Special relativity - Kinematics

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Pre-relativity mechanics

Special relativity has a lot to do with how different observers see physical phenomena. As we will see, the idea that, under certain conditions, the laws of physics should be *invariant* (that is, remain unchanged) for different observers plays a central role in many of the discussions we will have in this class. Indeed, *symmetry* and *invariance* are two of the deepest principles in physics, not only mechanics.

Pre-relativity (Newtonian) mechanics is based on the idea that the laws of mechanics should be the same in a frame of reference S' (Fig. 1) that moves at constant velocity V with respect to another reference frame S that is at rest.[1] Such frames are called *inertial reference frames*. Because defining what is *at rest* can be problematic (at rest with respect to what?), and to avoid circularity, we will define inertial reference frames as those where a free particle moves with uniform velocity when measured with the metersticks and clocks of your reference frame.

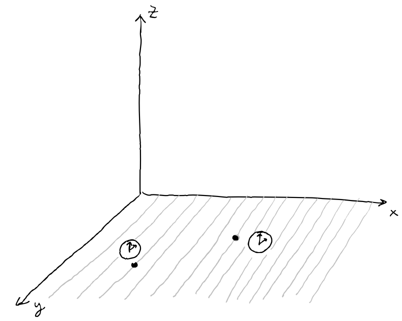


Figure 1: A reference frame (or frame of reference) is an abstract coordinate system whose origin, orientation, and scale are specified by a set of reference points. In special relativity we need to be precise about how positions and times are measured in a reference frame. In this sense, a reference frame can be thought of as a lattice of metersticks (represented as grey lines for the x axis) together with a set of synchronized clocks at every point.

[1] In fact, in pre-relativity physics, which reference frame is at rest and which one is moving is a matter of opinion *as long as we restrict ourselves to mechanics*, precisely because all the laws of mechanics are the same in both reference frames. As we will see, things could be different if we considered, for example, electromagnetism, which might be able to tell us which frame of reference is at rest and which one is moving at V.

Galilean frames of reference and transformations

So how do we measure space and time in pre-relativity mechanics? How do different observers in inertial reference frames describe the same points? Do they see, as we have postulated, the same laws of mechanics?

Consider two reference frames S and S', with S' moving at velocity V with respect to S. If it helps, you can think of S as the reference frame defined by a train station, and S' as the reference frame defined by a train that moves at constant velocity V with respect to the station. An observer in S (the station) will describe the position of a point X using coordinates (x, y, z). Similarly, an observer in S' (the train) will describe the position of the same point using coordinates (x', y', z'). Additionally, according to pre-relativity mechanics, both observers will assign the same times $t = t'$ to any event, that is, the clocks in S and S' will always be synchronized. For convenience, we arrange the axes of the reference frames so that the motion at velocity V happens along the x axis (Fig. 2). Since we are dealing with reference frames that move at constant velocity with respect to each other (and, of course, constant velocity implies constant direction) we can always do this without loss of generality.

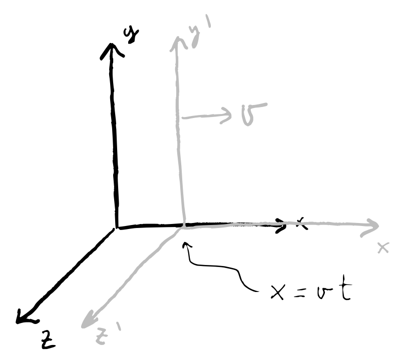


Figure 2: A frame S' moves with uniform velocity V along the x axis with respect to a *resting* frame S.

Now, consider two events A and B that happen in the train. For simplicity, both events happen at $y' = z' = 0$. From the perspective of an observer in the train S', A and B are separated spatially by a distance $\Delta x' = x'_B - x'_A$, and temporally by an interval $\Delta t' = t'_B - t'_A$. For an observer at the station S, the events are separated by an interval $\Delta t = \Delta t'$, because we have assumed that they all measure the same times. However the distance between events is transformed into

$$\Delta x = \Delta x' + V\Delta t' \tag{1}$$

because everything within the train is moving with it at velocity V. Eq. (1) together with $\Delta t = \Delta t'$ and the trivial $\Delta y = \Delta y' = \Delta z = \Delta z' = 0$ are the so-called *Galilean transformations*.

Invariance of Newton's laws under Galilean transformations

In the limit of very small intervals, the Galilean transformations can be written as

$$\begin{aligned} dt &= dt' \\ dx &= dx' + V dt' \end{aligned} \tag{2}$$

The principle of *Galilean invariance* tells us that the laws of mechanics should be invariant under Galilean transformations. Are Newton's laws invariant under these transformations?

From the differential form of the transformations in Eqs. (2), we see that velocities are transformed as

$$\frac{dx}{dt} = \frac{dx'}{dt} + V\frac{dt'}{dt} = \frac{dx'}{dt'} + V\ ,$$

and that accelerations are invariant under Galilean transformations

$$\frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx'}{dt'} + V\right) = \frac{d}{dt'}\left(\frac{dx'}{dt'} + V\right) = \frac{d^2x'}{dt'^2}\ .$$

Newton's laws are second order differential equations, and second derivatives of the position (accelerations) are invariant under Galilean transformations. Additionally, it is also plausible that forces felt by a particle do not depend on the inertial reference frame, so that $F = F'$. Therefore, **Newton's laws are, indeed, invariant under Galilean transformations**.

Non-invariance of electromagnetism and the speed of light

What about the laws of electromagnetism? Are they also invariant under Galilean transformations?

To start addressing this question, let's consider what happens to a ray of light as seen from different reference frames. In particular, let's go back to the moving train in Fig. 2, and imagine that a passenger on the train turns on a flashlight in the direction of the motion of the train. For this passenger, the speed of the light coming out of the flashlight is c'; what is the speed of light measured from the station S? Well, according the Galilean rule for transforming velocities, which we derived above, we have that

$$\frac{dx}{dt} = \frac{dx'}{dt'} + V\,.$$

Therefore, always according to this pre-relativity Newtonian/Galilean framework, the observer at rest at the station will see the light moving at speed $c = c' + V$. This simple and seemingly intuitive, almost trivial, result already spells trouble.

Let's see why. Consider Maxwell's equations, which, as you know, govern the behavior of electric $\vec{E}$ and magnetic $\vec{B}$ fields

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{E} + \partial_t\vec{B} = 0 \tag{3}$$
$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0} \qquad \vec{\nabla}\times\vec{B} - \mu_0\epsilon_0\partial_t\vec{E} = \mu_0\vec{j}\,. \tag{4}$$

These equations accept as a solution a wave that travels at speed $c = \sqrt{1/\epsilon_0\mu_0} = 3\times10^8\,\mathrm{m/s}$; "the speed of light is the speed of light." But as we have just seen, different inertial observers should see light traveling at different speeds! Taken together, these facts would suggest that Maxwell's equations only apply to a privileged reference frame (where $c = 3\times10^8\,\mathrm{m/s}$), but not to any other inertial frame that moves with respect to the privileged one. In other words, **in a Galilean world, the laws of electromagnetism, unlike the laws of mechanics, would *not* be the same in all inertial reference frames.**

Let's explore this more precisely. Again, let's assume that some scalar field $\phi'(x,t)$ satisfies the wave equation in the S' reference frame[2]:

$$\frac{\partial^2\phi'}{\partial x'^2} - \frac{1}{c'^2}\frac{\partial^2\phi'}{\partial t'^2}. \tag{5}$$

Now, what is the equation describing the same field ϕ as seen from from the station reference frame S? By applying the chain rule of derivation, we see that the differential operators transform as:

$$\begin{aligned}
\frac{\partial}{\partial x'} &= \frac{\partial}{\partial x}\\
\frac{\partial^2}{\partial x'^2} &= \frac{\partial^2}{\partial x^2}\\
\frac{\partial}{\partial t'} &= V\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\\
\frac{\partial^2}{\partial t'^2} &= V^2\frac{\partial^2}{\partial x^2} + 2V\frac{\partial^2}{\partial x\partial t} + \frac{\partial^2}{\partial t^2}
\end{aligned}$$

Additionally, the field measured at one point should be the same in all reference frames, so that $\phi = \phi'$.[3] Replacing all of these in Eq. (5)

[2] As you know from electromagnetism, the wave equation can be obtained by combining Maxwell's equations.

[3] The only difference is how we label the given point in each reference frame, but not *the value of the field there!*

we get the equation that the field obeys in the station S

$$\left(1-\frac{V^2}{c'^2}\right)\frac{\partial^2\phi}{\partial x^2}-\frac{1}{c'^2}\frac{\partial^2\phi}{\partial t^2}+\frac{2V}{c'^2}\frac{\partial^2\phi}{\partial x\partial t}=0\,,$$

which is **not a wave equation!**

Therefore, either by measuring the speed of light or by studying the physics of the propagation, one should be able to know if they are sitting in the privileged reference frame or not. To address this issue, physicists postulated the existence of a *luminiferous ether*, a medium that was supposed to be at rest at (or, rather, that defined) the privileged reference frame.

Michelson-Morley experiment

The Galilean dependence of the laws of electrodynamics on the reference frame was ugly enough to someone like Einstein. But, perhaps more compellingly for the rest of us, no one had ever been able to measure different speeds of light.

Willem de Sitter had pointed out a situation in which such differences should have been measurable, namely, in double-star systems. If light moves at a speed c with respect to the emitting object, light emitted from a star in a double-star system from different parts of the orbital path would travel towards us at different speeds (slower when the star is moving away from the Earth and faster when it is moving towards the Earth). Thus the laws that govern the motion of celestial bodies would apparently be distorted for a distant observer. None of this has ever been observed.

But perhaps the most elegant (and famous) experiment showing the invariance of the speed of light was carried out by Michelson and Morley in 1887. Specifically, their idea was to measure the speed of Earth with respect to the postulated ether.

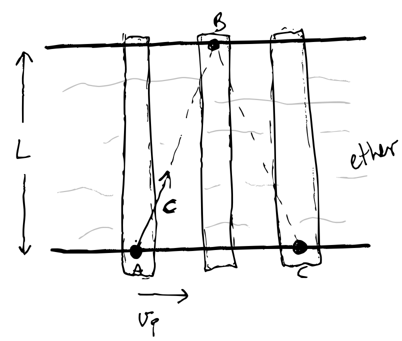


Figure 3: Perpendicular branch of the Michelson-Morley experiment.

Imagine two individuals sitting at the ends of a platform of length L that moves with velocity v_p through ether. Consider, first, that the platform moves in a direction that is perpendicular to itself (Fig. 3). The first individual emits a light ray (A); when the second individual receives the ray (B), she emits a light ray back to the her friend, who receives it some time later (C). Now, because the platform moves with respect to ether at v_p, the distance d_{AB} is longer than L. In fact, using similar triangles we get

$$\frac{d_{AB}}{L}=\frac{c}{\sqrt{c^2-v_p^2}}\implies d_{AB}=\frac{cL}{\sqrt{c^2-v_p^2}}\,.$$

Therefore, the time needed for the whole process AC is

$$t^{\text{perpendicular}}=\frac{2L}{\sqrt{c^2-v_p^2}}\,.$$

Now consider the simpler case, in which the movement at velocity v_p with respect to the ether is parallel to the platform itself. Now, it is as if the light traveled with velocity $c + v_p$ one way and $c - v_p$ on its way back (or vice versa), so the total time in this case would be

$$t^{\text{parallel}} = \frac{L}{c+v} + \frac{L}{c-v} = \frac{2Lc}{c^2 - v_p^2} \, .$$

The key observation from the above arguments is that $t^{\text{parallel}} \neq t^{\text{perpendicular}}$. Michelson and Morley set up an experiment in which a light beams is split so that it travels in two trajectories perpendicular to each other, which then recombine. The beams will interfere constructively or destructively, depending on whether the two light beams are in phase or out of phase when they recombine. This interference pattern is extremely delicate. The slightest change in travel times of the beams will cause the pattern to shift by an amount that can be measured. Of course, we don't know in which direction we would be moving with respect to the ether—for example, we could be moving in a direction 45° from both arms, in which case the times would be the same (by symmetry) and interference would be constructive; but by rotating the arms we should observe a shift in the interference pattern. However, when Michelson and Morley performed their experiment, they observed no interference shift as the apparatus was rotated around, which suggested that they were at rest with respect to ether. Even worse (or better), when they repeated the experiment half a year later, the result was exactly the same. So **there is no privileged reference frame, and the speed of light is the speed of light!**

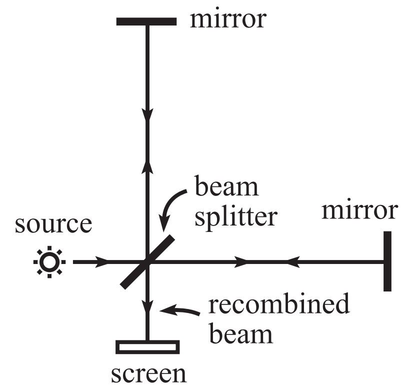


Figure 4: General setup of the Michelson-Morley experiment.

Postulates and consequences of special relativity

Postulates of special relativity

Special relativity assumes the following two postulates, which are consistent with the empirical evidence set forth by the Michelson-Morley experiment (as well as many other more recent experiments):

ALL LAWS OF PHYSICS are the same in any inertial reference frame. Unlike Newtonian mechanics, which only assumes that the laws of mechanics (but not electromagnetism, for example) are equivalent in all reference frames, special relativity assumes that all inertial reference frames are fully equivalent, so no physical experiment can tell, for example, whether one is at rest while the other is moving at a constant velocity with respect to the first. In fact, the expression "at rest" does not make sense any more; all motion is relative.

The speed of light c is a law of nature and therefore, by the first postulate, it is the same in any reference frame.

Minkowski diagrams and loss of simultaneity

We now explore the first striking and counter-intuitive consequence of the postulates of special relativity, namely, the fact that events that are simultaneous in one reference frame may not be simultaneous in another reference frame.

We do so by introducing a very useful tool in special relativity: the so-called Minkowski (or spacetime) diagram (Fig. 5).

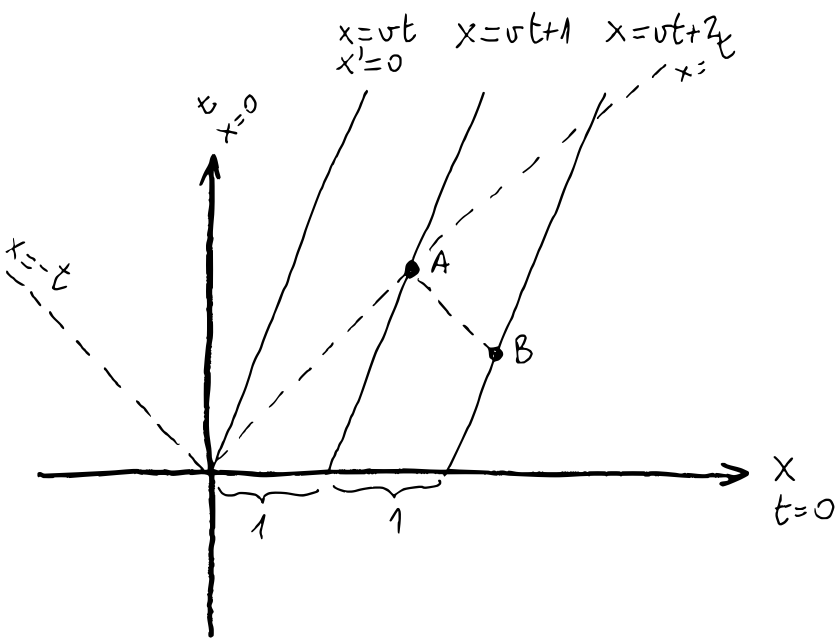


Figure 5: Minkowski diagram for three equally spaced points 0, 1, and 2 in S', which moves at velocity v with respect to S. Points 0 and 2 emit light rays towards 1 at $t' = 0$, which arrive at simultaneously at point A at t'_A. It is convenient (and thus, frequent) to use units such that $c = 1$, which is equivalent to considering that the temporal axis is ct, instead of just t.

We start by plotting the x and t axes according to reference frame S. Then we consider a frame S' that moves at constant velocity v with respect to S along the x axis. Next, we plot the location of three equally spaced points in S', which are separated a distance 1 (in $c = 1$ units) according to an observer in S. For convenience, we choose the first point to be at $x' = 0$ when $t' = 0$, and the origins of S and S' to coincide. The spacetime position of the three points in S is given by $x_0 = vt$, $x_1 = vt + 1$, and $x_2 = vt + 2$, respectively. Additionally, we note that light rays are also straight lines, with a slope $c = 1$ if they move towards the right and $-c = -1$ if they move towards the left.

Now, imagine that, at $t' = 0$ and moving with S', we emit light rays from points 0 and 2 towards 1. Because 1 is the midpoint between 0 and 2, and because $c = 1$ is the same in all directions, they will reach point 1 at $(x'_A = x'_1, t'_A)$ simultaneously. We call this event (both rays reaching x'_1) A.[4,5]

[4] From now on, we identify *events* with *spacetime locations* and use the concepts interchangeably.

[5] This, incidentally, is a way to synchronize clocks. If 0 and 2 emit at what they think is $t' = 0$ but the rays do not reach 1 at the same time, then the two clocks are not synchronized and need to be adjusted.

Now let's consider what we measure from the frame S. The light ray emitted from 0 will reach 1 when the lines $x = t$ (light ray seen from S, moving at $c = 1$) and $x = vt + 1$ (point 1 moving at v as seen from S) cross. That happens at $x_A = t_A = 1/(1 - v)$.

But for the light ray emitted from 2 to arrive at 1 at the same time as the one emitted from 0 (which has to happen also in S, because A is and *event*; same as if both rays were emitted *from* 1 towards 0 and 2), the former has to be emitted at B, which has $t_B \neq 0$. Therefore, and event that is simultaneous in S' (both rays were emitted simultaneously at $t' = 0$) is *not* simultaneous in S (one ray is emitted at $t = 0$ but the other at $t_B > 0$). Therefore, a first striking and counter-intuitive consequence of the postulates of special relativity is that *simultaneity is relative*, that is, that time is not the same in all frames of reference.

LET'S NOW COMPLETE the Minkowski diagram by adding to it the coordinate axes of S'. First, note that the x axis of the S frame (the horizontal axis) can be characterized as $t = 0$: the x axis is the set of points for which $t = 0$. Similarly, the t axis satisfies $x = 0$. Where are the t' (that is, $x' = 0$) and x' ($t' = 0$) axes in the Minkowski diagram? The first one is easy—the t' axis coincides with the trajectory of point 0 in the discussion above.

The x', $t' = 0$ axis is a bit more complicated but, in fact, we have already done most of the work. Indeed, we saw that, in S' both light rays are emitted at $t' = 0$. Therefore, the $t' = 0$ access must go through the origin (ray 0 emitted) and through B (ray 2 emitted). So all we need to do is to obtain the coordinates of B in S (Fig. 5).

We have already seen that $x_A = t_A = 1/(1 - v)$. Now, the line that goes through A and B has slope -1 and such that $x + t = \text{const}$. Because the line goes through A, we have that along this line $x + t = 2/(1 - v)$. B is the point where this line crosses $x = vt + 2$; by subtracting the second equation from the first and operating a bit we get $t_B = 2v/(1 - v^2)$. Finally, by replacing into either of the straight lines we get $x_B = 2/(1 - v^2)$. Therefore, the slope of the $t' = 0$ line is $t_B/x_B = v$ and we see that, in S, the x' axis corresponds simply to $t = vx$ (Fig 6). We see the symmetry between $x = vt$ for the t' axis ($x' = 0$) and $t = vx$ for the x' axis ($t' = 0$).[6]

[6] A word of **caution** on Minkowski diagrams. Minkowski diagrams are very useful to place events and quickly identify some relationships (for example, that in the x' axis we have $t' = 0$ and $t = vx/c^2$). However, we should *not* use Minkowski diagrams to calculate distances as in an Euclidean space, because spacetime is not Euclidean. More on this later.

The Lorentz transformation

We have already established, by means of the Minkowski diagram, how some points in the S' frame map into the S frame.[7] In particular,

[7] We continue to assume that the origins of S and S' coincide, and that S' moves with velocity v along the x axis with respect to S.

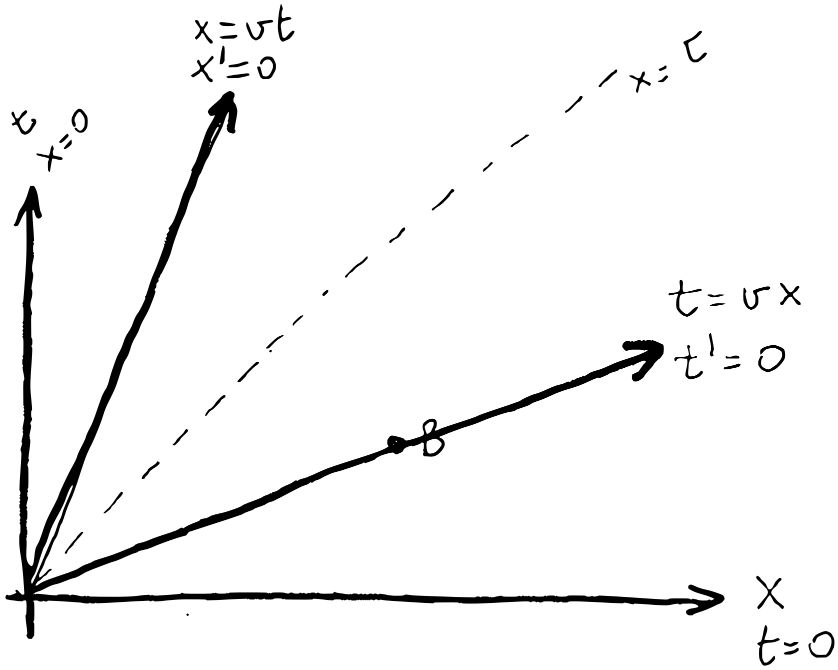


Figure 6: Minkowski diagram.

we have seen that the t' axis in S' ($x' = 0$) transforms into $x = vt$ in S; and that the x' axis in S' ($t' = 0$) transforms into $t = vx$ in S. Now, we will follow Einstein's arguments to see how all other points in S' transform into S.

We start from what we already know—again, that at $x' = 0$ we have $x = vt$. Additionally, the fact that spacetime is translationally invariant (that is, that we can place the origin anywhere we want without the equations of physics changing) implies that x' is linear in x. Altogether, the previous considerations imply that x' should be proportional to $x - vt$, just like the old Galilean transformations. We call the undetermined proportionality factor, which may depend on v, $f(v)$, so that

$$x' = (x - vt)f(v) \; . \tag{6}$$

Additionally, the proportionality factor $f(v)$ should not depend on whether we decide to make velocities positive towards the right or towards the left, so that we should have $f(v) = f(-v)$.

Similarly, we can surmise that

$$t' = (t - vx)g(v) \; , \tag{7}$$

where $g(v) = g(-v)$ is, in principle, different from $f(v)$.

Next, we use the second postulate of special relativity, namely, that the speed of light is the same in all reference frames. For a ray starting at the origin at $t = 0$, and with the units we have been using, the trajectory of the ray is given by $x = t$. But in S', the trajectory of the same ray is given by $x' = t'$. Imposing these, into the equations above we conclude that $f(v) = g(v)$.

FINALLY, WE USE SYMMETRY considerations to argue that, mutatis mutandi, we could have derived the following transformations to go from x' to x; the only difference is that, if S' moves towards the right with respect to S, then S moves towards the left with respect to S', so that

$$\begin{aligned} x &= (x' + vt')f(v)\,, \\ t &= (t' + vx')f(v)\,. \end{aligned} \tag{8}$$

By plugging Eq. (8) into Eq. (6), we conclude that the only acceptable form for $f(v)$ is

$$f(v) = \frac{1}{\sqrt{1-v^2}}\,,$$

so that we arrive at the so-called Lorentz transformations:

$$\begin{aligned} x' &= \frac{(x-vt)}{\sqrt{1-v^2}}\,, \\ t' &= \frac{(t-vx)}{\sqrt{1-v^2}}\,. \end{aligned} \tag{9}$$

Additionally, of course, since all the motion was assumed to happen along the x axis, we have that the other two space coordinates transform trivially as $y' = y$ and $z' = z$.

WE CAN NOW USE DIMENSIONAL ANALYSIS to reintroduce the missing c into the transformation, and get the more familiar form[8,9]

$$\begin{aligned} x' &= \frac{(x-vt)}{\sqrt{1-v^2/c^2}}\,, \\ y' &= y\,, \\ z' &= z\,, \\ t' &= \frac{(t-vx/c^2)}{\sqrt{1-v^2/c^2}}\,. \end{aligned} \tag{10}$$

[8] EXERCICE: Use dimensional analysis to reintroduce c. Prove that, in the limit $v \ll c$, the Lorentz transformation reduces to the good old Galilean transformation.

[9] Note that the transformation diverges when $v = c$ and gives stupid results when $v > c$. Thus, velocities larger that c are forbidden by the theory.

NOTATION. Besides using $c = 1$, it is common to use the following notation, which often makes math in special relativity more compact:

$$\begin{aligned} \vec{\beta} &= \frac{\vec{v}}{c}\,, \\ \gamma &= \frac{1}{\sqrt{1-v^2/c^2}} > 1\,. \end{aligned} \tag{11}$$

Additionally, since, as we see, space and time mix in special relativity, it is common to represent spacetime points using a single

4-vector[10] with greek indices x^μ, $\mu \in \{0,1,2,3\}$. Within this notation, the three components (for which we reserve Latin superscripts and arrow/vector indicators) x^i, $i \in \{1,2,3\}$, are the spatial (Galilean) *standard* position vectors, whereas $x^0 = ct$ is the temporal component.

[10] We will give a more precise definition of what 4-vectors are when we study dynamics in special relativity.

With this notation, the Lorentz transformation with $\vec{\beta}$ along the x axis takes a more compact form:

$$\begin{aligned} x^{0\prime} &= \gamma\left(x^0 - \beta x^1\right) , \\ x^{1\prime} &= \gamma\left(x^1 - \beta x^0\right) , \\ x^{2\prime} &= x^2 , \\ x^{3\prime} &= x^3 . \end{aligned} \tag{12}$$

THE LORENTZ TRANSFORMATION HAS A GEOMETRIC INTERPRETATION. Indeed, if we define $\beta = \tanh\phi$, the non-trivial part of Eq. (12) can be rewritten as[11]

[11] EXERCISE: Prove it.

$$\begin{aligned} x^{0\prime} &= x^0 \cosh\phi - x^1 \sinh\phi , \\ x^{1\prime} &= -x^0 \sinh\phi + x^1 \cosh\phi . \end{aligned} \tag{13}$$

This is reminiscent of a simple rotation in the plane, except for the hyperbolic functions replacing standard trigonometric functions and a change in sign. Indeed, this is a *hyperbolic rotation*. ϕ is referred to as the *rapidity*.

The general Lorentz transformation

From Eq. (13), for example, we see that the Lorentz transformation is a linear transformation of the coordinates *when the direction of motion is parallel to one of the spatial axes and the origins coincide*. What happens if we relax these two assumptions that we have been using so far? Or, in other words, what is the form of a *general* Lorentz transformation. A general Lorentz transformation is simply the composition of a translation (to align the origins), a spatial rotation[12] and a *simple* Lorentz transformation (such as the ones we have been discussing so far).

[12] Because a rotation is also a linear transformation, composing a rotation with a simple Lorentz transformation amounts simply to multiplying the corresponding *rotation* matrices.

Therefore, if we aim to prove, for example, that some law is invariant under a general Lorentz transformation, we only need to prove that it is invariant under translations, spatial rotations and the simple Lorentz transformation.

Length contraction

Next, we consider the second striking and counter-intuitive consequence of the postulates of special relativity—objects have different lengths in different reference frames, with lengths being shorter in any frame that moves with respect to the object.

Consider a bar of length L as measured in the reference frame S that is at rest with respect to the bar. What is the length L' measured from a frame S' that moves with velocity v with respect to S?[13] The situation is illustrated in Fig. 7.

[13] Measuring in S' means the following. All our meter sticks and synchronized clocks are arranged in S', and they all agree to measure the bar when $t' = 0$. At that instant (remember, they are all synchronized in S'), only one clock's position coincides with the beginning of the bar, and only one with the end. The length L' of the bar in S' is the difference between the locations of both clocks.

We are interested in calculating the length L' of the bar, which is given by point A. This point is at $x = L$ and, because it lays at $t' = 0$ (see note 13), we also have that $t = vx/c^2 = vL/c^2$. We then apply the Lorentz transformation with this conditions to get

$$L' = \frac{L - v^2 L/c^2}{\sqrt{1 - v^2/c^2}} = \frac{L}{\gamma} \tag{14}$$

Thus, indeed, since $\gamma > 1$ always, we have that the length in the moving frame S' is shorter than in the frame S in which the bar is at rest.[14]

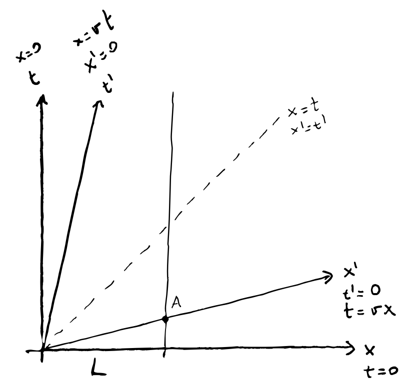
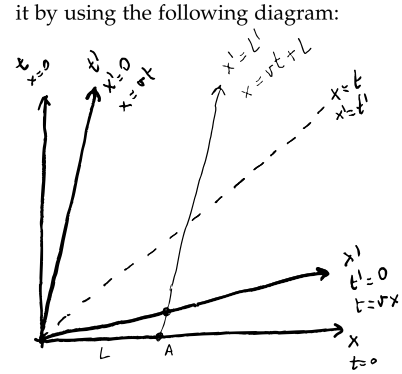


Figure 7: Minkowski diagram for the calculation of length contraction. A bar of length L sits at rest in S.

[14] Exercise: Prove that, if the bar is at rest in the moving frame S' (that is, it is moving with respect to us, who are at rest in S), then we have that $L = L'/\gamma$, so the bar is, again, shorter in the reference frame that moves with respect to the it. Do it by using the following diagram:



We end with a word of caution. Eq. 14 is only applicable to measures of length that involve the **simultaneous** measure of the two positions involved (beginning and end) in both reference frames. When two positions are measured at different times, the spacial distance between them in another reference frame should be calculated with the Lorentz transformation and, in general, will not coincide with Eq. (14).[15]

[15] Exercice: Prove it.

Time dilation

Conversely, the duration between two events is different in different reference frames, with the duration being longer in any frame that moves with respect to frame in which the events happen at the same point.

Consider two events happening at the same location $x = 0$ in the reference frame S. The first event happens at $t = 0$ and the second at $t = T$. In a frame S' that moves with v with respect to S, the first event will take place at $t' = 0$ (as always, we translate times as necessary for the origins to coincide) and the second at $t' = T'$. What is the value of T'?

Since both events happen at $x = 0$, we can immediately apply the Lorentz transformation to get

$$T' = \gamma \left(T - \beta x\right) = \gamma T > T \, . \tag{15}$$

Conversely, if the two events happen at $x' = 0$ and are separated by a time T' in the moving frame, the Lorentz transformations give us a duration T in the resting frame S

$$T = \gamma \left(T' + \beta x'\right) = \gamma T' > T' \, . \tag{16}$$

We end with a word of caution similar to the previous section. Eq. (16) is only applicable when the two events happen at the same point in one reference frame. Otherwise, the Lorentz transformation gives, in general, a different result.

Addition of velocities

We now consider how to add velocities in special relativity.

Longitudinal velocity addition. There is a train moving at constant v with respect to the train station. We call the respective reference frames S' (train) and S (station). Now, someone on the train throws a ball at velocity u parallel to and in the same direction as v. We assume that there is no gravity or any other force so that, once thrown, the ball keeps moving forward at constant velocity. What is the velocity w of the ball in the station frame S?

Pre-relativity, addition of velocities was straightforward—according to the Galilean transformations, the velocity of the ball measured in S would simply be $v + u$. However, we have already seen that this leads to violations of the second postulate of special relativity (the speed of light is always c). Indeed, if instead of a ball at velocity u we *threw* a light ray at velocity c, the Galilean rule leads to a velocity $c + v$ from S, which we know is incorrect.

Here we use the Lorentz transformation to calculate calculate w. First, we note that the velocity u in S' is given by the ratio between the spatial and time separation between two *events* in the trajectory of the ball, that is, $u = \Delta x'/\Delta t'$. Similarly, we can calculate the velocity in S using the same two events, but measured in S, that is, $w = \Delta x/\Delta t$. To map S and S' coordinates we use the Lorentz transformation with v:

$$\Delta x = \gamma_v(\Delta x' + v\Delta t') \quad \text{and} \quad \Delta t = \gamma_v(\Delta t' + v\Delta x'/c^2)\ . \tag{17}$$

It is then easy to obtain[16]

$$w = \frac{\Delta x}{\Delta t} = \frac{u + v}{1 + uv/c^2}\ . \tag{18}$$

Although it is easy to see that this expression behaves as expected in all the limits (in particular, $w = c$ when $u = c$), it may seem a bit awkward. Addition of velocities is much more intuitive using the rapidity introduced above. Indeed, if $u/c = \tanh\phi_u =$, $v/c = \tanh\phi_v =$, and $w/c = \tanh\phi_w =$, we have the simple addition rule $\phi_w = \phi_u + \phi_v$. [17]

Transverse velocity addition. Now consider that the ball inside the train is thrown, not exactly forward, but forward *and* upwards.

[16] Exercise: Do it.

[17] See this Twitter thread for a cool and graphical explanation of the relationship between velocity addition, rotations, and Lorentz transformations.

The forward component of the velocity with respect to the train is u_x; the upward component is u_y. The forward component transforms exactly like the longitudinal case that we have just seen

$$w_x = \frac{u_x + v}{1 + u_x v/c^2} \, .$$

What about w_y? For y we have $\Delta y = \Delta y'$ and $\Delta t = \gamma_v \left(\Delta t' + v\Delta x'/c^2\right)$, so that

$$w_y = \frac{\Delta y}{\Delta t} = \cdots = \frac{u_y}{\gamma_v \left(1 + u_x v/c^2\right)} \, .$$