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Vector Calculus

Fifth Edition

Chapter 3:
High-Order Derivatives: Maxima and Minima

3.2 Taylor's Theorem

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Key Points in this Section.

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1. The one-variable ***Taylor Theorem*** states that if f is C^{k+1}, then

$$f(x_0+h) = f(x_0)+f'(x_0)h+\frac{f''(x_0)}{2}h^2+\cdots+\frac{f^{(k)}(x_0)}{k!}h^k+R_k(x_0,h),$$

   where $R_k(x_0,h)/h^k \to 0$ as $h \to 0$

2. The idea of the proof is to start with the Fundamental Theorem of Calculus

$$f(x_0+h) = f(x_0) + \int_{x_0}^{x_0+h} f'(\tau)d\tau$$

   (which gives Taylors' theorem for $k = 0$) and integrating by parts.

3. For $f : U \subset \mathbb{R}^n \to \mathbb{R}$ of class C^3, the second-order ***Taylor Theorem*** states that

$$f(\mathbf{x}_0+\mathbf{h}) = f(\mathbf{x}_0)+\sum_{i=1}^{n} h_i \frac{\partial f}{\partial x_i}(\mathbf{x}_0)+\frac{1}{2}\sum_{i,j} h_i h_j \frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x}_0)+R_2(\mathbf{x}_0,\mathbf{h})$$

where $R_2(\mathbf{x}_0,\mathbf{h})/\|\mathbf{h}\|^2 \to 0$ as $\mathbf{h} \to \mathbf{0}$. Higher order versions are similar.

4. The idea of the proof is to apply the single-variable Taylor theorem to the function $g(t) = f(\mathbf{x}_0 + t\mathbf{h})$, expanded about $t_0 = 0$ with $h = 1$.

Sèrie de Taylor en una dimensió

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + R_{k,x_0,f}(x)$$

Residu en forma de Lagrange

$$R_{k,f,x_0}(x) = \frac{f^{(k+1)}(\xi)}{(k+1)!}(x - x_0)^{k+1}, \qquad \xi \in (x_0, x) \cup (x, x_0)$$

Residu en forma integral

$$R_{k,f,x_0}(x) = \int_{x_0}^{x} \frac{f^{(k+1)}(t)}{k!}(x - t)^k dt$$

Sèrie de Taylor en una dimensió

$$f(x_0+h) = f(x_0) + f'(x_0)\cdot h + \frac{f''(x_0)}{2}h^2 + \cdots + \frac{f^{(k)}(x_0)}{k!}h^k + R_k(x_0, h)$$

$$R_k(x_0, h) = \int_{x_0}^{x_0+h} \frac{(x_0+h-\tau)^k}{k!} f^{k+1}(\tau)\, d\tau$$

$$\lim_{h\to 0} \frac{R_k(x_0, h)}{h^k} = 0$$

THEOREM 2: First-Order Taylor Formula Let $f : U \subset \mathbb{R}^n \to \mathbb{R}$ be differentiable at $\mathbf{x}_0 \in U$. Then

$$f(\mathbf{x}_0 + \mathbf{h}) = f(\mathbf{x}_0) + \sum_{i=1}^{n} h_i \frac{\partial f}{\partial x_i}(\mathbf{x}_0) + R_1(\mathbf{x}_0, \mathbf{h}),$$

where $R_1(\mathbf{x}_0, \mathbf{h})/\|\mathbf{h}\| \to 0$ as $\mathbf{h} \to \mathbf{0}$ in $\mathbb{R}^n$.

THEOREM 3: Second-Order Taylor Formula Let $f\colon U \subset \mathbb{R}^n \to \mathbb{R}$ have continuous partial derivatives of third order.[3] Then we may write

$$f(\mathbf{x}_0 + \mathbf{h}) = f(\mathbf{x}_0) + \sum_{i=1}^{n} h_i \frac{\partial f}{\partial x_i}(\mathbf{x}_0) + \frac{1}{2} \sum_{i,j=1}^{n} h_i h_j \frac{\partial^2 f}{\partial x_i\, \partial x_j}(\mathbf{x}_0) + R_2(\mathbf{x}_0, \mathbf{h}),$$

where $R_2(\mathbf{x}_0, \mathbf{h})/\|\mathbf{h}\|^2 \to 0$ as $\mathbf{h} \to \mathbf{0}$ and the second sum is over all i's and j's between 1 and n (so there are n^2 terms).

$$
\begin{aligned}
f(\mathbf{x}_0 + \mathbf{h}) = {} & f(\mathbf{x}_0) + \left[\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right] \begin{bmatrix} h_1 \\ \vdots \\ h_n \end{bmatrix} \\
& + \frac{1}{2}[h_1, \ldots, h_n] \begin{bmatrix} \dfrac{\partial^2 f}{\partial x_1\, \partial x_1} & \dfrac{\partial^2 f}{\partial x_1\, \partial x_2} & \cdots & \dfrac{\partial^2 f}{\partial x_1\, \partial x_n} \\ \dfrac{\partial^2 f}{\partial x_2\, \partial x_1} & \dfrac{\partial^2 f}{\partial x_2\, \partial x_2} & \cdots & \dfrac{\partial^2 f}{\partial x_2\, \partial x_n} \\ \vdots & & & \\ \dfrac{\partial^2 f}{\partial x_n\, \partial x_1} & \dfrac{\partial^2 f}{\partial x_n\, \partial x_2} & \cdots & \dfrac{\partial^2 f}{\partial x_n\, \partial x_n} \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{bmatrix} \\
& + R_2(\mathbf{x}_0, \mathbf{h}),
\end{aligned}
$$

Third-order Taylor formula

$$f(\mathbf{x}_0+\mathbf{h}) = f(\mathbf{x}_0) + \sum_{i=1}^{n} h_i \frac{\partial f}{\partial x_i}(\mathbf{x}_0) + \frac{1}{2}\sum_{i,j=1}^{n} h_i h_j \frac{\partial^2 f}{\partial x_i\,\partial x_j}(\mathbf{x}_0)$$

$$+\frac{1}{3!}\sum_{i,j,k=1}^{n} h_i h_j h_k \frac{\partial^3 f}{\partial x_i\,\partial x_j\,\partial x_k}(\mathbf{x}_0) + R_3(\mathbf{x}_0,\mathbf{h}),$$

where $R_3(\mathbf{x}_0,\mathbf{h})/\|\mathbf{h}\|^3 \to 0$ as $\mathbf{h} \to \mathbf{0}$

Forms of the Remainder In Theorem 2,

$$R_1(\mathbf{x}_0, \mathbf{h}) = \sum_{i,j=1}^{n} \int_0^1 (1-t) \frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x}_0 + t\mathbf{h}) h_i h_j \, dt = \sum_{i,j=1}^{n} \frac{1}{2} \frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{c}_{ij}) h_i h_j, \tag{5}$$

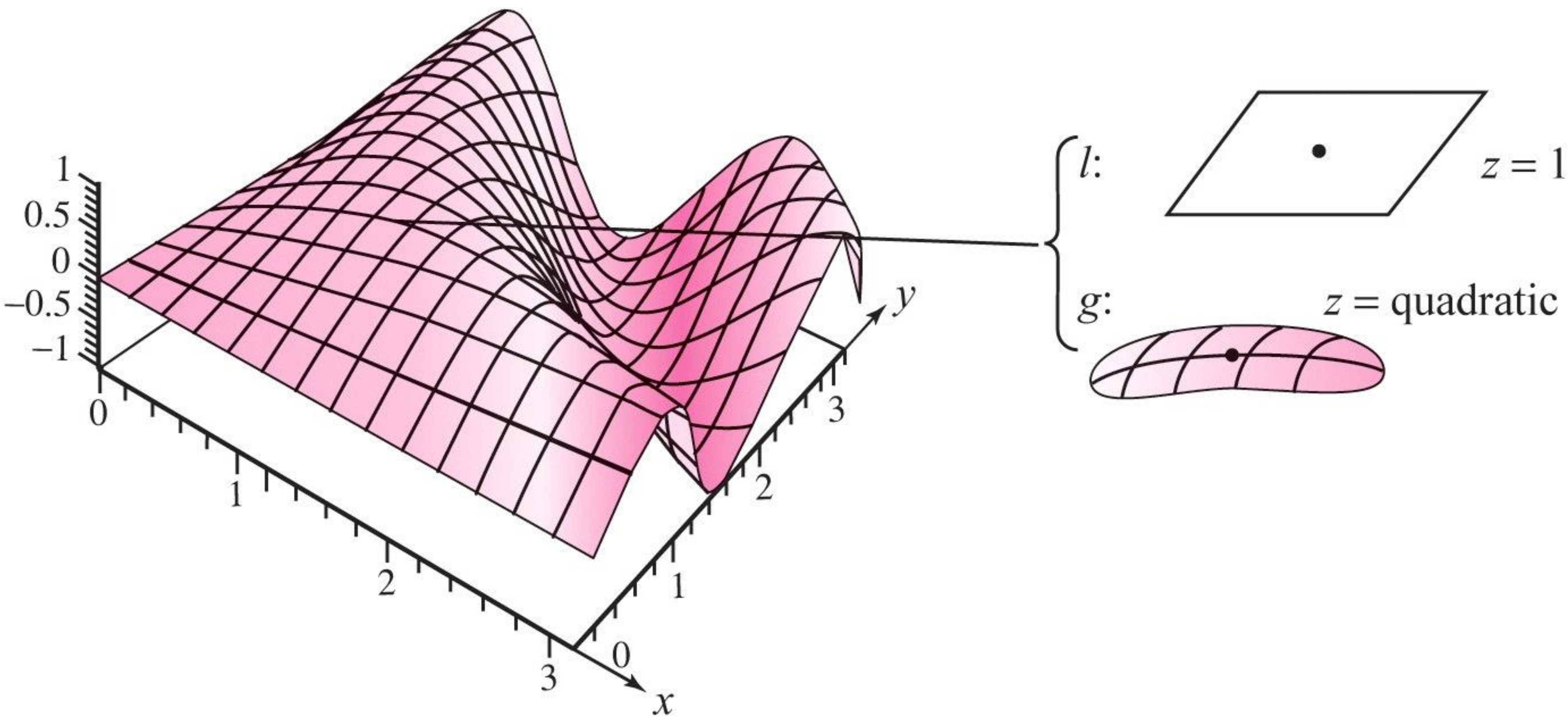
where $\mathbf{c}_{ij}$ lies somewhere on the line joining $\mathbf{x}_0$ to $\mathbf{x}_0 + \mathbf{h}$.

In Theorem 3,

$$\begin{aligned} R_2(\mathbf{x}_0, \mathbf{h}) &= \sum_{i,j,k=1}^{n} \int_0^1 \frac{(t-1)^2}{2} \frac{\partial^3 f}{\partial x_i \, \partial x_j \, \partial x_k}(\mathbf{x}_0 + t\mathbf{h}) h_i h_j h_k \, dt \\ &= \sum_{i,j,k=1}^{n} \frac{1}{3!} \frac{\partial^3 f}{\partial x_i \, \partial x_j \, \partial x_k}(\mathbf{c}_{ijk}) h_i h_j h_k, \end{aligned} \tag{5'}$$

where $\mathbf{c}_{ijk}$ lies somewhere on the line joining $\mathbf{x}_0$ to $\mathbf{x}_0 + \mathbf{h}$.

$$f(x, y) = \sin(xy) \qquad (x_0, y_0) = (1, \pi/2)$$



$$l(x) = 1$$

$$g(x) = 1 - \frac{\pi^2}{8}(x-1)^2 - \frac{\pi}{2}(x-1)\left(y - \frac{\pi}{2}\right) - \frac{1}{2}\left(y - \frac{\pi}{2}\right)^2$$