25. Change of coordinates: I

Definition 25.1. *A function $f\colon U \longrightarrow V$ between two open subsets of $\mathbb{R}^n$ is called a **diffeomorphism** if:*

(1) *f is a bijection,*
(2) *f is differentiable, and*
(3) *f^{-1} is differentiable.*

Almost be definition of the inverse function, $f \circ f^{-1}\colon V \longrightarrow V$ and $f^{-1} \circ f\colon U \longrightarrow U$ are both the identity function, so that

$$(f \circ f^{-1})(\vec{y}) = \vec{y} \qquad \text{and} \qquad (f^{-1} \circ f)(\vec{x}) = \vec{x}.$$

It follows that

$$Df(\vec{x})Df^{-1}(\vec{y}) = I_n \qquad \text{and} \qquad Df^{-1}(\vec{y})Df(\vec{x}) = I_n,$$

by the chain rule. Taking determinants, we see that

$$\det(Df)\det(Df^{-1}) = \det I_n = 1.$$

Therefore,

$$\det(Df^{-1}) = (\det(Df))^{-1}.$$

It follows that

$$\det(Df) \neq 0.$$

Theorem 25.2 (Inverse function theorem)**.** *Let $U \subset \mathbb{R}^n$ be an open subset and let $f\colon U \longrightarrow \mathbb{R}$ be a function.*

Suppose that

(1) *f is injective,*
(2) *f is $\mathcal{C}^1$, and*
(3) *$Df(\vec{x}) \neq 0$ for all $\vec{x} \in U$.*

Then $V = f(U) \subset \mathbb{R}^n$ is open and the induced map $f\colon U \longrightarrow V$ is a diffeomorphism.

Example 25.3. *Let $f(r,\theta) = (r\cos\theta, r\sin\theta)$. Then*

$$Df(r,\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix},$$

so that

$$\det Df(r,\theta) = r.$$

It follows that f defines a diffeomorphism $f\colon U \longrightarrow V$ between

$$U = (0,\infty)\times(0,2\pi) \qquad \text{and} \qquad V = \mathbb{R}^2 \setminus \{\, (x,y) \in \mathbb{R}^2 \,|\, y = 0, x \geq 0 \,\}.$$

Theorem 25.4. *Let $g\colon U \longrightarrow V$ be a diffeomorphism between open subsets of $\mathbb{R}^2$,*

$$g(u, v) = (x(u, v), y(u, v)).$$

Let $D^* \subset U$ be a region and let $D = f(D^*) \subset V$. Let $f\colon D \longrightarrow \mathbb{R}$ be a function.

Then

$$\iint_D f(x, y)\,\mathrm{d}x\,\mathrm{d}y = \iint_{D^*} f(x(u, v), y(u, v))|\det Dg(u, v)|\,\mathrm{d}u\,\mathrm{d}v.$$

It is convenient to use the following notation:

$$\frac{\partial(x, y)}{\partial(u, v)}(u, v) = \det Dg(u, v).$$

The LHS is called the **Jacobian**. Note that

$$\frac{\partial(x, y)}{\partial(u, v)}(u, v) = \left(\frac{\partial(u, v)}{\partial(x, y)}(x, y)\right)^{-1}.$$

Example 25.5. *There is no simple expression for the integral of e^{-x^2}. However it is possible to compute the following integral*

$$I = \int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x.$$

(In what follows, we will ignore issues relating to the fact that the integrals are improper; in practice all integrals converge). Instead of

computing I, we compute I^2,

$$\begin{aligned}
I^2 &= \left(\int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\,\mathrm{d}y\right) \\
&= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} e^{-x^2-y^2}\,\mathrm{d}x\right)\mathrm{d}y \\
&= \iint_{\mathbb{R}^2} e^{-x^2-y^2}\,\mathrm{d}x\,\mathrm{d}y \\
&= \iint_{\mathbb{R}^2} re^{-r^2}\,\mathrm{d}r\,\mathrm{d}\theta \\
&= \int_{0}^{\infty}\left(\int_{0}^{2\pi} re^{-r^2}\,\mathrm{d}\theta\right)\mathrm{d}r \\
&= \int_{0}^{\infty} re^{-r^2}\left(\int_{0}^{2\pi}\mathrm{d}\theta\right)\mathrm{d}r \\
&= 2\pi\int_{0}^{\infty} re^{-r^2}\mathrm{d}r \\
&= 2\pi\left[-\frac{e^{-r^2}}{2}\right]_0^{\infty} \\
&= \pi.
\end{aligned}$$

So $I = \sqrt{\pi}$.

Example 25.6. *Find the area of the region D bounded by the four curves*

$$xy = 1, \qquad xy = 3, \qquad y = x^3, \qquad \text{and} \qquad y = 2x^3.$$

Define two new variables,

$$u = \frac{x^3}{y} \qquad \text{and} \qquad v = xy.$$

Then D is a rectangle in uv-coordinates,

$$D^* = [1/2, 1] \times [1, 3]$$

Now for the Jacobian we have

$$\frac{\partial(u,v)}{\partial(x,y)}(x,y) = \begin{vmatrix} \frac{3x^2}{y} & -\frac{x^3}{y^2} \\ y & x \end{vmatrix} = \frac{4x^3}{y} = 4u.$$

It follows that

$$\frac{\partial(x,y)}{\partial(u,v)}(u,v) = \frac{1}{4u}.$$

This is nowhere zero. In fact note that we can solve for x and y explicitly in terms of u and v.

$$uv = x^4 \qquad \text{and} \qquad y = \frac{x}{v}.$$

So

$$x = (uv)^{1/4} \qquad \text{and} \qquad y = u^{-1/4}v^{3/4}.$$

Therefore

$$\begin{aligned}
\operatorname{area}(D) &= \iint_D \mathrm{d}x\,\mathrm{d}y \\
&= \iint_{D^*} \frac{1}{4u}\,\mathrm{d}u\,\mathrm{d}v \\
&= \frac{1}{4}\int_1^3 \left(\int_{1/2}^1 \frac{1}{u}\,\mathrm{d}u\right)\mathrm{d}v \\
&= \frac{1}{4}\int_1^3 [\ln u]_{1/2}^1\,\mathrm{d}v \\
&= \frac{1}{4}\int_1^3 \ln 2\,\mathrm{d}v \\
&= \frac{1}{2}\ln 2.
\end{aligned}$$

Theorem 25.7. *Let $g\colon U \longrightarrow V$ be a diffeomorphism between open subsets of $\mathbb{R}^3$,*

$$g(u, v, w) = (x(u, v, w), y(u, v, w), z(u, v, w)).$$

Let $W^* \subset U$ be a region and let $W = f(W^*) \subset V$. Let $f\colon W \longrightarrow \mathbb{R}$ be a function.

Then

$$\iiint_W f(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iiint_{W^*} f(x(u, v, w), y(u, v, w), z(u, v, w))|\det Dg(u, v, w)|\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w.$$

As before, it is convenient to introduce more notation:

$$\frac{\partial(x, y, z)}{\partial(u, v, w)}(u, v, w) = \det Dg(u, v, w).$$

26. Change of coordinates: II

Example 26.1. *Let D be the region bounded by the cardiod,*

$$r = 1 - \cos\theta.$$

If we multiply both sides by r, take $r\cos\theta$ over the other side, then we get

$$(x^2 + y^2 + x)^2 = x^2 + y^2.$$

We have

$$\begin{aligned}
\operatorname{area}(D) &= \iint_D \mathrm{d}x\,\mathrm{d}y \\
&= \iint_{D^*} r\,\mathrm{d}r\,\mathrm{d}\theta \\
&= \int_{-\pi}^{\pi} \left(\int_0^{1-\cos\theta} r\,\mathrm{d}r \right) \mathrm{d}\theta \\
&= \int_{-\pi}^{\pi} \left[\frac{r^2}{2} \right]_0^{1-\cos\theta} \mathrm{d}\theta \\
&= \int_{-\pi}^{\pi} \frac{(1-\cos\theta)^2}{2}\,\mathrm{d}\theta \\
&= \int_{-\pi}^{\pi} \frac{1}{2} - \cos\theta + \frac{\cos^2\theta}{2}\,\mathrm{d}\theta \\
&= \left[\frac{\theta}{2} - \sin\theta \right]_{-\pi}^{\pi} \frac{\pi}{2} \\
&= \frac{\pi}{2}.
\end{aligned}$$

In $\mathbb{R}^3$, we can either use cylindrical or spherical coordinates, instead of Cartesian coordinates.

Let's first do the case of cylindrical coordinates. Recall that

$$\begin{aligned}
x &= r\cos\theta \\
y &= r\sin\theta \\
z &= z.
\end{aligned}$$

So the Jacobian is given by

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)}(r,\theta,z) = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r.$$

So,
$$\iiint_W f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iiint_{W^*} f(r,\theta,z)\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z.$$

Example 26.2. *Consider a cone of height b and base radius a. Put the vertex of the cone at the point $(0,0,b)$, so that the base of the cone is the circle of radius a, centred at the origin, in the xy-plane. Note that at height z, we have a circle of radius*
$$a\left(1-\frac{z}{b}\right).$$

$$\begin{aligned}
\mathrm{vol}(W) &= \iiint_W \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\
&= \iiint_{W^*} r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z \\
&= \int_0^b \left(\int_0^{2\pi} \left(\int_0^{a(1-z/b} r\,\mathrm{d}r\right)\mathrm{d}\theta\right)\mathrm{d}z \\
&= \frac{1}{2}\int_0^b \left(\int_0^{2\pi} \left[r^2\right]_0^{a(1-z/b}\,\mathrm{d}\theta\right)\mathrm{d}z \\
&= \frac{1}{2}\int_0^b \left(\int_0^{2\pi} a^2\left(1-\frac{z}{b}\right)^2\mathrm{d}\theta\right)\mathrm{d}z \\
&= \pi a^2 \int_{-a}^{a} \left(1-\frac{z}{b}\right)^2\mathrm{d}z \\
&= -\pi a^2 b \int_1^0 u^2\,\mathrm{d}u \\
&= \pi a^2 b \int_0^1 u^2\,\mathrm{d}u \\
&= \frac{\pi a^2 b}{3}.
\end{aligned}$$

Example 26.3. *Consider a ball of radius a. Put the centre of the ball at the point $(0,0,0)$. Note that*
$$x^2+y^2+z^2=a^2,$$
translates to the equation
$$r^2+z^2=a^2,$$
so that
$$r=\sqrt{a^2-z^2}.$$

$$
\begin{aligned}
\operatorname{vol}(W) &= \iiint_W \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\
&= \iint_{W^*} r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z \\
&= \int_{-a}^{a}\left(\int_0^{2\pi}\left(\int_0^{\sqrt{a^2-z^2}} r\,\mathrm{d}r\right)\mathrm{d}\theta\right)\mathrm{d}z \\
&= \frac{1}{2}\int_{-a}^{a}\left(\int_0^{2\pi}\left[r^2\right]_0^{\sqrt{a^2-z^2}}\,\mathrm{d}\theta\right)\mathrm{d}z \\
&= \frac{1}{2}\int_{-a}^{a}\left(\int_0^{2\pi} a^2 - z^2\,\mathrm{d}\theta\right)\mathrm{d}z \\
&= \pi\int_{-a}^{a} a^2 - z^2\,\mathrm{d}z \\
&= \pi\left[a^2 z - \frac{z^3}{3}\right]_{-a}^{a} \\
&= \frac{4\pi a^3}{3}.
\end{aligned}
$$

Now consider using spherical coordinates. Recall that

$$
\begin{aligned}
x &= \rho\sin\phi\cos\theta \\
y &= \rho\sin\phi\sin\theta \\
z &= \rho\cos\phi.
\end{aligned}
$$

So

$$
\begin{aligned}
\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)}(\rho,\phi,\theta) &= \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix} \\
&= \rho^2\cos^2\phi\sin\phi + \rho^2\sin^3\phi = \rho^2\sin\phi.
\end{aligned}
$$

Notice that this is greater than zero, if $0 < \phi < \pi$. So,

$$
\iiint_W f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iiint_{W^*} f(\rho,\phi,\theta)\rho^2\sin\phi\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z.
$$

Example 26.4. *Consider a ball of radius a. Put the centre of the ball at the point* $(0,0,0)$.

$$\begin{aligned}
\mathrm{vol}(W) &= \iiint_W \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\
&= \iiint_{W^*} \rho^2 \sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta \\
&= \int_0^{2\pi} \left(\int_0^{\pi} \left(\int_0^{a} \rho^2 \sin\phi\,\mathrm{d}\rho \right) \mathrm{d}\phi \right) \mathrm{d}\theta \\
&= \int_0^{2\pi} \left(\int_0^{\pi} \sin\phi \left[\frac{\rho^3}{3} \right]_0^a \mathrm{d}\phi \right) \mathrm{d}\theta \\
&= \int_0^{2\pi} \left(\int_0^{\pi} \sin\phi \frac{a^3}{3}\,\mathrm{d}\phi \right) \mathrm{d}\theta \\
&= \frac{a^3}{3} \int_0^{2\pi} [-\cos\phi]_0^{\pi}\,\mathrm{d}\theta \\
&= \frac{2a^3}{3} \int_0^{2\pi} \mathrm{d}\theta \\
&= \frac{4\pi a^3}{3}.
\end{aligned}$$