

# **Lesson 8- Landau Theory**

# Phase transitions

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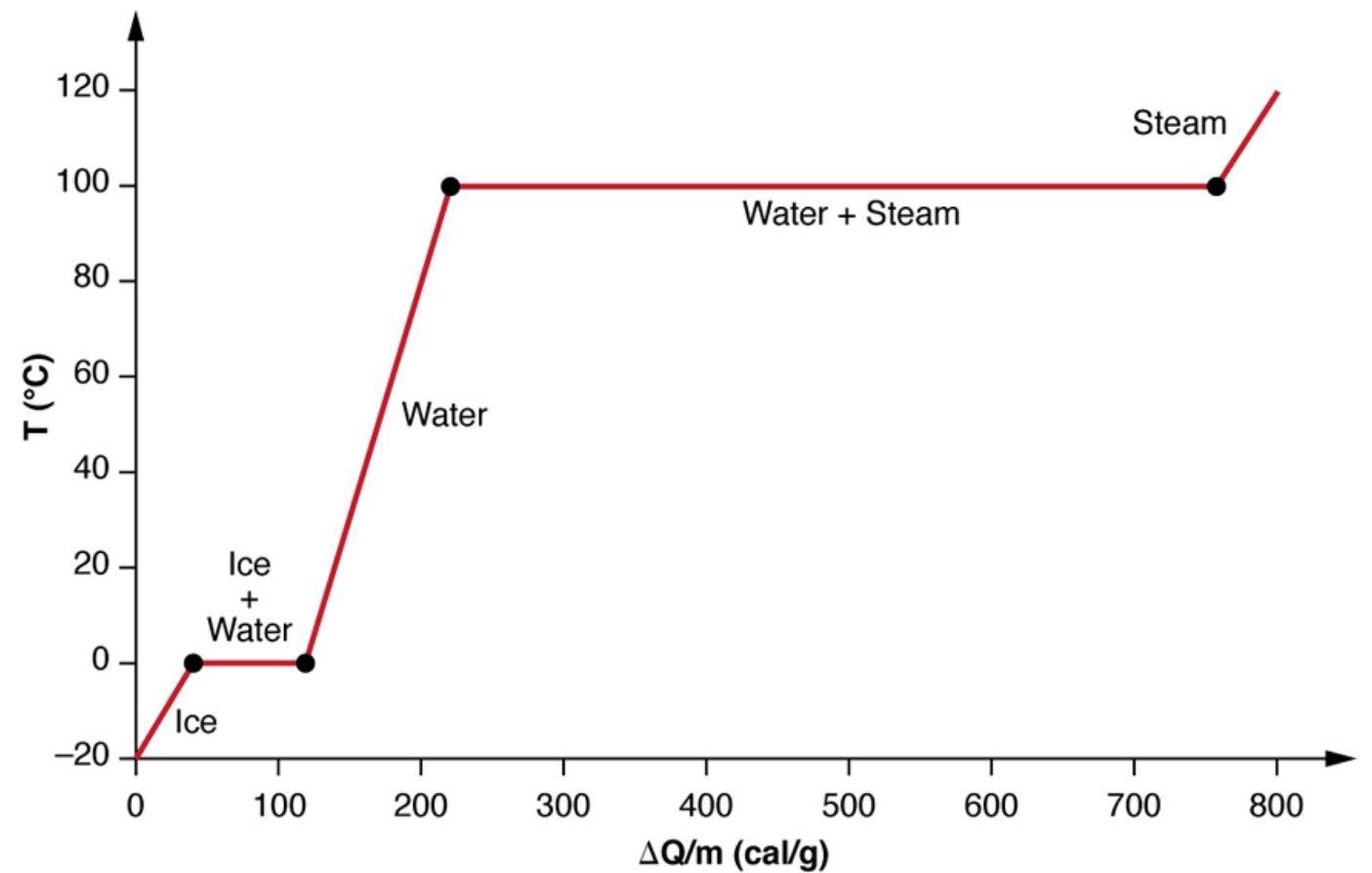
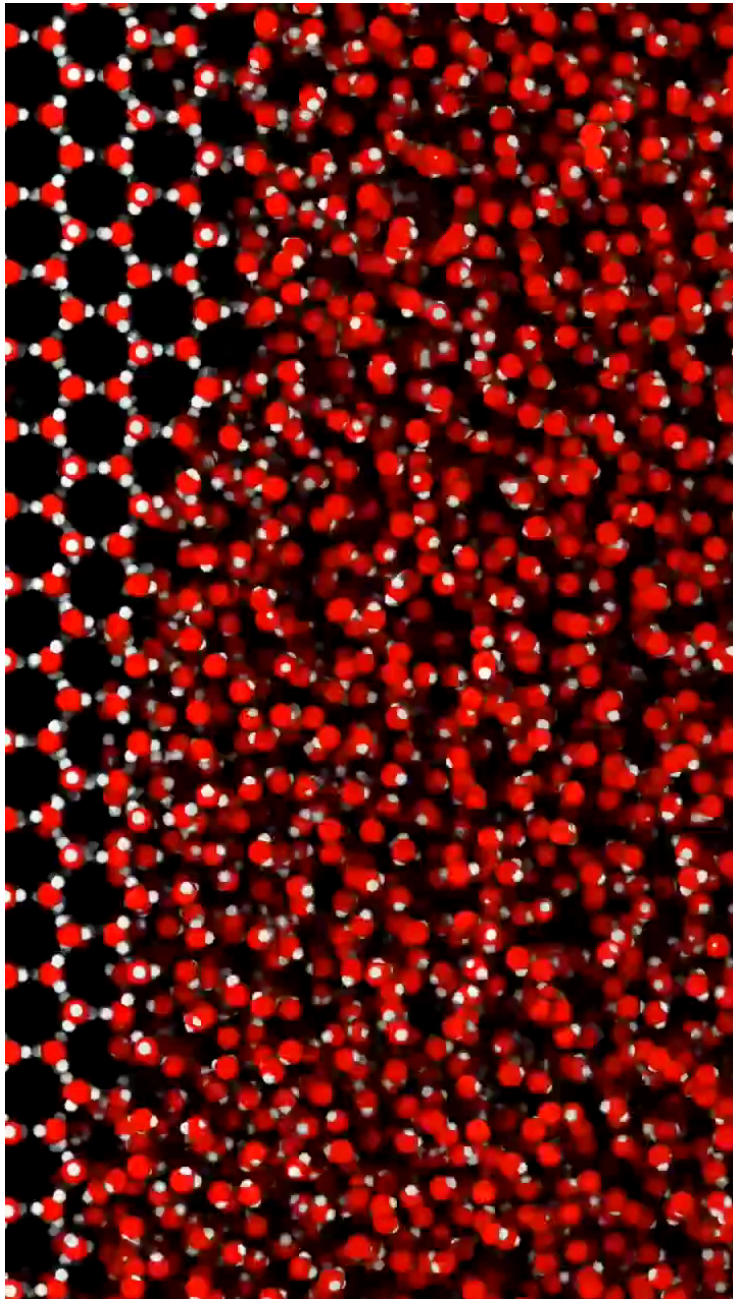
A **phase transition** (or phase change) is the physical process of transition between one state of a medium and another.

| Phase transitions of matter (V·T·E) |            |              |               |            |
|-------------------------------------|------------|--------------|---------------|------------|
| <div>To</div> <div>From</div>       | Solid      | Liquid       | Gas           | Plasma     |
| Solid                               |            | Melting      | Sublimation   |            |
| Liquid                              | Freezing   |              | Vaporization  |            |
| Gas                                 | Deposition | Condensation |               | Ionization |
| Plasma                              |            |              | Recombination |            |

Phase transitions occur when the thermodynamic **free energy** of a system is **non-analytic** for some choice of thermodynamic variables. This condition generally stems from the interactions of a large number of particles in a system, and does not appear in systems that are small. Phase transitions can occur for non-thermodynamic systems, where temperature is not a parameter. Examples include: quantum phase transitions, dynamic phase transitions, and topological (structural) phase transitions.

# Phase transitions

A phase transition of water into ice.



# Phase transitions

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## Ehrenfest Classification:

### First-Order Phase Transitions:

- Involves a **discontinuity in the first derivative** of the free energy.
- Examples: solid/liquid/gas transitions, where density undergoes a discontinuous change.

### Second-Order Phase Transitions:

- Characterized by continuous changes in the first derivative (order parameter) of the free energy.
- Exhibits a **discontinuity in a second derivative** of the free energy.
- Examples: ferromagnetic phase transitions like the Curie point in materials like iron.

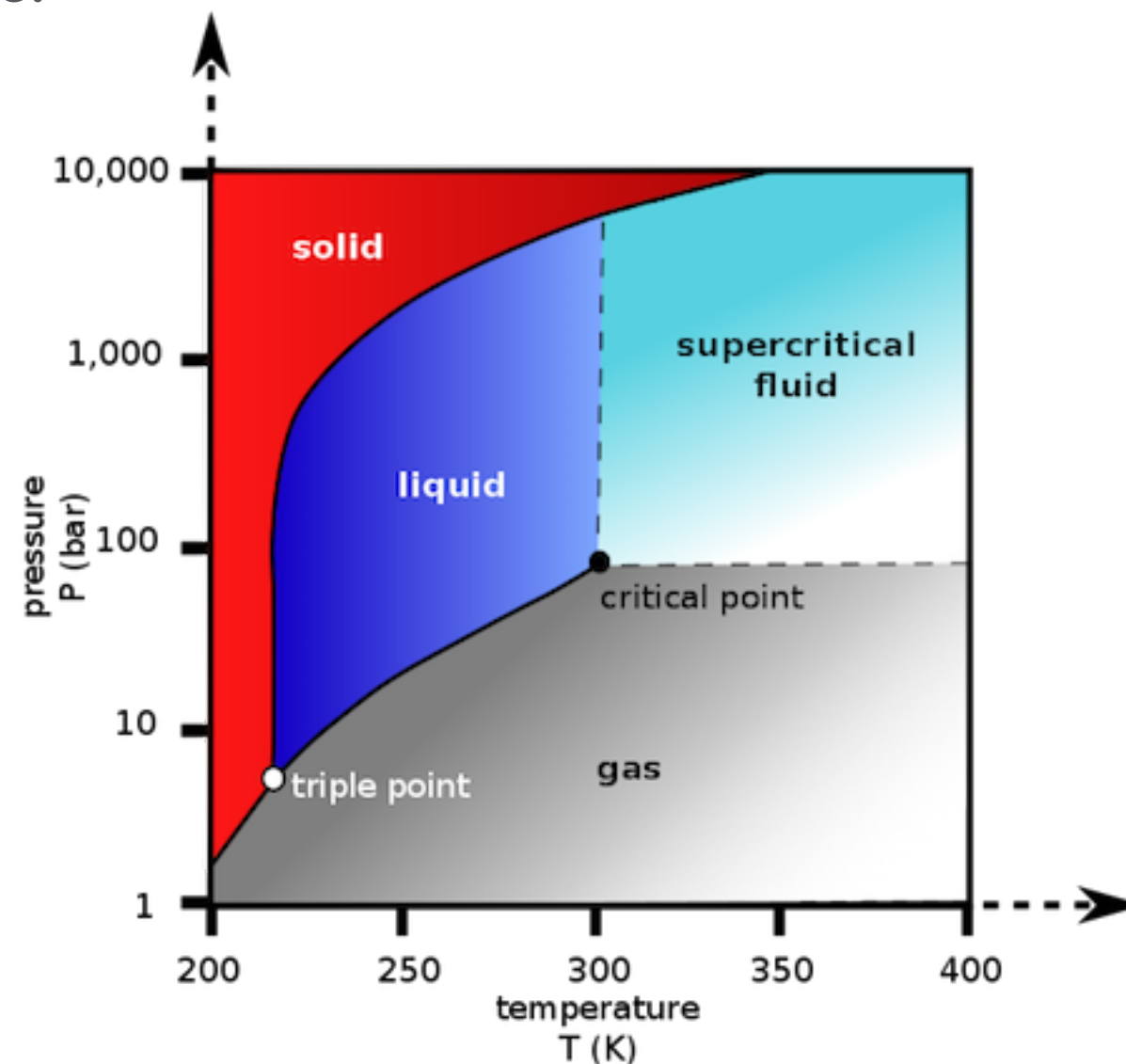
### Higher-Order Phase Transitions:

- Higher-order transitions involve **discontinuities in derivatives beyond the second order**.
- Examples: the Gross–Witten–Wadia phase transition in 2-D lattice quantum chromodynamics.

The classification system provides a structured way to understand and categorize phase transitions according to the thermodynamic free energy and its derivatives.

# Critical points

The critical point in the context of phase transitions refers to the **specific set of thermodynamic conditions** (such as temperature and pressure) **at which a substance undergoes a critical phase transition**. At the critical point, the distinction between the liquid and gaseous phases of a substance disappears, and the two phases become indistinguishable.



## Symmetry breaking

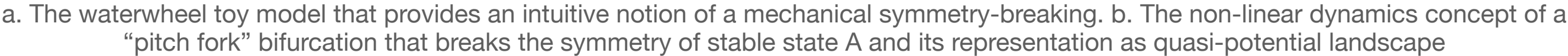
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Phase transitions frequently entail a process of **symmetry breaking**.

Examples:

- **Solidification:** when a fluid undergoes cooling and transforms into a crystalline solid, there is a breaking of continuous translation symmetry. In the fluid, each point possesses identical properties, whereas in a crystal, this uniformity is disrupted, unless the points are specifically chosen from the lattice points of the crystal lattice.
- **Ferromagnetism:** Above the Curie temperature,  $T_c$ , there is no magnetization; below  $T_c$ , magnetization spontaneously appears. The spatial isotropy is broken.
- **Rayleigh–Bénard convective instability:** If a fluid is enclosed between parallel planes and the lower plane is maintained at a higher temperature, the fluid develops convection cells, which break the Euclidean invariance of the liquid.
- Many more in particle physics...





## Order Parameter

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An **order parameter** is a measure of the **degree of order** across the boundaries in a phase transition system; it normally ranges between zero in one phase (usually above the critical point) and nonzero in the other. At the critical point, the order parameter susceptibility will usually diverge.

An example of an order parameter is the net magnetization  $m$  in a ferromagnetic system undergoing a phase transition. For liquid/gas transitions, the order parameter is the difference of the densities.

From a theoretical perspective, **order parameters arise from symmetry breaking**. When this happens, one needs to introduce one or more extra variables to describe the state of the system. For example, in the ferromagnetic phase, one must provide the net magnetization, whose direction was spontaneously chosen when the system cooled below the critical temperature (Curie point).



## Landau's Theory

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Landau's theory is an **approximate method** to describe the behavior of any system (whether a spin network, a gas, etc.) **near a critical point**. To introduce the method, we will first restrict ourselves to a spin system, like the Ising model but with any Hamiltonian, and then we will briefly discuss how it generalizes to other systems.

Consider a system of  $N$  spins with Hamiltonian  $H$  at temperature  $T$ . Given a state  $\sigma_1, \dots, \sigma_N$  of the system, the average magnetisation is:

$$m = \frac{1}{N} \sum_{i=1}^N \sigma_i$$

Notice that  $m$  can take  $N + 1$  different values,  $m \in \{-1, -1 + 2/N, \dots, 1 - 2/N, 1\}$ . We define the **Landau free energy**  $f(T, m)$  by the equation:

$$e^{-\beta N f(T, m)} \equiv \sum_{\sigma_i | m} e^{-\beta H(\sigma_1, \dots, \sigma_N)}$$

## Landau's Theory

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where the notation  $\sigma_i|m$  indicates that the sum is over all states  $\sigma_1, \dots, \sigma_N$  with magnetization  $m$ . We can write the canonical partition function in terms of Landau's free energy,

$$Z(T) = \sum_{\sigma_i} e^{-\beta H(\sigma_1, \dots, \sigma_N)} = \sum_m \sum_{\sigma_i|m} e^{-\beta H(\sigma_1, \dots, \sigma_N)} = \sum_m e^{-\beta N f(T, m)}$$

Since the last sum has  $N + 1$  terms, we know we can approximate it by its largest term,

$$Q(T) \approx e^{-\beta N f(T, \bar{m})} \Rightarrow F(T) \approx N f(T, \bar{m})$$

where  $\bar{m}$  is the minimum of  $f$ , and  $F$  denotes the Helmholtz free energy. The error we make in this approximation is  $\delta F = O(\log N)$ , and therefore very small compared to  $F$ .

## Landau's Theory

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If we repeat the argument by adding a magnetic field term  $h$  to the Hamiltonian and derive with respect to this last term, we realize that the minimum  $\bar{m}$  is also the average value of the magnetization.

Let's see what we can say about  $f$  without knowing the system's Hamiltonian. Suppose the Hamiltonian is invariant under the inversion of all spins,  $H(-\sigma_1, \dots, -\sigma_N) = H(\sigma_1, \dots, \sigma_N)$ , as occurs in the Ising model when there is no magnetic field. In this case, from definition of  $f$  must be an even function of  $m$ , i.e.  $f(T, -m) = f(T, m)$ . Also, suppose that, as occurs in the Ising model, the system undergoes a second-order phase transition at some critical temperature  $T_c$ , above which the average magnetization  $\bar{m}$  becomes zero and below which it takes a non-zero value.

## Landau's Theory

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Near the critical point, the average magnetization will be small, so it will only be necessary to know  $f$  for small values of  $m$ . Expanding in a Taylor series up to the fourth order and considering the parity of  $f$ , we obtain:

$$f(T, m) \approx f_0(T) + a(T)m^2 + b(T)m^4$$

where  $f_0$ ,  $a$  and  $b$  are unknown functions (their precise form will depend on the Hamiltonian). The first of these functions is usually ignored because it is irrelevant for determining the minimum  $\bar{m}$ , but here we will not do so. We assume that  $b > 0$ , because otherwise  $f$  would not have an absolute minimum at this order (it would become arbitrarily negative for large  $m$ ).

The minimum  $\bar{m}$  satisfies:

$$\left. \frac{\partial f}{\partial m} \right|_{m=\bar{m}} = 0 = 2\bar{m}(a + 2b\bar{m}^2)$$

## Landau's Theory

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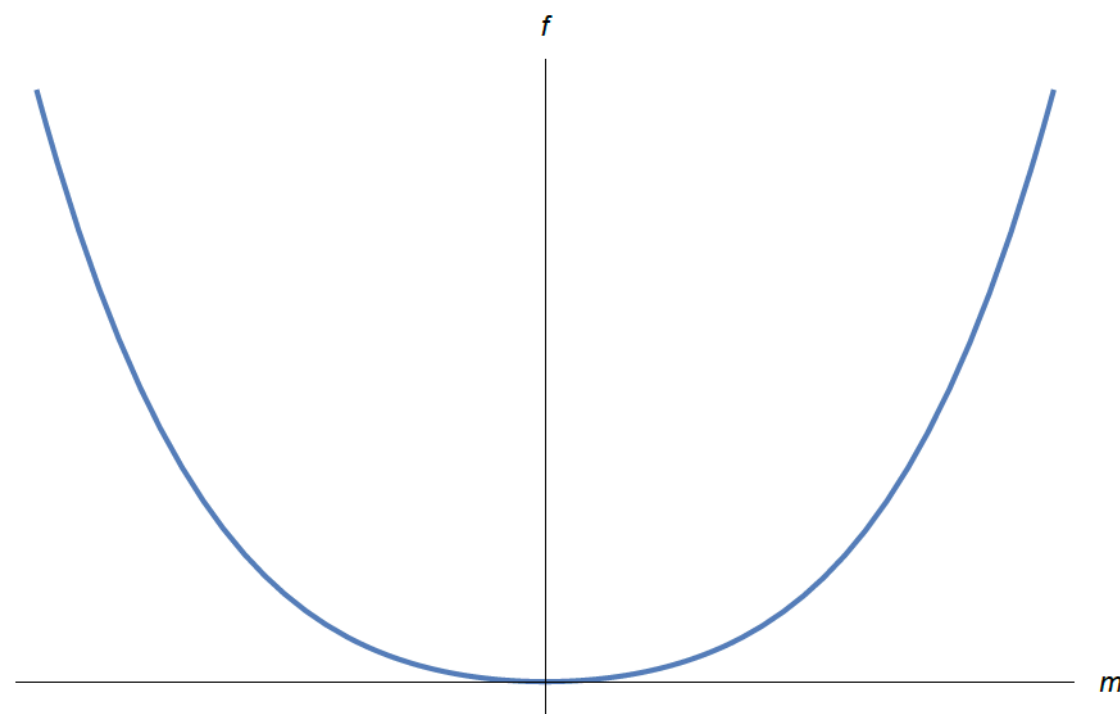
one solution is  $\bar{m} = 0$ . If  $a > 0$  this is the only solution and it is a minimum because  $f \rightarrow +\infty$  when  $m \rightarrow \pm\infty$ , so  $\bar{m} = 0$  when  $a > 0$ . In the case of  $a < 0$ , the equation has two additional solutions,  $\bar{m} = \pm\sqrt{-a/2b}$  and these are the minima because they are the first extremes coming from  $\pm\infty$ , where  $f$  tends to  $+\infty$  (we are ignoring the possibility that they are inflection points); in this case, the solution  $\bar{m} = 0$  is a maximum.

We know that  $\bar{m}$  has to be zero for  $T > T_c$  and different from zero for  $T < T_c$ . Therefore, from the previous equations we see that  $a(T)$  is positive for  $T > T_c$  and negative for  $T < T_c$ . Note that this implies  $a(T_c) = 0$  and  $a'(T_c) > 0$ .

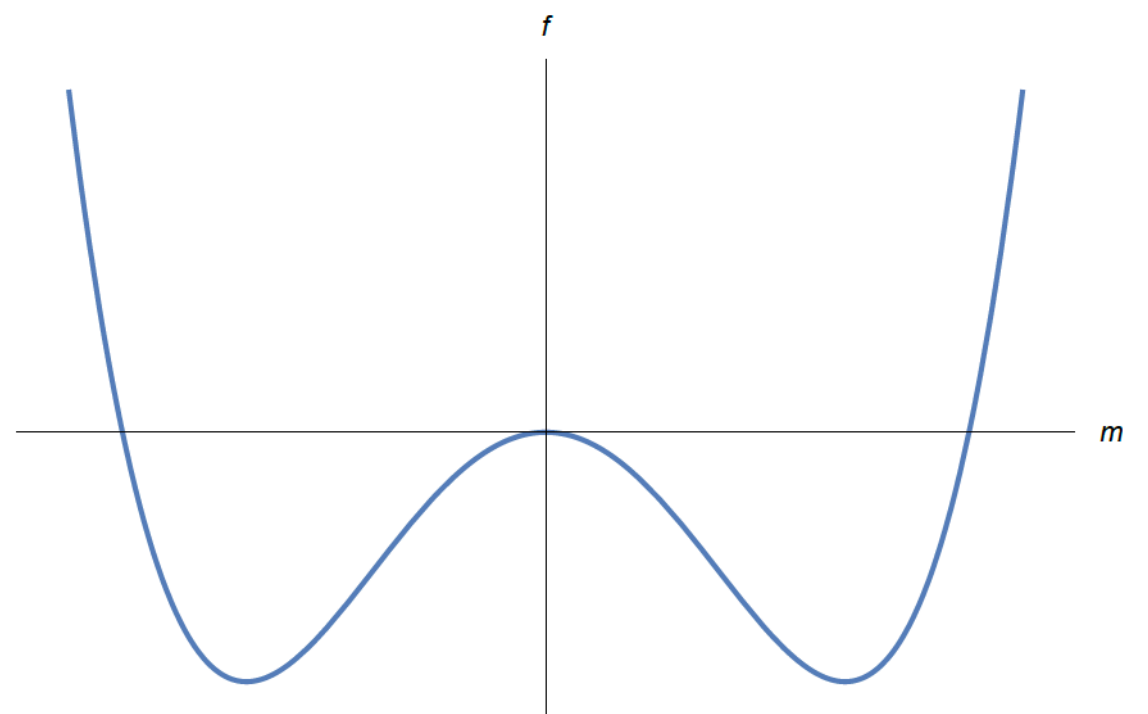
## Landau's Theory

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This figure shows the graph of  $f$  as a function of  $m$  for two temperatures, one higher and one lower than the critical temperature.



(a)  $T > T_c$



(b)  $T < T_c$



## Landau's Theory

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For temperatures very close to the critical temperature, we can write  $\bar{m}$  as a function of  $T$  more explicitly. Indeed, the properties of  $a$  and  $b$  that we just described imply that, at the lowest order around  $T_c$

$$a(T) \approx a_0(T - T_c) \quad b(T) \approx b_0$$

where  $a_0$  and  $b_0$  are positive constants. Replacing in equation for  $\bar{m}$  we obtain

$$\bar{m} = \begin{cases} 0 & \text{for } T > T_c \\ \sqrt{\frac{a_0}{2b_0}(T_c - T)} & \text{for } T < T_c \end{cases}$$

Note that  $\bar{m}$  is continuous at  $T = T_c$  but its derivative is not, which indicates that there is indeed a second-order phase transition at this temperature.

## Landau's Theory

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Now that we know the average value of the magnetization, we can obtain the Helmholtz free energy by replacing in the previous equations

$$\frac{F(T)}{N} = \begin{cases} f_0(T) & \text{for } T > T_c \\ f_0(T) - \frac{a_0^2}{4b_0}(T - T_c)^2 & \text{for } T < T_c \end{cases}$$

and from this result, we can calculate various quantities of interest (and remember, without knowing almost anything about the system's Hamiltonian!)

## Landau's Theory

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Definition: **Critical exponents** describe the behavior of physical quantities near continuous phase transitions. It is believed, **though not proven**, that they are universal, i.e. they do not depend on the details of the physical system, but only on some of its general features.

For the previous analysis, the critical exponents  $\alpha$ ,  $\beta$ ,  $\gamma$  and  $\delta$  are defined from the behavior of the average magnetization, specific heat, and magnetic susceptibility near the critical point by the equations

$$\bar{m} \sim (T_c - T)^\beta \quad (h = 0; T < T_c)$$

$$c_V \sim |T - T_c|^{-\alpha} \quad (h = 0)$$

$$\chi \sim |T - T_c|^{-\gamma} \quad (h = 0)$$

$$\bar{m} \sim h^{1/\delta} \quad (T = T_c)$$

where  $h$  denotes the magnetic field. From the equation we obtained for  $\bar{m}$  we immediately see that  $\beta = 1/2$ .

## Landau's Theory

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We also know that

$$c_V = \frac{T}{N} \frac{dS}{dT} = - \frac{T}{N} \frac{d^2F}{dT^2} = \begin{cases} c_{V,0} & T > T_c \\ c_{V,0} + \frac{a_0^2}{2b_0} T & T < T_c \end{cases}$$

where  $c_{V,0} = -Tf_0''$ . The only value of the critical exponent  $\alpha$  consistent with the previously discussed behavior of specific heat near the critical temperature is  $\alpha = 0$ .

To determine the other two critical exponents, we need to include a magnetic field  $h$ , that is, we need to consider the Hamiltonian

$$\tilde{H}(\sigma_1, \dots, \sigma_N) = H(\sigma_1, \dots, \sigma_N) - h \sum_{i=1}^N \sigma_i$$

From definition, it is immediate to see that the corresponding Landau free energy for this new Hamiltonian is

$$\tilde{f}(T, h, m) = f(T, m) - hm = f_0(T) + a(T)m^2 + b(T)m^4 - hm$$

## Landau's Theory

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Following the steps we had developed so far, **prove that** the

susceptibility  $\chi = \left. \frac{\partial \bar{m}}{\partial h} \right|_{h=0}$  satisfies:

$$\chi = \frac{1}{2(a + 6b\bar{m}^2)} = \begin{cases} \frac{1}{2a_0(T - T_c)} & T > T_c \\ \frac{1}{4a_0(T_c - T)} & T < T_c \end{cases}$$

from which  $\gamma = 1$ . And finally

$$\bar{m} = \sqrt[3]{\frac{h}{4b_0}}$$

and then  $\delta = 3$ .

## Landau's Theory

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### Exercise: Landau Theory for a Superfluid Transition

Use Landau theory to study the phase transition of a system into a superfluid state. Superfluidity is a phase of matter characterized by the absence of viscosity when flowing. The order parameter for this transition is the macroscopic wave function  $\psi$ , whose absolute square  $|\psi|^2$  represents the density of the superfluid component. The Landau free energy near the transition point can be expressed as:

$$F(T, \psi) = F_0(T) + \alpha(T) |\psi|^2 + \beta |\psi|^4$$

Here,  $F_0(T)$  is a temperature-dependent term,  $\alpha(T)$  changes sign at the critical temperature  $T_c$  and can be approximated as  $\alpha(T) = \alpha'(T - T_c)$  near  $T_c$ , with  $\alpha' > 0$ .  $\beta$  is a positive constant.

Tasks:

- Identify the stable states of the superfluid density  $|\psi|^2$  for temperatures above and below  $T_c$ .
- Discuss how the transition to superfluidity occurs as the temperature is lowered through  $T_c$  and the implications of this transition for the properties of the fluid, such as viscosity.



## Exercise: Landau Theory for a Superfluid Transition

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Task 1: To find the stable states, we need to minimize the Landau free energy  $F(T, \psi)$ . We do this by setting the derivative of  $F$  with respect to  $|\psi|^2$  equal to zero:

$$\frac{dF}{d|\psi|^2} = \alpha(T) + 2\beta|\psi|^2 = 0$$

For  $T > T_c$ ,  $\alpha(T) > 0$ , the only solution to this equation is  $|\psi|^2 = 0$ , which implies that the system does not exhibit superfluidity.

For  $T < T_c$ ,  $\alpha(T) < 0$ , solving the equation gives  $|\psi|^2 = -\frac{\alpha(T)}{2\beta}$ . Since  $\alpha(T) = \alpha'(T - T_c)$ , we

have:  $|\psi|^2 = \frac{\alpha'(T_c - T)}{2\beta}$

This positive value of  $|\psi|^2$  indicates the presence of a superfluid state.

Task 2: As the temperature lowers and crosses  $T_c$  from above, the coefficient  $\alpha(T)$  changes its sign from positive to negative. Right above  $T_c$ ,  $|\psi|^2 = 0$  indicating no superfluidity. As the temperature decreases below  $T_c$ ,  $|\psi|^2$  becomes positive, signaling the emergence of the superfluid phase. The transition is continuous as it changes smoothly from zero to a positive value, characteristic of a second-order phase transition. As the system undergoes a transition to the superfluid phase, the density of the superfluid component becomes non-zero. This marks the onset of superfluidity, a phase characterized by the absence of viscosity. The fluid can flow without friction, leading to phenomena like the absence of resistance to flow and the ability to climb the walls of its container (the so-called "creeping film" effect). The transition to superfluidity dramatically changes the fluid's behavior, exhibiting unique quantum mechanical properties at macroscopic scales.