Lagrangian mechanics - Symmetry

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So far, we have shown the the Lagrangian formalism is an elegant and practical alternative to Newtonian mechanics. It is elegant because all the physics is encapsulated in a single function, the Lagrangian. And it is practical because it allows us to easily find the equations of motion for systems that are difficult (or, at best, cumbersome) to solve using Newton's second law.

Here, we go beyond these "niceties" to discuss the full power of the Lagrangian formulation, namely, to uncover some of the profound implications of symmetry in physics.

Symmetry and conservation laws: Noether's theorem

Generalized coordinates and momenta

In the previous lectures we have seen that the laws of mechanics are all nicely encapsulated in the Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial q_i'}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = 0\,, i = 1, \ldots, n.$$

Here q_i are *n generalized coordinates*;[1] they can be the ordinary Cartesian coordinates typically used in Newtonian mechanics, but they can also be any other set of independent coordinates that provide a convenient description the problem at hand. Importantly, as we showed, the Euler-Lagrange equations are satisfied in *any* coordinate system.

[1] n is the number of degrees of freedom of the system, and is given by the number of coordinates needed to describe all the components of the system (typically, $3N$) minus the number of constraints that relate those coordinates to each other.

In the following, it is also useful to define *generalized momenta*. We have already seen this in some previous discussions and in exercises, so we will not spend too much time. Indeed, consider the term $\partial\mathcal{L}/\partial q_i'$. It is easy to see[2] that, for a free particle in Cartesian coordinates, $\partial\mathcal{L}/\partial x_i' = m x_i' = p_i$ is the ordinary linear momentum. We have also seen that, for systems with rotational symmetry described in polar (or cylindrical coordinates) $\partial\mathcal{L}/\partial\theta' = mr^2\theta'$ is the ordinary angular momentum. In general, we define the **generalized momentum** p_i conjugate to q_i as

[2] Exercise: Do it!

$$p_i = \frac{\partial \mathcal{L}}{\partial q_i'}\,.$$

Unlike in the examples discussed, generalized momenta may not have a straightforward interpretation, and it does not have fixed units, either.

Conservation of momentum

In many of the examples that we have discussed (especially when solving exercises) we have encountered instances of momentum conservation. Consider, for example, a problem with rotational symmetry in which a particle of mass m is subject to a potential $V(r)$ that only depends on the distance r from the vertical (let's say z) axis. Then, the Lagrangian can be written in cylindrical coordinates as

$$\mathcal{L} = \frac{m}{2}\left(r'^2 + r^2\theta'^2 + z'^2\right) - V(r)\,.$$

Because the Lagrangian does not depend explicitly on z, the Euler-Lagrange equation for this generalized coordinate becomes

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial z'}\right) = 0 \implies \frac{dp_z}{dt} = 0 \implies p_z = \text{constant},$$

that is, linear momentum in the z direction is conserved. Similarly, since $\mathcal{L}$ does not depend explicitly on θ, we have

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \theta'}\right) = 0 \implies \frac{dp_\theta}{dt} = 0 \implies p_\theta = \text{constant},$$

that is, angular momentum is conserved.

Variables that do not appear explicitly in the Lagrangian, like z and θ in the example above, are called *cyclical*. Therefore, the first important observation that we make here is that, **for each cyclical generalized coordinate q_i, the corresponding conjugate momentum $p_i = \partial_{q_i'}\mathcal{L}$ is conserved.**

Conservation of energy

What about time? What happens when the Lagrangian does not depend *explicitly* on time? In that case, the total derivative of the Lagrangian with respect to time is[3]

$$\frac{d\mathcal{L}}{dt} = \sum_i \frac{\partial \mathcal{L}}{\partial q_i} q_i' + \sum_i \frac{\partial \mathcal{L}}{\partial q_i'} q_i''$$

and, using the Euler-Lagrange equation on the first term, we get

$$\frac{d\mathcal{L}}{dt} = \sum_i q_i' \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial q_i'}\right) + \sum_i \frac{\partial \mathcal{L}}{\partial q_i'} q_i'' = \frac{d}{dt}\left(\sum_i q_i' \frac{\partial \mathcal{L}}{\partial q_i'}\right) = \frac{d}{dt}\left(\sum_i p_i q_i'\right)\,.$$

Therefore, when the Lagrangian does not depend explicitly on time, we have that[4]

$$\frac{d}{dt}\left(\sum_i p_i q_i' - \mathcal{L}\right) = 0 \implies \sum_i p_i q_i' - \mathcal{L} = \text{constant}.$$

[3] Note that the fact that the Lagrangian does not depend *explicitly* on time does not mean that it does not depend on time at all. It does! It depends on time because the coordinates $\mathbf{q}$ and their derivatives $\mathbf{q}'$ depend on time.

[4] Indeed, we already got this result before. Do you remember when?

But, what is the interpretation $\sum_i p_i q_i' - \mathcal{L}$? To answer, lets consider a simple case of a single particle of mass m moving in an arbitrary potential $V(\mathbf{x})$. In Cartesian coordinates we have that $p_i = m x_i'$, $q_i' = x_i'$ and $\mathcal{L} = m\mathbf{x}'^2/2 - V(\mathbf{x})$. Therefore

$$\sum_i p_i q_i' - \mathcal{L} = m\mathbf{x}'^2 - \frac{m}{2}\mathbf{x}'^2 + V(\mathbf{x}) = \frac{m}{2}\mathbf{x}'^2 + V(\mathbf{x}) = T + V\,,$$

that is, what is conserved in this case is the **total energy of the system**.[5] Indeed, as we will see in much more depth when we study Hamiltonian mechanics, we take $H = \sum_i p_i q_i' - \mathcal{L}$ as the *definition* of **generalized energy**.

[5] But—of course! We know, from relativity, that the relationship between energy and time is very similar to the relationship between momentum and spatial coordinates. Remember the energy-momentum 4-vector! Here we see the same paralelism again.

Beyond cyclical coordinates: Symmetry

Cyclical coordinates that lead to conservation of the corresponding conjugate momenta are trivial to spot in a Lagrangian. So is to see whether the Lagrangian depends explicitly on time, and thus to establish if the total energy is conserved or not. But not all conservation laws are so easy to identify.

Consider for example, the Lagrangian corresponding to two particles moving on a line with a potential energy that depends on their relative position (for example, on the distance between them)

$$\mathcal{L} = \frac{m}{2}\left(x_1'^2 + x_2'^2\right) - V(x_1 - x_2).$$

Now, neither x_1 nor x_2 are cyclical, so neither conjugate momentum is conserved. Does this mean that there are no conserved quantities? No, it does not. In fact, by writing the Lagrangian in terms of the new coordinates

$$x_+ = \frac{x_1 + x_2}{2} \quad \text{and} \quad x_- = \frac{x_1 - x_2}{2}$$

we get[6]

$$\mathcal{L} = m\left(x_+'^2 + x_-'^2\right) - V(x_-)\,.$$

[6] EXERCISE: Do it!

Therefore, x_+ is cyclical and the conjugate momentum (which turns out to be the total momentum $p_+ = m x_1' + m x_2'$) is conserved.[7]

[7] EXERCISE: Do it!

So cyclical coordinates are just the tip of the iceberg, the most obvious manifestation of something deeper. To go into conservation laws beyond cyclical coordinates, it is crucial to understand what it means for a coordinate to be absent from the Lagrangian. For this, consider an *active transformation*[8] in which the systems is translated from q_i to $q_i + \delta$. Now, if q_i is cyclical the Lagrangian will not change because of the transformation $q_i \to q_i + \delta$, that is, the system will not see any difference. Thus, **conservation arises from *symmetry*, that is, from the invariance of physics (encapsulated in the Lagrangian) under certain transformations.**

[8] So far, we have mostly thought of changes of coordinates as *passive transformations*—we considered that coordinates $\mathbf{x}$ and $\tilde{\mathbf{x}}$ referred to the same point in space, just labeled differently by two different observers. For the discussion on symmetry here, we consider changes of coordinates as *active transformations*, that is, as actual "moves" (translations, rotations...) of the system. That means, for example, that the potential energy changes with the transformation, that is, $V(\mathbf{x}) \neq V(\tilde{\mathbf{x}})$.

From this point of view, conservation of linear momentum arises from the translational invariance of a system or, equivalently, from the fact that space is homogeneous; conservation of angular momentum arises from rotational invariance, that is, from the equivalence between all directions of space (around a given point); and energy conservation arises from temporal invariance, that is, from the homogeneity of time.

Noether's theorem

We now present one of the most beautiful and useful theorems in physics. The theorem, due to Emmy Noether (Fig. 1), deals with the relationship between *continuous symmetries* and *conserved quantities*. It uses the concept of *continuous transformations*, which are those that can be obtained from the composition of infinitesimal transformations. This is useful because it allows us to explore all consequences of continuous symmetries by restricting our attention to the infinitesimal case. In this case, we say that there is a symmetry if the Lagrangian has no *first-order change* on the infinitesimal transformations.

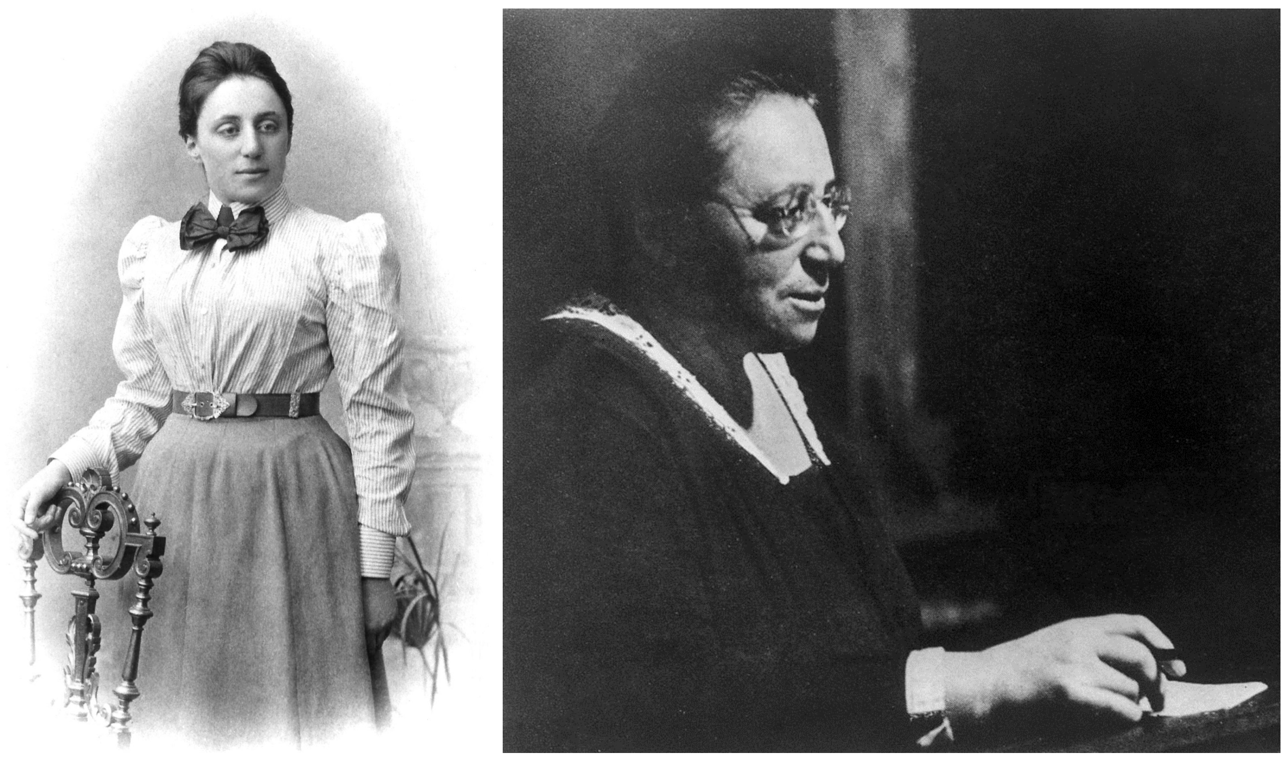

Figure 1: Emmy Noether c. 1900 (left) and c. 1930 (right).

NOETHER'S THEOREM. **For each continuous symmetry of the Lagrangian, there is a conserved quantity.**

To prove it, let us assume that the Lagrangian is invariant, to first order in the infinitesimal number ϵ, under the change of coordinates

$$q_i \to q_i + \epsilon K_i(\mathbf{q})\,.$$

That means that $d\mathcal{L}/d\epsilon = 0$, that is,

$$0 = \sum_i \left(\frac{\partial \mathcal{L}}{\partial q_i}\frac{\partial q_i}{\partial \epsilon} + \frac{\partial \mathcal{L}}{\partial q_i'}\frac{\partial q_i'}{\partial \epsilon} \right) = \sum_i \left(\frac{\partial \mathcal{L}}{\partial q_i} K_i + \frac{\partial \mathcal{L}}{\partial q_i'} K_i' \right)$$

and by using the Euler-Lagrange equations we get

$$\frac{d}{dt}\left(\sum_i p_i K_i \right) = 0 \implies P(\mathbf{q}, \mathbf{q}') \equiv \sum_i p_i K_i = \text{constant} \quad \square$$

$P(\mathbf{q}, \mathbf{q}')$ is given the generic name of **conserved momentum**.

LET'S SEE HOW THE THEOREM CAN BE APPLIED in practice. Consider the Lagrangian

$$\mathcal{L} = \frac{m}{2}\left(5x'^2 - 2x'y' + 2y'^2\right) - V(2x - y)\,.$$

Since the velocities are invariant under translations, we can focus on the potential energy to see immediately that the Lagrangian is symmetric under the transformation $x \to x + \epsilon$ and $y \to y + 2\epsilon$.[9] Therefore, $K_x = 1$ and $K_y = 2$, so the conserved quantity is $P = p_x + 2p_y = 3m(x' + y')$.[10] [11]

[9] In this case, the Langrangian is invariant, not only to first order in ϵ, but to any order.

[10] EXERCISE: Do it!

[11] Note that the prefactor $3m$ is irrelevant. Why?

Symmetry and the relativistic Lagrangian

To finish our study of Lagrangian mechanics, and to make the connection with the relativistic mechanics that we saw at the beginning of the course, we now derive the **relativistic Lagrangian of a free particle** from symmetry/invariance considerations.

Galilean invariance of the $\mathcal{L} = T - V$ Lagrangian

Consider the non-relativistic Lagrangian of a free particle as seen from an inertial reference frame, $\mathcal{L} = m\vec{v}^2/2$. Now, from Galilean invariance, we expect that the equations of motion should be the same in another reference frame that moves at constant velocity $\vec{V}$ with respect to the first one. In the new reference frame, the Lagrangian is[12]

$$\mathcal{L}' = \frac{m}{2}(\vec{v} + \vec{V})^2 = \frac{m}{2}\vec{v}^2 + m\vec{v}\cdot\vec{V} + \frac{m}{2}\vec{V}^2 = \mathcal{L} + \frac{dF}{dt}$$

[12] Note that primes now indicate the new reference frame, **not** time derivatives!

with

$$F = m\vec{r} \cdot \vec{V} + \frac{m}{2}\vec{V}^2 t \; .$$

Now, because both Lagrangians only differ in a quantity that is the total time derivative of a function F, then we know that the equations will be the same (see Exercise 3.1).

So, as we wanted to show, the equations of motion that result from the non-relativistic Lagrangian are invariant under Galilean transformations.

Relativistic Lagrangian

We now derive the relativistic Lagrangian of a free particle. We have seen that the non-relativistic Lagrangian leads to the same equations of motion in any inertial frame under Galilean transformations. When we think of an extension to relativistic settings, we start, precisely, from this consideration—whatever laws we derive from a hypothetical relativistic Lagrangian must be invariant under Lorentz transformations. This is the power, and the beauty, of symmetry considerations in physics.

So consider a particle of mass m moving in space. The easiest way to make sure that the equations of motion are Lorentz invariant is to propose a Langrangian that is invariant itself.[13] Now, what is invariant in the trajectory of a relativistic particle? Of course, *proper time*[14]

$$d\tau = \sqrt{dt^2 - dx^2 - dy^2 - dz^2} = dt\sqrt{1 - x'^2 - y'^2 - z'^2} \, .$$

Indeed, we know that observers sitting in different inertial frames will *not* agree on distances or times, but they will all measure the same quantity for $d\tau$, so we postulate the action

$$S = A \int_a^b dt \sqrt{1 - x'^2 - y'^2 - z'^2} \; ,$$

which is, except for the constant A, the total proper time of the trajectory. This looks pretty good—not only is it Lorentz invariant, but it is also quadratic in the velocities, as the good old $\mathcal{L} = T - V$. And, what about A? We have to choose it so that $\mathcal{L}$ has dimensions of energy.[15] By using the mass m of the particle and c (which, so far, we have set to $c = 1$, as usual), there is only one way to give $\mathcal{L}$ the right dimensions, namely,

$$\boxed{\mathcal{L} = -mc^2\sqrt{1 - \frac{x'^2 + y'^2 + z'^2}{c^2}} = -mc^2\sqrt{1 - \frac{v^2}{c^2}}} \, .$$

The minus sign in front of the Lagrangian is arbitrary, but we add it so that, in the limit $v \ll c$, this Lagrangian reduces to the non-

[13] Note that this is not a necessary condition for the equations of motion to be invariant, but it is sufficient.

[14] Note that, for the trajectory of a massive particle, which is time-like, it is proper time that is well defined, and not the invariant interval.

[15] Remember that the Lagrangian has units of energy like T and V.

relativistic Lagrangian of a free particle $\mathcal{L} = mv^2/2$, except for an irrelevant additive constant.[16]

[16] Prove that it does.

Relativistic momentum and energy from the Lagrangian

When we defined relativistic momentum and energy in the first block of the course, we had to make some choices that seemed somewhat arbitrary. By contrast, now we can use the general definitions that come out of the Lagrangian formalism to see if we recover the heuristic definitions that we used earlier.

Let's start with momentum. We have defined the generalized momentum as

$$p_i = \frac{\partial \mathcal{L}}{\partial x_i'} \,.$$

By using the relativistic Lagrangian,[17] we get precisely the definition that we had been using for the spatial components of the momentum 4-vector, namely

[17] EXERCISE: Do it!

$$p_i = \frac{m x_i'}{\sqrt{1 - v^2/c^2}} \,.$$

Similarly, we can get the relativistic energy by applying the definition above to the relativistic Lagrangian, namely[18]

[18] EXERCISE: Do it!

$$H = \sum_i p_i x_i' - \mathcal{L} = \frac{m \sum_i x_i'^2}{\sqrt{1 - v^2/c^2}} + mc^2 \sqrt{1 - v^2/c^2} = \frac{mc^2}{\sqrt{1 - v^2/c^2}} \,,$$

which, again, coincides with the result we had obtained earlier by making more or less educated guesses.