# *Exercises 1 - Special relativity - Kinematics*

1.1 Invariance of the wave equation under a Lorentz boost. As we saw in the first lectures, the one-dimensional wave equation given by

$$\frac{\partial^2\phi(x,t)}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2\phi(x,t)}{\partial t^2} = 0$$

is *not* invariant under Galilean transformations. Here, you are asked to prove that it *is* invariant under Lorentz boost with velocity $v$ along the $x$ direction:

(a) Use the chain rule of derivatives to prove that the relevant differential operators transform as

$$\partial_x = \gamma\partial_{x'} - \frac{v}{c^2}\gamma\partial_{t'} \qquad \text{and} \qquad \partial_t = -\gamma v\partial_{x'} + \gamma\partial_{t'} \ .$$

(b) Assuming that the scalar field is invariant,[1] $\phi(x,t) = \phi'(x',t')$, prove that the wave equation remains invariant.

[1] Well, of course. It is a scalar!

1.2 Loss of simultaneity. Two clocks are positioned at the front and back ends of a train of length L (as measured in its own frame). They are synchronized in the train frame. The train travels past you at speed $v$. It turns out that if you observe the clocks at simultaneous times in your frame, you will see the rear clock showing a higher reading than the front clock. By how much?

1.3 Loss of simultaneity. Two events $A$ and $B$ are simultaneous in a given reference frame $S$. In this frame, the separation between $A$ and $B$ is $L = 3 \times 10^8$ m. Find the velocities in the $AB$ direction at which another reference frame $S'$ should move for $A$ and $B$ to be separated by $T' = 0.1, 1$ and 10 s, respectively, in that frame.

1.4 Lorentz transformation. A (very fast) rocket with proper length $L_0 = 10$ m moves away from the Earth with $v = 0.8c$. At some point, we send a light signal from Earth towards the rocket.

(a) Calculate, both in the reference frame of the rocket and in our reference frame on Earth, how long does it take for the light signal to travel from the tail to the head of the rocket.

(b) When it reaches the head of the rocket, the light signal is reflected back towards Earth. Calculate, again in both reference frames, the interval betwen the instant in which the light signal first reached the tail of the rocket, and the instant in which the signal reaches the tail again in its way back to Earth after reflection.

1.5 LENGTH CONTRACTION. Two trains with identical proper length $L$ travel in opposite directions at constant velocity. Alice is at rest in one of the trains and measures an interval $\Delta T$ between the instant when the head of her train coincides with the tail of the other train, and the instant when the head of the other train coincides with the tail of her train.

(a) Calculate the relative velocity $v$ between the two trains.

(b) Find the reference frame from which the two events are simultaneous.

1.6 TIME DILATION AND CAUSALITY. Two events $A$ and $B$ happen at the same location but not simultaneously in a frame $S$. Without loss of generality, we assume that $A$ happens before $B$.

(a) Prove that $A$ happens before $B$ in any inertial frame of reference $S'$.

(b) Prove that the time separation between $A$ and $B$ in other frames $S'$ is unbounded and is minimal in $S$.

(c) Prove that there is no limit in the spatial separation that can be measured from other frames $S'$.

1.7 TIME DILATION. Muons are elementary particles identical to electrons, except that they are about 200 times as massive. They are created in the upper atmosphere when cosmic rays collide with air molecules. The muons have an average lifetime of about $2 \times 10^{-6}$ s (then they decay into electrons and neutrinos), and move at nearly the speed of light. Consider a certain muon created at a height of 50 km, which moves straight downward, has a speed $v = 0.99998c$, decays in exactly $T = 2 \times 10^{-6}$ s, and does not collide with anything on its way down.[2] Will the muon reach the earth before decaying?

[2] In the real world, the muons are created at various heights, move in different directions, have different speeds, decay in times that vary according to some probability distributions, and may very well bump into air molecules. So technically we've got everything wrong here. But that's no matter. This example will work just fine for the present purpose.

1.8 LENGTH CONTRACTION AND ADDITION OF VELOCITIES. Two trains, $A$ and $B$, each have proper length $L$ and move in the same direction. With respect to an observer $C$ on the ground, $A$'s speed is $v_A = 4c/5$ and $B$'s speed is $v_B = 3c/5$. $A$ starts behind $B$. How long, as viewed $C$, does it take for $A$ to overtake $B$? By this we mean the time between the front of $A$ passing the back of $B$, and the back of $A$ passing the front of $B$. Do the calculations from the frame of $C$, and then from the frame of the slower train.

1.9 ADDITION OF VELOCITIES. A train with proper length $L$ moves at speed $5c/13$ with respect to the ground. A ball is thrown from the back of the train to the front. The speed of the ball with respect to the train is $c/3$.

(a) As viewed by someone on the ground, how far does the ball travel before it reaches the front of the train?

(b) Again, as viewed by someone on the ground, how much time does the ball spend in the air? Calculate this using only the Lorentz transformation, and then using the relativistic rule for adding velocities.

(c) According to the ball itself, how much time does it spend in the air?

1.10 ADDITION OF VELOCITIES. Consider a 2-dimensional space, and the lines that go through the origin and are rotated by angles $\phi_1$, $\phi_2$ and $\phi_{12} = \phi_1 + \phi_2$ with respect to the $x$ axis (Fig. 1). Using the properties of the tangent, prove that the slope $m_{12}$ can be written as

$$m_{12} = \frac{m_1 + m_2}{1 - m_1 m_2} .$$

What does this remind you of? What is the same and what is different? Prove that, for small angles, this reduces to $m_{12} = m_1 + m_2$.

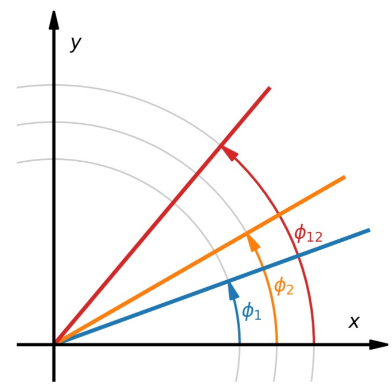


Figure 1: Exercise 10.