Exercises - Hamiltonian mechanics - Poisson brackets and canonical transformations

5.1 Poisson's theorem. Consider the functions $F(\mathbf{q}, \mathbf{p})$ and $G(\mathbf{q}, \mathbf{p})$, which do **not** depend explicitly on time and are, additionally, constants of motion. Prove, using only the properties of Poisson brackets, that $[F, G]$ is also a constant of motion.[1]

[1] In fact, Poisson's theorem also holds for $F(\mathbf{q}, \mathbf{p}, t)$ and $G(\mathbf{q}, \mathbf{p}, t)$ with an explicit time dependence, but for proving that one needs more than Poisson brackets.

5.2 Constants of motion. Consider the following Hamiltonians corresponding to two different dynamical systems with 4 and 3 degrees of freedom, respectively:

$$H = p_1^2 + \frac{p_2^2}{q_1} + \frac{(p_4 - q_4)^2}{q_1 \cos q_2} + \frac{p_3^2}{q_1 q_4 \cos q_2} - \frac{\sin^2 q_2}{q_1} + q_1^2$$

$$H = \frac{1}{2}\left(\frac{p_1}{q_1^2} + k q_1\right)^2 + \left(\frac{p_1}{q_1^2} + k q_1\right) q_3 t + \frac{p_3^2}{2} + (p_2 + B q_3)^2 \ .$$

Obtain as many constants of motion as possible by inspection of the structure of each Hamiltonian. Then, use Poisson's theorem from exercise 5.1, to see if you can obtain even more constants of motion.

5.3 Consider a system with n degrees of freedom, whose kinetic and potential energies are

$$T = \sum_{i=1}^{n} f_i(q_i) q_i'^2 \quad \text{and} \quad V = \sum_{i=1}^{n} V_i(q_i)$$

(a) Obtain the Lagrangian of the system.

(b) Prove that the Hamiltonian of the system can be written as $H = \sum_{i=1}^{n} H_i(q_i, p_i)$, and obtain $H_i(q_i, p_i)$.

(c) Is H a constant of motion? Are the H_i conserved independently?

5.4 Prove that, for a 1-dimensional harmonic oscillator with Hamiltonian

$$H = \frac{p^2}{2m} + \frac{k}{2} q^2 \,,$$

there exists a constant of motion $u(q, p, t)$ given by

$$u(q, p, t) = \ln(p + i m \omega q) - i \omega t \quad \text{with} \quad \omega = \sqrt{\frac{k}{m}} \,.$$

5.5 Show that the transformation

$$Q = p + iaq \quad \text{and} \quad P = \frac{p - iaq}{2ia}$$

(with $i = \sqrt{-1}$) is canonical and find the generating function F_2. Use the transformation to formulate the linear harmonic oscillator problem

$$H = \frac{p^2}{2m} + \frac{k}{2}q^2$$

in the variables (Q, P), and solve Hamilton's canonical equations for Q and P in the case $a = \sqrt{mk}$.

5.6 Find under which conditions the following transformation is canonical

$$Q = \alpha \frac{p}{x} \quad \text{and} \quad P = \beta x^2 \, .$$

Then, find a generating function F_1 for this transformation.

5.7 The Hamiltonian for a system consisting of a mass m sliding without friction on a cycloid curve (as in the brachistochrone problem!) is

$$H = \frac{p_\phi^2}{4ma^2(1 - \cos\phi)} - mag(1 - \cos\phi) \, .$$

(a) Prove that the following transformation is canonical[2]

$$s = -4a\cos(\phi/2) \quad \text{and} \quad p_s = \frac{p_\phi}{2a\sin(\phi/2)} \, .$$

(b) Obtain a generating function of type F_2. Is it possible to obtain a generating function of type F_1?

(c) Show that the transformed Hamiltonian $\tilde{H}(s, p_s)$ is the Hamiltonian of a harmonic oscillator.

[2] Note that s is simply the distance traveled by the mass.