Aplicacions de la integral definida

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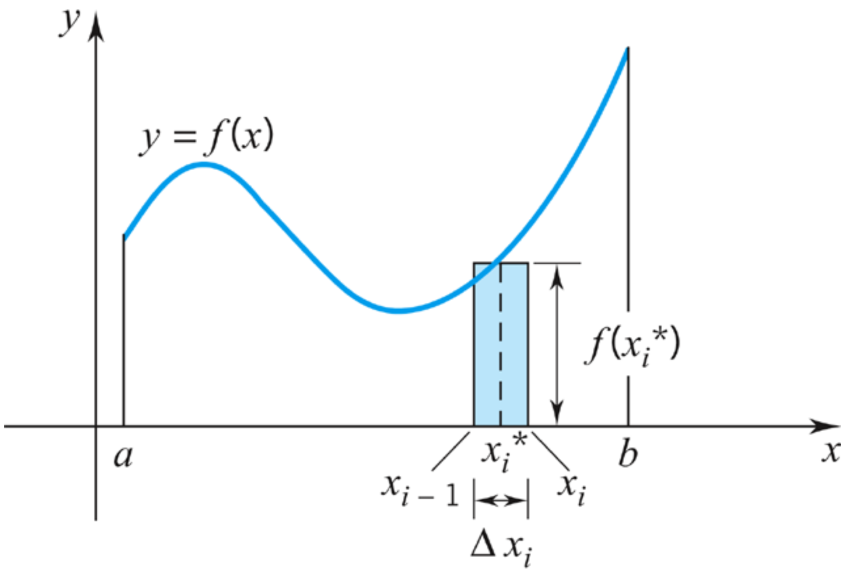
Area

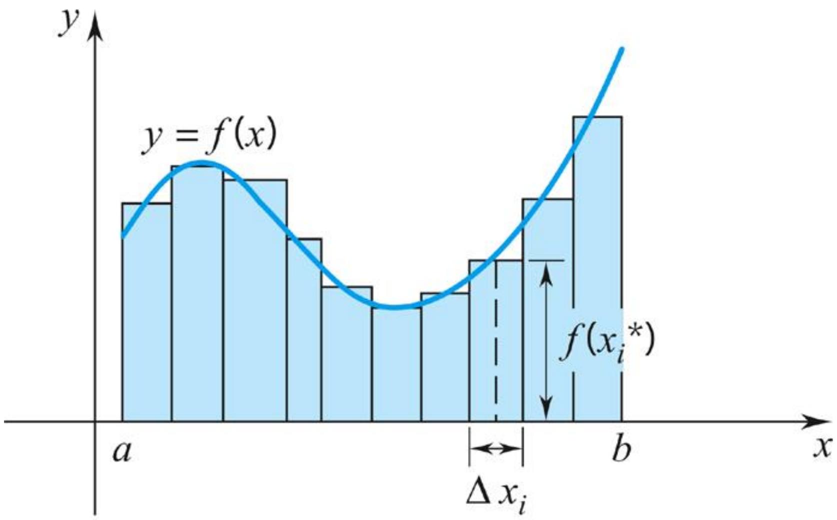
To calculate the **area between a curve f and the x-axis**, we already know how to express it in terms of Riemann sums

$$\int_a^b f(x)dx = \lim_{\|P\|\to 0} \left[f\left(x_1^*\right)\Delta x_1 + f\left(x_2^*\right)\Delta x_2 + \cdots + f\left(x_n^*\right)\Delta x_n\right].$$

which leads to

$$A = \int_a^b f(x)dx$$





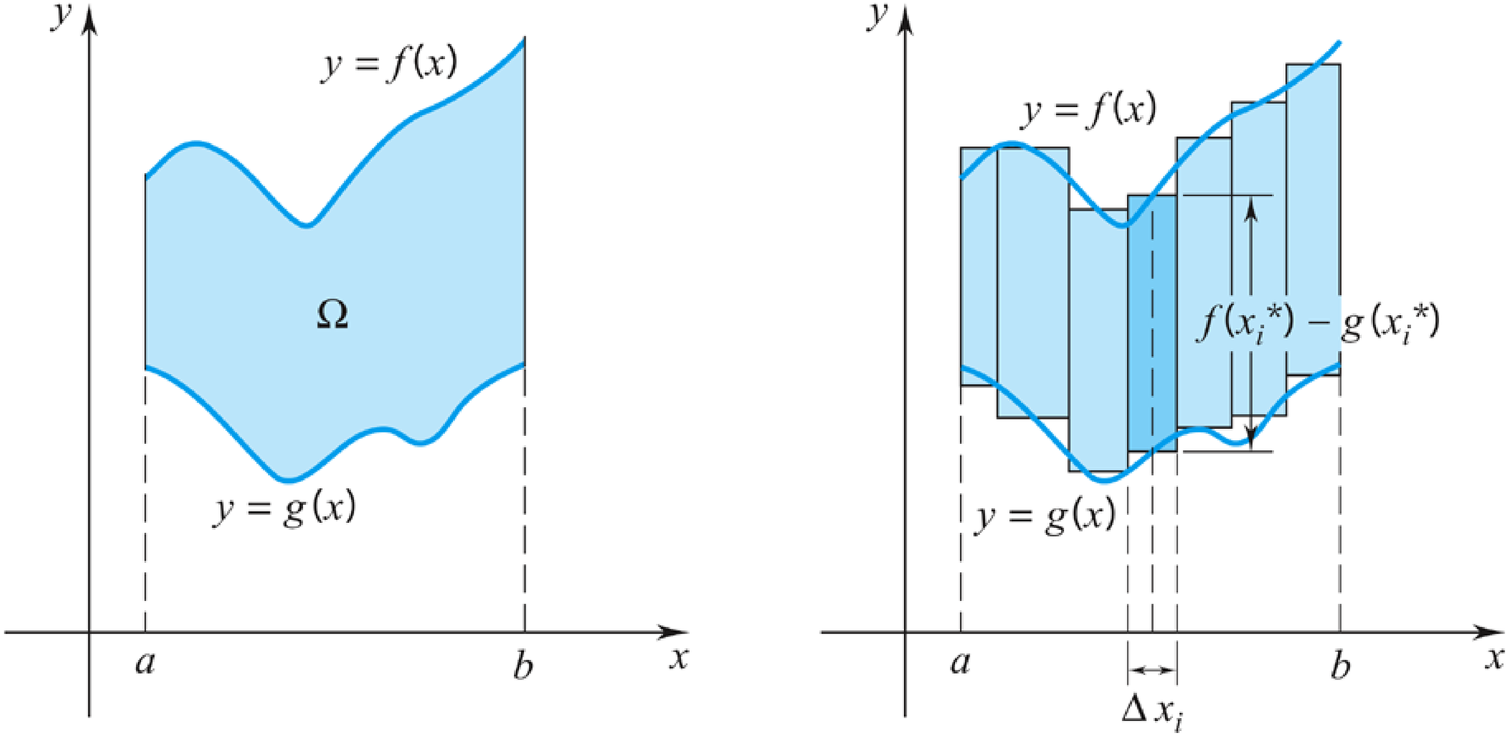
Area

To calculate the **area between two curves f and g**, the Riemann sums of the *vertical separations* read

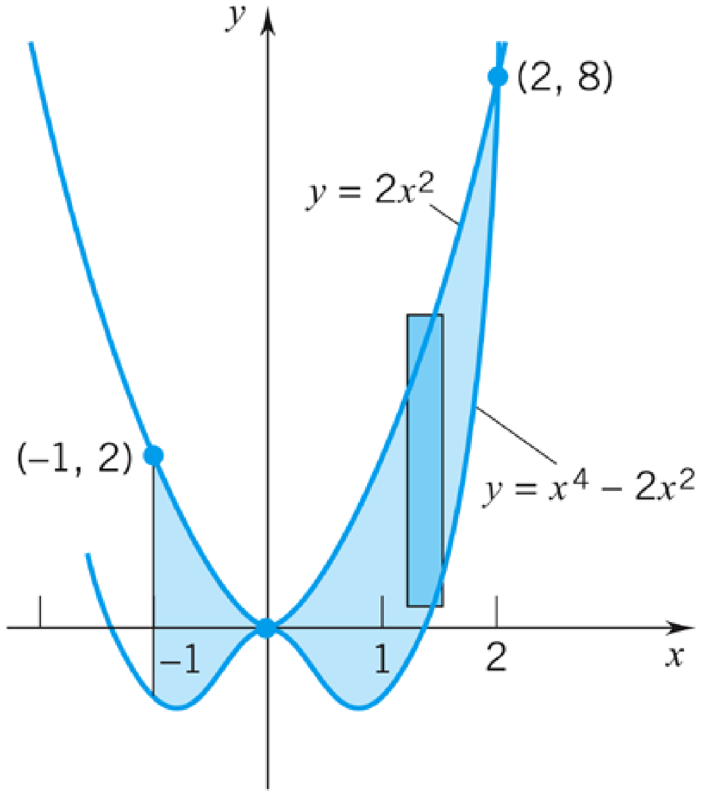
$$[f(x_1^*) - g(x_1^*)]\Delta x_1 + [f(x_2^*) - g(x_2^*)]\Delta x_2 + \cdots + [f(x_n^*) - g(x_n^*)]\Delta x_n.$$

which leads to

$$A = \int_a^b [f(x) - g(x)]dx$$



Area



Example

Find the area A of the shaded region

Solution

From $x = -1$ to $x = 2$ the vertical separation is the difference $2x^2 - (x^4 - 2x^2)$. Therefore

$$A = \int_{-1}^{2} [2x^2 - (x^4 - 2x^2)]\,dx = \int_{-1}^{2} (4x^2 - x^4)\,dx$$

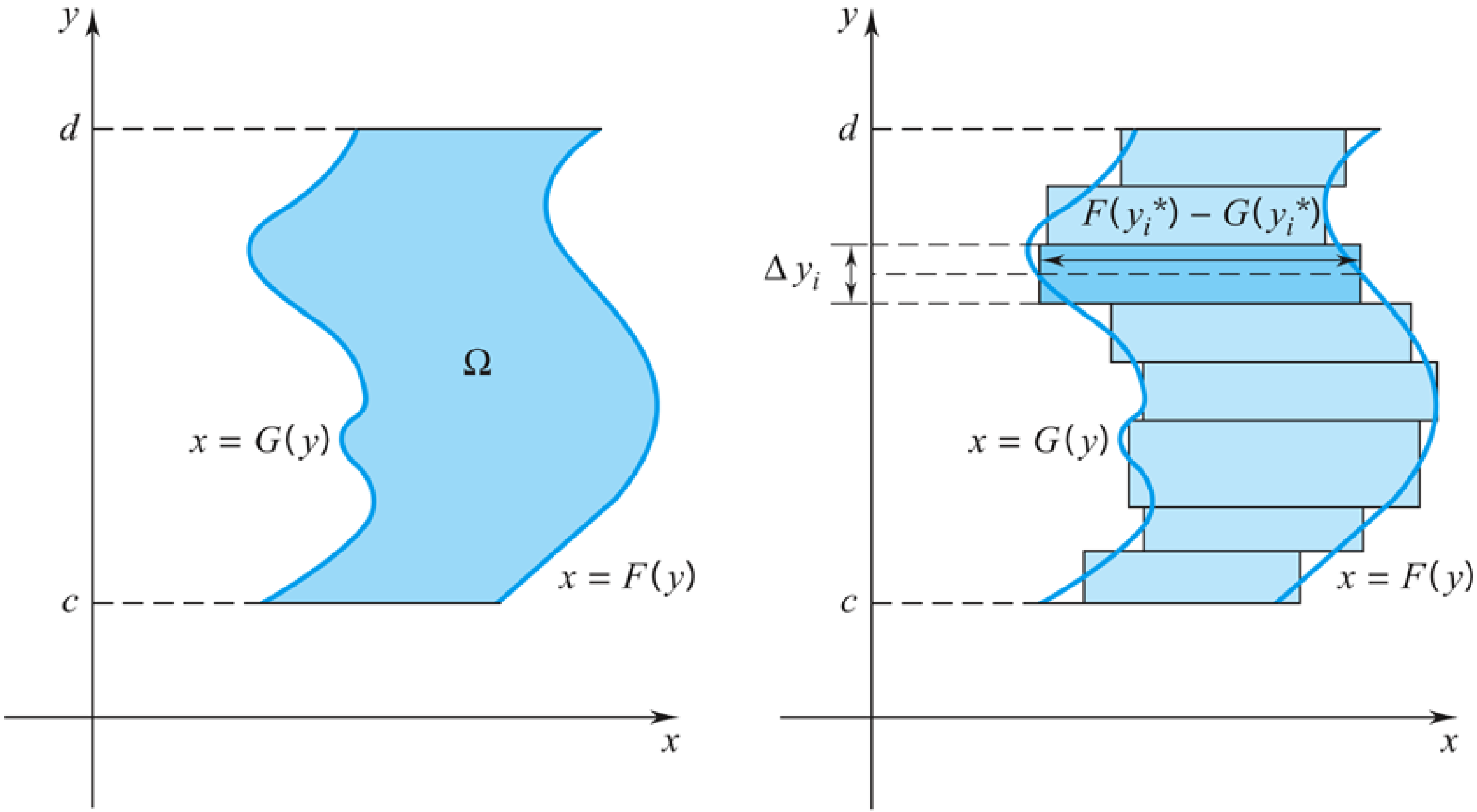
$$= \left[\tfrac{4}{3}x^3 - \tfrac{1}{5}x^5\right]_{-1}^{2} = \left[\tfrac{32}{3} - \tfrac{32}{5}\right] - \left[-\tfrac{4}{3} + \tfrac{1}{5}\right] = \tfrac{27}{5}$$

Area

Areas obtained by integration with respect to *y*

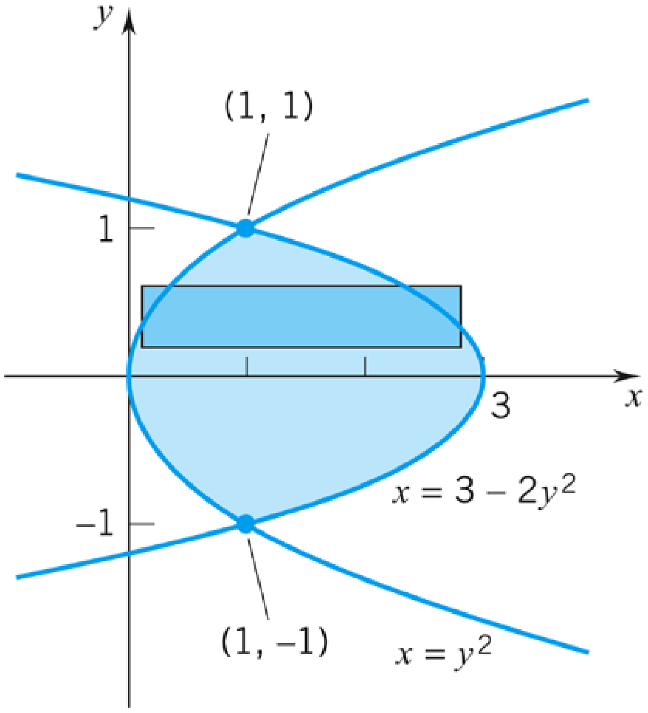
In this case we are integrating with respect to *y* the *horizontal separation* $F(y) - G(y)$ from $y = c$ to $y = d$

$$A = \int_c^d [F(y) - G(y)]\,dy$$



Area

Example
Find the area of the region bounded on the left by the curve $x = y^2$ and bounded on the right by the curve $x = 3 - 2y^2$.



Solution
The points of intersection can be found by solving the two equations simultaneously:

$$x = y^2 \quad \text{and} \quad x = 3 - 2y^2$$

together imply that

$$y = \pm 1.$$

The points of intersection are (1, 1) and (1,−1). The easiest way to calculate the area is to set our representative rectangles horizontally and integrate with respect to y. We then find the area by integrating the horizontal separation

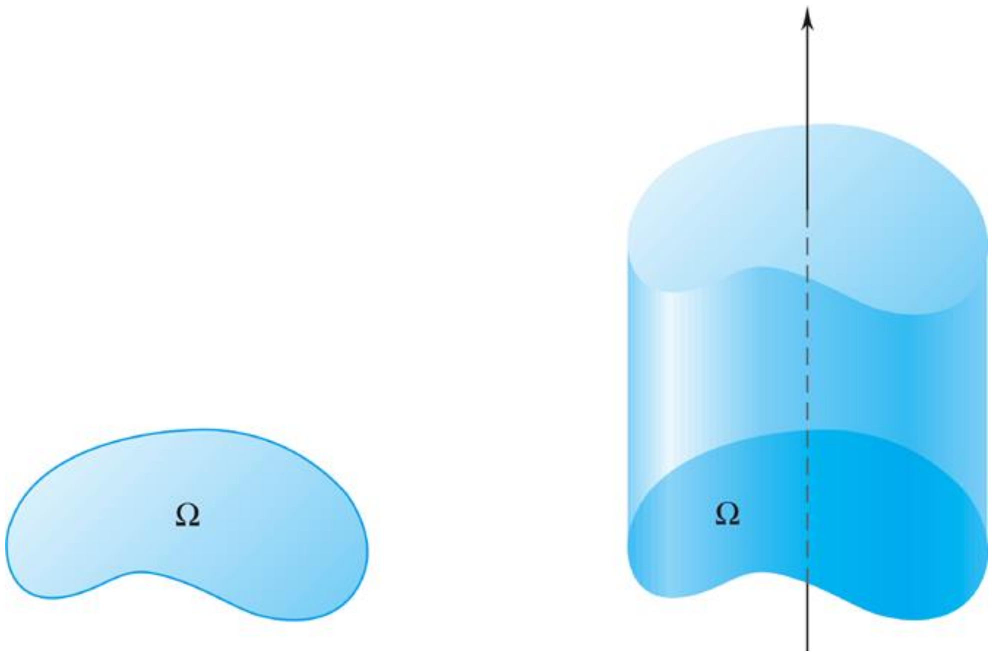
$$(3 - 2y^2) - y^2 = 3 - 3y^2$$

from $y = -1$ to $y = 1$:

$$A = \int_{-1}^{1} \left(3 - 3y^2\right) dy = \left[3y - y^3\right]_{-1}^{1} = 4$$

Volume by parallel cross sections

The figure shows a **plane region** Ω and a solid formed by translating Ω along a line perpendicular to the plane of Ω. Such a solid is called a **right cylinder with cross section** Ω.



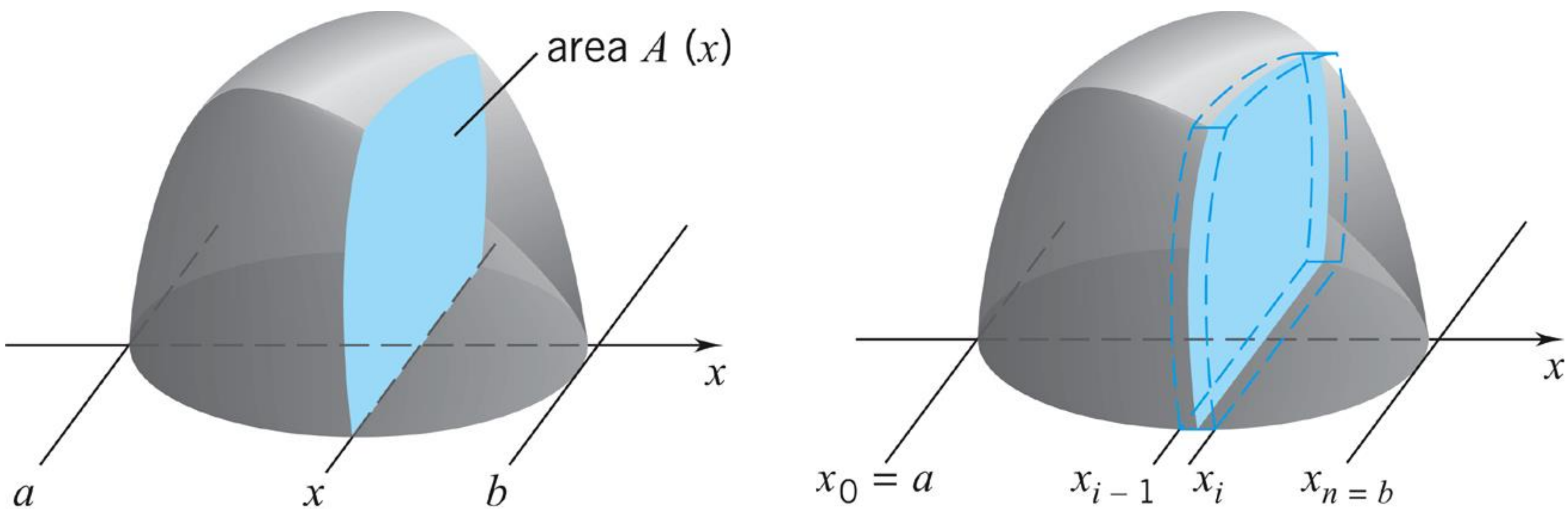
If Ω has area A and the solid has height h, then the volume of the solid is a simple product:

$$V = A \cdot h \qquad \text{(cross-sectional area} \cdot \text{height)}$$

Volume by parallel cross sections

If the **cross-sectional area** $A(x)$ varies continuously with x, then we can find the volume V of the solid by integrating $A(x)$ from $x = a$ to $x = b$:

$$\boxed{V = \int_a^b A(x)\,dx.}$$



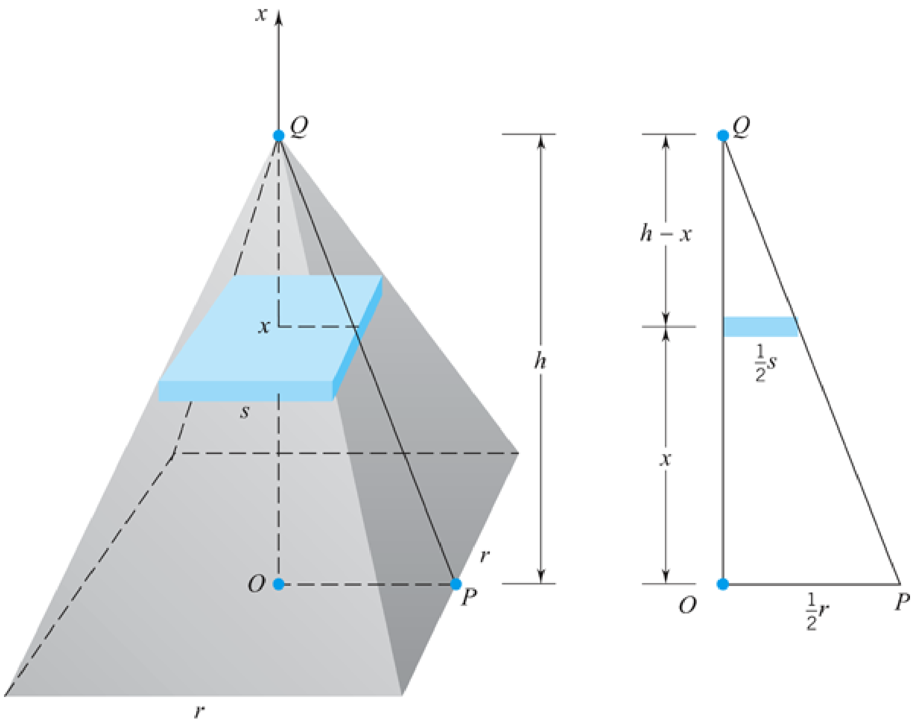
Volume by parallel cross sections

Example
Find the volume of the pyramid of height h given that the base of the pyramid is a square with sides of length r and the apex of the pyramid lies directly above the center of the base.

Solution
Set the x-axis as in the figure. The cross section at coordinate x is a square. Let s denote the length of the side of that square. By similar triangles

$$\frac{\frac{1}{2}s}{h-x} = \frac{\frac{1}{2}r}{h} \quad \text{and therefore} \quad s = \frac{r}{h}(h-x)$$

The area $A(x)$ of the square at coordinate x is $s^2 = (r^2/h^2)(h-x)^2$. Thus



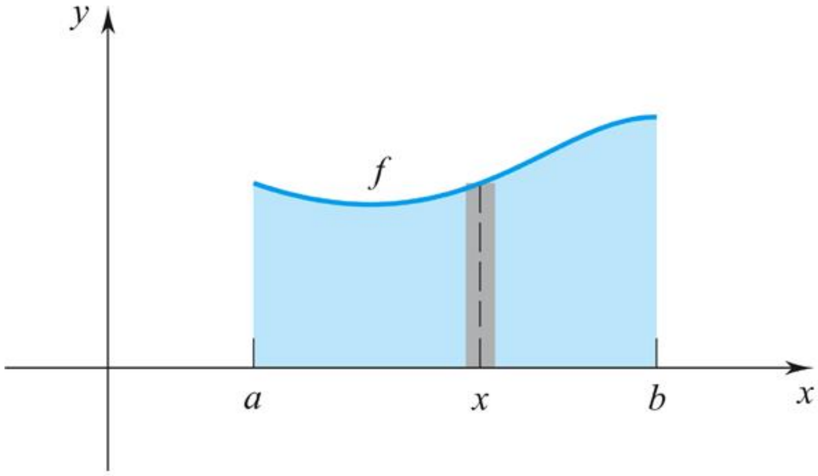
$$V = \int_0^h A(x)\,dx = \frac{r^2}{h^2}\int_0^h (h-x)^2\,dx = \frac{r^2}{h^2}\left[-\frac{(h-x)^3}{3}\right]_0^h = \tfrac{1}{3}r^2h.$$

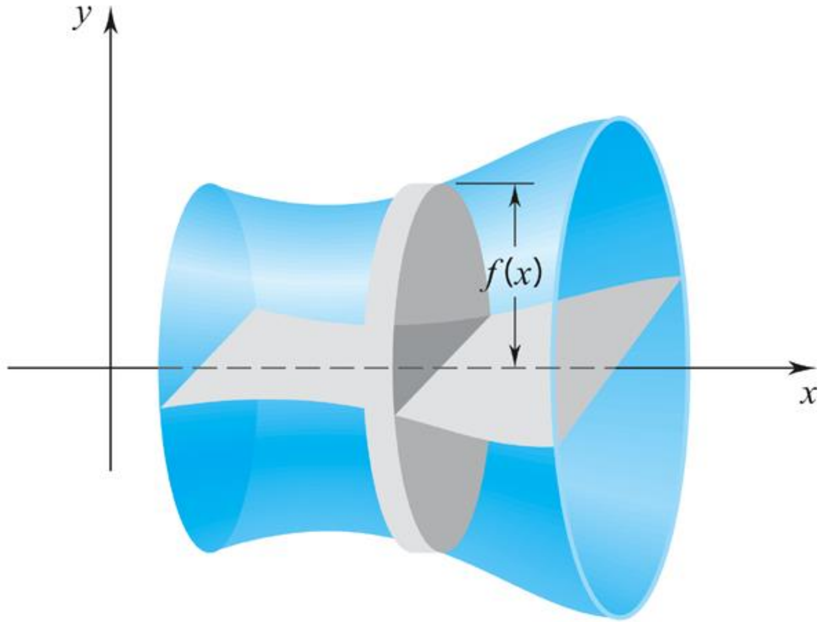
Volume by parallel cross sections

Solids of revolution: Disk method

A **solid of revolution** is obtained by **revolving a curve f about de x-axis**. The cross-sectional area is $A(x) = \pi[f(x)]^2$, thus the volume of this solid is given by

$$\boxed{V = \int_a^b \pi[f(x)]^2\,dx.}$$



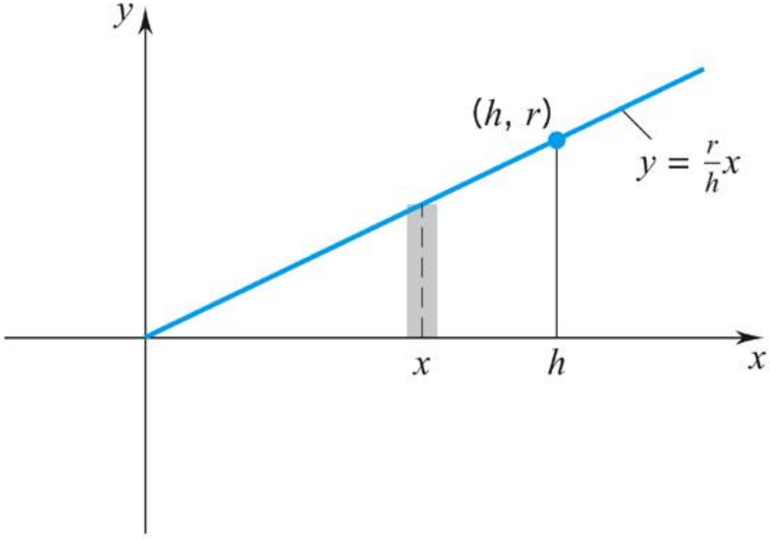


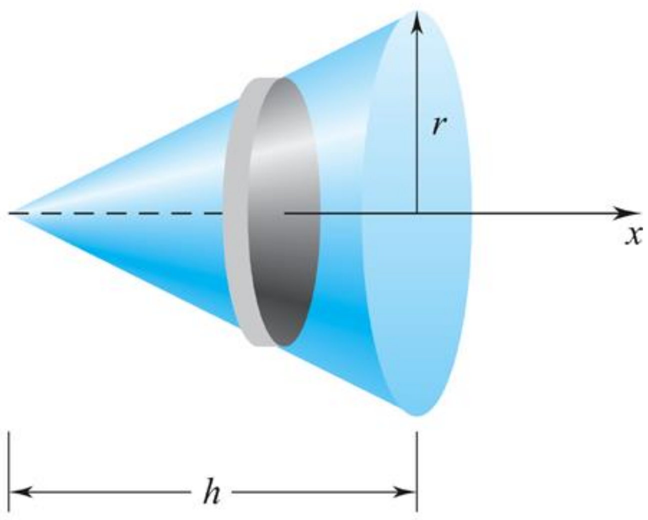
Volume by parallel cross sections

Example
We can generate a circular cone of base radius *r* and height *h* by revolving about the *x*-axis the region below the graph of

$$f(x) = \frac{r}{h}x, \qquad 0 \le x \le h$$

Volume of cone = $\displaystyle\int_0^h \pi\left[\frac{r}{h}x\right]^2 dx = \frac{\pi r^2}{h^2}\int_0^h x^2 dx = \frac{\pi r^2}{h^2}\left[\frac{x^3}{3}\right]_0^h = \frac{1}{3}\pi r^2 h$

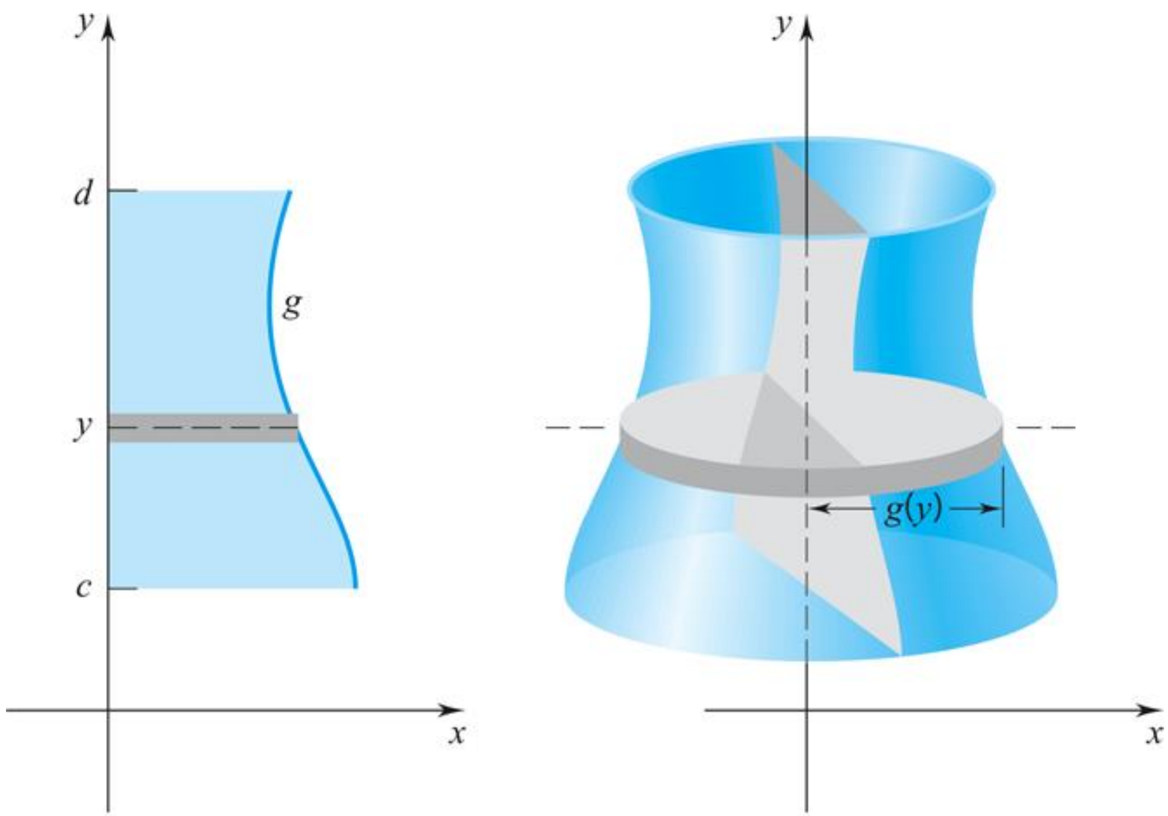




Volume by parallel cross sections

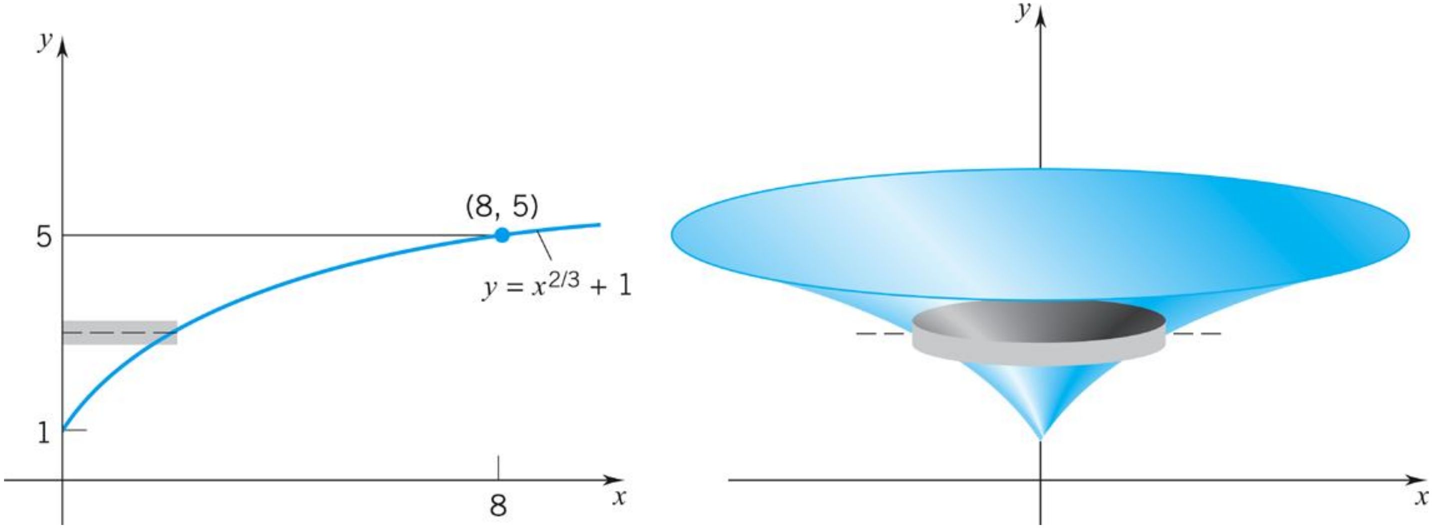
By **revolving about the *y*-axis** the region of the figure, we obtain a solid of cross-sectional area $A(y) = \pi[g(y)]^2$ and volume

$$V = \int_c^d \pi [g(y)]^2 \, dy.$$



Volume by parallel cross sections

Example
Let Ω be the region bounded below by the curve $y = x^{2/3} + 1$, bounded to the left by the y-axis, and above by the line $y = 5$. Find the volume of the solid generated by revolving Ω about the y-axis.



Solution We need to express the right boundary of Ω as a function of y:

$$y = x^{2/3} + 1 \quad \text{gives} \quad x^{2/3} = y - 1 \quad \text{and thus} \quad x = (y - 1)^{3/2}.$$

The volume of the solid obtained by revolving Ω about the y-axis is given by the integral

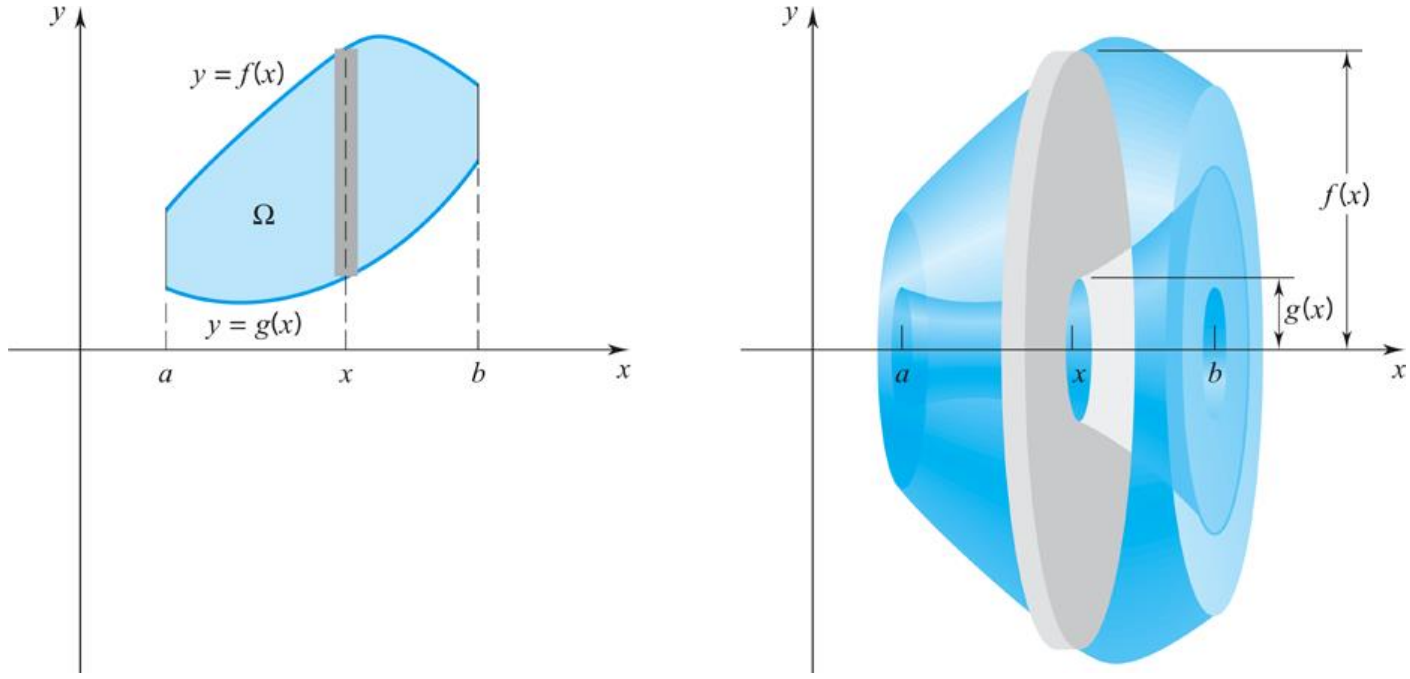
$$V = \int_1^5 \pi [g(y)]^2 dy = \pi \int_1^5 [(y-1)^{3/2}]^2 dy$$
$$= \pi \int_1^5 (y-1)^3 dy = \pi \left[\frac{(y-1)^4}{4} \right]_1^5 = 64\pi$$

Volume by parallel cross sections

Solids of Revolution: Washer Method

The **Washer method** is a slight generalization of the disk method. Suppose that f and g are nonnegative continuous functions with $g(x) \leq f(x)$ for all x in $[a, b]$. If we revolve the region Ω **about the *x*-axis**, we obtain a solid. The volume of this solid is

$$\boxed{V = \int_a^b \pi([f(x)]^2 - [g(x)]^2)\,dx.}$$

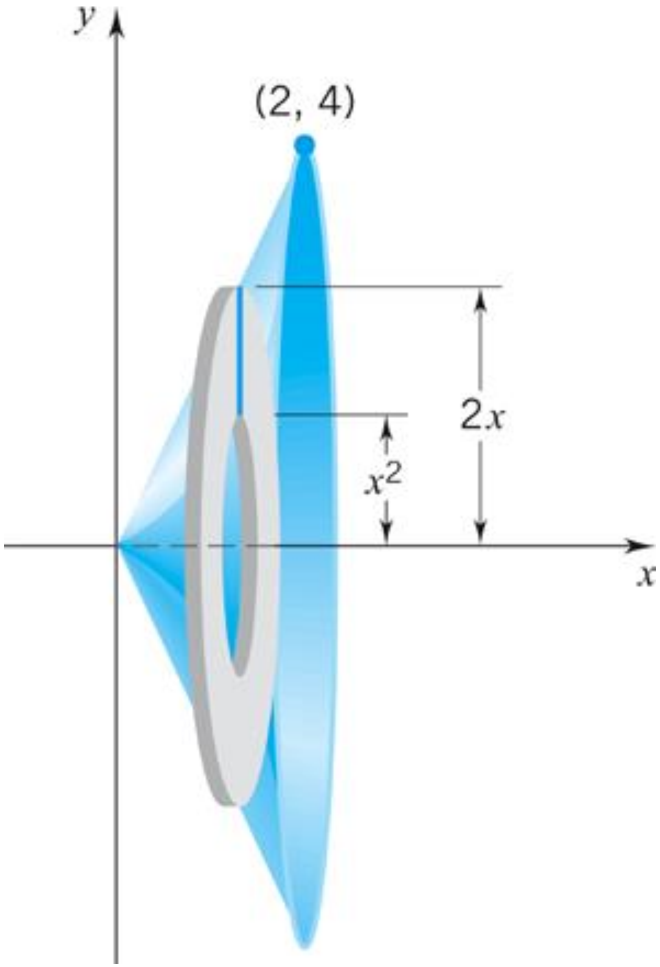


Volume by parallel cross sections

Example
Find the volume of the solid generated by revolving the region between

$$y = x^2 \text{ and } y = 2x$$

about the x-axis.



Solution
The curves intersect at the points (0, 0) and (2, 4).
For each x from 0 to 2, the x cross section is a washer of outer radius $2x$ and inner radius x^2.

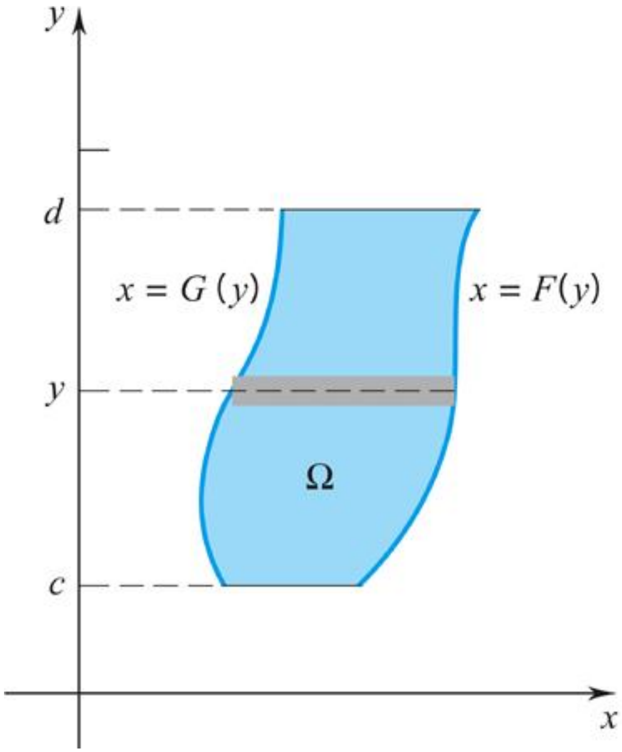
$$V = \int_0^2 \pi[(2x)^2 - (x^2)^2]\,dx = \pi \int_0^2 (4x^2 - x^4)\,dx$$

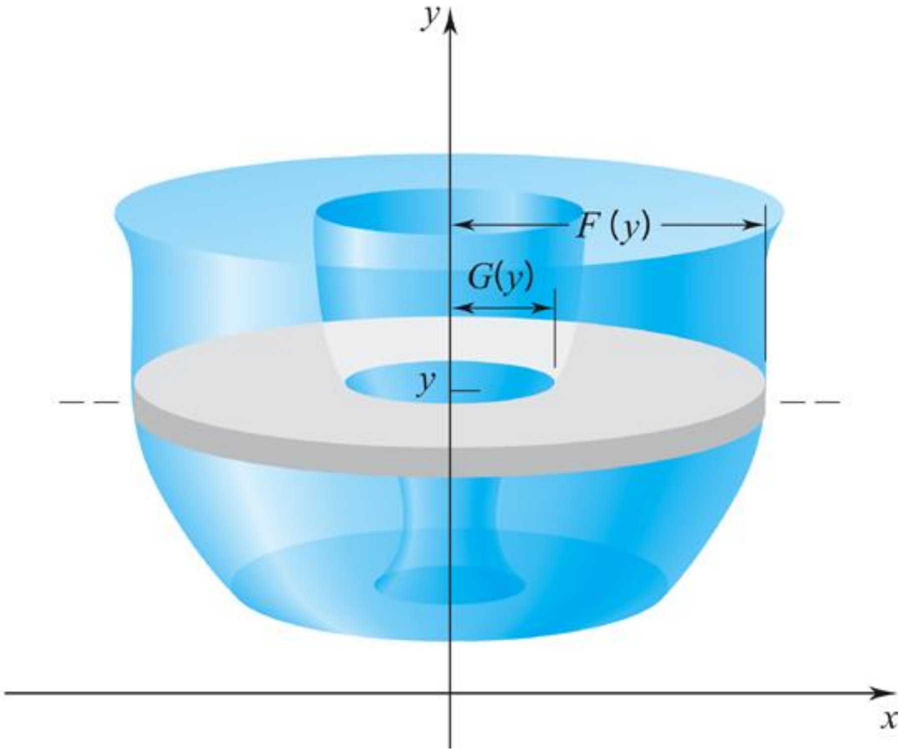
$$= \pi\left[\tfrac{4}{3}x^3 - \tfrac{1}{5}x^5\right]_0^2 = \tfrac{64}{15}\pi$$

Volume by parallel cross sections

We can interchange the roles played by *x* and *y*, and obtain the **washer method about the *y*-axis**

$$V = \int_c^d \pi([F(y)]^2 - [G(y)]^2)\,dy.$$





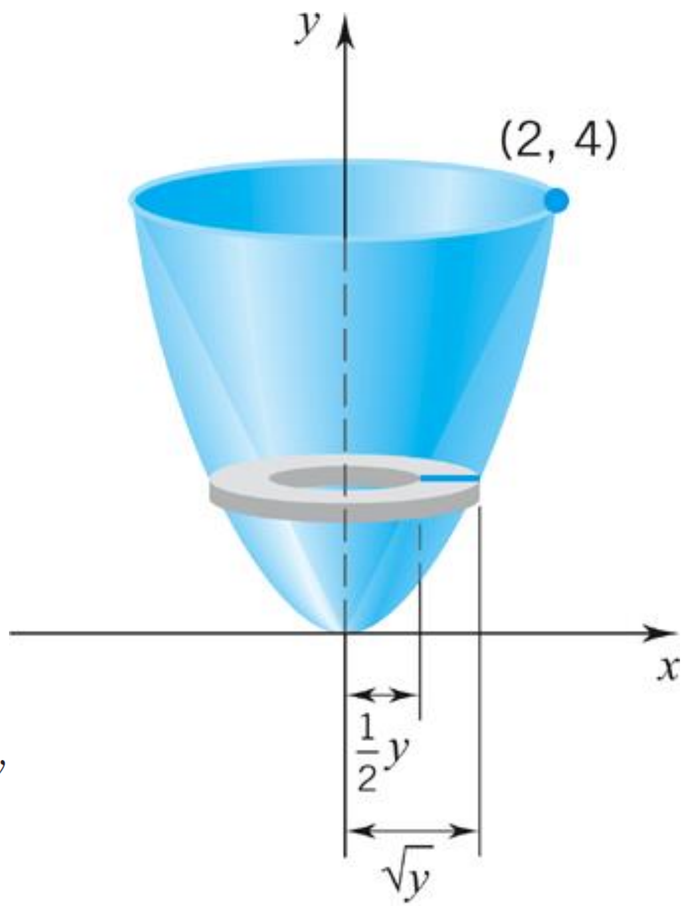
Volume by parallel cross sections

Example
Find the volume of the solid generated by revolving the region between

$$y = x^2 \text{ and } y = 2x$$

about the y-axis.

Solution
The curves intersect at the points (0, 0) and (2, 4).
For each y from 0 to 4, the y cross section is a washer of outer radius $\sqrt{y}$ and inner radius $\frac{1}{2}y$



$$V = \int_0^4 \pi\left[(\sqrt{y})^2 - \left(\tfrac{1}{2}y\right)^2\right]dy = \pi\int_0^4 \left(y - \tfrac{1}{4}y^2\right)dy$$

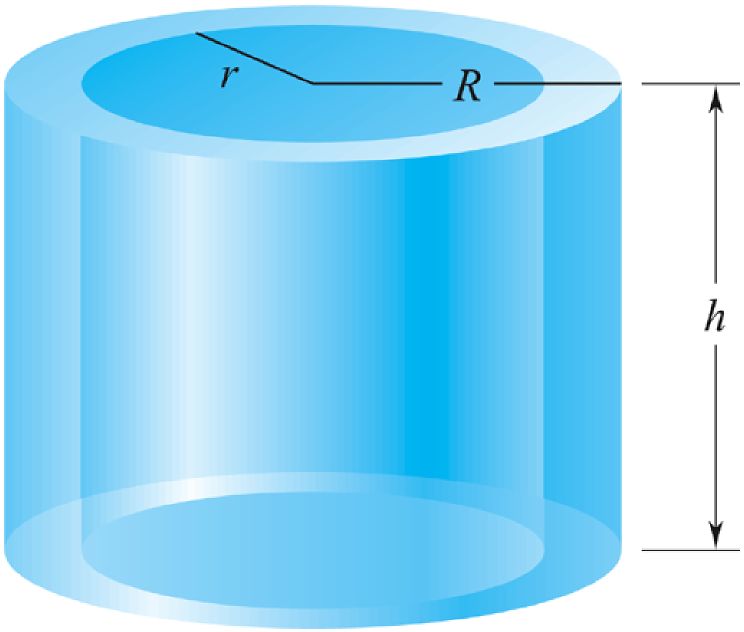
$$= \pi\left[\tfrac{1}{2}y^2 - \tfrac{1}{12}y^3\right]_0^4 = \tfrac{8}{3}\pi$$

Volume by shells

To describe the **shell method** of calculating volumes, we begin with a solid cylinder of radius R and height h, and from it we cut out a cylindrical core of radius r.

Since the original cylinder has volume $\pi R^2 h$ and the piece removed has volume $\pi r^2 h$, the cylindrical shell that remains has volume

$$\boxed{\pi R^2 h - \pi r^2 h = \pi h(R + r)(R - r).}$$



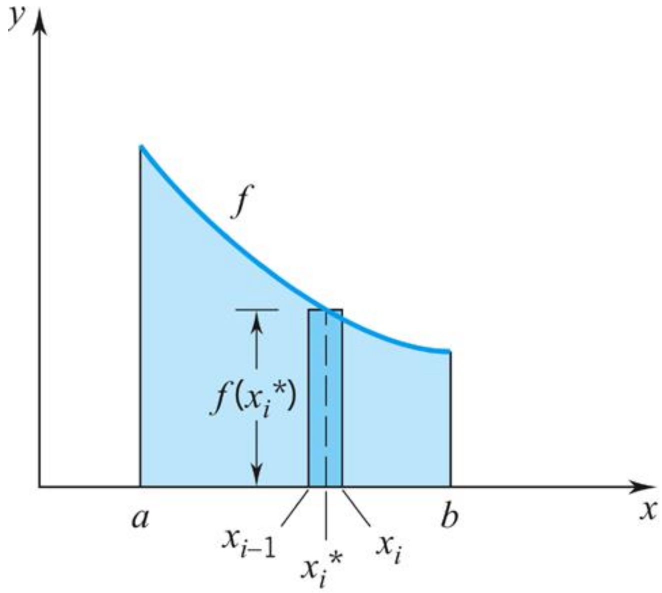
Volume by shells

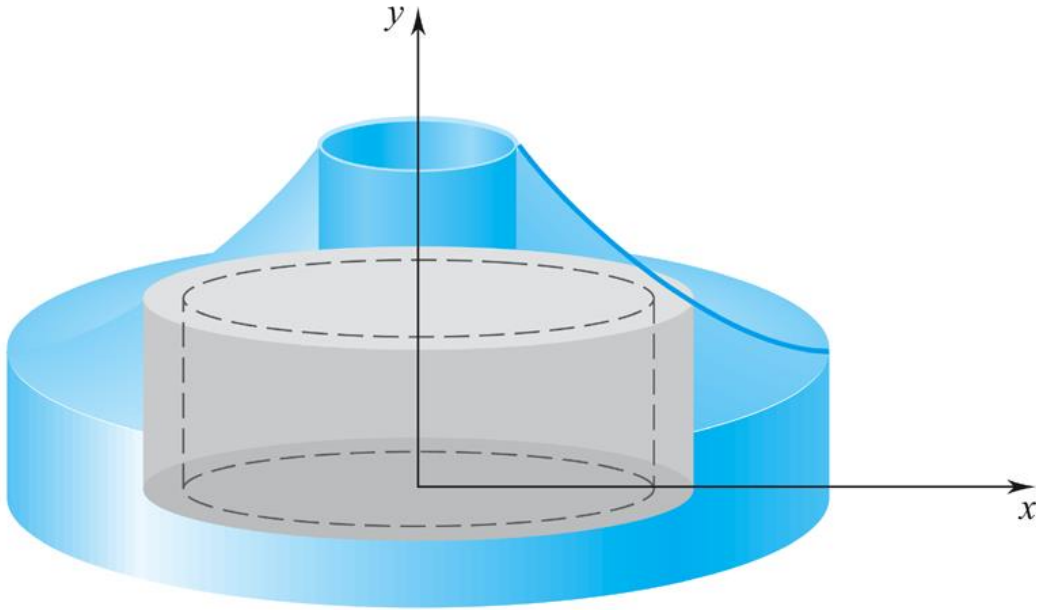
Now let $[a, b]$ be an interval with $a \geq 0$ and let f be a nonnegative function continuous on $[a, b]$. If the region bounded by the graph of f and the x-axis is revolved about the y-axis, then a solid is generated.

$$h = f\left(x_i^*\right) \qquad R - r = \Delta x_i \qquad R + r = x_i + x_{i-1} = 2x_i^*$$

$$\pi h(R+r)(R-r) = 2\pi x_i^* f\left(x_i^*\right)\Delta x_i$$

$$V \cong 2\pi x_1^* f\left(x_1^*\right)\Delta x_1 + 2\pi x_2^* f\left(x_2^*\right)\Delta x_2 + \cdots + 2\pi x_n^* f\left(x_n^*\right)\Delta x_n$$

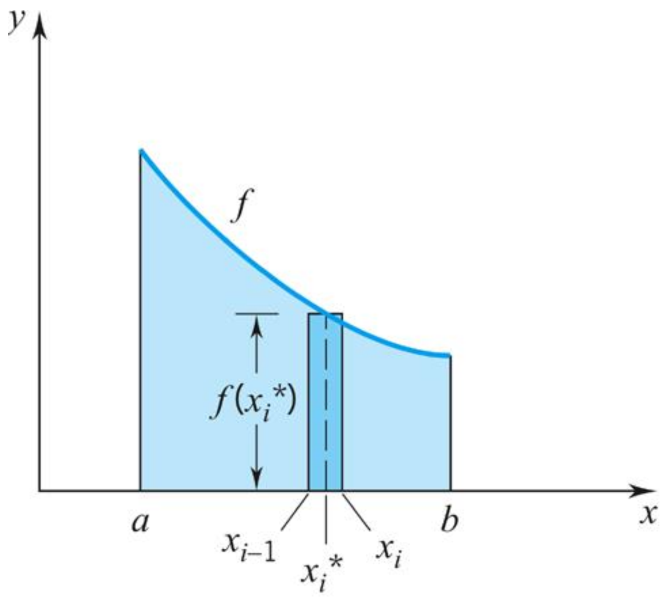


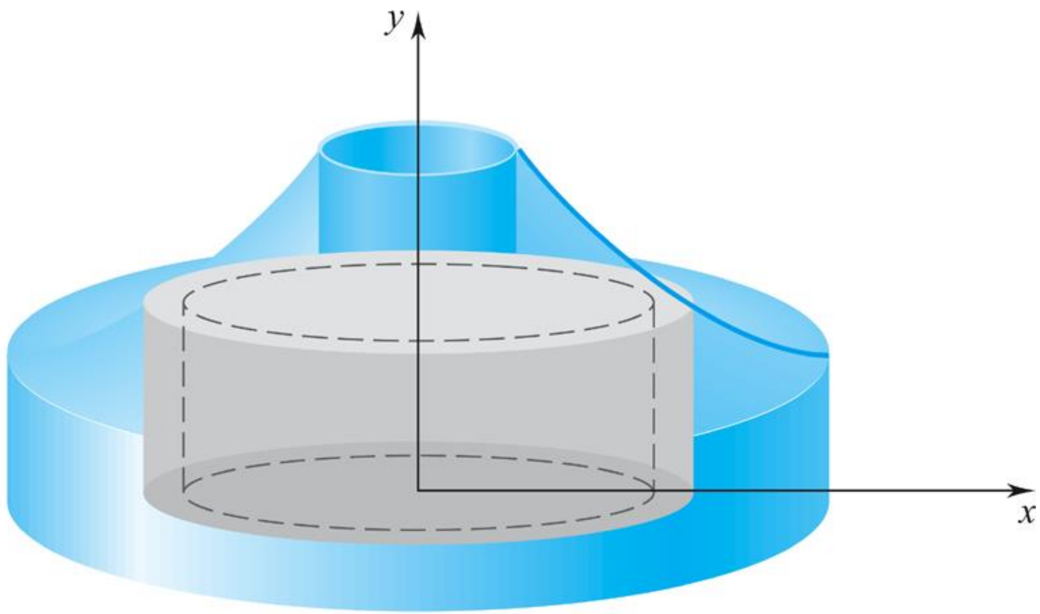


Volume by shells

Therefore, using the **shell method**, the volume of the solid generated by revolving about the y-axis the area between f and the x-axis is

$$\boxed{V = \int_a^b 2\pi x f(x)\,dx.}$$

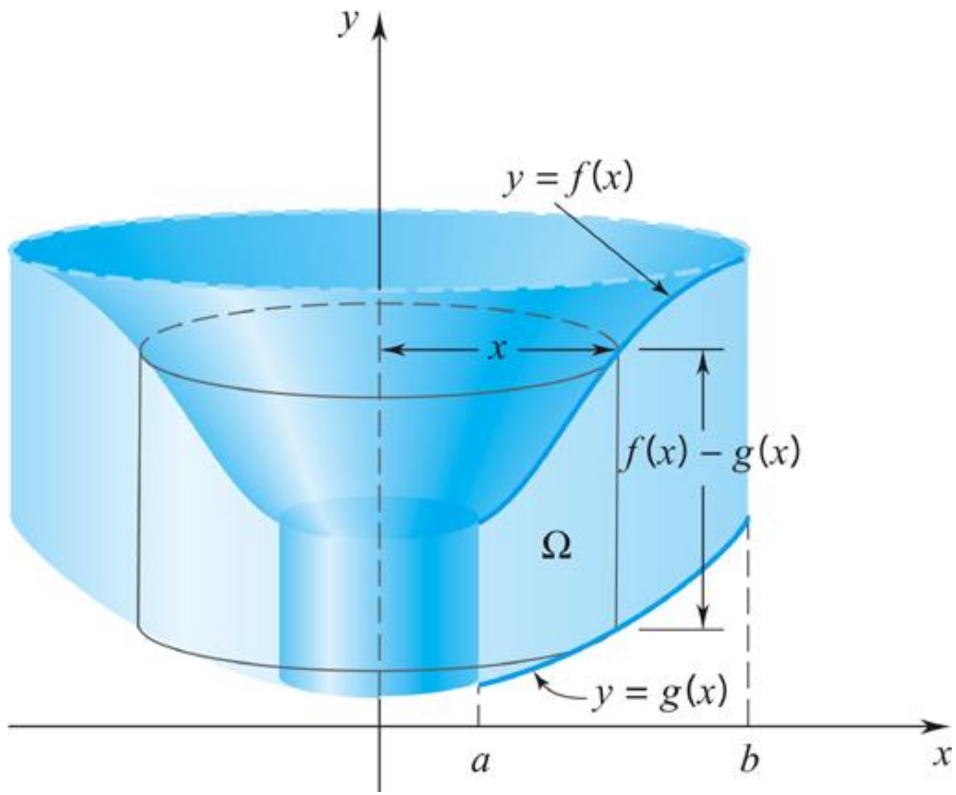




Volume by shells

If we replace the x-axis by another curve g, the volume generated by revolving Ω about the y-axis is given by the **shell method** as

$$\boxed{V = \int_a^b 2\pi x[f(x) - g(x)]\,dx.}$$

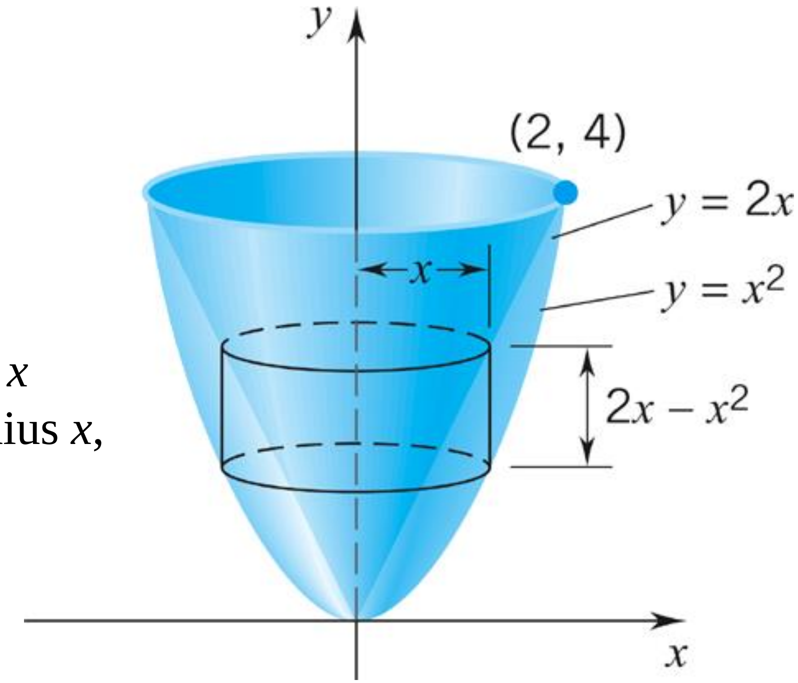


Volume by shells

Example
Find the volume of the solid generated by revolving the region between

$$y = x^2 \text{ and } y = 2x$$

about the y-axis.



Solution
The curves intersect at the points (0, 0) and (2, 4).

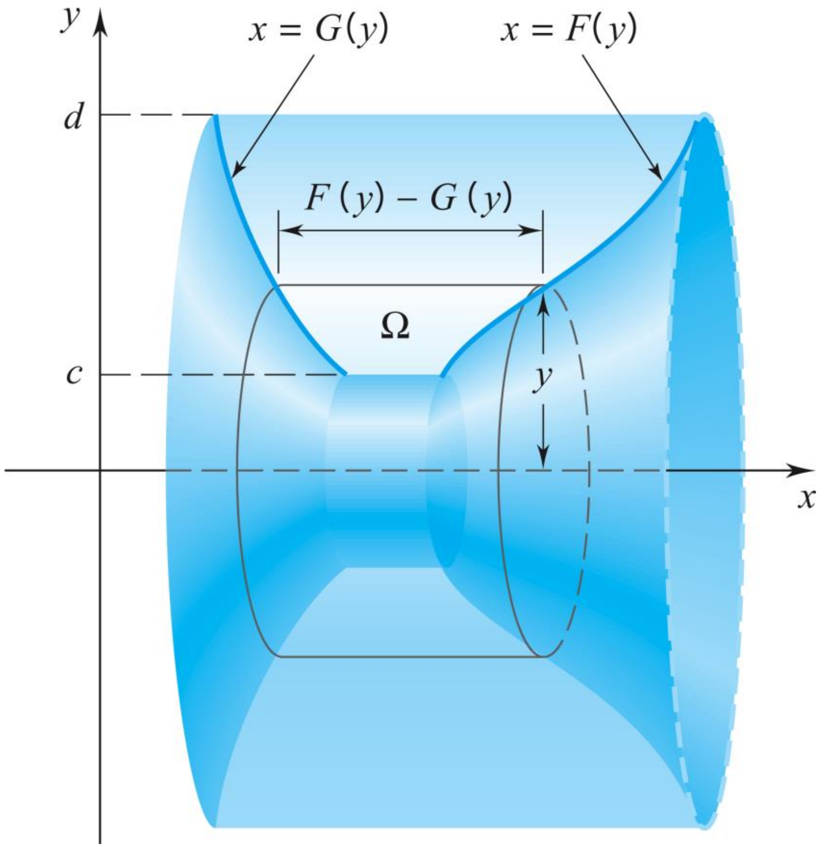
For each x from 0 to 2 the line segment at a distance x from the y-axis generates a cylindrical surface of radius x, height $(2x - x^2)$, and lateral area $2\pi x(2x - x^2)$

$$V = \int_0^2 2\pi x(2x - x^2)\,dx = 2\pi \int_0^2 (2x^2 - x^3)\,dx = 2\pi\left[\tfrac{2}{3}x^3 - \tfrac{1}{4}x^4\right]_0^2 = \tfrac{8}{3}\pi$$

Volume by shells

The volume generated by revolving Ω about the *x*-axis is given by

$$\boxed{V = \int_e^d 2\pi y[F(y) - G(y)]\,dy.}$$

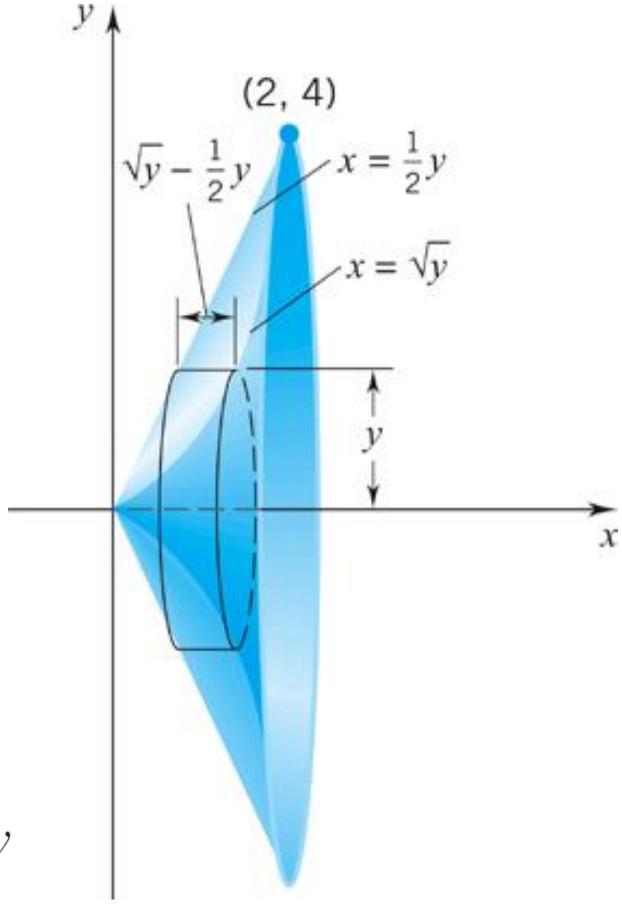


Volume by shells

Example
Find the volume of the solid generated by revolving the region between

$$y = x^2 \text{ and } y = 2x$$

about the x-axis.



Solution
The curves intersect at the points (0, 0) and (2, 4).
We begin by expressing the bounding curves as functions of y. We write $x = \sqrt{y}$ for the right boundary and $x = \frac{y}{2}$ for the left boundary.
For each y from 0 to 4 the line segment at a distance y from the x-axis generates a cylindrical surface of radius y, height $\left(\sqrt{y} - \frac{y}{2}\right)$, and lateral area $2\pi y\left(\sqrt{y} - \frac{y}{2}\right)$

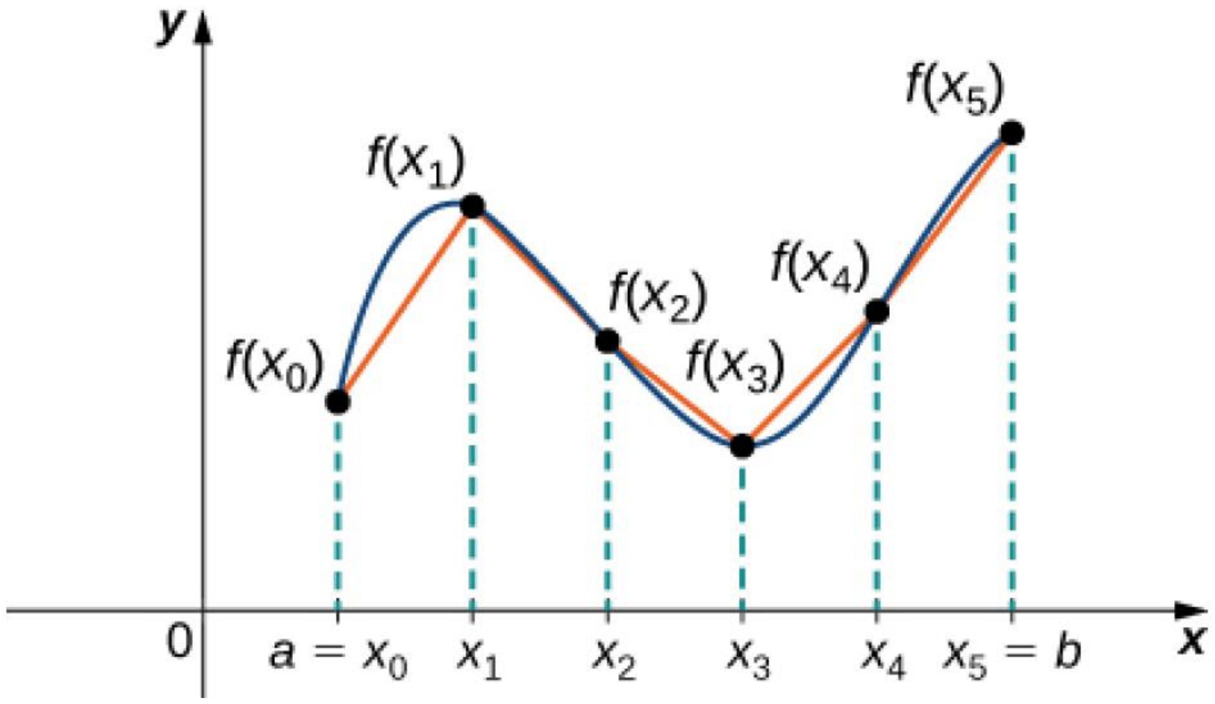
$$V = \int_0^4 2\pi y(\sqrt{y} - \tfrac{1}{2}y)\,dy = \pi \int_0^4 (2y^{3/2} - y^2)\,dy$$

$$= \pi\left[\tfrac{4}{5}y^{5/2} - \tfrac{1}{3}y^3\right]_0^4 = \tfrac{64}{15}\pi$$

Length

The length of a curve
To calculate the **length of a differentiable curve** $f(x)$ for x between a and b, we use a partition $P = \{x_0, x_1, \ldots, x_n\}$. The length L_i of segment i is $L_i = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$. Since $\Delta y_i = f'(x_i^*)\Delta x_i$, then

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}dx$$



Length

Example
Calculate the length of a circumference of radius R.

Solution
A circumference of radius R has equation $\mathrm{x}^2+\mathrm{y}^2+\mathrm{R}^2$. Thanks to the symmetry we only need to calculate the length in the first quadrant, $L/4$. We have $y=\sqrt{R^2-x^2}$, and

$$y'=\frac{-x}{\sqrt{R^2-x^2}}, \qquad (y')^2=\frac{x^2}{R^2-x^2}$$

Using the formula for the length

$$\frac{L}{4}=\int_0^R\sqrt{1+\frac{x^2}{R^2-x^2}}\,dx=\int_0^R\frac{1}{\sqrt{1-\left(\frac{x}{R}\right)^2}}\,dx$$

With the change of variable $x=Rt$ we get

$$\frac{L}{4}=R\int_0^1\frac{1}{\sqrt{1-t^2}}\,dt=R\arcsin t\Big|_0^1=R\,\frac{\pi}{2}$$

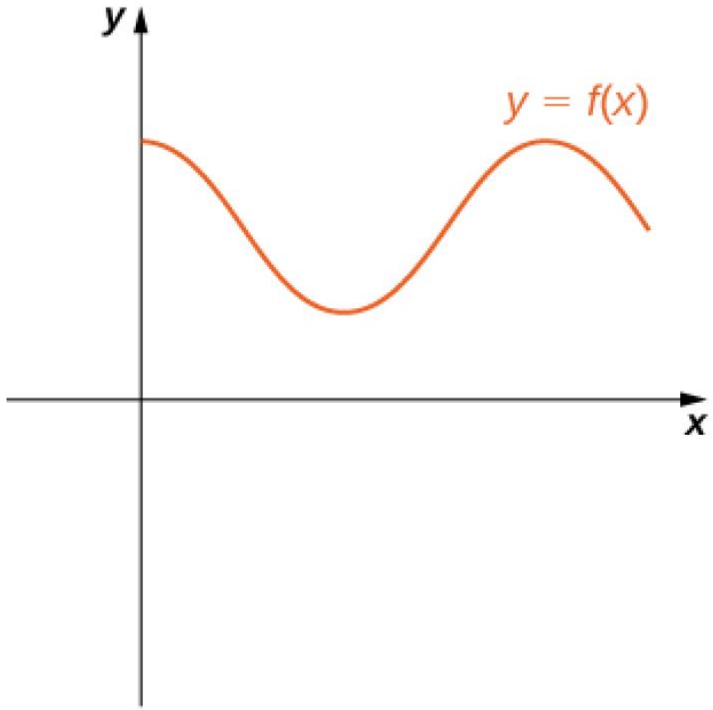
Thus, the full length of the circumference is

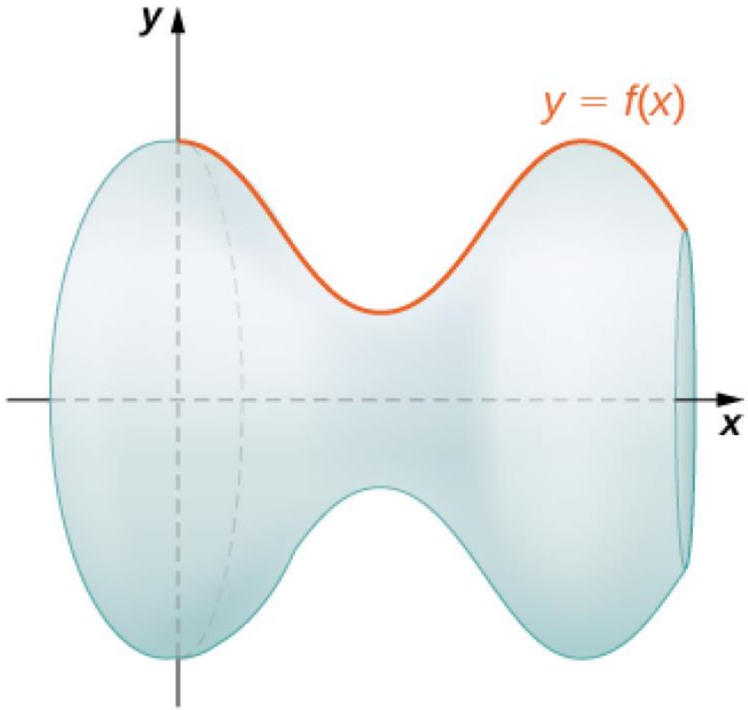
$$L=2\pi R$$

Area of surfaces of revolution

The area of the surface of revolution

We want to calculate the surface of a curve $f(x)$ when rotating about the x-axis.





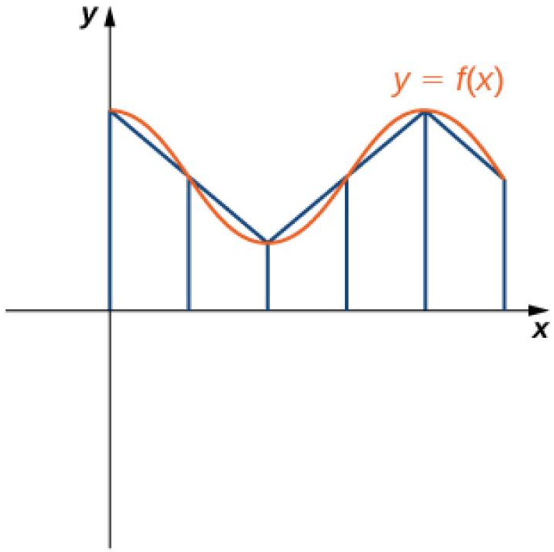
Area of surfaces of revolution

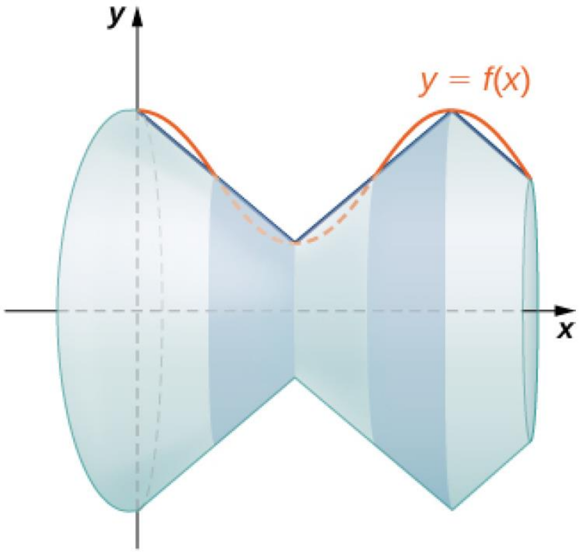
The area of the surface of revolution
We want to calculate the surface of a curve $f(x)$ when **rotating about the x-axis**. With a partition $P = \{x_0, x_1, \dots, x_n\}$, we need to sum the areas of the corresponding *fulcrums*. The area A_i of the fulcrum between x_i and x_{i-1} is $A_i = \pi[f(x_{i-1}) + f(x_i)]L_i$.
Summing them all and remembering that $L_i = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$ we get

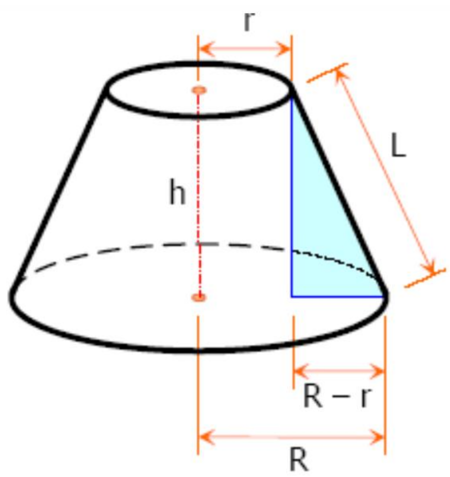
$$A = 2\pi \int_a^b f(x)\sqrt{1 + [f'(x)]^2}dx$$

If the **rotation** is **about the y-axis**, then

$$A = 2\pi \int_a^b x\sqrt{1 + [f'(x)]^2}dx$$







Area of surfaces of revolution

Example
Calculate the area of the surface of a sphere of radius R.

Solution
A circumference of radius R has equation $\mathrm{x}^2 + \mathrm{y}^2 + \mathrm{R}^2$. We have $y = \sqrt{R^2 - x^2}$, thus

$$y' = \frac{-x}{\sqrt{R^2 - x^2}}, \qquad (y')^2 = \frac{x^2}{R^2 - x^2}$$

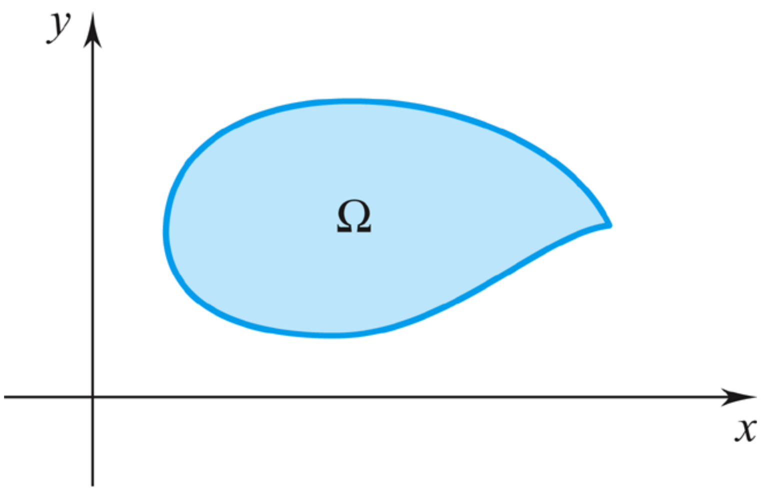
Using the formula for the area we get

$$A = 2\pi \int_{-R}^{R} \sqrt{R^2 - x^2} \sqrt{1 + \frac{x^2}{R^2 - x^2}}\, dx = 2\pi \int_{-R}^{R} R dx = 2\pi R\, x\Big|_{-R}^{R} = 4\pi R^2$$

Centroid

The centroid of a region

Suppose that we have a thin distribution of matter, a *plate*, laid out in the *xy*-plane in the shape of some region Ω. If the mass density of the plate varies from point to point, then the determination of the ***center of mass*** of the plate requires the evaluation of a double integral. If, however, the mass density of the plate is constant throughout Ω, then the center of mass depends only on the shape of Ω and falls on a point (x, y) that we call the ***centroid***. Unless Ω has a very complicated shape, we can locate the centroid by ordinary one-variable integration. We will use two guiding principles to locate the centroid of a plane region.



Centroid

Principle 1: Symmetry
If the region has an axis of symmetry, then the centroid $(\bar{x}, \bar{y})$ lies somewhere along that axis. In particular, if the region has a center, then the center is the centroid.

Principle 2: Additivity
If the region, having area A, consists of a finite number of pieces with areas $A_1, \ldots, A_n$ and centroids $(\bar{x}_1, \bar{y}_1), \ldots, (\bar{x}_n, \bar{y}_n)$, then

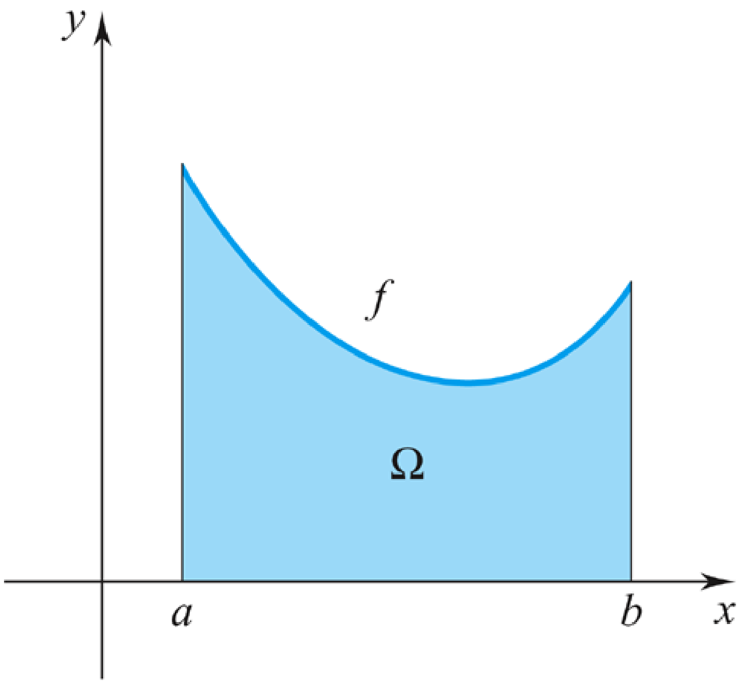
$$\boxed{\bar{x}A = \bar{x}_1 A_1 + \cdots + \bar{x}_n A_n \qquad \text{and} \qquad \bar{y}A = \bar{y}_1 A_1 + \cdots + \bar{y}_n A_n.}$$

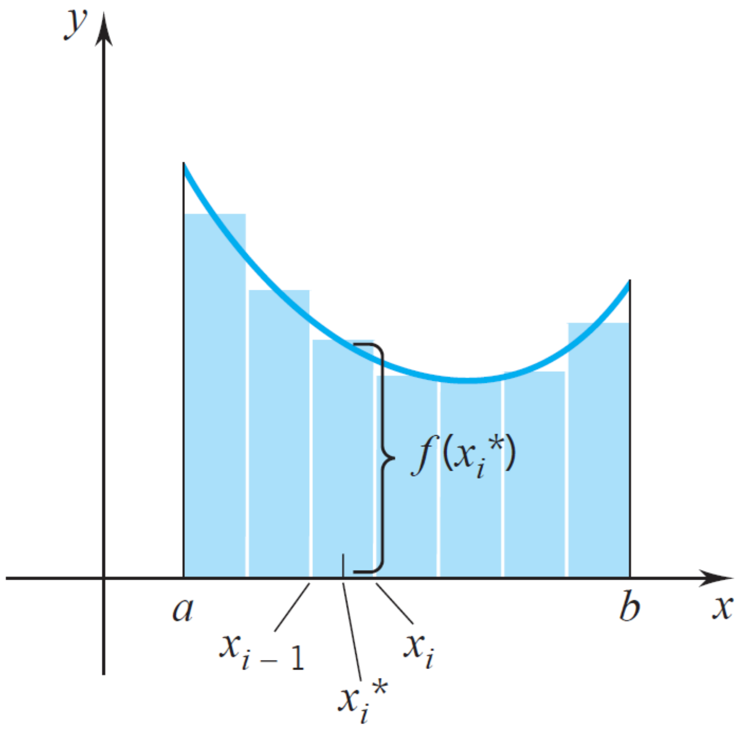
Centroid

Denote the area of Ω by A. The **centroid** $(\bar{x}, \bar{y})$ of Ω can be obtained using the additivity principle. Given a partition $P = \{x_0, x_1, \ldots, x_n\}$, each rectangle has area $A_i = f\left(x_i^*\right)\Delta x_i$ and centroid $(\bar{x}_i, \bar{y}_\mathrm{i}) = \left(x_i^*, \mathrm{f}\left(x_i^*\right)/2\right)$. Thus,

$$\bar{x}_p A_p = x_1^* f\left(x_1^*\right) \Delta x_1 + \cdots + x_n^* f\left(x_n^*\right) \Delta x_n,$$

$$\bar{y}_p A_p = \tfrac{1}{2}\left[f\left(x_1^*\right)\right]^2 \Delta x_1 + \cdots + \tfrac{1}{2}\left[f\left(x_n^*\right)\right]^2 \Delta x_n.$$

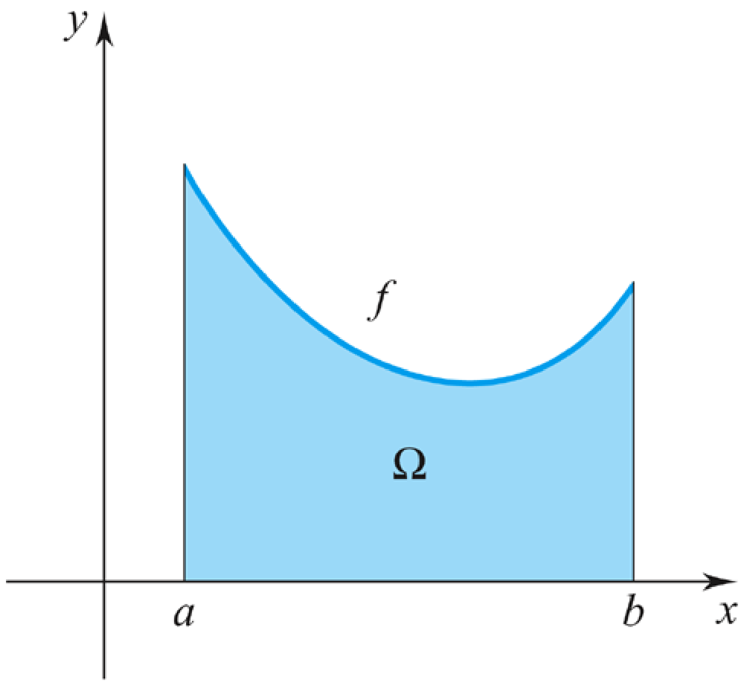




Centroid

The **centroid** $(\bar{x}, \bar{y})$ of Ω can be obtained from the following formulas:

$$\boxed{\bar{x}A = \int_a^b x f(x)\,dx, \quad \bar{y}A = \int_a^b \tfrac{1}{2}[f(x)]^2\,dx.}$$



Centroid

Example
Locate the centroid of the quarter-disk shown in Figure 6.4.4.

Solution
The quarter-disk is symmetric about the line $y = x$. Therefore, we know that $\bar{y} = \bar{x}$.
Here

$$\bar{y}A = \int_0^r \tfrac{1}{2}[f(x)]^2 dx = \int_0^r \tfrac{1}{2}(r^2 - x^2)\,dx = \tfrac{1}{2}\left[r^2 x - \tfrac{1}{3}x^3\right]_0^r = \tfrac{1}{3}r^3.$$

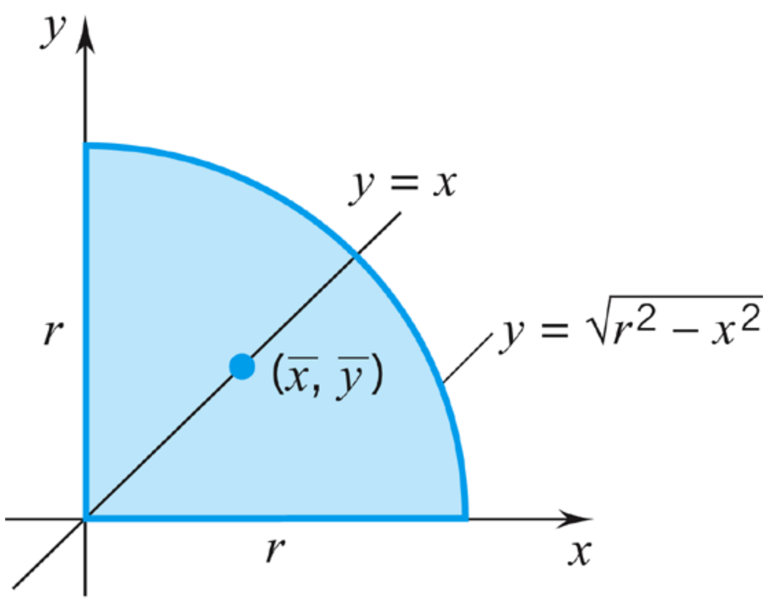
$b(x) = \sqrt{r^2 - x^2}$

Since $A = \frac{1}{4}\pi r^2$,

$$\bar{y} = \frac{\frac{1}{3}r^3}{\frac{1}{4}\pi r^2} = \frac{4r}{3\pi}$$

The centroid of the quarter-disk is the point

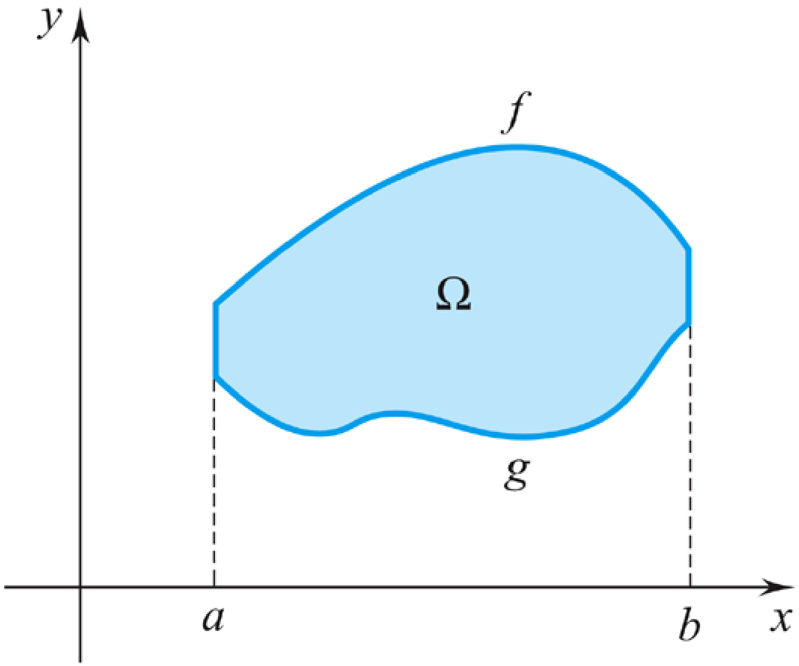
$$\left(\frac{4r}{3\pi}, \frac{4r}{3\pi}\right)$$



Centroid

Given a region Ω between the graphs of two continuous functions f and g, if Ω has area A and centroid $(\bar{x}, \bar{y})$, then

$$\boxed{\bar{x}A = \int_a^b x[f(x) - g(x)]\,dx, \quad \bar{y}A = \int_a^b \tfrac{1}{2}([f(x)]^2 - [g(x)]^2)\,dx.}$$



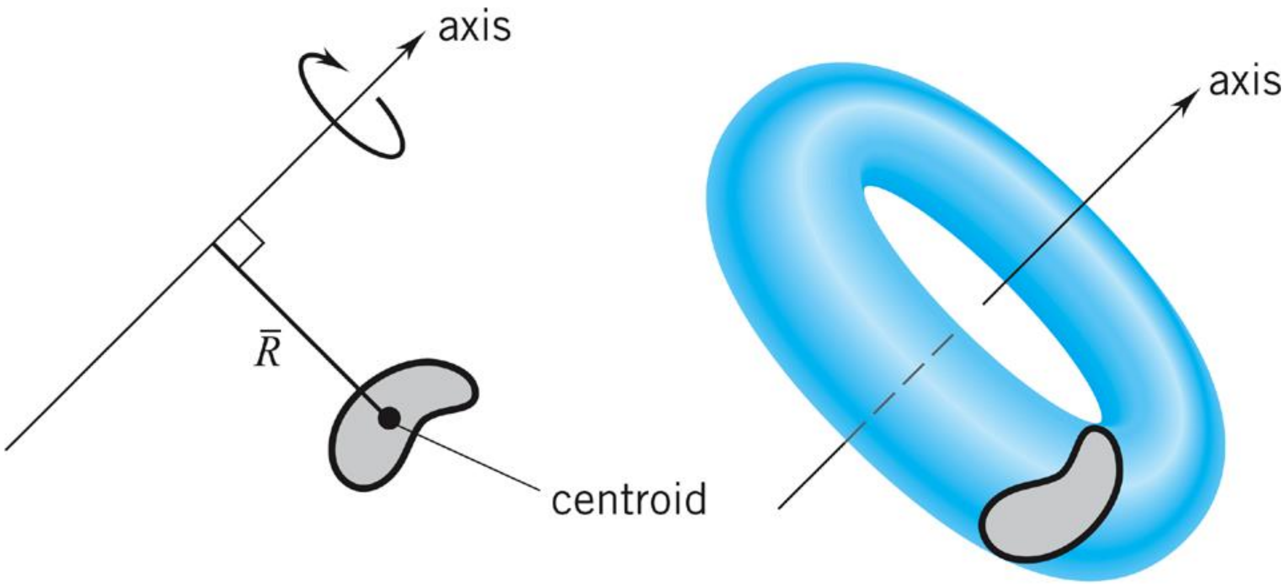
Pappus's Theorem on Volumes

THEOREM 6.4.4 PAPPUS'S THOREM ON VOLUMES[†]

A plane region is revolved about an axis that lies in its plane. If the region does not cross the axis, then the volume of the resulting solid of revolution is the area of the region multiplied by the circumference of the circle described by the centroid of the region:

$$V = 2\pi \overline{R} A$$

where A is the area of the region and $\overline{R}$ is the distance from the axis of revolution to the centroid of the region. (See Figure 6.4.8.)



Pappus's Theorem on Volumes

Applying Pappus's Theorem

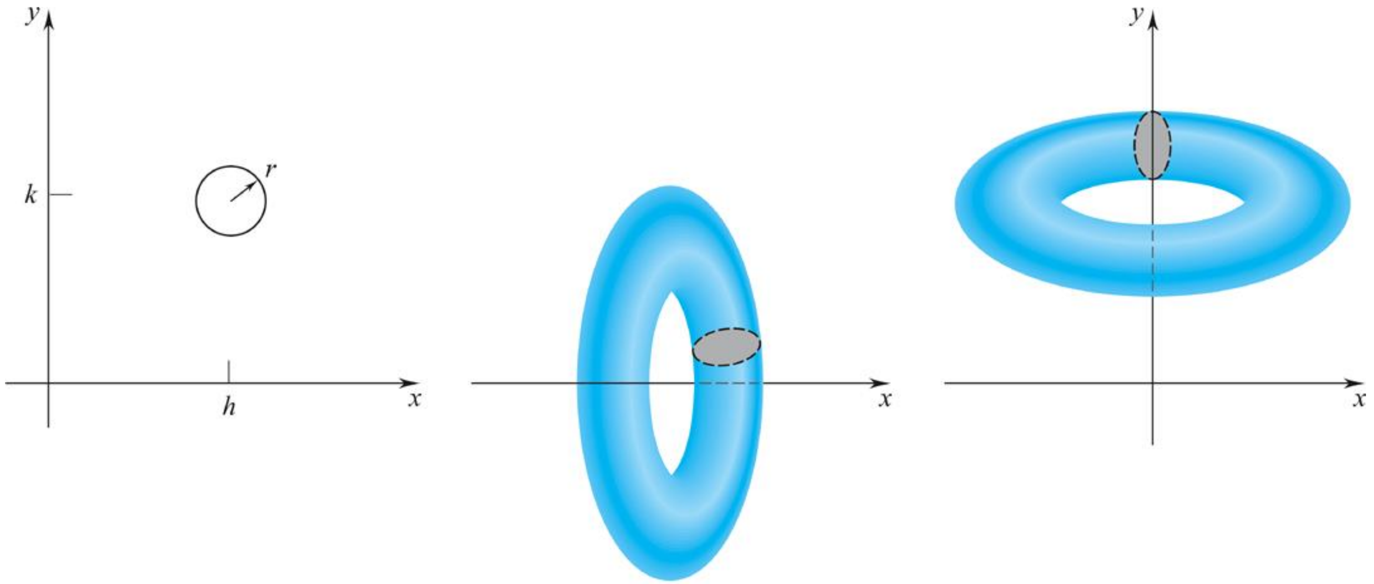
Example
Find the volume of the torus generated by revolving the circular disk

$$(x - h)^2 + (y - k)^2 \le r^2, \qquad h, k \ge r$$

(a) about the x-axis, **(b)** about the y-axis.

Solution The centroid of the disk is the center (h, k). This lies k units from the x-axis and h units from the y-axis. The area of the disk is πr^2. Therefore

(a) $V_x = 2\pi(k)(\pi r^2) = 2\pi^2 k r^2$ **(b)** $V_y = 2\pi(h)(\pi r^2) = 2\pi^2 h r^2$



Work

Suppose that an object moves along the x-axis from $x = a$ to $x = b$ subject to *constant* force F. The ***work*** done by F during the displacement is by definition the *force times the displacement:*

$$\boxed{W = F \cdot (b - a).}$$

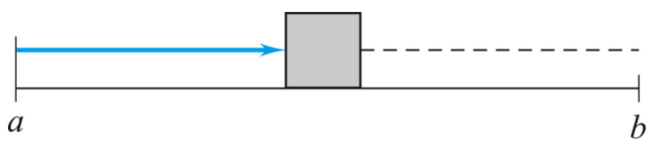
Work

If an object moves from $x = a$ to $x = b$ subject to a constant force F, then the work done by F is the constant value of F times $b - a$.

What is the work done by F if F does not remain constant but instead varies continuously as a function of x?

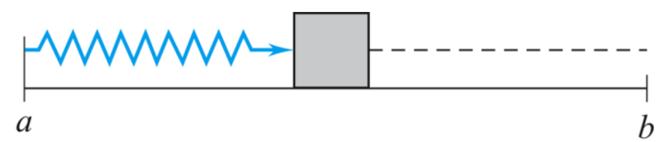
As you would expect, we then define the work done by F as the *average value of F times b − a*:

$$\boxed{W = \int_a^b f(x)\,dx.}$$



constant force

$W = F \bullet (b - a)$



variable force

$W = \int_a^b F(x)\,dx$

Work

Hooke's law
According to Hooke's law (Robert Hooke, 1635–1703), the force exerted by the spring can be written

$$F(x) = -kx$$

where k is a positive number, called *the spring constant*, and x is the displacement from the equilibrium position. The minus sign indicates that the spring force always acts in the direction opposite to the direction in which the spring has been deformed (the force always acts so as to restore the spring to its equilibrium state).

Remark Hooke's law is only an approximation, but it is a good approximation for small displacements.

Work

Hooke's law example
Stretched ⅓ meter beyond its natural length, a certain spring exerts a restoring force with a magnitude of 10 newtons. What work must be done to stretch the spring an additional ⅓ meter?

Solution
Place the spring on the x-axis so that the equilibrium point falls at the origin.
View stretching as a move to the right and assume Hooke's law: $F(x) = -kx$.

When the spring is stretched ⅓ meter, it exerts a force of −10 newtons (10 newtons to the left). Therefore, $-10 = -k(\tfrac{1}{3})$ and $k = 30$.

To find the work necessary to stretch the spring an additional ⅓ meter, we integrate the opposite force $-F(x) = 30x$ from $x = \tfrac{1}{3}$ to $x = \tfrac{2}{3}$:

$$W = \int_{1/3}^{2/3} 30x\,dx = 30\left[\frac{1}{2}x^2\right]_{1/3}^{2/3} = 5 \text{ joules}$$

Work

Counteracting the Force of Gravity
To lift an object, we must counteract the force of gravity. Therefore, the work required to lift an object is given by the equation

$$\boxed{\text{work} = (\text{weight of the object}) \times (\text{distance lifted}).^{\dagger}}$$

If an object is lifted from level $x = a$ to level $x = b$ and the weight of the object varies continuously with x—say the weight is $w(x)$—then the work done by the lifting force is given by the integral

$$\boxed{W = \int_a^b w(x)\,dx.}$$

Work

Example

A 150-pound bag of sand is hoisted from the ground to the top of a 50-foot building by a cable of negligible weight. Given that sand leaks out of the bag at the rate of 0.75 pounds for each foot that the bag is raised, find the work required to hoist the bag to the top of the building.

Solution

Once the bag has been raised x feet, the weight of the bag has been reduced to $150 - 0.75x$ pounds. Therefore

$$W = \int_0^{50} (150 - 0.75x)\, dx = \left[150x - \frac{1}{2}(0.75)x^2 \right]_0^{50}$$

$$= 150(50) - \frac{1}{2}(0.75)(50)^2 = 6562.5 \text{ foot-pounds}$$

Fluid Force

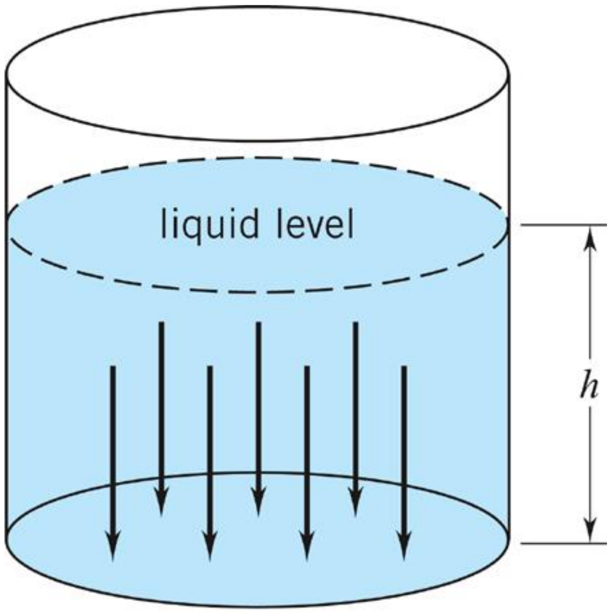
For any fluid, the weight per unit volume is called the *weight density* of the fluid. We'll denote this by the Greek letter σ.

An object submerged in a fluid experiences a *compressive force that acts at right angles to the surface of the body exposed to the fluid*.

Fluid in a container exerts a downward force on the base of the container. What is the magnitude of this force? It is the weight of the column of fluid directly above it.

If a container with base area A is filled to a depth h by a fluid of weight density σ, the downward force on the base of the container is given by the product

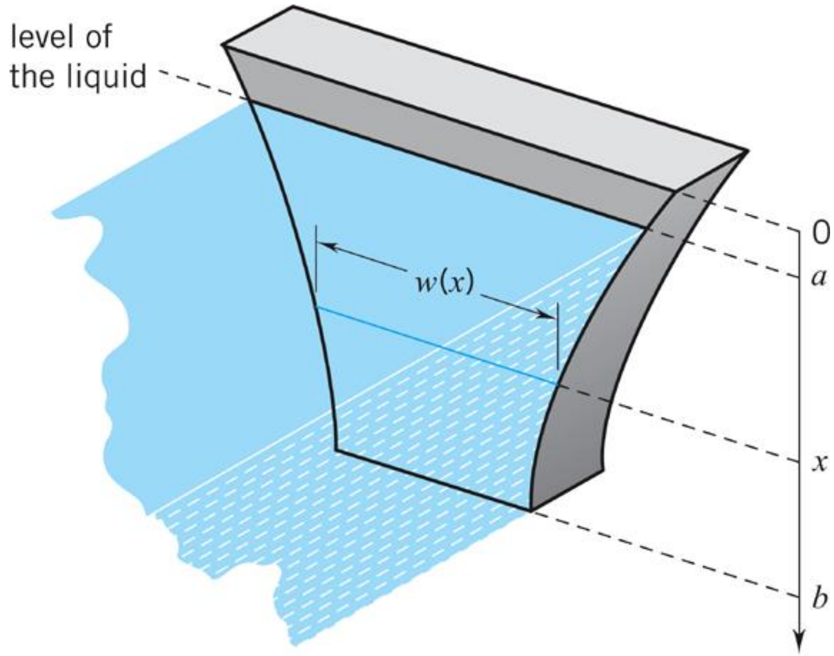
$$\boxed{F = \sigma h A.}$$



Fluid Force

Fluid force acts not only on the base of the container but also on the walls of the container. In Figure 6.6.2, we have depicted a vertical wall standing against a body of liquid. (Think of it as the wall of a container or as a dam at the end of a lake.) We want to calculate the force exerted by the liquid on this wall.

$$\text{fluid force against the wall} = \int_a^b \sigma x w(x)\, dx.$$



Fluid Force

Example

A cylindrical water main 6 feet in diameter is laid out horizontally. Given that the main is capped half-full, calculate the fluid force on the cap.

Solution

Here σ = 62.5 pounds per cubic foot. From the figure we see that $w(x) = 2\sqrt{9-x^2}$

The fluid force on the cap can be calculated as follows:

$$F = \int_0^3 (62.5)\, x\left(2\sqrt{9-x^2}\right) dx$$

$$= 62.5\int_0^3 2x\sqrt{9-x^2}\;dx$$

$$= 1125 \text{ pounds.}$$