8. Limits

Definition 8.1. *Let $P \in \mathbb{R}^n$ be a point. The* ***open ball of radius*** *$\epsilon > 0$* ***about*** *P is the set*

$$B_\epsilon(P) = \{\, Q \in \mathbb{R}^n \mid \|\overrightarrow{PQ}\| < \epsilon \,\}.$$

The ***closed ball of radius*** *$\epsilon > 0$* ***about*** *P is the set*

$$\{\, Q \in \mathbb{R}^n \mid \|\overrightarrow{PQ}\| \leq \epsilon \,\}.$$

Definition 8.2. *A subset $A \subset \mathbb{R}^n$ is called* ***open*** *if for every $P \in A$ there is an $\epsilon > 0$ such that the open ball of radius ϵ about P is entirely contained in A,*

$$B_\epsilon(P) \subset A.$$

We say that B is ***closed*** *if the complement of B is open.*

Put differently, an open set is a union of open balls. Open balls are open and closed balls are closed. $[0, 1)$ is neither open nor closed.

Definition 8.3. *Let $B \subset \mathbb{R}^n$. We say that $P \in \mathbb{R}^n$ is a* ***limit point*** *if for every $\epsilon > 0$ the intersection*

$$B_\epsilon(P) \cap B \neq \varnothing.$$

Example 8.4. *0 is a limit point of*

$$\{\, \frac{1}{n} \mid n \in \mathbb{N} \,\} \subset \mathbb{R}.$$

Lemma 8.5. *A subset $B \subset \mathbb{R}^n$ is closed if and only if B contains all of its limit points.*

Example 8.6. *$\mathbb{R}^n - \{0\}$ is open. One can see this directly from the definition or from the fact that the complement $\{0\}$ is closed.*

Definition 8.7. *Let $A \subset \mathbb{R}^n$ and let $P \in \mathbb{R}^n$ be a limit point. Suppose that $f\colon A \longrightarrow \mathbb{R}^m$ is a function.*

We say that f approaches L as Q approaches P and write

$$\lim_{Q \to P} f(Q) = L,$$

if for every $\epsilon > 0$ we may find $\delta > 0$ such that whenever $Q \in B_\delta(P) \cap A$ $f(Q) \in B_\epsilon(L)$. In this case we call L the ***limit****.*

It might help to understand the notion of a limit in terms of a game played between two people. Let's call the first player Larry and the second player Norman. Larry wants to show that L is the limit of $f(Q)$ as Q approaches P and Norman does not.

So Norman gets to choose $\epsilon > 0$. Once Norman has chosen $\epsilon > 0$, Larry has to choose $\delta > 0$. The smaller that Norman chooses $\epsilon > 0$, the harder Larry has to work (typically he will have to make a choice of $\delta > 0$ very small).

Proposition 8.8. *Let $f\colon A \longrightarrow \mathbb{R}^m$ and $g\colon A \longrightarrow \mathbb{R}^m$ be two functions. Let $\lambda \in \mathbb{R}$ be a scalar. If P is a limit point of A and*

$$\lim_{Q\to P} f(Q) = L \qquad \text{and} \qquad \lim_{Q\to P} g(Q) = M,$$

then

(1) $\lim_{Q\to P}(f+g)(Q) = L + M$, *and*
(2) $\lim_{Q\to P}(\lambda f)(Q) = \lambda L$.

Now suppose that $m = 1$.

(3) $\lim_{Q\to P}(fg)(Q) = LM$, *and*
(4) *if $M \neq 0$, then* $\lim_{Q\to P}(f/g)(Q) = L/M$.

Proof. We just prove (1). Suppose that $\epsilon > 0$. As L and M are limits, we may find δ_1 and δ_2 such that, if $\|Q - P\| < \delta_1$ and $Q \in A$, then $\|f(Q) - L\| < \epsilon/2$ and if $\|Q - P\| < \delta_2$ and $Q \in A$, then $\|g(Q) - L\| < \epsilon/2$.

Let $\delta = \min(\delta_1, \delta_2)$. If $\|Q - P\| < \delta$ and $Q \in A$, then

$$\begin{aligned}
\|(f+g)(Q) - L - M\| &= \|(f(Q) - L) + (g(Q) - M)\| \\
&\leq \|(f(Q) - L)\| + \|(g(Q) - M)\| \\
&\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon,
\end{aligned}$$

where we applied the triangle inequality to get from the second line to the third line. This is (1). (2-4) have similar proofs. □

Definition 8.9. *Let $A \subset \mathbb{R}^n$ and let $P \in A$. If $f\colon A \longrightarrow \mathbb{R}^m$ is a function, then we say that f is continuous at P, if*

$$\lim_{Q\to P} f(Q) = f(P).$$

We say that f is continuous, if it continuous at every point of A.

Theorem 8.10. *If $f\colon \mathbb{R}^n \longrightarrow \mathbb{R}$ is a polynomial function, then f is continuous.*

A similar result holds if f is a rational function (a quotient of two polynomials).

Example 8.11. *$f\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$ given by $f(x, y) = x^2 + y^2$ is continuous.*

Sometimes Larry is very lucky:

Example 8.12. *Does the limit*

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x - y},$$

exist? Here the domain of f is

$$A = \{\, (x, y) \in \mathbb{R}^2 \mid x \neq y \,\}.$$

Note $(0,0)$ is a limit point of A. Note that if $(x, y) \in A$, then

$$\frac{x^2 - y^2}{x - y} = x + y,$$

so that

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x - y} = \lim_{(x,y)\to(0,0)} x + y = 0.$$

So the limit does exist.

Norman likes the following result:

Proposition 8.13. *Let $A \subset \mathbb{R}^n$ and let $B \subset \mathbb{R}^m$. Let $f\colon A \longrightarrow B$ and $g\colon B \longrightarrow \mathbb{R}^l$.*

Suppose that P is a limit point of A, L is a limit point of B and

$$\lim_{Q\to P} f(Q) = L \qquad \text{and} \qquad \lim_{M\to L} g(M) = E.$$

Then

$$\lim_{Q\to P} (g \circ f)(Q) = E.$$

Proof. Let $\epsilon > 0$. We may find $\delta > 0$ such that if $\|M - L\| < \delta$, and $M \in B$, then $\|g(M) - E\| < \epsilon$. Given $\delta > 0$ we may find $\eta > 0$ such that if $\|Q - P\| < \eta$ and $Q \in A$, then $|f(Q) - L\| < \eta$. But then if $\|Q - P\| < \eta$ and $Q \in A$, then $M = f(Q) \in B$ and $\|M - L\| < \delta$, so that

$$\begin{aligned} \|(g \circ f)(Q) - E\| &= \|g(f(Q)) - E\| \\ &= \|g(M) - E\| \\ &< \epsilon. \end{aligned}$$

□

Example 8.14. *Does*

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + y^2}$$

exist? The answer is no.

To show that the answer is no, we suppose that the limit exists. Suppose we consider restricting to the x-axis. Let

$$f\colon \mathbb{R} \longrightarrow \mathbb{R}^2,$$

be given by $t \longrightarrow (t,0)$. As f is continuous, if we compose we must get a function with a limit,

$$\lim_{t\to 0} \frac{0}{t^2+0} = \lim_{t\to 0} 0 = 0.$$

Now suppose that we restrict to the line $y = x$. Now consider the function

$$f\colon \mathbb{R} \longrightarrow \mathbb{R}^2,$$

be given by $t \longrightarrow (t,t)$. As f is continuous, if we compose we must get a function with a limit,

$$\lim_{t\to 0} \frac{t^2}{t^2+t^2} = \lim_{t\to 0} \frac{1}{2} = \frac{1}{2}.$$

The problem is that the limit along two different lines is different. So the original limit cannot exist.

Example 8.15. *Does the limit*

$$\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2},$$

exist? Let us use polar coordinates. Note that

$$\frac{x^3}{x^2+y^2} = \frac{r^3\cos^3\theta}{r^2} = r\cos^3\theta.$$

So we guess the limit is zero.

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \left|\frac{x^3}{x^2+y^2}\right| &= \lim_{r\to 0} |r\cos^3\theta| \\ &\leq \lim_{r\to 0} |r| = 0.\end{aligned}$$

Example 8.16. *Does the limit*

$$\lim_{(x,y,z)\to(0,0,0)} \frac{xyz}{x^2+y^2+z^2},$$

exist? Same trick, but now let us use spherical coordinates.

$$\begin{aligned}\lim_{(x,y,z)\to(0,0,0)} \left|\frac{xyz}{x^2+y^2+z^2}\right| &= \lim_{\rho\to 0} \left|\frac{\rho^3\cos^2\phi\sin\phi\cos\theta\sin\theta}{\rho^2}\right| \\ &= \lim_{\rho\to 0} |\rho\cos^2\phi\sin\phi\cos\theta\sin\theta| \\ &\leq \lim_{\rho\to 0} |\rho| = 0.\end{aligned}$$

Sometimes Norman needs to restrict to more complicated curves than just lines:

Example 8.17. *Does the limit*

$$\lim_{(x,y)\to(0,0)} \frac{y}{y+x^2},$$

exist? If we restrict to the line $t \longrightarrow (at, bt)$*, then we get*

$$\lim_{t\to 0} \frac{bt}{bt + a^2t^2} = \lim_{t\to 0} \frac{b}{b+at} = 1.$$

But if we restrict to the conic $t \longrightarrow (t, at^2)$*, then we get*

$$\lim_{t\to 0} \frac{at^2}{at^2 + t^2} = \lim_{t\to 0} \frac{a}{1+a} = \frac{a}{1+a},$$

and the limit changes as we vary a*, so that the limit does not exist.*

Note that if we start with

$$\frac{y}{y+x^d},$$

then Norman even needs to use curves of degree d,

$$t \longrightarrow (t, at^d).$$