Numerical Methods

# Dynamical Systems ODE

- Ordinary Differential Equations
- Basic Notation

Course 2022-2023

## Dynamical Systems ODE

- The variables representing the system will vary continuously
- This variation is not random, but determined by some natural law
- Normally these laws are represented mathematically by relations between a function and its derivatives and we call them ***ordinary differential equations*** (***ODE***).

## Dynamical Systems ODE

- A differential equation will be a general relation of the form

$$y^{(n)} = f(t, y, y', \ldots, y^{(n-1)})$$

- Valid for some interval $a \le t \le b$ of the independent variable $t$. To obtain a particular solution we will also need to have at our disposal a set of initial conditions like:

$$y(a) = \alpha_0, \quad y'(a) = \alpha_1, \quad \ldots \quad , y^{(n-1)}(a) = \alpha_{n-1}$$

## Dynamical Systems ODE

- The branch of mathematics devoted to the study of differential equations is very old and it uses a particular notation that we will also follow.
- Normally, to represent the independent variable we use the symbol $t$, and many times it can be directly associated with the time in which our system is evolving.

## Dynamical Systems ODE

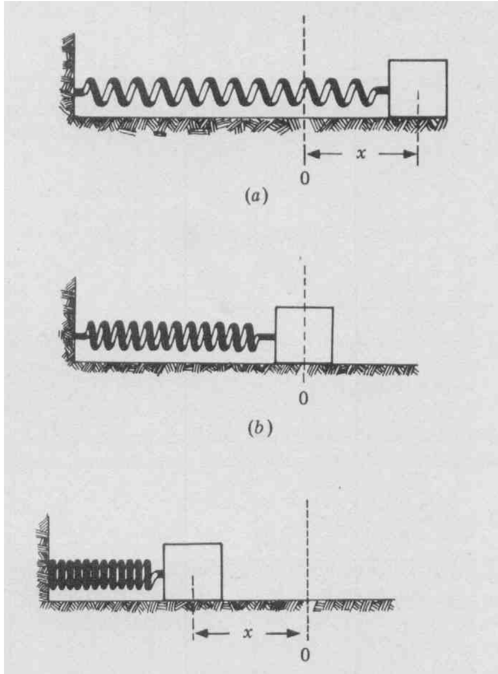
- The system variables will be represented as time functions: $y(t)$
- Their derivatives will be also time functions, but in this case, we will use a particular notation, using a dot over the first derivative, two dots over the second derivative and so on

$$y'(t) = \dot{y}(t) = \frac{dy(t)}{dt} = f(t, y)$$

# Harmonic Oscillator

- A very simple EDO, but very important in many applications is the harmonic oscillator:

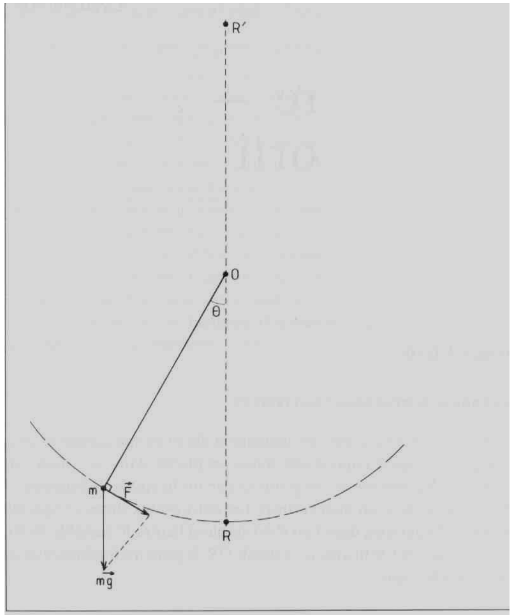
$$\boxed{\ddot{x}(t)+\frac{K}{m}x(t)=0}$$



## Simple Pendulum

- Another related example is the simple pendulum. For small oscillations of the angle $\theta$ we can write:

$$\ddot{\theta}(t) + \frac{g}{l}\theta(t) = 0$$



## Not so Simple Pendulum

- But the real equation is a bit more complicated

$$\ddot{\theta}(t) + \frac{g}{l}\sin\theta(t) = 0$$

- For this case we can find an analytic but complicated solution. The real damped pendulum, however, is still more difficult to solve and we must be solved numerically.

$$\ddot{\theta}(t) + a\dot{\theta}(t) + \frac{g}{l}\sin\theta(t) = 0$$

## Order of an ODE

- The ***order*** of the derivative with the highest degree in the relation gives the order of the ODE. Thus

$$\dot{x} + tx^2 = 0$$

- Is a first order ODE, while the harmonic oscillator is a second order ODE.

$$\ddot{x} + \omega^2 x = 0$$

## Reduction to First Order

- Any ODE of order $n$ can be written as a system of differential equations of first order
- We do not reduce the complexity of the problem. We change $n$ derivatives by $n$ variables
- This transformation is always necessary from the numeric point of view, as ***all the numerical algorithms must be applied on first order differential systems***.

# Reduction to First Order

- Thus, if we want to transform the ODE

$$y^{(n)} = f\left(t, y, y', \ldots, y^{(n-1)}\right)$$

- We can just make the following change of variables

$$\boxed{y = x_1, \qquad \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\vdots \\ \dot{x}_{n-1} &= x_n \\ \dot{x}_n &= f(t, x_1, x_2, \ldots, x_n) \end{aligned}}$$

# Harmonic Oscillator

- The ODE of the harmonic oscillator:

$$\ddot{x}(t) + \frac{K}{m} x(t) = 0$$

- May be written as a first order system

$$\left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{K}{m} x_1 \end{aligned}\right\}$$

- Which has an exact solution, being $\omega$ the oscillating frequency

$$x(t) = \cos(\omega t + \varphi_0), \quad \omega = \sqrt{\frac{K}{m}}$$

## Dynamical Systems ODE

- The ODE of $n$th degree is transformed in a differential system of equations, all of first order, $n$ variables
- If we need to use more than a single variable to describe the state of the system, the system can not be described a single differential equation. The system is described by a ***system of differential equations.***
- For each system variable we can use the same treatment and the dynamical system can still be analyzed as a system of first order ODEs.

## N-dimensional Systems

- If we need to use more than a single variable to describe the state of a system, we can use a vector notation.

$$\mathbf{x}(t) = \begin{pmatrix} x(t) \\ y(t) \\ \vdots \\ z(t) \end{pmatrix} = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}$$

## N-dimensional Systems

- Where each variable will satisfy its own differential equation:

$$\dot{\mathbf{x}}(t) = \begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_n(t) \end{pmatrix} = \begin{pmatrix} f_1(t, \mathbf{x}(t)) \\ f_2(t, \mathbf{x}(t)) \\ \vdots \\ f_n(t, \mathbf{x}(t)) \end{pmatrix}$$

# N-dimensional Systems

- The vector joining the functions which evaluate the derivatives is called the ***vector field of the ODE***. This gives a vector value for each point in the phase space.

$$\mathbf{f}(t,\mathbf{x}(t)) = \begin{pmatrix} f_1(t,\mathbf{x}(t)) \\ f_2(t,\mathbf{x}(t)) \\ \vdots \\ f_n(t,\mathbf{x}(t)) \end{pmatrix}$$

## N-dimensional Systems

- Let us assume that in the vector field $\boldsymbol{f}$, all its components $f_i$ are continuous functions. Moreover, if $\boldsymbol{f}$ is differentiable, we can write its first order derivative as the Jacobian matrix:

$$J(\mathbf{x}) = \frac{\partial \mathbf{f}}{\partial \mathbf{x}} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}$$

# Kepler's Problem

- This problem corresponds to the case of two bodies moving under the influence of their respective gravitational forces.
- Suppose that the bodies have masses $M$ and $m$, while their respective positions are $\boldsymbol{x_1}(t)$ and $\boldsymbol{x_2}(t)$. The force between them is given by Newton's law:

$$F = G\frac{Mm}{x_1^2 + x_2^2}$$

## Kepler's Problem

- The acceleration over each mass will be:

$$\ddot{x}_1 = -\frac{GMx_1}{\left(x_1^2 + x_2^2\right)^{3/2}}, \quad \ddot{x}_2 = -\frac{Gmx_2}{\left(x_1^2 + x_2^2\right)^{3/2}}$$

- Now if we write the velocities of each body as:

$$\dot{x}_1 = x_3, \quad \dot{x}_2 = x_4$$

## Kepler's Problem

- The reduction to a first order system results in the following system of ODEs:

$$\begin{aligned}
\dot{x}_1 &= x_3 \\
\dot{x}_2 &= x_4 \\
\dot{x}_3 &= -\frac{GMx_1}{\left(x_1^2 + x_2^2\right)^{3/2}} \\
\dot{x}_4 &= -\frac{Gmx_2}{\left(x_1^2 + x_2^2\right)^{3/2}}
\end{aligned}$$

## Vector Field

- Consider the following example used to describe the fall of weights within the atmosphere:

$$m\frac{dv}{dt} = mg - \gamma v$$

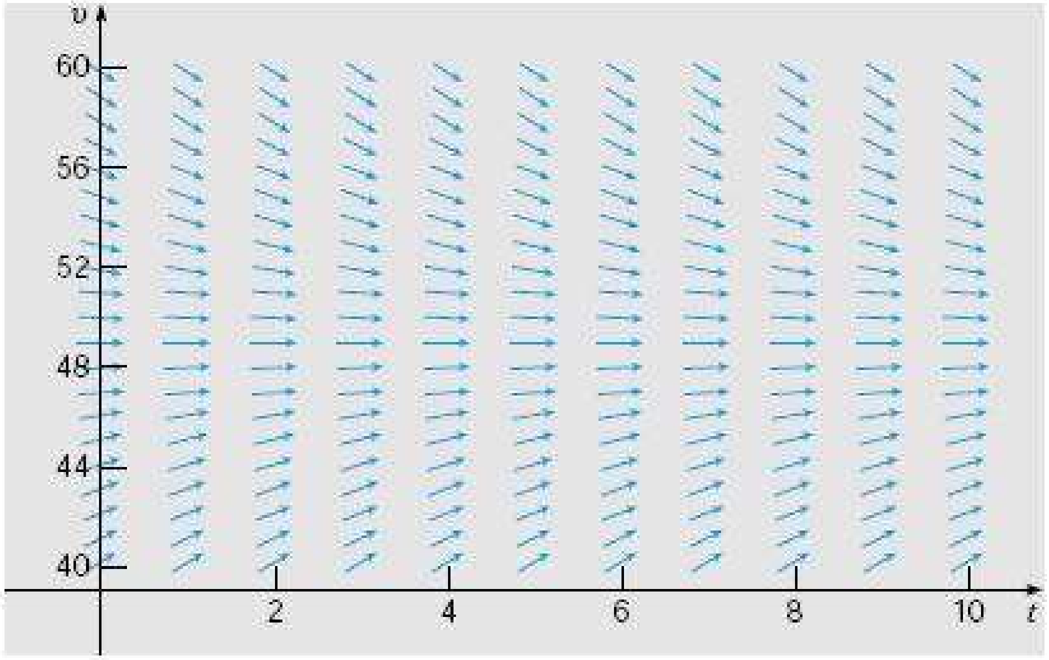
- The parameters $m$ and $\gamma$ are constants that may depend on diverse factors.

## Vector Field

- This is a first order ODE. When we fix the values of the constants, for each value of $t$ and each value of $v$, the derivative will give us the slope of the trajectory in each point. We can associate to each point a vector with the direction of the motion. In this way we get the vector field of the ODE

$$\dot{x}_1 = \frac{1}{m}\left(mg - \gamma x_1\right) = g - \frac{\gamma}{m} x_1$$

# Vector Field



## Vector Field

- This kind of diagrams can give much information before the numerical resolution of any ODE, which sometimes can be very CPU consuming.
- Note in this example that the vectors tend to flatten as time increases. Then the derivative of $v(t)$ goes to zero with time and the system tends to reach an equilibrium state, i.e., to fall with constant speed.

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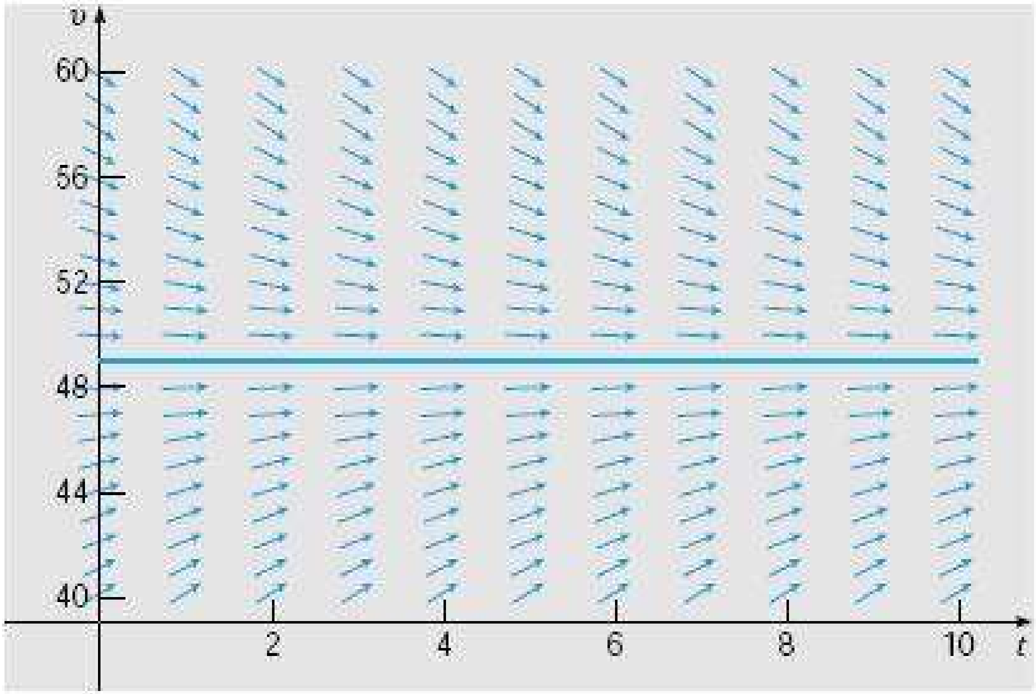
# Vector Field

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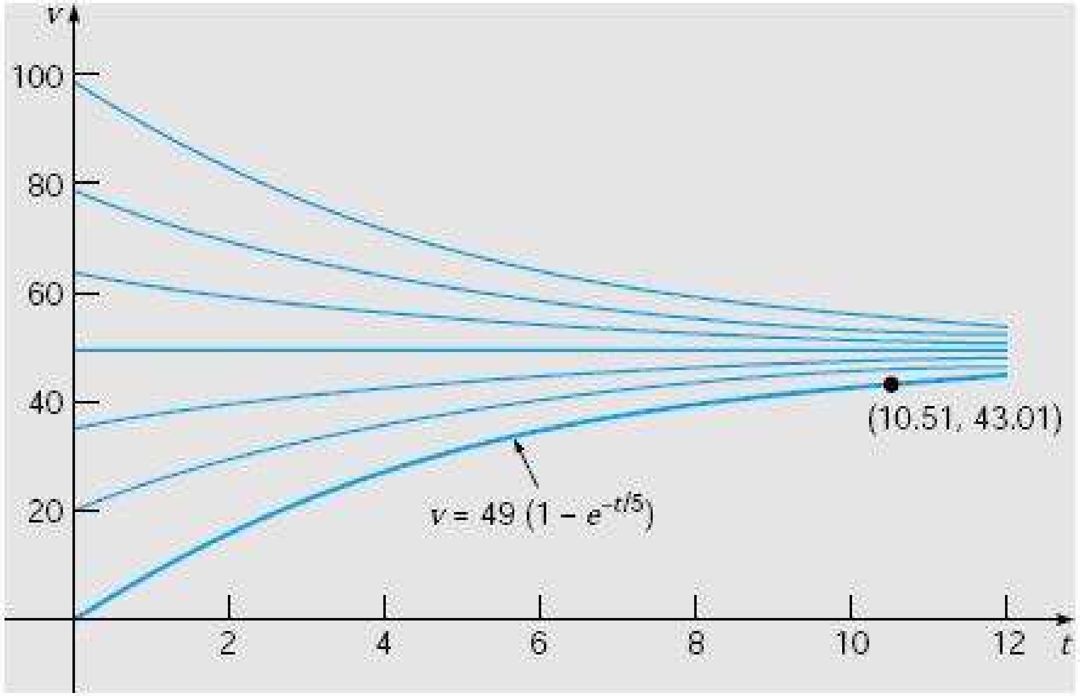
## Vector Field

- The solution of the ODEs often tend to these equilibrium solutions.
- In these cases, we can predict the behavior of the system without an accurate solution of the differential equation.
- In our example the speed of the weight will get smaller as time goes on and given enough time, the speed will reach a constant value. This is the limit speed of falling bodies.

Numerical Methods

# Vector Field

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## Phase Space

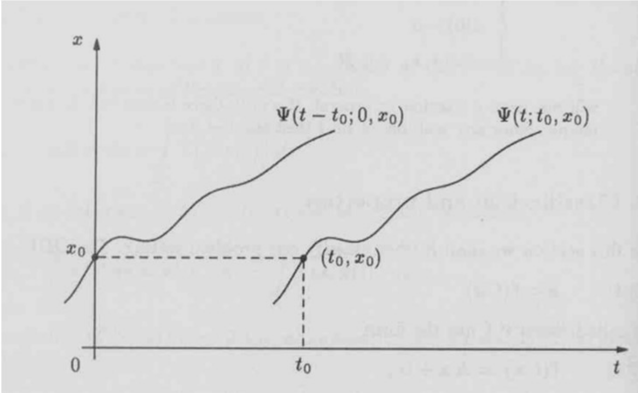
- The set of values that may take the system variables, that is the dominion of definition of these variables, is termed the ***phase space***.
- ***The orbit or trajectory*** of the system is the curve that the point, representing the state of the system, will describe within the phase space:

$$\{\phi(t;\mathbf{x}_0,t_0),t\in R\}$$

- For a $n$ dimensional system, the phase space will have $n+1$ dimensions

# Autonomous Systems

- For autonomous systems, the time origin is not important
- If $\boldsymbol{x}(t)$ is the solution of the equation in the interval $(a, b)$, then $\boldsymbol{x}(t+s)$ will be the solution in the interval $(a-s, b-s)$




$$\phi(t;\mathbf{x}_0,t_0)=\phi(t-t_0;\mathbf{x}_0,0)$$

# Initial Value Problem (IVP)

- We are often concerned with the evolution of a system from a particular starting point

$$\mathbf{x}(t_0) = \boldsymbol{\alpha}_0$$

- This problem is known as ***the initial value problem (IVP)***

$$\dot{x} = 2tx^2, \quad x(0) = 1$$

$$\ddot{x} + \dot{x} = 0, \quad x(1) = 0, \quad \dot{x}(1) = 1$$

# Initial Value Problem (IVP)

- The IVP may be written as:

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{f}(t, \mathbf{x}(t)) \\ \mathbf{x}(t_0) = \boldsymbol{\alpha}_0 \end{cases}$$

- If the vectorial field depends on time, we have a non autonomous system, otherwise the system is autonomous.

$$\begin{cases} \mathbf{x}(t) = \mathbf{f}(\mathbf{x}(t)) \\ \mathbf{x}(t_0) = \boldsymbol{\alpha}_0 \end{cases}$$

## ODE Systems

- We can guarantee that the evolution of the system from a given starting point will be unique under very mild conditions of the vector field. This means, that given an IVP, The evolution of the system will follow a single trajectory.
- The evolution of the system will be completely determined by the ODE. There are ***no random components***. We will deal with ***deterministic systems.***

# ODE Systems

- A different matter will be to compute the evolution of the system for any given time. Unless we have at our disposal an analytical solution, if we want to know the state of the system at a given time we will need to perform computations with high precision. The initial conditions need to be known also with high precision. Otherwise, the computation will depart more an more from the real orbit of the system as the computation unfolds. Our predictions will be dependent entirely on the precision of our computations and the initial conditions.

## Lipsitchz Condition

- Given two points in phase space $\boldsymbol{x}$ and $\boldsymbol{y}$, if the field satisfies

$$\left\|\mathbf{f}(\mathbf{t},\mathbf{x})-\mathbf{f}(\mathbf{t},\mathbf{y})\right\| \leq L\left\|\mathbf{x}-\mathbf{y}\right\|$$

- We say that $\boldsymbol{f}$ satisfies the Lipsitchz condition. All the continuous and differentiable functions satisfy the Lipsitchz condition

$$\left|\frac{\partial f}{\partial t}(t,\mathbf{x})\right| \approx \frac{f(t+h,\mathbf{x})-f(t,\mathbf{x})}{h} \leq L$$

# Uniqueness of the solutions

- **Theorem of uniqueness:** If the field ***f*** satisfies the Lipsitchz condition, then the IVP

$$\begin{cases} \vec{\mathbf{x}}(t_0) = \vec{\boldsymbol{\alpha}}_{\mathbf{0}} \\ \dot{\vec{\mathbf{x}}}(t) = \mathbf{f}(t, \vec{\mathbf{x}}(t)) \end{cases}$$

- has ***a unique solution***.

- Thus, ***the trajectories of a system will not cross themselves in phase space.***

## Boundary Value Problem (BVP)

- In some situations, for ODE systems of order greater than 2 $(n \geq 2)$ , we can consider the problem where we know, not just the initial conditions, but we know the state of the system at $n - 1$ points of a given trajectory in phase space.
- From this information, we want to recover the orbit of the system that goes through these $n - 1$ points.
- A common case corresponds to second order systems where we know the initial and final states.

## Boundary Value Problem (BVP)

- These are common problems in engineering. For instance, if we want to study the oscillations of a string fixed by the extremes, the differential system that we need to solve is a second order ODE system with the values fixed at the extremes of the string.

$$\ddot{y} + \lambda^2 y = 0, \quad y(0) = 0, \quad y(1) = 0$$

Numerical Methods

## Boundary Value Problem (BVP)

- This kind of problems is what we call ***boundary value problems (BVP),*** which can be represented generally by the ODE system:

$$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}), \ \mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \ \mathbf{f}(t, \mathbf{x}) = \begin{pmatrix} f_1(t, \mathbf{x}) \\ \vdots \\ f_1(t, \mathbf{x}) \end{pmatrix}$$

$$\mathbf{Ax(a)} + \mathbf{Bx(b)}, \ \mathbf{a}, \mathbf{b} \in \mathbb{R}^{\mathbf{n}}$$

## Boundary Value Problem (BVP)

- The boundary conditions my appear also in a separate form:

$$\mathbf{A}_1\mathbf{x}(\mathbf{a}) = \mathbf{c}_1, \quad \mathbf{B}_2\mathbf{x}(\mathbf{b}) = \mathbf{c}_2$$

- Or through complex expressions

$$\mathbf{r}(\mathbf{x}(\mathbf{a}), \mathbf{x}(\mathbf{b})) = 0$$

- Where $\boldsymbol{r}$ is a vector of $n$ components $r_i$ , each one defined by $2n$ variables

## Boundary Value Problem (BVP)

- These kind of problems are normally more complex than the IVP problems. In general, the IVP must have, under certain conditions, a unique solution.
- However, in the case of BVP problems we can not assure with the same certainty that the solution exists or that there is not more that one solution