Exercicis

3. Verify the first special case of the chain rule for the composition $f \circ \mathbf{c}$ in each of the cases:

(a) $f(x, y) = xy$, $\mathbf{c}(t) = (e^t, \cos t)$

(b) $f(x, y) = e^{xy}$, $\mathbf{c}(t) = (3t^2, t^3)$

(c) $f(x, y) = (x^2 + y^2) \log \sqrt{x^2 + y^2}$, $\mathbf{c}(t) = (e^t, e^{-t})$

(d) $f(x, y) = x \exp(x^2 + y^2)$, $\mathbf{c}(t) = (t, -t)$

6. Let $f\colon \mathbb{R}^3 \to \mathbb{R}$ be differentiable. Making the substitution

$$x = \rho \cos\theta \sin\phi, \qquad y = \rho \sin\theta \sin\phi, \qquad z = \rho \cos\phi$$

(spherical coordinates) into $f(x, y, z)$, compute $\partial f/\partial\rho$, $\partial f/\partial\theta$, and $\partial f/\partial\phi$ in terms of $\partial f/\partial x$, $\partial f/\partial y$, and $\partial f/\partial z$.

7. Let $f(u, v) = (\tan(u - 1) - e^v, u^2 - v^2)$ and $g(x, y) = (e^{x-y}, x - y)$. Calculate $f \circ g$ and $\mathbf{D}(f \circ g)(1, 1)$.

11. Let $f(x, y, z) = (3y + 2, x^2 + y^2, x + z^2)$. Let $\mathbf{c}(t) = (\cos(t), \sin(t), t)$.

(a) Find the path $\mathbf{p} = f \circ \mathbf{c}$ and the velocity vector $\mathbf{p}'(\pi)$.

(b) Find $\mathbf{c}(\pi)$, $\mathbf{c}'(\pi)$ and $\mathbf{D} f(-1, 0, \pi)$.

(c) Thinking of $\mathbf{D} f(-1, 0, \pi)$ as a linear map, find $\mathbf{D} f(-1, 0, \pi)\,(\mathbf{c}'(\pi))$.

13. Suppose that a duck is swimming in the circle $x = \cos t$, $y = \sin t$ and that the water temperature is given by the formula $T = x^2 e^y - xy^3$. Find dT/dt, the rate of change in temperature the duck might feel: (a) by the chain rule; (b) by expressing T in terms of t and differentiating.

15. Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2; (x, y) \mapsto (e^{x+y}, e^{x-y})$. Let $\mathbf{c}(t)$ be a path with $\mathbf{c}(0) = (0, 0)$ and $\mathbf{c}'(0) = (1, 1)$. What is the tangent vector to the image of $\mathbf{c}(t)$ under f at $t = 0$?

16. Let $f(x, y) = 1/\sqrt{x^2 + y^2}$. Compute $\nabla f(x, y)$.

17. Write out the chain rule for each of the following functions and justify your answer in each case using Theorem 11.

(a) $\partial h/\partial x$, where $h(x, y) = f(x, u(x, y))$

(b) dh/dx, where $h(x) = f(x, u(x), v(x))$

(c) $\partial h/\partial x$, where $h(x, y, z) = f(u(x, y, z), v(x, y), w(x))$

21. Dieterici's equation of state for a gas is

$$P(V - b)\, e^{a/RVT} = RT,$$

where a, b, and R are constants. Regard volume V as a function of temperature T and pressure P and prove that

$$\frac{\partial V}{\partial T} = \left(R + \frac{a}{TV}\right) \Big/ \left(\frac{RT}{V - b} - \frac{a}{V^2}\right).$$

33. Let $f\colon \mathbb{R}^4 \to \mathbb{R}$ and $\mathbf{c}(t)\colon \mathbb{R} \to \mathbb{R}^4$. Suppose $\nabla f(1, 1, \pi, e^6) = (0, 1, 3, -7)$, $\mathbf{c}(\pi) = (1, 1, \pi, e^6)$, and $\mathbf{c}'(\pi) = (19, 11, 0, 1)$. Find $\dfrac{d(f \circ \mathbf{c})}{dt}$ when $t = \pi$.

34. Suppose $f\colon \mathbb{R}^n \to \mathbb{R}^m$ and $g\colon \mathbb{R}^p \to \mathbb{R}^q$.

(a) What must be true about the numbers n, m, p, and q for $f \circ g$ to make sense?

(b) What must be true about the numbers n, m, p, and q for $g \circ f$ to make sense?

(c) When does $f \circ f$ make sense?

35. If $z = f(x - y)$, use the chain rule to show that $\dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y} = 0$.

Solucions

3. Compute each in two ways; the answers are

(a) $(f \circ \mathbf{c})'(t) = e^t(\cos t - \sin t)$

(b) $(f \circ \mathbf{c})'(t) = 15t^4 \exp(3t^5)$

(c) $(f \circ \mathbf{c})'(t) = (e^{2t} - e^{-2t})[1 + \log(e^{2t} + e^{-2t})]$

(d) $(f \circ \mathbf{c})'(t) = (1 + 4t^2)\exp(2t^2)$

7. $(f \circ g)(x, y) = (\tan(e^{x-y} - 1) - e^{x-y}, e^{2(x-y)} - (x-y)^2)$ and $\mathbf{D}(f \circ g)(1, 1) = \begin{bmatrix} 0 & 0 \\ 2 & -2 \end{bmatrix}$.

11. (a) $\mathbf{p}(t) = (3\sin(t) + 2, 1, \cos(t) + t^2)$,
$\mathbf{p}'(\pi) = (-3, 0, 2\pi)$

(b) $\mathbf{c}(\pi) = (-1, 0, \pi)$, $\mathbf{c}'(\pi) = (0, -1, 1)$,

$$\mathbf{D}f(-1, 0, \pi) = \begin{bmatrix} 0 & 3 & 0 \\ -2 & 0 & 0 \\ 1 & 0 & 2\pi \end{bmatrix}$$

(c) $(-3, 0, 2\pi)$

13. $-2\cos t \sin t e^{\sin t} + \sin^4 t + \cos^3 t e^{\sin t} - 3\cos^2 t \sin^2 t$ for both (a) and (b).

17. (a) $h(x, y) = f(x, u(x, y)) = f(p(x), u(x, y))$. We use p here solely as notation: $p(x) = x$.

Written out: $$\frac{\partial h}{\partial x} = \frac{\partial f}{\partial p}\frac{dp}{dx} + \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} = \frac{\partial f}{\partial p} + \frac{\partial f}{\partial u}\frac{\partial u}{\partial x}$$

because $$\frac{dp}{dx} = \frac{dx}{dx} = 1$$

Justification: Call (p, u) the variables of f. To use the chain rule we must express h as a composition of functions; that is, first find g such that $h(x, y) = f(g(x, y))$. Let $g(x, y) = (p(x), u(x, y))$. Therefore, $\mathbf{D}h = (\mathbf{D}f)(\mathbf{D}g)$. Then

$$\begin{bmatrix} \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial p} & \frac{\partial f}{\partial u} \end{bmatrix} \begin{bmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} \\ \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial f}{\partial p} & \frac{\partial f}{\partial u} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial f}{\partial p} + \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} & \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} \end{bmatrix},$$

and so $\frac{\partial h}{\partial x} = \frac{\partial f}{\partial p} + \frac{\partial f}{\partial u}\frac{\partial u}{\partial x}$. You may see $\frac{\partial h}{\partial x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial u}\frac{\partial u}{\partial x}$ as an answer. This requires careful interpretation because of possible ambiguity about the meaning of $\partial f/\partial x$, which is why the name p was used.

(b) $\frac{\partial h}{\partial x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial u}\frac{du}{dx} + \frac{\partial f}{\partial v}\frac{dv}{dx}$

(c) $\frac{\partial h}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial f}{\partial w}\frac{dw}{dx}$

21. Apply the chain rule to $\partial G/\partial T$, where $G(t(T, P), p(T, P), V(T, P)) = P(V - b)e^{a/RVT} - RT$ is identically 0; $t(T, P) = T$; and $p(T, P) = P$.

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35. Let $g(x, y) = x - y$, so that $z = f \circ g$. Then the chain rules implies $\frac{\partial z}{\partial x} = \frac{\partial f}{\partial g}\frac{\partial g}{\partial x} = \frac{\partial f}{\partial g}$ and $\frac{\partial z}{\partial y} = \frac{\partial f}{\partial g}\frac{\partial g}{\partial y} = -\frac{\partial f}{\partial g}$.