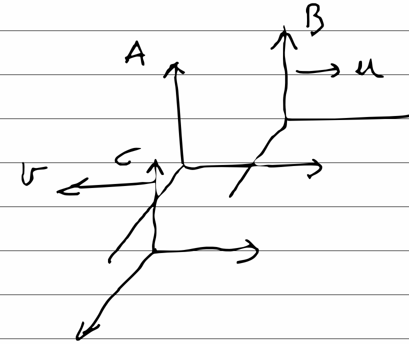
2.1



Velocity w of B with respect to C?

From A: B $V_B = (\gamma_u, \gamma_u u, 0, 0)$

C $V_C = (\gamma_v, -\gamma_v v, 0, 0)$

From C: B $V'_B = (\gamma_w, \gamma_w w, 0, 0)$

C $V'_C = (1, 0, 0, 0)$

$$-\gamma_u \gamma_v - \gamma_u \gamma_v uv = -\gamma_w \Rightarrow \gamma_w = \gamma_u \gamma_v (1 + uv) \Rightarrow$$

$$\frac{(1+uv)}{\gamma_w} = \frac{1}{\gamma_u \gamma_v} \Rightarrow (1+uv)^2 (1 - w^2) = (1-u^2)(1-v^2)$$

$$\Rightarrow (1+uv)^2 - (1-u^2)(1-v^2) = (1+uv)^2 w^2 \Rightarrow$$

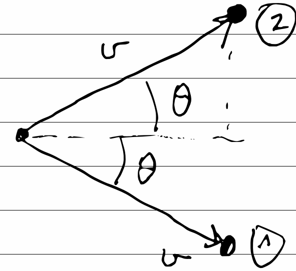
$$(1+uv)^2 w^2 = \cancel{1} + \cancel{(uv)^2} + 2uv - \cancel{1} + v^2 + u^2 - \cancel{(uv)^2} \Rightarrow$$

$$(1+uv)^2 w^2 = (u+v)^2 \Rightarrow w^2 = \left(\frac{u+v}{1+uv}\right)^2 \Rightarrow$$

$$\Rightarrow \boxed{w = \pm \frac{u+v}{1+uv}}$$

The sign depends on the orientation we choose for the x axis in C!

2.2



In the lab reference frame

$$V_1 = \gamma_v(1, v\cos\theta, v\sin\theta, 0)$$
$$V_2 = \gamma_v(1, v\cos\theta, -v\sin\theta, 0)$$

In 1's reference frame

$$V_1' = (1, 0, 0, 0)$$

$$V_2' = \gamma_u(1, u\cos 2\theta, u\sin 2\theta, 0)$$

$\cos^2\theta - \sin^2\theta = \cos 2\theta$

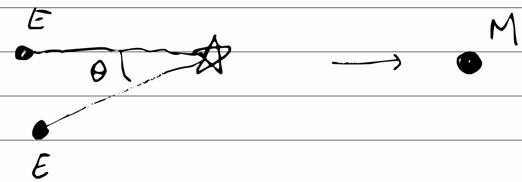
$$-\gamma_v^2\left(1 - v^2\cos^2\theta - v^2\sin^2\theta\right) = -\gamma_u \Rightarrow$$

$$\Rightarrow \gamma_u = \gamma_v^2\left(1 - v^2\cos 2\theta\right) \Rightarrow$$

$$\Rightarrow 1 - u^2 = \frac{(1-v^2)^2}{(1-v^2\cos 2\theta)^2} \Rightarrow u^2 = 1 - \frac{(1-v^2)^2}{(1-v^2\cos 2\theta)^2} \Rightarrow$$

$$\Rightarrow \boxed{u = \pm\sqrt{1 - \frac{(1-v^2)^2}{(1-v^2\cos 2\theta)^2}}}$$

2.3



Before: $P_1 = (E, p, 0, 0)$

$P_2 = (E, p\cos\theta, p\sin\theta, 0)$

What is p? $E^2 - p^2 = m^2 = 0 \Rightarrow |p| = |E|$

$\hookrightarrow m = 0$

We use $p^\mu p_\mu = m^2 c^4$

$$\Rightarrow \left.\begin{aligned} P_1 &= (E, E, 0, 0) \\ P_2 &= (E, E\cos\theta, E\sin\theta, 0) \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow P_{before} = (2E, E(1+\cos\theta), E\sin\theta, 0)$$

After: $P_{after} = P_{before} = (2E, E(1+\cos\theta), E\sin\theta, 0)$

$E_M^2 = (2E)^2 = 4E^2$

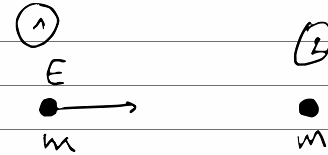
$\vec{P}_M^2 = E^2(1+\cos\theta)^2 + E^2\sin^2\theta = E^2 + E^2\cos^2\theta + E^2 2\cos\theta$
$+ E^2\sin^2\theta =$

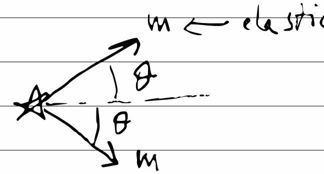
$= 2E^2 + E^2 2\cos\theta = 2E^2(1+\cos\theta)$

We use $P^\mu P_\mu = M^2$ $\quad M^2 = \underline{4E^2 - 2E^2 - 2E^2\cos\theta} = 2E^2(1-\cos\theta) \Rightarrow$

$$\Rightarrow \boxed{M = E\sqrt{2(1-\cos\theta)}}$$

2.4

Before 

After 

Before $P_1 = (E, p, 0, 0) = (E, \sqrt{E^2 - m^2}, 0, 0)$

$\to E^2 - p^2 = m^2 \Rightarrow$

$\Rightarrow p = \sqrt{E^2 - m^2}$

$P_2 = (E_2, 0, 0, 0) = (m, 0, 0, 0)$

$\to E_2^2 = m^2 \Rightarrow E = m$

$P_{before} = (E+m, \sqrt{E^2-m^2}, 0, 0)$

After

$P_1' = (E_1', p_1' \cos\theta, p_1' \sin\theta, 0)$

$P_2' = (E_2', p_2' \cos\theta, -p_2' \sin\theta, 0)$

By looking at the y component $\Rightarrow p_1' = p_2' \equiv p' \Rightarrow$

$\Rightarrow E_1' = E_2' \equiv E'$

$\Rightarrow P_{after} = (2E', 2p' \cos\theta, 0, 0)$

t component $E' = \dfrac{E+m}{2}$

x component $p' = \dfrac{\sqrt{E^2 - m^2}}{2\cos\theta}$

$\Rightarrow \left(\dfrac{E+m}{2}\right)^2 - \dfrac{E^2 - m^2}{(2\cos\theta)^2} = m^2$

$\Rightarrow (E+m)^2 - \dfrac{E^2 - m^2}{\cos^2\theta} = 4m^2 \Rightarrow E^2 + m^2 + 2Em - 4m^2 = \dfrac{E^2 - m^2}{\cos^2\theta} \Rightarrow$

$\Rightarrow \boxed{\cos^2\theta = \dfrac{(E+m)(E-m)}{E^2 + 2Em - 3m^2} = \dfrac{E+m}{E+3m}}$

Relativistic limit $E \gg m$

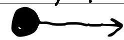
$$\cos^2\theta \approx 1 \Rightarrow \theta \approx 0$$

Non-relativistic limit $E \approx m$

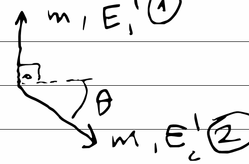
$$\cos^2\theta \approx \frac{1}{2} \Rightarrow \cos\theta \approx \frac{1}{\sqrt{2}} \Rightarrow \theta \approx 45^\circ$$

2.5

Before



After



Before $\quad P_M = (E, p, 0, 0) = (E, \sqrt{E^2 - M^2}, 0, 0)$

$\hookrightarrow E^2 - p^2 = M^2$

After $\quad P_1 = (E_1', 0, p_1', 0) = (E_1', 0, \sqrt{E_1'^2 - m^2}, 0)$

$P_2 = (E_2', p_2' \cos\theta, -p_2' \sin\theta, 0) = (E_2', \sqrt{E_2'^2 - m^2} \cos\theta, -\sqrt{} \sin\theta, 0)$

Component y $\quad \sqrt{E_1'^2 - m^2} = \sqrt{E_2'^2 - m^2} \sin\theta$

Component x $\quad \sqrt{E^2 - M^2} = \sqrt{E_2'^2 - m^2} \cos\theta$

$\Big\}\ ^2$ and sum

$$E_1'^2 - \cancel{m^2} + E^2 - M^2 = E_2'^2 - \cancel{m^2} \Rightarrow$$

$$E_1'^2 = M^2 - E^2 + E_2'^2$$

Component t $\quad E_1' + E_2' = E \Rightarrow E_1' = E - E_2' \Rightarrow$

$$\Rightarrow E_1'^2 = (E - E_2')^2$$

$\Big\} \Rightarrow$

$$M^2 - E^2 + \cancel{E_2'^2} = E^2 + \cancel{E_2'^2} - 2EE_2' \Rightarrow$$

$$\boxed{E_2' = \frac{2E^2 - M^2}{2E}}$$

$$\boxed{E_1' = E - E_2' = \frac{2E^2 - 2E^2 + M^2}{2E} = \frac{M^2}{2E}}$$

$\boxed{\text{Alternative}}$

as 4-vectors!

$$P_M = P_1 + P_2 \Rightarrow P_M - P_1 = P_2 \Rightarrow$$

$$\Rightarrow (P_M - P_1)^2 = P_2^2 \Rightarrow P_M^2 + P_1^2 - 2P_M P_1 = P_2^2$$

$$\Rightarrow M^2 + \cancel{m^2} - 2EE_1' = \cancel{m^2} \Rightarrow \boxed{E_1' = \frac{M^2}{2E}}$$

2.6 M, E $\qquad$ $m, \vec{P} = 0$ $\qquad$ Head on, elastic collision

Show that $E'_M = \dfrac{2mM^2 + E(m^2 + M^2)}{2Em + m^2 + M^2}$ using that $E' = E$ is a solution.

$$\left.\begin{aligned} \text{Before: } P_M &= \left(E, \sqrt{E^2 - M^2}\right) \\ P_m &= (m, 0) \end{aligned}\right\} \quad \begin{aligned} \text{After: } P'_M &= \left(E', \sqrt{E'^2 - M^2}\right) \\ P'_m &= \left(E'_m, \sqrt{E'^2_m - m^2}\right) \end{aligned}$$

$$t: E + m = E' + E'_m \Rightarrow E'_m = E + m - E' \Rightarrow$$

$$\Rightarrow P'_m = \left(E + m - E', \sqrt{(E + m - E')^2 - m^2}\right) =$$

$$= \left(E + m - E', \sqrt{E^2 + \cancel{m^2} + E'^2 + 2Em - 2EE' - 2mE' - \cancel{m^2}}\right)$$

$$= \left(E + m - E', \sqrt{E^2 + E'^2 + 2Em - 2EE' - 2mE'}\right)$$

$$x: \sqrt{E^2 - M^2} = \sqrt{E^2 + E'^2 + 2Em - 2EE' - 2mE'} + \sqrt{E'^2 - M^2}$$

$$\Rightarrow \sqrt{E^2 - M^2} = \sqrt{(E - E')^2 + 2m(E - E')} + \sqrt{E'^2 - M^2}$$

$$\Rightarrow \sqrt{E^2 - M^2} - \sqrt{E'^2 - M^2} = \sqrt{(E - E')^2 + 2m(E - E')} \Rightarrow$$

$$\cancel{E^2} - M^2 + \cancel{E'^2} - M^2 - 2\sqrt{(E^2 - M^2)(E'^2 - M^2)} =$$

$$= \cancel{E^2} + \cancel{E'^2} - 2EE' + 2m(E - E')$$

$$\Rightarrow -M^2 - \sqrt{(E^2 - M^2)(E'^2 - M^2)} = -EE' + m(E - E')$$

$$\Rightarrow \sqrt{(E^2 - M^2)(E'^2 - M^2)} = EE' - m(E - E') - M^2$$

$$\Rightarrow \sqrt{(E^2-M^2)(E'^2-M^2)} = EE' - m(E-E') - M^2$$

Square

$$(E^2-M^2)(E'^2-M^2) = \left(EE' - m(E-E') - M^2\right)^2$$

$$\cancel{E^2E'^2} - E^2M^2 - M^2E'^2 + \cancel{M^4} = \cancel{E^2E'^2} + m^2(E-E')^2 + \cancel{M^4} -$$

$$-2EE'm(E-E') - 2EE'M^2 + 2m(E-E')M^2 \Rightarrow$$

Group terms with $(E-E')$

$$\Rightarrow m^2(E-E')^2 + 2m(E-E')(M^2-EE') +$$

$$+M^2E^2 + M^2E'^2 - 2EE'M^2 = 0 \Rightarrow$$

$$m^2(E-E')^2 + 2m(E-E')(M^2-EE') + M^2(E-E')^2 = 0$$

Divide by $(E-E')$ $\rightarrow$ $E=E'$ solution

$$m^2(E-E') + 2m(M^2-EE') + M^2(E-E') = 0$$

$$m^2E - m^2E' + 2mM^2 - 2mEE' + M^2E - M^2E' = 0$$

$$\Rightarrow m^2E + 2mM^2 + M^2E = E'(M^2 + 2mE + m^2)$$

$$\Rightarrow E' = \frac{E(m^2+M^2) + 2mM^2}{2mE + M^2 + m^2}$$

q.e.d

Alternative

(t) $E + m = E' + e' \Rightarrow e' = E + m - E' = E - E' + m$

$$P'_m = \left(E - E' + m, \sqrt{(E-E')^2 + \cancel{m^2} + 2m(E-E') - \cancel{m^2}}\right) =$$

$$= \left(E - E' + m, \sqrt{(E-E')^2 + 2m(E-E')}\right)$$

$$P_M + P_m = P'_M + P'_m \Rightarrow$$

$$\Rightarrow P_M - P'_M + P_m = P'_m \Rightarrow$$

$$\Rightarrow (P_M - P'_M + P_m)^2 = P'^2_m \Rightarrow$$

$$P_M^2 + P'^2_M + P_m^2 - 2P_M P'_M + 2P_M P_m - 2P'_M P_m = -m^2$$

$$-2M^2 - \cancel{m^2} + 2EE' - 2\sqrt{E^2 - M^2}\sqrt{E'^2 - M^2} - 2Em + 2E'm = \cancel{-m^2}$$

$$2M^2 - 2EE' + 2\sqrt{(E^2 - M^2)(E'^2 - M^2)} + 2(E - E')m = 0$$

$$M^2 - EE' + (E - E')m = -\sqrt{(E^2 - M^2)(E'^2 - M^2)}$$

$$M^4 + (EE')^2 + (E - E')^2 m^2 - 2M^2 EE' + 2(E - E')mM^2 -$$

$$- 2EE'm(E - E') = (E^2 - M^2)(E'^2 - M^2)$$

$$M^4 + (EE')^2 + (E-E')^2 m^2 - 2M^2EE' + 2(E-E')mM^2 -$$
$$- 2EE'm(E-E') = (E^2-M^2)(E'^2-M^2)$$

$$\cancel{M^4} + (\cancel{EE'^2}) + (E-E')^2 m^2 - 2M^2EE' + 2(E-E')mM^2 - 2EE'm(E-E')$$
$$= \cancel{(EE')^2} - E^2M^2 - E'^2M^2 + \cancel{M^4} \Rightarrow$$

$$(E-E')^2 m^2 - 2M^2EE' + 2(E-E')mM^2 - 2EE'm(E-E')$$
$$+E^2M^2 + E'^2M^2 =$$
$$(E-E')^2 m^2 + 2(E-E')m(M^2 - EE') + M^2(E-E')^2$$

Divide by $(E-E')$

$$Em^2 - E'm^2 + 2mM^2 - 2mEE' + M^2E - M^2E' = 0$$

$$\Rightarrow \boxed{E'} = \frac{Em^2 + 2mM^2 + M^2E}{m^2 + 2mE + M^2} = \boxed{\frac{E(m^2+M^2)+2mM^2}{m^2+2mE+M^2}}$$

Limits

- $E \approx M$ $\quad E' \approx \dfrac{2mM^2 + Mm^2 + M^3}{2Mm + m^2 + M^2} = M$ (not moving)

- $M = m$ $\quad E' = \dfrac{2M^3 + 2EM^2}{2EM + 2M^2} = M$ (stops, m picks up all of E)

- $m \gg E\ (> M)$ $\quad E' \approx \dfrac{2mM^2 + Em^2}{2EM + m^2} \approx E$ (heavy mass m picks no energy)

- $(E >)\ M \gg m$ and $M^2 \gg Em$

$$E' \approx \frac{2mM^2 + EM^2}{2Em + M^2} \approx E$$ (as if m weren't there)

- $(E >)\ M \gg m$ and $Em \gg M^2$

$$E' \approx \frac{EM^2}{2Em} = \frac{M^2}{2m}$$ (Big against small: if thrown hard enough, final E' does not depend on E)

- $E \gg m \gg M$

$$E' \approx \frac{2mM^2 + Em^2}{2Em} \approx \frac{m}{2}$$ (Small against big; if thrown hard enough small one ends up with E' independent of E)

2.7 $V^\mu F_\mu = 0$?

$$V^\mu A_\mu = 0 \Rightarrow V^\mu \frac{F_\mu}{m} = 0 \Rightarrow V^\mu F_\mu = 0$$

$$V = (\gamma, \gamma \vec{v}) \quad ; \quad F = \left(\gamma \frac{dE}{dt}, \gamma \vec{f}\right)$$

↳ Newtonian force

$$0 = V^\mu F_\mu = -\gamma^2 \frac{dE}{dt} + \gamma^2 \vec{v} \cdot \vec{f} \Rightarrow \boxed{\frac{dE}{dt} = \vec{v} \cdot \vec{f}}$$