Exercises in Probability	**Ver. 2.1, Dec. 14, 2023**
Hebert Pérez Rosés	**Univ. Rovira i Virgili**

Try to answer these questions as if it were an exam, that is, without the aid of any electronic devices (calculators, mobile phones, etc.).
All answers must include the supporting deductions and calculations (if the calculations are too heavy then they can be left implied).

BASIC PRINCIPLES OF PROBABILITY

1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $A, B \in \mathcal{F}$. Prove that $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$.
   Hint: Use the fact that $A \cup B$ is the disjoint union of $A \cap B$, $A \cap \overline{B}$ and $\overline{A} \cap B$.
2. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $A_1, A_2, \ldots, A_n \in \mathcal{F}$ be a countable collection of events. Prove Bonferroni's inequality:
$$\mathbb{P}\left(\bigcap_{i=1}^{n} A_i\right) \geq \sum_{i=1}^{n} \mathbb{P}(A_i) - (n-1)$$
   Hint: Prove the inequality for $n = 1$ and $n = 2$ and then use induction on n.
3. (2nd partial exam, 2022–2023) Let $\Omega = \{a, b, c, d\}$ and $\mathcal{F} = \{\emptyset, \{a\}, \{b, c\}, \{d\}, \{a, b, c\}, \{b, c, d\}, \{a, d\}, \Omega\}$.
   (a) Prove that $(\Omega, \mathcal{F})$ is a measurable space.
   (b) Let $\mathbb{P} : \mathcal{F} \longrightarrow [0; 1]$ be defined as follows: $\mathbb{P}(\{a\}) = \frac{1}{3}$, $\mathbb{P}(\{b, c\}) = \frac{1}{4}$ and $\mathbb{P}(\{d\}) = \alpha$. Determine the value of α so that $\mathbb{P}$ is a probability measure over $(\Omega, \mathcal{F})$.
   (c) Calculate $\mathbb{P}(\{b, c, d\})$ and $\mathbb{P}(\{a, d\})$.
4. (2nd partial exam, 2022–2023) Let $\Omega = \{a, b, c, d\}$ and $\mathcal{F} = \{\emptyset, \{a\}, \{b, c\}, \{d\}, \{a, b, c\}, \{b, c, d\}, \{a, d\}, \Omega\}$.
   (a) Prove that $(\Omega, \mathcal{F})$ is a measurable space.
   (b) Let $\mathbb{P} : \mathcal{F} \longrightarrow [0; 1]$ be defined as follows: $\mathbb{P}(\{a\}) = \frac{2}{5}$, $\mathbb{P}(\{b, c\}) = \frac{1}{3}$ and $\mathbb{P}(\{d\}) = \alpha$. Determine the value of α so that $\mathbb{P}$ is a probability measure over $(\Omega, \mathcal{F})$.
   (c) Calculate $\mathbb{P}(\{b, c, d\})$ and $\mathbb{P}(\{a, d\})$.
5. The probability that Tom passes the Probability exam is 0.8, the probability that Jerry passes the same exam is 0.4, and the probability that both pass the exam is 0.3. Calculate the probability that
   (a) At least one of them passes the exam.
   (b) None of them passes the exam.

(c) Only one of them passes the exam.

6. In a gym there are 7 women and 12 men. The instructor chooses 3 people at random for a particular exercise. Find the probability that two women and a man are chosen.

7. Tom's lives in a 7-story building. There is one apartment on every floor. Two strangers arrive at the building and are waiting for the lift.

   (a) What is the probability that both visitors are going to the same floor?

   (b) What would be the probability for three visitors?

   (c) What is the probability that both visitors are going to Tom's apartment?

   (d) What would be the probability for three visitors?

8. A game consists of rolling two fair cubical dice. The player who rolls the dice wins the game if the difference between the two dice is a positive even number (different from zero). If the difference is odd, then the player loses the game. If the difference is zero then the result is undefined. Find the probability that on a given round the player

   (a) wins the game

   (b) loses the game

   Is the game fair? (i.e. are the probabilities of winning and losing the same?). If not, what can be done to make it fair?

9. The same game is now played with two fair dodecahedral dice (see Figure 9). Again, the player who rolls the dice wins if the difference between the two dice is a positive even number. If the difference is odd, then the player loses the game. If the difference is zero then the result is undefined. Find the probability that on a given round the player

   (a) wins the game

   (b) loses the game

   Is the game fair? (i.e. are the probabilities of winning and losing the same?). If not, what can be done to make it fair?

Figure 1: Dodecahedral die, with 12 pentagonal sides

10. The coefficients a, b and c of a quadratic equation $ax^2 + bx + c = 0$ are chosen by tossing a fair cubical die thrice.

    (a) What is the probability that the equation has real roots?

(b) What is the probability that it has a double root?

11. The coefficients a, b and c of a quadratic equation $ax^2 + bx + c = 0$ are chosen by tossing a fair dodecahedral die thrice.

    (a) What is the probability that the equation has real roots?

    (b) What is the probability that it has a double root?

12. Antoine Gombaud, also know as the *Chevalier de Méré*, was a French dandy and gambling enthusiast of the 17-th Century. He asked his friend, the mathematician Blaise Pascal, the following questions:

    (a) If we roll two fair cubical dice n times, what is the probability p that a double six will be obtained at least once?

    (b) What should be the value of n so that $p \geq \frac{1}{2}$?

13. Now suppose we have two fair dodecahedral dice.

    (a) If we roll both dice n times, what is the probability p that a double twelve will be obtained at least once?

    (b) What should be the value of n so that $p \geq \frac{1}{2}$?

GEOMETRIC PROBABILITY

1. Let $\overline{AB}$ be a line segment of length 4 units. A point P is chosen at random on $\overline{AB}$. What is the probability that P is closer to the midpoint of $\overline{AB}$ than to either endpoint?
2. Suppose that two numbers are chosen at random in the interval $[0;1]$.
   (a) What is the probability that $x+y \geq \frac{3}{2}$?
   (b) What is the probability that $xy \geq \frac{1}{2}$?
   (c) Let $a = \min(x,y)$ and $b = \max(x,y)$. What is the probability that the three segments determined by x and y, i.e. $\overline{0a}$, $\overline{ab}$ and $\overline{b1}$, are the sides of some triangle?
   (d) What is the probability that all three segments, $\overline{0a}$, $\overline{ab}$ and $\overline{b1}$, have a length greater than $\frac{1}{4}$.
3. Tom and Jerry go every Saturday to a bar where there is a dart board for throwing darts (see Figure 3). Jerry throws two darts at the board. What is the probability that he gets at least 10 points?

   Note: We assume that all darts end up inside the square board (i.e. the probability of a dart landing outside the square board is negligible).

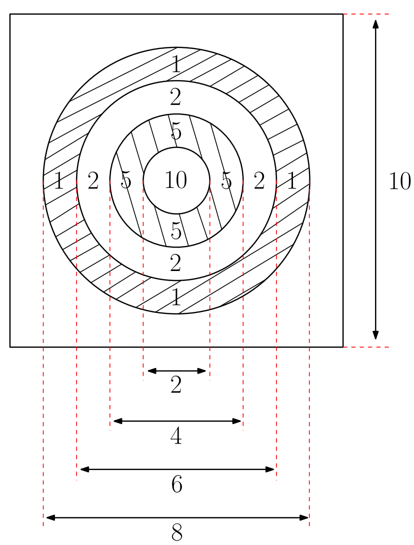

Figure 2: Dart board

4. The floor of Tom's house is covered by square tiles of size $L \times L$. Tom tosses a coin of radius R (with $R < \frac{L}{2}$). What is the probability that the coin lands completely within one tile? (i.e. the coin does not intersect any line).

5. (2nd partial exam, 2022–2023) Suppose that the floor of our house is covered with rectangular tiles of size $2a \times a$, as in Figure 3. If we flip a coin of radius 1 on the floor, there is a probability p that the coin **does not** intersect any of the lines. What should be the value of a so that $p > \frac{1}{2}$?

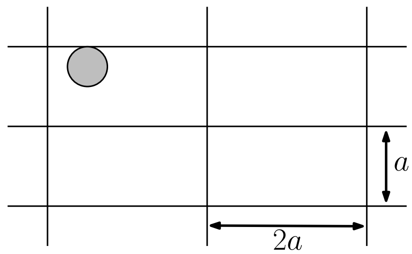

Figure 3: Tiled floor and coin of radius 1

6. (2nd partial exam, 2022–2023) Suppose that the floor of our house is covered with rectangular tiles of size $a \times 3a$, as in Figure 4. If we flip a coin of radius 1 on the floor, there is a probability p that the coin **does not** intersect any of the lines. What should be the value of a so that $p > \frac{1}{2}$?

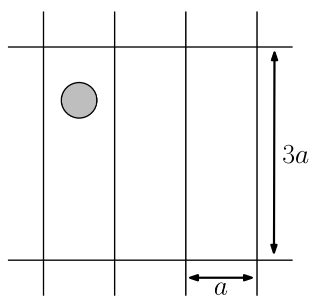

Figure 4: Tiled floor and coin of radius 1

7. On a circumference we choose three points at random: A, B and C,
   (a) What is the probability that the triangle $\triangle ABC$, contains the center of the circumference?
   (b) What is the probability that the triangle $\triangle ABC$ has no internal angle greater than $90°$?
8. Let $A(0;1)$, $B(3;4)$, $Q(0;0)$ and $R(3;0)$. Now we choose a point P at random on the segment $\overline{AB}$. What is the probability that the area of the triangle $\triangle PQR$ (the shaded triangle in Figure 8) is at least 2?

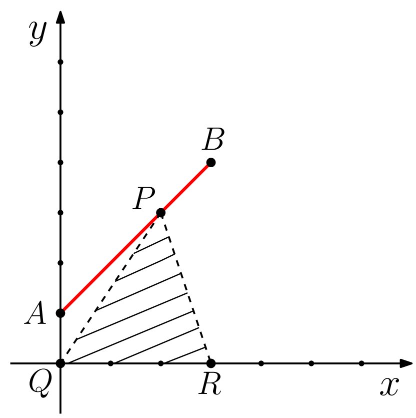

Figure 5: Random triangle

CONDITIONAL PROBABILITY, INDEPENDENCE AND BAYES' RULE

1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $A, B, C \in \mathcal{F}$. Prove that
$$\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A) \cdot \mathbb{P}(B \mid A) \cdot \mathbb{P}(C \mid A \cap B).$$
2. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $A, B, C \in \mathcal{F}$ be three independent events. Prove that
   (a) The events $A \cup B$ and C are also independent
   (b) The events $A \cap B$ and C are also independent
3. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $A, B \in \mathcal{F}$ be two arbitrary events. Determine if the following assertions are true or false. In case an assertion is true, then prove it. Otherwise provide a counterexample.
   (a) $\mathbb{P}(A \mid B) \leq \mathbb{P}(A)$
   (b) $\mathbb{P}(A \mid B) \geq \mathbb{P}(A)$
   (c) If $0 < \mathbb{P}(B) < 1$ then $\mathbb{P}(A \mid \overline{B}) = 1 - \mathbb{P}(A \mid B)$
4. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $A, B \in \mathcal{F}$, with $\mathbb{P}(A) > \mathbb{P}(B)$. Show that $\mathbb{P}(A \mid B) > \mathbb{P}(B \mid A)$.
5. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $A, B \in \mathcal{F}$ such that $\mathbb{P}(A) = 0.4$ and $\mathbb{P}(A \cup B) = 0.8$. Calculate $\mathbb{P}(B)$ in the following cases:
   (a) A and B are independent
   (b) A and B are incompatible
   (c) $\mathbb{P}(A \mid B) = 0.5$
6. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $A, B \in \mathcal{F}$ be two <u>independent</u> events such that $\mathbb{P}(A) = \mathbb{P}(B) = 0.5$. Calculate
$$\mathbb{P}(A \Delta B) = \mathbb{P}\left((A - B) \cup (B - A)\right)$$
7. Suppose that a family has three children, and define the following events:
   - $A \equiv$ "In the family there are children of both genders"
   - $B \equiv$ "There is at most one boy in the family"

   Are these two events independent?
   Note: This time we will assume that the probability of conceiving a boy is p and the probability of conceiving a girl is $q = 1 - p$ (not necessarily equal).
8. At the University of Pandora 25% of the students fail the Math test, 15% fail the Chemistry test, and 10% fail both the Math test and the Chemistry test. Suppose that we pick a random student.
   (a) What is the probability that this student has failed Math **or** Chemistry?
   (b) Knowing that the student has failed the Math test, what is the probability that he/she has also failed the Chemistry test?

(c) Knowing that the student has failed the Chemistry test, what is the probability that he/she has also failed the Math test?

9. (2nd partial exam, 2022–2023) We have two urns, the first one containing 4 black balls and 2 white balls, while the second box contains 3 white balls and 2 black balls. An experiment consists of selecting a box at random, and then extracting two balls at once from the chosen box, also at random.

   (a) Calculate the probability that at least one of the extracted balls is white.

   (b) Suppose that we don't know which box was chosen. If the two balls turn out to be white, calculate the probability that the first box was chosen.

10. (2nd partial exam, 2022–2023) We have two urns, the first one containing 4 black balls and 2 white balls, while the second box contains 3 white balls and 2 black balls. An experiment consists of selecting a box at random, and then extracting two balls at once from the chosen box, also at random.

    (a) Calculate the probability that at least one of the extracted balls is black.

    (b) Suppose that we don't know which box was chosen. If the two balls turn out to be white, calculate the probability that the second box was chosen.

RANDOM VARIABLES

1. Five thousand lottery tickets are sold for €1 each. One ticket will win €1 000, two tickets will win €500 each, and ten tickets will win €100 each. Let the random variable χ denote the net gain from the purchase of a randomly selected ticket.
   (a) Construct the Probability Mass Function (PMF) of χ.
   (b) Compute the expected value of χ.
   (c) Compute the variance of χ.
2. Suppose that we roll a fair cubical die repeatedly until a number larger than 3 is observed. Let χ be the random variable that represents the total number of times that we have to roll the die.
   (a) Find the PMF of χ, i.e. $p_\chi(k) = \mathbb{P}(\chi = k)$, for $k = 1, 2, 3, \dots$
   (b) Compute the expected value and the variance of χ.
   (c) Calculate the probability that $\chi \geq 4$.
3. Now suppose that we roll a fair dodecahedral die repeatedly until a number larger than 6 is observed. Let χ be the random variable that represents the total number of times that we have to roll the die.
   (a) Find the PMF of χ, i.e. $p_\chi(k) = \mathbb{P}(\chi = k)$, for $k = 1, 2, 3, \dots$
   (b) Compute the expected value and the variance of χ.
   (c) Calculate the probability that $\chi \geq 7$.
4. Let χ be a discrete random variable with the following PMF:

$$p_\chi(k) = \begin{cases} \frac{1}{4} \text{ for } k = -2 \\ \frac{1}{8} \text{ for } k = -1 \\ \frac{1}{8} \text{ for } k = 0 \\ \frac{1}{4} \text{ for } k = 1 \\ \frac{1}{4} \text{ for } k = 2 \\ 0 \text{ otherwise.} \end{cases}$$

   (a) Compute the expected value and the variance of χ.
   (b) We define a new random variable $\psi = (\chi + 1)^2$. Construct the PMF of ψ.
   (c) Compute the expected value and the variance of ψ.
   (d) Calculate the probability that $\psi \geq 4$.
5. (3rd partial exam, 2022–2023) Let χ be a discrete random variable that takes positive integer values with probabilities

$$p_k = \mathbb{P}(\chi = k) = M \left(\frac{2}{3}\right)^k,$$

   for $k = 1, 2, 3, \dots$ and for some constant $M \in \mathbb{R}$.
   (a) Determine the value of M.

(b) Find the Probability Generating Function (PGF) associated with the sequence $\langle p_k \rangle_{k=1}^{\infty}$.

(c) Calculate the expected value $\mathbb{E}(\chi)$ and the variance $\mathbb{V}(\chi)$.

6. (3rd partial exam, 2022–2023) Let χ be a discrete random variable that takes positive integer values with probabilities

$$p_k = \mathbb{P}(\chi = k) = M \left(\frac{4}{5}\right)^k,$$

for $k = 1, 2, 3, \ldots$ and for some constant $M \in \mathbb{R}$.

   (a) Determine the value of M.

   (b) Find the Probability Generating Function (PGF) associated with the sequence $\langle p_k \rangle_{k=1}^{\infty}$.

   (c) Calculate the expected value $\mathbb{E}(\chi)$ and the variance $\mathbb{V}(\chi)$.

7. Let χ be a continuous random variable with Probability Density Function (PDF)

$$f(x) = |1 - x|$$

for $0 < x < 2$.

   (a) Compute the expected value and the variance of χ.

   (b) Determine the Cumulative Distribution Function (CDF) of χ.

   (c) Calculate the probability of the events $[0; 1]$, $[-2; 2]$ and $\left[\frac{1}{2}; +\infty\right)$.

8. The random variable χ indicates the gross annual income of the employees of the company Puke Sound, Inc. (in thousands of euros). Its Cumulative Distribution Function (CDF) is given by

$$F_\chi(x) = \begin{cases} 0 \text{ if } x < 15, \\ 1 - \left(\frac{15}{x}\right)^3 \text{ for } x \geq 15. \end{cases}$$

   (a) Verify that $F_\chi(x)$ is indeed a CDF.

   (b) Obtain the corresponding Probability Density Function (PDF).

   (c) Calculate the probability that the gross annual income of a random employee is greater than 20 000 euros.

   (d) If the company has 10 000 employees, how many of them are expected to earn between 18 000 and 20 000 euros.

9. Let χ be a continuous random variable with Probability Density Function (PDF)

$$f(x) = \frac{c}{1 + x^2}$$

for all $x \in \mathbb{R}$.

   (a) Determine the value of c.

   (b) Calculate $\mathbb{P}(\chi \geq 0)$.

   (c) Compute the expected value and the variance of χ (if they exist).

(d) Determine the Cumulative Distribution Function (CDF) of χ.

10. Let χ be a continuous random variable with Cumulative Distribution Function (CDF) given by

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{1}{2}x & \text{if } 0 \leq x \leq 2 \\ 1 & \text{if } x > 2. \end{cases}$$

Now let $\psi = \chi^2$.

(a) Verify that $F(x)$ is indeed a CDF, and determine the Probability Density Function (PDF) $f(x)$ of χ.

(b) Determine the Probability Density Function (PDF) $g(x)$ and the Cumulative Distribution Function (CDF) $G(x)$ of ψ.

(c) Compute the expected value and the variance of ψ.

(d) Calculate the probabilities $\mathbb{P}(\psi \leq \chi)$, $\mathbb{P}(\chi + \psi \leq \frac{3}{4})$ and $\mathbb{P}(\chi \leq 2\psi)$.

11. (3rd partial exam, 2022–2023) Let χ be a discrete random variable with the following PMF:

$$p_\chi(k) = \begin{cases} \dfrac{1}{3} & \text{for } k = 0 \\ \\ \dfrac{1}{6} & \text{for } k = 1, -1, 2, -2 \\ \\ 0 & \text{otherwise.} \end{cases}$$

(a) We define a new random variable $\psi = (\chi + 1)^2$. Construct the PMF of ψ.

(b) Compute the expected value and the variance of ψ.

(c) Calculate $\mathbb{P}(\psi \geq 4)$ and $\mathbb{P}(\psi \leq 1)$.

12. (3rd partial exam, 2022–2023) Let χ be a discrete random variable with the following PMF:

$$p_\chi(k) = \begin{cases} \dfrac{1}{3} & \text{for } k = 0 \\ \\ \dfrac{1}{6} & \text{for } k = 1, -1, 2, -2 \\ \\ 0 & \text{otherwise.} \end{cases}$$

(a) We define a new random variable $\psi = (\chi - 1)^2$. Construct the PMF of ψ.

(b) Compute the expected value and the variance of ψ.

(c) Calculate $\mathbb{P}(\psi \geq 4)$ and $\mathbb{P}(\psi \leq 1)$.

13. (3rd partial exam, 2022–2023) Let χ be a continuous random variable with Probability Density Function (PDF)

$$f(x) = \begin{cases} 0 & \text{for } x \in (-\infty; 1) \\ \\ \dfrac{c}{x^4} & \text{for } x \in [1; \infty) \end{cases}$$

(a) Determine the value of c and the Cumulative Distribution Function (CDF) $F(x)$ of χ.

(b) Compute the expected value of χ (if it exists).

(c) Compute the variance of χ (if it exists).

14. (3rd partial exam, 2022–2023) Let χ be a continuous random variable with Probability Density Function (PDF)

$$f(x) = \begin{cases} 0 \text{ for } x \in (-\infty; 1) \\ \\ \dfrac{c}{x^5} \text{ for } x \in [1; \infty) \end{cases}$$

(a) Determine the value of c and the Cumulative Distribution Function (CDF) $F(x)$ of χ.

(b) Compute the expected value of χ (if it exists).

(c) Compute the variance of χ (if it exists).

15. If we toss a fair coin three times, what is the probability that we will get

(a) Three HEADS?

(b) Two TAILS and one HEAD?

(c) At least one HEAD?

(d) Not more than one TAIL?

16. If we toss a fair cubical die five times, what is the probability that

(a) The number 3 appears twice?

(b) The number 3 appears at most once?

(c) The number 3 appears at least two times?

Hint: Define a random variable χ as the number of times that the number 3 appears in five tosses of the die. This random variable has a binomial distribution.