Exercicis

3. Verify the first special case of the chain rule for the composition $f \circ \mathbf{c}$ in each of the cases:

(a) $f(x, y) = xy$, $\mathbf{c}(t) = (e^t, \cos t)$

(b) $f(x, y) = e^{xy}$, $\mathbf{c}(t) = (3t^2, t^3)$

(c) $f(x, y) = (x^2 + y^2) \log \sqrt{x^2 + y^2}$, $\mathbf{c}(t) = (e^t, e^{-t})$

(d) $f(x, y) = x \exp(x^2 + y^2)$, $\mathbf{c}(t) = (t, -t)$

6. Let $f\colon \mathbb{R}^3 \to \mathbb{R}$ be differentiable. Making the substitution

$$x = \rho \cos\theta \sin\phi, \qquad y = \rho \sin\theta \sin\phi, \qquad z = \rho \cos\phi$$

(spherical coordinates) into $f(x, y, z)$, compute $\partial f/\partial \rho$, $\partial f/\partial \theta$, and $\partial f/\partial \phi$ in terms of $\partial f/\partial x$, $\partial f/\partial y$, and $\partial f/\partial z$.

7. Let $f(u, v) = (\tan(u - 1) - e^v, u^2 - v^2)$ and $g(x, y) = (e^{x-y}, x - y)$. Calculate $f \circ g$ and $\mathbf{D}(f \circ g)(1, 1)$.

11. Let $f(x, y, z) = (3y + 2, x^2 + y^2, x + z^2)$. Let $\mathbf{c}(t) = (\cos(t), \sin(t), t)$.

(a) Find the path $\mathbf{p} = f \circ \mathbf{c}$ and the velocity vector $\mathbf{p}'(\pi)$.

(b) Find $\mathbf{c}(\pi)$, $\mathbf{c}'(\pi)$ and $\mathbf{D}f(-1, 0, \pi)$.

(c) Thinking of $\mathbf{D}f(-1, 0, \pi)$ as a linear map, find $\mathbf{D}f(-1, 0, \pi)\,(\mathbf{c}'(\pi))$.

13. Suppose that a duck is swimming in the circle $x = \cos t$, $y = \sin t$ and that the water temperature is given by the formula $T = x^2 e^y - xy^3$. Find dT/dt, the rate of change in temperature the duck might feel: (a) by the chain rule; (b) by expressing T in terms of t and differentiating.

15. Let $f: \mathbb{R}^2 \to \mathbb{R}^2; (x, y) \mapsto (e^{x+y}, e^{x-y})$. Let $\mathbf{c}(t)$ be a path with $\mathbf{c}(0) = (0, 0)$ and $\mathbf{c}'(0) = (1, 1)$. What is the tangent vector to the image of $\mathbf{c}(t)$ under f at $t = 0$?

16. Let $f(x, y) = 1/\sqrt{x^2 + y^2}$. Compute $\nabla f(x, y)$.

17. Write out the chain rule for each of the following functions and justify your answer in each case using Theorem 11.

(a) $\partial h/\partial x$, where $h(x, y) = f(x, u(x, y))$

(b) dh/dx, where $h(x) = f(x, u(x), v(x))$

(c) $\partial h/\partial x$, where $h(x, y, z) = f(u(x, y, z), v(x, y), w(x))$

21. Dieterici's equation of state for a gas is

$$P(V - b)\, e^{a/RVT} = RT,$$

where a, b, and R are constants. Regard volume V as a function of temperature T and pressure P and prove that

$$\frac{\partial V}{\partial T} = \left(R + \frac{a}{TV}\right) \Big/ \left(\frac{RT}{V - b} - \frac{a}{V^2}\right).$$

33. Let $f: \mathbb{R}^4 \to \mathbb{R}$ and $\mathbf{c}(t): \mathbb{R} \to \mathbb{R}^4$. Suppose $\nabla f(1, 1, \pi, e^6) = (0, 1, 3, -7)$, $\mathbf{c}(\pi) = (1, 1, \pi, e^6)$, and $\mathbf{c}'(\pi) = (19, 11, 0, 1)$. Find $\dfrac{d(f \circ \mathbf{c})}{dt}$ when $t = \pi$.

34. Suppose $f: \mathbb{R}^n \to \mathbb{R}^m$ and $g: \mathbb{R}^p \to \mathbb{R}^q$.

(a) What must be true about the numbers n, m, p, and q for $f \circ g$ to make sense?

(b) What must be true about the numbers n, m, p, and q for $g \circ f$ to make sense?

(c) When does $f \circ f$ make sense?

35. If $z = f(x - y)$, use the chain rule to show that $\dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y} = 0$.