Nonlinear Equations

Bisection Method

1) Find the root of the equation

$$f(x) = 2\ln(x) + \sqrt{x} - 2$$

in the interval $[1,2]$ using the bisection method.

2) Let

$$f(x) = x^4 - 3x^3 - 2x^2 + 12x - 8$$

Note that $f(1) = f(2) = 0.$

a) Find the roots of the polinomial using the Bisecton method.

3) Find a root of the polynomial:

$$x^8 - 36x^7 + 546x^6 - 4536x^5 + 22449x^4 - 67284x^3 + 118124x^2 - 109584x + 40320 = 0$$

in the interval $[5.5, 6.5]$. Change -36 to -36.001 and repeat.

4) a) If you borrow L dollars at an annual interest rate of r, for a period of m years, then the size of the monthly payment, M, is given by the annuity equation:

$$M = \frac{12L}{r}\left[1 - \left(1 + \frac{r}{12}\right)^{-12m}\right]$$

You need to borrow 150.000€ to buy the new house that you want, but you can only afford to pay 600€ per month. Assuming a 30 year mortgage, use the bisection method to determine what interest rate you can afford to pay.

b) Repeat the above, assuming that the mortgage is only 15 years in length.

5) Consider the problem of modeling the position of the liquid-solid boundary in a substance that is melting due to the application of heat at one end. In a simplified model, if the initial position of the interface is taken to be $x = 0$, then the interfaces moves according to

$$x = 2\beta\sqrt{t}$$

where β satisfies the nonlinear equation

$$\frac{(T_M - T_0)k}{\lambda\sqrt{\pi}} e^{-\beta^2/k} = \beta\,\mathrm{erf}\left(\frac{\beta}{\sqrt{k}}\right).$$

Here T_M is the melting themperature (absolute scale), $T_0 < T_M$ is the applied temperature, k and λ are parameters dependent on the material properties of the substance involved and $\mathrm{erf}(x)$ is the error function, defined by

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt.$$

MATLAB has an intrinsic error function, *erf*. If you can also approximate $\mathrm{erf}(x)$ by the expression:

$$E(x) = 1 - (a_1\xi + a_2\xi^2 + a_3\xi^3)e^{-x^2},$$

where $\xi = 1/(1 + px)$ and

$$p = 0.47047, \qquad a_1 = 0.3480242, \qquad a_2 = -0.0958798, \qquad a_3 = 0.747856.$$

a) Show that finding β is equivalent to finding the root α of the one-parameter family of functions defined by

$$f(x) = \theta e^{-x^2} - x\,\mathrm{erf}(x).$$

What is θ? How it α related to β?

b) Find the value of α corresponding to $\theta = 0.01, 0.1, 1, 10.$ Use the bisection method, and either MATLAB *erf* command or $E(x)$ to approximate the error function.

Regula Falsi or False Position Method

1) False Position Method. The False Position is a bracketing algorithm that can be considered a variant of the bisection method. Instead of dividing the interval using the midpoint as in the bisection method, the false position uses the chord that passes through the points $(a, f(a))$ and $(b, f(b))$ and the new interval is selected as to keep the extremes where the function changes sign.

Write an algorithm that uses the false position and solve the equation

$$x^2 - 78.8 = 0$$

in order to find its positive root, using both the bisection and the false position methods.

2) Let

$$f(x) = x^4 - 3x^3 - 2x^2 + 12x - 8$$

Note that $f(1) = f(2) = 0.$

a) Find the roots of the polinomial using the Regula Falsi method.

Newton-Raphson and Similar Methods.

1) Newton's method is applied to the solution of

$$e^x - x - 2 = 0.$$

Show that if the starting value is positive, the iteration converges to the positive solution, and if the starting value is negative it converges to the negative solution.

a) Obtain approximate expressions for the solutions.

b) What happens if $i)\ x_0 = 100$ and $ii)\ x_0 = -100.$ About how many iterations would be required to obtain the solution to six decimal digits in these two cases?

2) The function

$$f(x) = x^3 - 7x$$

has three simple roots, namely $\alpha = 0, \alpha = \pm\sqrt{7}$

a) What happens if we try the initial condition $x_0 = 1.19$ on Newton Method?

b) Find these solutions, if possible using the Newton or the Bisection method.

3) Find the smallest postive zero of the polynomial

$$f(x) = x^4 - 6.4x^3 + 6.45x^2 + 20.538x - 31.752$$

Use the Newton method and the "robust" Newton method.

4) Consider the Steffensen's method

$$x_{k+1} = x_k - \frac{[f(x_k)]^2}{f(x_k + f(x_k)) - f(x_k)}, \quad k = 0, 1, 2, \ldots$$

for the solution of $f(x) = 0.$ Apply this method to find the two roots of the equation:

$$e^x - x - 2 = 0.$$

and verify quadratic convergence

b) What happens if we begin our iterations from $x_0 = 10$ and from $x_0 = -10.$

5) Use Newton's Method, Secant and Steffensen methods to compute the negative

zero of the function:

$$f(x) = e^x - 1.5 - \tan^{-1}(x)$$

The derivative is easy to compute and

$$f'(x) = e^x - \frac{1}{(1+x^2)}$$

6) Halley's method uses the second derivative of the funciton to increase the speed of convergence:

$$x_{k+1} = x_k - \frac{2f(x_k)f'(x_k)}{2[f'(x_k)]^2 - f(x_k)f''(x_k)}, \quad k = 0,1,2,\ldots$$

Use the Halley and Newton methods to compute the real roots of the polynomial:

$$P_5(x) = x^5 - 1.6x^4 - 4.65x^3 - 1.0x^2 + 1.6x + 4.65$$

Iteration

1) Find the only positive solution of the equation

$$x^3 - x^2 - x - 1 = 0$$

by iteration, writing it in the form

$$x = 1 + \frac{1}{x} + \frac{1}{x^2}$$

beginning with $x_0 = 1$.

2) Which is the value of

$$s = \sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots}}}$$

Compute the value of s as the limit of the sequence $\{x_n\}$ generated by

$$x = g(x)$$

where

$$g(x) = \sqrt{2+x}$$

and $x_0 = 0$. Show that g satisfies, in a suitable interval, the conditions of convergence.

3) Population Dynamics. In the study of populations, the equation

$$x_{n+1} = x_n R(x_n)$$

establishes a link between the number of individuals in a generation x_n and the number of individuals in the following generation. The function $R(x)$ models de variation rate of the considered population and can be chosen in different ways:

1) Matlhus' model (Thomas Malthus, 1766-1834):

$$R(x) = R_M(x) = r, \qquad r > 0$$

2) Growth with limited resources. Beverton-Holt's or Discrete Verhulst's model:

$$R(x) = R_V(x) = \frac{r}{x+A}, \qquad r > 0, K > 0$$

3) Predator-Prey model with saturation:

$$R(x) = R_p(x) = \frac{rx}{x^2+B} \qquad r > 0, K > 0$$

which represents the evolution of Beverton-Holt's model in the presence of an antagonist population.

Study the evoluion of these models by means of fixed point iteration. In which of these models is possible to reach a stable population?

General Problems

1) The volume of liquid V in a hollow horizontal cylinder of radius r and length L is related to the depth of the liquid h by

$$V = \left[r^2 \cos^{-1}\left(\frac{r-h}{r}\right) - (r-h)\sqrt{2rh - h^2}\ \right] L$$

determine h when $r = 2\ m,\ L = 5\ m^3,$ and $V = 8\ m^3.$

2) A relationship for the compressibility factor c of real gases has the form

$$c = \frac{1 + x + x^2 - x^3}{(1-x)^3}$$

If $c = 0.9,$ find the value of x in the interval $[-0.1, 0].$

3) L.L. Vant-Hull (Solar Energy, 18,33 (1976)) derived the following equation for the geometrical concentration factor, $C,$ in the study of solar energy

$$C = \frac{\pi (h/\cos A)^2 F}{0.5\pi D^2 (1 + \sin A - 0.5\cos A)}$$

If $h = 290,\ C = 1100,\ F = 0.7$ and $D = 13,$ find the value of $A.$

4) Real mechanical systems may involve the deflection of nonlinear springs. If a bloc of mass m is released a distance h above a nonlinear spring, the resistance force F of the spring is given by

$$F = -(k_1 d + k_2 d^{3/2}).$$

Then conservation of energy can be used to show that

$$0 = \frac{2k_2 d^{5/2}}{5} + \frac{1}{2}k_1 d^2 - mgd - mgh.$$

Solve for the compression of the spring $d,$ given the following parameters: $k_1 = 40.000$ $g/s^2,\ k_2 = 40\ g/(s^2 m^5),\ m = 95\ g,\ g = 9.81\ m/s^2$ and $h = 0.43.$

5) In a chemical engineering process, water vapor (H_2O) is heated to sufficiently high temperatures that a significant portion of the water dissociates, or splits apart, to form oxygen (O_2) and hydrogen (H_2) :

$$H_2O \rightleftarrows H_2 + \frac{1}{2}O_2$$

If it is assumed that this is the only reaction involved, the mole fraction x of H_2O that dissociates can be represented by

$$K = \frac{x}{1-x}\sqrt{\frac{2p_t}{2+x}}$$

where K is the reaction's equilibrium constant and p_t is the total pressure of the mixture. If $p_t = 3\ atm.$ and $K = 0.05,$ determine the value of x with a precision of $\epsilon = 10^{-7}.$

6) The Redlich-Kwong equation of state is given by

$$p = \frac{RT}{v-b} - \frac{a}{v(v+b)\sqrt{T}}$$

where $R = 0.518\ KJ/(Kg\ K)$ is the universal gas constant, T the absolute temperature (K), p the absolute pressure (kPa), and v the volume of a Kg of gas (m^3/Kg). The parameters a and b are calculated by

$$a = 0.427\frac{R^2 T_c^{0.25}}{p_c}, \qquad b = 0.0866R\frac{T_c}{p_c}$$

where $p_c = 4600\ kPa$ and $T_c = 191\ K$. As a chemical engineer, you are asked to determine the amount of methane fuel that can held in a $3\ m^3$ tank at a temperature of $-40\ ^oC$ with a pressure of $65.000\ kPa$. Use a root-locating method to calculate v and then the mass of methane contained in the tank.

7) Planck's radiaton law for a black body is

$$E(\lambda, T) = \left(\frac{8\pi hc}{\lambda^5}\right)\frac{1}{e^{(hc/\lambda kT)} - 1}.$$

where $h = 6,62559 \times 10^{-27}\ erg/s$ is Planck's constant, $c = 2,997925 \times 10^{10}\ cm/s$ is the speed of light and $k = 1,38054 \times 10^{-16}\ erg/K$ is Boltzmann's constant. Knowing that the Sun has a surface temperature of $T = 5.778\ K$, find the wavelength of the maximum of the emmited radiation if the Sun is considered a black body.

8) A uniform hydro cable $C = 80\ m$ long with mass per unit length $\rho = 0.5\ kg/m$ is hung from two supports at the same level $L = 70\ m$ apart. The tension T in the cable at its lowest point must satisfy the equation

$$\frac{\rho g\, C}{T} = e^{\rho g L/(2T)} - e^{-\rho g L/(2T)}$$

where $g = 9.81\ m/s^2$. Find the value of the tension T with a precision of seven decimal places.

9) A relationship betwen V, P and T of gases is given by the Beattle-Bridgeman equation of state

$$P = \frac{RT}{V} + \frac{a_2}{V^2} + \frac{a_3}{V^3} + \frac{a_4}{V^4}$$

where P is the pressure, V is the volume, T is the temperature and $a_2 = -1.06$, $a_3 = 0.057$, $a_4 = -0.0001$, $R = 0.082\ liter\text{-}atm/K\text{-}g$ mole. If $T = 293\ K$ and $P = 25\ atm$, find V.

10) Archimedes' Law states that when a solid of density δ_L is placed in a liquid of density δ_L, it sinks to a depth h at which the amount of liquid displaced weight as much as the solid. For a sphere of radius r, Archimedes' law becomes:

$$\frac{\pi}{3}(3rh^2 - h^3)\delta_L = \frac{4}{3}\pi r^3 \delta_S$$

if $r = 500\ ft$ and $\delta_L = 62.5\ lb/ft^3$ (density of water). Find the value of h when a) $\delta_L = 20\ lb/ft^3$, b) $\delta_L = 45\ lb/ft^3$, c) $\delta_L = 70\ lb/ft^3$

11) Based on the work of Frank-Kamenetski (Diffusion and Heat Exchange in Chemical Kinetics, 1955), temperatures in the interior of a material with imbedded heat sources can be determined solving the equation:

$$e^{-(1/2)T}\cosh^{-1}(e^{(1/2)T}) = \sqrt{k/2}.$$

Given that $k = 0.67$, find the temperature T.