Jerrold E. Marsden and Anthony J. Tromba

Vector Calculus

Fifth Edition

Chapter 2:
Differentiation

2.5 Properties of the Derivative

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Key Points in this Section.

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1. The ***constant multiple rule***, the ***sum rule***, ***product rule*** and ***quotient rule*** are all analogous to their counterparts in single-variable calculus.

2. The ***chain rule*** states that

$$\mathbf{D}(f \circ g)(\mathbf{x}_0) = \mathbf{D}f(\mathbf{y}_0)\mathbf{D}g(\mathbf{x}_0)$$

   where $g : U \subset \mathbb{R}^n \to \mathbb{R}^m$ and $f : V \subset \mathbb{R}^m \to \mathbb{R}^p$ are differentiable, with $g(U) \subset V$ so that the ***composition*** $f \circ g$ is defined and where $\mathbf{D}f(\mathbf{y}_0)\mathbf{D}g(\mathbf{x}_0)$ is the $p \times n$ matrix that is the product of the $p \times m$ matrix $\mathbf{D}f(\mathbf{y}_0)$ with the $m \times n$ matrix $\mathbf{D}g(\mathbf{x}_0)$.

3. Special cases of the chain rule are, firstly,

$$\frac{dh}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

where $h(t) = f(x(t), y(t), z(t))$ and secondly,

$$\frac{\partial h}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial x},$$

where $h(x, y, z) = f(u(x, y, z), v(x, y, z), w(x, y, z))$.

THEOREM 10: Sums, Products, Quotients

(i) **Constant Multiple Rule**. Let $f\colon U \subset \mathbb{R}^n \to \mathbb{R}^m$ be differentiable at $\mathbf{x}_0$ and let c be a real number. Then $h(\mathbf{x}) = cf(\mathbf{x})$ is differentiable at $\mathbf{x}_0$ and

$$\mathbf{D}h(\mathbf{x}_0) = c\mathbf{D}f(\mathbf{x}_0) \qquad \text{(equality of matrices).}$$

(ii) **Sum Rule**. Let $f\colon U \subset \mathbb{R}^n \to \mathbb{R}^m$ and $g\colon U \subset \mathbb{R}^n \to \mathbb{R}^m$ be differentiable at $\mathbf{x}_0$. Then $h(\mathbf{x}) = f(\mathbf{x}) + g(\mathbf{x})$ is differentiable at $\mathbf{x}_0$ and

$$\mathbf{D}h(\mathbf{x}_0) = \mathbf{D}f(\mathbf{x}_0) + \mathbf{D}g(\mathbf{x}_0) \qquad \text{(sum of matrices).}$$

(iii) **Product Rule**. Let $f\colon U \subset \mathbb{R}^n \to \mathbb{R}$ and $g\colon U \subset \mathbb{R}^n \to \mathbb{R}$ be differentiable at $\mathbf{x}_0$ and let $h(\mathbf{x}) = g(\mathbf{x})f(\mathbf{x})$. Then $h\colon U \subset \mathbb{R}^n \to \mathbb{R}$ is differentiable at $\mathbf{x}_0$ and

$$\mathbf{D}h(\mathbf{x}_0) = g(\mathbf{x}_0)\mathbf{D}f(\mathbf{x}_0) + f(\mathbf{x}_0)\mathbf{D}g(\mathbf{x}_0).$$

(Note that each side of this equation is a $1 \times n$ matrix; a more general product rule is presented in Exercise 29 at the end of this section.)

(iv) **Quotient Rule**. With the same hypotheses as in rule (iii), let $h(\mathbf{x}) = f(\mathbf{x})/g(\mathbf{x})$ and suppose g is never zero on U. Then h is differentiable at $\mathbf{x}_0$ and

$$\mathbf{D}h(\mathbf{x}_0) = \frac{g(\mathbf{x}_0)\mathbf{D}f(\mathbf{x}_0) - f(\mathbf{x}_0)\mathbf{D}g(\mathbf{x}_0)}{[g(\mathbf{x}_0)]^2}.$$

Proof of Sum rule

(ii) By the triangle inequality, we may write

$$\frac{\| h(\mathbf{x}) - h(\mathbf{x}_0) - [\mathbf{D}f(\mathbf{x}_0) + \mathbf{D}g(\mathbf{x}_0)](\mathbf{x} - \mathbf{x}_0) \|}{\|\mathbf{x} - \mathbf{x}_0\|}$$
$$= \frac{\| f(\mathbf{x}) - f(\mathbf{x}_0) - [\mathbf{D}f(\mathbf{x}_0)](\mathbf{x} - \mathbf{x}_0) + g(\mathbf{x}) - g(\mathbf{x}_0) - [\mathbf{D}g(\mathbf{x}_0)](\mathbf{x} - \mathbf{x}_0) \|}{\|\mathbf{x} - \mathbf{x}_0\|}$$
$$\leq \frac{\| f(\mathbf{x}) - f(\mathbf{x}_0) - [\mathbf{D}f(\mathbf{x}_0)](\mathbf{x} - \mathbf{x}_0) \|}{\|\mathbf{x} - \mathbf{x}_0\|} + \frac{\| g(\mathbf{x}) - g(\mathbf{x}_0) - [\mathbf{D}g(\mathbf{x}_0)](\mathbf{x} - \mathbf{x}_0) \|}{\|\mathbf{x} - \mathbf{x}_0\|},$$

and each term approaches 0 as $\mathbf{x} \to \mathbf{x}_0$. Hence, rule (ii) holds. ■

example

$$f(x, y, z) = x^2 + y^2 + z^2 \text{ and } g(x, y, z) = x^2 + 1.$$

$$h(x, y, z) = \frac{x^2 + y^2 + z^2}{x^2 + 1}$$

example

$$f(x, y, z) = x^2 + y^2 + z^2 \text{ and } g(x, y, z) = x^2 + 1.$$

$$h(x, y, z) = \frac{x^2 + y^2 + z^2}{x^2 + 1}$$

$$\mathbf{D}h(x, y, z) = \left[\frac{\partial h}{\partial x}, \frac{\partial h}{\partial y}, \frac{\partial h}{\partial z}\right] = \left[\frac{(x^2+1)2x - (x^2+y^2+z^2)2x}{(x^2+1)^2}, \frac{2y}{x^2+1}, \frac{2z}{x^2+1}\right]$$

$$= \left[\frac{2x(1-y^2-z^2)}{(x^2+1)^2}, \frac{2y}{x^2+1}, \frac{2z}{x^2+1}\right].$$

$$\mathbf{D}h = \frac{g\mathbf{D}f - f\mathbf{D}g}{g^2} = \frac{(x^2+1)[2x, 2y, 2z] - (x^2+y^2+z^2)[2x, 0, 0]}{(x^2+1)^2}$$

THEOREM 11: Chain Rule Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be open sets. Let $g: U \subset \mathbb{R}^n \to \mathbb{R}^m$ and $f: V \subset \mathbb{R}^m \to \mathbb{R}^p$ be given functions such that g maps U into V, so that $f \circ g$ is defined. Suppose g is differentiable at $\mathbf{x}_0$ and f is differentiable at $\mathbf{y}_0 = g(\mathbf{x}_0)$. Then $f \circ g$ is differentiable at $\mathbf{x}_0$ and

$$\mathbf{D}(f \circ g)(\mathbf{x}_0) = \mathbf{D}f(\mathbf{y}_0)\mathbf{D}g(\mathbf{x}_0). \tag{1}$$

The right-hand side is the matrix product of $\mathbf{D}f(\mathbf{y}_0)$ with $\mathbf{D}g(\mathbf{x}_0)$.

$\boldsymbol{D}f(\boldsymbol{y}_0)$ matriu $p \times m$
$\boldsymbol{D}g(\boldsymbol{x}_0)$ matriu $m \times n$
$\boldsymbol{D}(f \circ g)(\boldsymbol{x}_0)$ matriu $p \times n$

First special case of the Chain Rule

Suppose $\mathbf{c}\colon \mathbb{R} \to \mathbb{R}^3$ is a differentiable path and $f\colon \mathbb{R}^3 \to \mathbb{R}$. Let $h(t) = f(\mathbf{c}(t)) = f(x(t), y(t), z(t))$, where $\mathbf{c}(t) = (x(t), y(t), z(t))$. Then

$$\frac{dh}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}. \tag{2}$$

That is,

$$\frac{dh}{dt} = \nabla f(\mathbf{c}(t)) \cdot \mathbf{c}'(t),$$

where $\mathbf{c}'(t) = (x'(t), y'(t), z'(t))$.

Proof

By definition,

$$\frac{dh}{dt}(t_0) = \underset{t \to t_0}{\text{limit}} \frac{h(t) - h(t_0)}{t - t_0}.$$

Adding and subtracting two terms, we write

$$\begin{aligned}
\frac{h(t) - h(t_0)}{t - t_0} &= \frac{f(x(t), y(t), z(t)) - f(x(t_0), y(t_0), z(t_0))}{t - t_0} \\
&= \frac{f(x(t), y(t), z(t)) - f(x(t_0), y(t), z(t))}{t - t_0} \\
&\quad + \frac{f(x(t_0), y(t), z(t)) - f(x(t_0), y(t_0), z(t))}{t - t_0} \\
&\quad + \frac{f(x(t_0), y(t_0), z(t)) - f(x(t_0), y(t_0), z(t_0))}{t - t_0}.
\end{aligned}$$

Now we invoke the *mean-value theorem* from one-variable calculus, which states: *If* $g \colon [a, b] \to \mathbb{R}$ *is continuous and is differentiable on the open interval* (a, b), *then there is a point* c *in* (a, b) *such that* $g(b) - g(a) = g'(c)(b - a)$. Applying this to f as a function of x, we can assert that for some c between x and x_0,

$$f(x, y, z) - f(x_0, y, z) = \left[\frac{\partial f}{\partial x}(c, y, z)\right](x - x_0).$$

Proof

In this way, we find that

$$\frac{h(t)-h(t_0)}{t-t_0} = \left[\frac{\partial f}{\partial x}(c, y(t), z(t))\right]\frac{x(t)-x(t_0)}{t-t_0} + \left[\frac{\partial f}{\partial y}(x(t_0), d, z(t))\right]\frac{y(t)-y(t_0)}{t-t_0}$$
$$+ \left[\frac{\partial f}{\partial z}(x(t_0), y(t_0), e)\right]\frac{z(t)-z(t_0)}{t-t_0},$$

where c, d, and e lie between $x(t)$ and $x(t_0)$, between $y(t)$ and $y(t_0)$, and between $z(t)$ and $z(t_0)$, respectively. Taking the limit $t \to t_0$, using the continuity of the partials $\partial f/\partial x$, $\partial f/\partial y$, $\partial f/\partial z$, and the fact that c, d, and e converge to $x(t_0)$, $y(t_0)$, and $z(t_0)$, respectively, we obtain formula (2). ■

$$h(t) = f\left(x_1(t), \ldots, x_m(t)\right)$$

$$\frac{dh}{dt} = \sum_{k=1}^{m} \frac{\partial f}{\partial x_k} \frac{dx_k}{dt}$$

Second special case of the Chain Rule

Let $f\colon \mathbb{R}^3 \to \mathbb{R}$ and let $g\colon \mathbb{R}^3 \to \mathbb{R}^3$. Write

$$g(x, y, z) = (u(x, y, z), v(x, y, z), w(x, y, z))$$

and define $h\colon \mathbb{R}^3 \to \mathbb{R}$ by setting

$$h(x, y, z) = f(u(x, y, z), v(x, y, z), w(x, y, z)).$$

In this case, the chain rule states that

$$\begin{bmatrix} \dfrac{\partial h}{\partial x} & \dfrac{\partial h}{\partial y} & \dfrac{\partial h}{\partial z} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial f}{\partial u} & \dfrac{\partial f}{\partial v} & \dfrac{\partial f}{\partial w} \end{bmatrix} \begin{bmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial z} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial z} \\ \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} & \dfrac{\partial w}{\partial z} \end{bmatrix}.$$

$$h(\boldsymbol{x}) = f\big(u_1(\boldsymbol{x}), \dots, u_m(\boldsymbol{x})\big)$$

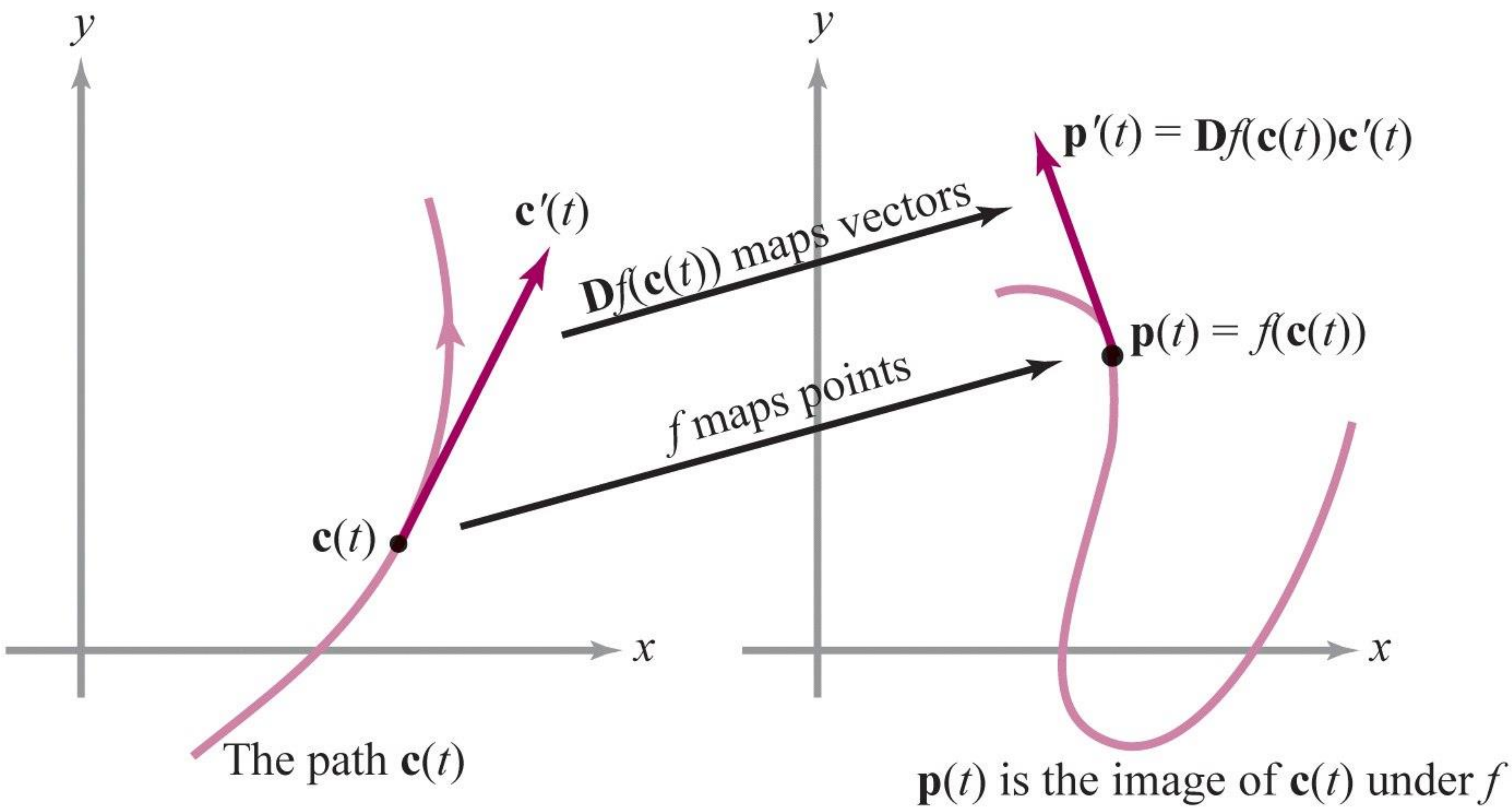
$$\frac{\partial h}{\partial x_i} = \sum_{k=1}^{m} \frac{\partial f}{\partial u_k} \frac{\partial u_k}{\partial x_i}$$

General case of the Chain Rule

$$h(\boldsymbol{x}) = \Big(f_1\big(u_1(\boldsymbol{x}), \dots, u_m(\boldsymbol{x})\big), \dots, f_p\big(u_1(\boldsymbol{x}), \dots, u_m(\boldsymbol{x})\big)\Big)$$

$$\frac{\partial h_j}{\partial x_i} = \sum_{k=1}^{m} \frac{\partial f_j}{\partial u_k} \frac{\partial u_k}{\partial x_i}$$

Geometric interpretation of Chain Rule for $\boldsymbol{p}(t) = f(\boldsymbol{c}(t))$



example

$$f(u, v, w) = u^2 + v^2 - w$$

$$u(x, y, z) = x^2 y, \qquad v(x, y, z) = y^2, \qquad w(x, y, z) = e^{-xz}.$$

$$h(x, y, z) = f(u(x, y, z), v(x, y, z), w(x, y, z))$$

$$\frac{\partial h}{\partial x}$$

example

$$f(u, v, w) = u^2 + v^2 - w$$

$$u(x, y, z) = x^2 y, \qquad v(x, y, z) = y^2, \qquad w(x, y, z) = e^{-xz}.$$

$$h(x, y, z) = f(u(x, y, z), v(x, y, z), w(x, y, z))$$

Opció 1

$$\begin{aligned} h(x, y, z) &= f(u(x, y, z), v(x, y, z), w(x, y, z)) \\ &= (x^2 y)^2 + y^4 - e^{-xz} = x^4 y^2 + y^4 - e^{-xz} \end{aligned}$$

$$\frac{\partial h}{\partial x} = 4x^3 y^2 + z e^{-xz}$$

example

$$f(u, v, w) = u^2 + v^2 - w$$

$$u(x, y, z) = x^2 y, \qquad v(x, y, z) = y^2, \qquad w(x, y, z) = e^{-xz}.$$

$$h(x, y, z) = f(u(x, y, z), v(x, y, z), w(x, y, z))$$

Opció 2

$$\frac{\partial h}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial x} = 2u(2xy) + 2v \cdot 0 + (-1)(-ze^{-xz})$$
$$= (2x^2 y)(2xy) + ze^{-xz} = 4x^3 y^2 + ze^{-xz}$$

example

Given $g(x, y) = (x^2 + 1, y^2)$ and $f(u, v) = (u + v, u, v^2)$, compute the derivative of $f \circ g$ at the point $(x, y) = (1, 1)$ using the chain rule.

example

Given $g(x, y) = (x^2 + 1, y^2)$ and $f(u, v) = (u + v, u, v^2)$, compute the derivative of $f \circ g$ at the point $(x, y) = (1, 1)$ using the chain rule.

$$\mathbf{D}f(u, v) = \begin{bmatrix} \dfrac{\partial f_1}{\partial u} & \dfrac{\partial f_1}{\partial v} \\ \dfrac{\partial f_2}{\partial u} & \dfrac{\partial f_2}{\partial v} \\ \dfrac{\partial f_3}{\partial u} & \dfrac{\partial f_3}{\partial v} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 2v \end{bmatrix} \quad \text{and} \quad \mathbf{D}g(x, y) = \begin{bmatrix} 2x & 0 \\ 0 & 2y \end{bmatrix}$$

When $(x, y) = (1, 1)$, note that $g(x, y) = (u, v) = (2, 1)$. Hence,

$$\mathbf{D}(f \circ g)(1, 1) = \mathbf{D}f(2, 1)\mathbf{D}g(1, 1) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 0 \\ 0 & 4 \end{bmatrix}$$