Exercicis

Part A

3. Evaluate the following iterated integrals:

(a) $\int_{-1}^{1}\int_{0}^{1}(x^4 y + y^2)\, dy\, dx$

(b) $\int_{0}^{\pi/2}\int_{0}^{1}(y\cos x + 2)\, dy\, dx$

(c) $\int_{0}^{1}\int_{0}^{1}(xye^{x+y})\, dy\, dx$

(d) $\int_{-1}^{0}\int_{1}^{2}(-x\log y)\, dy\, dx$

5. Use Cavalieri's principle to show that the volumes of two cylinders with the same base and height are equal (see Figure 5.1.10).

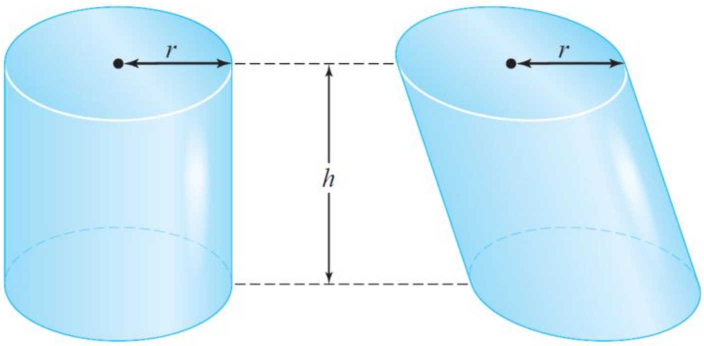

figure **5.1.10** Two cylinders with the same base and height have the same volume.

R is the rectangle $[0, 2] \times [-1, 0]$

9. $\iint_R (x^2 y^2 + x)\, dy\, dx$

10. $\iint_R \left(|y| \cos \frac{1}{4}\pi x\right) dy\, dx$

11. $\iint_R \left(-xe^x \sin \frac{1}{2}\pi y\right) dy\, dx$

13. Evaluate the iterated integral:

$$\int_0^1 \int_0^1 (3x + 2y)^7 \, dx \, dy.$$

Part B

13. Consider the integral in 2(a) as a function of m and n; that is,

$$f(m, n) := \iint\limits_R x^m y^n \, dx \, dy.$$

Evaluate $\lim_{m,n\to\infty} f(m, n)$.

14. Let:

$$f(m, n) := \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \cos nx \sin my \, dx \, dy.$$

Show that $\lim_{m,n\to\infty} f(m, n) = 0$.

15. Let f: $[0, 1] \times [0, 1] \to \mathbb{R}$ be defined by

$$f(x, y) = \begin{cases} 1 & x \text{ rational} \\ 2y & x \text{ irrational.} \end{cases}$$

Show that the iterated integral $\int_0^1 \left[\int_0^1 f(x, y) \, dy \right] dx$ exists but that f is not integrable.

Part C

3. Evaluate the following iterated integrals and draw the regions D determined by the limits. State whether the regions are x-simple, y-simple, or simple.

(a) $\displaystyle\int_0^1 \int_0^{x^2} dy \, dx$

(b) $\displaystyle\int_1^2 \int_{2x}^{3x+1} dy \, dx$

(c) $\displaystyle\int_0^1 \int_1^{e^x} (x + y) \, dy \, dx$

(d) $\displaystyle\int_0^1 \int_{x^3}^{x^2} y \, dy \, dx$

11. Let D be the region given as the set of (x, y), where $1 \leq x^2 + y^2 \leq 2$ and $y \geq 0$. Is D an elementary region? Evaluate $\iint_D f(x, y)\, dA$, where $f(x, y) = 1 + xy$.

19. Let D be a region given as the set of (x, y) with $-\phi(x) \leq y \leq \phi(x)$ and $a \leq x \leq b$, where ϕ is a nonnegative continuous function on the interval $[a, b]$. Let $f(x, y)$ be a function on D such that $f(x, y) = -f(x, -y)$ for all $(x, y) \in D$. Argue that $\iint_D f(x, y)\, dA = 0$.

Part D

5. Change the order of integration and evaluate:

$$\int_0^1 \int_{\sqrt{y}}^1 e^{x^3}\, dx\, dy.$$

7. If $f(x, y) = e^{\sin(x+y)}$ and $D = [-\pi, \pi] \times [-\pi, \pi]$, show that

$$\frac{1}{e} \leq \frac{1}{4\pi^2} \iint_D f(x, y)\, dA \leq e.$$

9. If $D = [-1, 1] \times [-1, 2]$, show that

$$1 \leq \iint_D \frac{dx\, dy}{x^2 + y^2 + 1} \leq 6.$$

10. Using the mean-value inequality, show that

$$\frac{1}{6} \leq \iint_D \frac{dA}{y - x + 3} \leq \frac{1}{4},$$

where D is the triangle with vertices $(0, 0)$, $(1, 1)$, and $(1, 0)$.

Part E

15. $\displaystyle\int_0^1 \int_1^2 \int_2^3 \cos[\pi(x+y+z)]\, dx\, dy\, dz$

16. $\displaystyle\int_0^1 \int_0^x \int_0^y (y+xz)\, dz\, dy\, dx$

17. $\displaystyle\iiint_W (x^2+y^2+z^2)\, dx\, dy\, dz$; W is the region bounded by $x+y+z=a$ (where $a>0$), $x=0$, $y=0$, and $z=0$.

18. $\displaystyle\iiint_W z\, dx\, dy\, dz$; W is the region bounded by the planes $x=0$, $y=0$, $z=0$, $z=1$, and the cylinder $x^2+y^2=1$, with $x\geq 0$, $y\geq 0$.

19. $\displaystyle\iiint_W x^2 \cos z\, dx\, dy\, dz$; W is the region bounded by $z=0$, $z=\pi$, $y=0$, $y=1$, $x=0$, and $x+y=1$.

For the regions in Exercises 25 to 28, find the appropriate limits $\phi_1(x)$, $\phi_2(x)$, $\gamma_1(x,y)$, and $\gamma_2(x,y)$, and write the triple integral over the region W as an iterated integral in the form

$$\iiint_W f\, dV = \int_a^b \left\{ \int_{\phi_1(x)}^{\phi_2(x)} \left[\int_{\gamma_1(x,y)}^{\gamma_2(x,y)} f(x,y,z)\, dz \right] dy \right\} dx.$$

25. $W = \{(x,y,z) \mid \sqrt{x^2+y^2} \leq z \leq 1\}$

26. $W = \{(x,y,z) \mid \frac{1}{2} \leq z \leq 1 \text{ and } x^2+y^2+z^2 \leq 1\}$

27. $W = \{(x,y,z) \mid x^2+y^2 \leq 1,\ z \geq 0 \text{ and } x^2+y^2+z^2 \leq 4\}$

28. $W = \{(x,y,z) \mid |x| \leq 1,\ |y| \leq 1,\ z \geq 0 \text{ and } x^2+y^2+z^2 \leq 1\}$

Solutions

Part A

3. (a) $\dfrac{13}{15}$ (b) $\pi + \dfrac{1}{2}$

(c) 1 (d) $\log 2 - \dfrac{1}{2}$

5. To show that the volumes of the two cylinders are equal, show that their area functions are equal.

9. $\dfrac{26}{9}$

11. $(2/\pi)(e^2+1)$

13. $\dfrac{35795}{8}$

Part B

13. By Exercise 2(a), we have:

$$f(m,n) = \iint_R x^m y^n \, dx\, dy = \left(\frac{1}{m+1}\right)\left(\frac{1}{n+1}\right).$$

Then, as $m, n \to \infty$, we see that $\lim f(m,n) = 0$.

15. Because $\int_0^1 dy = \int_0^1 2y\, dy = 1$, we have $\int_0^1 [\int_0^1 f(x,y)\, dy]\, dx = 1$. In any partition of $R = [0,1] \times [0,1]$, each rectangle R_{jk} contains points $\mathbf{c}_{jk}^{(1)}$ with x rational and $\mathbf{c}_{jk}^{(2)}$ with x irrational. If in the regular partition of order n, we choose $\mathbf{c}_{jk} = \mathbf{c}_{jk}^{(1)}$ in those rectangles with $0 \le y \le \frac{1}{2}$ and $\mathbf{c}_{jk} = \mathbf{c}_{jk}^{(2)}$ when $y > \frac{1}{2}$, the approximating sums are the same as those for

$$g(x,y) = \begin{cases} 1 & 0 \le y \le \frac{1}{2} \\ 2y & \frac{1}{2} < y < 1. \end{cases}$$

Because g is integrable, the approximating sums must converge to $\int_R g\, dA = 7/8$. However, if we had picked all $\mathbf{c}_{ij} = \mathbf{c}_{jk}^{(1)}$, all approximating sums would have the value 1.

Part C

3. (a) $1/3$, both.

(b) $5/2$, both.

(c) $(e^2 - 1)/4$, both.

(d) $1/35$, both.

11. y-simple; $\pi/2$.

19. Compute the integral with respect to y first. Split that into integrals over $[-\phi(x), 0]$ and $[0, \phi(x)]$ and change variables in the first integral, or use symmetry.

Part D

5. $\frac{1}{3}(e - 1)$

7. Note that the maximum value of f on D is e and the minimum value of f on D is $1/e$. Use the ideas in the proof of Theorem 4 to show that

$$\frac{1}{e} \leq \frac{1}{4\pi^2} \iint f(x, y)\ dA \leq e.$$

9. The smallest value of $f(x, y) = 1/(x^2 + y^2 + 1)$ on D is $\frac{1}{6}$, at $(1, 2)$, and so

$$\iint_D f(x, y)\ dx\, dy \geq \frac{1}{6} \cdot \text{area } D = 1.$$

The largest value is 1, at $(0, 0)$, and so

$$\iint_D f(x, y)\ dx\, dy \leq 1 \cdot \text{area } D = 6.$$

Part E

15. 0

17. $a^5/20$

19. 0

25. $\displaystyle\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^{1} f(x, y, z)\, dz\, dy\, dx$

27. $\displaystyle\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{0}^{\sqrt{4-x^2-y^2}} f(x, y, z)\, dz\, dy\, dx$