Anàlisi Matemàtica 1 (AM1)
GEMiF

E3.2 Exercicis: Derivació

1. Solucions:

35. $\frac{dy}{dx} = 5\sec^2(5x)$

37. $y' = 3x\sin(1-3x) + \cos(1-3x)$ **39.** $\frac{dy}{dt} = 2(4t+9)^{-1/2}$

41. $y' = 4(\sin x - 3x^2)(x^3 + \cos x)^{-5}$ **43.** $y' = \frac{\sqrt{2}\cos 2x}{2\sqrt{\sin 2x}}$

45. $\frac{dy}{dz} = (z+1)^3(2z-1)^2(14z+2)$

47. $y' = \frac{1}{2x^2\sqrt{x+1}}(3x^3 + 2x^2 - x - 2)$ **49.** $y' = \frac{x\cos(x^2) - 3\sin 6x}{\sqrt{\cos 6x + \sin(x^2)}}$

51. $y' = 3(x^2\sec^2(x^3) + \sec^2 x\tan^2 x)$ **53.** $\frac{dy}{dz} = \frac{-1}{\sqrt{z+1}(z-1)^{3/2}}$

55. $\frac{dy}{dx} = \frac{\sin(-1) - \sin(1+x)}{(1+\cos x)^2}$ **57.** $\frac{dy}{dx} = -35x^4\cot^6(x^5)\csc^2(x^5)$

59. $\frac{dy}{dx} = 30x(1 + (x^2+2)^5)^2(x^2+2)^4$

61. $\frac{dy}{dx} = -4e^{-x} - 14e^{-2x}$

63. $\frac{dy}{dx} = 24(2e^{3x} + 3e^{-2x})^3(e^{3x} - e^{-2x})$

65. $\frac{dy}{dt} = e^{-2t}(2t-1)\sin(te^{-2t})$

67. $\frac{dy}{dx} = 4(x+1)(x^2+2x+3)e^{(x^2+2x+3)^2}$

2. Solucions

(a)

Prenem el logaritme dels dos costats: $\ln y = x\ln x$

Derivem els dos costats: $\frac{1}{y}\frac{dy}{dx} = \ln x + x\frac{1}{x} = 1 + \ln x$

Queda: $y' = y(1+\ln x) = x^x(1+\ln x)$

(b)

Prenem el logaritme dels dos costats: $\ln y = x^2\ln x$

Derivem els dos costats: $\frac{1}{y}\frac{dy}{dx} = 2x\ln x + x^2\frac{1}{x} = x + 2x\ln x = x(1+2\ln x)$

Queda: $y' = y\,x(1+2\ln x) = x^{x^2+1}(1+2\ln x)$

3. Solucions:

(a)

$f(x) = (x+3)^5$

$f'(x) = 5(x+3)^4$

$f'(x+3) = 5(x+6)^4$

(b)

$f(x+3) = x^5$

$f(x) = (x-3)^5$

$f'(x) = 5(x-3)^4$

$f'(x+3) = 5x^4$

(c)

$f(x+3) = (x+5)^7$

$f(x) = (x+2)^7$

$f'(x) = 7(x+2)^6$

$f'(x+3) = 7(x+5)^6$

4. Solucions:

(19)

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x) = -\sin(x^2+1)(2x) = -2x\sin(x^2+1).$$

$$\frac{d}{dx} g(f(x)) = g'(f(x))f'(x) = 2(\cos x)(-\sin x) = -2\sin x\cos x.$$

(20)

$(f\circ g)(x) = (x+1)^{-3}$ $\quad (f\circ g)'(x) = -3(x+1)^{-4} = -\frac{3}{(x+1)^4}$

$(g\circ f)(x) = (x^3+1)^{-1}$ $\quad (g\circ f)'^{(x)} = -(x^3+1)^{-2}(3x^2) = -\frac{3x^2}{(x^3+1)^2}$

5. Solució:

$$\frac{df}{dx} = \frac{df}{du}\frac{du}{dx} = 2\cdot 6 = 12$$

6. Solucions:

(a) Si $s(t) = c\,t^2$, aleshores $s'(t) = 2c\,t$. Es demana si existeix un k tal que, per tot t, $s(t) = k\,s'(t)$, és dir, $c\,t^2 = k\,2c\,t$. Aquesta equació només té solució per dos valors de t, mai per tot t, per tant, la hipòtesi era incorrecta.

(b)

(i) Si $s(t) = \left(\frac{a}{2}\right)t^2$, aleshores $s'(t) = at$, i $s''(t) = a$.

(ii) $[s'(t)]^2 = a^2t^2 = 2a\,s(t)$.

(c) $s(t) = 16t^2 = 400$ peus implica $t = 5$s. Després de 5s la velocitat és $s'(5) = 32\cdot 5 = 160$ peus/s. La velocitat era la meitat quan $s'(t) = 32\cdot t = 80$, és dir, als $5/2 = 2.5$s. L'altura als 2.5s és $400 - s(2.5) = 400 - 16\cdot 2.5^2 = 300$ peus.

7. Solucions:

$$V(r) = \frac{4}{3}\pi\, r^3,\ \ r(t) = 2\,t$$
$$\frac{dV(r)}{dr} = 4\pi r^2,\ \ \frac{dr(t)}{dt} = 2$$
$$\frac{dV(r(t))}{dt} = 4\pi r^2 \cdot 2 = 8\pi r^2$$

(a) $\frac{dV}{dt}(r = 3) = 8\pi 3^2 = 72\pi$ cm^3/s

(b) $r(3) = 2 \cdot 3 = 6$ cm, $\frac{dV}{dt}(r = 6) = 8\pi 6^2 = 288\pi$ cm^3/s

8. Solucions:

$$\frac{dP}{dT} = 5.256\left(\frac{T + 273.1}{288.08}\right)^{4.256}\left(\frac{1}{288.08}\right) = 6.21519 \times 10^{-13}\,(273.1 + T)^{4.256}$$

and $\frac{dT}{dh} = -0.000649°\text{C/m}$. $\frac{dP}{dh} = \frac{dP}{dT}\frac{dT}{dh}$, so

$$\frac{dP}{dh} = \left(6.21519 \times 10^{-13}\,(273.1 + T)^{4.256}\right)(-0.000649) = -4.03366 \times 10^{-16}\,(288.14 - 0.000649\,h)^{4.256}.$$

When $h = 3000$,

$$\frac{dP}{dh} = -4.03366 \times 10^{-16}(286.193)^{4.256} = -1.15 \times 10^{-5}\ \text{kPa/m};$$

therefore, for each additional meter of altitude,

$$\Delta P \approx -1.15 \times 10^{-5}\ \text{kPa} = -1.15 \times 10^{-2}\ \text{Pa}.$$

9. Solucions:

(a) $x(t) = L\sin(2\pi f t)$
$v(t) = x'(t) = 2\pi f L\cos(2\pi f t)$
$a(t) = v'(t) = -4\pi^2 f^2\, L\sin(2\pi f t) = -4\pi^2 f^2 x(t)$

$$F = -kx(t) = ma(t) = -m4\pi^2 f^2 x(t) \ \Rightarrow\ k = m4\pi^2 f^2 \ \Rightarrow\ 2\pi f = \sqrt{\frac{k}{m}}$$

(b) $E(t) = K(t) + U(t) = \frac{1}{2}mv(t)^2 + \frac{1}{2}kx(t)^2$
$= \frac{1}{2}m[2\pi f L\cos(2\pi f t)]^2 + \frac{1}{2}k[L\sin(2\pi f t)]^2$
$= \frac{1}{2}kL^2\cos^2(2\pi f t) + \frac{1}{2}kL^2\sin^2(2\pi f t) = \frac{1}{2}kL^2$ no depèn de t, per tant és constant, i $\frac{dE}{dt} = 0$.

10. Solucions (manca indicar el tipus d'indeterminació que surt en cada pas):

1. $\lim_{x\to 0+} \frac{\sin x}{\sqrt{x}} \overset{\star}{=} \lim_{x\to 0+} 2\sqrt{x}\cos x = 0$

2. $\lim_{x\to 1} \frac{\ln x}{1-x} \overset{\star}{=} \lim_{x\to 1} \frac{1/x}{-1} = -1$

3. $\lim_{x\to 0} \frac{e^x-1}{\ln(1+x)} \overset{\star}{=} \lim_{x\to 0}(1+x)e^x = 1$

4. $\lim_{x\to 4} \frac{\sqrt{x}-2}{x-4} \overset{\star}{=} \lim_{x\to 4} \frac{\frac{1}{2\sqrt{x}}}{1} = \frac{1}{4}$

5. $\lim_{x\to \pi/2} \frac{\cos x}{\sin 2x} \overset{\star}{=} \lim_{x\to \pi/2} \frac{-\sin x}{2\cos 2x} = \frac{1}{2}$

6. $\lim_{x\to a} \frac{x-a}{x^n-a^n} \overset{\star}{=} \lim_{x\to a} \frac{1}{nx^{n-1}} = \frac{1}{na^{n-1}}$

7. $\lim_{x\to 0} \frac{2^x-1}{x} \overset{\star}{=} \lim_{x\to 0} 2^x \ln\, 2 = \ln\, 2$

8. $\lim_{x\to 0} \frac{\tan^{-1} x}{x} \overset{\star}{=} \lim_{x\to 0} \frac{\frac{1}{1+x^2}}{1} = 1$

9. $\lim_{x\to 1} \frac{x^{1/2}-x^{1/4}}{x-1} \overset{\star}{=} \lim_{x\to 1} \left(\frac{1}{2}x^{-1/2} - \frac{1}{4}x^{-3/4}\right) = \frac{1}{4}$

10. $\lim_{x\to 0} \frac{e^x-1}{x(1+x)} \overset{\star}{=} \lim_{x\to 0} \frac{e^x}{1+2x} = 1$

11. $\lim_{x\to 0} \frac{e^x-e^{-x}}{\sin x} \overset{\star}{=} \lim_{x\to 0} \frac{e^x+e^{-x}}{\cos x} = 2$

12. $\lim_{x\to 0} \frac{1-\cos x}{3x} \overset{\star}{=} \lim_{x\to 0} \frac{\sin x}{3} = 0$

13. $\lim_{x\to 0} \frac{x+\sin \pi x}{x-\sin \pi x} \overset{\star}{=} \lim_{x\to 0} \frac{1+\pi\cos \pi x}{1-\pi\cos \pi x} = \frac{1+\pi}{1-\pi}$

14. $\lim_{x\to 0} \frac{a^x-(a+1)^x}{x} \overset{\star}{=} \lim_{x\to 0} \frac{a^x\ln a-(a+1)^x\ln(a+1)}{1} = \ln\left(\frac{a}{a+1}\right)$

15. $\lim_{x\to 0} \frac{e^x+e^{-x}-2}{1-\cos 2x} \overset{\star}{=} \lim_{x\to 0} \frac{e^x-e^{-x}}{2\sin 2x} \overset{\star}{=} \lim_{x\to 0} \frac{e^x+e^{-x}}{4\cos 2x} = \frac{1}{2}$

16. $\lim_{x\to 0} \frac{x-\ln(x+1)}{1-\cos 2x} \overset{\star}{=} \lim_{x\to 0} \frac{1-\frac{1}{x+1}}{2\sin 2x} \overset{\star}{=} \lim_{x\to 0} \frac{\frac{1}{(x+1)^2}}{4\cos 2x} = \frac{1}{4}$

17. $\lim_{x\to 0} \frac{\tan \pi x}{e^x-1} \overset{\star}{=} \lim_{x\to 0} \frac{\pi\sec^2 \pi x}{e^x} = \pi$

18. $\lim_{x\to 0} \frac{\cos x-1+x^2/2}{x^4} \overset{\star}{=} \lim_{x\to 0} \frac{-\sin x+x}{4x^3} \overset{\star}{=} \lim_{x\to 0} \frac{-\cos x+1}{12x^2} \overset{\star}{=} \lim_{x\to 0} \frac{\sin x}{24x} = \frac{1}{24}$

19. $\lim_{x\to 0} \frac{1+x-e^x}{x(e^x-1)} \overset{\star}{=} \lim_{x\to 0} \frac{1-e^x}{xe^x+e^x-1} \overset{\star}{=} \lim_{x\to 0} \frac{-e^x}{xe^x+2e^x} = -\frac{1}{2}$

20. $\lim_{x\to 0} \frac{\ln(\sec x)}{x^2} \overset{\star}{=} \lim_{x\to 0} \frac{\tan x}{2x} \overset{\star}{=} \lim_{x\to 0} \frac{\sec^2 x}{2} = \frac{1}{2}$

21. $\lim_{x\to 0} \frac{x-\tan x}{x-\sin x} \overset{\star}{=} \lim_{x\to 0} \frac{1-\sec^2 x}{1-\cos x} \overset{\star}{=} \lim_{x\to 0} \frac{-2\sec^2 x\tan x}{\sin x} = \lim_{x\to 0} \frac{-2\sec^2 x}{\cos x} = -2$

22. $\lim_{x\to 0} \frac{xe^{nx}-x}{1-\cos nx} \overset{\star}{=} \lim_{x\to 0} \frac{e^{nx}+nxe^{nx}-1}{n\sin nx} \overset{\star}{=} \lim_{x\to 0} \frac{e^{nx}(2n+n^2x)}{n^2\cos nx} = \frac{2}{n}$

23. $\lim_{x\to 1^-} \frac{\sqrt{1-x^2}}{\sqrt{1-x^3}} = \lim_{x\to 1^-} \sqrt{\frac{1-x^2}{1-x^3}} = \sqrt{\frac{2}{3}} = \frac{1}{3}\sqrt{6}$ since $\lim_{x\to 1^-} \frac{1-x^2}{1-x^3} \overset{\star}{=} \lim_{x\to 1^-} \frac{2x}{3x^2} = \frac{2}{3}$

24. $\lim_{x\to 0} \dfrac{2x-\sin \pi x}{4x^2-1} = 0$

25. $\lim_{x\to \pi/2} \dfrac{\ln(\sin x)}{(\pi-2x)^2} \overset{\star}{=} \lim_{x\to \pi/2} \dfrac{-\cot x}{4(\pi-2x)} \overset{\star}{=} \lim_{x\to \pi/2} \dfrac{\csc^2 x}{-8} = -\dfrac{1}{8}$

26. $\lim_{x\to 0^+} \dfrac{\sqrt{x}}{\sqrt{x}+\sin\sqrt{x}} \overset{\star}{=} \lim_{x\to 0^+} \dfrac{\frac{1}{2\sqrt{x}}}{\frac{1}{2\sqrt{x}}+\frac{\cos\sqrt{x}}{2\sqrt{x}}} = \lim_{x\to 0^+} \dfrac{1}{1+\cos\sqrt{x}} = \dfrac{1}{2}$

27. $\lim_{x\to 0} \dfrac{\cos x-\cos 3x}{\sin(x^2)} \overset{\star}{=} \lim_{x\to 0} \dfrac{-\sin x+3\sin 3x}{2x\cos(x^2)} \overset{\star}{=} \lim_{x\to 0} \dfrac{-\cos x+9\cos 3x}{2\cos(x^2)-4x^2\sin(x^2)} = 4$

28. $\lim_{x\to 0} \dfrac{\sqrt{a+x}-\sqrt{a-x}}{x} \overset{\star}{=} \lim_{x\to 0} \dfrac{\frac{1}{2\sqrt{a+x}}+\frac{1}{2\sqrt{a-x}}}{1} = \dfrac{1}{\sqrt{a}} = \dfrac{\sqrt{a}}{a}$

29.
$$\lim_{x\to \pi/4} \frac{\sec^2 x-2\tan x}{1+\cos 4x} \overset{\star}{=} \lim_{x\to \pi/4} \frac{2\sec^2 x\ \tan x-2\sec^2 x}{-4\sin 4x}$$
$$\overset{\star}{=} \lim_{x\to \pi/4} \frac{2\sec^4 x+4\sec^2 x\tan^2 x-4\sec^2 x\tan x}{-16\cos 4x} = \frac{1}{2}$$

30. $\lim_{x\to 0} \dfrac{x-\arcsin x}{\sin^3 x} \overset{\star}{=} \lim_{x\to 0} \dfrac{1-\frac{1}{\sqrt{1-x^2}}}{3\sin^2 x\cos x} \overset{\star}{=} \lim_{x\to 0} \dfrac{\frac{-x}{(1-x^2)^{3/2}}}{6\sin x\cos^2 x-3\sin^3 x}$

$\overset{\star}{=} \lim_{x\to 0} \dfrac{\frac{-1-2x^2}{(1-x^2)^{5/2}}}{6\cos^3 x-21\sin^2 x\cos x} = -\dfrac{1}{6}$

31. $\lim_{x\to 0} \dfrac{\tan^{-1} x}{\tan^{-1} 2x} \overset{\star}{=} \lim_{x\to 0} \dfrac{1}{1+x^2}\dfrac{1+4x^2}{2} = \dfrac{1}{2}$

32. $\lim_{x\to 0} \dfrac{\sin^{-1} x}{x} \overset{\star}{=} \lim_{x\to 0} \dfrac{\frac{1}{\sqrt{1-x^2}}}{1} = 1$

11. Solució:

$$\lim_{x\to 0}(2+x+\sin x)\neq 0, \quad \lim_{x\to 0}(x^3+x-\cos x)\neq 0$$

12. Solucions (manca indicar el tipus d'indeterminació que surt en cada pas):

(a)
The limit does not exist if $b\neq 1$. Therefore, $b=1$.

$$\lim_{x\to 0} \frac{\cos ax-1}{2x^2} \overset{\star}{=} \lim_{x\to 0} \frac{-a\sin ax}{4x} \overset{\star}{=} \lim_{x\to 0} \frac{-a^2\cos ax}{4} = -\frac{a^2}{4}$$

Now, $-\dfrac{a^2}{4} = -4$ implies $a=\pm 4$.

(b)

$$\lim_{x\to 0}\frac{\sin 2x + ax + bx^3}{x^3} \overset{\star}{=} \lim_{x\to 0}\frac{2\cos 2x + a + 3bx^2}{3x^2} \quad \text{need } a=-2 \text{ to keep numerator } 0$$

$$\overset{\star}{=} \lim_{x\to 0}\frac{-4\sin 2x + 6bx}{6x} \overset{\star}{=} \lim_{x\to 0}\frac{-8\cos 2x + 6b}{6} = 0 \quad \text{if} \quad 6b = 8$$

$$\Longrightarrow \quad a=-2, \quad b = \frac{4}{3}$$

13. Solució (manca indicar el tipus d'indeterminació que surt en cada pas):

(a)

Si $\alpha > 0$ surt $\frac{\infty}{\infty}$, i per tant es pot aplicar l'Hôpital:

$$\lim_{x\to\infty}\frac{\ln x}{x^\alpha} \overset{*}{=} \lim_{x\to\infty}\frac{1/x}{\alpha x^{\alpha-1}} = \lim_{x\to\infty}\frac{1}{\alpha x^\alpha} = 0.$$

Si $\alpha = 0$ el denominador és 1, i per tant el límit és $+\infty$

Si $\alpha < 0$ el denominador és 0, i per tant el límit és $+\infty$

(b)

$$\lim_{x\to\infty}\frac{e^x}{x^k} \overset{*}{=} \lim_{x\to\infty}\frac{e^x}{kx^{k-1}} \overset{*}{=} \lim_{x\to\infty}\frac{e^x}{k(k-1)x^{k-2}} \overset{*}{=} \cdots \overset{*}{=} \lim_{x\to\infty}\frac{e^x}{k!} = \infty.$$

(c)

$$\lim_{x\to 0^+}\sqrt{x}\ln x = \lim_{x\to 0^+}\frac{\ln x}{1/\sqrt{x}} \overset{*}{=} \lim_{x\to 0^+}\frac{1/x}{-\frac{1}{2}x^{-3/2}} = \lim_{x\to 0^+} -2\sqrt{x} = 0.$$

(d)

$$\tan x - \sec x = \frac{\sin x}{\cos x} - \frac{1}{\cos x} = \frac{\sin x - 1}{\cos x}$$

$$\lim_{x\to(\pi/2)^-}\frac{\sin x - 1}{\cos x} \overset{*}{=} \lim_{x\to(\pi/2)^-}\frac{\cos x}{-\sin x} = \frac{0}{-1} = 0.$$

(e)

$$\lim_{x\to 0^+}\ln(x^x) = \lim_{x\to 0^+}(x\ln x) = \lim_{x\to 0^+}\frac{\ln x}{1/x} \overset{*}{=} \lim_{x\to 0^+}\frac{1/x}{-1/x^2} = \lim_{x\to 0^+}(-x) = 0.$$

$$\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{x\ln x} = e^0 = 1$$

(f)

$$\lim_{x\to 0^+}\ln(1+x)^{1/x} = \lim_{x\to 0^+}\frac{\ln(1+x)}{x} \overset{*}{=} \lim_{x\to 0^+}\frac{1}{1+x} = 1.$$

$$\lim_{x\to 0^+}(1+x)^{1/x} = e^1 = e$$

(g)

$$\lim_{x\to\infty} \ln (a^x + b^x)^{1/x} = \lim_{x\to\infty} \frac{\ln (a^x + b^x)}{x}$$

$$\overset{*}{=} \lim_{x\to\infty} \frac{a^x \ln a + b^x \ln b}{a^x + b^x} = \lim_{x\to\infty} \frac{(a/b)^x \ln a + \ln b}{(a/b)^x + 1} = \ln b$$

$$\lim_{x\to 0^+} (a^x + b^x)^{1/x} = e^{\ln b} = b$$

14. Solució (manca indicar el tipus d'indeterminació que surt en cada pas):

1. $$\lim_{x\to-\infty} \frac{x^2+1}{1-x} \overset{\star}{=} \lim_{x\to-\infty} \frac{2x}{-1} = \infty$$

2. $$\lim_{x\to\infty} \frac{20x}{x^2+1} = 0$$

3. $$\lim_{x\to\infty} \frac{x^3}{1-x^3} = \lim_{x\to\infty} \frac{1}{1/x^3 - 1} = -1$$

4. $$\lim_{x\to\infty} \frac{x^3-1}{2-x} = -\infty$$

5. $$\lim_{x\to\infty} x^2 \sin\frac{1}{x} = \lim_{h\to 0+} \left[\left(\frac{1}{h}\right)\left(\frac{\sin h}{h}\right)\right] = \infty$$

6. $$\lim_{x\to\infty} \frac{\ln(x^k)}{x} \overset{\star}{=} \lim_{x\to\infty} \frac{k/x}{1} = 0$$

7. $$\lim_{x\to\frac{\pi}{2}^-} \frac{\tan 5x}{\tan x} = \lim_{x\to\frac{\pi}{2}^-} \left[\left(\frac{\sin 5x}{\sin x}\right)\left(\frac{\cos x}{\cos 5x}\right)\right] = \frac{1}{5} \qquad \text{since}$$

$$\lim_{x\to\frac{\pi}{2}^-} \frac{\sin 5x}{\sin x} = 1 \quad \text{and} \quad \lim_{x\to\frac{\pi}{2}^-} \frac{\cos x}{\cos 5x} \overset{\star}{=} \lim_{x\to\frac{\pi}{2}^-} \frac{\sin x}{5\sin 5x} = \frac{1}{5}$$

8. $$\lim_{x\to 0}(x\ln|\sin x|) = \lim_{x\to 0} \frac{\ln|\sin x|}{1/x} \overset{\star}{=} \lim_{x\to 0} \frac{\frac{\cos x}{\sin x}}{-\frac{1}{x^2}} = \lim_{x\to 0} \left(-\frac{x}{\sin x}\right)(x\cos x) = 0$$

9. $$\lim_{x\to 0+} x^{2x} = \lim_{x\to 0+} (x^x)^2 = 1^2 = 1 \qquad \text{[see (11.6.4)]}$$

10. $$\lim_{x\to\infty} x\sin\frac{\pi}{x} = \lim_{t\to 0+} \frac{\sin \pi t}{t} \overset{\star}{=} \lim_{t\to 0+} \frac{\pi\cos \pi t}{1} = \pi$$

11. $$\lim_{x\to 0} x(\ln|x|)^2 = \lim_{x\to 0} \frac{(\ln|x|)^2}{1/x} \overset{\star}{=} \lim_{x\to 0} \frac{2\ln|x|}{-1/x} \overset{\star}{=} \lim_{x\to 0} \frac{2}{1/x} = \lim_{x\to 0} 2x = 0$$

12. $$\lim_{x\to 0+} \frac{\ln x}{\cot x} \overset{\star}{=} \lim_{x\to 0+} \frac{1/x}{-\csc^2 x} = \lim_{x\to 0+} -\frac{\sin^2 x}{x} = \lim_{x\to 0+} -\frac{\sin x}{x}\cdot \sin x = 0$$

14. $$\lim_{x\to\infty} \frac{\sqrt{1+x^2}}{x} = \lim_{x\to\infty} \sqrt{\frac{1}{x^2}+1} = 1$$

15. $$\lim_{x\to 0}\left[\frac{1}{\sin^2 x}-\frac{1}{x^2}\right]=\lim_{x\to 0}\frac{x^2-\sin^2 x}{x^2\sin^2 x}\overset{\star}{=}\lim_{x\to 0}\frac{2x-2\sin x\cos x}{2x^2\sin x\cos x+2x\sin^2 x}$$
$$=\lim_{x\to 0}\frac{2x-\sin 2x}{x^2\sin 2x+2x\sin^2 x}\overset{\star}{=}\lim_{x\to 0}\frac{2-2\cos 2x}{2x^2\cos 2x+4x\sin 2x+2\sin^2 x}$$
$$\overset{\star}{=}\lim_{x\to 0}\frac{4\sin 2x}{-4x^2\sin 2x+12x\cos 2x+6\sin 2x}$$
$$\overset{\star}{=}\lim_{x\to 0}\frac{8\cos 2x}{-8x^2\cos 2x-32x\sin 2x+24\cos 2x}=\frac{1}{3}$$

16. Since $\lim\limits_{x\to 0}\ln(|\sin x|^x)=\lim\limits_{x\to 0}(x\ln|\sin x|)=0$ by Exercise 8, $\lim\limits_{x\to 0}|\sin x|^x=e^0=1$

17. $\lim\limits_{x\to 1}x^{1/(x-1)}=e$ since $\lim\limits_{x\to 1}\ln\left[x^{1/(x-1)}\right]=\lim\limits_{x\to 1}\dfrac{\ln x}{x-1}\overset{\star}{=}\lim\limits_{x\to 1}\dfrac{1}{x}=1$

18. Take log:
$$\lim_{x\to 0^+}\ln\left(x^{\sin x}\right)=\lim_{x\to 0^+}(\sin x\,\ln x)=\lim_{x\to 0^+}\left(\frac{\ln x}{\csc x}\right)$$
$$\overset{\star}{=}\lim_{x\to 0^+}\frac{1/x}{-\csc x\cot x}=\lim_{x\to 0^+}\frac{-\sin^2 x}{x\cos x}=0,\quad\text{so}\quad\lim_{x\to 0^+}x^{\sin x}=e^0=1$$

19. $$\lim_{x\to\infty}\left(\cos\frac{1}{x}\right)^x=1\quad\text{since}\quad\lim_{x\to\infty}\ln\left[\left(\cos\frac{1}{x}\right)^x\right]=\lim_{x\to\infty}\frac{\ln\left(\cos\frac{1}{x}\right)}{(1/x)}$$
$$\overset{\star}{=}\lim_{x\to\infty}\left(-\frac{\sin(1/x)}{\cos(1/x)}\right)=0$$

20. Take log:
$$\lim_{x\to\pi/2}\ln\left(|\sec x|^{\cos x}\right)=\lim_{x\to\pi/2}\cos x\ln|\sec x|=\lim_{x\to\pi/2}\frac{\ln|\sec x|}{\sec x}$$
$$\overset{\star}{=}\lim_{x\to\pi/2}\frac{\tan x}{\sec x\tan x}=\lim_{x\to\pi/2}\cos x=0,\quad\text{so}\quad\lim_{x\to\pi/2}|\sec x|^{\cos x}=e^0=1$$

21. $$\lim_{x\to 0}\left[\frac{1}{\ln(1+x)}-\frac{1}{x}\right]=\lim_{x\to 0}\frac{x-\ln(1+x)}{x\ln(1+x)}\overset{\star}{=}\lim_{x\to 0}\frac{x}{x+(1+x)\ln(1+x)}$$
$$\overset{\star}{=}\lim_{x\to 0}\frac{1}{1+1+\ln(1+x)}=\frac{1}{2}$$

22. Take log: $\lim\limits_{x\to\infty}\ln(x^2+a^2)^{(1/x)^2}=\lim\limits_{x\to\infty}\dfrac{\ln(x^2+a^2)}{x^2}\overset{\star}{=}\lim\limits_{x\to\infty}\dfrac{\frac{2x}{x^2+a^2}}{2x}=0,$

so $\lim\limits_{x\to\infty}(x^2+a^2)^{(1/x)^2}=e^0=1$

23. $$\lim_{x\to 0}\left[\frac{1}{x}-\cot x\right]=\lim_{x\to 0}\frac{\sin x-x\cos x}{x\sin x}\overset{\star}{=}\lim_{x\to 0}\frac{x\sin x}{\sin x+x\cos x}$$
$$\overset{\star}{=}\lim_{x\to 0}\frac{\sin x+x\cos x}{2\cos x-x\sin x}=0$$

24. $\displaystyle\lim_{x\to\infty} \ln\left(\frac{x^2-1}{x^2+1}\right)^3 = 3\lim_{x\to\infty} \ln\left(\frac{x^2-1}{x^2+1}\right) = 0$

25. $$\lim_{x\to\infty}\left(\sqrt{x^2+2x}-x\right) = \lim_{x\to\infty}\left[\left(\sqrt{x^2+2x}-x\right)\left(\frac{\sqrt{x^2+2x}+x}{\sqrt{x^2+2x}+x}\right)\right]$$
$$= \lim_{x\to\infty}\frac{2x}{\sqrt{x^2+2x}+x} = \lim_{x\to\infty}\frac{2}{\sqrt{1+2/x}+1} = 1$$

26. $\displaystyle\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{bx} = \lim_{x\to\infty}\left[\left(1+\frac{a}{x}\right)^x\right]^b = (e^a)^b = e^{ab}.$

27. $\displaystyle\lim_{x\to\infty}(x^3+1)^{1/\ln x} = e^3$ since
$$\lim_{x\to\infty}\ln\left[(x^3+1)^{1/\ln x}\right] = \lim_{x\to\infty}\frac{\ln(x^3+1)}{\ln x} \stackrel{\star}{=} \lim_{x\to\infty}\frac{\left(\dfrac{3x^2}{x^3+1}\right)}{1/x} = \lim_{x\to\infty}\frac{3}{1+1/x^3} = 3.$$

28. Take log: $\displaystyle\lim_{x\to\infty}\frac{\ln(e^x+1)}{x} \stackrel{\star}{=} \lim_{x\to\infty}\frac{\frac{e^x}{e^x+1}}{1} \stackrel{\star}{=} \lim_{x\to\infty}\frac{e^x}{e^x} = 1,$

so $\displaystyle\lim_{x\to\infty}(e^x+1)^{1/x} = e$

29. $\displaystyle\lim_{x\to\infty}(\cosh x)^{1/x} = e$ since
$$\lim_{x\to\infty}\ln[(\cosh x)^{1/x}] = \lim_{x\to\infty}\frac{\ln(\cosh x)}{x} \stackrel{\star}{=} \lim_{x\to\infty}\frac{\sinh x}{\cosh x} = 1.$$

30. Take log: $\displaystyle\lim_{x\to\infty} 3x\ln\left(1+\frac{1}{x}\right) = 3\lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} \stackrel{\star}{=} 3\lim_{x\to\infty}\frac{\frac{-1/x^2}{1+1/x}}{-\frac{1}{x^2}} = 3,$

so $\displaystyle\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{3x} = e^3$

31. $$\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right) = \lim_{x\to 0}\frac{x-\sin x}{x\,\sin x} \stackrel{\star}{=} \lim_{x\to 0}\frac{1-\cos x}{\sin x + x\,\cos x}$$
$$\stackrel{\star}{=} \lim_{x\to 0}\frac{\sin x}{2\cos x - x\,\sin x} = 0$$

32. Take log: $\displaystyle\lim_{x\to 0}\frac{\ln(e^x+3x)}{x} \stackrel{\star}{=} \lim_{x\to 0}\frac{\frac{e^x+3}{e^x+3x}}{1} = 4,$ so $\displaystyle\lim_{x\to 0}(e^x+3x)^{1/x} = e^4$

33. $$\lim_{x\to 1}\left(\frac{1}{\ln x}-\frac{x}{x-1}\right) = \lim_{x\to 1}\frac{x-1-x\ln x}{(x-1)\ln x} \stackrel{\star}{=} \lim_{x\to 1}\frac{-\ln x}{(x-1)(1/x)+\ln x}$$
$$= \lim_{x\to 1}\frac{-x\,\ln x}{x-1+x\,\ln x} \stackrel{\star}{=} \lim_{x\to 1}\frac{-\ln x - 1}{2+\ln x} = -\frac{1}{2}$$

34. $\sqrt{2}$: take log: $\displaystyle\lim_{x\to 0}\frac{\ln(1+2^x)-\ln 2}{x} \stackrel{\star}{=} \lim_{x\to 0}\frac{\frac{2^x\ln 2}{1+2^x}}{1} = \frac{\ln 2}{2};\quad \lim_{x\to 0}\left(\frac{1+2^x}{2}\right)^{1/x} = e^{\frac{1}{2}\ln 2} = \sqrt{2}$

15. Solució:

(a)

$$\lim_{k\to 0^+} v(t) = \lim_{k\to 0^+} \frac{mg\left(1-e^{-(k/m)t}\right)}{k} \stackrel{\star}{=} \lim_{k\to 0^+} \frac{gte^{-(k/m)t}}{1} = gt$$

(b)

$$\frac{dv}{dt} = g \implies v(t) = gt + C; \quad v(0) = 0 \implies C = 0 \quad \text{and} \quad v(t) = gt.$$

16. Solució:

$$\begin{aligned}
[P(x)]^{1/n} - x &= \left([P(x)]^{1/n} - x\right) \cdot \frac{[P(x)]^{(n-1)/n} + x[P(x)]^{(n-2)/n} + \cdots + x^{n-2}[P(x)]^{1/n} + x^{n-1}}{[P(x)]^{(n-1)/n} + x[P(x)]^{(n-2)/n} + \cdots + x^{n-2}[P(x)]^{1/n} + x^{n-1}} \\
&= \frac{P(x) - x^n}{[P(x)]^{(n-1)/n} + x[P(x)]^{(n-2)/n} + \cdots + x^{n-2}[P(x)]^{1/n} + x^{n-1}} \\
&= \frac{b_1 x^{n-1} + b_2 x^{n-2} + \cdots + b_n}{[P(x)]^{(n-1)/n} + x[P(x)]^{(n-2)/n} + \cdots + x^{n-2}[P(x)]^{1/n} + x^{n-1}} \to \frac{b_1}{n} \quad \text{as} \quad x \to \infty
\end{aligned}$$