Anàlisi Matemàtica 1 (AM1)
GEMiF

E2.1 Exercicis: Límits de funcions

1. Solucions

15. 12 **16.** 5 **17.** 1 **18.** does not exist

19. $\frac{3}{2}$ **20.** does not exist **21.** does not exist **22.** does not exist

23. $\lim\limits_{x\to 3} \dfrac{2x-6}{x-3} = \lim\limits_{x\to 3} 2 = 2$

24. $\lim\limits_{x\to 3} \dfrac{x^2-6x+9}{x-3} = \lim\limits_{x\to 3} \dfrac{(x-3)^2}{x-3} = \lim\limits_{x\to 3}(x-3) = 0$

25. $\lim\limits_{x\to 3} \dfrac{x-3}{x^2-6x+9} = \lim\limits_{x\to 3} \dfrac{x-3}{(x-3)^2} = \lim\limits_{x\to 3} \dfrac{1}{x-3}$; does not exist

26. $\lim\limits_{x\to 2} \dfrac{x^2-3x+2}{x-2} = \lim\limits_{x\to 2} \dfrac{(x-1)(x-2)}{x-2} = \lim\limits_{x\to 2}(x-1) = 1$

27. $\lim\limits_{x\to 2} \dfrac{x-2}{x^2-3x+2} = \lim\limits_{x\to 2} \dfrac{x-2}{(x-1)(x-2)} = \lim\limits_{x\to 2} \dfrac{1}{x-1} = 1$

28. does not exist **29.** does not exist

30. $\lim\limits_{x\to 1} \left(x+\dfrac{1}{x}\right) = 2$ **31.** $\lim\limits_{x\to 0} \dfrac{2x-5x^2}{x} = \lim\limits_{x\to 0}(2-5x) = 2$

32. $\lim\limits_{x\to 3} \dfrac{x-3}{6-2x} = \lim\limits_{x\to 3} \dfrac{x-3}{2(3-x)} = \lim\limits_{x\to 3} \left(-\dfrac{1}{2}\right) = -\dfrac{1}{2}$

33. $\lim\limits_{x\to 1} \dfrac{x^2-1}{x-1} = \lim\limits_{x\to 1} \dfrac{(x-1)(x+1)}{x-1} = \lim\limits_{x\to 1}(x+1) = 2$

34. $\lim\limits_{x\to 1} \dfrac{x^3-1}{x-1} = \lim\limits_{x\to 1} \dfrac{(x-1)(x^2+x+1)}{(x-1)} = \lim\limits_{x\to 1} x^2+x+1 = 3$

35. 0 **36.** does not exist

2. Solucions

37. 1

38. 3

39. 16

40. 0

41. does not exist

42. 2

43. 4

44. 0

45. does not exist

46. 2

47.

$$\begin{aligned}\lim_{x\to 1}\frac{\sqrt{x^2+1}-\sqrt{2}}{x-1} &= \lim_{x\to 1}\frac{(\sqrt{x^2+1}-\sqrt{2}\,)(\sqrt{x^2+1}+\sqrt{2}\,)}{(x-1)(\sqrt{x^2+1}+\sqrt{2}\,)}\\ &= \lim_{x\to 1}\frac{x^2-1}{(x-1)(\sqrt{x^2+1}+\sqrt{2}\,)} = \lim_{x\to 1}\frac{x+1}{\sqrt{x^2+1}+\sqrt{2}} = \frac{2}{2\sqrt{2}} = \frac{1}{\sqrt{2}}\end{aligned}$$

48.

$$\begin{aligned}\lim_{x\to 5}\frac{\sqrt{x^2+5}-\sqrt{30}}{x-5} &= \lim_{x\to 5}\frac{(\sqrt{x^2+5}-\sqrt{30}\,)(\sqrt{x^2+5}+\sqrt{30}\,)}{(x-5)(\sqrt{x^2+5}+\sqrt{30}\,)}\\ &= \lim_{x\to 5}\frac{x^2-25}{(x-5)(\sqrt{x^2+5}+\sqrt{30}\,)} = \lim_{x\to 5}\frac{x+5}{\sqrt{x^2+5}+\sqrt{30}} = \frac{10}{2\sqrt{30}} = \frac{5}{\sqrt{30}}\end{aligned}$$

49.

$$\begin{aligned}\lim_{x\to 1}\frac{x^2-1}{\sqrt{2x+2}-2} &= \lim_{x\to 1}\frac{x^2-1}{\sqrt{2x+2}-2}\,\frac{\sqrt{2x+2}+2}{\sqrt{2x+2}+2}\\ &= \lim_{x\to 1}\frac{(x-1)(x+1)\left(\sqrt{2x+2}+2\right)}{2x+2-4} = \lim_{x\to 1}\frac{(x-1)(x+1)\left(\sqrt{2x+2}+2\right)}{2(x-1)}\\ &= \lim_{x\to 1}\frac{(x+1)\left(\sqrt{2x+2}+2\right)}{2} = \frac{2\cdot 4}{2} = 4\end{aligned}$$

3. Solucions

1. $\frac{1}{2}$

2. $\lim_{x\to 0}\frac{x^2(1+x)}{2x} = \lim_{x\to 0}\frac{1}{2}x(1+x) = 0$

3. $\lim_{x\to 0}\frac{x(1+x)}{2x^2} = \lim_{x\to 0}\frac{1+x}{2x}$; does not exist

4. $\frac{4}{3}$

5. $\lim_{x\to 1}\frac{x^4-1}{x-1} = \lim_{x\to 1}(x^3+x^2+x+1) = 4$

6. does not exist

7. does not exist

8. $\lim_{x\to 1}\frac{x^2-1}{x^2-2x+1} = \lim_{x\to 1}\frac{(x-1)(x+1)}{(x-1)^2} = \lim_{x\to 1}\frac{x+1}{x-1}$; does not exist

9. -1

10. does not exist

11. does not exist

12. -1

13. 0

14. 0

15. $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+}(x^2-x) = 2$

16. $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} 1 = 1$

17. 1

18. 9

19. 1

20. 10

4. Solucions

23. $\delta = \frac{1}{2}\epsilon = .05$

24. $\delta = \frac{1}{5}\epsilon = 0.1$

25. $2\epsilon = .02$

26. $\delta = 5\,\epsilon = 0.5$

5. Solucions

37. Since

$$|(6x-7)-11| = |6x-18| = 6|x-3|,$$

we can take $\delta = \frac{1}{6}\epsilon$:

$$\text{if} \quad 0 < |x-3| < \tfrac{1}{6}\epsilon \quad \text{then} \quad |(6x-7)-11| = 6|x-3| < \epsilon.$$

38. Since

$$|(2-5x)-2| = |-5x| = 5|x|,$$

we can take $\delta = \frac{1}{5}\epsilon$:

$$\text{if} \quad 0 < |x| < \tfrac{1}{5}\epsilon \quad \text{then,} \quad |(2-5x)-2| = 5|x| < \epsilon.$$

39. Since

$$\big||1-3x|-5\big| = \big||3x-1|-5\big| \le |3x-6| = 3|x-2|,$$

we can take $\delta = \frac{1}{3}\epsilon$:

$$\text{if} \quad 0 < |x-2| < \tfrac{1}{3}\epsilon \quad \text{then} \quad \big||1-3x|-5\big| \le 3|x-2| < \epsilon.$$

40. Take $\delta = \epsilon$: if $0 < |x-2| < \epsilon$ then $||x-2|-0| = |x-2| < \epsilon$

50. Take $\delta =$ minimum of 1 and $\epsilon/5$. If $0 < |x-2| < \delta$, then $|x-2| < \epsilon/5$, $|x+2| < 5$ and Therefore $\left|x^2-4\right| = |x-2|\,|x+2| < (\epsilon/5)(5) = \epsilon$.

$$|x^3-1| = |x^2+x+1|\,|x-1| < 7|x-1| < 7(\epsilon/7) = \epsilon.$$

51. Take $\delta =$ minimum of 1 and $\epsilon/7$. If $0 < |x-1| < \delta$, then $0 < x < 2$ and $|x-1| < \epsilon/7$. Therefore

$$|x^3-1| = |x^2+x+1|\,|x-1| < 7|x-1| < 7(\epsilon/7) = \epsilon.$$

52. Take $\delta = 2\epsilon$. If $0 < |x-3| < \delta$, then

$$\left|\sqrt{x+1}-2\right| = \frac{|x-3|}{\sqrt{x+1}+2} < \frac{1}{2}|x-3| < \epsilon$$

53. Set $\delta = \epsilon^2$. If $3-\epsilon^2 < x < 3$, then $-\epsilon^2 < x-3$, $0 < 3-x < \epsilon^2$ and therefore $|\sqrt{3-x}-0| < \epsilon$.

6. Solucions

Les afirmacions (b), (e), (g) i (i) són necessàriament certes

7. Solucions

Suppose, on the contrary, that $\lim_{x \to c} f(x) = L$ for some particular c. Taking $\epsilon = \frac{1}{2}$, there must exist $\delta > 0$ such that

$$\text{if} \quad 0 < |x - c| < \delta, \quad \text{then} \quad |f(x) - L| < \tfrac{1}{2}.$$

Let x_1 be a rational number satisfying $0 < |x_1 - c| < \delta$ and x_2 an irrational number satisfying $0 < |x_2 - c| < \delta$. (That such numbers exist follows from the fact that every interval contains both rational and irrational numbers.) Now $f(x_1) = L$ and $f(x_2) = 0$. Thus we must have both

$$|1 - L| < \tfrac{1}{2} \quad \text{and} \quad |0 - L| < \tfrac{1}{2}.$$

From the first inequality we conclude that $L > \frac{1}{2}$. From the second, we conclude that $L < \frac{1}{2}$. Clearly no such number L exists.

8. Solucions

Para cualquier número a, con $0 < a < 1$, la función f tiende hacia 0 en a. Para demostrar esto, considérese un número cualquiera $\varepsilon > 0$. Sea n un número natural suficientemente grande para que $1/n \leq \varepsilon$. Obsérvese que los únicos números x para los que pudiera ser falso $|f(x) - 0| < \varepsilon$ son:

$$\frac{1}{2}; \frac{1}{3}, \frac{2}{3}; \frac{1}{4}, \frac{3}{4}; \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}; \ldots; \frac{1}{n}, \ldots, \frac{n-1}{n}.$$

(Si a es racional, entonces a podría ser uno de estos números.) Por muchos de tales números que pueda haber, son en todo caso en número finito. Por lo tanto, entre todos estos números habrá uno que será el más próximo a a; es decir, $|p/q - a|$ es mínimo para algún p/q entre estos números. (Si ocurre que a es uno de estos números, considérense entonces sólo los valores $|p/q - a|$ para $p/q \neq a$. Puede elegirse como δ esta distancia mínima. Pues si $0 < |x - a| < \delta$, entonces x *no es ninguno* de los

$$\frac{1}{2}, \ldots, \frac{n-1}{n}$$

y por lo tanto *se cumple* $|f(x) - 0| < \varepsilon$. Obsérvese que nuestra descripción del δ que va bien para un ε determinado es del todo adecuada; no es, en ningún modo, preciso dar una fórmula para δ en términos de ε.