EXERCISES - QUANTUM STATISTICS

Exercise 1. For a 2 -dimensional medium in which the lowest-lying vibrational modes of wavelength λ have frequencies given by a dispersion relation of the form $\omega \sim \lambda^{-s}$ where $\lambda = \frac{1}{k}$ ($\vec{k}$ is the wavevector), find expressions for the internal energy and the specific heat at constant volume using the Debye approximation. What is the behaviour of $C_V(T)$ near absolute zero?

Solution: $\lim_{T\to 0} C_V \propto T^{2/s}$

Exercise 2. Suppose a Bose gas such that the energy of the ground state is $\epsilon_0 = 0$, $\epsilon_i \in (0, +\infty)$.

a) At T approaches 0, all particles are in the ground state. If the gas has *N* particles, find the expression of the chemical potential at low T such that $\langle N_0 \rangle = N$ for large N.

b) Assuming that the density of states is $g(\epsilon) = a\epsilon^2$, $\epsilon \geq 0$, obtain an expression for the expected number of particles that are not in the ground state using the large N limit using the expression of μ computed in a).

c) The condition for the onset of Bose-Einstein condensation is that all particles are in excited states (states other than the ground state). At temperatures below this 'critical temperature' T_c, Bose-Einstein condensation takes place. What is the temperature at which all particles are in excited states T_c?

NOTE:

Express the result in b) and c) using the definition of Riemann's ζ function:

$$\zeta(s) = \frac{1}{\Gamma(x)} \int_0^\infty \frac{x^{s-1}}{e^x - 1} dx$$

Solution: a) $\beta\mu = -\frac{1}{N}$**; b)** $N_{\text{ext}} = \frac{2a}{\beta^3}\zeta(3)$**; c)** $T_c = \frac{1}{k_B}\left(\frac{N}{2a\zeta(3)}\right)^{1/3}$

Exercise 3. Suppose an ideal Bose gas with particles with spins s (i.e. for each energy particles can be in 2s+1 different states) in a box of volume $V = L^3$ and chemical potential μ, for which the energy is given by $\epsilon(\vec{k}) = \frac{\hbar^2 k^2}{2m}$.

a) Compute the density of states assuming $\vec{k} = \frac{\pi}{L}\vec{n}$.

b) Compute the occupation of the ground state

c) Compute the equation of state using the grand canonical function potential.

Solution: a) $g(\epsilon) = \dfrac{(2s+1)\, Vm^{3/2}\sqrt{\epsilon}}{\sqrt{2}\hbar^3\pi^2}$; b) $\langle N_o \rangle = (2s+1)\dfrac{z}{1-z}$; c) $\beta P = \dfrac{(2s+1)}{\lambda^3} g_{5/2}(z)$, where $z = e^{\beta\mu}$,$\lambda = \sqrt{\dfrac{2\pi\beta\hbar^2}{m}}$ and $g_n(z) = \sum \dfrac{z^n}{n^{5/2}}$ is the PolyLogarithm function.

NOTE: To obtain the last result you have to perform a Taylor expansion for the integral of $\log \mathscr{Z}$

Exercise 4. Electrons in a piece of Cu metal can be assumed to behave as an ideal Fermi gas with $\vec{p} = \hbar\vec{k}$, and $\epsilon = \dfrac{p^2}{2m}$ two possible states per energy level ($\uparrow$, $\downarrow$). Cu metal in the solid state has mass density $\rho = 9\,\mathrm{gr/cm^3}$. Assume that each Cu atom donates an electron to the Fermi gas. Assume the system is at $T = 0\,K$ (that is, that $f(\epsilon)$ is a step function).
a) Compute the density of states $g(\epsilon)$
b) Compute the Fermi energy, ϵ_F, and the Fermi temperature, T_F, of the electron gas, by imposing the density of electrons $n_e = \dfrac{N}{V}$ in a Cu metal.

Solution: a) $g(\epsilon) = \dfrac{\sqrt{2}V}{\pi^2\hbar^3} m^{3/2}\sqrt{\epsilon} = g_0\sqrt{\epsilon}$; **b)** $\epsilon_F = \left(\dfrac{3n_e}{2g_0}\right)^{2/3}$, $T_F = \epsilon_F / k_B$

Exercise 5. The density of states of an ideal Fermi-Dirac gas is $g(\epsilon) = D, \epsilon > 0$ and 0 otherwise, where D is a constant. Compute the the chemical potential at finite temperature imposing that there are N gas particles. Check the consistency of your result result by calculating the chemical potential at $T \to 0$.

Solution: $\mu = k_B T \log\left(-1 + e^{\frac{\beta N}{D}}\right)$; for $T \to 0, \mu = N/D$

Exercise 6. Suppose a gas of N spin 1/2 fermions of mass m in a surface of area A. Obtain an explicit expression for the chemical potential of this gas a s a function of temperature.

Solution. $\mu = k_B T \log\left(e^{\epsilon_F/k_B T} - 1\right)$