**1.1**

$$\frac{\partial^2\phi}{\partial x^2} - \frac{\partial^2\phi}{\partial t^2} = 0 \qquad x' = \gamma(x - vt)$$
$$t' = \gamma(t - vx)$$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial t'}{\partial x}\frac{\partial}{\partial t'} =$$

$$= \gamma\frac{\partial}{\partial x'} - \gamma v\frac{\partial}{\partial t'}$$

$$\frac{\partial^2}{\partial x^2} = \left(\gamma\frac{\partial}{\partial x'} - \gamma v\frac{\partial}{\partial t'}\right)\left(\gamma\frac{\partial}{\partial x'} - \gamma v\frac{\partial}{\partial t'}\right) =$$

$$= \gamma^2\frac{\partial^2}{\partial x'^2} + \gamma^2 v^2\frac{\partial^2}{\partial t'^2} - 2\gamma^2 v\frac{\partial^2}{\partial x'\partial t'}$$

$$\frac{\partial}{\partial t} = \frac{\partial x'}{\partial t}\frac{\partial}{\partial x'} + \frac{\partial t'}{\partial t}\frac{\partial}{\partial t'} = -\gamma v\frac{\partial}{\partial x'} + \gamma\frac{\partial}{\partial t'}$$

$$\frac{\partial^2}{\partial t'^2} = \left(-\gamma v\frac{\partial}{\partial x'} + \gamma\frac{\partial}{\partial t'}\right)\left(-\gamma v\frac{\partial}{\partial x'} + \gamma\frac{\partial}{\partial t'}\right) =$$

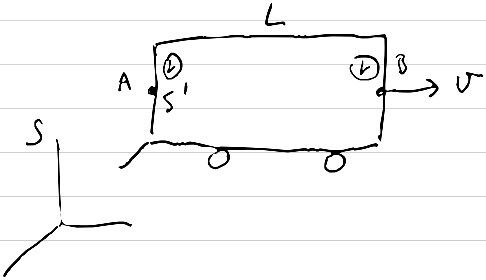
$$= \gamma^2 v^2\frac{\partial^2}{\partial x'^2} + \gamma^2\frac{\partial^2}{\partial t'^2} - 2\gamma^2 v\frac{\partial^2}{\partial x'\partial t'}$$

$$0 = \frac{\partial^2\phi}{\partial x^2} - \frac{\partial^2\phi}{\partial t^2} = \gamma^2\frac{\partial^2\phi}{\partial x'^2} + \gamma^2 v^2\frac{\partial^2\phi}{\partial t'^2} - \cancel{2\gamma^2 v\frac{\partial^2\phi}{\partial x'\partial t'}} -$$

$$-\gamma^2 v^2\frac{\partial^2\phi}{\partial x'^2} - \gamma^2\frac{\partial^2}{\partial t'^2} + \cancel{2\gamma^2 v\frac{\partial^2}{\partial x'\partial t'}} =$$

$$= \boxed{\cancel{\gamma^2(1-v^2)}\frac{\partial^2\phi}{\partial x'^2} - \cancel{\gamma^2(1-v^2)}\frac{\partial^2\phi}{\partial t'^2} = 0} \quad \square$$

# 1.2




We observe A and B from S at the same time t. What is the difference in times in S'?

Let's set $t=0$ and $t'_A=0$, $x=x'_A=0$

A

$x'_A=0$

$x_A=0$

$t_A=0$

$t'_A=0$

B

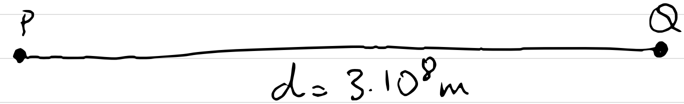
$x'_B=L$

$x_B=\frac{x'_B}{\gamma_v}=\frac{L}{\gamma_v}$

$$\boxed{t'_B}=\gamma_v\left(t_B-\frac{v x_B}{c^2}\right)=\gamma_v\left(\cancel{t_B}^{\,0}-\frac{vL}{\gamma_v c^2}\right)=$$

$$=\boxed{-\frac{vL}{c^2}}$$

# 1.3




P & Q simultaneous in $S$

Get the velocity of $S'$ with respect to $S$ so that P & Q are separated by $T' = 0.1s, 1s, 10s$.

$\boxed{\text{In } S}$: P $x_P = 0$, $t_P = 0$ Q $x_Q = L$, $t_Q = 0$

$\boxed{S'}$, $v$ P $x'_P = 0$, $t'_P = 0$ Q $\underbrace{t'_Q}_{T'} = \gamma_v \left(\overset{0}{\cancel{t_Q}} - \frac{vL}{c^2}\right) \Rightarrow$

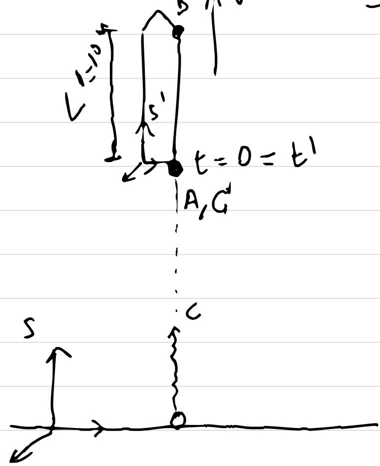
$$\Rightarrow \sqrt{1-\frac{v^2}{c^2}} = \frac{vL}{T'c^2} \Rightarrow$$

$$\Rightarrow \sqrt{\frac{1}{v^2}-\frac{1}{c^2}} = -\frac{L}{T'c^2} \Rightarrow$$

$$\Rightarrow \frac{1}{v^2} = \frac{1}{c^2} + \frac{L^2}{T'^2c^4} \Rightarrow \boxed{v = \pm \frac{c}{\sqrt{1+\frac{L^2}{T'^2c^2}}}}$$

# 1.4

$v = 0.8c \Rightarrow \beta = 0.8$




a) $S' \rightarrow \boxed{t'_B = \frac{L'}{c}} = \boxed{3.33 \cdot 10^{-8}\,s}$

$x'_B = L'$

$S$

$\boxed{t_B} = \gamma_v \left( t'_B + \frac{v x'_B}{c^2} \right) = \dfrac{\frac{L'}{c} + \frac{vL'}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} =$

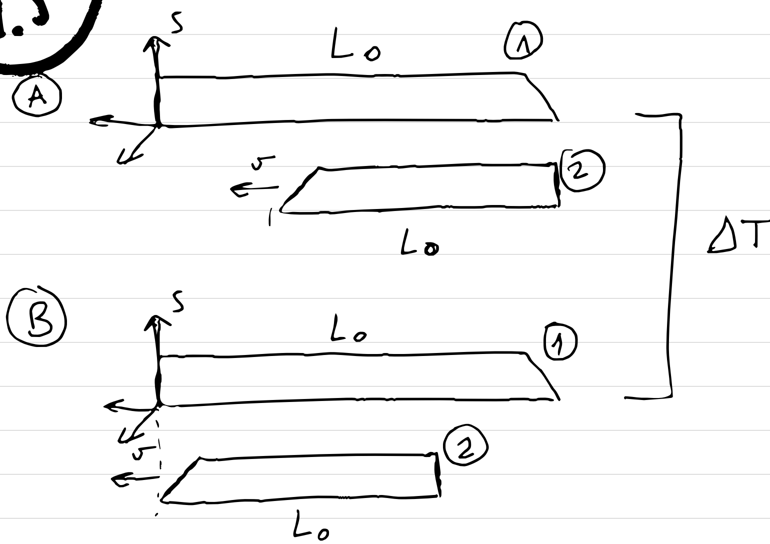
$= \frac{L'}{c} \dfrac{\left(1 + \frac{v}{c}\right)}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{L'}{c} \dfrac{1+\beta}{\sqrt{(1-\beta)(1+\beta)}} =$

$= \frac{L'}{c} \sqrt{\frac{1+\beta}{1-\beta}} = \boxed{1{,}0 \cdot 10^{-7}\,s}$

b) $S'$ $x'_C = 0$ $\boxed{t'_C} = \frac{2L'}{c} = \boxed{6.66 \cdot 10^{-8}\,s}$

$\boxed{t_G} = \gamma_v \left( t'_G - \cancel{\frac{v x'_C}{c^2}} \right) = \dfrac{2L'}{c\sqrt{1 - \frac{v^2}{c^2}}} = \dfrac{2L'}{\sqrt{c^2 - v^2}} = \boxed{1{,}11 \cdot 10^{-7}\,s}$

# 1.5




(a) From $S$, the lengths of the trains are

$$L_1 = L_0$$
$$L_2 = \frac{L_0}{\gamma_v}$$

From $S$, $\Delta L = L_0 - \frac{L_0}{\gamma} = \frac{L_0}{\gamma}(\gamma - 1)$

and $v$ must be $v\Delta T = \Delta L \Rightarrow$

$$v = \frac{\Delta L}{\Delta T} = \frac{L_0}{\Delta T}\left(1 - \frac{1}{\gamma}\right) = \frac{L_0}{\Delta T}\left(1 - \sqrt{1 - \frac{v^2}{c^2}}\right) \Rightarrow$$

$$\left(1 - \frac{\Delta T v}{L_0}\right)^2 = 1 - \frac{v^2}{c^2} \Rightarrow \cancel{1} - \frac{2v\Delta T}{L_0} + \frac{v^2 \Delta T^2}{L_0^2} = \cancel{1} - \frac{v^2}{c^2}$$

$$\Rightarrow v^2\left(\frac{1}{c^2} + \frac{\Delta T^2}{L_0^2}\right) - \frac{2v\Delta T}{L_0} = 0 \Rightarrow$$

$$v^2\left(\frac{1}{c^2}+\frac{\Delta T^2}{L_0^2}\right)-\frac{2v\Delta T}{L_0}=0 \Rightarrow$$

$$\Rightarrow v\left[v\left(\frac{1}{c^2}+\frac{\Delta T^2}{L_0^2}\right)-\frac{2\Delta T}{L_0}\right]=0$$

$$\Rightarrow \begin{cases} v=\dfrac{2\Delta T}{L_0\left(\frac{1}{c^2}+\frac{\Delta T^2}{L_0^2}\right)}=\dfrac{2\Delta T L_0}{\frac{L_0^2}{c^2}+\Delta T^2}=\dfrac{2L_0}{\frac{L_0^2}{\Delta T c^2}+\Delta T} \\ v=0 \text{ (can be understood as a limit, if } \Delta T \text{ is finite} \end{cases}$$

(b) Ref. frame in which A and B are simultaneous
$\hookrightarrow S'$

$\boxed{\text{From } S}$

$t_A=0 \qquad t_B=\Delta T$

$x_A=0 \qquad x_B=L_0$

$\boxed{\text{From } S'}$

$t'_A=t'_B$

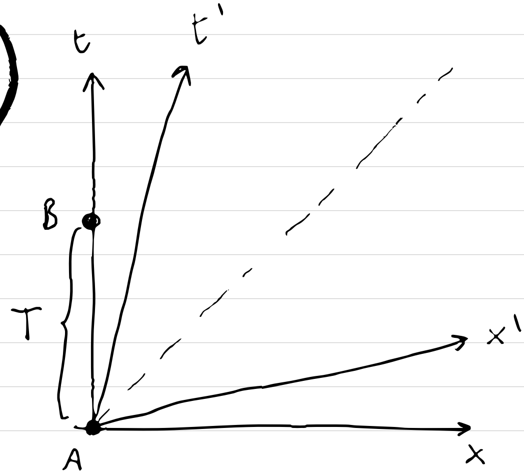
$$t'_A=\gamma_u\left(t_A-\frac{x_A u}{c^2}\right)=0$$

$$t'_B=\gamma_u\left(t_B-\frac{x_B u}{c^2}\right)=\gamma_u\left(\Delta T-\frac{L_0 u}{c^2}\right)=0 \Rightarrow$$

$$\Rightarrow \boxed{u=\frac{\Delta T c^2}{L_0}}$$

Note that $\dfrac{u+u}{1+\frac{u^2}{c^2}}=v$ !!

# 1.6




a) Prove $t'_B > t'_A$ in any $S'$

b) Prove $\infty < t'_B - t'_A \leq t_B - t_A$ (= in $S$)

c) $x'_B - x'_A$ is unbounded

A $\quad x = x' = 0$, $t = t' = 0$

B $\quad x = 0$, $t = T$

$$x'_B = \gamma_v (x_B - v t_B) = -v \gamma_v t_B$$

$$T' = \gamma_v \left(T - \frac{v x_B}{c^2}\right) = \gamma_v T = \frac{T}{\sqrt{1 - \frac{v^2}{c^2}}}$$

a $\quad \mathrm{sign}(T') = \mathrm{sign}(T)$

$$\frac{dT'}{dv} = T\left(-\frac{1}{2}\right)\left(1 - \frac{v^2}{c^2}\right)^{-3/2}\left(-\frac{2v}{c^2}\right) =$$

$$= \frac{Tv}{\left(1 - \frac{v^2}{c^2}\right)^{3/2}} \Rightarrow \boxed{\frac{dT'}{dv} = 0 \iff v = 0}$$

b

## 1.7

$T = 2 \times 10^{-6}$ s $\qquad$ $c = 3 \times 10^{8}$ s

$h = 5 \times 10^{4}$ m $\qquad$ $v = 0{,}99998\,c$

→ Earth frame

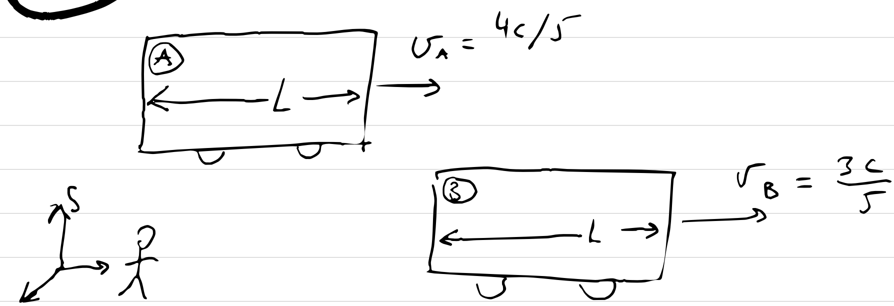
$$T' = \gamma_v T = \frac{2 \times 10^{-6}\,\text{s}}{\sqrt{1-(0.99998)^2}} = 3{,}16 \cdot 10^{-4}\,\text{s}$$

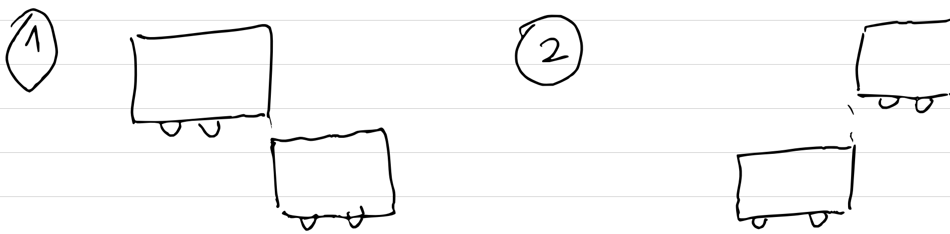
$$\boxed{L'} = T' v = 3{,}16 \cdot 10^{-4} \cdot 0.99998 \cdot 3 \times 10^{8}\,\text{m} =$$

$$= \boxed{94{,}9\,\text{km}}$$

It will reach the surface of the Earth.

# 1.8



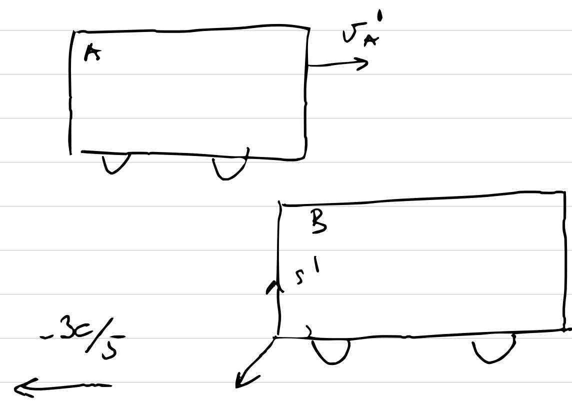

From $S$: $$L_A = \frac{L}{\gamma_A} = L\left(1 - \frac{16}{25}\right)^{1/2} = \frac{3L}{5}$$

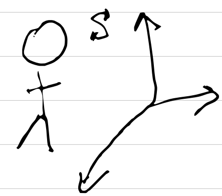
$$L_B = \frac{L}{\gamma_B} = L\left(1 - \frac{9}{25}\right) = \frac{4L}{5}$$

From $S$, A has to travel $L_A + L_B$ to overtake B, at a relative velocity that is, <u>from $S$</u>, $\frac{4c}{5} - \frac{3c}{5} = \frac{c}{5}$. Thus, the time needed is

$$\boxed{\Delta t = \frac{L_A + L_B}{\frac{c}{5}} = \frac{7L/5}{c/5} = \frac{7L}{c}}$$

Let's try to do everything from the point of view of the slow train B, S'!




$$\beta_A = \frac{4}{5}$$

$$\beta_A = \frac{\beta_B + \beta_A'}{1 + \beta_B \beta_A'} \qquad \beta_B = \frac{3}{5}$$

$$\Rightarrow \beta_A' = \beta_A (1 + \beta_B \beta_A') - \beta_B$$

$$\beta_A' (1 - \beta_A \beta_B) = \beta_A - \beta_B \Rightarrow \beta_A' = \frac{\beta_A - \beta_B}{1 - \beta_A \beta_B} \Rightarrow$$

$$\boxed{\beta_A' = \frac{1/5}{1 - 12/25} = \frac{1/5}{13/25} = \frac{5}{13}}$$

Now we work on S'

$$L_A' = \frac{L}{\gamma_A'} = L \left(1 - \beta_A'^2\right)^{1/2} = L \left(1 - \frac{25}{169}\right)^{1/2} = \frac{12 L}{13}$$

So now, A has to travel $L + \frac{12}{13}L = \frac{25}{13}L$ at a speed $\frac{5c}{13}$, so

$$\Delta t' = \frac{25/13\, L}{5/13\, c} = \frac{5L}{c}$$

So, back in S we have:

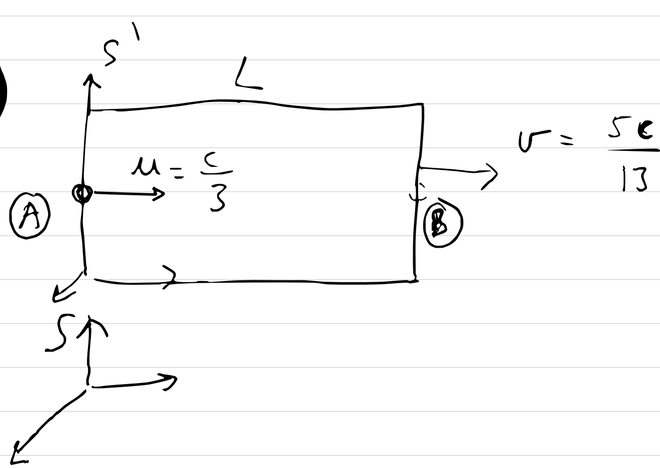
Event 1: $t_1' = 0$, $x_1' = 0$ Event 2: $t_2' = \frac{5L}{c}$, $x_2' = L$

$t_1 = 0$, $x_1 = 0$

Event 2: in S; $\boxed{t_2} = \gamma_B \left( t_2' + \frac{x_2' v_B}{c^2} \right) =$ !

$$= \frac{\frac{5L}{c} + \frac{3L}{5c}}{\sqrt{1 - \frac{9}{25}}} = \frac{L}{c} \frac{28/5}{4/5} = \boxed{\frac{7L}{c}}$$

# 1.9




a) From S, how far does the ball travel

| | S | S' |
|---|---|---|
| A | $x_A = t_A = 0$ | $x'_A = t'_A = 0$ |
| B | $X_B$? | $x'_B = L$ ; $t'_B = \frac{L}{u} = \frac{3L}{c}$ |

$$\boxed{X_B} = \gamma(x'_B + v\,t'_B) = \frac{L + \frac{5\cancel{c}}{13}\frac{3L}{\cancel{c}}}{\sqrt{1-\left(\frac{5}{13}\right)^2}} = \frac{28L/\cancel{13}}{\frac{\sqrt{13^2-5^2}}{\cancel{13}}} =$$

$$= \frac{28L}{12} = \boxed{\frac{7L}{3}}$$

b) Lorentz

$$\boxed{t_D} = \gamma_v\left(t'_B + \frac{v\,x'_B}{c^2}\right) = \frac{\frac{3L}{c} + \frac{5\cancel{c}L}{13c^2}}{\sqrt{1-\left(\frac{5}{13}\right)^2}} =$$

$$= \frac{44L/\cancel{13}c}{\frac{12}{\cancel{13}}} = \frac{44L}{12c} = \boxed{\frac{11}{3}\frac{L}{c}}$$

## Relative velocity

Velocity of the ball with respect to S:

$$W = \frac{v+u}{1+\frac{uv}{c^2}} = \frac{\frac{5c}{13}+\frac{c}{3}}{1+\frac{5\cancel{c^2}}{13\cdot 3\,\cancel{c^2}}} = \frac{\frac{15c+13c}{13\cdot 3}}{\frac{44}{13\cdot 3}} =$$

$$= \frac{28c}{44} = \frac{7c}{11}$$

$$\boxed{t_B} = \frac{x_B}{W} = \frac{7L/3}{7c/11} = \boxed{\frac{11}{3}\frac{L}{c}}$$

c) S'' is the frame of the ball

$$\boxed{t''_B} = \gamma_u\left(t'_B - \frac{u\,x'_B}{c^2}\right) = \frac{\frac{3L}{c} - \frac{cL}{3c^2}}{\sqrt{1-\frac{(c/3)^2}{c^2}}} =$$

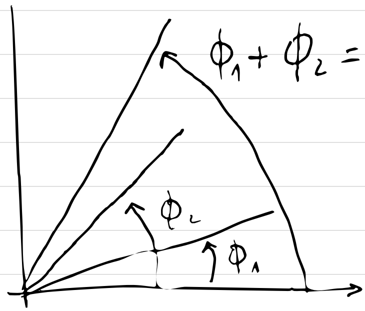
$$= \frac{8L/3c}{\sqrt{8/9}} = \boxed{\sqrt{8}\,\frac{L}{c}}$$

## Time dilation

$$\gamma_u = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}} = \frac{1}{\sqrt{1-\frac{\cancel{c^2}}{3^2\cancel{c^2}}}} = \sqrt{\frac{9}{8}} = \frac{3}{\sqrt{8}}$$

$$\boxed{t''_B} = \frac{t'_B}{\gamma} = \frac{3L\sqrt{8}}{c\,3} = \boxed{\frac{\sqrt{8}\,L}{c}}$$

**1.10**




$m_1 = tg\,\phi_1$
$m_2 = tg\,\phi_2$

$$\boxed{m_{12}} = tg(\phi_1 + \phi_2) = \frac{tg\,\phi_1 + tg\,\phi_2}{1 - tg\,\phi_1\, tg\,\phi_2} = \boxed{\frac{m_1 + m_2}{1 - m_1 m_2}}$$