Numerical Methods

Interpolation

•Parametric Curves

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[D∑IM]

Parametric Interpolation

- The algorithms discussed so far are appropriate to interpolate functions in 2D. They will not be useful however, to interpolate curves that cross themselves in the plane.
- Moreover, these algorithm are not able to generate more general curves in 3D, essential for modern graphics.
- We need to reformulate these algorithms in parametric form.

Parametric Interpolation

- We will describe points in the 3D space or in the 2D plane by capital letters $\mathbf{P} = \mathbf{P}(x, y, z)$ while vectors will be described by lower case letters $\mathbf{v} = \mathbf{v}(x, y, z)$
- The difference of two points $P_0(x_0, y_0, z_0)$ and $P_1(x_1, y_1, z_1)$ is well defined and gives a vector pointing from the first point to the second

$$\mathbf{v} = \mathbf{P}_1 - \mathbf{P}_0 = (x_1 - x_0, y_1 - y_0, z_1 - z_0)$$

Parametric Interpolation

- The sum of a point and a vector is well defined and is a point

$$\mathbf{P_0} = \mathbf{P_1} + \mathbf{v} \Leftrightarrow \mathbf{v} = \mathbf{P_0} - \mathbf{P_1}$$

- The sum $\mathbf{P} + \mathbf{v}$ and be interpreted as moving away from $\mathbf{P}$ in the direction of $\mathbf{v}$ a certain amount, thereby bringing us to another point.

Parametric Interpolation

- Consider any two points, say, $\mathbf{P_0}$ and $\mathbf{P_2}$, then the expression $\mathbf{P_0} + \alpha(\mathbf{P_2} - \mathbf{P_0})$ is the sum of a point and a vector, so it is a point $\mathbf{P_1}$ on the line connecting $\mathbf{P_0}$ to $\mathbf{P_2}$. The points $\mathbf{P_0}, \mathbf{P_1}$ and $\mathbf{P_2}$ will be ***collinear***.
- The expression $\mathbf{P_1} = \mathbf{P_0} + \alpha(\mathbf{P_2} - \mathbf{P_0})$ can also be written as $\mathbf{P_1} = (1 - \alpha)\mathbf{P_0} + \alpha\mathbf{P_2}$ showing that $\mathbf{P_1}$ is a ***linear combination*** of $\mathbf{P_0}$ and $\mathbf{P_2}$

Barycentric Sum

- The sum of points is not well defined as it will depend on the center of coordinates used.
- There is, however, one important special case where the sum of points is well defined, the so called ***barycentric sum***. This is the case where we add a set of points, each one multiplied by a certain weight w_i, with the condition

$$\sum_{i=0}^{n} w_i = 1$$

Barycentric Sum

- The barycentric sum of points is affinely invariant, i.e., it is a valid point.

$$\sum_{i=0}^{n} w_i \mathbf{P_i} = \mathbf{P}_0 + \sum_{i=1}^{n} w_i \mathbf{P}_i - (1 - w_0)\mathbf{P_0}$$
$$= \mathbf{P}_0 + w_1\mathbf{P_1} + w_2\mathbf{P}_2 + \cdots + w_n\mathbf{P_n} - (w_1 + \cdots + w_n)\mathbf{P_0}$$
$$= \mathbf{P}_0 + w_1\left(\mathbf{P_1} - \mathbf{P_0}\right) + w_2\left(\mathbf{P}_2 - \mathbf{P_0}\right) + \cdots + w_n\left(\mathbf{P_n} - \mathbf{P_0}\right)$$
$$= \mathbf{P}_0 + \sum_{i=1}^{n} w_i\left(\mathbf{P_i} - \mathbf{P_0}\right)$$

- Note that some of the weights can take negative values as long as the total sum is the unity.

Barycentric Sum

- A special case is the ***barycentric sum*** of two points, written as:

$$\mathbf{P}(t) = (1-t)\mathbf{P}_0 + t\mathbf{P}_1$$

- This represents a point on the line from $\mathbf{P_0}$ to $\mathbf{P_1}$. Clearly, $\mathbf{P}(0) = \mathbf{P_0}$ and $\mathbf{P}(1) = \mathbf{P_1}$. Also, since $\mathbf{P}(t)$ is a linear function. The tangent vector $d\mathbf{P}/dt$ is the constant vector $\mathbf{P_1} - \mathbf{P_0}$.
- The two numbers $1-t$ and t are termed the ***barycentric coordinates*** of the point $\mathbf{P}(t)$.

The Isotropic Principle

- Given a curve that is constructed as the sum

$$\mathbf{P}(t) = \sum_{i=0}^{n} w_i \mathbf{P_i} + \sum_{j=0}^{m} u_i \mathbf{v_i}$$

- Where $\mathbf{P_i}$ are points and $\mathbf{v_i}$ are vectors, the curve is independent of the system used if and only if the weights w_i are barycentric. There is no similar requirement for the u_i weights. This statement is known as the ***isotropic principle***.

Parametric Blending

- ***Parametric blending*** is used to vary the value of some quantity in small steps. If P_1 and P_2 are points the expression $\alpha P_1 + \beta P_2$ is a blend of the two points.
- Given several points, a curve can be created as a weighted sum of the points. It has the form $\sum w_i(t) P_i$, where weights $w_i(t)$ are barycentric. Such a curve is a ***blend*** of the points. It is possible to blend vectors in addition to points as part of the curve.

Parametric Blending

- A special case of curve construction is the linear blending of two points, which can be expressed as:

$$(1-t)\mathbf{P}_1 + t\mathbf{P}_2 \qquad 0 \le t \le 1$$

- It is possible to blend points in a nonlinear way. A quadratic blending $P(t) = (1-t)^2 P_1 + t^2 P_2$, will not be useful as it will depend on the coordinate axis since it is not barycentric. It turns out that

$$(1-t)^2 + 2t(1-t) + t^2 = 1$$

- As a result, we can use quadratic blending of three points but not two.

Parametric Curves

- In computer graphics we want to display curves and surfaces that look real. These are specified by the user in terms of points and are constructed in an interactive process.
- We may want the curve to pass through all the points. Such points are called ***data points or nodes*** and the curve is an ***interpolating curve***
- We may want the point to control the shape of the curve by exerting a “pull” on it. Generally, the curve does not pass through these points. Such points are called ***control points***.

Parametric Curves

- The curve representation used in practice is the parametric representation:

$$\mathbf{P}(t) = (x(t), y(t), z(t))$$

- The points are obtained when the parameter t is varied over a certain interval $[a, b]$, normally the interval unity $[0,1]$.
- The first derivative is the ***tangent vector*** to the curve at any point and is denoted by

$$\dot{\mathbf{P}} = \frac{d\mathbf{P}}{dt} = (\dot{x}(t), \dot{y}(t), \dot{z}(t))$$

Parametric Curves

- The ***derivative is a vector*** and not a point because it is the limit of the difference

$$\frac{\mathbf{P}(t+\Delta)-\mathbf{P}(t)}{\Delta}$$

- And the difference of points is a vector.
- The tangent possesses a ***direction*** (the direction of the curve at that point) and a ***magnitude*** (the ***speed*** of the curve at that point.

Parametric Curves

- The tangent, however, is not the ***slope*** of the curve. The tangent is given by a pair of numbers, whereas the slope is a single number. The slope equals $tan\theta$, where θ is the angle between the tangent vector and one of the axis. For a two-dimensional curve, the slope is

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\dot{x}(t)}{\dot{y}(t)}$$

Curve Continuity

- A Complete curve is of often made up of segments, so it is important to understand how individual segments can be connected. There are two types of ***curve continuities: geometric and parametric***.
- If two consecutive segments meet at a point, the total curve is said to have G^0 geometric continuity. If, in addition the direction of the tangent vectors of the two segments are the same the curve has G^1 geometric continuity. The two segments will connect smoothly

Curve Continuity

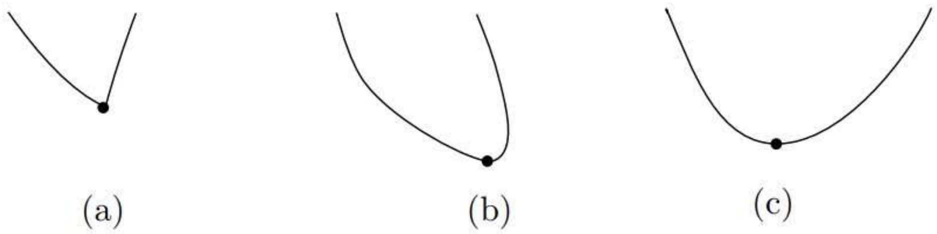

Figure 1.8: (a) G^0 Continuity (a Sharp Corner). (b) G^1 Continuity (a Smooth Connection). (c) G^2 Continuity (a Tight Curve).

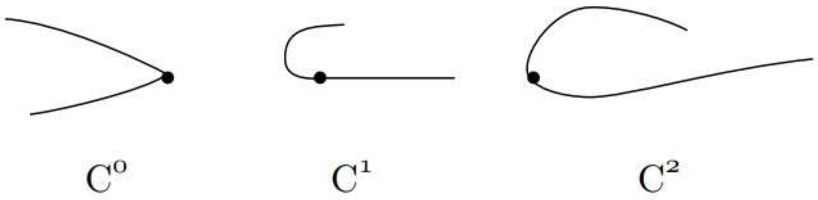

Figure 1.9: Three Curve Continuities.

Curve Continuity

- In general, a curve has geometric continuity G^n at a joint point if every pair of the first n derivatives of the two segments have the same direction at the point. If the same derivatives also have identical magnitudes at the point, then the curve is said to have ***C^n parametric continuity*** at the point.
- We refer to C^0, C^1, and C^2 as ***point, tangent, and curvature continuities,*** respectively.

Parameter Substitution

- Instead of naming the parameter t, we can give a different name. Moreover, we can use a function of t as the parameter. If $g(t)$ is a function that increases monotonically with t, then $P(g(t))$ will have the same shape as $P(t)$, although $g(t)$ will normally have to vary in a different range than t.

Parameter Substitution

- The reason for having two types of continuities has to do with parameter substitution. Given a curve segment $\mathbf{P}(t)$, where $0 \le t \le 1$, we can substitute $T = t^2$. The new segment $\mathbf{Q}(T)$ will have the same shape. However, their calculated tangent vectors have different magnitudes

$$\frac{d\mathbf{Q}(t^2)}{dt} = 2t\frac{d\mathbf{Q}(t)}{dt} = 2t\frac{d\mathbf{P}(t)}{dt}$$

PC Curves

- Parametric curves used in computer graphics are based on polynomials. A polynomial of degree 3 (cubic) has the form:

$$\mathbf{P_3}(t) = \mathbf{A}t^3 + \mathbf{B}t^2 + \mathbf{C}t + \mathbf{D}$$

- This is the simplest curve that can have complex shapes and can also be a space curve. The complexity is limited. It can have at most one loop and at most two inflection points. Polynomials of higher degrees are more difficult to control.

PC Curves

- As a result, a complete curve is often constructed from segments, each a ***parametric cubic polynomial*** (also called a ***PC***). The complete curve is a piecewise polynomial curve, sometimes also called a spline.
- A single PC segment is determined by means of points (data or control) or tangent vectors. Continuity considerations are also used to constrain the curve.

PC Curves

- The segment always have the form:

$$\mathbf{P}_3(t) = \mathbf{A}t^3 + \mathbf{B}t^2 + \mathbf{C}t + \mathbf{D}$$

- Thus, four unknown coefficients must be calculated, which requires four equations, which must depend on four known quantities, points or vectors $\mathbf{G_1}, \dots, \mathbf{G_4}$. The PC segment is expressed as the product

$$\mathbf{P}(t) = (t^3, t^2, t, 1)\begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{pmatrix}\begin{pmatrix} \mathbf{G_1} \\ \mathbf{G_2} \\ \mathbf{G_3} \\ \mathbf{G_4} \end{pmatrix} = \mathbf{T}(t)\cdot\mathbf{M}\cdot\mathbf{G}$$

PC Curves

- **M** is the basis matrix that depends on the method used and **G** is the geometry vector, consisting of the four given quantities. The segment can also be written as the weighted sum

$$
\begin{aligned}
\mathbf{P}(t) &= \left(t^3 m_{11} + t^2 m_{21} + t m_{31} + m_{41}\right)\mathbf{G_1} + \left(t^3 m_{12} + t^2 m_{22} + t m_{32} + m_{42}\right)\mathbf{G_2} \\
&+ \left(t^3 m_{13} + t^2 m_{23} + t m_{33} + m_{43}\right)\mathbf{G}_3 + \left(t^3 m_{14} + t^2 m_{24} + t m_{34} + m_{44}\right)\mathbf{G}_4 \\
&= B_1(t)\mathbf{G_1} + B_2(t)\mathbf{G_2} + B_3(t)\mathbf{G_3} + B_4(t)\mathbf{G_4} = \mathbf{B}(t)\cdot\mathbf{G} = \mathbf{T}(t)\cdot\mathbf{N}\cdot\mathbf{G}
\end{aligned}
$$

- $B_i(t)$ are the weight called also blending functions, since they blend the four given quantities

PC Curves

- A PC segment can also be written in the form

$$
\begin{aligned}
\mathbf{P}(t) &= \mathbf{A}t^3 + \mathbf{B}t^2 + \mathbf{C}t + \mathbf{D} \\
&= (t^3, t^2, t, 1)\begin{pmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \\ D_x & D_y & D_z \end{pmatrix} = \mathbf{T}(t)\cdot\mathbf{C}
\end{aligned}
$$

- Where $\mathbf{A} = (A_x, A_y, A_z)$ and similarly for **B**,**C** and **D**.

- Its first derivative is:

$$\frac{d\mathbf{P}(t)}{dt} = \frac{d\mathbf{T}(t)}{dt} \cdot \mathbf{C} = (3t^2, 2t, 1, 0) \cdot \mathbf{C}$$

- Which is the tangent vector of the curve. This vector points in the direction of the tangent to the curve, but its magnitude describes the speed of this curve. This represents rather the distance covered on the curve when t is incremented in equal steps.

Lagrange's Interpolation

- Given the set of $n+1$ data points or nodes expressed in the form $\mathbf{P_0} = (x_0, y_0), \mathbf{P_1} = (x_1, y_1), \dots, \mathbf{P_n} = (x_n, y_n)$. The problem is to find a function $y = f(x)$ that will pas through all of them.
- Lagrange's algorithm seeks for an expression of the form:

$$P(x) = \sum_{i=0}^{n} y_i L_i^n(x)$$

- Which is a weighted sum of the coordinates y_i and where the weights depend on the x_i coordinates

Lagrange's Interpolation

- This weighted sum must be satisfied for all the points $\mathbf{P_i}$, then we must have that

$$L_i^n(x) = \begin{cases} 1, & x = x_i \\ 0 & \text{else} \end{cases}$$

- And the functions $L_i^n(x)$ must be of the form

$$L_i^n(x) = \frac{\prod_{j \neq i} (x - x_j)}{\prod_{j \neq i} (x_i - x_j)}$$

$$= \frac{(x - x_0)(x - x_1) \cdots (x - x_{i-1})(x - x_{i+1}) \cdots (x - x_n)}{(x_i - x_0)(x_i - x_1) \cdots (x_i - x_{i-1})(x_i - x_{i+1}) \cdots (x_i - x_n)}$$

Lagrange's Interpolation

- The Lagrange weights $L_i^n(x)$ are barycentric. The interpolating polynomial is unique. Consider the interpolating polynomial for $f(x) \equiv 1$. The Lagrange polynomial is

$$LP(x) = \sum_{i=0}^{n} y_i L_i^n(x) = \sum_{i=0}^{n} 1 \cdot L_i^n(x) = \sum_{i=0}^{n} L_i^n(x) = 1$$

Lagrange's Polynomial

- The Lagrange polynomial can also be written in parametric form.
- Giving the data $n + 1$ data points $\mathbf{P_0}, \mathbf{P_1}, \dots, \mathbf{P}_n$ in the plane or in the space, we want to find a polynomial $\mathbf{P}(t)$, such that $\mathbf{P}(t_0) = \mathbf{P_0}, \mathbf{P}(t_1) = \mathbf{P_1}, \dots, \mathbf{P}(t_n) = \mathbf{P_n}$ where we have $t_0 = 0$, $t_n = 1$ while from t_1 through t_n take certain values between 0 and 1 (knot values).

Lagrange's Interpolation

- The polynomial is written as a weighted barycentric sum of the individual points

$$\mathbf{P}(t) = \sum_{i=0}^{n} \mathbf{P}_\mathbf{i} L_i^n(t) = \sum_{i=0}^{n} \mathbf{P}_i \frac{\prod_{j \neq i} (t - t_j)}{\prod_{j \neq i} (t_i - t_j)}$$

Lagrange's Interpolation

- Consider the case with three data points, i.e., a quadratic polynomial. The weight functions are:

$$L_0^2(t) = \frac{\prod_{j\neq 0}^{2}(t - t_j)}{\prod_{j\neq 0}^{2}(t_0 - t_j)} = \frac{(t - t_1)(t - t_2)}{(t_0 - t_1)(t_0 - t_2)}$$

$$L_1^2(t) = \frac{\prod_{j\neq 0}^{2}(t - t_j)}{\prod_{j\neq 0}^{2}(t_1 - t_j)} = \frac{(t - t_0)(t - t_1)}{(t_1 - t_0)(t_1 - t_1)}$$

$$L_2^2(t) = \frac{\prod_{j\neq 0}^{2}(t - t_j)}{\prod_{j\neq 0}^{2}(t_2 - t_j)} = \frac{(t - t_0)(t - t_2)}{(t_2 - t_0)(t_1 - t_2)}$$

Lagrange's Interpolation

- The interpolating polynomial reads:

$$\mathbf{P_2}(t) = \sum_{i=0}^{2} \mathbf{P_i} L_i^2(t)$$

- As the data points $\mathbf{P_i}$, are known its is only necessary to evaluate the nodes t_i to evaluate the polynomial. We can build the Lagrange polynomial taking the values t_0=0, t_1=1 i t_2=2

Lagrange's Interpolation

- Then

$$\mathbf{P}_{\mathbf{2u}}(t) = \frac{t^2 - 3t + 2}{2}\mathbf{P}_0 - (t^2 - 2t)\mathbf{P}_1 + \frac{t^2 - t}{2}\mathbf{P}_2$$

$$= \left(t^2, t, 1\right)\begin{pmatrix} 1/2 & -1 & 1/2 \\ -3/2 & 2 & -1/2 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} \mathbf{P}_0 \\ \mathbf{P}_1 \\ \mathbf{P}_2 \end{pmatrix},\ 0 \le t \le 2$$

- Note that there is a similar expression for each coordinate, There will be a similar expression
- For each data set $\mathbf{P_i}, \mathbf{P_{i+1}}, \mathbf{P_{i+2}}$ we change only the point coordinates.

Lagrange's Interpolation

- In the case of 4 points, the standard third-degree Lagrange polynomial is obtained taking the values $t_0=0$, $t_1=1/3$, $t_2=2/3$ and $t_3=1$,

$$\mathbf{P}_{\mathbf{3u}}(t)=\left(t^3,t^2,t,1\right)\begin{pmatrix} -9/2 & 27/2 & -27/2 & 9/2 \\ 9 & -45/2 & 18 & -9/2 \\ -11/2 & 9 & -9/2 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \mathbf{P_0} \\ \mathbf{P_1} \\ \mathbf{P_2} \\ \mathbf{P_3} \end{pmatrix}$$

Newton's Interpolation

- Newton's polynomial can be written also in a parametric form. Suppose that we have the $n+1$ data points $\boldsymbol{P_0}, \dots, \boldsymbol{P_n}$ and we have the knot values

$$t_0 = 0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = 1$$

- We want an expression of the form

$$\mathbf{P}(t) = \sum_{i=0}^{n} N_i(t)\mathbf{A}_i$$

Newton's Interpolation

- The basis functions, $N_i(t)$ depend only in the value of nodes and not on the data points. Only the unknown coefficients $\mathbf{A_i}$ must depend on the data point values
- In this way, if we add a new point, P_{n+1} we will need to calculate only a new coefficient and one basis function, $N_{n+1}(t)$.

Newton's Interpolation

- The basis functions are defined as

$$N_0(t) = 1$$
$$N_i(t) = (t - t_0)(t - t_1)\cdots(t - t_{i-1}), \quad i = 1, \ldots, n$$

- In order to calculate the coefficients, we impose the condition that the polynomial must satisfy all the data points, and this must be true for any set of data points

Newton's Interpolation

- Then it must be true that

$$\mathbf{P_0} = \mathbf{P}(t_0) = \mathbf{A_0},$$
$$\mathbf{P_1} = \mathbf{P}(t_1) = \mathbf{A_0} + \mathbf{A_1}(t - t_0)$$
$$\mathbf{P_2} = \mathbf{P}(t_2) = \mathbf{A_0} + \mathbf{A_1}(t - t_0) + \mathbf{A_2}(t - t_0)(t - t_1)$$
$$\vdots$$
$$\mathbf{P_n} = \mathbf{P}(t_n) = \mathbf{A_0} + \cdots$$

- An then we can calculate the coefficients recursively

Newton's Interpolation

- Then we have:

$$\mathbf{A_0} = \mathbf{P_0}$$

$$\mathbf{A_1} = \frac{\mathbf{P_1} - \mathbf{P_0}}{t_1 - t_0}$$

$$\mathbf{A_2} = \frac{\mathbf{P_2} - \mathbf{P_0} - \dfrac{(\mathbf{P_1} - \mathbf{P_0})(t_2 - t_0)}{t_1 - t_0}}{(t_2 - t_0)(t_2 - t_1)} = \frac{\dfrac{\mathbf{P_2} - \mathbf{P_1}}{t_2 - t_1} - \dfrac{\mathbf{P_1} - \mathbf{P_0}}{t_1 - t_0}}{t_2 - t_0}$$

$$\vdots$$

- If we introduce the notation

$$\left[t_i t_k\right] = \frac{P_i - P_k}{t_i - t_k}$$

Newton's Interpolation

- We obtain finally the recursive relations

$$\mathbf{A_0} = \mathbf{P_0}$$

$$\mathbf{A_1} = \frac{\mathbf{P_1 - P_0}}{t_1 - t_0} = [t_1 t_0]$$

$$\mathbf{A_2} = [t_2 t_1 t_0] = \frac{[t_2 t_1] - [t_1 t_0]}{t_2 - t_0}$$

$$\mathbf{A_3} = [t_3 t_2 t_1 t_0] = \frac{[t_3 t_2 t_1] - [t_2 t_1 t_0]}{t_3 - t_0}$$

$$\vdots$$

$$\mathbf{A_n} = [t_n \ldots t_1 t_0] = \frac{[t_n \ldots t_1] - [t_{n-1} \ldots t_0]}{t_n - t_1}$$