4.1

$$A(x,y) = (1+x^2)y^2$$

a) Legendre transform $B(x,z)$

$$z = \frac{\partial A}{\partial y} = 2(1+x^2)y \Rightarrow y = \frac{z}{2(1+x^2)}$$

$$B(x,z) = yz - A(x,y) = \frac{z^2}{2(1+x^2)} - \cancel{(1+x^2)}\frac{z^2}{4(1+x^2)^{\cancel{2}}} =$$

$$= \frac{z^2}{4(1+x^2)} \quad \square$$

b) $B(x,z) = \dfrac{z^2}{4(1+x^2)}$ $\qquad y = \dfrac{\partial B}{\partial z} = \dfrac{z}{2(1+x^2)} \Rightarrow z = 2y(1+x^2)$

$$A(x,y) = yz - B(x,z) = y\,2y(1+x^2) - \frac{4y^2(1+x^2)^{\cancel{2}}}{4\cancel{(1+x^2)}} =$$

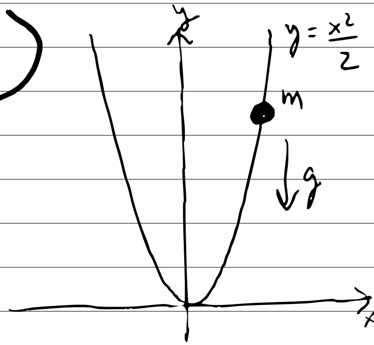
$$= y^2(1+x^2) \quad \square$$

c) Show we cannot choose arbitrary transformation $y = z$

$$B(x,z) = yz - A(x,y) = z^2 - (1+x^2)z^2 = -x^2z^2$$

Now, if we try to reverse transform, we get

4.2



a) $\mathcal{L}(x, \dot{x})$?

$T = \frac{m}{2}(\dot{x}^2 + \dot{y}^2)$

$V = mgy$

$y = \frac{x^2}{2}$, $\dot{y} = x\dot{x}$; $\dot{y}^2 = x^2\dot{x}^2$

$$\boxed{\mathcal{L}} = \frac{m}{2}(\dot{x}^2 + x^2\dot{x}^2) - \frac{m}{2} g x^2 = \boxed{\frac{m}{2}\dot{x}^2(1+x^2) - \frac{m}{2} g x^2}$$

b) $\boxed{p} = \frac{\partial \mathcal{L}}{\partial \dot{x}} = \boxed{m\dot{x}(1+x^2)}$

c) $\dot{x} = \frac{p}{m(1+x^2)}$

$$\boxed{H} = p\dot{x} - \mathcal{L} = \frac{p^2}{m(1+x^2)} - \frac{\cancel{m}}{2}\frac{p^2}{m^2(1+x^2)^2}\cancel{(1+x^2)} + \frac{m}{2} g x^2$$

$$= \boxed{\frac{p^2}{2m(1+x^2)} + \frac{m}{2} g x^2}$$

d) $\boxed{\dot{x}} = \frac{\partial H}{\partial p} = \boxed{\frac{p}{m(1+x^2)}}$; $\boxed{\dot{p}} = -\frac{\partial H}{\partial x} = -\frac{p^2 x}{m(1+x^2)^2} - mgx =$

$$= \boxed{-x\left(\frac{p^2}{m(1+x^2)^2} + m g\right)}$$

e)

$$\frac{p^2}{2m(1+x^2)} + \frac{m}{2} g x^2 = E \Rightarrow \boxed{p = \pm\sqrt{\left(E - \frac{m}{2} g x^2\right) 2m(1+x^2)}}$$

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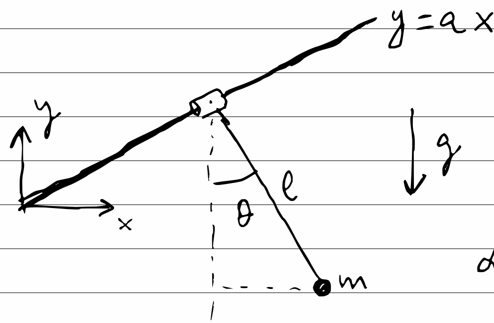


$$T = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2\right)$$

$$V = +mgy$$

$$\mathcal{L} = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2\right) - mgy$$

The coordinates of the support are $(X, Y = aX)$. We describe the system in terms of X and θ.

$$x = X + \ell \sin\theta \;;\quad \dot{x} = \dot{X} + \ell\cos\theta\,\dot{\theta} \;;\quad \dot{x}^2 = \dot{X}^2 + \ell^2\cos^2\theta\,\dot{\theta}^2 + 2\dot{X}\dot{\theta}\ell\cos\theta$$

$$y = Y - \ell\cos\theta \;;\quad \dot{y} = a\dot{X} + \ell\sin\theta\,\dot{\theta} \;;\quad \dot{y}^2 = a^2\dot{X}^2 + \ell^2\sin^2\theta\,\dot{\theta}^2 + 2a\dot{X}\dot{\theta}\ell\sin\theta$$

$$\mathcal{L} = \frac{m}{2}\left[(1+a^2)\dot{X}^2 + \ell^2\dot{\theta}^2 + 2\ell\dot{X}\dot{\theta}(\cos\theta + a\sin\theta)\right] - mg(aX - \ell\cos\theta)$$

Get p_X, p_θ

$$\begin{cases} p_X = \dfrac{\partial\mathcal{L}}{\partial\dot{X}} = \overbrace{m(1+a^2)}^{\equiv B}\dot{X} + \overbrace{m\ell}^{\equiv C}\dot{\theta}\overbrace{(\cos\theta + a\sin\theta)}^{} \equiv B\dot{X} + C\dot{\theta} \\[2ex] p_\theta = \dfrac{\partial\mathcal{L}}{\partial\dot{\theta}} = \underbrace{m\ell^2}_{\equiv A}\dot{\theta} + \underbrace{m\ell}_{\equiv C}\dot{X}\underbrace{(\cos\theta + a\sin\theta)}_{} \equiv C\dot{X} + A\dot{\theta} \end{cases}$$

Get $\dot{X}, \dot{\theta}$

$$\begin{cases} A p_X - C p_\theta = (AB - C^2)\dot{X} \Rightarrow \dot{X} = \dfrac{A p_X - C p_\theta}{AB - C^2} \\[2ex] C p_X - B p_\theta = (C^2 - BA)\dot{\theta} \Rightarrow \dot{\theta} = \dfrac{B p_\theta - C p_X}{AB - C^2} \end{cases}$$

$$\mathcal{L} = \frac{B}{2}\dot{X}^2 + \frac{A}{2}\dot{\theta}^2 + C\dot{X}\dot{\theta} - mg(aX - \ell\cos\theta)$$

$$\boxed{H} = \dot{\theta} p_\theta + \dot{X} p_x - \mathcal{L} = \frac{B p_\theta^2 - C p_\theta p_x}{AB - C^2} + \frac{A p_x^2 - C p_\theta p_x}{AB - C^2} -$$

$$- \frac{B}{2} \frac{A^2 p_x^2 + C^2 p_\theta^2 - 2AC p_x p_\theta}{(AB - C^2)^2} - \frac{A}{2} \frac{B^2 p_\theta^2 + C^2 p_x^2 - 2BC p_x p_\theta}{(AB - C^2)^2}$$

$$- C \frac{(A p_x - C p_\theta)(B p_\theta - C p_x)}{(AB - C^2)^2} + mg(aX - l\cos\theta)$$

$$= \frac{B p_\theta^2 + A p_x^2 - 2C p_\theta p_x}{AB - C^2} + mg(aX - l\cos\theta)$$

$$- \frac{1}{2}\left[\frac{BA^2 p_x^2 + B\cancel{C^2} p_\theta^2 - 2ABC p_x p_\theta + AB^2 p_\theta + A\cancel{C^2} p_x^2 - \cancel{2ABC p_x p_\theta}}{(AB - C^2)^2} + \right.$$

$$\left. \frac{+ \cancel{2CAB p_x p_\theta} - \cancel{2}C^2 A p_x^2 - \cancel{2}C^2 B p_\theta^2 + 2C^3 p_x p_\theta}{(AB - C^2)^2} \right] =$$

$$= mg(aX - l\cos\theta) + \frac{B p_\theta^2 + A p_x^2 - 2C p_x p_\theta}{AB - C^2} -$$

$$- \frac{1}{2}\left[\frac{A p_x (\cancel{AB - C^2}) + B p_\theta (\cancel{AB - C^2}) - 2C p_x p_\theta (\cancel{AB - C^2})}{(AB - C^2)^{\cancel{2}}} \right] =$$

$$\boxed{= mg(aX - l\cos\theta) + \frac{1}{AB - C^2}\left[\frac{A}{2} p_x^2 + \frac{B}{2} p_\theta^2 - C p_x p_\theta \right]} \square$$

4.4

$$x = R\sin\theta\cos\phi$$
$$y = R\sin\theta\sin\phi$$
$$z = R(1-\cos\theta)$$

(See Exercise 3.6 for the expression of the kinetic energy on spherical coordinates)

a) $$\mathcal{L} = \frac{m}{2}\left(\dot r^2 + r^2\dot\theta^2 + r^2\dot\phi^2\sin^2\theta\right) - mgr\cos\theta$$

$r = R =$ constant $\Rightarrow$ $$\boxed{\mathcal{L} = \frac{m}{2}\left(R^2\dot\theta^2 + R^2\dot\phi^2\sin^2\theta\right) + mgR\cos\theta}$$

+ constant

b) $$p_\phi = \frac{\partial\mathcal{L}}{\partial\dot\phi} = mR^2\dot\phi\sin^2\theta \Rightarrow \dot\phi = \frac{p_\phi}{mR^2\sin^2\theta}$$

$$p_\theta = \frac{\partial\mathcal{L}}{\partial\dot\theta} = mR^2\dot\theta \Longrightarrow \dot\theta = \frac{p_\theta}{mR^2}$$

c) $$\boxed{H} = \dot\theta p_\theta + \dot\phi p_\phi - \mathcal{L} =$$

$$= \frac{p_\theta^2}{mR^2} + \frac{p_\phi^2}{mR^2\sin^2\theta} - \left[\frac{m}{2}\left(R^2\frac{p_\theta^2}{m^2R^4} + R^2\sin^2\theta\frac{p_\phi^2}{m^2R^4\sin^4\theta}\right)\right.$$

$$\left. + mgR\cos\theta\Big)\right] =$$

$$= \frac{p_\theta^2}{mR^2} + \frac{p_\phi^2}{mR^2\sin^2\theta} - \frac{p_\theta^2}{2mR^2} - \frac{p_\phi^2}{2mR^2\sin^2\theta} - mgR\cos\theta =$$

$$\boxed{= \frac{p_\theta^2}{2mR^2} + \frac{p_\phi^2}{2mR^2\sin^2\theta} - mgR\cos\theta}$$

d) $\dot{\theta} = \dfrac{\partial H}{\partial p_\theta} = \dfrac{p_\theta}{mR^2}$

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = -\frac{p_\phi^2}{2mR^2}\left(-2\sin^{-3}\theta\right)\cos\theta - mgR\sin\theta =$$

$$= \frac{p_\phi^2 \cos\theta}{mR^2\sin^3\theta} - mgR\sin\theta$$

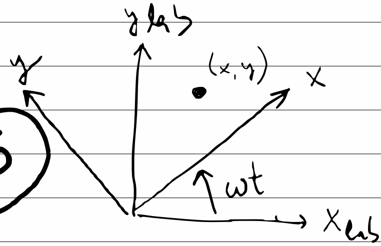
$$\dot{\phi} = \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{mR^2\sin^2\theta}$$

$$\dot{p}_\phi = -\frac{\partial H}{\partial \phi} = 0 \Rightarrow p_\phi = \text{constant}$$

e) $\boxed{\ddot{\theta}} = \dfrac{\dot{p}_\theta}{mR^2} = \dfrac{1}{mR^2}\left[\dfrac{p_\phi^2\cos\theta}{mR^2\sin^3\theta} - mgR\sin\theta\right] =$

$$= \boxed{\frac{p_\phi^2\cos\theta}{m^2R^4\sin^3\theta} - \frac{g\sin\theta}{R}}$$

4.5



$$x_{lab} = x\cos\omega t - y\sin\omega t$$

$$y_{lab} = x\sin\omega t + y\cos\omega t$$

$$\dot{x}_{lab} = \dot{x}\cos\omega t - x\omega\sin\omega t - \dot{y}\sin\omega t - y\omega\cos\omega t$$

$$\dot{y}_{lab} = \dot{x}\sin\omega t + x\omega\cos\omega t + \dot{y}\cos\omega t - y\omega\sin\omega t$$

$$\dot{x}_{lab}^2 = \dot{x}^2\cos^2 + x^2\omega^2\sin^2 + \dot{y}^2\sin^2 + y^2\omega^2\cos^2 - 2\dot{x}x\cancel{\omega}\cos\sin - 2\dot{x}\dot{y}\cancel{\sin}\cos - 2\dot{x}y\omega\cos^2 +$$
$$+ 2x\dot{y}\omega\sin^2 + 2xy\omega^2\cancel{\sin}\cos + 2y\dot{y}\cancel{\omega}\sin\cos$$

$$\dot{y}_{lab}^2 = \dot{x}^2\sin^2 + x^2\omega^2\cos^2 + \dot{y}^2\cos^2 + y^2\omega^2\sin^2 + 2x\cancel{\dot{x}\omega}\sin\cos + 2\dot{x}\cancel{\dot{y}}\sin\cos - 2\dot{x}y\omega\sin^2 +$$
$$+ 2x\dot{y}\omega\cos^2 - 2xy\omega^2\cancel{\sin}\cos - 2y\dot{y}\omega\cancel{\sin}\cos$$

$$\boxed{\mathcal{L}} = \frac{m}{2}\left(\dot{x}^2 + x^2\omega^2 + \dot{y}^2 + y^2\omega^2 - 2\dot{x}y\omega + 2x\dot{y}\omega\right) - V(x,y) =$$

$$\boxed{= \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2\right) + \frac{m\omega^2}{2}\left(x^2 + y^2\right) + m\omega\left(x\dot{y} - \dot{x}y\right) - V(x,y)}$$

b) $$p_x = \frac{\partial\mathcal{L}}{\partial\dot{x}} = m(\dot{x} - \omega y) \Rightarrow \dot{x} = \frac{p_x + \omega y m}{m}$$

$$p_y = \frac{\partial\mathcal{L}}{\partial\dot{y}} = m(\dot{y} + \omega x) \Rightarrow \dot{y} = \frac{p_y - \omega x m}{m}$$

c) $\boxed{H} = p_x \dot{x} + p_y \dot{y} - \mathcal{L} = \frac{p_x^2}{m} + \omega y \, p_x + \frac{p_y^2}{m} - \omega x \, p_y + V(x,y)$

$$- \frac{m}{2}\left(\frac{p_x^2 + \omega^2 y^2 m^2 + 2 p_x m \omega y + p_y^2 + \omega^2 x^2 m^2 - 2 p_y m \omega x}{m^2} \right) - \frac{m\omega^2}{2}(x^2+y^2)$$

$$- m\omega\left(\frac{x p_y}{m} - \omega x^2 - \frac{y p_x}{m} - \omega y^2 \right) =$$

$$= \frac{p_x^2}{m} + \omega y \, p_x + \frac{p_y^2}{m} - \omega x \, p_y - V(x,y) - \frac{p_x^2}{2m} - \frac{m}{2}\omega^2 y^2$$

$$- \cancel{p_x \omega y} - \frac{p_y^2}{2m} - \cancel{\frac{m\omega^2}{2} x^2} + \cancel{p_y \omega x} - \cancel{\frac{m\omega^2 x^2}{2}} - \cancel{\frac{m\omega^2}{2} y^2}$$

$$- \cancel{\omega x p_y} + \cancel{m\omega^2 x^2} + \cancel{\omega y p_x} + \cancel{m\omega^2 y^2} + V(x,y)$$

$$= \boxed{\frac{p_x^2 + p_y^2}{2m} + \omega\left(y \, p_x - x \, p_y\right) + V(x,y)} \quad \square$$

4.6

a) $K(p,\dot{p},t) = \mathcal{L}(q,\dot{q},t) - p\dot{q} - q\dot{p} =$

$= \mathcal{L}(q,\dot{q},t) - \frac{d}{dt}(pq)$

so they only differ in a total time derivative $\square$.

b) $\mathcal{L} = p\dot{q} - H$, $H = \frac{1}{2}(p^2 + \omega^2 q^2)$

$\dot{q} = \frac{\partial H}{\partial p} = p$

$\boxed{\mathcal{L}} = \dot{q}^2 - \frac{\dot{q}^2}{2} - \frac{\omega^2 q^2}{2} = \boxed{\frac{\dot{q}^2}{2} - \frac{\omega^2 q^2}{2}}$

c) $p = \frac{\partial \mathcal{L}}{\partial \dot{q}} = \dot{q}$

$\dot{p} = -\frac{\partial H}{\partial q} = -\omega^2 q \Rightarrow q = -\frac{\dot{p}}{\omega^2}$

d) $\boxed{K(p,\dot{p},t)} = \mathcal{L} - p\dot{q} - q\dot{p} =$

$= \frac{\dot{q}^2}{2} - \frac{\omega^2 q^2}{2} - p \cdot p + \frac{\dot{p}}{\omega^2}\dot{p} =$

$= \frac{p^2}{2} - \frac{\omega^2 \dot{p}^2}{2\omega^4} - p^2 + \frac{\dot{p}^2}{\omega^2} = -\frac{p^2}{2} + \frac{\dot{p}^2}{2\omega^2} =$

$= \boxed{\frac{\dot{p}^2}{2\omega^2} - \frac{p^2}{2}}$