Numerical Methods

Hermite Interpolation

- Parametric Hermite Curves

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[DΣIM]

Hermite Interpolation

- Using polynomials is not difficult to build a curve satisfying a set of data points
- But the building of the classical polynomial is not an interactive process. If the curve obtained in not as we wish, the only way out is to add still more points, increasing the computation cost with no assurance that the curve can become better.

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Hermite Interpolation

- A good interpolation method should be interactive, allowing to user to add new data points to generate the curve intuitively. The Hermite interpolation allows this kind of procedure.
- In order to generate the curve from the point $\mathbf{P}_1$ to the point $\mathbf{P}_2$ this method uses also the vector tangent to the curve in these points.

Hermite Interpolation

- Changing the magnitude of this tangent vectors we can obtain curves with very different shapes. These curves can have spikes and even loops in its shape.
- Charles Hermite developed this method in 1870. Their first aim was not to draw different curves, but to predict certain future values from an initial point and the rate of variation of the magnitude (derivative)

Hermite Interpolation

- In this figure we can appreciate how the curve can change depending the direction of the tangent vectors.

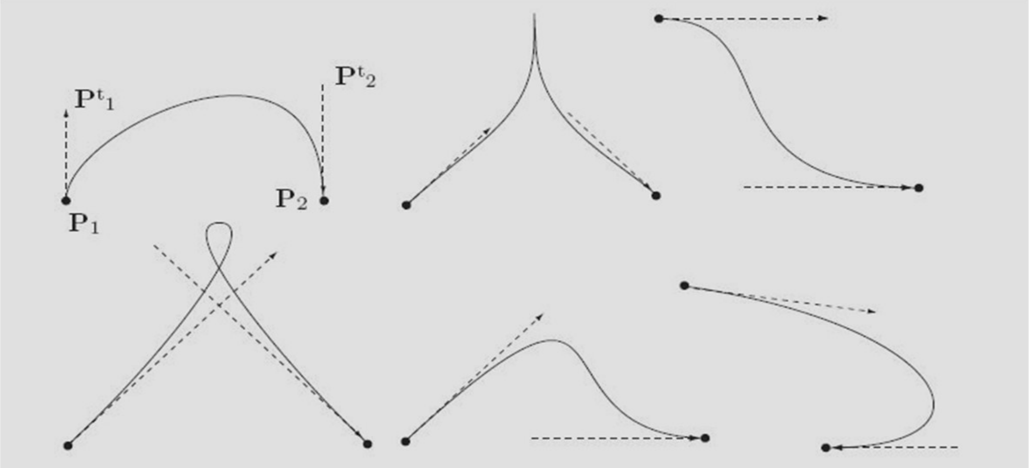


Figure 4.1: Various Hermite Curve Segments.

Hermite Interpolation

- Not only the direction, also the magnitudes of the tangent vectors can change the shape of the curve.

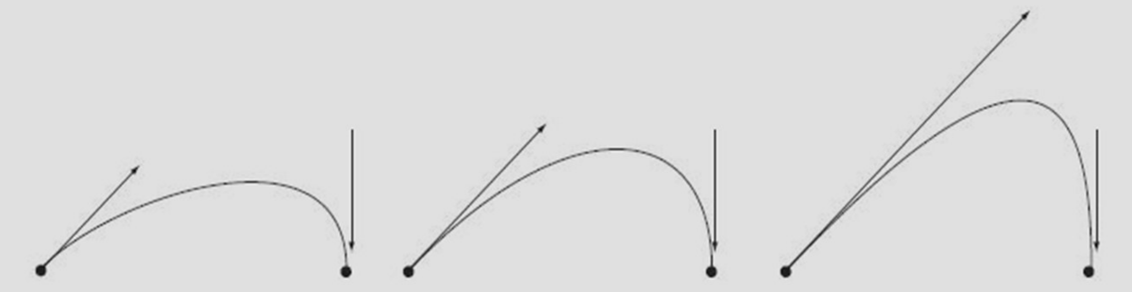


Figure 4.2: Effects of Varying the Tangent's Magnitude.

Hermite Interpolation

- The magnitude of the tangent vectors is also necessary. If we want to draw a parametric curve in 3D using cubic polynomials, then we will need one such polynomial for each dimension giving a total 12 constant coefficients.
- The two endpoints supply six known quantities, and the two tangents should supply the other 6. If we consider only the direction of the vector only two of its components are independent. We would need two additional independent quantities.

Hermite Curve

- The Hermite curve segment is a cubic polynomial with the following shape

$$\mathbf{Q}(t) = \mathbf{a}t^3 + \mathbf{b}t^2 + \mathbf{c}t + \mathbf{d} = (t^3, t^2, t, 1)\begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \\ \mathbf{d} \end{pmatrix} = \mathbf{T}(t) \cdot \mathbf{A}$$

- This is the algebraic representation of the curve, in which the four coefficients are still unknown.

Hermite Curve

- The tangent to the curve, **Q**(t), its derivative, can be written as

$$\dot{\mathbf{Q}}(t) = 3\mathbf{a}t^2 + 2\mathbf{b}t + \mathbf{c}$$

- If we represent our two points by $\mathbf{P_1}$ and $\mathbf{P_2}$ and their tangent vectors by $\mathbf{v_1}$ and $\mathbf{v_2}$, we have 4 vectorial equations to determine all the constants of the curve

Hermite Curve

- These equations follow when imposing the conditions that $\mathbf{Q}(0) = \mathbf{P_1}$ and $\mathbf{Q}(1) = \mathbf{P_2}$ and that $\dot{\mathbf{Q}}(0) = \mathbf{v_1}$ and $\dot{\mathbf{Q}}(1) = \mathbf{v_2}$

$$\mathbf{a} \cdot 0^3 + \mathbf{b} \cdot 0^2 + \mathbf{c} \cdot 0 + \mathbf{d} = \mathbf{P_1}$$

$$\mathbf{a} \cdot 1^3 + \mathbf{b} \cdot 1^2 + \mathbf{c} \cdot 1 + \mathbf{d} = \mathbf{P_2}$$

$$3\mathbf{a} \cdot 0^2 + 2\mathbf{b} \cdot 0 + \mathbf{c} = \mathbf{v}_1$$

$$3\mathbf{a} \cdot 1^2 + 2\mathbf{b} \cdot 1 + \mathbf{c} = \mathbf{v}_2$$

Hermite Curve

- Solving these equations, we obtain the value of the constants as a function of the data points and the tangent vectors to the curve at these points.

$$\mathbf{a} = 2\mathbf{P}_1 - 2\mathbf{P}_2 + \mathbf{v}_1 + \mathbf{v}_2$$

$$\mathbf{b} = -3\mathbf{P}_1 + 3\mathbf{P}_2 - 2\mathbf{v}_1 - \mathbf{v}_2$$

$$\mathbf{c} = \mathbf{v}_1$$

$$\mathbf{d} = \mathbf{P}_1$$

Hermite Interpolation

- The polynomial is then,
- $$\mathbf{Q}(t) = \mathbf{P_1} + \mathbf{v_1}t + \left[3(\mathbf{P_2} - \mathbf{P}_1) - 2\mathbf{v_1} - \mathbf{v_2}\right]t^2$$
$$+\left[2\left(\mathbf{P}_1 - \mathbf{P_2}\right) + \mathbf{v_1} + \mathbf{v_2}\right]t^3$$
- And after grouping similar terms we have:

$$\mathbf{Q}(t) = (2t^3 - 3t^2 + 1)\mathbf{P_1} + (-2t^3 + 3t^2)\mathbf{P_2}$$
$$+(t^3 - 2t^2 + t)\mathbf{v_1} + (t^3 - t^2)\mathbf{v_2}$$
$$= F_1(t)\mathbf{P_1} + F_2(t)\mathbf{P_2} + F_3(t)\mathbf{v_1} + F_4(t)\mathbf{v_2}$$

Hermite Interpolation

- $F_i(t)$ are the Hermite blending functions. They create any point on the curve as a blend of the quantities $\mathbf{P_1}, \mathbf{P_2}, \mathbf{v_1}, \mathbf{v_2}$.
- $F_1(t) + F_2(t) \equiv 1$. These two functions blend points, not tangent vectors and should therefore be barycentric.

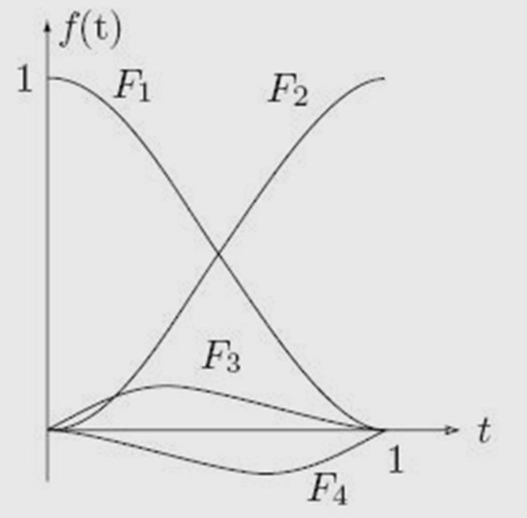


Figure 4.3: Hermite Weight Functions

Hermite Curve

- If we write this expression in matrix form, we have:

$$
\begin{aligned}
\mathbf{Q}(t) &= \\
&= F_1(t)\mathbf{P_1} + F_2(t)\mathbf{P_2} + F_3(t)\mathbf{v_1} + F_4(t)\mathbf{v_2} \\
&= \left(F_1(t), F_2(t), F_3(t), F_4(t)\right) \bullet \begin{pmatrix} \mathbf{P_1} \\ \mathbf{P_2} \\ \mathbf{v_1} \\ \mathbf{v_2} \end{pmatrix} \\
&= \mathbf{F}(t) \bullet \mathbf{B}
\end{aligned}
$$

Hermite Curve

- Note that we can write the first function as

$$F_1(t) = (t^3, t^2, t, 1) \bullet \left(2, -3, 0, 1\right)^T$$

- With similar relations for the rest of the functions. Then:

$$\mathbf{F}(t) = \left(t^3, t^2, t, 1\right) \begin{pmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \mathbf{T}(t) \bullet \mathbf{H}$$

Hermite Curve

- The Hermite curve can be written as

$$\mathbf{Q}(t) = \mathbf{F}(t) \bullet \mathbf{B} = \mathbf{T}(t) \bullet \mathbf{H} \bullet \mathbf{B}$$

$$= (t^3, t^2, t, 1) \begin{pmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{P_1} \\ \mathbf{P_2} \\ \mathbf{v_1} \\ \mathbf{v_2} \end{pmatrix}$$

- All the necessary information is in the matrix **B**. The matrix name **H**, is called the Hermite matrix basis.

Hermite Curve with Tension

- The Hermite curve segments with tension are generated by non uniform curves. In order to do so, a parameter *t,* is varied within the interval $[0,\Delta]$ where Δ can be any positive number. The set of equations is then

$$\mathbf{a}\cdot 0^3+\mathbf{b}\cdot 0^2+\mathbf{c}\cdot 0+\mathbf{d}=\mathbf{P_1}$$

$$\mathbf{a}\cdot \Delta^3+\mathbf{b}\cdot \Delta^2+\mathbf{c}\cdot \Delta+\mathbf{d}=\mathbf{P_2}$$

$$3\mathbf{a}\cdot 0^2+2\mathbf{b}\cdot 0+\mathbf{c}=\mathbf{v}_1$$

$$3\mathbf{a}\cdot \Delta^2+2\mathbf{b}\cdot \Delta+\mathbf{c}=\mathbf{v}_2$$

Hermite Curve with Tension

- While the solutions are in this case:

$$\mathbf{a} = \frac{2(\mathbf{P_1} - \mathbf{P_2})}{\Delta^3} + \frac{\mathbf{v_1} + \mathbf{v_2}}{\Delta^2}$$

$$\mathbf{b} = \frac{3(\mathbf{P_2} - \mathbf{P_1})}{\Delta^2} - \frac{2\mathbf{v_1} + \mathbf{v_2}}{\Delta}$$

$$\mathbf{c} = \mathbf{v_1}$$

$$\mathbf{d} = \mathbf{P_1}$$

Hermite Curve with Tension

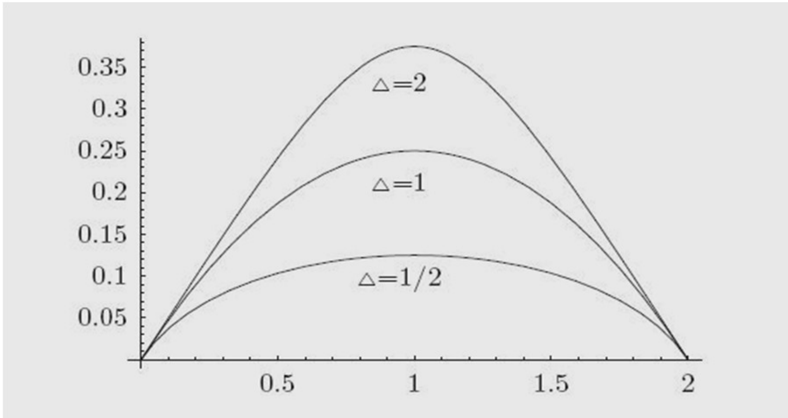
- Which can be expressed in matrix for as:

$$\mathbf{Q}_\Delta(t) = \left(t^3, t^2, t, 1\right)\begin{pmatrix} \frac{2}{\Delta^3} & \frac{-2}{\Delta^3} & \frac{1}{\Delta^2} & \frac{1}{\Delta^2} \\ \frac{-3}{\Delta^2} & \frac{3}{\Delta^2} & \frac{-2}{\Delta} & \frac{-1}{\Delta} \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \mathbf{P_1} \\ \mathbf{P_2} \\ \mathbf{v_1} \\ \mathbf{v_2} \end{pmatrix}$$

$$= \mathbf{T}_\Delta(t)\mathbf{H}_\Delta\mathbf{B}$$

Hermite Curve with Tension

- Note that matrix H_Δ is reduced to matrix H when $\Delta = 1$. In the following figure we can see the effect of a variable Δ on the final shape of the curve



Hermite Curve with Tension

- The effect of increasing the value of the parameter Δ is equivalent to increase the modulus of the tangent vectors in the extremes of the curve. Sometimes this curve is written as

$$\mathbf{Q}(t) = \mathbf{F}(t) \bullet \mathbf{B} = \mathbf{T}(t) \bullet \mathbf{H} \bullet \mathbf{B}$$

$$= (t^3, t^2, t, 1)\begin{pmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \mathbf{P_1} \\ \mathbf{P_2} \\ \Delta\mathbf{v_1} \\ \Delta\mathbf{v_2} \end{pmatrix}$$

Hermite Curve with Tension

- This non uniform Hermite's curve is an especial case of the Hermite's uniform curve. We can change from one to another adjusting the values of the tangent vectors or by adjusting the value of the parameter t.
- This variation is understood as a change in the ***tension*** of the curve

Hermite Curve of Degree 5

- The ideas from Hermite's curve can be extended to polynomials of higher degree. The higher the degree, the more initial data will be necessary to set the polynomial constants
- Normally in order to increase the degree of the polynomial we use data on higher derivatives, not only on the first derivative (tangent vectors)

Hermite Curve of Degree 5

- If we use the data points $\mathbf{P_1}$ and $\mathbf{P_2}$ and the tangent vectors in these points $\mathbf{v_1}$ and $\mathbf{v_2}$, and we add the second derivatives of the tangents, we will be using also the acceleration vectors $\mathbf{a_1}$ and $\mathbf{a_2}$. We will have then enough information to build a polynomial curve of degree 5:

$$\mathbf{P}(t) = \mathbf{A}t^5 + \mathbf{B}t^4 + \mathbf{C}t^3 + \mathbf{D}t^2 + \mathbf{E}t + \mathbf{F}$$

Hermite's Curve of Degree 5

- When imposing the initial conditions, we obtain:

$$\mathbf{P}(0) = \mathbf{F} = \mathbf{P}_1$$

$$\mathbf{P}(1) = \mathbf{A} + \mathbf{B} + \mathbf{C} + \mathbf{D} + \mathbf{E} + \mathbf{F} = \mathbf{P}_2$$

$$\mathbf{v}_1 = \mathbf{E}$$

$$\mathbf{v}_2 = 5\mathbf{A} + 4\mathbf{B} + 3\mathbf{C} + 2\mathbf{D} + \mathbf{E}$$

$$\mathbf{a}_1 = 2\mathbf{D}$$

$$\mathbf{a}_2 = 20\mathbf{A} + 12\mathbf{B} + 6\mathbf{C} + 2\mathbf{D}$$

Hermite Curve of Degree 5

- And finally in matrix form:

$$\mathbf{P}(t) =$$

$$= \begin{pmatrix} t^5, & t^4, & t^3, & t^2, & t, & 1 \end{pmatrix} \begin{pmatrix} -6 & 6 & -3 & -3 & -1/2 & 1/2 \\ 15 & -15 & 8 & 7 & 3/2 & -1 \\ -10 & 10 & -6 & -4 & -3/2 & 1/2 \\ 0 & 0 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{P}_1 \\ \mathbf{P}_2 \\ \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{a}_1 \\ \mathbf{a}_2 \end{pmatrix}$$

Hermite Derivatives

- One of the uses of the classical polynomial is the use of its derivatives as an approximation of the derivatives of the real curve. We can use for the same purposes the Hermite's curve
- This curve is written in compact form as

$$\mathbf{Q}(t) = \mathbf{T}(t)\mathbf{HB}$$

Hermite Derivatives

- The matrix **H** is a constant matrix while vector **B** contains our geometrical information. Only the vector **T**(t) is dependent on the parameter t. Then the derivative of Hermite's curve can be computed as:

$$\dot{\mathbf{Q}}(t) = \dot{\mathbf{T}}(t)\mathbf{HB} = \left(3t^2, 2t, 1, 0\right)\mathbf{HB}$$

- But as **Q**(t)=**F**(t)**B,** where **F**(t) is the vector containing the blending functions

Hermite Derivatives

- The Hermite derivative can be written as

$$\dot{\mathbf{Q}}(t) = \dot{\mathbf{F}}(t) = \left(\dot{F}_1(t), \dot{F}_2(t), \dot{F}_3(t), \dot{F}_4(t)\right)\mathbf{B}$$

- Each of the blending functions $F_i(t)$ can be derived separately:

$$\dot{F}_1(t) = 6t^2 - 6t \quad \dot{F}_2(t) = -6t^2 + 6t$$

$$\dot{F}_3(t) = 3t^2 - 4t + 1 \quad \dot{F}_3(t) = 3t^2 - 2t$$

Hermite derivatives

- We can finally write the derivative in matrix form:

$$\dot{\mathbf{Q}}(t) = \left(t^3, t^2, t, 1\right) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 6 & -6 & 3 & 3 \\ -6 & 6 & -4 & -2 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{P}_1 \\ \mathbf{P}_2 \\ \mathbf{v}_1 \\ \mathbf{v}_2 \end{pmatrix}$$

- Remember that the result of a derivative in parametric form will be always a vector

Hermite derivatives

- This process can be continued. If we derivate again the blending functions, we can obtain the second derivative of the curve in any point:

$$\ddot{\mathbf{Q}}(t) = \left(t^3, t^2, t, 1\right) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 12 & -12 & 6 & 6 \\ -6 & 6 & -4 & -2 \end{pmatrix} \begin{pmatrix} \mathbf{P}_1 \\ \mathbf{P}_2 \\ \mathbf{v}_1 \\ \mathbf{v}_2 \end{pmatrix}$$