Combinatorics and Probability

Part II: Recurrences

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Linear recurrences with constant coefficients - Introductory example

Example 0 (introductory example): Suppose we have an alphabet Σ consisting of the letters A, B and C. We can construct strings or words with the letters of this alphabet, respecting a single rule: Any non-terminal A has to be immediately followed by a B. What is the number of distinct words of length n that can be constructed, for each $n \geq 0$?

Let us denote by a_n the number of words of length n. We can start by constructing all the admissible words for small values of n:

- For $n = 0$ there is a single word (the empty word). Hence $a_0 = 1$.
- For $n = 1$ we have the words A, B and C. Hence $a_1 = 3$.
- For $n = 2$ we have the words AB, BA, BB, BC, CA, CB and CC. Hence $a_2 = 7$.

Linear recurrences with constant coefficients - Example 0 (continued)

For larger values of n it becomes increasingly difficult to enumerate all the admissible words. However, we may reason as follows:

Suppose that $n > 2$. The initial letter of the word could be A, B or C. If we choose B or C, then the second letter can again be A, B or C. So, we are left with a string of length $n-1$ that has to follow the same rule.

On the other hand, if we choose A as the first letter, then the second letter must be a B, so we are left with a string of length $n-2$ that must obey the same rule as before. Considering the two options, we get the equation

$$a_n = 2a_{n-1} + a_{n-2}.$$

This is a **recurrence equation** with initial values $a_0 = 1$ and $a_1 = 3$.

Linear recurrences with constant coefficients

We define a *recurrence equation* (or simply, a recurrence) as an equation of the form

$$a_n = f(a_{n-1}, a_{n-2}, \ldots, a_{n-k}),$$

with some initial conditions $a_0 = b_0, a_1 = b_1, \ldots, a_{k-1} = b_{k-1}$, where $b_0, b_1, \ldots, b_{k-1}$ are constant (in our setting we may assume that $b_0, b_1, \ldots, b_{k-1} \in \mathbb{R}$).

A *(homogeneous) linear recurrence with constant coefficients* is an equation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k},$$

with the initial conditions $a_0 = b_0, a_1 = b_1, \ldots, a_{k-1} = b_{k-1}$, where the coefficients $b_0, b_1, \ldots, b_{k-1}, c_1, c_2, \ldots c_k$ are real constants.

Special case: Second order homogeneous linear recurrences with constant coefficients

$$a_n = \begin{cases} b \text{ if } n = 0 \\ c \text{ if } n = 1 \\ p\, a_{n-1} + q\, a_{n-2} \text{ otherwise} \end{cases}$$

where $b, c, p, q \in \mathbb{R}$.

Now, we can represent the recurrence in matrix form:

$$\begin{pmatrix} a_{n+1} \\ a_n \end{pmatrix} = \begin{pmatrix} p & q \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_n \\ a_{n-1} \end{pmatrix}$$

So we have:

Second order homogeneous linear recurrences with constant coefficients

$$\begin{pmatrix} a_{n+1} \\ a_n \end{pmatrix} = \begin{pmatrix} p & q \\ 1 & 0 \end{pmatrix}^2 \begin{pmatrix} a_{n-1} \\ a_{n-2} \end{pmatrix} = \ldots = \begin{pmatrix} p & q \\ 1 & 0 \end{pmatrix}^k \begin{pmatrix} a_{n-(k-1)} \\ a_{n-k} \end{pmatrix}$$

For $k = n$ we have:

$$\begin{pmatrix} a_{n+1} \\ a_n \end{pmatrix} = \begin{pmatrix} p & q \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} a_1 \\ a_0 \end{pmatrix} = \begin{pmatrix} p & q \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} c \\ b \end{pmatrix}$$

In order to compute the powers of the matrix we can try to diagonalize it. First we compute the eigenvalues by solving the equation:

$$\begin{vmatrix} p-x & q \\ 1 & -x \end{vmatrix} = x^2 - px - q = 0$$

Second order homogeneous linear recurrences with constant coefficients

whose solutions are

$$\lambda = \frac{1}{2}\left(p + \sqrt{p^2 + 4q}\right), \quad \text{and} \quad \mu = \frac{1}{2}\left(p - \sqrt{p^2 + 4q}\right).$$

Let us assume that λ and μ are real and different, i.e. $\lambda > \mu$. Then $Q = S \cdot D \cdot S^{-1}$, where

$$D = \begin{pmatrix} \mu & 0 \\ 0 & \lambda \end{pmatrix}, \quad S = \begin{pmatrix} \mu & \lambda \\ 1 & 1 \end{pmatrix}, \quad \text{and} \quad S^{-1} = \frac{1}{\lambda - \mu}\begin{pmatrix} -1 & \lambda \\ 1 & -\mu \end{pmatrix}.$$

Second order homogeneous linear recurrences with constant coefficients

Therefore,

$$\begin{pmatrix} p & q \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} \mu & \lambda \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \mu^n & 0 \\ 0 & \lambda^n \end{pmatrix} \begin{pmatrix} -1 & \lambda \\ 1 & -\mu \end{pmatrix} \frac{1}{\lambda - \mu}$$

$$\begin{pmatrix} p & q \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} \frac{\lambda^{n+1}-\mu^{n+1}}{\lambda-\mu} & \frac{\lambda\mu^{n+1}-\mu\lambda^{n+1}}{\lambda-\mu} \\ \frac{\lambda^n-\mu^n}{\lambda-\mu} & \frac{\lambda\mu^n-\mu\lambda^n}{\lambda-\mu} \end{pmatrix}$$

$$\begin{pmatrix} a_{n+1} \\ a_n \end{pmatrix} = \begin{pmatrix} \frac{\lambda^{n+1}-\mu^{n+1}}{\lambda-\mu} & \frac{\lambda\mu^{n+1}-\mu\lambda^{n+1}}{\lambda-\mu} \\ \frac{\lambda^n-\mu^n}{\lambda-\mu} & \frac{\lambda\mu^n-\mu\lambda^n}{\lambda-\mu} \end{pmatrix} \begin{pmatrix} c \\ b \end{pmatrix}$$

Second order homogeneous linear recurrences with constant coefficients

Finally,

$$a_n = \frac{\lambda^n - \mu^n}{\lambda - \mu} c + \frac{\lambda\mu^n - \mu\lambda^n}{\lambda - \mu} b$$

$$= \alpha\lambda^n + \beta\mu^n,$$

for some constants α and β, which can be determined from the initial conditions.

Exercise: Determine α and β in general.

Second order recurrences – Binet's formula and Fibonacci numbers

In particular, if $b = 0$ and $c = 1$ we get Binet's formula:

$$a_n = \frac{\lambda^n - \mu^n}{\lambda - \mu}$$

Example 1 – Fibonnaci numbers: The Fibonacci numbers are given by the recurrence $F_n = F_{n-1} + F_{n-2}$, with $F_0 = 0$ and $F_1 = 1$. The characteristic equation $x^2 - x - 1 = 0$ has two distinct solutions, namely $\lambda = \frac{1}{2}(1 + \sqrt{5})$ and $\mu = \frac{1}{2}(1 - \sqrt{5})$, hence by Binet's formula

$$F_n = \frac{\lambda^n - \mu^n}{\lambda - \mu} = \frac{\lambda^n - \mu^n}{\sqrt{5}}$$

Second order recurrences – Fibonacci numbers

Example 1 (Fibonnaci numbers, continued):

The dominant root $\lambda \approx 1.618$ is usually denoted as φ, and it is known as *the golden ratio*, or *the divine proportion*, and it arises in many unexpected scenarios, not only in Mathematics, but also in nature and art.

The other root, $\mu \approx -0.618$, turns out to be $1 - \varphi = -\dfrac{1}{\varphi}$. Since $|\mu| < 1$ we have $\lim_{n \longrightarrow \infty} \mu^n = 0$. Therefore, F_n is the closest integer to $\dfrac{\varphi^n}{\sqrt{5}}$.

Second order homogeneous linear recurrences with constant coefficients – Example 2

Note: Our derivation is also valid when λ and μ are complex conjugates. This case gives rise to periodic solutions.

Example 2: Let $a_n = -a_{n-2}$, with $a_0 = 0$ and $a_1 = 1$. The characteristic equation $x^2 + 1 = 0$ has roots $\pm i$. Then $a_n = \alpha i^n + \beta(-i)^n$ for some constants α and β. In order to find α and β we solve the linear system

$$\begin{aligned} a_0 &= \alpha i^0 + \beta(-i)^0 = \alpha + \beta = 0 \\ a_1 &= \alpha i + \beta(-i) = \alpha i - \beta i = 1, \end{aligned}$$

which yields $\alpha = -\frac{1}{2} i$ and $\beta = \frac{1}{2} i$. Hence, $a_n = -\frac{1}{2}\left(i^{n+1} + (-i)^{n+1}\right)$.

Second order homogeneous linear recurrences – Example 2, continued

Example 2 (continued): By looking at some terms of the sequence we can conclude that

$$a_n = \begin{cases} 0 \ \text{ if } \ n = 2k \\ 1 \ \text{ if } \ n = 4k+1 \\ -1 \ \text{ if } \ n = 4k+3, \end{cases}$$

with $k = 0, 1, 2, \ldots$, which reveals the periodicity of the sequence. Alternatively, if we plot the points we can see that the ensuing graph has a **sinusoidal form**, and we can conclude that $a_n = \sin\left(\frac{n\pi}{2}\right)$.

We can also find this compact representation with the aid of the trigonometric form of complex numbers, as follows.

Second order homogeneous linear recurrences – Example 2, continued

Example 2 (continued): We have

$$
\begin{aligned}
a_n &= -\frac{1}{2}\left(i^{n+1} + (-i)^{n+1}\right) \\
&= -\frac{1}{2}\left[\cos\frac{(n+1)\pi}{2} + i\sin\frac{(n+1)\pi}{2}\right] \\
&\quad - \frac{1}{2}\left[\cos\left(-\frac{(n+1)\pi}{2}\right) + i\sin\left(-\frac{(n+1)\pi}{2}\right)\right] \\
&= -\frac{1}{2}\left[\cos\frac{(n+1)\pi}{2} + \cancel{i\sin\frac{(n+1)\pi}{2}}\right] \\
&\quad - \frac{1}{2}\left[\cos\frac{(n+1)\pi}{2} - \cancel{i\sin\frac{(n+1)\pi}{2}}\right].
\end{aligned}
$$

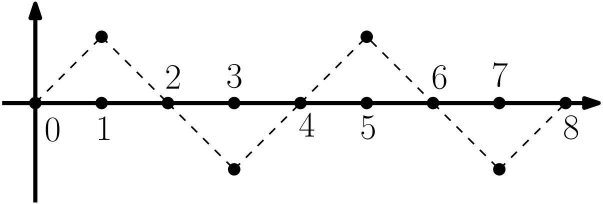
Second order homogeneous linear recurrences – Example 2, continued

Example 2 (continued): So,

$$a_n = -\cos\frac{(n+1)\pi}{2} = -\cos\left(\frac{n\pi}{2} + \frac{\pi}{2}\right) = \sin\left(\frac{n\pi}{2}\right),$$

because

$$\cos\left(\alpha + \frac{\pi}{2}\right) = -\sin\alpha.$$



Second order homogeneous linear recurrences with constant coefficients

Let us now suppose that the characteristic equation $x^2 - px - q = 0$ has a single root of multiplicity 2, i.e. $\lambda = \mu = \dfrac{p}{2}$. In that case the matrix

$$Q = \begin{pmatrix} p & q \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} p & -\frac{p^2}{4} \\ 1 & 0 \end{pmatrix}$$

is not diagonalizable, but we can still find its Jordan decomposition: $Q = S \cdot J \cdot S^{-1}$, where

$$J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}, \quad S = \begin{pmatrix} \lambda & 1 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad S^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -\lambda \end{pmatrix}.$$

(See Jordan decomposition)

Second order homogeneous linear recurrences with constant coefficients

Therefore,

$$\begin{aligned}
\begin{pmatrix} p & q \\ 1 & 0 \end{pmatrix}^n &= \begin{pmatrix} \lambda & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \lambda^n & n \cdot \lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & -\lambda \end{pmatrix} \\
&= \lambda^{n-1} \begin{pmatrix} \lambda & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \lambda & n \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & -\lambda \end{pmatrix} \\
&= \lambda^{n-1} \begin{pmatrix} \lambda(n+1) & -\lambda^2 n \\ n & \lambda(1-n) \end{pmatrix}
\end{aligned}$$

where $\lambda = \dfrac{p}{2}$

Second order homogeneous linear recurrences with constant coefficients

Hence,

$$\begin{pmatrix} a_{n+1} \\ a_n \end{pmatrix} = \begin{pmatrix} (n+1)\lambda^n & -n\lambda^{n+1} \\ n\lambda^{n-1} & (1-n)\lambda^n \end{pmatrix} \begin{pmatrix} c \\ b \end{pmatrix},$$

so that

$$a_n \;=\; P_1(n)\lambda^n + P_2(n)\lambda^{n-1} \;=\; Q(n)\lambda^n,$$

where $P_1(n)$, $P_2(n)$ and $Q(n)$ are polynomials of degree ≤ 1 in the indeterminate n, whose coefficients can be calculated from b and c. Note that some of these terms may vanish, so that the actual result may be simpler. For example, if $b = 0$ and $c = 1$ we get

$$a_n = n\lambda^{n-1}.$$

Cassini's identity

The matrix representation of recurrence relations is useful for obtaining other identities, e.g. Cassini's identity of Fibonacci numbers. Recall that

$$\begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix}, \quad \text{hence}$$

$$\begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} F_1 \\ F_0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}. = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \quad \text{Thus}$$

$$\begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n-1} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n$$

Taking determinants on both sides we get Cassini's identity:

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n$$

Homogeneous linear recurrences of arbitrary order

More generally, the characteristic equation associated with the linear recurrence

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k},$$

is

$$z^k - c_1 z^{k-1} - c_2 z^{k-2} - \cdots - c_{k-1} z - c_k = 0$$

The most general case is

$$c_0 a_n + c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} = 0,$$

whose characteristic equation is

$$c_0 z^k + c_1 z^{k-1} + c_2 z^{k-2} + \cdots + c_k = 0.$$

Homogeneous linear recurrences of arbitrary order

Theorem (Lueker, 1980)

Let $p(z)$ be the characteristic polynomial of the above equation, having roots $r_1, \ldots, r_t$, with respective multiplicities $m_1, \ldots, m_t$. Then, any solution of the general homogeneous recurrence is of the form

$$a_n = \sum_{i=1}^{t} \left(r_i^n \sum_{j=0}^{m_i-1} c_{ij} n^j \right),$$

where the c_{ij} are constants that can be determined from the initial conditions.

Example 3: 2nd order equation with double root

$$a_n = \begin{cases} b \text{ if } \ n = 0 \\ c \text{ if } \ n = 1 \\ 4\, a_{n-1} - 4\, a_{n-2} \ \text{ otherwise} \end{cases}$$

The characteristic equation $z^2 - 4z + 4 = 0$ has a single root $z = 2$ of multiplicity two. Then by the previous theorem, the general solution is of the form $a_n = (c_1 n + c_2)2^n$. The solutions for some particular values of b and c are:

If $b = 0, c = 1$ we get: $a_n \ = \ \frac{1}{2} n 2^n \ = \ n 2^{n-1}$

If $b = 1, c = 0$ we get: $a_n \ = \ (1 - n)2^n$

If $b = 1, c = 1$ we get: $a_n \ = \ \frac{1}{2}(2 - n)2^n \ = \ (2 - n)2^{n-1}$

Go to Example 4

Non-homogeneous linear recurrences

A non-homogeneous linear recurrence with constant coefficients is an equation of the form

$$c_0 a_n + c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} = f(n),$$

with the corresponding initial conditions $a_0 = b_0, a_1 = b_1, \ldots, a_{k-1} = b_{k-1}$, where the coefficients $b_0, b_1, \ldots, b_{k-1}, c_0, c_1, c_2, \ldots c_k$ are real constants, and $f(n)$ is a non-zero function of n.

Non-homogeneous linear recurrences

Any sequence a_n which satisfies the above non-homogeneous recurrence is called a *particular solution*. On the other hand, any sequence a_n which satisfies the corresponding homogeneous recurrence

$$c_0 a_n + c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} = 0.$$

is called a *homogeneous solution*

Theorem (Lueker, 1980)

If we start with any particular solution a_n and add any homogeneous solution we obtain another particular solution. Moreover the difference between any two particular solutions is always a homogeneous solution.

Method of undetermined coefficients for solving non-homogeneous linear recurrences

Let us again consider the non-homogeneous linear recurrence

$$c_0 a_n + c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} = f(n), \tag{1}$$

and its corresponding homogeneous recurrence

$$c_0 a_n + c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} = 0. \tag{2}$$

Then we have

Method of undetermined coefficients for solving non-homogeneous linear recurrences

Theorem (Lueker, 1980, reformulated)

The general solution $\langle a_n \rangle$ of Equation (1) must have the form

$$a_n = h(n) + p(n),$$

where $h(n)$ is the general solution of the homogeneous equation, i.e. Equation (2), and $p(n)$ is a particular solution of Equation (1).

The particular solution $p(n)$ can be found by judicious guessing, taking into account the form of the term $f(n)$.

Method of undetermined coefficients for solving non-homogeneous linear recurrences

Case 1:

$f(n)$ is a polynomial of degree t, i.e.

$$f(n) = b_t n^t + b_{t-1} n^{t-1} + \cdots + b_1 n + b_0$$

Try with

$$p(n) = c_t n^t + c_{t-1} n^{t-1} + \cdots + c_1 n + c_0,$$

where $c_0, \ldots, c_t$ are constants to be determined.

Method of undetermined coefficients for solving non-homogeneous linear recurrences

Case 2.1:

$f(n) = B\alpha^n$, where α is **not** a root of the characteristic polynomial of Equation (2). In this case we can try with $p(n) = C\alpha^n$, where C is a constant to be determined.

Case 2.2:

$f(n) = B\alpha^n$, where α is a root of multiplicity m of the characteristic polynomial of Equation (2). In this case try with $p(n) = Cn^m\alpha^n$, where C is a constant to be determined.

Method of undetermined coefficients – Example 4

Example 4 – Sequel of Example 3 and Case 2.2:

Find the general solution of the recurrence

$$a_n = \begin{cases} 1 \ \text{ if } \ n = 0 \ \text{ or } \ n = 1 \\ 4\ a_{n-1} - 4\ a_{n-2} + 2^n \ \text{ otherwise} \end{cases}$$

The characteristic equation is $z^2 - 4z + 4 = 0$, which has a single root $z = 2$ with multiplicity 2. In a previous example we saw that $h(n) = 2^n - n2^{n-1}$. Now, $p(n) = Cn^2 2^n$ for some constant C. In order to determine C we plug $p(n)$ into the recurrence, simplify and solve for C, as follows:

Method of undetermined coefficients – Example 4

$$\begin{aligned}
p(n) &= 4p(n-1) - 4p(n-2) + 2^n \\
Cn^2 2^n &= 4C(n-1)^2 2^{n-1} - 4C(n-2)^2 2^{n-2} + 2^n \\
4Cn^2 2^{n-2} &= 8C(n^2-2n+1)2^{n-2} - 4C(n^2-4n+4)2^{n-2} + 4 \cdot 2^{n-2} \\
8C2^{n-2} &= 42^{n-2} \\
&\text{whence } C = \frac{1}{2}.
\end{aligned}$$

Thus,

$$a_n = h(n) + p(n) = 2^n - n2^{n-1} + n^2 2^{n-1}.$$

Method of undetermined coefficients – Generalization of Case 2.2

Suppose that the particular solution that we want to try has the form $p(n) = p_1(n) + p_2(n) + \cdots p_s(n)$, and suppose further that one of the terms, say $p_i(n)$, is already a solution of the associated homogeneous equation, i.e. Equation (2). In this case we have to multiply $p_i(n)$ by n^m, where m is the smallest integer such that $n^m p_i(n)$ is *not* a solution of Equation (2).

Note: If $f(n)$ is not among the cases considered above, then the method of undetermined coefficients may not be applicable.

Method of undetermined coefficients – Principle of superposition

Principle of superposition:

If $f(n)$ is the sum (or product) of the cases considered before, then try with a $p(n)$ that is a sum (resp. product) of the corresponding candidate particular solutions.

Example 5: Let $f(n) = n^2 3^n$, i.e. the product of a second-degree polynomial and the exponential 3^n (where 3 is not a solution of the characteristic equation). Then our candidate particular solution $p(n)$ will have the form $(c_2 n^2 + c_1 n + c_0)3^n$.

General strategy for solving non-homogeneous linear recurrences

Apply transformations on both sides of the equation

$$c_0 a_n + c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} = f(n),$$

so that $f(n)$ vanishes. Then solve the resulting homogeneous equation.

Example 6: Solve $a_n - 2a_{n-1} = 2^{n+1}$, with $a_0 = 2$. From the equation we can find that $a_1 = 8$. Now we can add

$$\begin{aligned} a_n - 2a_{n-1} &= 2^{n+1} \\ -2(a_{n-1} - 2a_{n-2} &= 2^n) \end{aligned}$$

to get the homogeneous equation

$$a_n - 4a_{n-1} + 4a_{n-2} = 0.$$

Annihilators for various types of sequences

We define the operators $\mathbb{E}\langle a_n \rangle = \langle a_{n+1} \rangle$, and $c\langle a_n \rangle = \langle c a_n \rangle$, and we define addition and multiplication of operators as

$$(A+B)\langle a_n \rangle = A\langle a_n \rangle + B\langle a_n \rangle$$
$$(AB)\langle a_n \rangle = A(B\langle a_n \rangle)$$

Then we have

Sequence	Annihilator
$\langle c \rangle$	$\mathbb{E} - 1$
$\langle$ poly. in n of degree k $\rangle$	$(\mathbb{E} - 1)^{k+1}$
$\langle c^n \rangle$	$\mathbb{E} - c$
$\langle c^n \times$ poly. in n of degree k $\rangle$	$(\mathbb{E} - c)^{k+1}$

Example 7: 2nd order non-homogeneous recurrence

$$a_n = \begin{cases} 5 & \text{if } n = 0, \\ 7 & \text{if } n = 1, \\ 5\,a_{n-1} - 6\,a_{n-2} + 4 & \text{otherwise.} \end{cases}$$

The characteristic equation $z^2 - 5z + 6 = 0$ has roots $z = 2$ and $z = 2$. In the notation of annihilators the equation $a_n - 5a_{n-1} + 6a_{n-2} = 4$ can be written as

$$(\mathbb{E}^2 - 5\mathbb{E} + 6)\langle a_n \rangle = \langle 4 \rangle, \text{ or}$$
$$(\mathbb{E} - 2)(\mathbb{E} - 3)\langle a_n \rangle = \langle 4 \rangle$$

Now we apply $(\mathbb{E} - 1)$ on both sides to annihilate the 4, and we get

$$(\mathbb{E} - 1)(\mathbb{E} - 2)(\mathbb{E} - 3)\langle a_n \rangle = \langle 0 \rangle.$$

The new homogeneous recurrence has solution $a_n = 2 + 4 \cdot 2^n - 3^n$.

1. Recurrence equations
   - Homogeneous linear recurrences with constant coefficients
   - Non-homogeneous linear recurrences with constant coefficients
   - Non-linear recurrences

2. Appendix: Jordan normal form

Non-linear recurrences: range transformation

Example 8: Solve the equation $a_n = 3a_{n-1}^2$ with $a_0 = 1$. We can define the sequence $b_n = \log_2 a_n$, so that

$$\begin{aligned} b_n &= 2b_{n-1} + \log_2 3, \\ b_0 &= 0. \end{aligned}$$

The latter recurrence is linear; its solution is

$$\begin{aligned} b_n &= (2^n - 1)\log_2 3, \quad \text{whence} \\ a_n &= 2^{(2^n-1)\log_2 3} \\ &= \left(2^{\log_2 3}\right)^{2^n-1} = 3^{2^n-1} \end{aligned}$$

Non-linear recurrences: domain transformation

Example 9: Solve the equation

$$T_n = \begin{cases} 0 \text{ if } n = 1, \\ 2T_{\frac{n}{2}} + n - 1 \text{ for } n = 2^k,\ k \geq 1. \end{cases}$$

Here we may define $a_k = T_n = T_{2^k}$, so we get the recurrence

$$a_k = \begin{cases} 0 \text{ if } k = 0, \\ 2a_{k-1} + 2^k - 1 \text{ for } k \geq 1. \end{cases}$$

The latter is a linear recurrence whose solution is

$$a_k = (k-1)2^k + 1, \text{ whence}$$
$$T_n = (\log_2 n - 1)n + 1$$

Bibliography Recurrences

1. Balakrishnan, V.K.: *Combinatorics – Including concepts of Graph Theory*. Schaum's Outline Series, McGraw-Hill, Inc. 1995 (in English)
2. Lueker, George S.: "Some techniques for solving recurrences", *Computing Surveys*, vol. 12, num. 4, 1980.
3. Rodríguez Velázquez, Juan Alberto. *Matemática Discreta*, Universitat Rovira i Virgili 2021 (in Spanish).
4. Weintraub, Steven H.: *Jordan Canonical Form – Theory and Practice*. Morgan & Claypool 2009 (in English)

Jordan normal form

A Jordan matrix is a block-diagonal matrix J of the form

$$J = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_p \end{bmatrix}$$

where each block J_i is associated with a particular eigenvalue λ_i, and has the form

$$J_i = \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix}$$

Jordan normal form

- Every square matrix M with real (or complex) coefficients is similar to a matrix J in Jordan form, i.e. there exists an invertible matrix S such that

$$M = S \cdot J \cdot S^{-1}$$

- The Jordan normal form of M is unique up to the reordering of the Jordan blocks.
- A diagonal matrix is a special case of the Jordan normal form. Note that the Jordan matrix is either diagonal or *almost* diagonal.
- If we know the Jordan form decomposition of a matrix M, it is now feasible to compute the powers of M, as well as other functions of M. In particular,

$$M^n = S \cdot J^n \cdot S^{-1}$$

Jordan normal form - Computing powers

Given a Jordan block B, associated with the eigenvalue λ, its n-th power can be calculated as follows:

$$\begin{bmatrix} \lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix}^n = \begin{bmatrix} \lambda^n & \binom{n}{1}\lambda^{n-1} & \binom{n}{2}\lambda^{n-2} & \cdots\cdots & \binom{n}{k-1}\lambda^{n-k+1} \\ & \lambda^n & \binom{n}{1}\lambda^{n-1} & \cdots\cdots & \binom{n}{k-2}\lambda^{n-k+2} \\ & & \ddots & \ddots \quad \vdots & \vdots \\ & & & \ddots \quad \ddots & \vdots \\ & & & \lambda^n & \binom{n}{1}\lambda^{n-1} \\ & & & & \lambda^n \end{bmatrix}$$

Now, the n-th power of the Jordan matrix J can be calculated blockwise.

Jordan normal form - Example

$$\underbrace{\begin{pmatrix} -10 & 1 & 7 \\ -7 & 2 & 3 \\ -16 & 2 & 12 \end{pmatrix}}_{M} = \underbrace{\begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 1 & 2 & 0 \end{pmatrix}}_{S} \cdot \underbrace{\begin{pmatrix} -2 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}}_{J} \cdot \underbrace{\begin{pmatrix} 2 & 0 & -1 \\ -1 & 0 & 1 \\ -3 & 1 & 2 \end{pmatrix}}_{S^{-1}}$$

Now,

$$M^n = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 1 & 2 & 0 \end{pmatrix} \cdot \begin{pmatrix} (-2)^n & 0 & 0 \\ 0 & 3^n & n \cdot 3^{n-1} \\ 0 & 0 & 3^n \end{pmatrix} \cdot \begin{pmatrix} 2 & 0 & -1 \\ -1 & 0 & 1 \\ -3 & 1 & 2 \end{pmatrix}$$

Back to second-order linear recurrences.