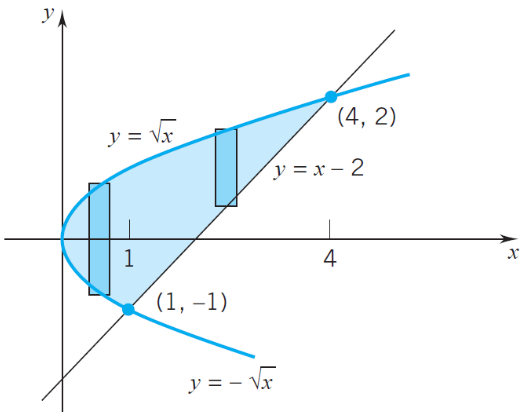
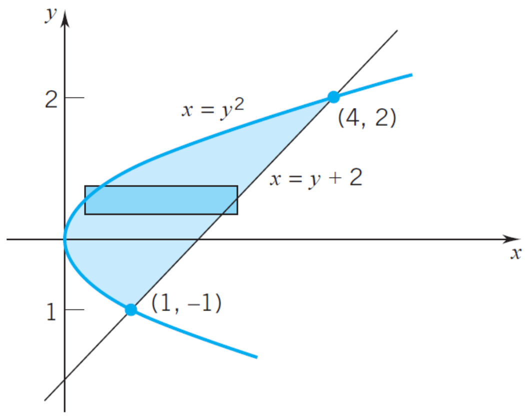
Anàlisi Matemàtica 1 (AM1)
GEMiF

E5.4 Exercicis: Aplicacions de la integral definida

1. Calcula l'àrea de la regió delimitada per les corbes $x = y^2$ i $x - y = 2$
    a) Per integració respecte x
    b) Per integració respecte y

Els punts d'intersecció es troben resolent $y^2 - y = 2$, i són $(1, -1)$ i $(4,2)$





a) Per integració respecte x

$$A = \int_0^1 [\sqrt{x} - (-\sqrt{x})]\,dx + \int_1^4 \sqrt{x} - (x-2)\,dx$$

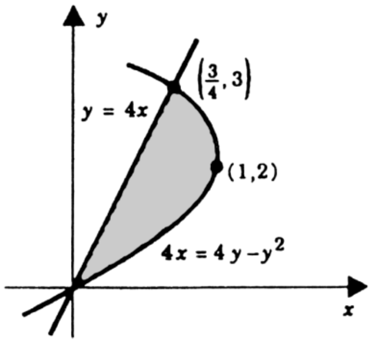
$$= 2\int_0^1 \sqrt{x}\,dx + \int_1^4 (\sqrt{x} - x + 2)\,dx = \left[\tfrac{4}{3}x^{3/2}\right]_0^1 + \left[\tfrac{2}{3}x^{3/2} - \tfrac{1}{2}x^2 + 2x\right]_1^4 = \tfrac{9}{2}$$

b) Per integració respecte y

$$A = \int_{-1}^2 [(y+2) - y^2]\,dy = \left[\tfrac{1}{2}y^2 + 2y - \tfrac{1}{3}y^3\right]_{-1}^2 = \tfrac{9}{2}$$

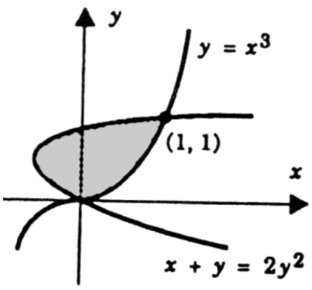
2. Calcula l'àrea de la regió delimitada per les corbes
   a) $4x = 4y - y^2$, $4x - y = 0$
   b) $x + y = 2y^2$, $y = x^3$
   c) $8x = y^3$, $8x = 2y^3 + y^2 - 2y$
   d) $y = \cos x$, $y = \sec^2 x$, $x \in [-\pi/4, \pi/4]$
   e) $y = 2\cos x$, $y = \sin 2x$, $x \in [-\pi, \pi]$
   f) $y = 6 - x^2$, $y = -|x|$
   g) $x^2 = 4py$, $y^2 = 4px$, $p > 0$
   h) Els cercles de radi 2 i centres (0,0) i (2,2)

a) $4x = 4y - y^2$, $4x - y = 0$



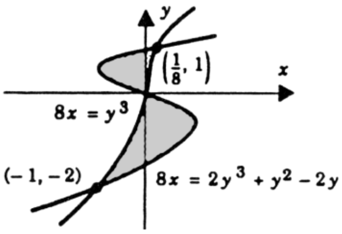
$$A = \int_0^3 \left[\left(\frac{4y - y^2}{4}\right) - \left(\frac{y}{4}\right)\right] dy$$
$$= \int_0^3 \left(\frac{3}{4}y - \frac{1}{4}y^2\right) dy$$
$$= \left[\tfrac{3}{8}y^2 - \tfrac{1}{12}y^3\right]_0^3 = \tfrac{9}{8}$$

b) $x + y = 2y^2$, $y = x^3$



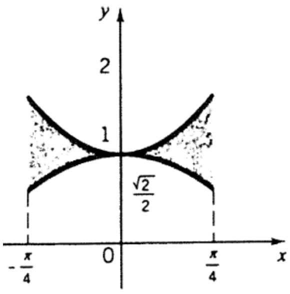
$$A = \int_0^1 \left[y^{1/3} - (2y^2 - y)\right] dy$$
$$= \int_0^1 (y^{1/3} + y - 2y^2)\, dy$$
$$= \left[\frac{3}{4}y^{4/3} + \frac{y^2}{2} - \frac{2}{3}y^3\right]_0^1 = \frac{7}{12}$$

c) $8x = y^3$, $8x = 2y^3 + y^2 - 2y$



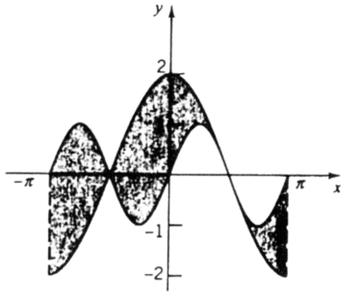
$$A = \int_{-2}^0 \left[\frac{1}{8}(2y^3 + y^2 - 2y) - \frac{1}{8}y^3\right] dy$$
$$+ \int_0^1 \left[\frac{1}{8}y^3 - \frac{1}{8}(2y^3 + y^2 - 2y)\right] dy$$
$$= \frac{1}{8}\left[\frac{y^4}{4} + \frac{y^3}{3} - y^2\right]_{-2}^0 + \frac{1}{8}\left[-\frac{y^4}{4} - \frac{y^3}{3} + y^2\right]_0^1 = \frac{37}{96}$$

d) $y = \cos x,\, y = \sec^2 x,\;\; x \in [-\pi/4, \pi/4]$



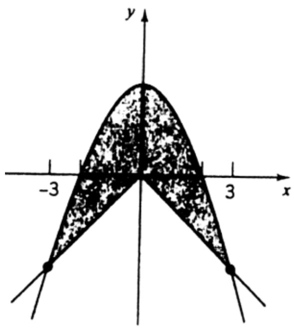
$$A = \int_{-\pi/4}^{\pi/4} \left[\sec^2 x - \cos x\right]\, dx$$
$$= 2\int_{0}^{\pi/4} \left[\sec^2 x - \cos x\right]\, dx$$
$$= 2\left[\tan x + \sin x\right]_0^{\pi/4} = 2\left[1 + \sqrt{2}/2\right] = 2 + \sqrt{2}$$

e) $y = 2\cos x,\, y = \sin 2x,\;\; x \in [-\pi, \pi]$



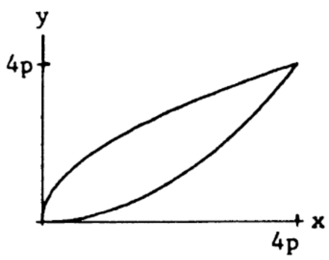
$$A = \int_{-\pi}^{-\pi/2} \left[\sin 2x - 2\cos x\right]\, dx + \int_{-\pi/2}^{\pi/2} \left[2\cos x - \sin 2x\right]\, dx$$
$$+ \int_{\pi/2}^{\pi} \left[\sin 2x - 2\cos x\right]\, dx$$
$$= \left[-\tfrac{1}{2}\,\cos 2x - 2\sin x\right]_{-\pi}^{-\pi/2} + \left[2\sin x + \tfrac{1}{2}\,\cos 2x\right]_{-\pi/2}^{\pi/2}$$
$$+ \left[-\tfrac{1}{2}\,\cos 2x - 2\sin x\right]_{\pi/2}^{\pi} = 8$$

f) $y = 6 - x^2,\, y = -|x|$



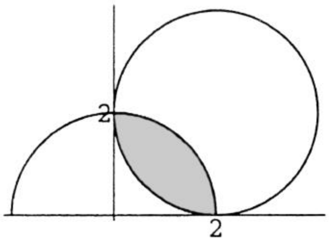
$$A = \int_{-3}^{0} \left[6 - x^2 - x\right]\, dx + \int_{0}^{3} \left[6 - x^2 - (-x)\right]\, dx$$
$$= \left[6x - \frac{1}{3}\,x^3 - \frac{1}{2}\,x^2\right]_{-3}^{0} + \left[6x - \frac{1}{3}\,x^3 + \frac{1}{2}\,x^2\right]_{0}^{3} = 27$$

g) $x^2 = 4py,\, y^2 = 4px,\, p > 0$



$$A = \int_0^{4p} \left(\sqrt{4px} - \frac{x^2}{4p}\right)\, dx$$
$$= \left[\sqrt{4p}\,\frac{2}{3}x^{3/2} - \frac{x^3}{12p}\right]_0^{4p} = \frac{16}{3}p^2$$

h) Els cercles de radi 2 i centres (0,0) i (2,2)



Cercle de centre (0,0)

$$x^2 + y^2 = 4 \;\Rightarrow\; y = \pm\sqrt{4 - x^2}$$

Cercle de centre (2,2)

$$(x-2)^2 + (y-2)^2 = 4 \;\Rightarrow\; y - 2 = \pm\sqrt{4 - (x-2)^2}$$
$$\Rightarrow\; y = 2 \pm \sqrt{4x - x^2}$$

$$A = \int_0^2 \left[\sqrt{4 - x^2} - \left(2 - \sqrt{4x - x^2}\right)\right] dx$$
$$= \int_0^2 \sqrt{4 - x^2}dx - 2[x]_0^2 + \int_0^2 \sqrt{4x - x^2}dx$$
$$= \int_0^2 \sqrt{4 - x^2}dx + \int_0^2 \sqrt{4x - x^2}dx - 4$$

La primera integral, per parts

$$u = \sqrt{4 - x^2} \qquad \Rightarrow \qquad u' = -\frac{x}{\sqrt{4 - x^2}}$$
$$v' = 1 \qquad\qquad v = x$$

$$\int_0^2 \sqrt{4 - x^2}dx = \left[x\sqrt{4 - x^2}\right]_0^2 + \int_0^2 \frac{x^2}{\sqrt{4 - x^2}}dx = 0 - \int_0^2 \frac{4 - x^2 - 4}{\sqrt{4 - x^2}}dx$$
$$= -\int_0^2 \left(\sqrt{4 - x^2} - \frac{4}{\sqrt{4 - x^2}}\right) dx$$

$$\int_0^2 \sqrt{4 - x^2}dx = \frac{1}{2}\int_0^2 \frac{4}{\sqrt{4 - x^2}}dx = \int_0^2 \frac{1}{\sqrt{1 - \left(\frac{x}{2}\right)^2}}dx = 2\left[\arcsin\frac{x}{2}\right]_0^2 = \pi$$

La segona integral

$$4x - x^2 = 4 - (x-2)^2$$

Fem canvi de variable

$t = 2 - x, \qquad dt = -dx$

$x = 0 \Rightarrow t = 2$

$x = 2 \Rightarrow t = 0$

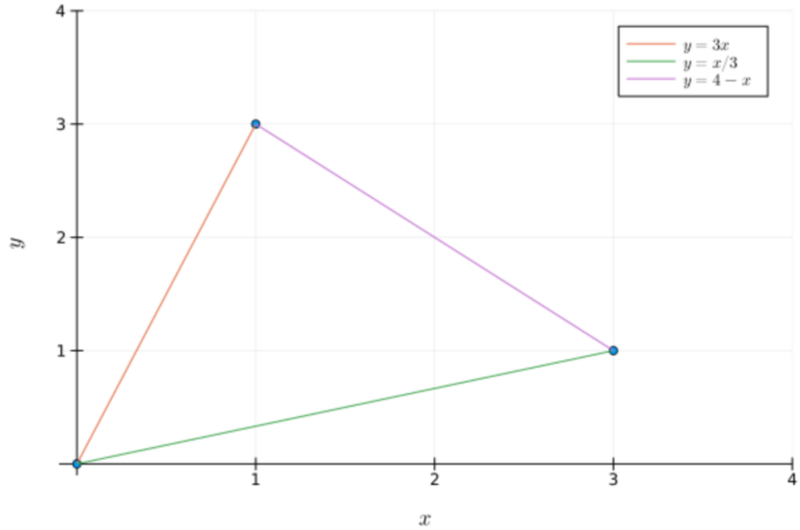
$$\int_0^2 \sqrt{4x - x^2}dx = -\int_2^0 \sqrt{4 - t^2}dx = \int_0^2 \sqrt{4 - x^2}dx = \pi$$

Queda

$A = 2\pi - 4$

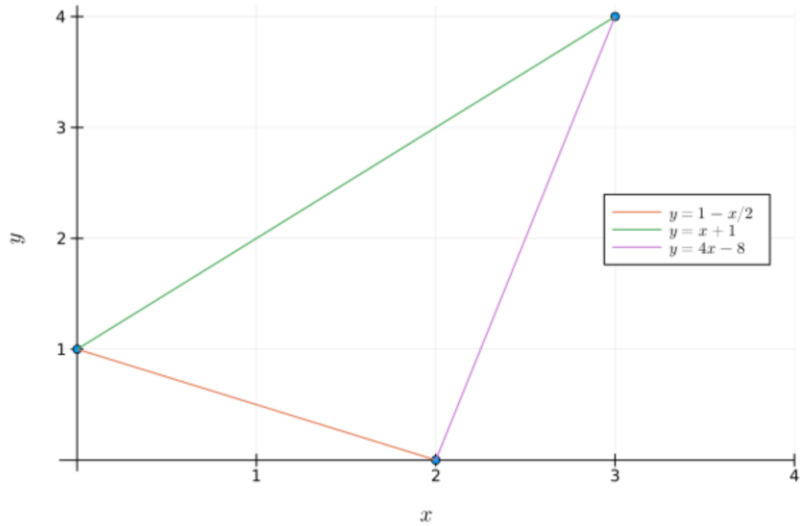
3. Calcula l'àrea dels polígon de vèrtexs
   a) $(0,0), (1,3), (3,1)$
   b) $(0,1), (2,0), (3,4)$
   c) $(-2,-2), (1,1), (5,1)$, $(7,-2)$

a) $(0,0), (1,3), (3,1)$



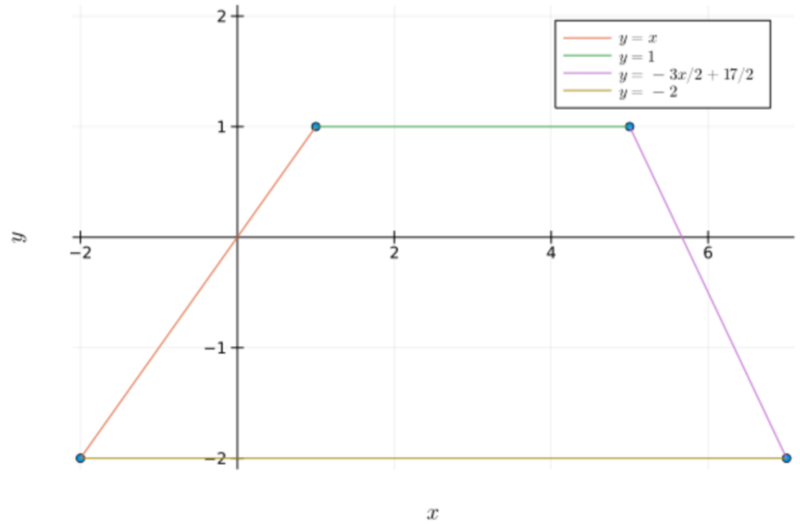
$$A = \int_0^1 \left[3x - \frac{1}{3}x\right] dx + \int_1^3 \left[-x + 4 - \frac{1}{3}x\right] dx = \left[\frac{4}{3}x^2\right]_0^1 + \left[-\frac{2}{3}x^2 + 4x\right]_1^3 = 4$$

b) $(0,1), (2,0), (3,4)$



$$A = \int_0^2 \left[(x+1) - \left(1 - \frac{x}{2}\right)\right] dx + \int_2^3 [(x+1) - (4x-8)] dx = \left[\frac{x^2}{4}\right]_0^2 + \left[-\frac{3x^2}{2} + 9x\right]_2^3 = \frac{5}{2}$$

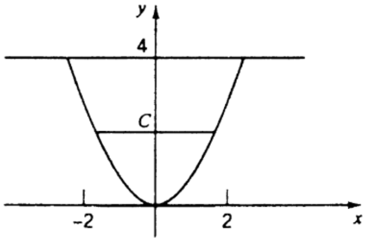
c) $(-2,-2)$, $(1,1)$, $(5,1)$, $(7,-2)$



$$A = \int_{-2}^{1} [x - (-2)]\, dx + \int_{1}^{5} [1 - (-2)]\, dx + \int_{5}^{7} \left[-\frac{3}{2}x + \frac{17}{2} - (-2)\right] dx$$

$$= \left[\tfrac{1}{2}x^2 + 2x\right]_{-2}^{1} + \left[3x\right]_{1}^{5} + \left[-\tfrac{3}{4}x^2 + \tfrac{21}{2}x\right]_{5}^{7} = \frac{39}{2}$$

4. La regió delimitada per $y = x^2$ i $y = 4$ es divideix en dues subregions de la mateixa àrea amb la línia $y = c$. Troba el valor de c.



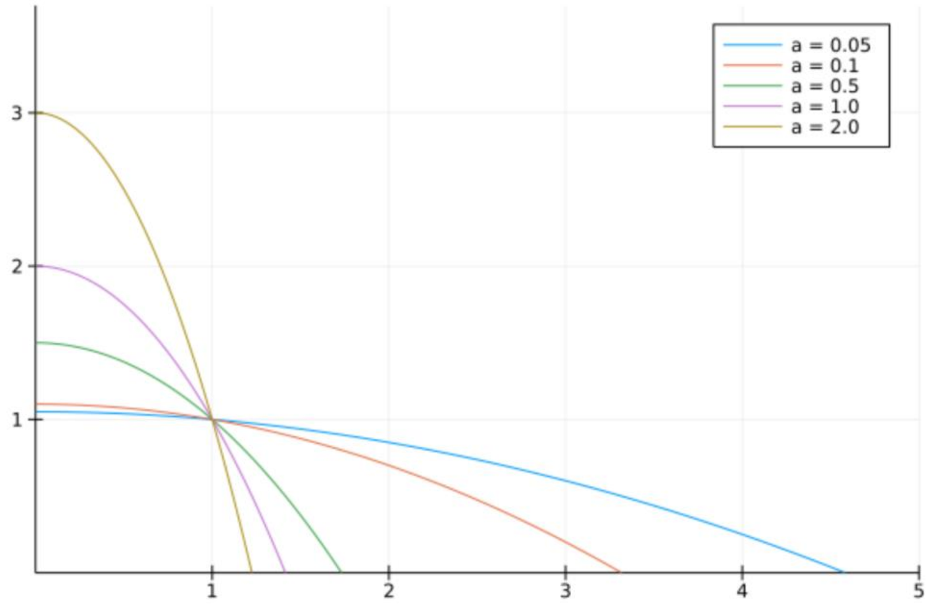
$$\int_0^{\sqrt{c}} \left[c - x^2\right]\, dx = \frac{1}{2} \int_0^2 \left[4 - x^2\right]\, dx$$

$$\left[cx - \tfrac{1}{3}\, x^3\right]_0^{\sqrt{c}} = \tfrac{1}{2}\, \left[4x - \tfrac{1}{3}\, x^3\right]_0^2$$

$$\tfrac{2}{3}\, c^{3/2} = \tfrac{8}{3} \quad \text{and} \quad c = 4^{2/3}$$

$$c = \sqrt[3]{16}$$

5. Calcula l'àrea de la regió delimitada per la corba $y = 1 + a - ax^2$, $a > 0$, al primer quadrant. Quin valor d'a fa que l'àrea sigui mínima?



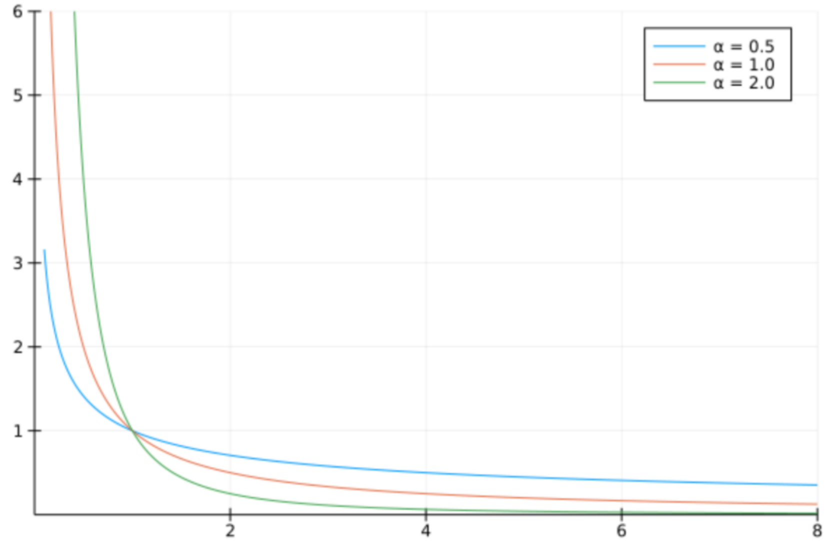
$$A(a) = \int_0^{\sqrt{\frac{1+a}{a}}} (1 + a - ax^2)dx = \left[(1 + a)x - \frac{ax^3}{3}\right]_0^{\sqrt{\frac{1+a}{a}}}$$

$$= (1 + a)\sqrt{\frac{1 + a}{a}} - \frac{a}{3}\sqrt{\frac{(1 + a)^3}{a^3}} = \frac{2}{3}\sqrt{\frac{(1 + a)^3}{a}} = \frac{2\sqrt{(1 + a)^3}}{3\sqrt{a}}$$

$$A'(a) = \frac{2}{3}\left[\frac{\frac{3}{2}(1 + a)^{\frac{1}{2}}a^{\frac{1}{2}} - (1 + a)^{\frac{3}{2}}\frac{1}{2}a^{-\frac{1}{2}}}{a}\right] = \frac{1}{3}\sqrt{1 + a}\frac{3a - (1 + a)}{\sqrt{a^3}}$$

$$A'(a) = 0 \quad \Rightarrow \quad a = \frac{1}{2}$$

6. Volem calcular l'àrea delimitada per les corbes del tipus $f(x) = x^{-\alpha}$, $\alpha > 0$, en el primer quadrant.
    a) Calcula l'àrea a l'interval $[1, +\infty)$, en funció del paràmetre α
    b) Calcula l'àrea a l'interval $(0,1)$, en funció del paràmetre α
    c) Calcula l'àrea a l'interval $(0, +\infty)$, en funció del paràmetre α



La integral indefinida és

$$\int \frac{1}{x^\alpha} dx = \begin{cases} \ln x\,, & \alpha = 1 \\ \dfrac{1}{(1-\alpha)x^{\alpha-1}}, & \alpha \neq 1 \end{cases}$$

a) Calcula l'àrea a l'interval $[1, +\infty)$, en funció del paràmetre α

$$A = \int_1^\infty \frac{1}{x^\alpha} dx = \lim_{L\to+\infty} \int_1^L \frac{1}{x^\alpha} dx$$

$$\int_1^L \frac{1}{x^\alpha} dx = \begin{cases} \ln L\,, & \alpha = 1 \\ \dfrac{1}{1-\alpha}\left(\dfrac{1}{L^{\alpha-1}} - 1\right), & \alpha \neq 1 \end{cases}$$

$$A = \begin{cases} +\infty, & \alpha < 1 \\ +\infty, & \alpha = 1 \\ \dfrac{1}{\alpha - 1}, & \alpha > 1 \end{cases}$$

b) Calcula l'àrea a l'interval $(0,1)$, en funció del paràmetre α

$$A = \int_0^1 \frac{1}{x^\alpha} dx = \lim_{\epsilon \to 0^+} \int_\epsilon^1 \frac{1}{x^\alpha} dx$$

$$\int_\epsilon^1 \frac{1}{x^\alpha} dx = \begin{cases} -\ln \epsilon, & \alpha = 1 \\ \dfrac{1}{1-\alpha}\left(1 - \dfrac{1}{\epsilon^{\alpha-1}}\right), & \alpha \neq 1 \end{cases}$$

$$A = \begin{cases} \dfrac{1}{1-\alpha}, & \alpha < 1 \\ +\infty, & \alpha = 1 \\ +\infty, & \alpha > 1 \end{cases}$$

c) Calcula l'àrea a l'interval $(0, +\infty)$, en funció del paràmetre α

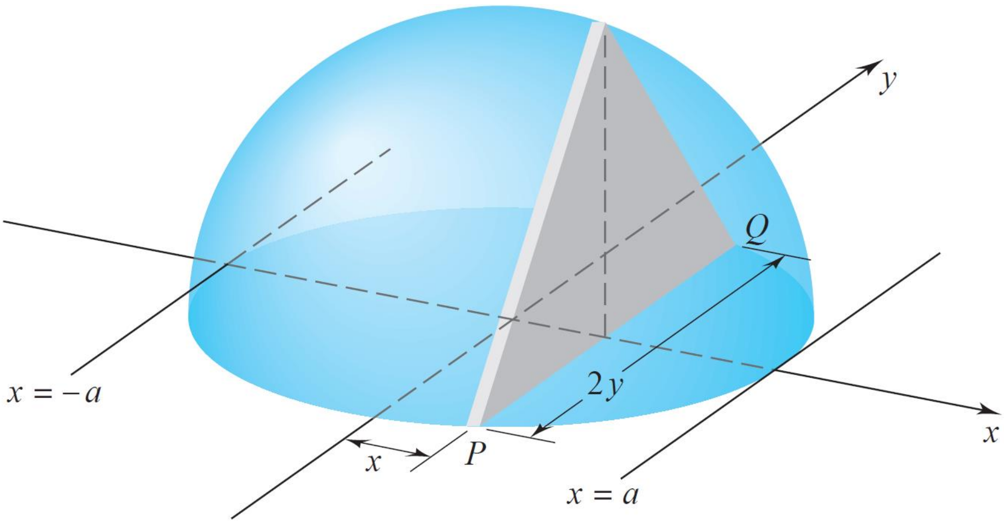
$$A = \int_0^\infty \frac{1}{x^\alpha} dx = \int_0^1 \frac{1}{x^\alpha} dx + \int_1^\infty \frac{1}{x^\alpha} dx$$

Per a tot $\alpha > 0$ resulta $A \to +\infty$

7. La base d'un sòlid és la regió envoltada per l'el·lipsi

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

Troba el volum del sòlid tal que les seccions perpendiculars a l'eix x són triangles isòsceles amb vèrtexs de la base sobre la el·lipsi i alçada la meitat que la base.



Base del triangle per un cert valor de l'abscissa x

$$\overline{PQ}=2y=\frac{2b}{a}\sqrt{a^2-x^2}$$

Àrea del triangle corresponent

$$A(x)=\frac{1}{2}\overline{PQ}\,\frac{\overline{PQ}}{2}=\frac{b^2}{a^2}(a^2-b^2)$$

El volum és (utilitzant la simetria del sòlid)

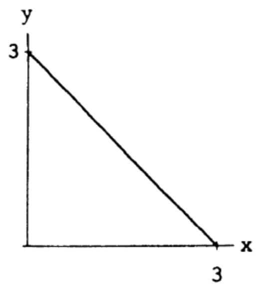
$$V=\int_{-a}^{a}A(x)dx=2\int_{0}^{a}A(x)dx=\frac{2b^2}{a^2}\int_{0}^{a}(a^2-b^2)dx=\frac{2b^2}{a^2}\left[a^2x-\frac{x^3}{3}\right]_0^a$$

$$=\frac{2b^2}{a^2}\left(a^3-\frac{a^3}{3}\right)=\frac{4}{3}ab^2$$

8. Troba el volum del sòlid de revolució obtingut en girar la regió Ω delimitada per les corbes indicades al voltant de l'eix x

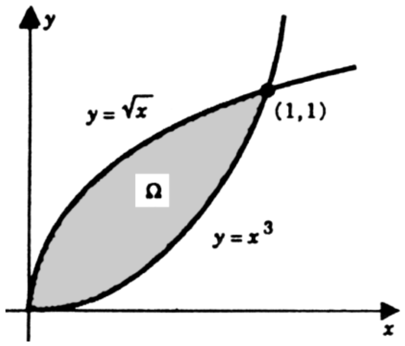
   a) $x + y = 3,\ y = 0,\ x = 0$
   b) $y = \sqrt{x},\ y = x^3$
   c) $y = x^3,\ x + y = 10,\ y = 1$
   d) $y = \cos x,\ y = x + 1,\ y = \pi/2$
   e) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,\ y = 0$

a) $x + y = 3,\ y = 0,\ x = 0$



$$V = \int_0^3 \pi\,(3-x)^2\,dx = \left[-\frac{\pi(3-x)^3}{3}\right]_0^3 = 9\pi$$

b) $y = \sqrt{x},\ y = x^3$

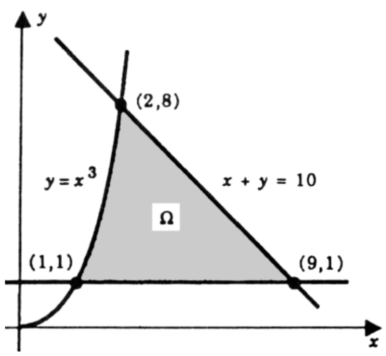


$$V = \int_0^1 \pi\left[\left(\sqrt{x}\,\right)^2 - \left(x^3\right)^2\right]\,dx$$

$$= \int_0^1 \pi\left(x - x^6\right)\,dx$$

$$= \pi\left[\frac{1}{2}x^2 - \frac{1}{7}x^7\right]_0^1 = \frac{5\pi}{14}$$

c) $y = x^3,\ x + y = 10,\ y = 1$

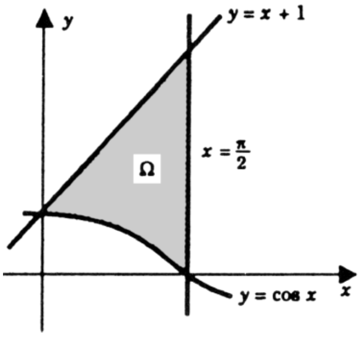


$$V = \int_1^2 \pi\left[\left(x^3\right)^2 - (1)^2\right]\,dx + \int_2^9 \pi\left[(10-x)^2 - (1)^2\right]$$

$$= \int_1^2 \pi\left(x^6 - 1\right)\,dx + \int_2^9 \pi\left(99 - 20x + x^2\right)\,dx$$

$$= \pi\left[\frac{1}{7}x^7 - x\right]_1^2 + \pi\left[99x - 10x^2 + \frac{1}{3}x^3\right]_2^9 = \frac{3790\pi}{21}$$

d) $y = \cos x,\, y = x + 1,\, y = \pi/2$



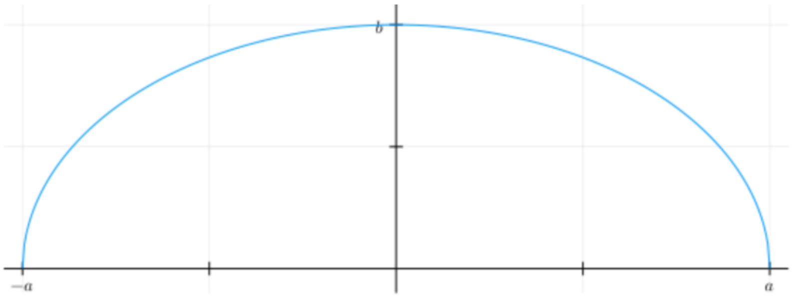
$$V = \int_0^{\pi/2} \pi \left[(x+1)^2 - (\cos x)^2 \right] dx$$

$$= \int_0^{\pi/2} \pi \left[(x+1)^2 - \left(\frac{1}{2} + \frac{1}{2}\cos 2x \right) \right] dx$$

$$= \pi \left[\frac{1}{3}(x+1)^3 - \frac{1}{2}x - \frac{1}{4}\sin 2x \right]_0^{\pi/2}$$

$$= \tfrac{\pi^2}{24}\left(\pi^2 + 6\pi + 6\right)$$

e) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,\, y = 0$

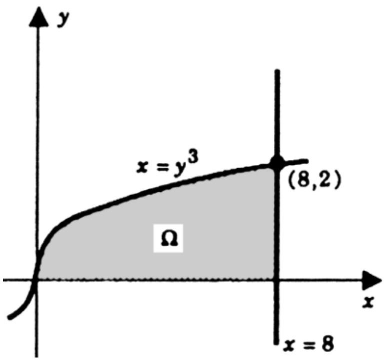


$$V = \int_{-a}^{a} \pi \left(b\sqrt{1 - \frac{x^2}{a^2}} \right)^2 dx = \frac{2\pi b^2}{a^2} \int_0^a \pi \left(a^2 - x^2\right) dx = \frac{2\pi b^2}{a^2}\pi \left[a^2 x - \frac{1}{3}x^3 \right]_0^a$$

$$= \frac{2\pi b^2}{a^2}\left(\frac{2}{3}a^3\right) = \frac{4}{3}\pi a b^2$$

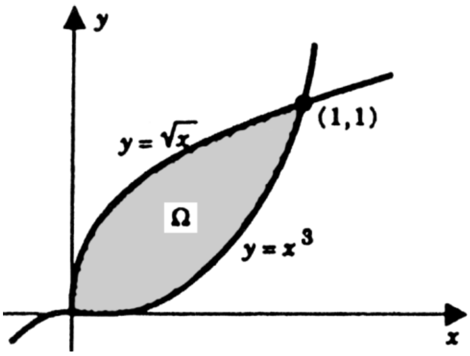
9. Troba el volum del sòlid de revolució obtingut en girar la regió Ω delimitada per les corbes indicades al voltant de l'eix y
   a) $x = y^3, x = 8, y = 0$
   b) $y = \sqrt{x}, y = x^3$
   c) $y = x, y = 2x, x = 4$
   d) $x = \sqrt{9 - y^2}, x = 0$
   e) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, x = 0$

a) $x = y^3, x = 8, y = 0$



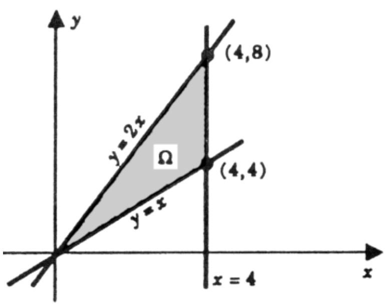
$$V = \int_0^2 \pi \left[(8)^2 - \left(y^3\right)^2 \right] dy$$
$$= \int_0^2 \pi \left(64 - y^6\right) dy$$
$$= \pi \left[64y - \tfrac{1}{7}y^7\right]_0^2 = \tfrac{768}{7}\pi$$

b) $y = \sqrt{x}, y = x^3$



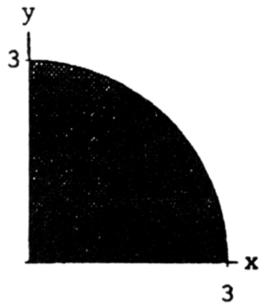
$$V = \int_0^1 \pi \left[\left(y^{1/3}\right)^2 - \left(y^2\right)^2 \right] dy$$
$$= \int_0^1 \pi \left[y^{2/3} - y^4 \right] dy$$
$$= \pi \left[\tfrac{3}{5}y^{5/3} - \tfrac{1}{5}y^5\right]_0^1 = \tfrac{2}{5}\pi$$

c) $y = x, y = 2x, x = 4$



$$V = \int_0^4 \pi \left[y^2 - \left(\frac{y}{2}\right)^2 \right] dy + \int_4^8 \pi \left[4^2 - \left(\frac{y}{2}\right)^2 \right] dy$$
$$= \int_0^4 \pi \left[\frac{3}{4}y^2 \right] dy + \int_4^8 \pi \left[16 - \frac{1}{4}y^2 \right] dy$$
$$= \pi \left[\tfrac{1}{4}y^3\right]_0^4 + \pi \left[16y - \tfrac{1}{12}y^3\right]_4^8 = \tfrac{128}{3}\pi$$

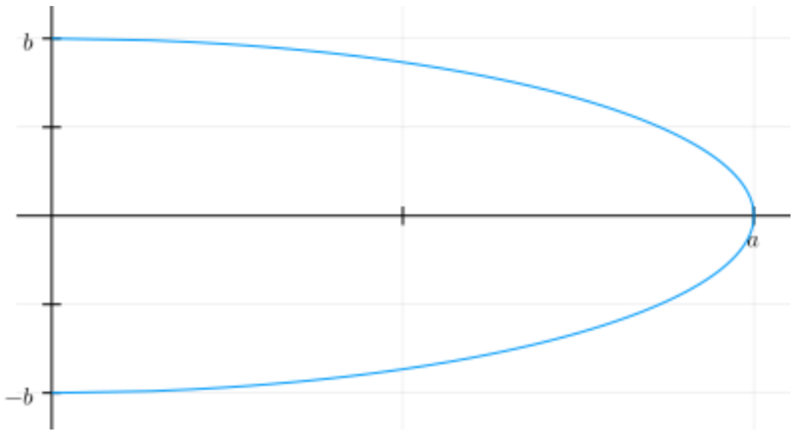
d) $x = \sqrt{9 - y^2}$, $x = 0$



$$V = \int_0^3 \pi(9 - y^2)\,dy = \pi\left[9y - \frac{y^3}{3}\right]_0^3 = 18\pi$$

És un hemisferi de radi 3

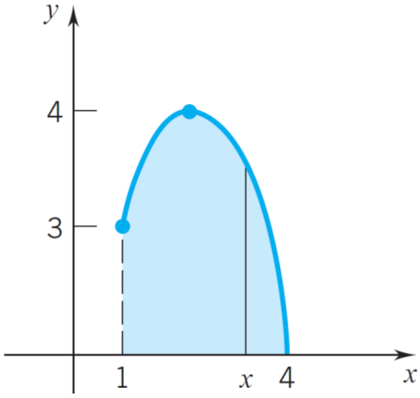
e) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $y = 0$



$$V = \int_{-b}^{b} \pi\left(\frac{a}{b}\sqrt{b^2 - y^2}\right)^2 dy = \frac{\pi a^2}{b^2}\int_{-b}^{b}(b^2 - y^2)\,dy$$

$$= \frac{\pi a^2}{b^2}\left[b^2 y - \frac{y^3}{3}\right]_{-b}^{b} = \frac{4}{3}\pi a^2 b$$

10. Troba el volum del sòlid de revolució obtingut en girar la regió delimitada per $x = 1$, $x = 4$, $y = 4x - x^2$ al voltant de l'eix y



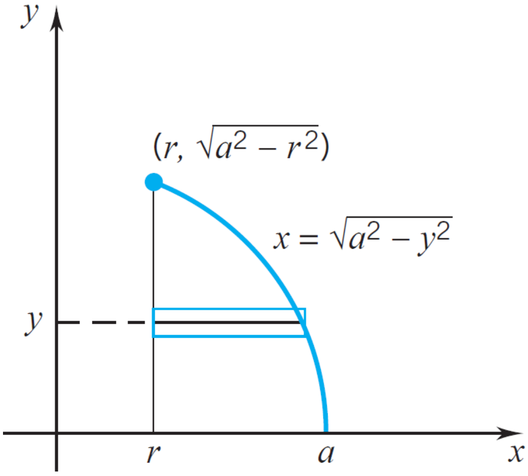
$$V = \int_1^4 2\pi x(4x - x^2)\,dx = 2\pi \int_1^4 (4x^2 - x^3)\,dx = 2\pi\left[\tfrac{4}{3}x^3 - \tfrac{1}{4}x^4\right]_1^4 = \tfrac{81}{2}\pi$$

11. Donat un hemisferi de radi a, se li fa un forat cilíndric de radi r perpendicularment a la base i d'eix que passa pel centre de la base. Calcula el volum que queda:

a) Pel Washer method

b) Pel Shell method

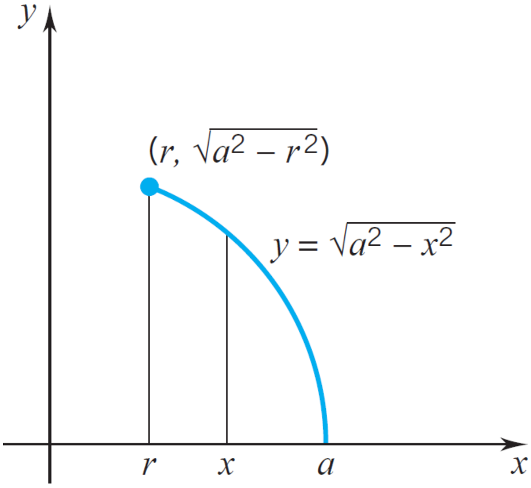
a) Pel Washer method



$$V = \int_0^{\sqrt{a^2-r^2}} \pi\left(\left[\sqrt{a^2-y^2}\right]^2 - r^2\right) dy = \pi\int_0^{\sqrt{a^2-r^2}} (a^2 - r^2 - y^2)\, dy$$

$$= \pi\left[(a^2-r^2)y - \tfrac{1}{3}y^3\right]_0^{\sqrt{a^2-r^2}} = \tfrac{2}{3}\pi(a^2-r^2)^{3/2}\ .$$

b) Pel Shell method

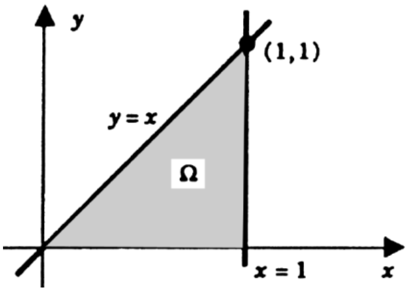


$$V = \int_r^a 2\pi x\sqrt{a^2-x^2}\, dx = -\pi\int_{a^2-r^2}^{0} u^{1/2} du = \pi\int_0^{a^2-r^2} u^{1/2} du$$

$$= \pi\left[\tfrac{2}{3}u^{3/2}\right]_0^{a^2-r^2} = \tfrac{2}{3}\pi(a^2-r^2)^{3/2}.$$

12. Troba el volum del sòlid de revolució obtingut en girar la regió Ω delimitada per les corbes indicades al voltant de l'eix y utilitzant el mètodes de capes
    a) $y = x, x = 1, y = 0$
    b) $y = \sqrt{x}, x = 4, y = 0$
    c) $y = \sqrt{x}, y = x^3$

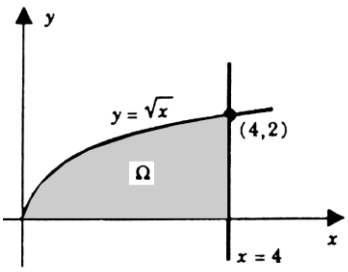
a) $y = x, x = 1, y = 0$



$$V = \int_0^1 2\pi x\,[x - 0]\,dx = 2\pi \int_0^1 x^2\,dx$$

$$= 2\pi\left[\frac{1}{3}x^3\right]_0^1 = \frac{2\pi}{3}$$

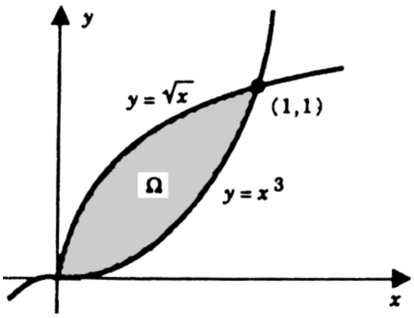
b) $y = \sqrt{x}, x = 4, y = 0$



$$V = \int_0^4 2\pi x\,[\sqrt{x} - 0]\,dx = 2\pi \int_0^4 x^{3/2}\,dx$$

$$= 2\pi\left[\tfrac{2}{5}x^{5/2}\right]_0^4 = \frac{128}{5}\pi$$

c) $y = \sqrt{x}, y = x^3$



$$V = \int_0^1 2\pi x\,[\sqrt{x} - x^3]\,dx$$

$$= 2\pi \int_0^1 \left(x^{3/2} - x^4\right)\,dx$$

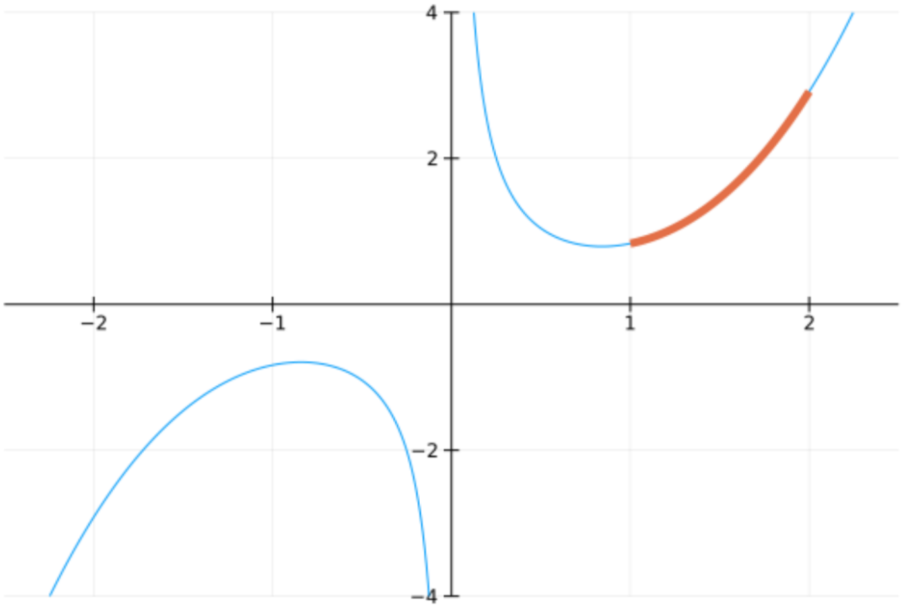
$$= 2\pi\left[\frac{2}{5}x^{5/2} - \frac{1}{5}x^5\right]_0^1 = \frac{2\pi}{5}$$

13. Troba la longitud de les següents corbes

a) $y = \frac{1}{6}x^3 + \frac{1}{2x}$, entre $x = 1$ i $x = 2$

b) $y = a\cosh\frac{x}{a}$, $x = -L$, $y = L$ (una catenària)

a) $y = \frac{1}{6}x^3 + \frac{1}{2x}$, entre $x = 1$ i $x = 2$

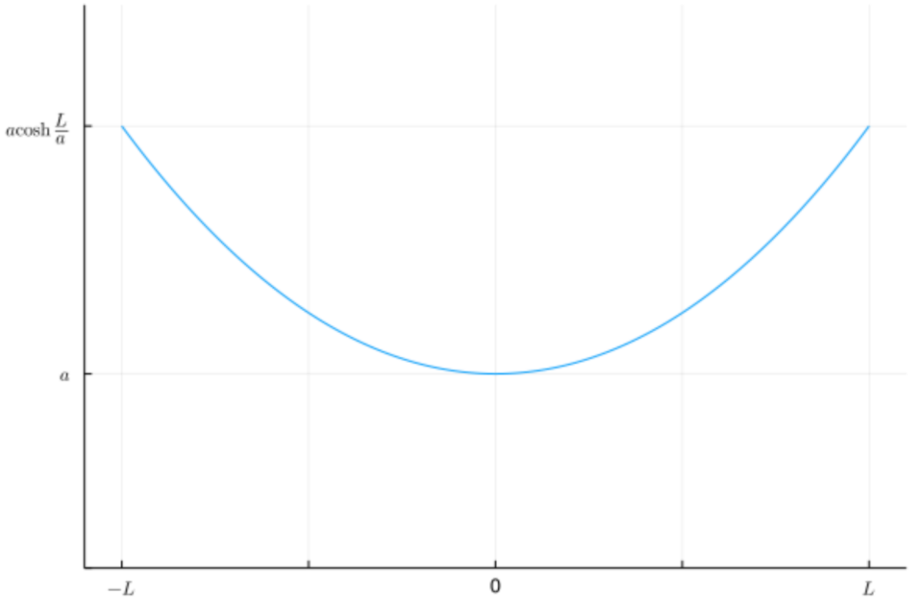


$$L = \int_1^2 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_1^2 \sqrt{1 + \left(\frac{x^2}{2} - \frac{1}{2x^2}\right)^2}\, dx$$

$$= \int_1^2 \sqrt{1 + \frac{x^4}{4} - \frac{1}{2} + \frac{1}{4x^4}}\, dx = \frac{1}{2}\int_1^2 \sqrt{x^4 + 2 + \frac{1}{x^4}}\, dx$$

$$= \frac{1}{2}\int_1^2 \sqrt{\left(x^2 + \frac{1}{x^2}\right)^2}\, dx = \frac{1}{2}\left[\frac{x^3}{3} - \frac{1}{2x}\right]_1^2 = \frac{17}{12}$$

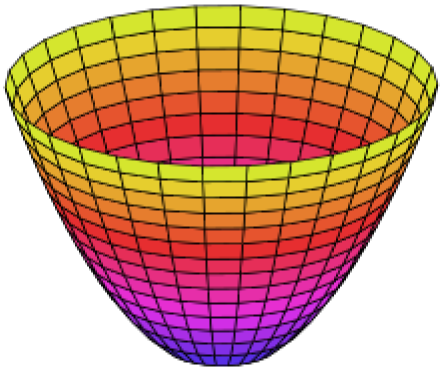
b) $y = a\cosh\frac{x}{a}$, $x = -L$, $y = L$ (una catenària)



$$L = \int_{-L}^{L} \sqrt{1 + \left(\sinh\frac{x}{a}\right)^2}\, dx = 2\int_0^L \sqrt{\cosh^2\frac{x}{a}}\, dx = 2\int_0^L \cosh\frac{x}{a}\, dx$$

$$= 2\left[a\sinh\frac{x}{a}\right]_0^L = 2a\cosh\frac{L}{a}$$

14. Troba l'àrea de la superfície generada per una paràbola $y = x^2$ entre $x = 0$ i $x = 1$ en girar respecte l'eix y (paraboloide de revolució)



$$A = \int_0^1 2\pi x\sqrt{1+(2x)^2}dx = 2\pi\int_0^1 x\sqrt{1+4x^2}dx$$

Per canvi de variable

$t = 1 + 4x^2$
$dt = 8xdx$
Si $x = 0 \;\Rightarrow\; t = 1$
Si $x = 1 \;\Rightarrow\; t = 5$

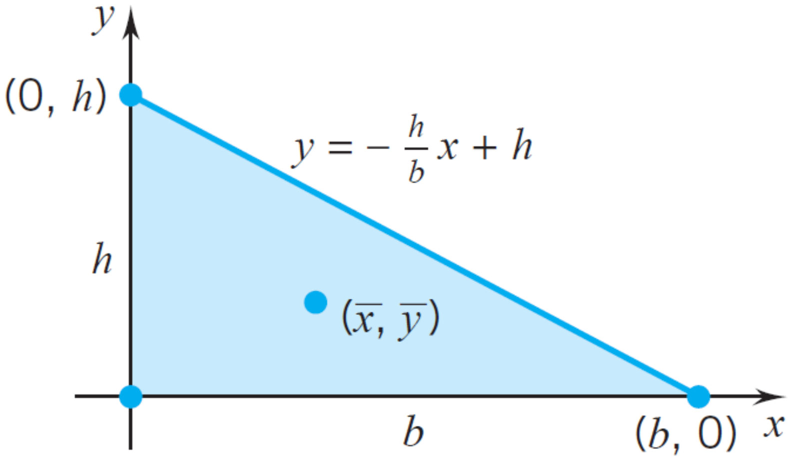
$$A = 2\pi\int_0^1 x\sqrt{1+4x^2}dx = \frac{\pi}{4}\int_1^5 \sqrt{t}dt = \frac{\pi}{4}\left[\frac{2\sqrt{t^3}}{3}\right]_1^5 = \frac{\pi}{6}\left(5\sqrt{5}-1\right)$$

15. Troba el centroide de les següents regions

a) Triangle rectangle de base b i alçada h

b) Regió delimitada per $y = x^2$, $y = 2x$

a) Triangle rectangle de base b i alçada h



$$A = \tfrac{1}{2}bh$$

$$\overline{x}A = \int_0^b xf(x)\,dx = \int_0^b \left(-\frac{h}{b}x^2 + hx\right)dx = \frac{1}{6}b^2h$$

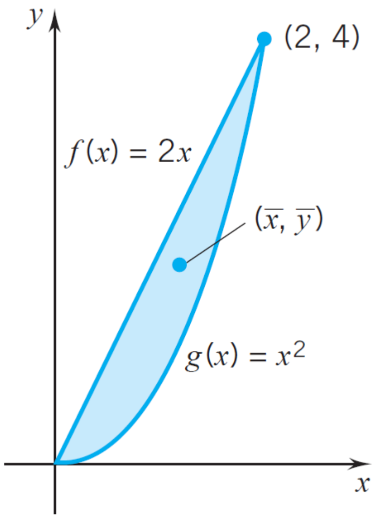
$$\overline{y}A = \int_0^b \tfrac{1}{2}[f(x)]^2\,dx = \tfrac{1}{2}\int_0^b \left(\frac{h^2}{b^2}x^2 - \frac{2h^2}{b}x + h^2\right)dx = \tfrac{1}{6}bh^2$$

$$\overline{x} = \frac{\frac{1}{6}b^2h}{\frac{1}{2}bh} = \tfrac{1}{3}b \qquad \text{and} \qquad \overline{y} = \frac{\frac{1}{6}bh^2}{\frac{1}{2}bh} = \tfrac{1}{3}h$$

Per tant, el centroide té coordenades

$$(\bar{x}, \bar{y}) = \left(\frac{b}{3}, \frac{h}{3}\right)$$

b) Regió delimitada per $y = x^2$, $y = 2x$



$$A = \int_0^2 [f(x) - g(x)]\,dx = \int_0^2 (2x - x^2)\,dx = \left[x^2 - \tfrac{1}{3}x^3\right]_0^2 = \tfrac{4}{3},$$

$$\overline{x} = \int_0^2 x[f(x) - g(x)]\,dx = \int_0^2 (2x^2 - x^3)\,dx = \left[\tfrac{2}{3}x^3 - \tfrac{1}{4}x^4\right]_0^2 = \tfrac{4}{3},$$

$$\overline{y}A = \int_0^2 \tfrac{1}{2}([f(x)]^2 - [g(x)]^2)\,dx = \tfrac{1}{2}\int_0^2 (4x^2 - x^4)\,dx = \tfrac{1}{2}\left[\tfrac{4}{3}x^3 - \tfrac{1}{5}x^5\right]_0^2 = \tfrac{32}{15}.$$

$\overline{x} = \frac{4}{3}/\frac{4}{3} = 1$ and $\overline{y} = \frac{32}{15}/\frac{4}{3} = \frac{8}{5}$

Per tant, el centroide té coordenades

$$(\bar{x}, \bar{y}) = \left(1, \frac{8}{5}\right)$$

16. Utilitzant el teorema de Pappus, calcula el volum del cos de revolució en fer girar el triangle de l'exercici 15a al voltant del següent eixos
    a) Eix x
    b) Eix y
    c) Recta $x = -b$
    d) Recta $y = -x$

a) Eix x

Sabem les coordenades del centroide: $(\bar{x}, \bar{y}) = \left(\frac{b}{3}, \frac{h}{3}\right)$
La distància del centroide a l'eix x és $\bar{R} = h/3$
Per tant, pel teorema de Pappus, el volum és

$$V = 2\pi\bar{R}A = 2\pi\frac{h}{3}\frac{bh}{2} = \frac{1}{3}\pi h^2 b$$

b) Eix y

Sabem les coordenades del centroide: $(\bar{x}, \bar{y}) = \left(\frac{b}{3}, \frac{h}{3}\right)$
La distància del centroide a l'eix y és $\bar{R} = b/3$
Per tant, pel teorema de Pappus, el volum és

$$V = 2\pi\bar{R}A = 2\pi\frac{b}{3}\frac{bh}{2} = \frac{1}{3}\pi b^2 h$$

c) Recta $x = -b$

Sabem les coordenades del centroide: $(\bar{x}, \bar{y}) = \left(\frac{b}{3}, \frac{h}{3}\right)$
La distància del centroide a la recta $x = -b$ és $\bar{R} = b + b/3 = 4b/3$
Per tant, pel teorema de Pappus, el volum és

$$V = 2\pi\bar{R}A = 2\pi\frac{4b}{3}\frac{bh}{2} = \frac{4}{3}\pi b^2 h$$

d) Recta $y = -x$

Sabem les coordenades del centroide: $(\bar{x}, \bar{y}) = \left(\frac{b}{3}, \frac{h}{3}\right)$
La distància del centroide a la recta $x + y = 0$ és

$$\bar{R} = \frac{\left|\frac{b}{3}1 + \frac{h}{3}1\right|}{\sqrt{1^2 + 1^2}} = \frac{(b+h)\sqrt{2}}{6}$$

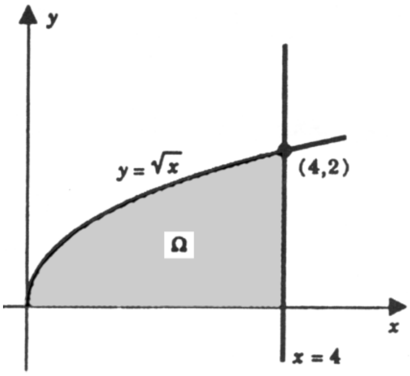
Per tant, pel teorema de Pappus, el volum és

$$V = 2\pi\bar{R}A = 2\pi\frac{(b+h)\sqrt{2}}{6}\frac{bh}{2} = \frac{\sqrt{2}}{6}\pi bh(b+h)$$

17. Calcula el volum del cos de revolució en fer girar les següents regions respecte els eixos x i y

    a) $y=\sqrt{x}$, $x=4$, $y=0$
    b) $y=x^2$, $x=y^3$
    c) $y=2x$, $x=3$, $y=2$
    d) $x+y=1$, $\sqrt{x}+\sqrt{y}=1$
    e) $y=x$, $x+y=6$, $y=1$

a) $y=\sqrt{x}$, $x=4$, $y=0$



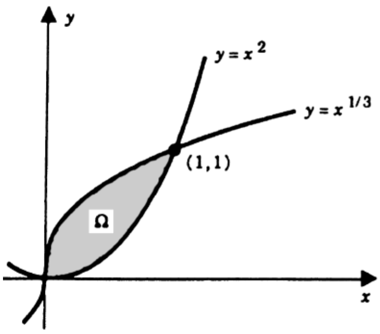
$$A=\int_0^4 \sqrt{x}\,dx=\frac{16}{3}$$

$$\overline{x}A=\int_0^4 x\sqrt{x}\,dx=\frac{64}{5},\quad \overline{x}=\frac{12}{5}$$

$$\overline{y}A=\int_0^4 \frac{1}{2}\left(\sqrt{x}\right)^2\,dx=4,\quad \overline{y}=\frac{3}{4}$$

$$V_x=2\pi\overline{y}A=8\pi,\quad V_y=2\pi\overline{x}A=\frac{128}{5}\pi$$

b) $y=x^2$, $x=y^3$



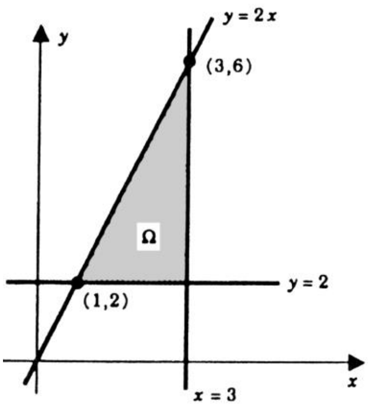
$$A=\int_0^1 \left(x^{1/3}-x^2\right)\,dx=\frac{5}{12}$$

$$\overline{x}A=\int_0^1 x\left(x^{1/3}-x^2\right)\,dx=\frac{5}{28},\quad \overline{x}=\frac{3}{7}$$

$$\overline{y}A=\int_0^1 \frac{1}{2}\left[\left(x^{1/3}\right)^2-\left(x^2\right)^2\right]\,dx=\frac{1}{5},\quad \overline{y}=\frac{12}{25}$$

$$V_x=2\pi\overline{y}A=\tfrac{2}{5}\pi,\quad V_y=2\pi\overline{x}A=\tfrac{5}{14}\pi$$

c) $y=2x$, $x=3$, $y=2$



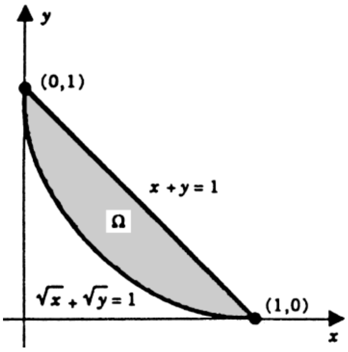
$$A=\int_1^3 (2x-2)\,dx=4$$

$$\overline{x}A=\int_1^3 x\,(2x-2)\,dx=\frac{28}{3},\quad \overline{x}=\frac{7}{3}$$

$$\overline{y}A=\int_1^3 \frac{1}{2}\left[(2x)^2-(2)^2\right]\,dx=\frac{40}{3},\quad \overline{y}=\frac{10}{3}$$

$$V_x=2\pi\overline{y}A=\tfrac{80}{3}\pi,\quad V_y=2\pi\overline{x}A=\tfrac{56}{3}\pi$$

d) $x + y = 1, \sqrt{x} + \sqrt{y} = 1$



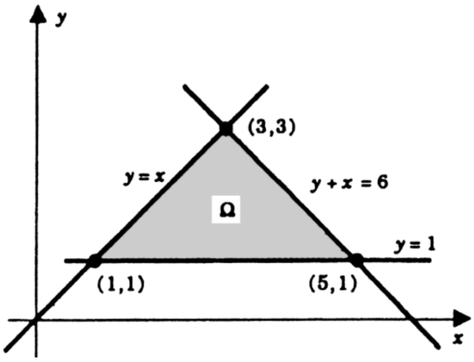
$$A = \int_0^1 \left[(1-x) - (1-\sqrt{x})^2\right] dx = \frac{1}{3}$$

$$\overline{x}A = \int_0^1 x\left[(1-x) - (1-\sqrt{x})^2\right] dx = \frac{2}{15}, \quad \overline{x} = \frac{2}{5}$$

$\overline{y} = \frac{2}{5}$ by symmetry

$V_x = 2\pi\overline{y}A = \frac{4}{15}\pi, \quad V_y = \frac{4}{15}\pi$ by symmetry

e) $y = x, x + y = 6, y = 1$



$A = \frac{1}{2}bh = 4;$ by symmetry, $\overline{x} = 3$

$$\overline{y}A = \int_1^3 y\left[(6-y) - y\right] dy = \frac{20}{3}, \quad \overline{y} = \frac{5}{3}$$

$V_x = 2\pi\overline{y}A = \frac{40}{3}\pi, \quad V_y = 2\pi\overline{x}A = 24\pi$

18. Calcula el volum del con de gelat de la figura A, i la coordenada x del centroide de la figura B

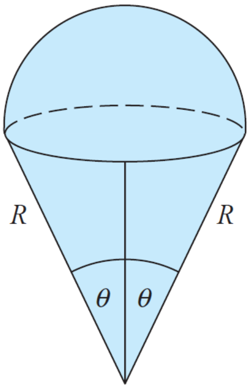


Figure A

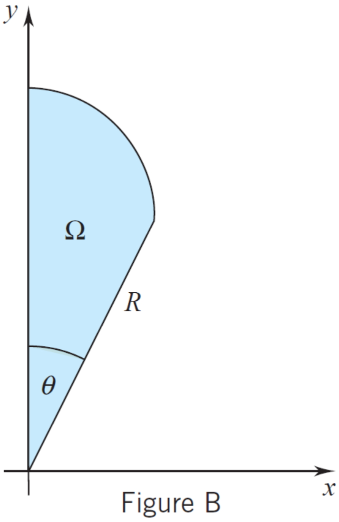


Figure B

L'hemisferi té radi $r = R\sin\theta$, per tant el seu volum és

$$V_h = \frac{1}{2}\frac{4}{3}\pi r^3 = \frac{2}{3}\pi R^3 \sin^3\theta$$

El con té base de radi $r = R\sin\theta$ i alçada $h = R\cos\theta$, per tant el seu volum és

$$V_c = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi R^3 \sin^2\theta \cos\theta$$

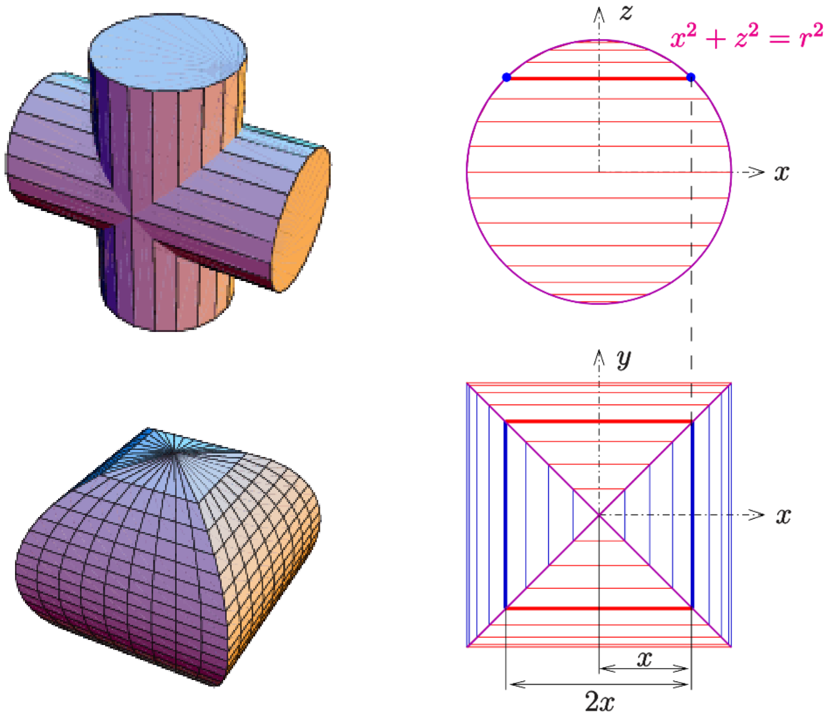
El volum total és

$$V = V_h + V_c = \frac{1}{3}\pi R^3 \sin^2\theta\,(2\sin\theta + \cos\theta)$$

La coordenada $\bar{x}$ és

$$\bar{x} = \frac{V}{2\pi A} = \frac{\frac{1}{3}\pi R^3 \sin^2\theta\,(2\sin\theta + \cos\theta)}{2\pi\left(\frac{1}{2}R^2\sin\theta\cos\theta + \frac{1}{4}\pi R^2\sin^2\theta\right)} = \frac{2R\sin\theta\,(2\sin\theta + \cos\theta)}{3(\pi\sin\theta + 2\cos\theta)}$$

19. Calcula el volum de la intersecció entre dos cilindres de radi r amb eixos perpendiculars i que es troben en el mateix pla.



Les seccions paral·leles al pla que conté els eixos dels cilindres són quadrats, de volum $dV = (2x)^2 dz$. Com els costats d'aquests quadrats estan delimitats per una circumferència, es compleix $x^2 + z^2 = r^2$. Per tant, el volum es pot expressar com

$$V = \int_{-r}^{r} 4(r^2 - z^2)dz = 4\left[r^2 z - \frac{z^3}{3}\right]_{-r}^{r} = 4\left(r^3 - \frac{r^3}{3}\right) - 4\left(-r^3 + \frac{r^3}{3}\right)$$
$$= 8\frac{2r^3}{3} = \frac{16}{3}r^3$$