Jerrold E. Marsden and Anthony J. Tromba

Vector Calculus

Fifth Edition

Chapter 2:
Differentiation

2.6 Gradients and Directional Derivatives

2.6 Gradients and Directional Derivatives

Key Points in this Section.

1. The***gradient*** of a differentiable function $f : U \subset \mathbb{R}^3 \to \mathbb{R}$ is

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}.$$

2. The ***directional derivative*** of f in the direction of a *unit* vector $\mathbf{v}$ at the point $\mathbf{x}$ is

$$\left.\frac{d}{dt} f(\mathbf{x} + t\mathbf{v})\right|_{t=0} = \nabla f(\mathbf{x}) \cdot \mathbf{v}$$

3. The direction in which f is *increasing the fastest* at $\mathbf{x}$ is the direction parallel to $\nabla f(\mathbf{x})$. The direction of fastest *decrease* is parallel to $-\nabla f(\mathbf{x})$.

4. For $f : U \subset \mathbb{R}^3 \to \mathbb{R}$ a C^1 function, with $\nabla f(x_0, y_0, z_0) \neq \mathbf{0}$, the vector $\nabla f(x_0, y_0, z_0)$ is perpendicular to the level set $f(x, y, z) = f(x_0, y_0, z_0)$. Thus, the ***tangent plane*** to this level set is

$$\nabla f(x_0, y_0, z_0) \cdot (x - x_0, y - y_0, z - z_0) = 0.$$

5. The gravitational force field

$$\mathbf{F} = -\frac{GMm}{r^3}\mathbf{r} = -\frac{GMm}{r^2}\mathbf{n}$$

(the inverse square law), where $\mathbf{n} = \mathbf{r}/r$, $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $r = \|\mathbf{r}\|$, is a gradient. Namely,

$$\mathbf{F} = -\nabla V,$$

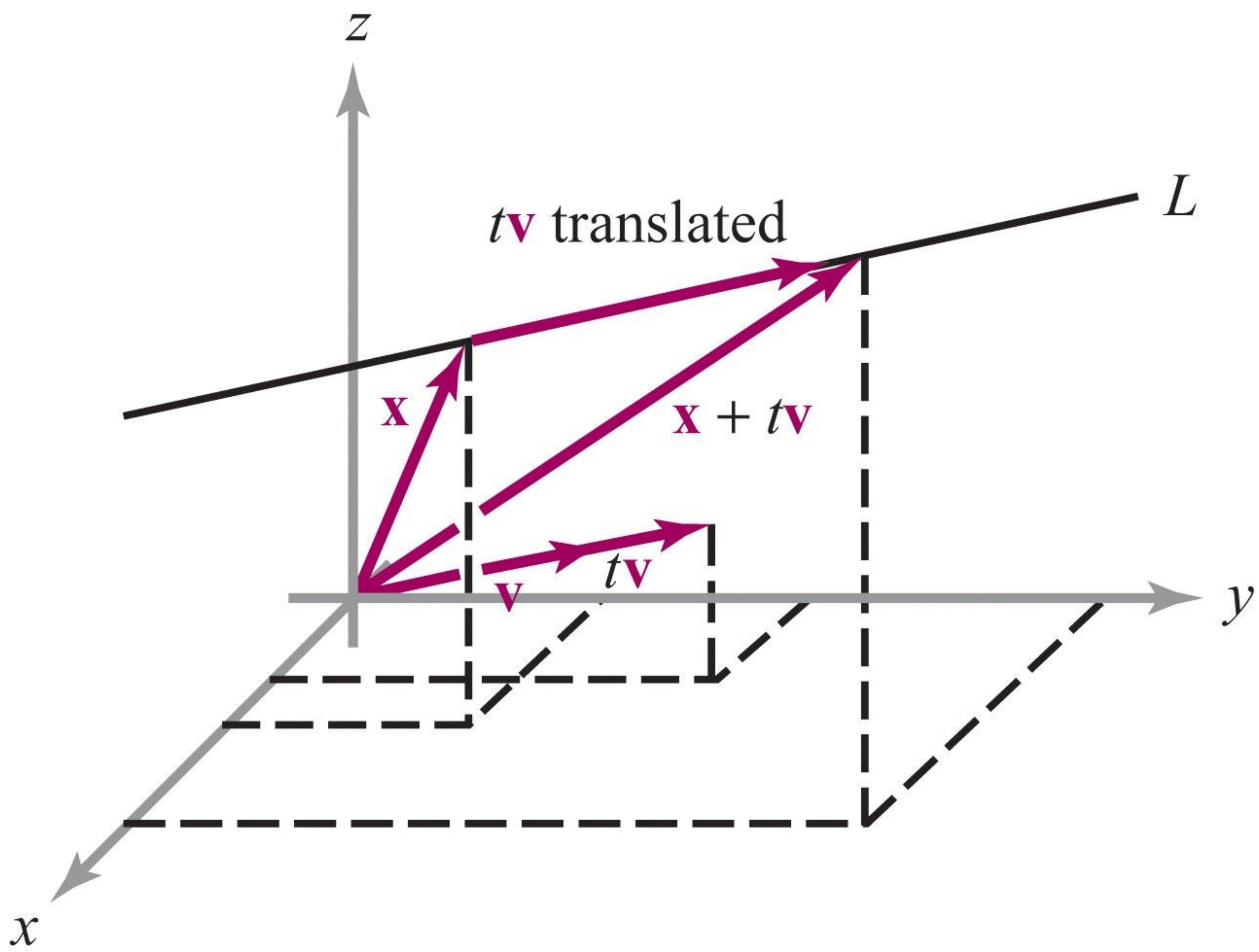
where

$$V = -\frac{GMm}{r}.$$

DEFINITION: The Gradient If $f : U \subset \mathbb{R}^3 \to \mathbb{R}$ is differentiable, the ***gradient*** of f at (x, y, z) is the vector in space given by

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right).$$

This vector is also denoted $\nabla f(x, y, z)$. Thus, ∇f is just the matrix of the derivative $\mathbf{D}f$, written as a vector.



z
L
tv translated
x
x + tv
tv
v
y
x

DEFINITION: Directional Derivatives If $f\colon \mathbb{R}^3 \to \mathbb{R}$, the ***directional derivative*** of f at $\mathbf{x}$ along the vector $\mathbf{v}$ is given by

$$\frac{d}{dt} f(\mathbf{x} + t\mathbf{v})\bigg|_{t=0}$$

if this exists.

In the definition of a directional derivative, we normally choose $\mathbf{v}$ to be a *unit* vector. In this case we are moving in the direction $\mathbf{v}$ with unit speed and we refer to $\nabla f(\mathbf{x}) \cdot \mathbf{v}$ as the ***directional derivative of f in the direction*** $\mathbf{v}$.

THEOREM 12 If $f\colon \mathbb{R}^3 \to \mathbb{R}$ is differentiable, then all directional derivatives exist. The directional derivative at $\mathbf{x}$ in the direction $\mathbf{v}$ is given by

$$\mathbf{D}f(\mathbf{x})\mathbf{v} = \operatorname{grad} f(\mathbf{x}) \cdot \mathbf{v} = \nabla f(\mathbf{x}) \cdot \mathbf{v} = \left[\frac{\partial f}{\partial x}(\mathbf{x})\right] v_1 + \left[\frac{\partial f}{\partial y}(\mathbf{x})\right] v_2 + \left[\frac{\partial f}{\partial z}(\mathbf{x})\right] v_3,$$

where $\mathbf{v} = (v_1, v_2, v_3)$.

THEOREM 13 Assume $\nabla f(\mathbf{x}) \neq \mathbf{0}$. Then $\nabla f(\mathbf{x})$ points in the direction along which f is increasing the fastest.

THEOREM 13 Assume $\nabla f(\mathbf{x}) \neq \mathbf{0}$. Then $\nabla f(\mathbf{x})$ points in the direction along which f is increasing the fastest.

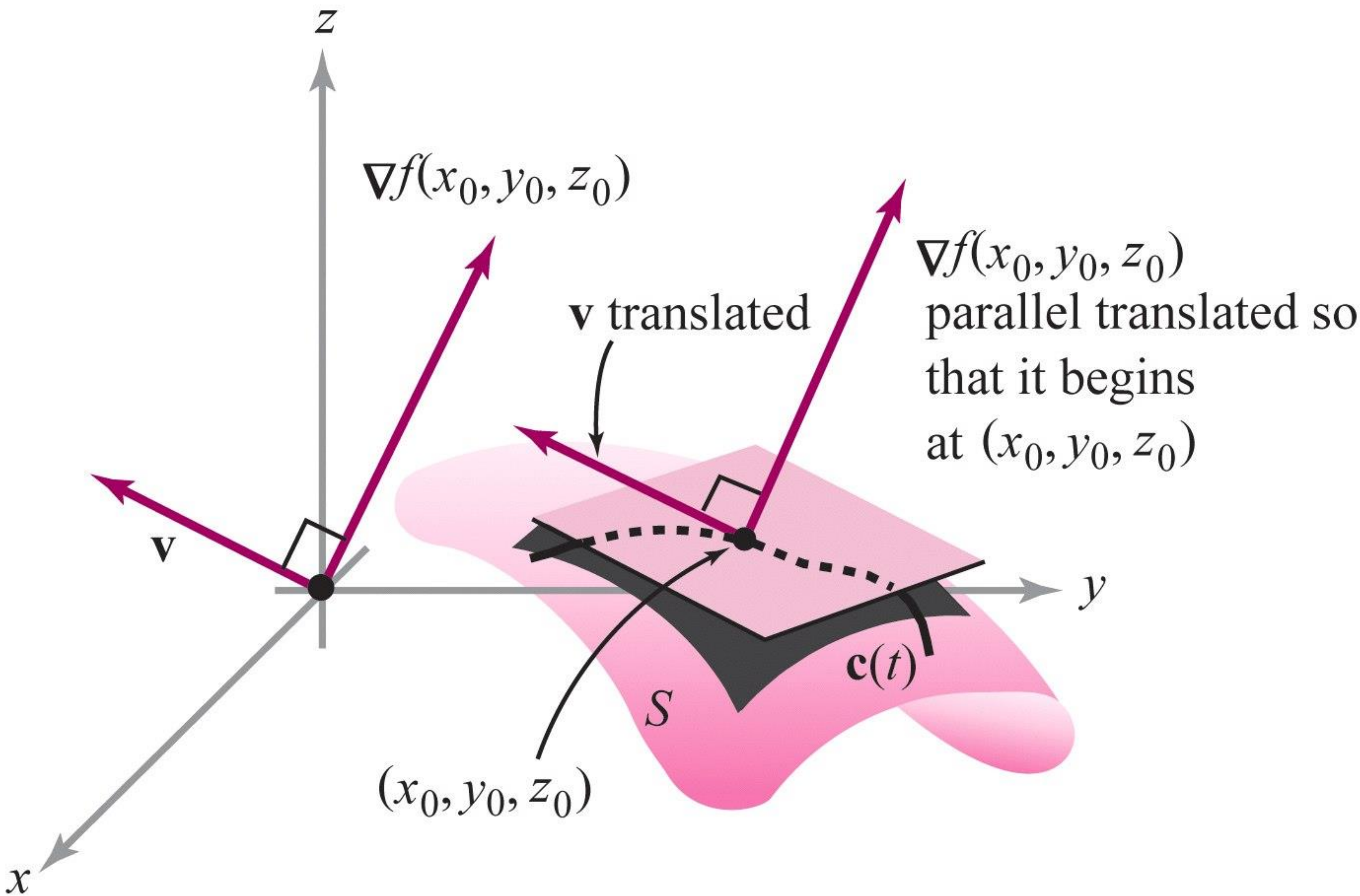
proof If $\mathbf{n}$ is a unit vector, the rate of change of f in direction $\mathbf{n}$ is given by $\nabla f(\mathbf{x}) \cdot \mathbf{n} = \|\nabla f(\mathbf{x})\| \cos\theta$, where θ is the angle between $\mathbf{n}$ and $\nabla f(\mathbf{x})$. This is maximum when $\theta = 0$; that is, when $\mathbf{n}$ and ∇f are parallel. [If $\nabla f(\mathbf{x}) = \mathbf{0}$ this rate of change is 0 for any $\mathbf{n}$.] ∎

THEOREM 14: The Gradient is Normal to Level Surfaces Let $f\colon \mathbb{R}^3 \to \mathbb{R}$ be a C^1 map and let (x_0, y_0, z_0) lie on the level surface S defined by $f(x, y, z) = k$, for k a constant. Then $\nabla f(x_0, y_0, z_0)$ is normal to the level surface in the following sense: If $\mathbf{v}$ is the tangent vector at $t = 0$ of a path $\mathbf{c}(t)$ in S with $\mathbf{c}(0) = (x_0, y_0, z_0)$, then $\nabla f(x_0, y_0, z_0) \cdot \mathbf{v} = 0$ (see Figure 2.6.2).

THEOREM 14: The Gradient is Normal to Level Surfaces Let $f\colon \mathbb{R}^3 \to \mathbb{R}$ be a C^1 map and let (x_0, y_0, z_0) lie on the level surface S defined by $f(x, y, z) = k$, for k a constant. Then $\nabla f(x_0, y_0, z_0)$ is normal to the level surface in the following sense: If $\mathbf{v}$ is the tangent vector at $t = 0$ of a path $\mathbf{c}(t)$ in S with $\mathbf{c}(0) = (x_0, y_0, z_0)$, then $\nabla f(x_0, y_0, z_0) \cdot \mathbf{v} = 0$ (see Figure 2.6.2).

proof Let $\mathbf{c}(t)$ lie in S; then $f(\mathbf{c}(t)) = k$. Let $\mathbf{v}$ be as in the hypothesis; then $\mathbf{v} = \mathbf{c}'(0)$. Hence, the fact that $f(\mathbf{c}(t))$ is constant in t, and the chain rule give

$$0 = \left.\frac{d}{dt} f(\mathbf{c}(t))\right|_{t=0} = \nabla f(\mathbf{c}(0)) \cdot \mathbf{v}. \qquad \blacksquare$$

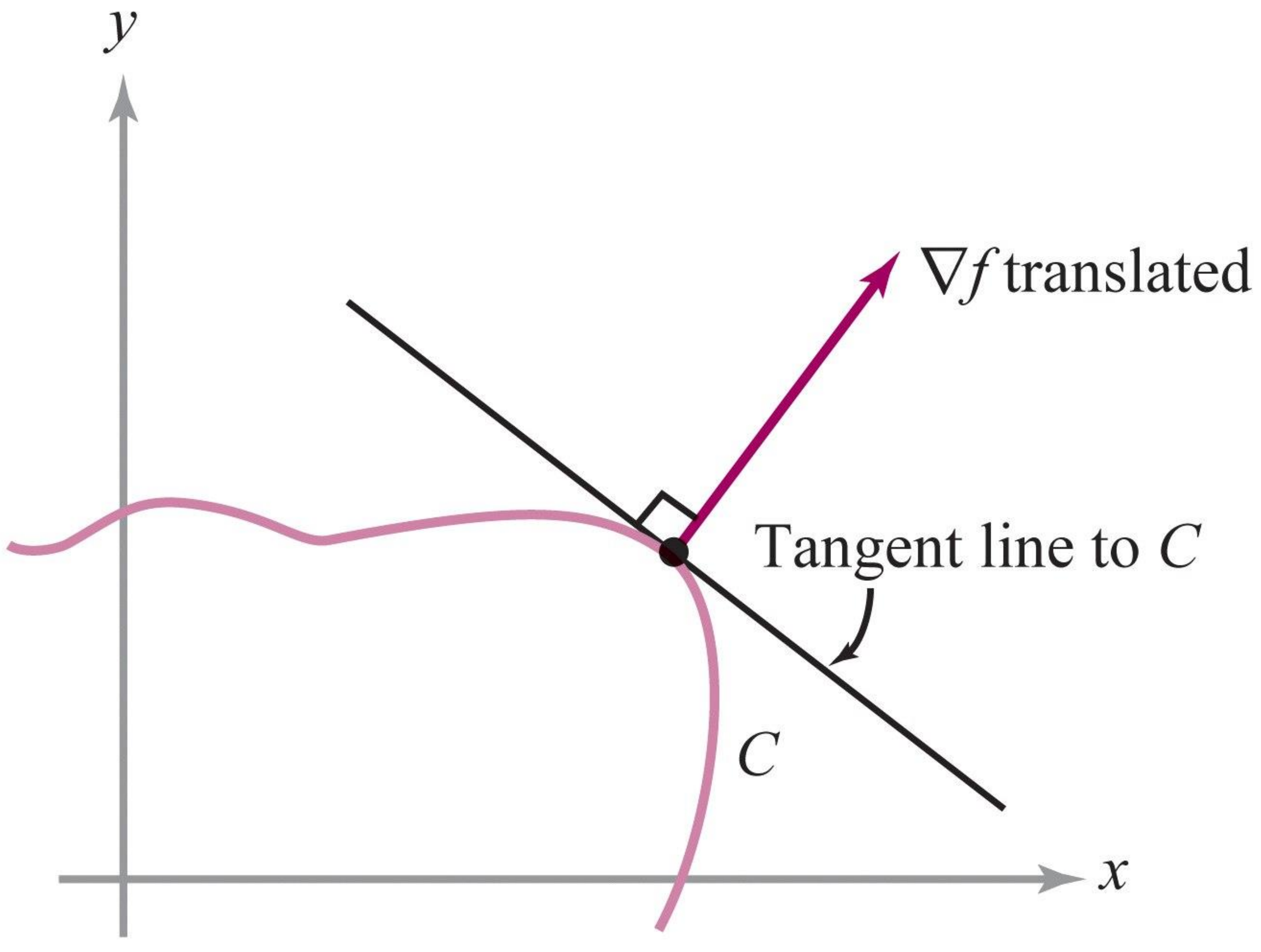


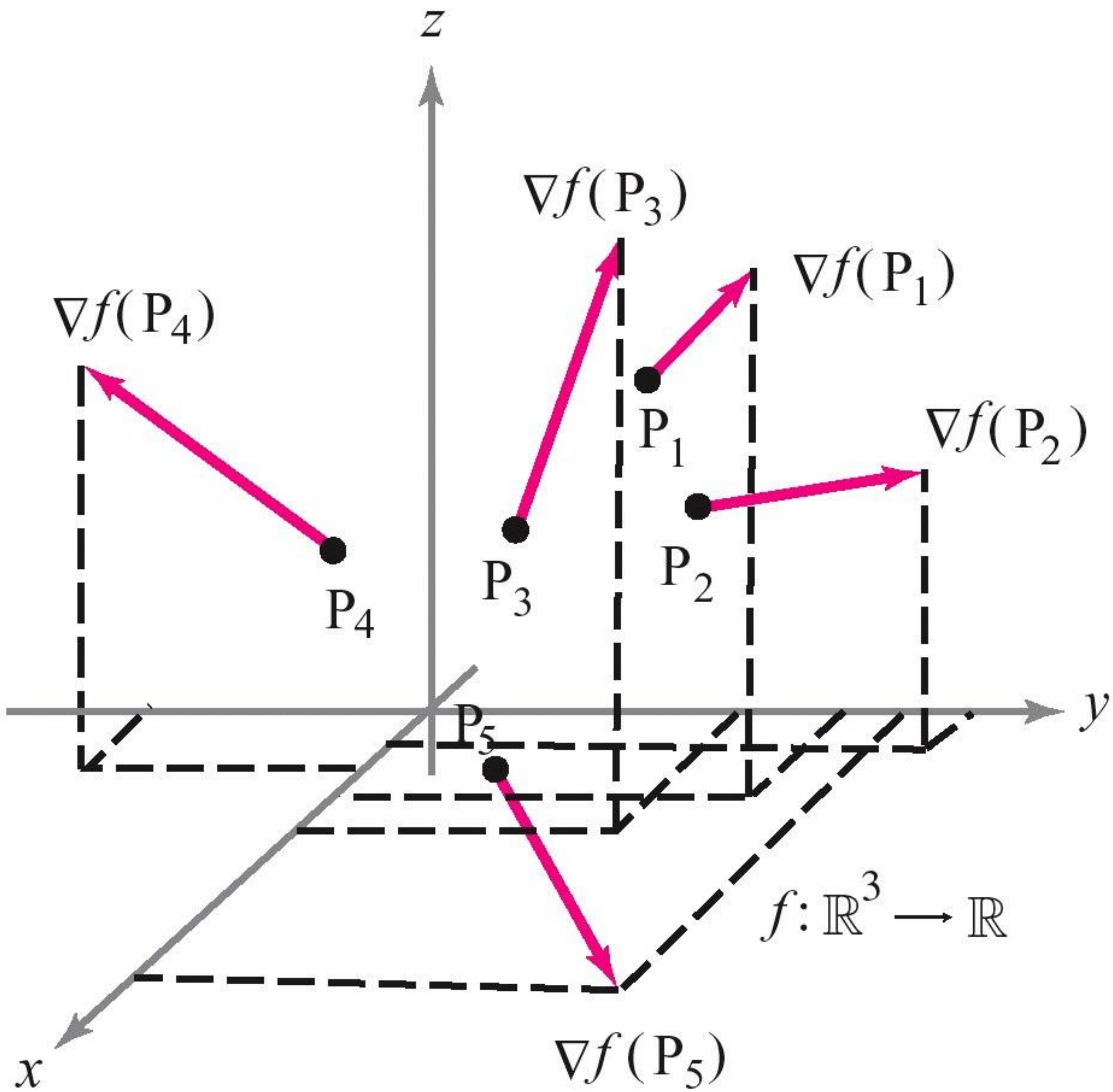
z
$\nabla f(x_0, y_0, z_0)$
$\nabla f(x_0, y_0, z_0)$
parallel translated so
that it begins
at (x_0, y_0, z_0)
v translated
v
y
$\mathbf{c}(t)$
S
(x_0, y_0, z_0)
x

DEFINITION: Tangent Planes to Level Surfaces Let S be the surface consisting of those (x, y, z) such that $f(x, y, z) = k$, for k a constant. The ***tangent plane*** of S at a point (x_0, y_0, z_0) of S is defined by the equation

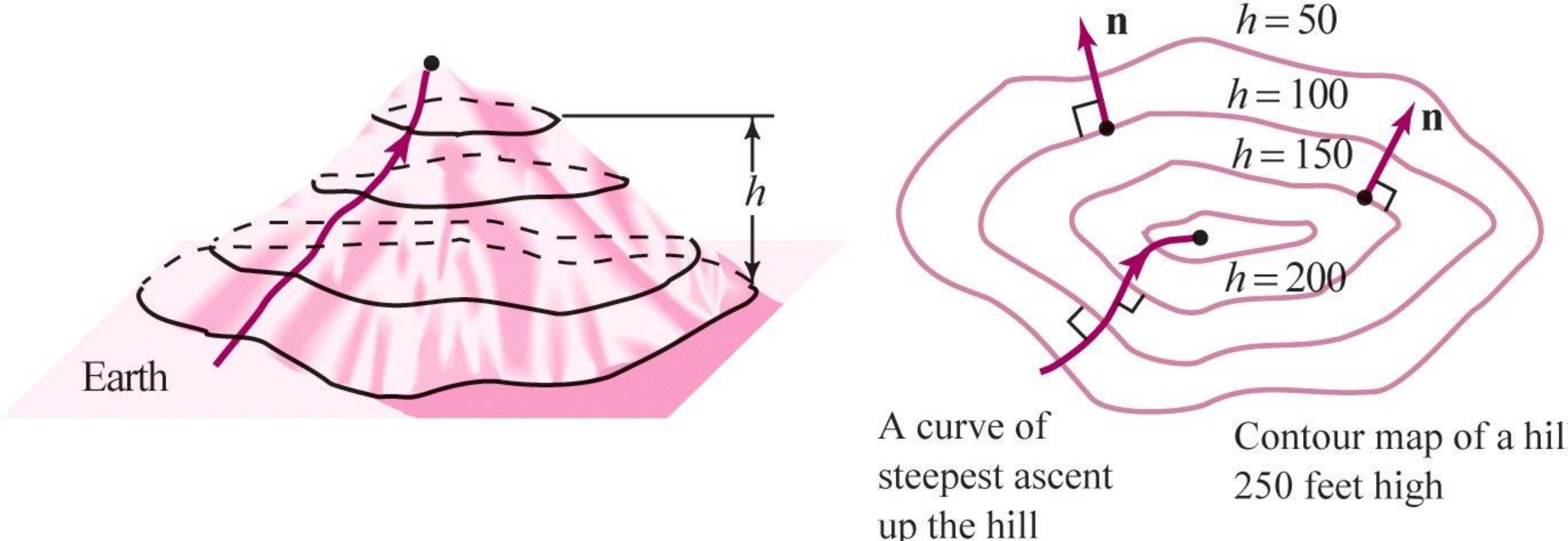
$$\nabla f(x_0, y_0, z_0) \cdot (x - x_0, y - y_0, z - z_0) = 0 \tag{1}$$

if $\nabla f(x_0, y_0, z_0) \neq \mathbf{0}$. That is, the tangent plane is the set of points (x, y, z) that satisfy equation (1).


y
x
∇f translated
Tangent line to C
C

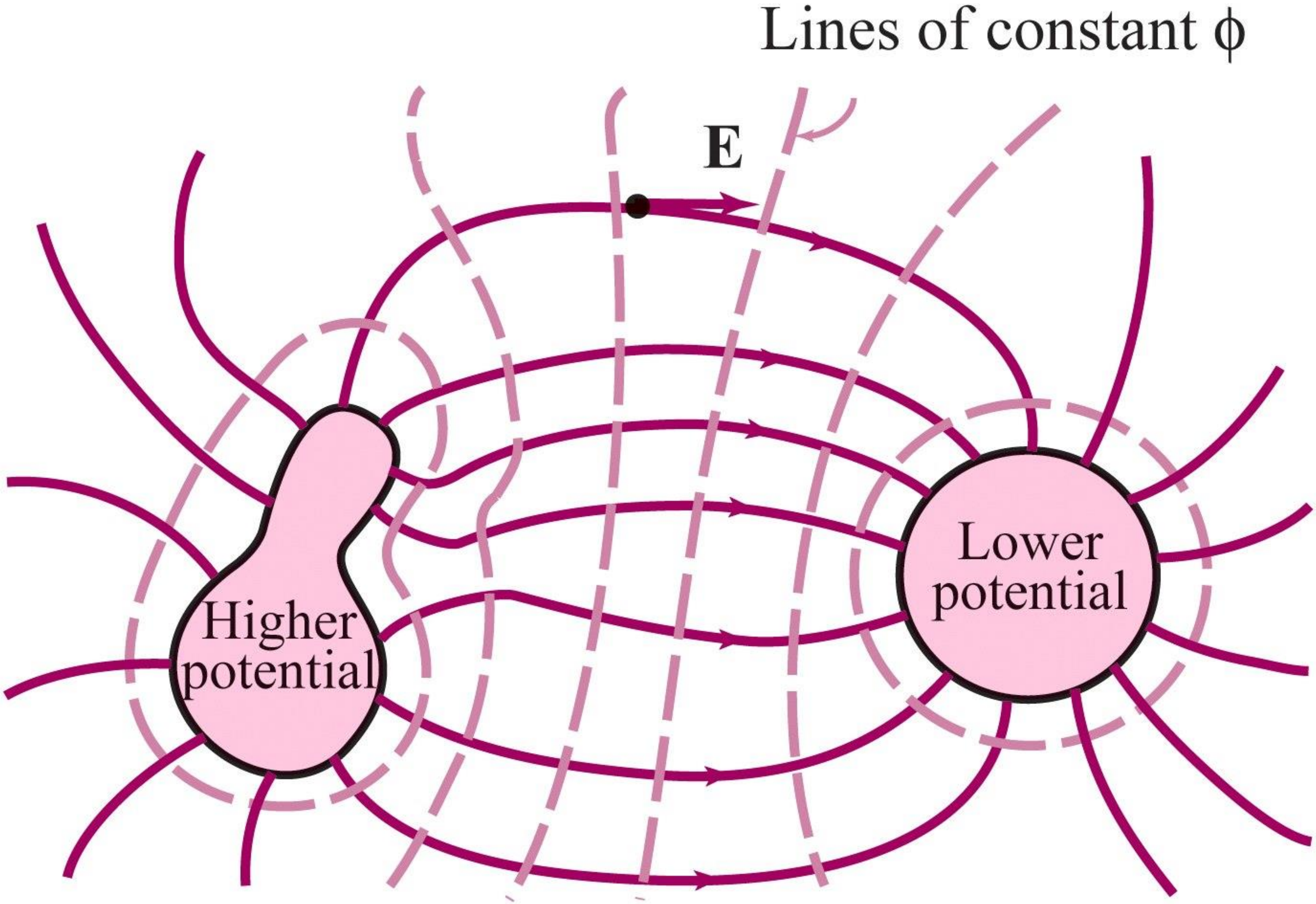


z
∇f(P3)
∇f(P1)
∇f(P4)
P1
∇f(P2)
P4
P3
P2
y
P5
f: ℝ3 → ℝ
x
∇f(P5)



Earth
h
n
$h = 50$
$h = 100$
$h = 150$
$h = 200$
A curve of steepest ascent up the hill
Contour map of a hill 250 feet high

(a) (b)


Lines of constant ϕ
E
Higher potential
Lower potential

Example

The gravitational force on a unit mass m at (x, y, z) produced by a mass M at the origin in $\mathbb{R}^3$ is, according to Newton's law of gravitation, given by

$$\mathbf{F} = -\frac{GmM}{r^2}\mathbf{n},$$

where G is a constant; $r = \|\mathbf{r}\| = \sqrt{x^2 + y^2 + z^2}$, which is the distance of (x, y, z) from the origin; and $\mathbf{n} = \mathbf{r}/r$, the unit vector in the direction of $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, which is the position vector from the origin to (x, y, z).

Note that $\mathbf{F} = \nabla(GmM/r) = -\nabla V$; that is, $\mathbf{F}$ is the negative of the gradient of the gravitational potential $V = -GmM/r$. This can be verified as in Example 1. Notice that $\mathbf{F}$ is directed inward toward the origin. Also, the level surfaces of V are spheres. The gradient vector field $\mathbf{F}$ is normal to these spheres, which confirms the result of Theorem 14. ▲

Exercise

Find a unit vector normal to the surface S given by $z = x^2 y^2 + y + 1$ at the point $(0, 0, 1)$.

Exercise

Find a unit vector normal to the surface S given by $z = x^2 y^2 + y + 1$ at the point $(0, 0, 1)$.

Solution

Let $f(x, y, z) = x^2 y^2 + y + 1 - z$, and consider the level surface defined by $f(x, y, z) = 0$. Because this is the set of points (x, y, z) with $z = x^2 y^2 + y + 1$, we see that this level set coincides with the surface S. The gradient is given by

$$\nabla f(x, y, z) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k} = 2xy^2\mathbf{i} + (2x^2 y + 1)\mathbf{j} - \mathbf{k},$$

and so

$$\nabla f(0, 0, 1) = \mathbf{j} - \mathbf{k}.$$

This vector is perpendicular to S at $(0, 0, 1)$, and so to find a unit normal $\mathbf{n}$ we divide this vector by its length to obtain

$$\mathbf{n} = \frac{\nabla f(0, 0, 1)}{\|\nabla f(0, 0, 1)\|} = \frac{1}{\sqrt{2}}(\mathbf{j} - \mathbf{k}). \quad \blacktriangle$$