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Vector Calculus

Fifth Edition

Chapter 3:
High-Order Derivatives: Maxima and Minima

3.1 Iterated Partial Derivatives

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3.1 Iterated Partial Derivatives

Key Points in this Section.

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1. **Equality of Mixed Partials.** If $f(x, y)$ is C^2 (has continuous 2nd partial derivatives), then

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}.$$

2. The idea of the proof is to apply the mean value theorem to the "difference of differences" written in the two ways

$$\begin{aligned} S(h, k) &= \{S(x + h, y + k) - S(x + h, y)\} - \{S(x, y + k) - S(x, y)\} \\ &= \{S(x + h, y + k) - S(x, y + k)\} - \{S(x + h, y) - S(x, y)\} \end{aligned}$$

3. Higher order partials are also symmetric; for example, for $f(x, y, z)$,

$$\frac{\partial^4 f}{\partial x \partial^2 z \partial y} = \frac{\partial^4 f}{\partial x \partial y \partial^2 z}$$

4. Many important equations describing nature involve partial derivatives, such as the ***heat equation*** for the temperature $T(x, y, z, t)$:

$$\frac{\partial T}{\partial t} = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right).$$

example 1 Find all second partial derivatives of $f(x, y) = xy + (x + 2y)^2$.

example 2 Find all second partial derivatives of $f(x, y) = \sin x \sin^2 y$.

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$$\frac{\partial f}{\partial x} = y + 2(x + 2y) \qquad \frac{\partial^2 f}{\partial x^2} = 2, \qquad \frac{\partial^2 f}{\partial y^2} = 8$$

$$\frac{\partial f}{\partial y} = x + 4(x + 2y) \qquad \frac{\partial^2 f}{\partial x \partial y} = 5, \qquad \frac{\partial^2 f}{\partial y \partial x} = 5.$$

example 2 Find all second partial derivatives of $f(x, y) = \sin x \sin^2 y$.

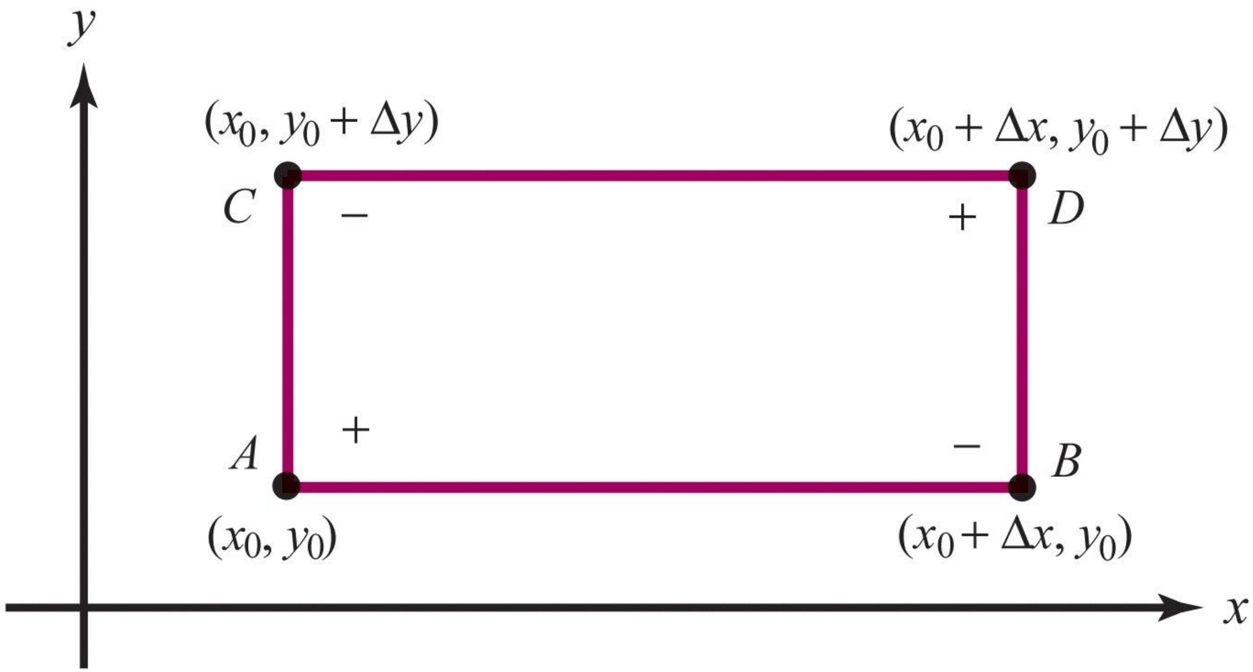
$$\frac{\partial f}{\partial x} = \cos x \sin^2 y \qquad \frac{\partial f}{\partial y} = 2 \sin x \sin y \cos y = \sin x \sin 2y$$

$$\frac{\partial^2 f}{\partial x^2} - = -\sin x \sin^2 y, \qquad \frac{\partial^2 f}{\partial y^2} = 2 \sin x \cos 2y;$$

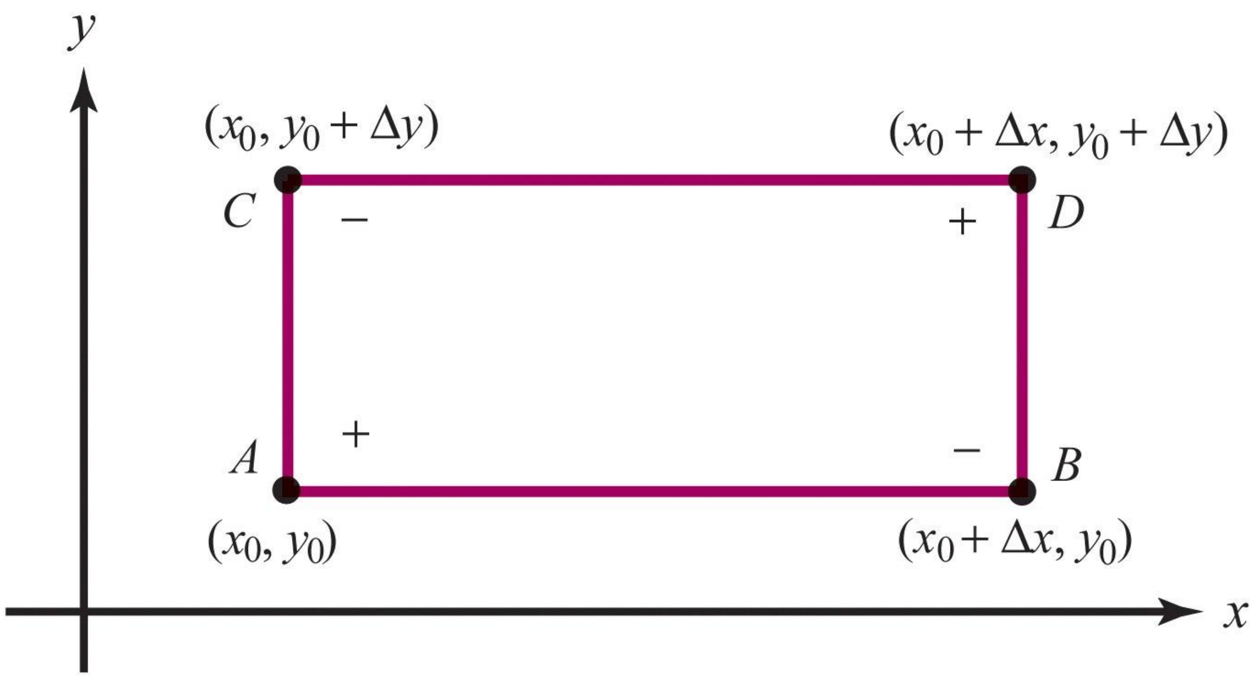
$$\frac{\partial^2 f}{\partial x \partial y} = \cos x \sin 2y, \qquad \frac{\partial^2 f}{\partial y \partial x} = 2 \cos x \sin y \cos y = \cos x \sin 2y.$$

THEOREM 1: Equality of Mixed Partials If $f(x, y)$ is of class C^2 (is twice continuously differentiable), then the mixed partial derivatives are equal; that is,

$$\frac{\partial^2 f}{\partial x\, \partial y} = \frac{\partial^2 f}{\partial y\, \partial x}.$$



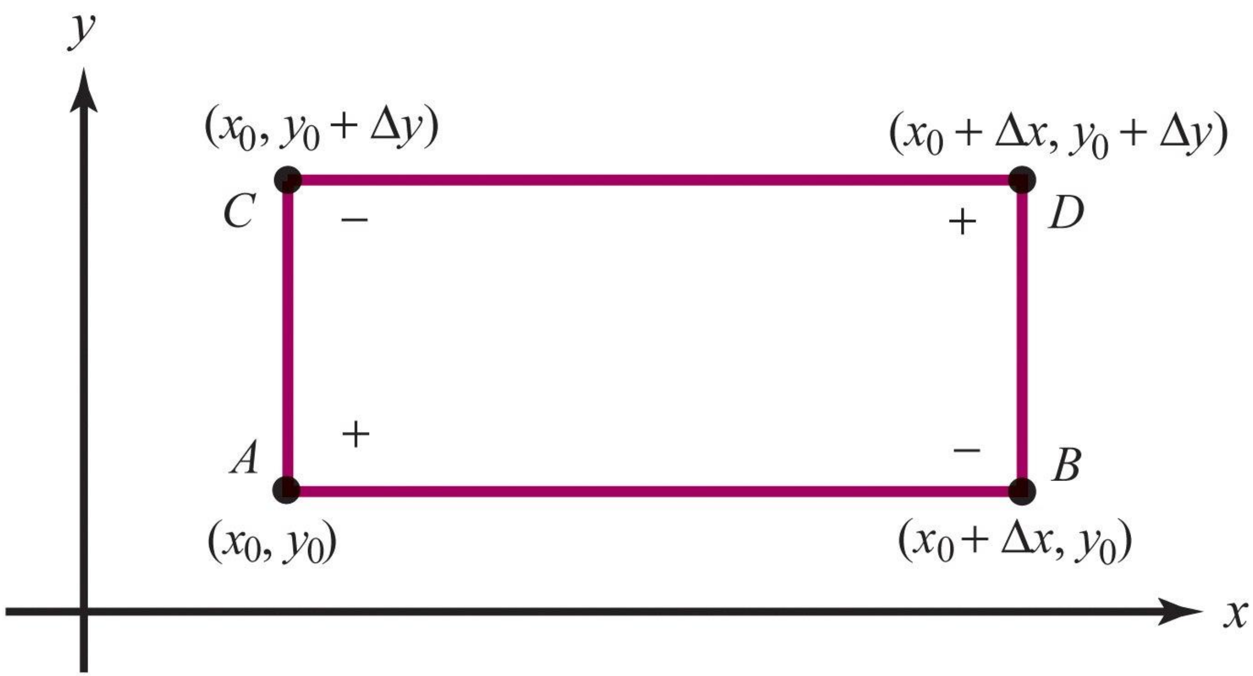
$$\begin{aligned} S(\Delta x, \Delta y) = {} & f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0 + \Delta x, y_0) \\ & - f(x_0, y_0 + \Delta y) + f(x_0, y_0). \end{aligned}$$



$$S(\Delta x, \Delta y) = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0 + \Delta x, y_0) \\ - f(x_0, y_0 + \Delta y) + f(x_0, y_0).$$

$$g(x) = f(x, y_0 + \Delta y) - f(x, y_0)$$

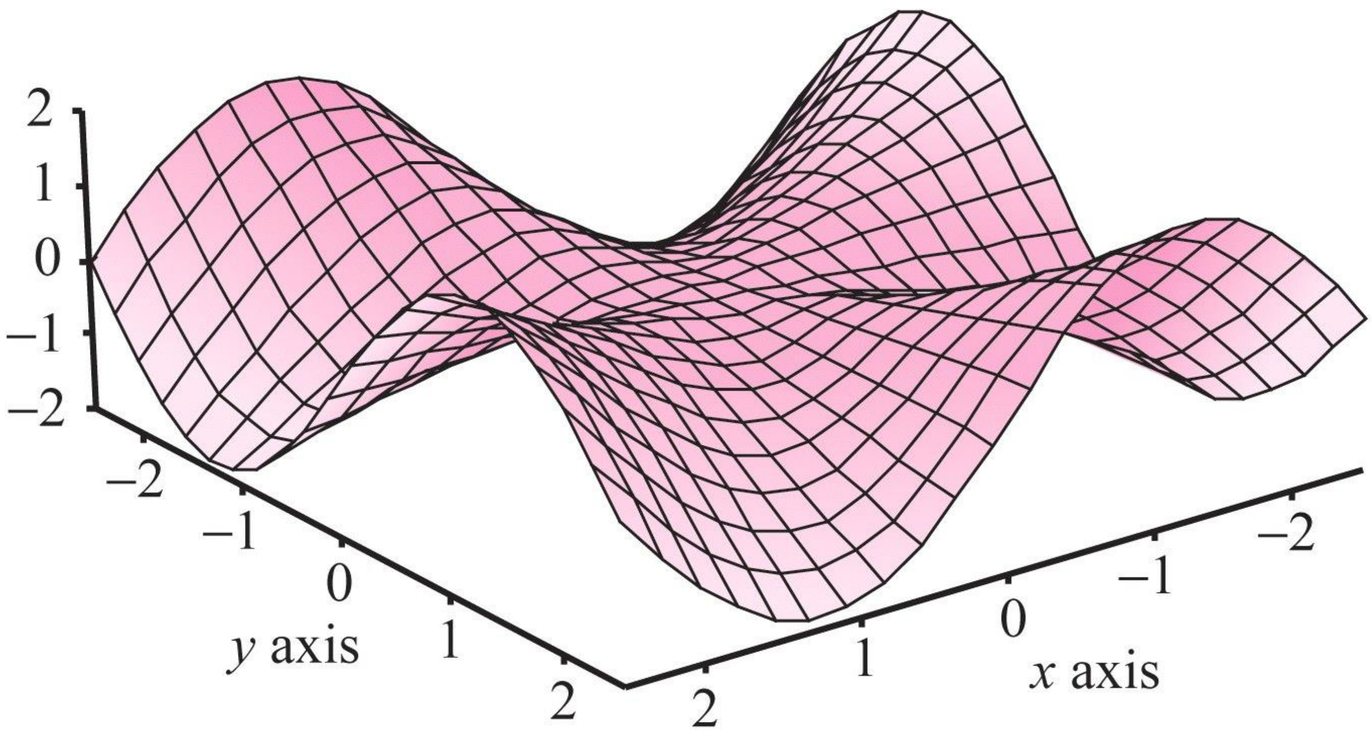
$$S(\Delta x, \Delta y) = g(x_0 + \Delta x) - g(x_0) = g'(\bar{x})\,\Delta x \\ = \left[\frac{\partial f}{\partial x}(\bar{x}, y_0 + \Delta y) - \frac{\partial f}{\partial x}(\bar{x}, y_0)\right]\Delta x = \frac{\partial^2 f}{\partial y\,\partial x}(\bar{x}, \bar{y})\,\Delta x\,\Delta y.$$



$$S(\Delta x, \Delta y) = \frac{\partial^2 f}{\partial y \, \partial x}(\bar{x}, \bar{y}) \, \Delta x \, \Delta y.$$

$$\frac{\partial^2 f}{\partial y \, \partial x}(x_0, y_0) = \operatorname*{limit}_{(\Delta x, \Delta y) \to (0,0)} \frac{1}{\Delta x \Delta y}[S(\Delta x, \Delta y)]$$

$$f(x, y) = \begin{cases} xy(x^2 - y^2)/(x^2 + y^2), & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$



$$\left.\frac{\partial^2 f}{\partial x \partial y}\right|_{(0,0)} \neq \left.\frac{\partial^2 f}{\partial y \partial x}\right|_{(0,0)}$$