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Vector Calculus

Fifth Edition

Chapter 6:
The Change of Variables Formula and Applications of Integration

6.4 Improper Integrals

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Key Points in this Section.

1. Improper integrals occur when either (a) the function being integrated is unbounded in an elementary region D or (b) the region itself is unbounded. In case (a), if $f : D \to \mathbb{R}$ is unbounded at parts of the boundary of D, then we find a sequence of smaller regions, say $D_{\eta,\delta}$ obtained by "backing off" by an amount η from the sides and δ from the top and bottom. Then we define

$$\iint_D f\,dA = \lim_{(\eta,\delta)\to(0,0)} \iint_{D_{\eta,\delta}} f\,dA$$

if the limit exists. For y-simple regions,

$$\iint_{D_{\eta,\delta}} f\,dA = \int_{a+\eta}^{b-\eta} \int_{\phi_1(x)+\delta}^{\phi_2(x)-\delta} f(x,y)\,dy\,dx.$$

In case (b) one similarly finds a family of bounded regions expanding to the given region and again takes the limit of the integrals over the bounded regions.

2. **Fubini's Theorem.** If f is a function, satisfying $f \geq 0$, continuous except possibly on the boundary of a y-simple region D, and if the iterated (improper) integral

$$\int_a^b \int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy\,dx$$

exists, then f itself is integrable and $\iint_D f\,dA$ equals the iterated integral. Here, for each x,

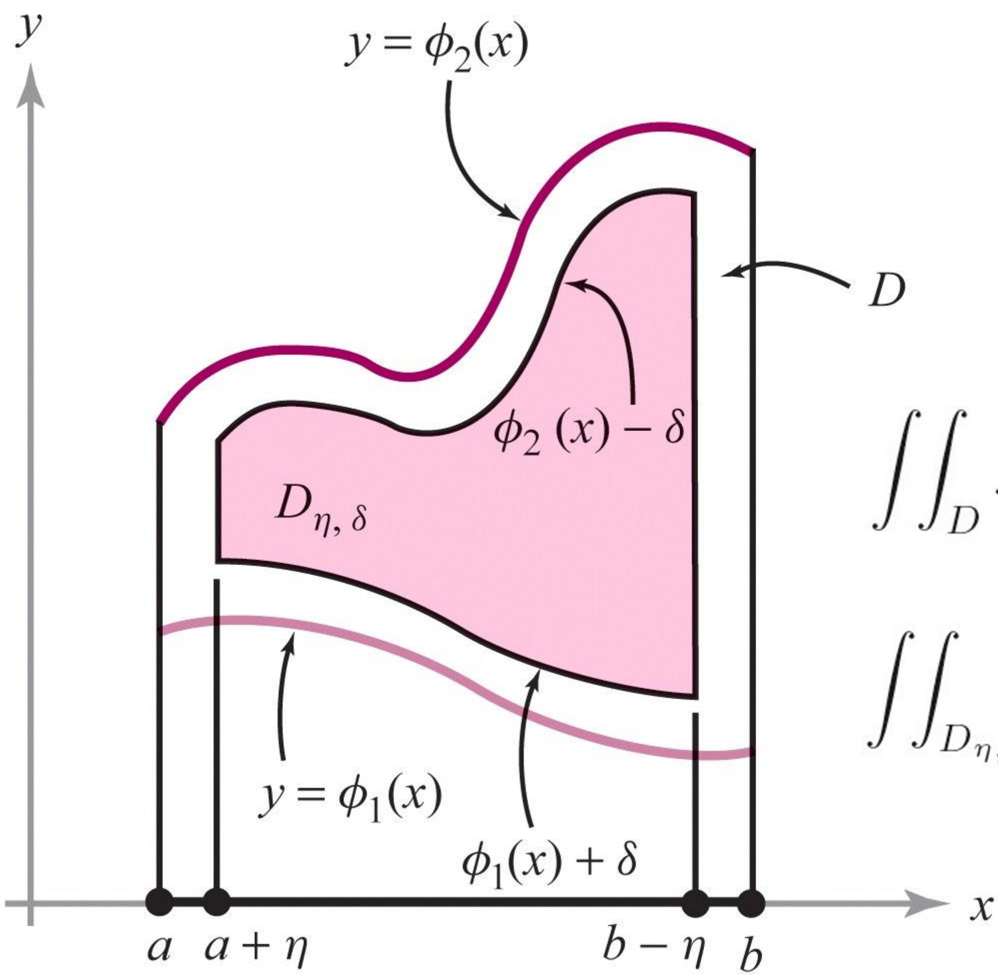
$$g(x) = \int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy = \lim_{\alpha \to 0+} \int_{\phi_1(x)+\alpha}^{\phi_2(x)-\alpha} f(x,y)\,dy,$$

and $\int_a^b g(x)\,dx = \lim_{\beta \to 0+} \int_{a+\beta}^{b-\beta} g(x)\,dx$, as in one variable calculus. There is a similar statement for x-simple regions.

The subtlety here is that for positive functions, two *single limits* can be replaced by one *double limit.* Exercise 18 shows that positivity of f is essential, or this result is not true.

Integrals impròpies

Cas 1: funció no fitada a la vora de la regió *D*



$$\iint_D f\, dA = \lim_{(\eta,\delta)\to(0,0)} \iint_{D_{\eta,\delta}} f\, dA$$

$$\iint_{D_{\eta,\delta}} f\, dA = \int_{a+\eta}^{b-\eta} \int_{\phi_1(x)+\delta}^{\phi_2(x)-\delta} f(x,y)\, dy\, dx$$

THEOREM 3: Fubini's Theorem Let D be an elementary region in the plane and $f \geq 0$ a function continuous except for points possibly on the boundary of D. If either of the integrals

$$\iint_D f(x, y)\, dA,$$

$$\int_a^b \int_{\phi_1(x)}^{\phi_2(x)} f(x, y)\, dy\, dx, \qquad \text{for } y\text{-simple regions}$$

$$\int_c^d \int_{\psi_1(y)}^{\psi_2(y)} f(x, y)\, dx\, dy \qquad \text{for } x\text{-simple regions}$$

exist as improper integrals, f is integrable and they are all equal.

Integrals impròpies

Cas 2: funció no fitada en un punt aïllat (x_0, y_0) de l'interior de la regió D

$$D_\delta = D_\delta(x_0, y_0)$$

$$\lim_{\delta \to 0} \iint_{D \setminus D_\delta} f \, dA$$

Integrals impròpies

Cas 3: funció fitada en una regió *D* no fitada

$$D_1 \subset D_2 \subset \cdots \subset D_n \subset \cdots \subset D$$

$$\lim_{n\to\infty} D_n = D$$

$$\iint_D f(x,y)\,dxdy = \lim_{n\to\infty} \iint_{D_n} f(x,y)\,dxdy$$

Exercici

Demostrar que el valor de la integral gaussiana és

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}.$$

Passos:

- Calcular $\iint_{D_a} e^{-(x^2+y^2)}\, dx\, dy$ on D_a és el disc $x^2 + y^2 \leq a^2$
- Fer el límit $a \to \infty$
- Fer la mateixa integral doble en un domini $D_a = [-a, a]^2$