**EXERCISES - PROBABILITY THEORY**

1. **Stirling's approximation.** In its crudest form, Stirling's approximation states that $\log n! \approx n(\log n - 1)$ for $n \gg 1$.
   - Show that if we take a continuous approximation to compute $\log n!$, we obtain the above approximation.
   - Write a short program that computes the error of the approximation. For which values of $n$ can you consider that the approximation is acceptable?

2. **Discrete probability distributions.** We have a large number of atoms in a system. Each atom decays at a rate such that after a time $t$, there is a probability $\lambda t$ for the atom to have decayed. The probability that $n$ atoms have decayed at time $t$ is $p_n = (\lambda t)^n \dfrac{e^{-\lambda t}}{n!}$. What is the average number of atoms $\langle n \rangle$ that will have decayed at time $t$?

3. **Continuous probability distributions.** Consider that the velocity of the molecules within a gas follows Maxwell's distribution $p(\vec{v})\, d\vec{v} = Ae^{-\beta \frac{mv^2}{2}} d\vec{v}$
   - Compute the probability density of $v_x$
   - Find the expected value and the most probable value of $v_x$
   - Find the probability density of $\mathbf{v}$, the modulus of $\vec{v}$.
   - Find the expected value and the most probable value of $\mathbf{v}$.
   - Find the probability density of the momentum $\vec{p} = m\vec{v}$
   - Find the probability density of the kinetic energy of the particles.

4. **Marginals.** Consider the following probability distribution of two random variables $x, y$, $p(x, y) = Bx \exp\left(-\dfrac{x^2}{2} - yx\right)$ with $x, y \geq 0$. What is the probability that $x = 1$?

5. **Convolution.** The stochastic variables $X$ and $Y$ are independent and exponentially distributed with rate parameters $\lambda_x$, $\lambda_y$, respectively. Find the distribution of the stochastic variable $Z = X + Y$. What changes if the two rate parameters are the same?

6. **Moments and cumulants.** The stochastic variables $X$ and $Y$ are independent and Gaussian distributed with first moment $\langle X \rangle = \langle Y \rangle = 0$ and standard deviation $s_x = s_y = 1$. Find the distribution of the stochastic variable

$Z = X^2 + Y^2$, and compute the moments $\langle Z \rangle, \langle Z^2 \rangle, \langle Z^3 \rangle$. Find the first three cumulants.
HINT: use cylindrical coordinates and the property of the delta distribution $\int f(x)\ \delta(g(x))\, dx = \sum_i \frac{f(x_i)}{|g'(x_i)|}$ where $x_i$ are the roots of $g(x)$.

7. **Combinatorics.** Find the number of permutations of the letters in the word MONOTONOUS. In how many ways are 4 O's together? In how many ways are (only) 3 O's together?

8. **Combinatorics.** A book with 700 misprints contains 1400 pages.
   - What is the probability that one page contains no mistakes?
   - What is the probability that one pages contains 2 mistakes?

9. **Loaded dice.** A dice is loaded so that even numbers occur 3 times as often as odd numbers.
   1. If we roll the dice $N = 12$ times, what is the probability that odd numbers occur 3 times? If we roll it $N = 120$, what is the probability that odd numbers occur 30 times? Use the binomial distribution.
   2. Compute the same quantities as in part 1, but use the Gaussian distribution
   3. Compare answers (1) and (2). Plot the binomial and Gaussian distributions for the case $N = 12$.

10. **Quantum dice.** Your are given two unusual three-sided dice, which, when rolled, show either one, two or three spots. There are three games played with these dice: D*istinguishable, Bosons and Fermions.* In each game, each player rolls the dice twice. In *Distinguishable* all the rolls are legal. In *Bosons,* a roll is legal only if the second of the two dices shows a number that is larger or equal than the one show by the first dice. In *Fermions*, a roll is legal only if the second number is strictly larger than the preceding number. (In fact these dice rules are the same as those that govern the quantum statistics of noninteracting particles).
    1. Presume the dice are fair: each of the three numbers appear the same number of times. For a legal turn in each of the three games, what's the probability of $p(4)$ of rolling a 4?
    2. For a legal turn in the three games, what is the probability of of rolling a double?
    3. In a turn of three rolls, what is the enhancement of probability of getting triples in *Bosons* over that in *Distinguishable*? In a turn of M rolls, what is the enhancement of probability for generating an M-tuple (all rolls having the same number of dots showing)?

11. **Particle Statistics.** We want to place $n$ particles in $N \gg n$ boxes. Compute the probability that there is at most a particle in each box in the three following cases;
    1. The particles are distinguishable and there is no restriction in the number of particles (classical statistics)
    2. The particles are indistinguishable and there is no restriction in the number of particles that can be in the same box (Bose-Einstein statistics)
    3. The particles are identical and each box can contain at most one particle. (Fermi–Dirac statistics).