Solution of Linear Systems

Iterative Methods

1) Write MATLAB scripts to implement both the Gauss-Seidel and the Jacobi method and use them to solve, with an accuracy of $\epsilon = 0.000005$ the system $\mathbf{Ax} = \mathbf{b}$ where the elements of $\mathbf{A}$ are

$$\begin{cases} a_{ii} = -4 \\ a_{ij} = 1 \text{ if } |i-j| = 1 \\ a_{ij} = 0 \text{ if } |i-j| \geq 2, \text{ where } i,j = 1,2,\ldots,10 \end{cases}$$

and

$$\mathbf{b}^T = (2,3,4,\ldots,11)$$

What happens if we change the matrix A by the matrix?

$$\begin{cases} a_{ii} = -4 \\ a_{ij} = 2 \text{ if } |i-j| = 1 \\ a_{ij} = 0 \text{ if } |i-j| \geq 2, \text{ where } i,j = 1,2,\ldots,10 \end{cases}$$

2) Solve the following linear system using both Jacobi and Gauss-Seidel methods.

$$\begin{cases} x_1 + 2x_2 - 2x_3 = 1 \\ x_1 + x_2 + x_3 = 1 \\ 2x_1 + 2x_2 + x_3 = 1 \end{cases}$$

Write in each case the respectiye iteration matrix. Compute the eigenvalues of the iteration matrices with the MATLAB command *eig*. Explain the results.

3) Using the MATLAB *eig* function compute the spectral radius of the iterative matrices of the Jacobi and Gauss-Seidel method for the matrix:

$$\mathbf{A} = \begin{pmatrix} 3 & 1 & -1 \\ 4 & -10 & 1 \\ 2 & 1 & 5 \end{pmatrix}$$

a) Show that the iterative schemes of Jacobi and Gauss-Seidel will converge.

b) Give a plot of the spectral radius of the iterative matrix for the SOR method. Give an estimation of the optimum value of ω.

4) Given the linear system

$$\begin{pmatrix} 1 & -a \\ -a & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

where $a \in \mathbb{R}$, it can be solved using the SOR method:

$$\begin{pmatrix} 1 & 0 \\ -\omega a & 1 \end{pmatrix} \mathbf{x}^{(k+1)} = \begin{pmatrix} 1-\omega & -\omega a \\ 0 & 1-\omega \end{pmatrix} \mathbf{x}^{(k)} + \omega \mathbf{b}$$

1. Which is the range of values of a for which the method is convergent if $\omega = 1$?

2. Take $a = 0.5$. Solve the system by the SOR method for $\omega \in (0.85, 125)$

3. Show that in this case, the optimum value of ω is

$$\omega = \frac{2}{a^2}\left(1 - \sqrt{1 - a^2}\right).$$

5) Consider the following linear system with $\mathbf{Ax} = \mathbf{b}$, where

$$\mathbf{A} = \begin{pmatrix} 4 & -2 & 0 & 0 \\ -2 & 3 & 2 & 0 \\ 0 & 2 & 3 & -1 \\ 0 & 0 & -1 & 4 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ -1 \\ -1 \end{pmatrix}$$

Use the Jacobi, Gauss-Seidel and SOR methods (taking $\omega = 1.007$). Compute the spectral radius in each case and get the solution accurate to $4, 6,$ and 8 significant digits, with

$$x^{(0)} = (2.5, 5.5, -4.5, -0.5)^T.$$

6) Given the linear system $\mathbf{Ax} = \mathbf{b}$ with

$$\mathbf{A} = \begin{pmatrix} 4 & -1 & 0 & 0 \\ -1 & 4 & -1 & 0 \\ 0 & -1 & 4 & -1 \\ 0 & 0 & -1 & 4 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 2 \end{pmatrix}$$

a) Compute the spectral radius of the Jacobi and Gauss-Seidel Iterative matrices.

b) Use these results to obtain the optimal value to perform an SOR iterative process and show that with this value we have

$$\rho(B_\omega) < \rho(B_{GS}) < \rho(B_J)$$

c) Solve the system using the three methods with a precision of $\epsilon < 10^{-5}$

7) Ostrowski-Reich theorem. Consider the linear system $\mathbf{Ax} = \mathbf{b}$, with:

$$\mathbf{A} = \begin{pmatrix} 4 & 3 & 0 \\ 3 & 4 & -1 \\ 0 & -1 & 4 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Compute the spectral radius of the Jacobi iterative matrix $\rho(B_J)$. Explore the values $\omega \in (0.2, 1.6)$ with steps $\Delta\omega = 0.05$ and show thatthe optimum value for the SOR iterative method is:

$$\omega = \frac{2}{1 + \sqrt{1 - [\rho(B_J)]^2}}$$

while the spectral radius is:

$$\rho(B_\omega) = \omega - 1.$$

This is always the case for positive definite tridiagonal matrices.

8) Consider the systems $Ax = b$ with

a)

$$A = H_n$$

b)

$$\mathbf{T}_n = (t_{ij}) = \begin{cases} 4, & i = j \\ 1, & |i-j| = 1 \\ 0, & \text{else} \end{cases},$$

Compute the condition numbers of the matrices for $n = 5 : 15$ and solve the systems with

$$b = (1,1,\ldots,1)^T$$

Compute the norm of the residual for each case.

9) Consider the linear system $\mathbf{Ax} = \mathbf{b}$, defined by :

$$\mathbf{A} = \begin{pmatrix} 4 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 4 & -1 & 0 & 0 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 4 & -1 & 0 & 0 & -1 & 0 & \ddots & \vdots & 0 \\ 0 & 0 & -1 & 4 & 0 & 0 & 0 & -1 & \ddots & \ddots & \vdots \\ -1 & 0 & 0 & 0 & 4 & -1 & 0 & 0 & \ddots & \ddots & \vdots \\ 0 & -1 & \ddots & \ddots & -1 & 4 & -1 & 0 & \ddots & \ddots & 0 \\ \vdots & 0 & \ddots & \ddots & 0 & \ddots & \ddots & -1 & \ddots & \ddots & -1 \\ \vdots & \ddots & \ddots & \ddots & 0 & 0 & \ddots & \ddots & -1 & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & -1 & 4 & -1 & 0 \\ 0 & \ddots & \ddots & \ddots & 0 & -1 & 0 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & \cdots & \cdots & 0 & -1 & 0 & 0 & -1 & 4 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ \vdots \\ \vdots \\ \vdots \\ 38 \\ 39 \\ 40 \end{pmatrix}$$

Such matrices are very common when using finite difference methods to solve differential equations.

a) Compute the condition number of the matrix using the MATLAB command *cond*

b) Solve the system using the backslash operator \ and compute the residual. Use the residual and the condition number to give an estimate of the error in the solution.

c) Solve the system using an iterative method and repeat the error analysis.

Iterative Inverses

1) An approximation for the inverse of $(\mathbf{I} - \mathbf{A})$ where $\mathbf{I}$ is an $n \times n$ unit matrix and A is an $n \times n$ matrix is given by

$$(\mathbf{I} - \mathbf{A})^{-1} = \mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \mathbf{A}^3 + \cdots$$

This seris only converges an the approximation is only valid if the maximum eigenvalue of $\mathbf{A}$ is less than 1. Write a MATLAB function *invapprox(A,k)* that obtains an approximation to $(\mathbf{I} - \mathbf{A})^{-1}$ using k terms of the given series. The function must find all eigenvalues of $\mathbf{A}$ using the MATLAB function *eig.* If the largest eigenvalue is greater that one, then a message will be output indicationg that the method fails. Otherwise, the function will compute an approximation to $(\mathbf{I} - \mathbf{A})^{-1}$ using k terms of the series expansion given. Taking $k = 4$, test the inverse with the matrices

$$\begin{pmatrix} 0.2 & 0.3 & 0 \\ 0.3 & 0.2 & 0.3 \\ 0 & 0.3 & 0.2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1.0 & 0.3 & 0 \\ 0.3 & 1.0 & 0.3 \\ 0 & 0.3 & 1.0 \end{pmatrix}$$

Use the *norm* function to compare the accuracy of the inverse of the matrix $(I - A)$ found using the MATLAB *inv* function and the function *invapprox(A,k)* form $k = 4, 8, 16$.

2) It can be proved that the series

$$(\mathbf{I} - \mathbf{A})^{-1} = \mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \mathbf{A}^3 + \cdots$$

where $\mathbf{A}$ is an $n \times n$ matrix, converges if the eigenvalues $\mathbf{A}$ are all less than unity. The following $n \times n$ matrix satisfies this condition if $a + 2b < 1$ and a and b are positive.

$$\begin{pmatrix} a & b & 0 & \cdots & 0 & 0 \\ b & a & b & \ddots & \vdots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \vdots & \ddots & b & a & b \\ 0 & 0 & \cdots & 0 & b & a \end{pmatrix}$$

Experiment with this matrix for various values of n, a and b to illustrate that the series converges under the condition stated.

3) Let $\mathbf{A}$ be an $n \times n$ matrix and suppose that $\mathbf{X}_0$ is an approximation to $\mathbf{A}^{-1}$. Consider the succession of matrices given by the recurrence

$$\mathbf{E}_k = \mathbf{I} - \mathbf{A}\mathbf{X}_k$$
$$\mathbf{X}_{k+1} = \mathbf{X}_k(\mathbf{I} + \mathbf{E}_k + \mathbf{E}_k^2), \quad (k \geq 0).$$

1. Show that $\mathbf{E}_{k+1} = \mathbf{E}_k^3$ and that $\mathbf{X}_k = \mathbf{A}^{-1}(\mathbf{I} - \mathbf{E}_0^{3^k}), \ (k \geq 0)$.

2. Give necessary and sufficient conditions for the convergence of $\mathbf{X}_k$ to $\mathbf{A}^{-1}$

$$\lim_{k \to \infty} \mathbf{X}_k = \mathbf{A}^{-1}$$

3. If we take $\mathbf{X}_0$ such that

$$\mathbf{E}_0 = \begin{pmatrix} 0.2 & 0.2 & 0.3 \\ 0.2 & 0.3 & 0.1 \\ 0.3 & 0.1 & 0.2 \end{pmatrix}$$

How many iterations do we need to obtain $\|\mathbf{E}_k\|_\infty < 10^{-12}$?