Anàlisi Matemàtica 1 (AM1)
GEMiF

E5.3 Exercicis: Mètodes d'integració

1. Calcula les següents integrals per canvi de variable:

a)
$$\int x(2x-3)^{99}dx$$

Fem canvi de variables
$$t = 2x-3, \qquad x = \frac{t+3}{2}$$
$$dt = 2dx$$

$$\int x(2x-3)^{99}dx = \int \frac{t+3}{2} t^{99}\frac{dt}{2} = \frac{1}{4}\int (t^{100}+3t^{99})dt = \frac{t^{101}}{4\cdot 101}+\frac{3t^{100}}{4\cdot 100}$$
$$= \frac{1}{4\cdot 101}(2x-3)^{101}+\frac{3}{4\cdot 100}(2x-3)^{100}$$
$$= \frac{1}{4}(2x-3)^{100}\left[\frac{2x-3}{101}+\frac{3}{100}\right]$$
$$= \frac{(2x-3)^{100}[2\cdot 100x+3]}{4\cdot 100\cdot 101}+C$$

b)
$$\int x(4x^2+1)(2x^4+x^2+1)^k dx\,, \quad k \neq -1$$

$$\int x(4x^2+1)(2x^4+x^2+1)^k dx = \int (4x^3+x)(2x^4+x^2+1)^k dx$$

Fem canvi de variables
$$t = 2x^4+x^2+1$$
$$dt = (8x^3+2x)dx = 2(4x^3+x)dx$$

$$\int (4x^3+x)(2x^4+x^2+1)^k dx = \frac{1}{2}\int t^k dt = \frac{t^{k+1}}{2(k+1)}$$
$$= \frac{(2x^4+x^2+1)^{k+1}}{2(k+1)}+C$$

c)
$$\int a^x e^x dx$$

$$\int a^x e^x dx = \int (a\ e)^x dx = \frac{(a\ e)^x}{\ln\ (a\ e)} = \frac{a^x e^x}{1 + \ln\ a} + C$$

d)
$$\int \frac{x}{\sqrt{x-1}} dx$$

Fem canvi de variables

$$t = x - 1$$
$$dt = dx$$

$$\int \frac{x}{\sqrt{x-1}} dx = \int \frac{t+1}{\sqrt{t}} dt = \int \left(t^{1/2} + t^{-1/2}\right) dt = \frac{2}{3} t^{3/2} + 2t^{1/2}$$
$$= \frac{2}{3} t^{\frac{1}{2}} (t+3) = \frac{2}{3} (x+2)\sqrt{x-1} + C$$

e)
$$\int \frac{x}{(x+1)^2 + 4} dx$$

Fem canvi de variables

$$t = x + 1$$
$$dt = dx$$

$$\int \frac{x}{(x+1)^2 + 4} dx = \int \frac{t-1}{t^2+4} dt = \frac{1}{2} \int \frac{2t}{t^2+4} dt - \int \frac{1}{t^2+4} dt$$
$$= \frac{1}{2} \ln|t^2 + 4| - \frac{1}{4} \int \frac{1}{\left(\frac{t}{2}\right)^2 + 1} dt$$

Fem canvi de variables

$$u = t/2$$
$$du = 1/2dt$$

$$\int \frac{1}{\left(\frac{t}{2}\right)^2 + 1} dt = 2 \int \frac{1}{u^2+1} du = 2 \arctan u = 2 \arctan \frac{x+1}{2}$$

Per tant

$$\int \frac{x}{(x+1)^2 + 4} dx = \frac{1}{2} \ln|(x+1)^2 + 4| - \frac{1}{2} \arctan \frac{x+1}{2} + C$$

f)

$$\int \frac{\ln x}{x} dx$$

Fem canvi de variables

$$t = \ln x$$

$$dt = \frac{1}{x} dx$$

$$\int \frac{\ln x}{x} dx = \int t\, dt = \frac{1}{2} t^2 = \frac{1}{2} (\ln x)^2 + C$$

g)

$$\int \frac{\cos(\ln(x^2))}{x} dx$$

$$\int \frac{\cos(\ln(x^2))}{x} dx = \int \frac{\cos(2 \ln x)}{x} dx$$

Fem canvi de variables

$$t = \ln x$$

$$dt = \frac{1}{x} dx$$

$$\int \frac{\cos(\ln(x^2))}{x} dx = \int \frac{\cos(2 \ln x)}{x} dx = \int \cos(2t)\, dt = \frac{1}{2} \sin(2t)$$
$$= \frac{1}{2} \sin(2 \ln x) = \frac{1}{2} \sin(\ln(x^2)) + C$$

h)

$$\int \frac{\cos^2(\ln x)}{x} dx$$

Fem canvi de variables

$$t = \ln x$$

$$dt = \frac{1}{x} dx$$

$$\int \frac{\cos^2(\ln x)}{x} dx = \int \cos^2 t\, dt = \frac{1}{2} \int (1 + \cos(2t))\, dt = \frac{1}{2} t + \frac{1}{2} \sin(2t)$$
$$= \frac{1}{2} [\ln x + \sin(\ln(x^2))] + C$$

i)

$$\int \frac{\arcsin(2x)}{\sqrt{1-4x^2}} dx$$

Fem canvi de variables

$$t = \arcsin(2x)$$
$$dt = \frac{2}{\sqrt{1-4x^2}} dx$$

$$\int \frac{\arcsin(2x)}{\sqrt{1-4x^2}} dx = \frac{1}{2}\int t\, dt = \frac{1}{4}t^2 = \frac{1}{4}[\arcsin(2x)]^2 + C$$

j)

$$\int x^2 \cos^2(x^3) \sin(x^3)\, dx$$

Fem canvi de variables

$$t = \cos(x^3)$$
$$dt = -3x^2 \sin(x^3)\, dx$$

$$\int x^2 \cos^2(x^3) \sin(x^3)\, dx = -\frac{1}{3}\int t^2\, dt = -\frac{1}{9}t^3 = -\frac{1}{9}\cos^3(x^3) + C$$

k)

$$\int_0^1 x^2(1+3x^3)^5 dx$$

Fem canvi de variables

$$t = 1 + 3x^3$$
$$dt = 9x^2 dx$$
$$x = 0 \;\Rightarrow\; t = 1$$
$$x = 1 \;\Rightarrow\; t = 4$$

$$\int_0^1 x^2(1+3x^3)^5 dx = \frac{1}{9}\int_1^4 t^5\, dt = \frac{1}{54}[t^6]_1^4 = \frac{1}{54}(4^6 - 1^6) = \frac{455}{6}$$

l)

$$\int_0^1 xe^{4x^2+3}dx$$

Fem canvi de variables

$$t = 4x^2 + 3$$
$$dt = 8x\,dx$$
$$x = 0 \;\Rightarrow\; t = 3$$
$$x = 1 \;\Rightarrow\; t = 7$$

$$\int_0^1 xe^{4x^2+3}dx = \frac{1}{8}\int_3^7 e^t\,dt = \frac{1}{8}[e^t]_3^7 = \frac{1}{8}(e^7 - e^3) = \frac{e^3}{8}(e^4 - 1)$$

2. Calcula les següents integrals per integració per parts:

a)
$$\int x2^x dx$$

Fem integració per parts

$$\begin{aligned} u &= x \\ v' &= 2^x \end{aligned} \Rightarrow \begin{aligned} u' &= 1 \\ v &= 2^x/\ln 2 \end{aligned}$$

$$\int x2^x dx = \frac{1}{\ln 2}x2^x - \frac{1}{\ln 2}\int 2^x dx = \frac{1}{\ln 2}x2^x - \frac{1}{(\ln 2)^2}2^x$$
$$= \frac{2^x}{(\ln 2)^2}(x\ln 2 - 1) = +C$$

b)
$$\int \ln x\, dx$$

Fem integració per parts

$$\begin{aligned} u &= \ln x \\ v' &= 1 \end{aligned} \Rightarrow \begin{aligned} u' &= 1/x \\ v &= x \end{aligned}$$

$$\int \ln x\, dx = x\ln x - \int \frac{x}{x}dx = x\ln x - x = x(\ln x - 1) + C$$

c)
$$\int \arctan x\, dx$$

Fem integració per parts

$$\begin{aligned} u &= \arctan x \\ v' &= 1 \end{aligned} \Rightarrow \begin{aligned} u' &= 1/(1+x^2) \\ v &= x \end{aligned}$$

$$\int \arctan x\, dx = x\arctan x - \int \frac{x}{1+x^2}dx = x\arctan x - \frac{1}{2}\ln|1+x^2| + C$$

d)
$$\int x^k \ln x\, dx\,, \qquad k \neq -1$$

Fem integració per parts

$$\begin{aligned} u &= \ln x \\ v' &= x^k \end{aligned} \Rightarrow \begin{aligned} u' &= 1/x \\ v &= x^{k+1}/(k+1) \end{aligned}$$

$$\int x^k \ln x\, dx = \frac{x^{k+1}}{k+1}\ln x - \int \frac{x^k}{k+1}dx = \frac{x^{k+1}}{k+1}\ln x - \frac{x^{k+1}}{(k+1)^2}$$
$$= \frac{x^{k+1}[(k+1)\ln x - 1]}{(k+1)^2} + C$$

e)

$$\int \cos(\ln x)\,dx\,,\quad k\neq -1$$

Fem integració per parts

$$\begin{aligned} u &= \cos(\ln x) \\ v' &= 1 \end{aligned} \quad\Rightarrow\quad \begin{aligned} u' &= -\sin(\ln x)/x \\ v &= x \end{aligned}$$

$$I=\int \cos(\ln x)\,dx = x\cos(\ln x) + \int \sin(\ln x)\,dx$$

Fem ara

$$\begin{aligned} u &= \sin(\ln x) \\ v' &= 1 \end{aligned} \quad\Rightarrow\quad \begin{aligned} u' &= \cos(\ln x)/x \\ v &= x \end{aligned}$$

$$I = x\cos(\ln x) + x\sin(\ln x) - \int \cos(\ln x)\,dx = x[\cos(\ln x)+\sin(\ln x)] - I$$

Queda

$$2I = x[\cos(\ln x)+\sin(\ln x)]$$

És dir

$$I=\int \cos(\ln x)\,dx = \frac{x}{2}[\cos(\ln x)+\sin(\ln x)] + C$$

f)

$$\int \frac{\cos x}{e^x}\,dx$$

Fem integració per parts

$$\begin{aligned} u &= \cos x \\ v' &= e^{-x} \end{aligned} \quad\Rightarrow\quad \begin{aligned} u' &= -\sin x \\ v &= -e^{-x} \end{aligned}$$

$$I=\int e^{-x}\cos x\,dx = -e^{-x}\cos x - \int e^{-x}\sin x\,dx$$

Fem ara

$$\begin{aligned} u &= \sin x \\ v' &= e^{-x} \end{aligned} \quad\Rightarrow\quad \begin{aligned} u' &= \cos x \\ v &= -e^{-x} \end{aligned}$$

$$I = -e^{-x}\cos x - \left[-e^{-x}\sin x + \int e^{-x}\cos x\,dx\right] = e^{-x}[\sin x - \cos x] - I$$

Queda

$$2I = e^{-x}[\sin x - \cos x]$$

És dir

$$I=\int e^{-x}\cos x\,dx = \frac{e^{-x}}{2}[\sin x - \cos x] + C$$

g)

$$\int \frac{\ln(x+1)}{\sqrt{x+1}} dx$$

Fem integració per parts

$$\begin{aligned} u &= \ln(x+1) \\ v' &= (x+1)^{-1/2} \end{aligned} \quad \Rightarrow \quad \begin{aligned} u' &= 1/(x+1) \\ v &= 2(x+1)^{1/2} \end{aligned}$$

$$\int \frac{\ln(x+1)}{\sqrt{x+1}} dx = 2\sqrt{x+1}\ln(x+1) - 2\int (x+1)^{-\frac{1}{2}} dx$$
$$= 2\sqrt{x+1}\ln(x+1) - 4\sqrt{x+1} = 2\sqrt{x+1}[\ln(x+1) - 2] + C$$

h)

$$\int e^x \sin x \, dx$$

Fem integració per parts

$$\begin{aligned} u &= \sin x \\ v' &= e^x \end{aligned} \quad \Rightarrow \quad \begin{aligned} u' &= \cos x \\ v &= e^x \end{aligned}$$

$$I = \int e^x \sin x \, dx = e^x \sin x - \int e^x \cos x \, dx$$

Fem ara

$$\begin{aligned} u &= \cos x \\ v' &= e^x \end{aligned} \quad \Rightarrow \quad \begin{aligned} u' &= -\sin x \\ v &= e^x \end{aligned}$$

$$I = e^x \sin x - \left[e^x \cos x + \int e^x \sin x \, dx\right] = e^x[\sin x - \cos x] - I$$

Queda

$$2I = e^x[\sin x - \cos x]$$

És dir

$$I = \int e^x \sin x \, dx = \frac{e^x}{2}[\sin x - \cos x] + C$$

i)

$$\int \sin x \sinh x \, dx$$

Fem integració per parts

$$\begin{aligned} u &= \sin x \\ v' &= \sinh x \end{aligned} \quad \Rightarrow \quad \begin{aligned} u' &= \cos x \\ v &= \cosh x \end{aligned}$$

$$I = \int \sin x \sinh x \, dx = \sin x \cosh x - \int \cos x \cosh x \, dx$$

Fem ara

$$\begin{aligned} u &= \cos x \\ v' &= \cosh x \end{aligned} \quad \Rightarrow \quad \begin{aligned} u' &= -\sin x \\ v &= \sinh x \end{aligned}$$

$$\begin{aligned} I = \sin x \cosh x - \left[\cos x \sinh x + \int \sin x \sinh x \, dx\right] \\ = \sin x \cosh x - \cos x \sinh x - I \end{aligned}$$

Queda

$$2I = \sin x \cosh x - \cos x \sinh x$$

És dir

$$I = \int \sin x \sinh x \, dx = \frac{1}{2}[\sin x \cosh x - \cos x \sinh x] + C$$

j)

$$\int_{-\pi}^{\pi} \cos x \cosh(2x) \, dx$$

Fem integració per parts

$$\begin{aligned} u &= \cosh(2x) \\ v' &= \cos x \end{aligned} \quad \Rightarrow \quad \begin{aligned} u' &= 2\sinh(2x) \\ v &= \sin x \end{aligned}$$

$$I = \int \cos x \cosh(2x) \, dx = \sin x \cosh(2x) - 2\int \sin x \sinh(2x) \, dx$$

Fem ara

$$\begin{aligned} u &= \sinh(2x) \\ v' &= \sin x \end{aligned} \quad \Rightarrow \quad \begin{aligned} u' &= 2\cosh(2x) \\ v &= -\cos x \end{aligned}$$

$$\begin{aligned} I = \sin x \cosh(2x) - 2\left[-\cos x \sinh(2x) + 2\int \cos x \cosh(2x) \, dx\right] \\ = \sin x \cosh(2x) + 2\cos x \sinh(2x) - 4I \end{aligned}$$

Queda

$$5I = \sin x \cosh(2x) + 2\cos x \sinh(2x)$$

És dir

$$I = \int \cos x \cosh(2x) \, dx = \frac{1}{5}[\sin x \cosh(2x) + 2\cos x \sinh(2x)] + C$$

$$\int_{-\pi}^{\pi} \cos x \cosh(2x) \, dx = \frac{1}{5}[0 - 2\sinh(2\pi) - 0 - 2\sinh(2\pi)] = -\frac{4}{5}\sinh(2\pi)$$

k)
$$\int e^{ax}\sin(bx)\,dx$$
l)
$$\int e^{ax}\cos(bx)\,dx$$

Les fem juntes. Definim

$$I_s = \int e^{ax}\sin(bx)\,dx$$
$$I_c = \int e^{ax}\cos(bx)\,dx$$

Fem I_s per parts

$$\begin{matrix} u = \sin(bx) \\ v' = e^{ax} \end{matrix} \quad \Rightarrow \quad \begin{matrix} u' = b\cos(bx) \\ v = e^{ax}/a \end{matrix}$$

$$\begin{aligned} I_s = \int e^{ax}\sin(bx)\,dx &= \frac{1}{a}e^{ax}\sin(bx) - \frac{b}{a}\int e^{ax}\cos(bx)\,dx \\ &= \frac{1}{a}e^{ax}\sin(bx) - \frac{b}{a}I_c \end{aligned}$$

Fem I_c per parts

$$\begin{matrix} u = \cos(bx) \\ v' = e^{ax} \end{matrix} \quad \Rightarrow \quad \begin{matrix} u' = -b\sin(bx) \\ v = e^{ax}/a \end{matrix}$$

$$\begin{aligned} I_c = \int e^{ax}\cos(bx)\,dx &= \frac{1}{a}e^{ax}\cos(bx) + \frac{b}{a}\int e^{ax}\sin(bx)\,dx \\ &= \frac{1}{a}e^{ax}\cos(bx) + \frac{b}{a}I_s \end{aligned}$$

Substituint I_c en I_s

$$I_s = \frac{1}{a}e^{ax}\sin(bx) - \frac{b}{a}\left[\frac{1}{a}e^{ax}\cos(bx) + \frac{b}{a}I_s\right]$$
$$\left(1 + \frac{b^2}{a^2}\right)I_s = \frac{1}{a}e^{ax}\sin(bx) - \frac{b}{a^2}e^{ax}\cos(bx)$$
$$I_s = \int e^{ax}\sin(bx)\,dx = \frac{e^{ax}}{a^2+b^2}[a\sin(bx) - b\cos(bx)] + C$$

Substituint I_s en I_c

$$I_c = \frac{1}{a}e^{ax}\cos(bx) + \frac{b}{a}\left[\frac{1}{a}e^{ax}\sin(bx) - \frac{b}{a}I_c\right]$$
$$\left(1 + \frac{b^2}{a^2}\right)I_c = \frac{1}{a}e^{ax}\cos(bx) + \frac{b}{a^2}e^{ax}\sin(bx)$$
$$I_c = \int e^{ax}\cos(bx)\,dx = \frac{e^{ax}}{a^2+b^2}[a\cos(bx) + b\sin(bx)] + C$$

3. Calcula les següents integrals:

a)

$$\int \frac{1}{\cos x} dx$$

$$\int \frac{1}{\cos x} dx = \int \frac{\cos\ x}{\cos^2 x} dx = \int \frac{\cos\ x}{1 - \sin^2 x} dx$$

Fem canvi de variables

$t = \sin x$

$dt = \cos x\, dx$

$$\int \frac{1}{\cos x} dx = \int \frac{1}{1 - t^2} dt$$

Descomponem l'integrand

$$\frac{1}{1 - t^2} = \frac{1}{(1 + t)(1 - t)} = \frac{A}{1 + t} + \frac{B}{1 - t} = \frac{A(1 - t) + B(1 + t)}{(1 + t)(1 - t)}$$

$A(1 - t) + B(1 + t) = 1$

Per a $t = 1$: $2B = 1 \Rightarrow B = 1/2$

Per a $t = -1$: $2A = 1 \Rightarrow A = 1/2$

$$\int \frac{1}{\cos x} dx = \int \frac{1}{1 - t^2} dt = \frac{1}{2} \int \frac{1}{1 + t} dt + \frac{1}{2} \int \frac{1}{1 - t} dt$$
$$= \frac{1}{2} \ln|1 + t| - \frac{1}{2} \ln|1 - t| = \frac{1}{2} \ln \left|\frac{1 + t}{1 - t}\right| = \frac{1}{2} \ln \left|\frac{1 + \sin x}{1 - \sin x}\right| + C$$

b)

$$\int \tan^2 x\, dx$$

$$\int \tan^2 x\, dx = \int (\sec^2 x - 1) dx = \tan x - x + C$$

c)

$$\int e^{ax}\cosh(bx)\,dx$$

$$\begin{aligned}\int e^{ax}\cosh(bx)\,dx &= \frac{1}{2}\int e^{ax}(e^{bx}+e^{-bx})dx = \frac{1}{2}\int (e^{(a+b)x}+e^{(a-b)x})dx \\ &= \frac{1}{2(a+b)}e^{(a+b)x}+\frac{1}{2(a-b)}e^{(a-b)x} \\ &= \frac{e^{ax}}{2(a^2-b^2)}[(a-b)e^{bx}+(a+b)e^{-bx}] \\ &= \frac{e^{ax}}{2(a^2-b^2)}[a(e^{bx}+e^{-bx})-b(e^{bx}-e^{-bx})] \\ &= \frac{e^{ax}}{a^2-b^2}[a\cosh(bx)-b\sinh(bx)]+C\end{aligned}$$

d)

$$\int \sin^3 x\,dx$$

$$\int \sin^3 x\,dx = \int \sin^2 x \sin x\,dx = \int (1-\cos^2 x)\sin x\,dx$$

Fem canvi de variables

$$t = \cos x$$
$$dt = -\sin x\,dx$$

$$\int \sin^3 x\,dx = -\int (1-t^2)dt = \frac{t^3}{3} - t = \frac{1}{3}\cos^3 x - \cos x + C$$

e)

$$\int \cos^4 x\,dx$$

$$\begin{aligned}\int \cos^4 x\,dx &= \frac{1}{4}\int [1+\cos(2x)]^2 dx = \frac{1}{4}\int [1+2\cos(2x)+\cos^2(2x)]dx \\ &= \frac{1}{4}\left[x+\sin(2x)+\frac{1}{2}\int [1+\cos(4x)]dx\right] \\ &= \frac{1}{4}\left[x+\sin(2x)+\frac{1}{2}\left(x+\frac{1}{4}\sin(4x)\right)\right] \\ &= \frac{3}{8}x+\frac{1}{4}\sin(2x)+\frac{1}{32}\sin(4x)+C\end{aligned}$$

f)

$$\int \sec^3(\pi x)\,dx$$

$$\int \sec^3(\pi x)\,dx = \int \frac{1}{\cos^3(\pi x)}dx = \int \frac{\cos(\pi x)}{\cos^4(\pi x)}dx = \int \frac{\cos(\pi x)}{(1-\sin^2(\pi x))^2}dx$$

Fem canvi de variables

$$t = \sin(\pi x)$$
$$dt = \pi\cos(\pi x)\,dx$$

$$\int \sec^3(\pi x)\,dx = \frac{1}{\pi}\int \frac{1}{(1-t^2)^2}dt$$

Descomposem l'integrand

$$\frac{1}{(1-t^2)^2} = \frac{1}{(1+t)^2(1-t)^2} = \frac{A}{1+t} + \frac{B}{(1+t)^2} + \frac{C}{1-t} + \frac{D}{(1-t)^2}$$
$$= \frac{A(1+t)(1-t)^2 + B(1-t)^2 + C(1-t)(1+t)^2 + D(1+t)^2}{(1+t)^2(1-t)^2}$$

$A(1+t)(1-t)^2 + B(1-t)^2 + C(1-t)(1+t)^2 + D(1+t)^2 = 1$

Per a $t = 1$: $4D = 1 \Rightarrow D = \frac{1}{4}$

Per a $t = -1$: $4B = 1 \Rightarrow B = \frac{1}{4}$

Per a $t = 0$: $A + \frac{1}{4} + C + \frac{1}{4} = 1 \Rightarrow A + C = \frac{1}{2}$

Per a $t = 2$: $3A + \frac{1}{4} - 9C + \frac{9}{4} = 1 \Rightarrow 3A - 9C = -\frac{3}{2}$

Solucionant el sistema en A i C: $A = C = \frac{1}{4}$

$$\frac{1}{(1-t^2)^2} = \frac{1}{4}\left(\frac{1}{1+t} + \frac{1}{(1+t)^2} + \frac{1}{1-t} + \frac{1}{(1-t)^2}\right)$$

$$\int \sec^3(\pi x)\,dx = \frac{1}{4\pi}\int \left(\frac{1}{1+t} + \frac{1}{(1+t)^2} + \frac{1}{1-t} + \frac{1}{(1-t)^2}\right)dt$$
$$= \frac{1}{4\pi}\left(\ln|1+t| - \frac{1}{1+t} - \ln|1-t| + \frac{1}{1-t}\right)$$
$$= \frac{1}{4\pi}\left(\ln\left|\frac{1+t}{1-t}\right| + \frac{2t}{1-t^2}\right)$$
$$= \frac{1}{4\pi}\left(\ln\left|\frac{1+\sin(\pi x)}{1-\sin(\pi x)}\right| + \frac{2\sin(\pi x)}{\cos^2(\pi x)}\right) + C$$

g)

$$\int \frac{1}{x^2\sqrt{4-x^2}}dx$$

Fem canvi de variables

$$x = 2\sin t$$
$$dx = 2\cos t\, dt$$

$$\int \frac{1}{x^2\sqrt{4-x^2}}dx = \int \frac{2\cos t}{4\sin^2 t\sqrt{4-4\sin^2 t}}dt = \frac{1}{4}\int \frac{\cos t}{\sin^2 t\cos t}dt$$
$$= \frac{1}{4}\int \frac{1}{\sin^2 t}dt = -\frac{1}{4}\cot t = -\frac{\cos t}{4\sin t} = -\frac{\sqrt{1-\sin^2 t}}{4\sin t}$$
$$= -\frac{\sqrt{1-\left(\frac{x}{2}\right)^2}}{2x} = -\frac{\sqrt{4-x^2}}{4x} + C$$

h)

$$\int \frac{x^3}{\sqrt{9+x^2}}dx$$

Fem canvi de variables

$$x = 3\tan t$$
$$dx = \frac{3}{\cos^2 t}dt$$

$$\int \frac{x^3}{\sqrt{9+x^2}}dx = \int \frac{27\tan^3 t}{\sqrt{9+9\tan^2 t}}\frac{3}{\cos^2 t}dt = 27\int \frac{\tan^3 t}{\cos^2 t\sqrt{\sec^2 t}}\,dt$$
$$= 27\int \frac{\tan^3 t}{\cos t}\,dt = 27\int \frac{\sin^3 t}{\cos^4 t}\,dt$$
$$= 27\int \frac{(1-\cos^2 t)\sin t}{\cos^4 t}\,dt$$

Fem ara canvi de variables

$$u = \cos t$$
$$du = -\sin t\, dt$$

$$\int \frac{x^3}{\sqrt{9+x^2}}dx = -27\int \frac{1-u^2}{u^4}\,dt = -27\left(-\frac{1}{3u^3}+\frac{1}{u}\right) = \frac{9}{u}\left(\frac{1}{u^2}-3\right)$$

Per a posar-ho en funció de x

$$\frac{1}{u} = \sec t = \sqrt{1+\tan^2 t} = \sqrt{1+\left(\frac{x}{3}\right)^2} = \frac{1}{3}\sqrt{9+x^2}$$

$$\int \frac{x^3}{\sqrt{9+x^2}}dx = 3\sqrt{9+x^2}\left(\frac{1}{9}(9+x^2)-3\right) = \frac{1}{3}(x^2-18)\sqrt{9+x^2} + C$$

i)

$$\int \frac{x}{(x^2+2x+5)^2}dx$$

$$\begin{aligned}\int \frac{x}{(x^2+2x+5)^2}dx &= \frac{1}{2}\int \frac{2x+2-2}{(x^2+2x+5)^2}dx \\ &= \frac{1}{2}\int \frac{2x+2}{(x^2+2x+5)^2}dx - \int \frac{1}{(x^2+2x+5)^2}dx \\ &= -\frac{1}{2}(x^2+2x+5)^{-1} - \int \frac{1}{((x+1)^2+4)^2}dx\end{aligned}$$

$$\int \frac{1}{((x+1)^2+4)^2}dx = \frac{1}{16}\int \frac{1}{\left(\left(\frac{x+1}{2}\right)^2+1\right)^2}dx$$

Fem ara canvi de variables

$$t = \frac{x+1}{2}$$
$$dt = \frac{1}{2}dx$$

$$\begin{aligned}\int \frac{1}{\left(\left(\frac{x+1}{2}\right)^2+1\right)^2}dx &= 2\int \frac{1}{(t^2+1)^2}dt = 2\int \frac{1+t^2-t^2}{(t^2+1)^2}dt \\ &= 2\int \frac{1}{1+t^2}dt - 2\int \frac{t^2}{(t^2+1)^2}dt \\ &= 2\arctan t - 2\int \frac{t^2}{(t^2+1)^2}dt\end{aligned}$$

Fem la integral que queda per parts

$$\begin{matrix} u = t & & u' = 1 \\ v' = \dfrac{t}{(t^2+1)^2} & \Rightarrow & v = -\dfrac{1}{2(t^2+1)} \end{matrix}$$

$$\int \frac{t^2}{(t^2+1)^2}dt = -\frac{t}{2(t^2+1)} + \frac{1}{2}\int \frac{1}{1+t^2}dt = -\frac{t}{2(t^2+1)} + \frac{1}{2}\arctan t$$

Queda

$$\begin{aligned}\int \frac{x}{(x^2+2x+5)^2}dx &= -\frac{1}{2}(x^2+2x+5)^{-1} - \frac{1}{8}\left[\frac{1}{2}\arctan t + \frac{t}{2(t^2+1)}\right] \\ &= -\frac{1}{2(x^2+2x+5)} - \frac{1}{16}\arctan\frac{x+1}{2} - \frac{x+1}{8(x^2+2x+5)} \\ &= -\frac{x+5}{5(x^2+2x+5)} - \frac{1}{16}\arctan\frac{x+1}{2}\end{aligned}$$

j)

$$\int \frac{1}{\sqrt{x^2+a^2}}dx$$

Fem canvi de variables

$$x = a\tan t$$

$$dx = \frac{a}{\cos^2 t}dt$$

$$\int \frac{1}{\sqrt{x^2+a^2}}dx = \int \frac{1}{a\sqrt{\tan^2 t+1}}\,\frac{a}{\cos^2 t}dt = \int \frac{1}{\sec t\cos^2 t}dt = \int \frac{1}{\cos t}dt$$

Aquesta integral es va fer a l'exercici 3a, per tant

$$\int \frac{1}{\cos t}dt = \frac{1}{2}\ln\left|\frac{1+\sin t}{1-\sin t}\right|$$

Cal expressar $\sin t$ en funció de $\tan t$

$$\sin t = \tan t\cos t = \frac{\tan t}{\sqrt{1+\tan^2 t}}$$

$$1 \pm \sin t = \frac{\sqrt{1+\tan^2 t} \pm \tan t}{\sqrt{1+\tan^2 t}}$$

Queda

$$\int \frac{1}{\sqrt{x^2+a^2}}dx = \frac{1}{2}\ln\left|\frac{\sqrt{1+\tan^2 t}+\tan t}{\sqrt{1+\tan^2 t}-\tan t}\right| = \frac{1}{2}\ln\left|\frac{\sqrt{1+\left(\frac{x}{a}\right)^2}+\frac{x}{a}}{\sqrt{1+\left(\frac{x}{a}\right)^2}-\frac{x}{a}}\right|$$

$$= \frac{1}{2}\ln\left|\frac{\sqrt{a^2+x^2}+x}{\sqrt{a^2+x^2}-x}\right| + C$$

k)

$$\int \frac{1}{\sqrt{x^2-a^2}}dx$$

Fem canvi de variables

$$x = a\sec t$$
$$dx = \frac{a\sin t}{\cos^2 t}dt$$

$$\int \frac{1}{\sqrt{x^2-a^2}}dx = \int \frac{1}{a\sqrt{\sec^2 t-1}}\frac{a\sin t}{\cos^2 t}dt = \int \frac{\sin t}{\tan t\cos^2 t}dt = \int \frac{1}{\cos t}dt$$

Aquesta integral es va fer a l'exercici 3a, per tant

$$\int \frac{1}{\cos t}dt = \frac{1}{2}\ln\left|\frac{1+\sin t}{1-\sin t}\right|$$

Cal expressar $\sin t$ en funció de $\sec t$

$$\sin t = \sqrt{1-\frac{1}{\sec^2 t}} = \frac{\sqrt{\sec^2 t-1}}{\sec t}$$
$$1 \pm \sin t = \frac{\sec t \pm \sqrt{\sec^2 t-1}}{\sec t}$$

Queda

$$\int \frac{1}{\sqrt{x^2-a^2}}dx = \frac{1}{2}\ln\left|\frac{\sec t+\sqrt{\sec^2 t-1}}{\sec t-\sqrt{\sec^2 t-1}}\right| = \frac{1}{2}\ln\left|\frac{\frac{x}{a}+\sqrt{\left(\frac{x}{a}\right)^2-1}}{\frac{x}{a}-\sqrt{\left(\frac{x}{a}\right)^2-1}}\right|$$
$$= \frac{1}{2}\ln\left|\frac{x+\sqrt{x^2-a^2}}{x-\sqrt{x^2-a^2}}\right| + C$$

4. Superposició d'ones (desenvolupament en sèrie de Fourier):

a) Calcula $\int_0^{2\pi} \sin^2(nx)\,dx, \quad n \in \mathbb{N}$

Si $n = 0$

$$\int_0^{2\pi} \sin^2(0)\,dx = 0$$

Si $n > 0$

$$\int_0^{2\pi} \sin^2(nx)\,dx = \frac{1}{2}\int_0^{2\pi} (1 - \cos(2nx))dx = \frac{1}{2}\left[x - \frac{1}{2n}\sin(2nx)\right]_0^{2\pi}$$
$$= \frac{1}{2}\left[\left(2\pi - \frac{1}{2n}\sin(2\pi n)\right) - \left(0 - \frac{1}{2n}\sin 0\right)\right] = \pi$$

b) Calcula $\int_0^{2\pi} \cos^2(nx)\,dx, \quad n \in \mathbb{N}$

Si $n = 0$

$$\int_0^{2\pi} \cos^2(0)\,dx = \int_0^{2\pi} dx = 2\pi$$

Si $n > 0$

$$\int_0^{2\pi} \cos^2(nx)\,dx = \int_0^{2\pi} (1 - \sin^2(nx))dx = [x]_0^{2\pi} - \pi = \pi$$

c) Calcula $\int_0^{2\pi} \sin(nx)\cos(nx)\,dx, \quad n \in \mathbb{N}$

Si $n = 0$

$$\int_0^{2\pi} \sin(0)\cos(0)\,dx = 0$$

Si $n > 0$

$$\int_0^{2\pi} \sin(nx)\cos(nx)\,dx = \frac{1}{2}\int_0^{2\pi} \sin(2nx)\,dx = -\frac{1}{4n}[\cos(2nx)]_0^{2\pi}$$
$$= -\frac{1}{4n}(\cos(4\pi n) - \cos(0)) = -\frac{1}{4n}(1 - 1) = 0$$

d) Calcula $\int_0^{2\pi} \sin(nx)\sin(mx)\,dx, \quad n, m \in \mathbb{N}, \quad m \neq n$

Si $n = 0$ o $m = 0$ (sigui $n = 0, m > 0$)

$$\int_0^{2\pi} \sin(0)\sin(mx)\,dx = 0$$

Si $n > 0$ i $m > 0$

$$\int_0^{2\pi} \sin(nx)\sin(mx)\,dx = \frac{1}{2}\int_0^{2\pi} [\cos((n-m)x) - \cos((n+m)x))]dx$$
$$= \frac{1}{2}\left[\frac{1}{n-m}\sin((n-m)x) - \frac{1}{n+m}\sin((n+m)x)\right]_0^{2\pi} = 0$$

e) Calcula $\int_0^{2\pi} \cos(nx)\cos(mx)\,dx, \quad n, m \in \mathbb{N}, \quad m \neq n$

Si $n = 0$ o $m = 0$ (sigui $n = 0, m > 0$)

$$\int_0^{2\pi} \cos(0)\cos(mx)\,dx = \frac{1}{m}[\sin(mx)]_0^{2\pi} = 0$$

Si $n > 0$ i $m > 0$

$$\int_0^{2\pi} \cos(nx)\cos(mx)\,dx = \frac{1}{2}\int_0^{2\pi} [\cos((n-m)x) + \cos((n+m)x))]dx$$
$$= \frac{1}{2}\left[\frac{1}{n-m}\sin((n-m)x) + \frac{1}{n+m}\sin((n+m)x)\right]_0^{2\pi} = 0$$

f) Calcula $\int_0^{2\pi} \sin(nx)\cos(mx)\,dx, \quad n, m \in \mathbb{N}, \quad m \neq n$

Si $n = 0$ o $m = 0$

$$\int_0^{2\pi} \sin(nx)\cos(mx)\,dx = 0$$

Si $n > 0$ i $m > 0$

$$\int_0^{2\pi} \sin(nx)\cos(mx)\,dx = \frac{1}{2}\int_0^{2\pi} [\sin((n-m)x) + \sin((n+m)x))]dx$$
$$= -\frac{1}{2}\left[\frac{1}{n-m}\cos((n-m)x) + \frac{1}{n+m}\cos((n+m)x)\right]_0^{2\pi} = 0$$

g) Sigui f una funció definida a l'interval $[0,2\pi]$ tal que

$$f(x) = a_0 + a_1 \cos x + a_2 \cos 2x + \cdots + a_n \cos nx$$
$$+ b_1 \sin x + b_2 \sin 2x + \cdots + b_n \sin nx$$

Determina els coeficients a_k i b_k en funció d'integrals de f

Ho escrivim amb sumatoris

$$f(x) = a_0 + \sum_{p=1}^{n} a_p \cos px + \sum_{p=1}^{n} b_p \sin px$$

Multipliquem tot per $\sin kx$ i integrem entre 0 i 2π

$$\int_0^{2\pi} f(x) \sin kx \, dx$$
$$= a_0 \int_0^{2\pi} \sin kx \, dx + \sum_{p=1}^{n} a_p \int_0^{2\pi} \cos px \sin kx \, dx$$
$$+ \sum_{p=1}^{n} b_p \int_0^{2\pi} \sin px \sin kx \, dx$$

Tots els termes són 0 llevat el de b_k

$$\int_0^{2\pi} f(x) \sin kx \, dx = b_k \int_0^{2\pi} \sin^2 kx \, dx = b_k \pi$$

De la mateixa manera, multiplicant per $\cos kx$ i integrant entre 0 i 2π queda

$$\int_0^{2\pi} f(x) \cos kx \, dx = a_k \int_0^{2\pi} \cos^2 kx \, dx = a_k \pi, \quad k > 0$$
$$\int_0^{2\pi} f(x) \cos 0x \, dx = a_0 \int_0^{2\pi} \cos 0x \, dx = a_0 2\pi, \quad k = 0$$

Per tant

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(x) dx$$
$$a_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos kx \, dx, \quad k > 0$$
$$b_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin kx \, dx$$

5. Calcula les següents integrals racionals:

a)

$$\int \frac{x^5+2}{x^2-1}dx$$

És racional i el numerador té grau superior al denominador, per tant primer dividim

$$\begin{array}{rrrr|l} x^5 & & & +2 & x^2-1 \\ \hline -(x^5 & -x^3 & &) & x^3+x \\ \hline & x^3 & & +2 & \\ & -(x^3 & -x &) & \\ \hline & & x & +2 & \end{array}$$

Per tant

$$\frac{x^5+2}{x^2-1} = x^3 + x + \frac{x+2}{x^2-1}$$

Descomposem el darrer terme

$$\frac{x+2}{x^2-1} = \frac{x+2}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1} = \frac{A(x-1)+B(x+1)}{(x+1)(x-1)}$$

Queda

$A(x-1)+B(x+1) = x+2$

Per a $x = 1$: $2B = 3 \Rightarrow B = \frac{3}{2}$

Per a $x = -1$: $-2A = 1 \Rightarrow A = -\frac{1}{2}$

$$\frac{x+2}{x^2-1} = \frac{3}{2(x-1)} - \frac{1}{2(x+1)}$$

Ja podem integrar

$$\begin{aligned} \int \frac{x^5+2}{x^2-1}dx &= \int \left(x^3 + x + \frac{3}{2(x-1)} - \frac{1}{2(x+1)}\right)dx \\ &= \frac{x^4}{4} + \frac{x^2}{2} + \frac{3}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| \\ &= \frac{1}{4}x^4 + \frac{1}{2}x^2 + \frac{1}{2}\ln\left|\frac{(x-1)^3}{x+1}\right| + C \end{aligned}$$

b)
$$\int \frac{2x^3+3}{x(x-1)^2}\,dx$$

És racional i el numerador té grau igual al denominador, per tant primer dividim

$$\begin{array}{rr|l} 2x^3 \qquad\qquad\qquad +3 & & x^3-2x^2+x \\ \cline{3-3} \underline{-(2x^3-4x^2+2x\qquad)} & & 2 \\ 4x^2-2x+3 & & \end{array}$$

Per tant
$$\frac{2x^3+3}{x(x-1)^2} = 2 + \frac{4x^2-2x+3}{x(x-1)^2}$$

Descomposem el darrer terme
$$\frac{4x^2-2x+3}{x(x-1)^2} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2} = \frac{A(x-1)^2 + Bx(x-1) + Cx}{x(x-1)^2}$$

Queda

$A(x-1)^2 + Bx(x-1) + Cx = 4x^2 - 2x + 3$
Per a $x = 1$: $C = 5$
Per a $x = 0$: $A = 3$
Per a $x = 2$: $A + 2B + 2C = 15 \Rightarrow 2B = 2 \Rightarrow B = 1$

$$\frac{4x^2-2x+3}{x(x-1)^2} = \frac{3}{x} + \frac{1}{x-1} + \frac{5}{(x-1)^2}$$

Ja podem integrar

$$\int \frac{2x^3+3}{x(x-1)^2}\,dx = \int \left(2 + \frac{3}{x} + \frac{1}{x-1} + \frac{5}{(x-1)^2}\right) dx$$
$$= 2x + 3\ln|x| + \ln|x-1| - \frac{5}{x-1} + C$$

c)

$$\int \frac{x^2+5x+2}{x^3+x^2+x+1}dx$$

El grau del numerador és inferior al del denominador. Cal descomposar el denominador en producte de polinomis de grau 1 i 2 elevats a potències. Com ja es veu que -1 és una arrel, dividim per $x+1$ utilitzant Ruffini

$$\begin{array}{c|cccc} & 1 & 1 & 1 & 1 \\ -1 & & -1 & 0 & -1 \\ \hline & 1 & 0 & 1 & | \; 0 \end{array}$$

Per tant, $x^3+x^2+x+1=(x+1)(x^2+1)$, i podem fer la descomposició

$$\frac{x^2+5x+2}{x^3+x^2+x+1}=\frac{x^2+5x+2}{(x+1)(x^2+1)}=\frac{A}{x+1}+\frac{Bx+C}{x^2+1}$$
$$=\frac{A(x^2+1)+(Bx+C)(x+1)}{(x+1)(x^2+1)}$$

$$A(x^2+1)+(Bx+C)(x+1)=(A+B)x^2+(B+C)x+(A+C)$$
$$=x^2+5x+2$$

Queda el sistema

$$\begin{cases} A+B=1 \\ B+C=5 \\ A+C=2 \end{cases} \Rightarrow \begin{cases} A=1-B \\ B+C=5 \\ -B+C=1 \end{cases} \Rightarrow \begin{cases} A=-1 \\ B=2 \\ C=3 \end{cases}$$

$$\frac{x^2+5x+2}{x^3+x^2+x+1}=-\frac{1}{x+1}+\frac{2x+3}{x^2+1}$$

Ja podem integrar

$$\int \frac{x^2+5x+2}{x^3+x^2+x+1}dx=\int\left(-\frac{1}{x+1}+\frac{2x+3}{x^2+1}\right)dx$$
$$=-\ln|x+1|+\int\frac{2x}{x^2+1}dx+3\int\frac{1}{x^2+1}dx$$
$$=-\ln|x+1|+\ln|x^2+1|+3\arctan x$$
$$=\ln\left|\frac{x^2+1}{x+1}\right|+3\arctan x+C$$

d)

$$\int \frac{2x^5 + 9x^3 + x^2 + 7x + 3}{(x^2+1)(x^2+3)} dx$$

És racional i el numerador té grau superior al denominador, per tant primer dividim per $(x^2+1)(x^2+3) = x^4 + 4x^2 + 3$

$$\begin{array}{rrrrr|l} 2x^5 & +9x^3 & +x^2 & +7x & +3 & x^4+4x^2+3 \\ \hline -(2x^5 & +8x^3 & & +6x &) & 2x \\ \hline & x^3 & +x^2 & +x & +3 & \end{array}$$

Per tant

$$\frac{2x^5 + 9x^3 + x^2 + 7x + 3}{(x^2+1)(x^2+3)} = 2x + \frac{x^3 + x^2 + x + 3}{(x^2+1)(x^2+3)}$$

Podem fer la descomposició

$$\frac{x^3 + x^2 + x + 3}{(x^2+1)(x^2+3)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+3} = \frac{(Ax+B)(x^2+3) + (Cx+D)(x^2+1)}{(x^2+1)(x^2+3)}$$

$$(Ax+B)(x^2+3) + (Cx+D)(x^2+1) = (A+C)x^3 + (B+D)x^2 + (3A+C)x + (3B+D) = x^3 + x^2 + x + 3$$

Queda el sistema

$$\begin{cases} A + C = 1 \\ B + D = 1 \\ 3A + C = 1 \\ 3B + D = 3 \end{cases} \Rightarrow \begin{cases} A = 0 \\ B = 1 \\ C = 1 \\ D = 0 \end{cases}$$

$$\frac{x^3 + x^2 + x + 3}{(x^2+1)(x^2+3)} = \frac{1}{x^2+1} + \frac{x}{x^2+3}$$

Ja podem integrar

$$\int \frac{2x^5 + 9x^3 + x^2 + 7x + 3}{(x^2+1)(x^2+3)} dx = \int \left(2x + \frac{1}{x^2+1} + \frac{x}{x^2+3}\right) dx = x^2 + \arctan x + \frac{1}{2}\ln|x^2+3| + C$$

6. Calcula les següents integrals:

a)

$$\int x^2\sqrt{1+x}dx$$

Fem el canvi de variable

$$t = 1 + x$$
$$dt = dx$$

$$\int (t-1)^2\sqrt{t}dt = \int (t^2 - 2t + 1)t^{1/2}dt = \int \left(t^{5/2} - 2t^{3/2} + t^{1/2}\right)dt$$
$$= \frac{2}{7}t^{\frac{7}{2}} - \frac{4}{5}t^{\frac{5}{2}} + \frac{2}{3}t^{\frac{3}{2}} = \frac{2}{7}(1+x)^{\frac{7}{2}} - \frac{4}{5}(1+x)^{\frac{5}{2}} + \frac{2}{3}(1+x)^{\frac{3}{2}}$$
$$= \sqrt{(x+1)^3}\left(\frac{2}{7}(x+1)^2 - \frac{4}{5}(x+1) + \frac{2}{3}\right)$$
$$= \frac{2}{105}\sqrt{(x+1)^3}(15x^2 - 12x + 8) + C$$

b)

$$\int \frac{\sqrt{x}}{1+x}dx$$

Fem el canvi de variable

$$t = \sqrt{x}$$
$$dt = \frac{1}{2\sqrt{x}}dx = \frac{\sqrt{x}}{2x}dx \quad \Rightarrow \quad \sqrt{x}dx = 2t^2dt$$

$$\int \frac{\sqrt{x}}{1+x}dx = 2\int \frac{t^2}{1+t^2}dt = 2\int \frac{1+t^2-1}{1+t^2}dt = 2\int \left(1 - \frac{1}{1+t^2}\right)dt$$
$$= 2t - 2\arctan t = 2\sqrt{x} - 2\arctan\sqrt{x} + C$$

c)

$$\int \sqrt{1+e^x}dx$$

Fem el canvi de variable

$$t = \sqrt{1+e^x} \quad \Rightarrow \quad t^2 = 1+e^x$$

$$dt = \frac{e^x}{2\sqrt{1+e^x}}dx = \frac{e^x\sqrt{1+e^x}}{2(1+e^x)}dx \quad \Rightarrow \quad \sqrt{1+e^x}dx = \frac{2t^2}{t^2-1}dt$$

$$\int \sqrt{1+e^x}dx = \int \frac{2t^2}{t^2-1}dt = 2\int \frac{t^2-1+1}{t^2-1}dt = 2\int \left(1+\frac{1}{t^2-1}\right)dt$$
$$= 2t + 2\int \frac{1}{t^2-1}dt$$

Descomposem

$$\frac{1}{t^2-1} = \frac{1}{(t-1)(t+1)} = \frac{A}{t-1} + \frac{B}{t+1} = \frac{A(t+1)+B(t-1)}{(t-1)(t+1)}$$

$A(t+1)+B(t-1) = 1$

Per a $t = 1$: $A = \frac{1}{2}$

Per a $t = -1$: $B = -\frac{1}{2}$

$$\frac{1}{t^2-1} = \frac{1}{2(t-1)} - \frac{1}{2(t+1)}$$

Queda

$$\int \sqrt{1+e^x}dx = 2t + \int \left(\frac{1}{t-1} - \frac{1}{t+1}\right)dt = 2t + \ln|t-1| - \ln|t+1|$$
$$= 2\sqrt{1+e^x} + \ln\left|\frac{\sqrt{1+e^x}-1}{\sqrt{1+e^x}+1}\right|$$
$$= 2\sqrt{1+e^x} + \ln\left|\frac{\sqrt{1+e^x}-1}{\sqrt{1+e^x}+1}\,\frac{\sqrt{1+e^x}-1}{\sqrt{1+e^x}-1}\right|$$
$$= 2\sqrt{1+e^x} + \ln\left|\frac{\left(\sqrt{1+e^x}-1\right)^2}{1+e^x-1}\right|$$
$$= 2\sqrt{1+e^x} + 2\ln\left|\sqrt{1+e^x}-1\right| - x + C$$

d)

$$\int \frac{1}{x(\sqrt[3]{x}-1)}dx$$

Fem el canvi de variable

$$t = \sqrt[3]{x} \quad \Rightarrow \quad t^3 = x$$

$$dt = \frac{1}{3\sqrt[3]{x^2}}dx \quad \Rightarrow \quad dx = 3\sqrt[3]{(t^3)^2}dt = 3t^2 dt$$

$$\int \frac{1}{x(\sqrt[3]{x}-1)}dx = 3\int \frac{t^2}{t^3(t-1)}dt = 3\int \frac{1}{t(t-1)}dt$$

Descomposem

$$\frac{1}{t(t-1)} = \frac{A}{t} + \frac{B}{t-1} = \frac{A(t-1)+Bt}{t(t-1)}$$

$A(t-1)+Bt = 1$
Per a $t = 0$: $A = -1$
Per a $t = 1$: $B = 1$

$$\frac{1}{t(t-1)} = -\frac{1}{t} + \frac{1}{t-1}$$

Queda

$$\int \frac{1}{x(\sqrt[3]{x}-1)}dx = 3\int \left(\frac{1}{t-1} - \frac{1}{t}\right)dt = 3\ln|t-1| - 3\ln|t|$$
$$= 3\ln\left|\sqrt[3]{x}-1\right| - 3\ln\left|\sqrt[3]{x}\right| = 3\ln\left|\sqrt[3]{x}-1\right| - \ln|x| + C$$

e)

$$\int \frac{x}{\sqrt{1+x}}dx$$

Fem el canvi de variable

$$t = 1 + x$$
$$dt = dx$$

$$\int \frac{x}{\sqrt{1+x}}dx = \int \frac{t-1}{\sqrt{t}}dt = \int \left(t^{\frac{1}{2}} - t^{-\frac{1}{2}}\right)dt = \frac{2}{3}t^{\frac{3}{2}} - 2t^{\frac{1}{2}}$$
$$= \frac{2}{3}\sqrt{(1+x)^3} - 2\sqrt{1+x} = \frac{2}{3}\sqrt{1+x}(1+x-3)$$
$$= \frac{2}{3}(x-2)\sqrt{1+x} + C$$

f)

$$\int \frac{1-e^x}{1+e^x}dx$$

Fem el canvi de variable

$$t = 1 + e^x$$
$$dt = e^x dx \quad \Rightarrow \quad dx = \frac{1}{t-1}dt$$

$$\int \frac{1-e^x}{1+e^x}dx = \int \frac{1-(t-1)}{t(t-1)}dt = \int \left(\frac{1}{t(t-1)} - \frac{1}{t}\right)dt$$

Ja hem vist a apartat d) que

$$\frac{1}{t(t-1)} = -\frac{1}{t} + \frac{1}{t-1}$$

Per tant

$$\int \frac{1-e^x}{1+e^x}dx = \int \left(-\frac{2}{t} + \frac{1}{t-1}\right)dt = -2\ln|t| + \ln|t-1|$$
$$= -2\ln|1+e^x| + \ln|e^x| = x - 2\ln|1+e^x| + C$$

g)

$$\int \frac{1}{1+\cos x - \sin x}dx$$

Fem el canvi de variable

$$t = \tan\frac{x}{2}, \qquad \sin x = \frac{2t}{1+t^2}, \qquad \cos x = \frac{1-t^2}{1+t^2}$$
$$dx = \frac{2}{1+t^2}dt$$

$$\int \frac{1}{1+\cos x - \sin x}dx = \int \frac{1}{1+\frac{1-t^2}{1+t^2} - \frac{2t}{1+t^2}} \frac{2}{1+t^2}dt$$
$$= \int \frac{2}{(1+t^2)+(1-t^2)-(2t)}dt = \int \frac{1}{1-t}dt = -\ln|t-1|$$
$$= -\ln\left|\tan\frac{x}{2} - 1\right| + C$$

h)

$$\int \frac{1}{2+\sin x}\,dx$$

Fem el canvi de variable

$$t = \tan\frac{x}{2}, \qquad \sin x = \frac{2t}{1+t^2}$$

$$dx = \frac{2}{1+t^2}\,dt$$

$$\int \frac{1}{2+\sin x}\,dx = \int \frac{1}{2+\frac{2t}{1+t^2}}\,\frac{2}{1+t^2}\,dt = \int \frac{2}{2(1+t^2)+2t}\,dt$$

$$= \int \frac{1}{t^2+t+1}\,dt = \int \frac{1}{\left(t+\frac{1}{2}\right)^2-\frac{1}{4}+1}\,dt$$

$$= \int \frac{1}{\left(t+\frac{1}{2}\right)^2+\frac{3}{4}}\,dt = \frac{4}{3}\int \frac{1}{\left(\frac{2}{\sqrt{3}}\left(t+\frac{1}{2}\right)\right)^2+1}\,dt$$

Fem el canvi de variable

$$u = \frac{2}{\sqrt{3}}\left(t+\frac{1}{2}\right)$$

$$du = \frac{2}{\sqrt{3}}\,dt$$

$$\int \frac{1}{2+\sin x}\,dx = \frac{4}{3}\int \frac{1}{u^2+1}\,\frac{\sqrt{3}}{2}\,du = \frac{2\sqrt{3}}{3}\arctan u$$

$$= \frac{2\sqrt{3}}{3}\arctan\left(\frac{2}{\sqrt{3}}\left(t+\frac{1}{2}\right)\right)$$

$$= \frac{2\sqrt{3}}{3}\arctan\left(\frac{\sqrt{3}}{3}\left(2\tan\frac{x}{2}+1\right)\right)+C$$

i)

$$\int \frac{1}{\sin x + \tan x}\,dx$$

Fem el canvi de variable

$$t = \tan\frac{x}{2}, \qquad \sin x = \frac{2t}{1+t^2}, \qquad \cos x = \frac{1-t^2}{1+t^2}, \qquad \tan x = \frac{2t}{1-t^2}$$

$$dx = \frac{2}{1+t^2}\,dt$$

$$\int \frac{1}{\sin x + \tan x}\,dx = \int \frac{1}{\frac{2t}{1+t^2} + \frac{2t}{1-t^2}} \; \frac{2}{1+t^2}\,dt = \int \frac{1}{t + \frac{t(1+t^2)}{1-t^2}}\,dt$$

$$= \int \frac{1-t^2}{t(1-t^2)+t(1+t^2)}\,dt = \int \frac{1-t^2}{2t}\,dt = \frac{1}{2}\ln|t| - \frac{1}{4}t^2$$

$$= \frac{1}{2}\ln\left|\tan\frac{x}{2}\right| - \frac{1}{4}\tan^2\frac{x}{2} + C$$