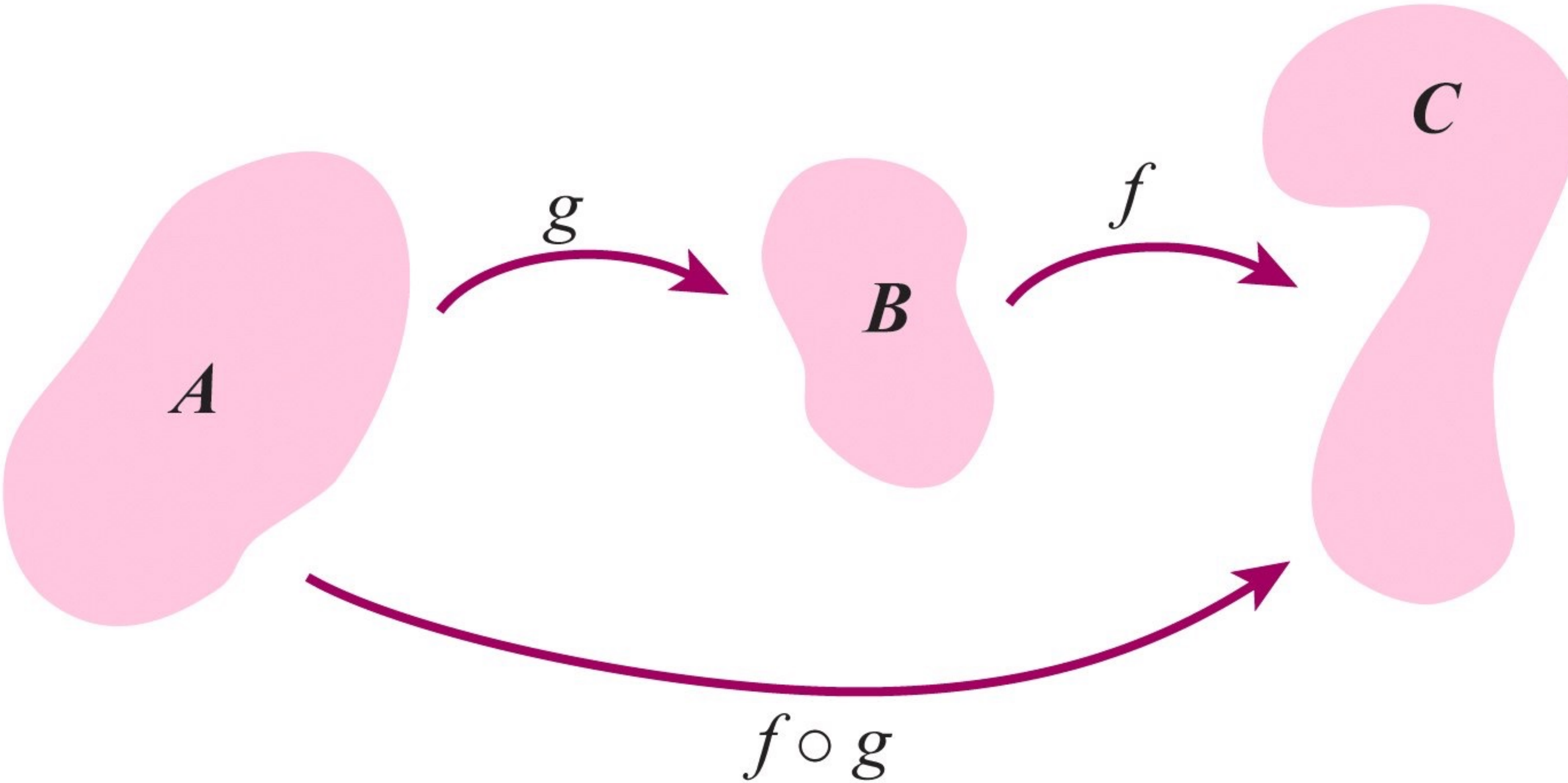
(i) If f is continuous at $\mathbf{x}_0$, so is cf, where $(cf)(\mathbf{x}) = c[f(\mathbf{x})]$.

(ii) If f and g are continuous at $\mathbf{x}_0$, so is $f + g$, where the sum of f and g is defined by $(f + g)(\mathbf{x}) = f(\mathbf{x}) + g(\mathbf{x})$.

(iii) If f and g are continuous at $\mathbf{x}_0$ and $m = 1$, then the product function fg defined by $(fg)(\mathbf{x}) = f(\mathbf{x})g(\mathbf{x})$ is continuous at $\mathbf{x}_0$.

(iv) If $f\colon A \subset \mathbb{R}^n \to \mathbb{R}$ is continuous at $\mathbf{x}_0$ and nowhere zero on A, then the quotient $1/f$ is continuous at $\mathbf{x}_0$, where $(1/f)(\mathbf{x}) = 1/f(\mathbf{x})$.

(v) If $f\colon A \subset \mathbb{R}^n \to \mathbb{R}^m$ and $f(\mathbf{x}) = (f_1(\mathbf{x}), \ldots, f_m(\mathbf{x}))$, then f is continuous at $\mathbf{x}_0$ if and only if each of the real-valued functions $f_1, \ldots, f_m$ is continuous at $\mathbf{x}_0$.


g
f
A
B
C
f ∘ g

THEOREM 5: Continuity of Compositions Let $g\colon A \subset \mathbb{R}^n \to \mathbb{R}^m$ and let $f\colon B \subset \mathbb{R}^m \to \mathbb{R}^p$. Suppose $g(A) \subset B$, so that $f \circ g$ is defined on A. If g is continuous at $\mathbf{x}_0 \in A$ and f is continuous at $\mathbf{y}_0 = g(\mathbf{x}_0)$, then $f \circ g$ is continuous at $\mathbf{x}_0$.

Proof. We use the ε-δ criterion for continuity. Thus, given $\varepsilon > 0$, we must find $\delta > 0$ such that for $\mathbf{x} \in A$.

$$\|\mathbf{x} - \mathbf{x}_0\| < \delta \text{ implies } \|(g \circ f)(\mathbf{x}) - (g \circ f)(\mathbf{x}_0)\| < \varepsilon.$$

Since g is continuous at $f(\mathbf{x}_0) = \mathbf{y}_0 \in B$, there is a $\gamma > 0$ such that for $\mathbf{y} \in B$,

$$\|\mathbf{y} - \mathbf{y}_0\| < \gamma \text{ implies } \|g(\mathbf{y}) - g(f(\mathbf{x}_0))\| < \varepsilon.$$

Since f is continuous $\mathbf{x}_0 \in A$, there is, for this γ, a $\delta > 0$ such that for $\mathbf{x} \in A$,

$$\|\mathbf{x} - \mathbf{x}_0\| < \delta \text{ implies } \|f(\mathbf{x}) - f(\mathbf{x}_0)\| < \gamma,$$

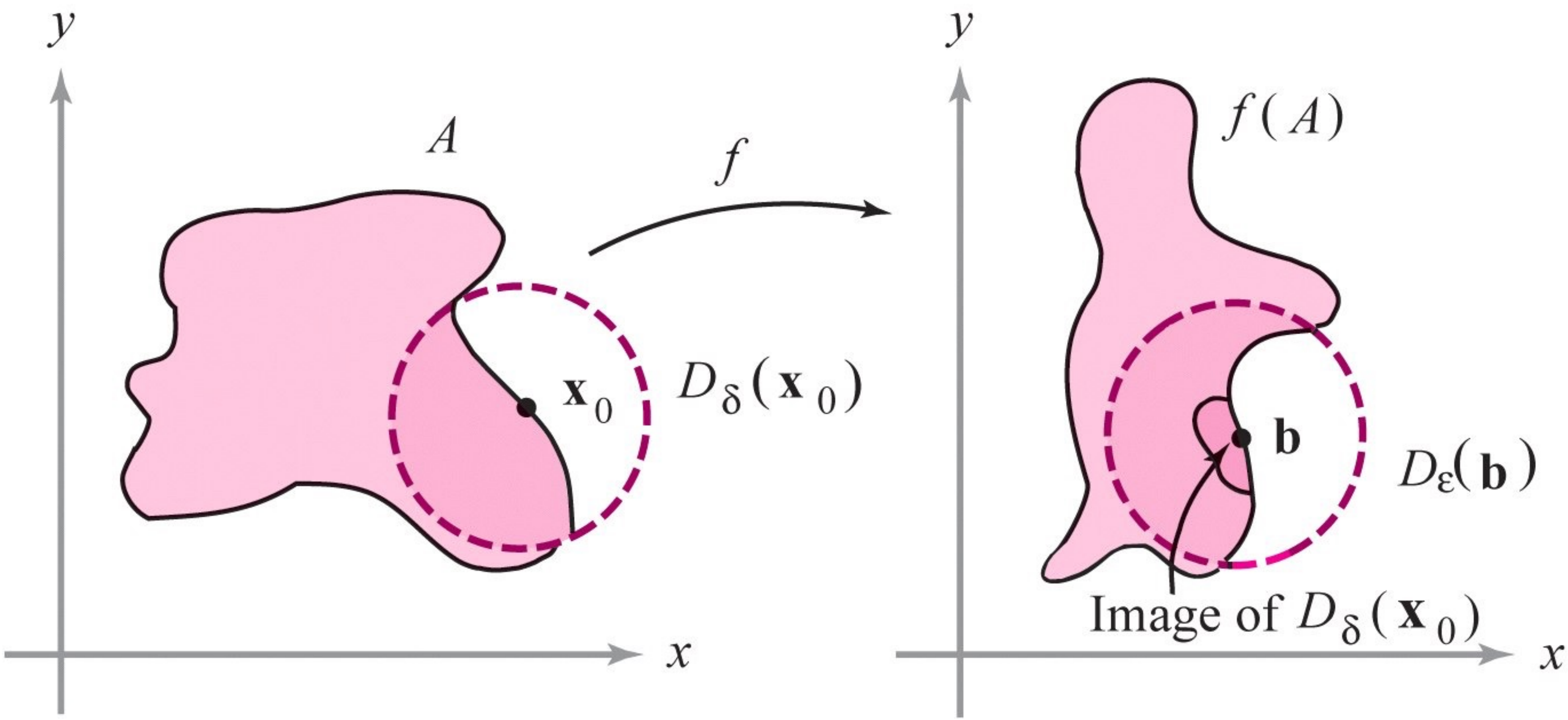
which in turn implies

$$\|g(f(\mathbf{x})) - g(f(\mathbf{x}_0))\| < \varepsilon,$$

which is the desired conclusion. ■

In terms of ε's and δ's

THEOREM 6 Let $f\colon A \subset \mathbb{R}^n \to \mathbb{R}^m$ and let $\mathbf{x}_0$ be in A or be a boundary point of A. Then $\displaystyle\operatorname*{limit}_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = \mathbf{b}$ if and only if for every number $\varepsilon > 0$ there is a $\delta > 0$ such that for any $\mathbf{x} \in A$ satisfying $0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta$, we have $\|f(\mathbf{x}) - \mathbf{b}\| < \varepsilon$ (see Figure 2.2.16).



y
A
f
x
$\mathbf{x}_0$
$D_\delta(\mathbf{x}_0)$
y
$f(A)$
$\mathbf{b}$
$D_\varepsilon(\mathbf{b})$
Image of $D_\delta(\mathbf{x}_0)$
x

Show that $\text{limit}_{(x,y)\to(0,0)}\, x = 0$ using the ε-δ method.

Show that $\text{limit}_{(x,y)\to(0,0)}\, x = 0$ using the ε-δ method.

Note that if $\delta > 0$, $\|(x, y) - (0, 0)\| = \sqrt{x^2 + y^2} < \delta$ implies $|x - 0| = |x| = \sqrt{x^2} \leq \sqrt{x^2 + y^2} < \delta$. Thus, if $\|(x, y) - (0, 0)\| < \delta$, then $|x - 0|$ is also less than δ. Given $\varepsilon > 0$, we are required to find a $\delta > 0$ (generally depending on ε) with the property that $0 < \|(x, y) - (0, 0)\| < \delta$ implies $|x - 0| < \varepsilon$. What are we to pick as our δ? From the preceding calculation, we see that if we choose $\delta = \varepsilon$, then $\|(x, y) - (0, 0)\| < \delta$ implies $|x - 0| < \varepsilon$. This shows that $\text{limit}_{(x,y)\to(0,0)}\, x = 0$. Given $\varepsilon > 0$, we could have also chosen $\delta = \varepsilon/2$ or $\varepsilon/3$, but it suffices to find just one δ satisfying the requirements of the definition of a limit. ▲

Show that

$$\operatorname*{limit}_{(x,y)\to(0,0)} \frac{x^2}{\sqrt{x^2+y^2}} = 0.$$

Show that

$$\lim_{(x,y)\to(0,0)} \frac{x^2}{\sqrt{x^2+y^2}} = 0.$$

We must show that $x^2/\sqrt{x^2+y^2}$ is small when (x, y) is close to the origin. To do this, we use the following inequality:

$$\begin{aligned} 0 \le \frac{x^2}{\sqrt{x^2+y^2}} &\le \frac{x^2+y^2}{\sqrt{x^2+y^2}} \qquad \text{(because } y^2 \ge 0\text{)} \\ &= \sqrt{x^2+y^2}. \end{aligned}$$

Given $\varepsilon > 0$, choose $\delta = \varepsilon$. Then $\|(x, y) - (0, 0)\| = \|(x, y)\| = \sqrt{x^2+y^2}$, and so $\|(x, y) - (0, 0)\| < \delta$ implies that

$$\left|\frac{x^2}{\sqrt{x^2+y^2}} - 0\right| = \frac{x^2}{\sqrt{x^2+y^2}} \le \sqrt{x^2+y^2} = \|(x, y) - (0, 0)\| < \delta = \varepsilon.$$

Thus, the conditions of Theorem 6 have been fulfilled and the limit is verified. ▲

Show that $\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$ *do not exists*

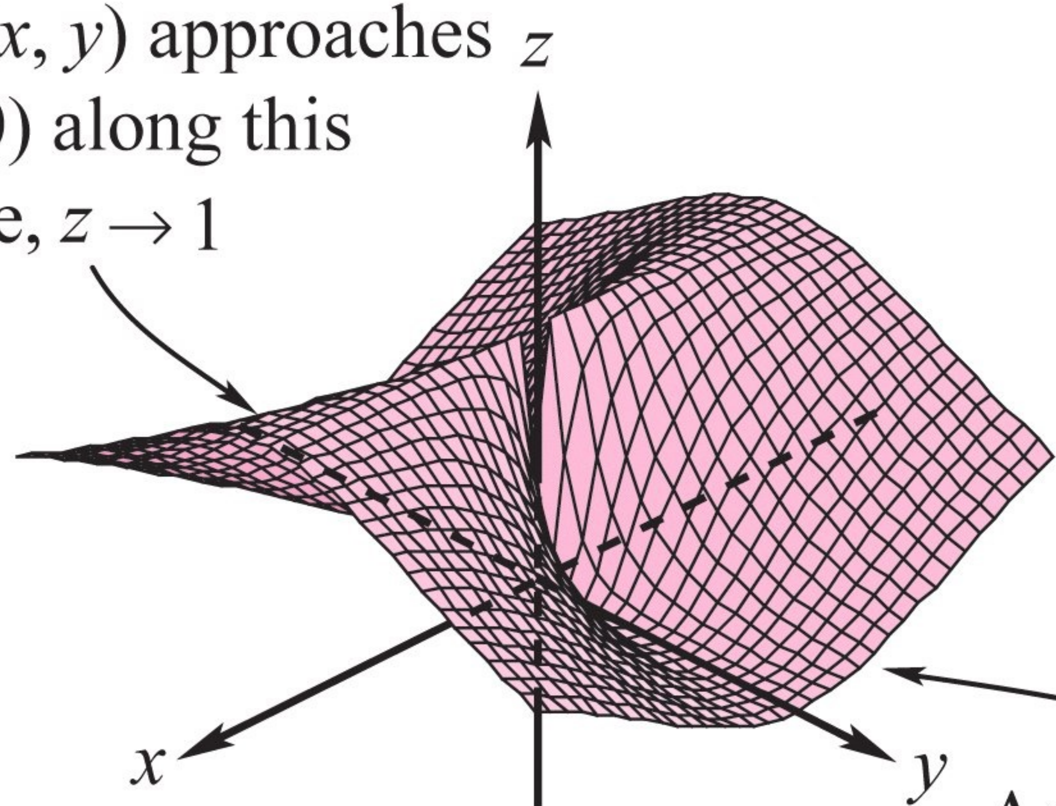
Show that $\lim_{(x,y)\to(0,0)} \dfrac{x^2}{x^2+y^2}$ *do not exists*

(a) If the limit exists, $x^2/(x^2+y^2)$ should approach a definite value, say a, as (x, y) gets near $(0, 0)$. In particular, if (x, y) approaches zero along any given path, then $x^2/(x^2+y^2)$ should approach the limiting value a. If (x, y) approaches $(0, 0)$ along the line $y = 0$, the limiting value is clearly 1 (just set $y = 0$ in the preceding expression to get $x^2/x^2 = 1$). If (x, y) approaches $(0, 0)$ along the line $x = 0$, the limiting value is

$$\lim_{y\to 0} \frac{0^2}{0^2+y^2} = 0 \neq 1.$$

Hence, $\text{limit}_{(x,y)\to(0,0)}\, x^2/(x^2+y^2)$ does not exist.

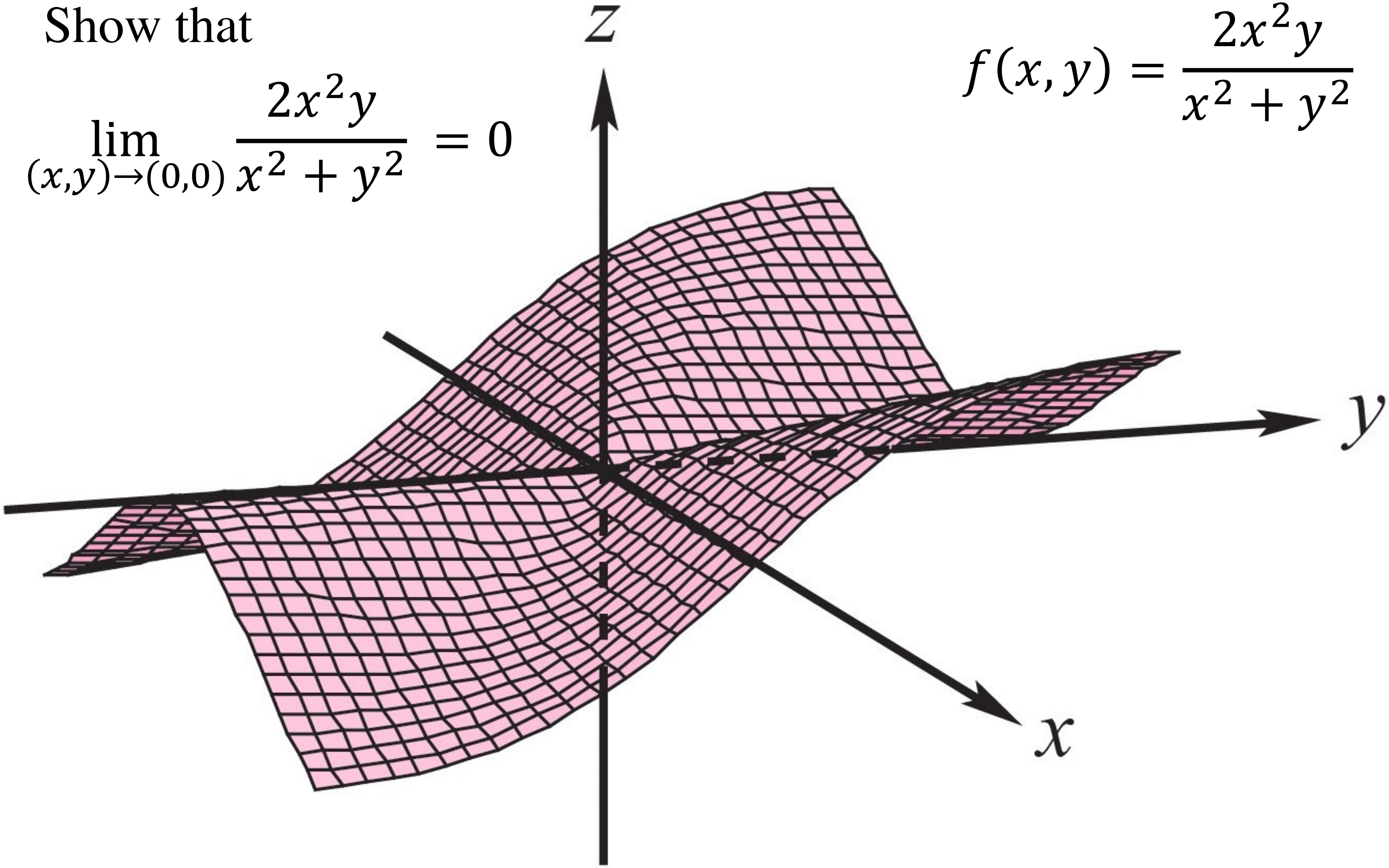
$$f(x, y) = \frac{x^2}{x^2 + y^2}$$

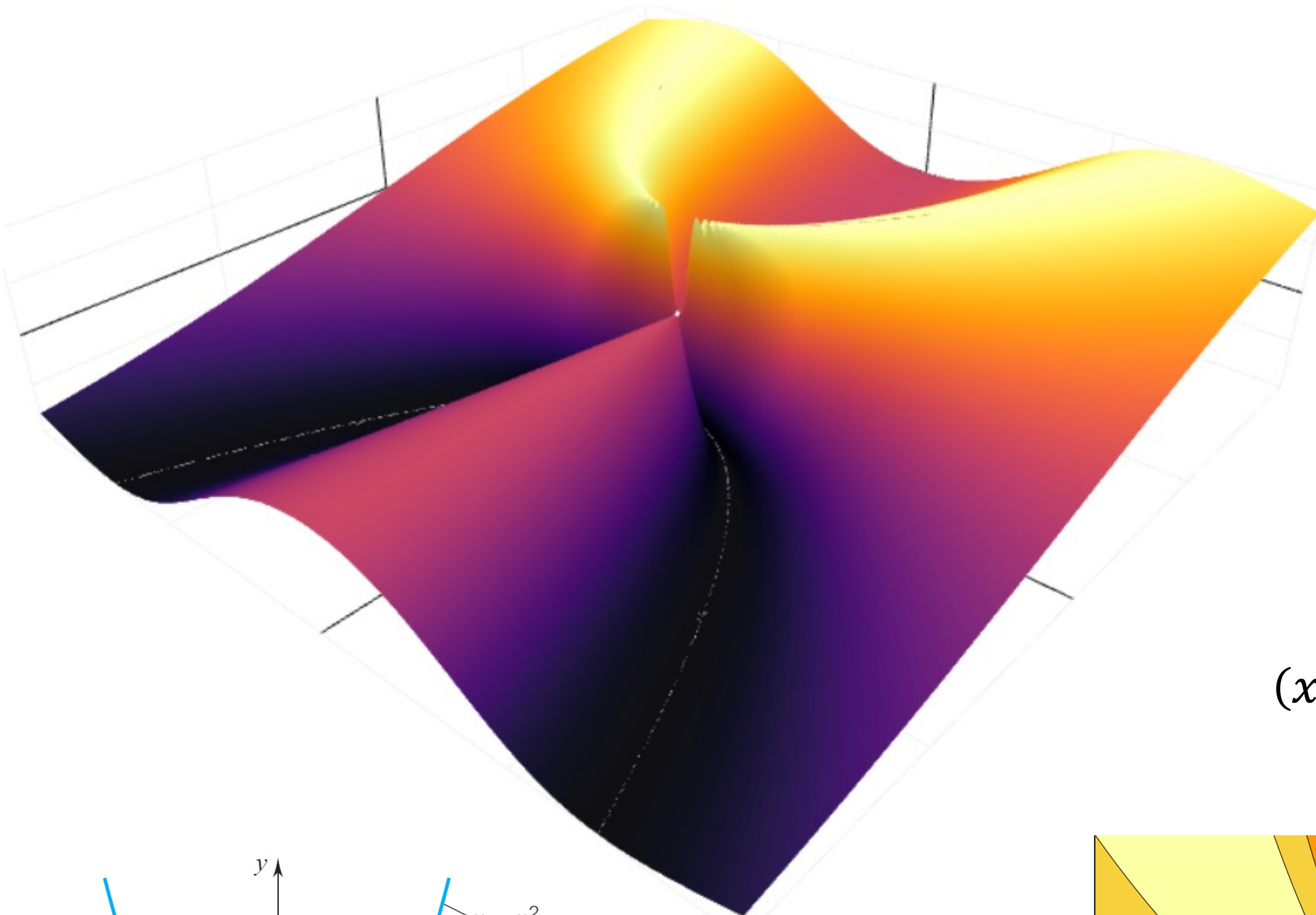


Show that

$$\lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^2+y^2} = 0$$

$$f(x,y) = \frac{2x^2y}{x^2+y^2}$$

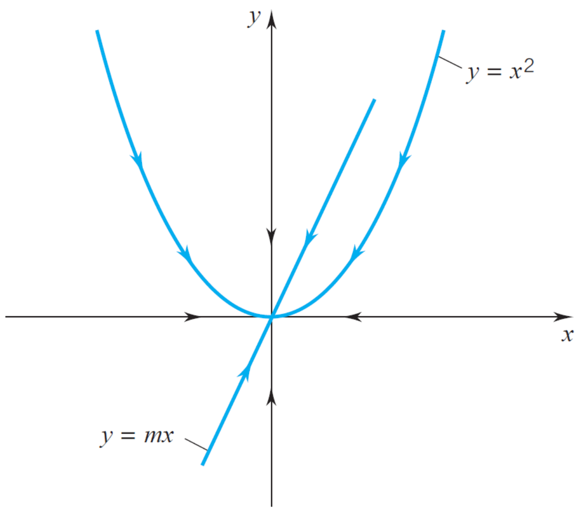


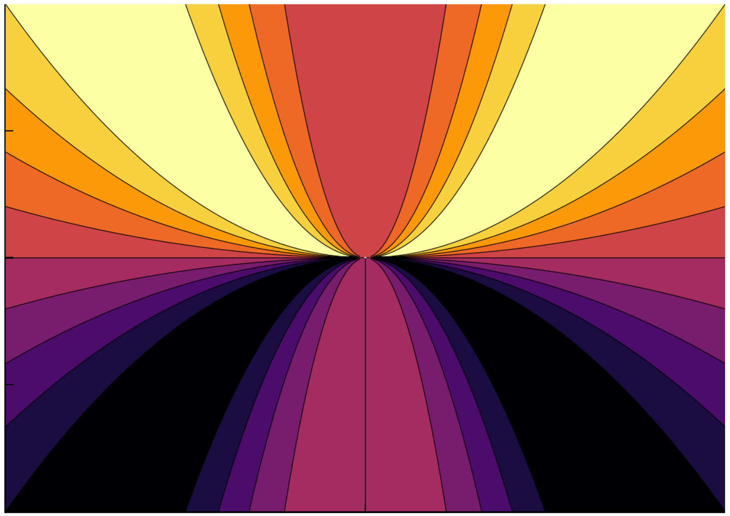


$$f(x,y) = \frac{x^2 y}{x^4 + y^2}$$

Check

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^4 + y^2}$$





THEOREM 7 Let $f\colon A \subset \mathbb{R}^n \to \mathbb{R}^m$ be given. Then f is continuous at $\mathbf{x}_0 \in A$ if and only if for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that

$$\mathbf{x} \in A \qquad \text{and} \qquad \|\mathbf{x} - \mathbf{x}_0\| < \delta \qquad \text{implies} \qquad \|f(\mathbf{x}) - f(\mathbf{x}_0)\| < \varepsilon.$$

2.2 Limits and Continuity

Key Points in this Section.

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6. **Continuity**. Let $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$ and $\mathbf{x}_0 \in A$. We say f is ***continuous at*** $\mathbf{x}_0$ provided

$$\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = f(\mathbf{x}_0).$$

   If f is continuous at every point of A, we just say f is ***continuous***.

7. The sum of continuous functions is continuous. The same is true of products and quotients of real-valued functions (if the denominator is non-zero).

8. The composition of continuous functions is continuous. ***Compositions*** $f \circ g$ are defined by $(f \circ g)(\mathbf{x}) = f(g(\mathbf{x}))$.

9. The usual functions of one-variable calculus, such as polynomials, trigonometric, and exponential functions are continuous and these can be used to build up continuous functions of several variables. For instance, $f(x, y) = e^{xy}/(1 - x^2 - y^2)$ is continuous on $\mathbb{R}^2$ minus the unit circle.

10. If $f(x, y)$ has different limits as $(0, 0)$ is approached along two different rays (such as the x- and y-axes), then f is not continuous at $(0, 0)$.