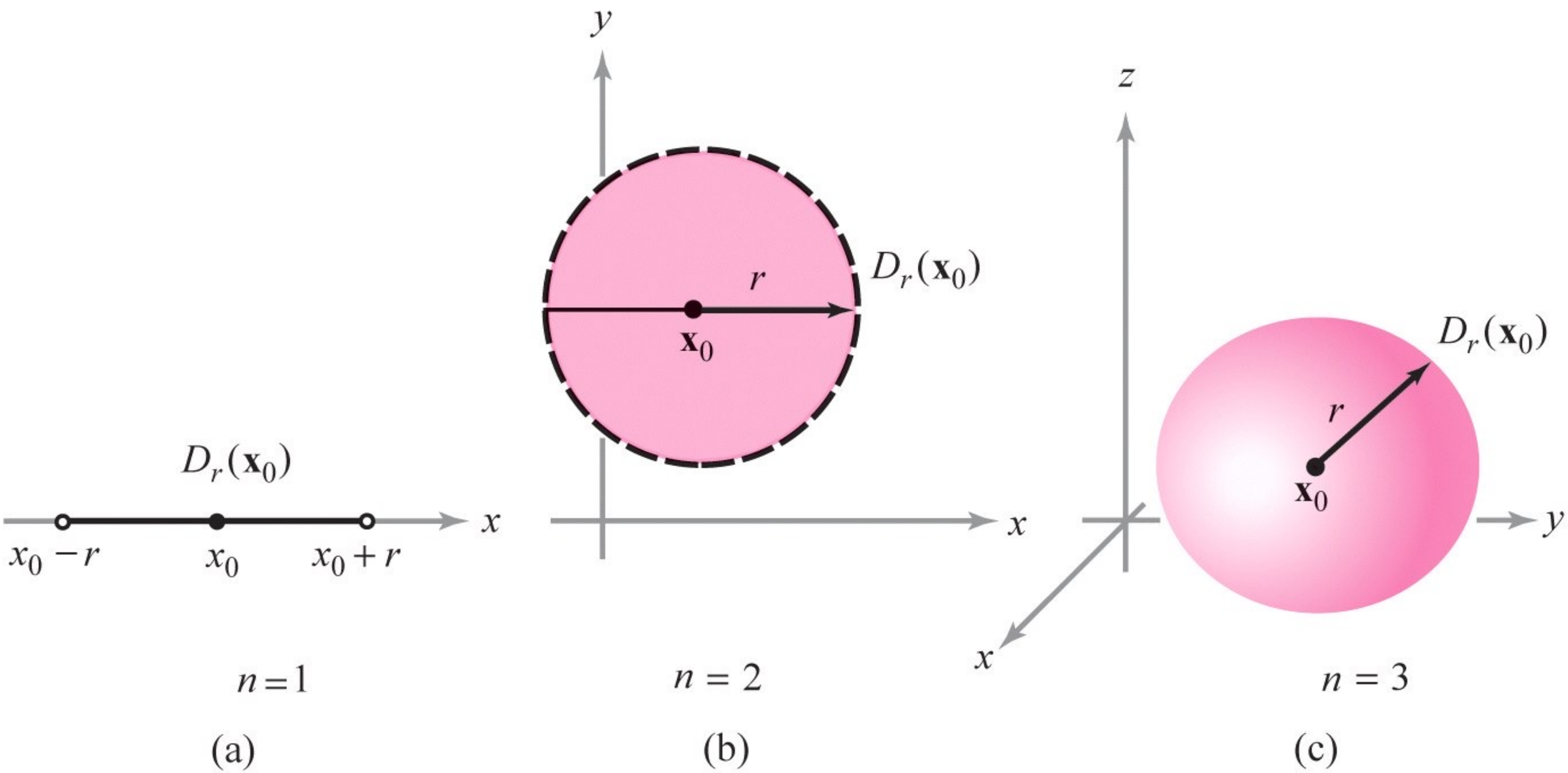
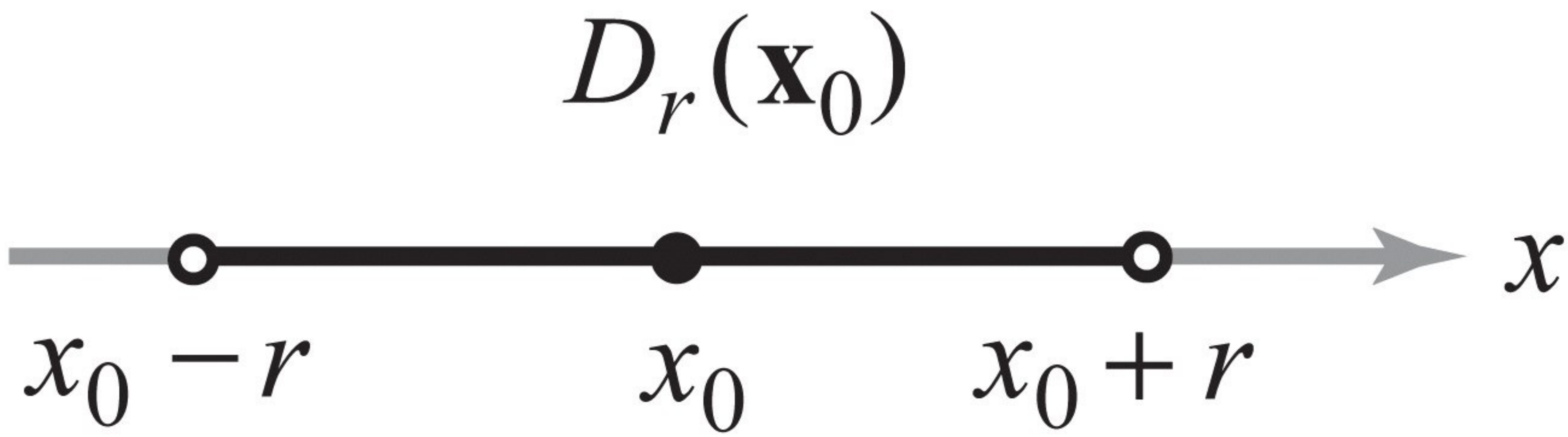
Chapter 2: Differentiation

2.2 Limits and Continuity

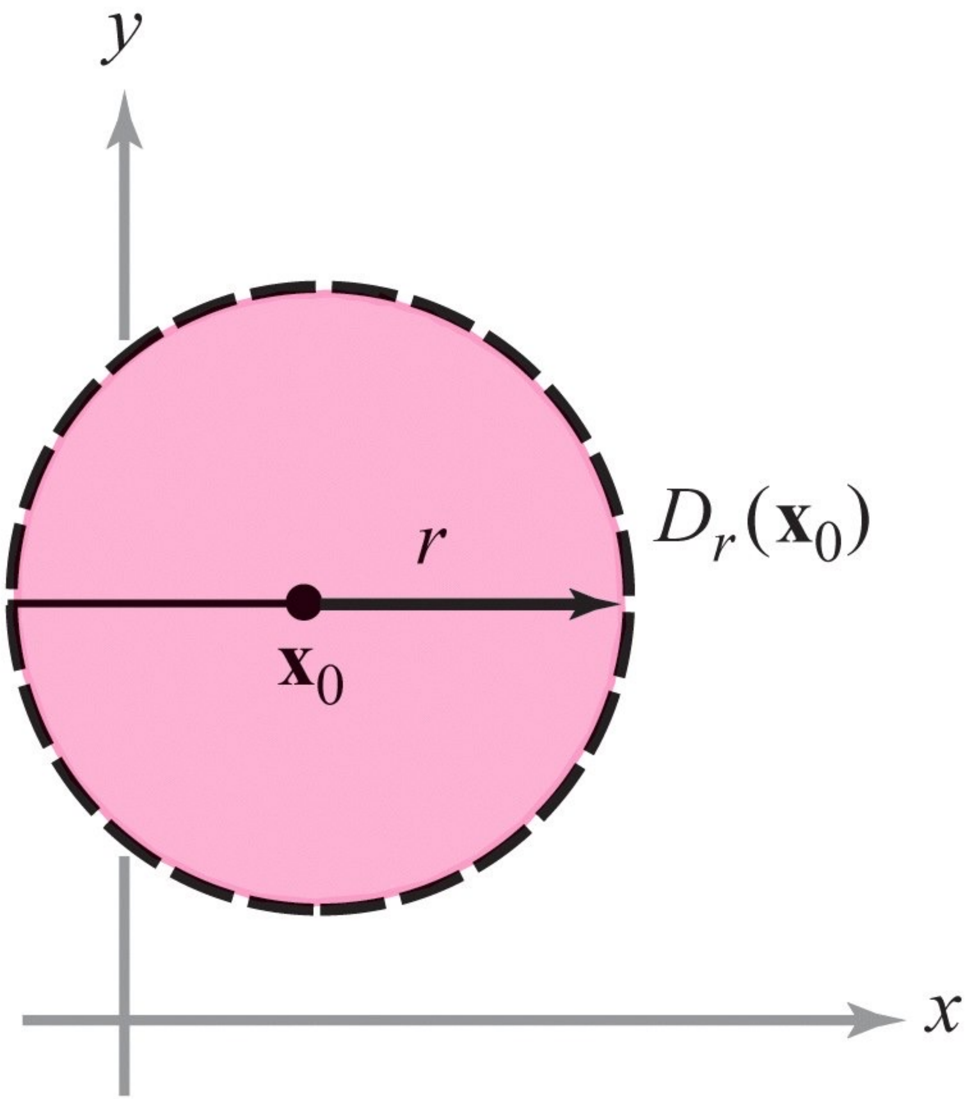
DEFINITION: Open Sets Let $U \subset \mathbb{R}^n$ (that is, let U be a subset of $\mathbb{R}^n$). We call U an ***open set*** when for every point $\mathbf{x}_0$ in U there exists some $r > 0$ such that $D_r(\mathbf{x}_0)$ is contained within U; symbolically, we write $D_r(\mathbf{x}_0) \subset U$ (see Figure 2.2.2).



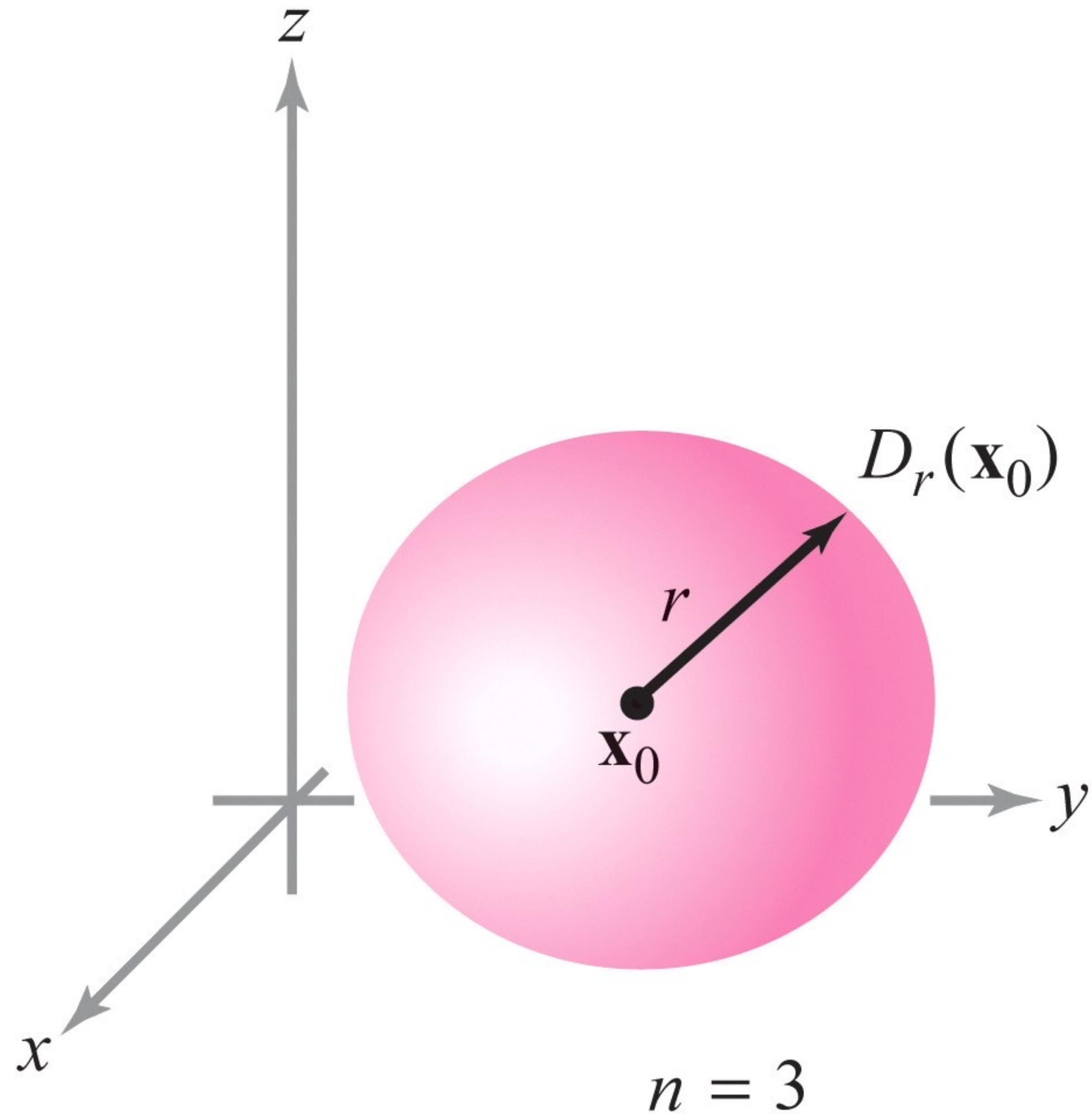
$D_r(\mathbf{x}_0)$
x
$x_0 - r$
x_0
$x_0 + r$
$n = 1$
(a)
y
r
$D_r(\mathbf{x}_0)$
$\mathbf{x}_0$
x
$n = 2$
(b)
z
$D_r(\mathbf{x}_0)$
r
$\mathbf{x}_0$
y
x
$n = 3$
(c)

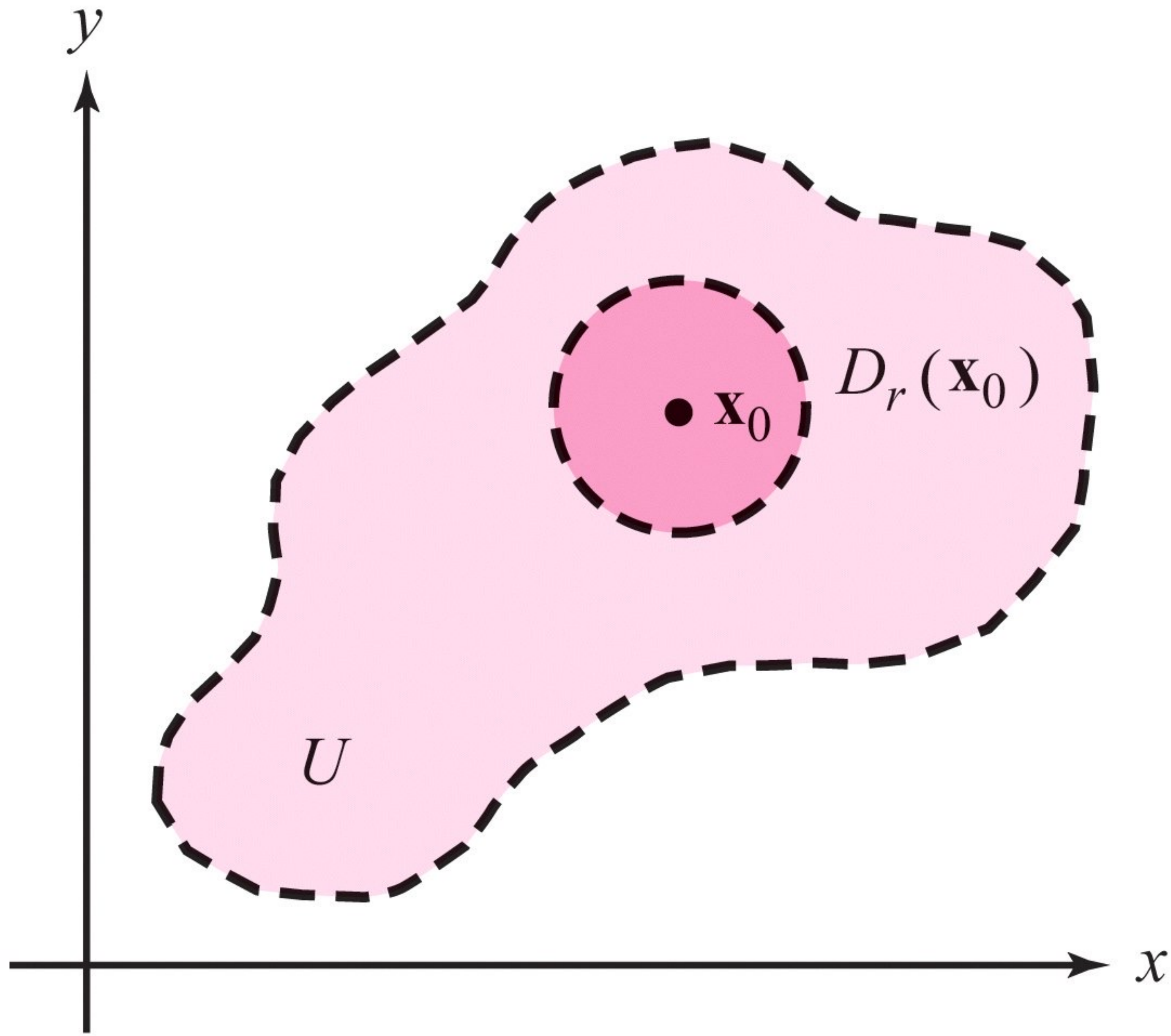


$n = 1$



$n = 2$


z
$D_r(\mathbf{x}_0)$
r
$\mathbf{x}_0$
y
x
$n = 3$



y
$\mathbf{x}_0$
$D_r(\mathbf{x}_0)$
U
x

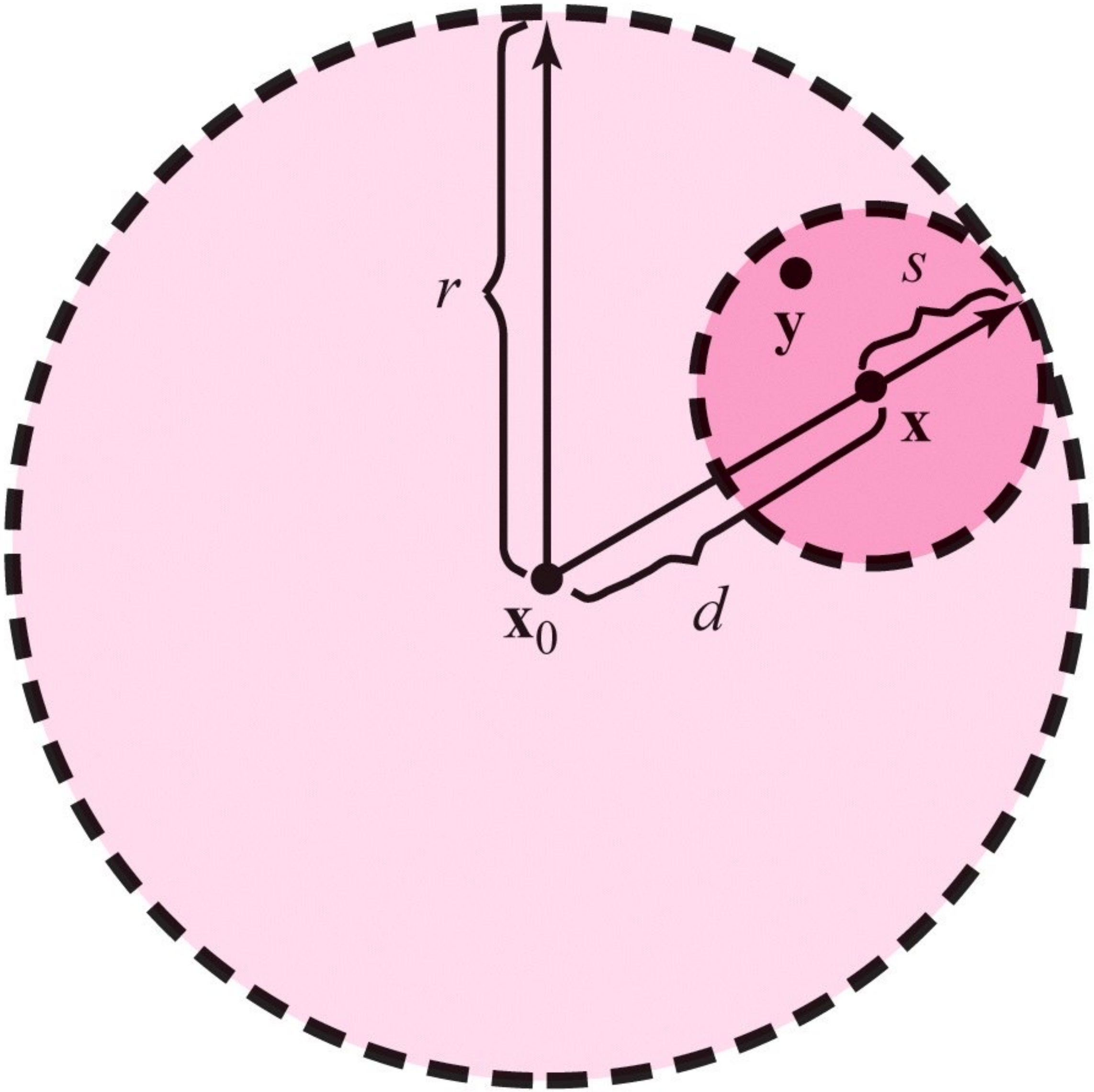
THEOREM 1 For each $\mathbf{x}_0 \in \mathbb{R}^n$ and $r > 0$, $D_r(\mathbf{x}_0)$ is an open set.

proof Let $\mathbf{x} \in D_r(\mathbf{x}_0)$; that is, let $\|\mathbf{x} - \mathbf{x}_0\| < r$. According to the definition of an open set, we must find an $s > 0$ such that $D_s(\mathbf{x}) \subset D_r(\mathbf{x}_0)$. Referring to Figure 2.2.3, we see that $s = r - \|\mathbf{x} - \mathbf{x}_0\|$ is a reasonable choice; note that $s > 0$, but that s becomes smaller if $\mathbf{x}$ is nearer the edge of $D_r(\mathbf{x}_0)$.

To prove that $D_s(\mathbf{x}) \subset D_r(\mathbf{x}_0)$, let $\mathbf{y} \in D_s(\mathbf{x})$; that is, let $\|\mathbf{y} - \mathbf{x}\| < s$. We want to prove that $\mathbf{y} \in D_r(\mathbf{x}_0)$ as well. Proving this, in view of the definition of an r-disk, entails showing that $\|\mathbf{y} - \mathbf{x}_0\| < r$. This is done by using the triangle inequality for vectors in $\mathbb{R}^n$:

$$\|\mathbf{y} - \mathbf{x}_0\| = \|(\mathbf{y} - \mathbf{x}) + (\mathbf{x} - \mathbf{x}_0)\| \le \|\mathbf{y} - \mathbf{x}\| + \|\mathbf{x} - \mathbf{x}_0\| < s + \|\mathbf{x} - \mathbf{x}_0\| = r.$$

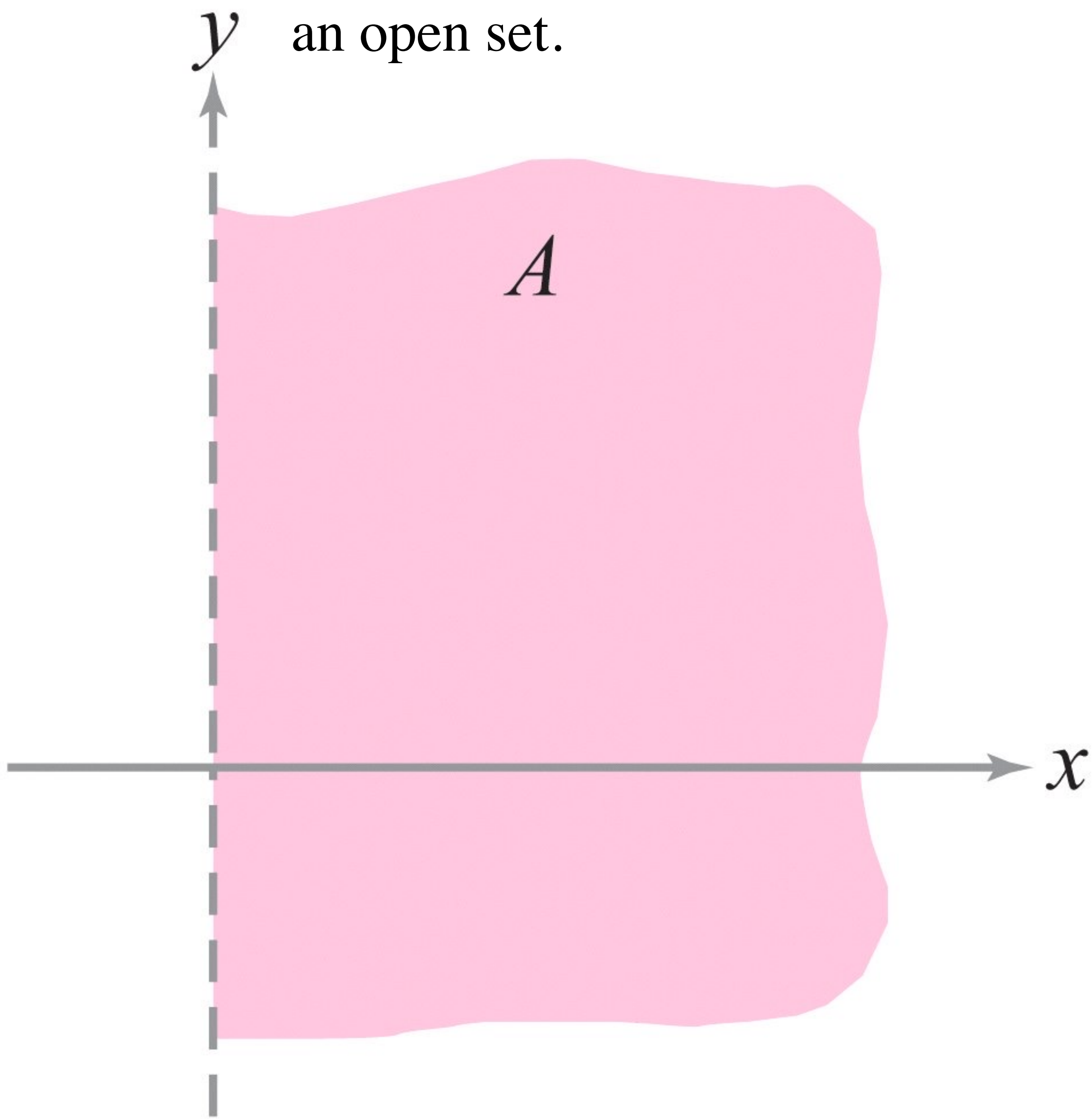
Hence, $\|\mathbf{y} - \mathbf{x}_0\| < r$. ■



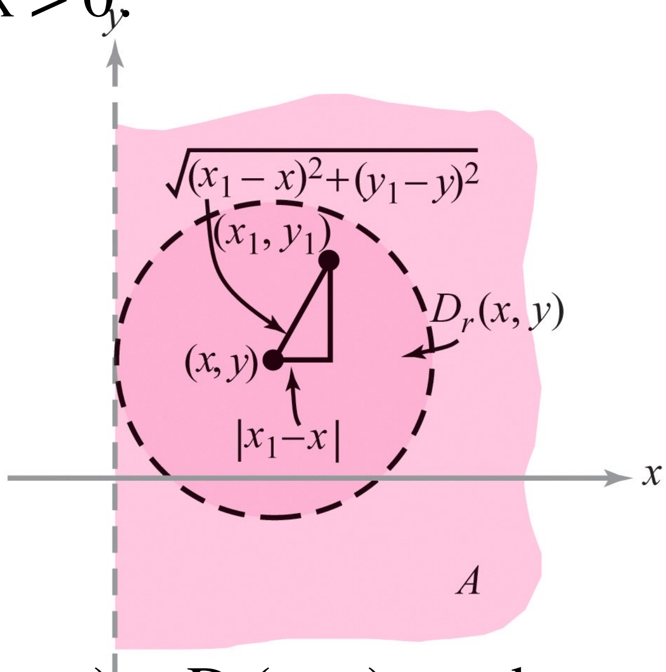
$$d = \|\mathbf{x} - \mathbf{x}_0\|$$

$$s = r - \|\mathbf{x} - \mathbf{x}_0\|$$

Exercise: Prove that $A = \{(x, y) \in R^2 \mid x > 0\}$ is an open set.



To prove that A is open, we have to show that for every point $(x, y) \in A$ there exists an $r > 0$ such that $D_r(x, y) \subset A$. If $(x, y) \in A$, then $x > 0$.



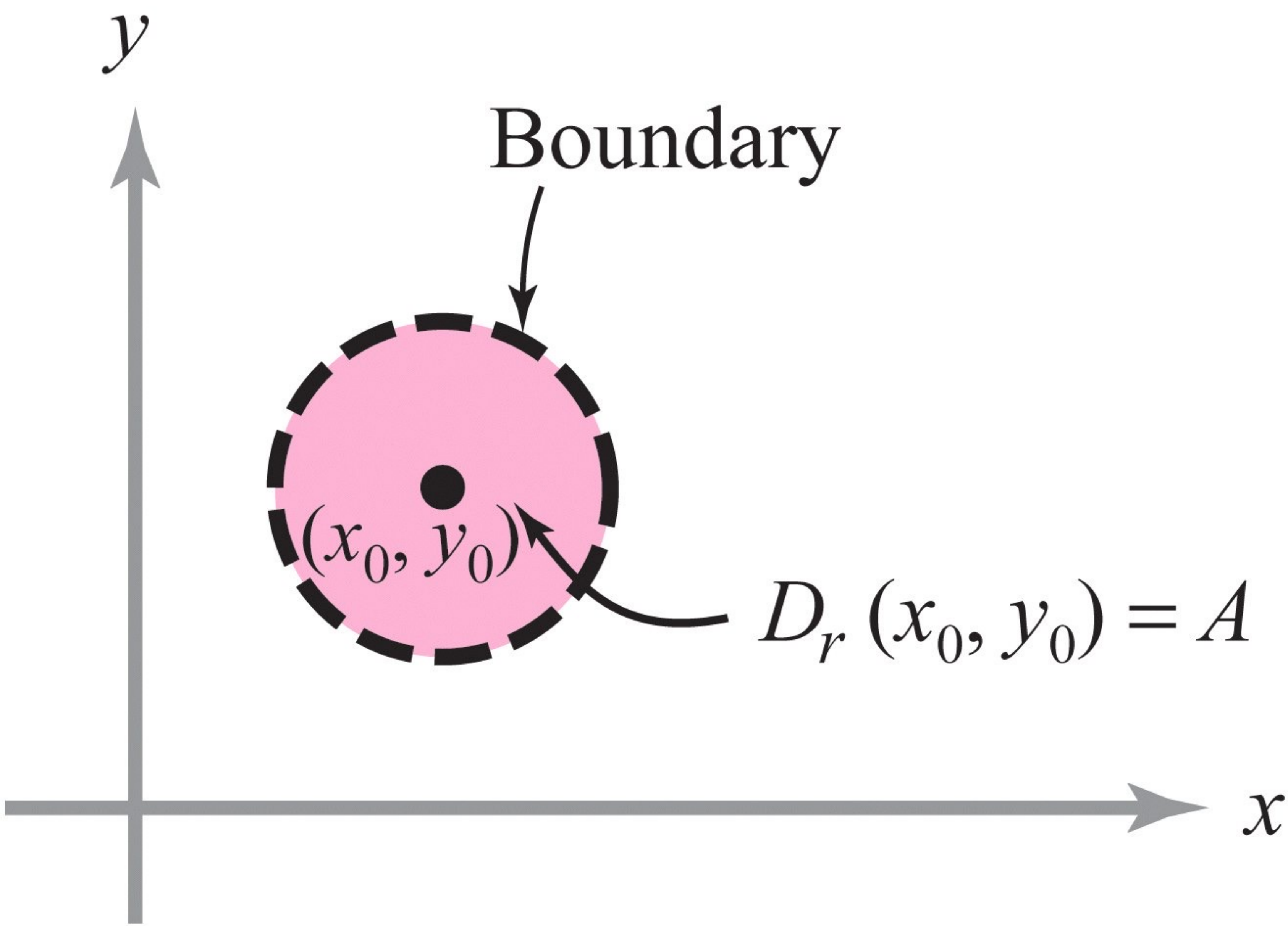
Choose $r = x$. If $(x_1, y_1) \in D_r(x, y)$, we have

$$|x_1 - x| = \sqrt{(x_1 - x)^2} \le \sqrt{(x_1 - x)^2 + (y_1 - y)^2} < r = x,$$

and so $x_1 - x < x$ and $x - x_1 < x$. The latter inequality implies $x_1 > 0$, that is, $(x_1, y_1) \in A$.

DEFINITION: Boundary Points Let $A \subset \mathbb{R}^n$. A point $\mathbf{x} \in \mathbb{R}^n$ is called a ***boundary point*** of A if every neighborhood of $\mathbf{x}$ contains at least one point in A and at least one point not in A.

Boundary points
a
b
x



y
Boundary
(x_0, y_0)
$D_r(x_0, y_0) = A$
x

DEFINITION: Limit Let $f\colon A \subset \mathbb{R}^n \to \mathbb{R}^m$, where A is an open set. Let $\mathbf{x}_0$ be in A or be a boundary point of A, and let N be a neighborhood of $\mathbf{b} \in \mathbb{R}^m$. We say f is ***eventually in*** N ***as*** $\mathbf{x}$ ***approaches*** $\mathbf{x}_0$ if there exists a neighborhood U of $\mathbf{x}_0$ such that $\mathbf{x} \neq \mathbf{x}_0$, $\mathbf{x} \in U$, and $\mathbf{x} \in A$ imply $f(\mathbf{x}) \in N$. [The geometric meaning of this assertion is illustrated in Figure 2.2.8; note that $\mathbf{x}_0$ need not be in the set A, so that $f(\mathbf{x}_0)$ is not necessarily defined.] We say $f(\mathbf{x})$ ***approaches*** $\mathbf{b}$ as $\mathbf{x}$ approaches $\mathbf{x}_0$, or, in symbols,

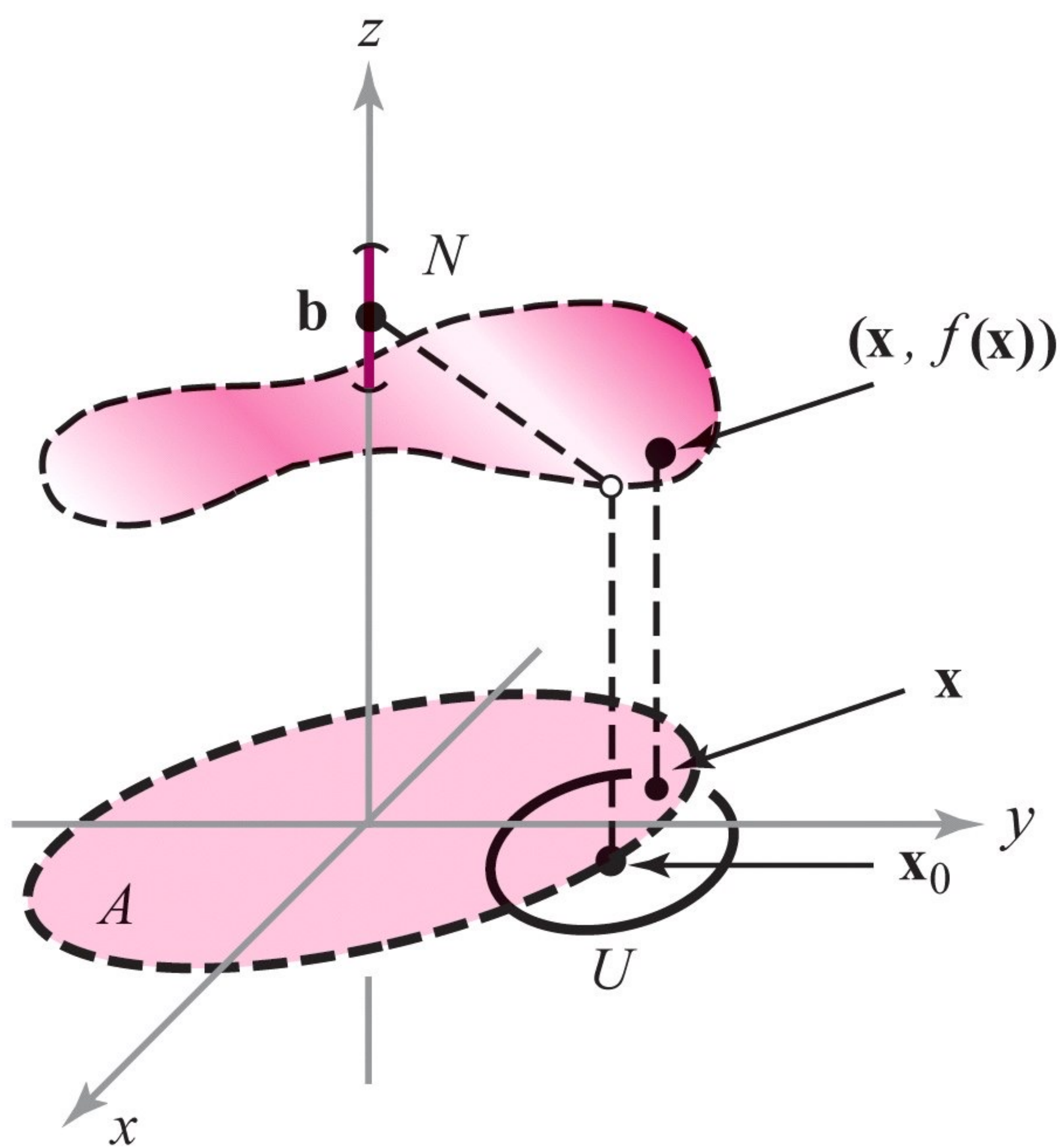
$$\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = \mathbf{b} \qquad \text{or} \qquad f(\mathbf{x}) \to \mathbf{b} \quad \text{as} \quad \mathbf{x} \to \mathbf{x}_0,$$

when, given *any* neighborhood N of $\mathbf{b}$, f is eventually in N as $\mathbf{x}$ approaches $\mathbf{x}_0$ [that is, "$f(\mathbf{x})$ is close to $\mathbf{b}$ if $\mathbf{x}$ is close to $\mathbf{x}_0$"]. It may be that as $\mathbf{x}$ approaches $\mathbf{x}_0$, the values $f(\mathbf{x})$ do not get close to any particular number. In this case, we say that $\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x})$ ***does not exist***.

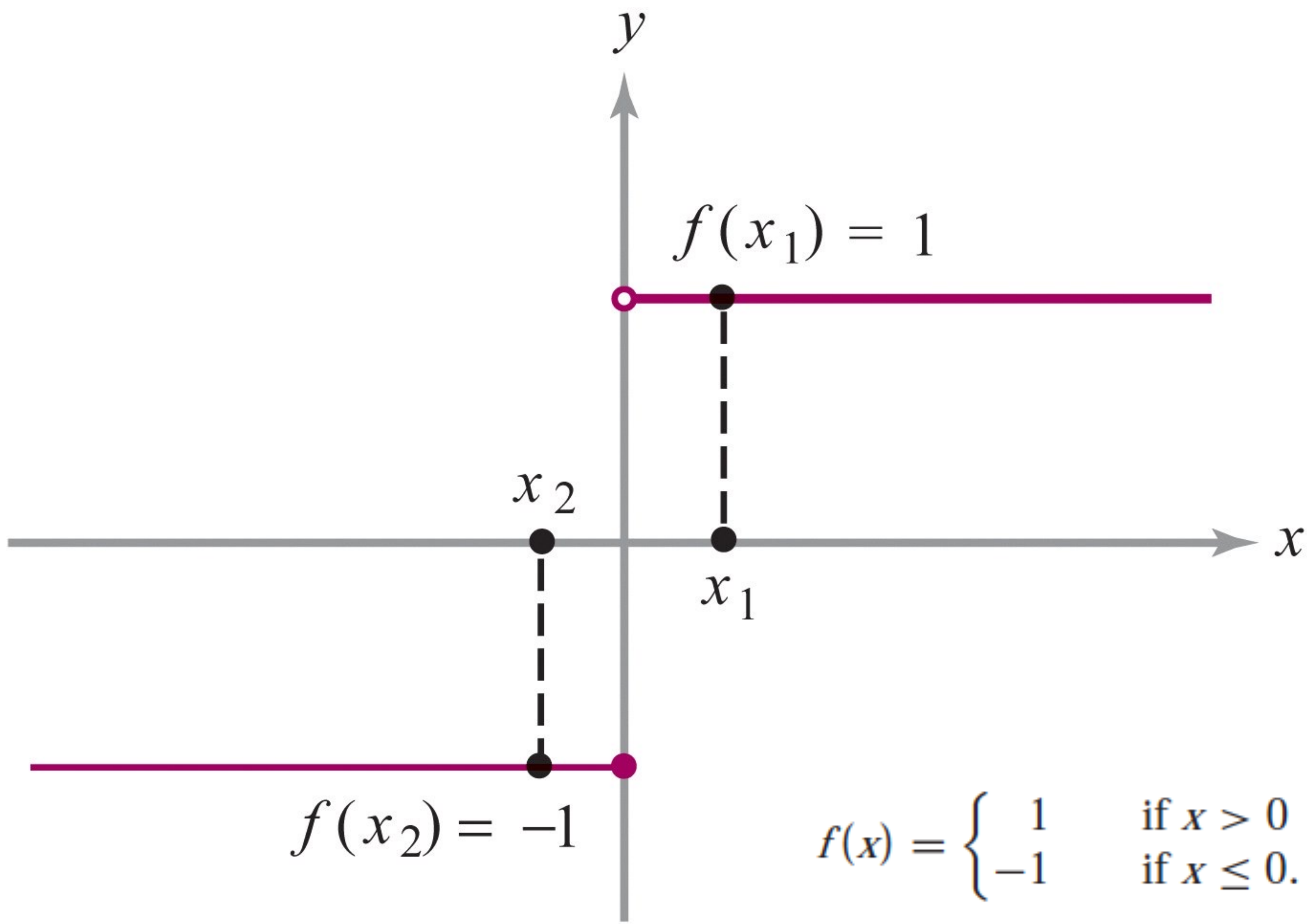
Theorem 6. *Let $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$ and let $\mathbf{x}_0$ be in A or be a boundary point of A. Then $\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = \mathbf{b}$ if and only if for every number $\varepsilon > 0$ there is a $\delta > 0$ such that for any $\mathbf{x} \in A$ satisfying $0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta$, we have $\|f(\mathbf{x}) - \mathbf{b}\| < \varepsilon$.*

Proof. First let us assume that $\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = \mathbf{b}$. Let $\varepsilon > 0$ be given, and consider the ε neighborhood $N = D_\varepsilon(\mathbf{b})$, the ball or disk of radius ε with center $\mathbf{b}$. By the definition of a limit, f is eventually in $D_\varepsilon(\mathbf{b})$, as $\mathbf{x}$ approaches $\mathbf{x}_0$, which means there is a neighborhood U of $\mathbf{x}_0$ such that $f(\mathbf{x}) \in D_\varepsilon(\mathbf{b})$ if $\mathbf{x} \in U$, $\mathbf{x} \in A$, and $\mathbf{x} \neq \mathbf{x}_0$. Now since U is open and $\mathbf{x}_0 \in U$, there is a $\delta > 0$ such that $D_\delta(\mathbf{x}_0) \subset U$. Consequently, $0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta$ and $\mathbf{x} \in A$ implies $\mathbf{x} \in D_\delta(\mathbf{x}_0) \subset U$. Thus $f(\mathbf{x}) \in D_\varepsilon(\mathbf{b})$, which means that $\|f(\mathbf{x}) - \mathbf{b}\| < \varepsilon$. This is the ε-δ assertion we wanted to prove.

We now prove the converse. Assume that for every $\varepsilon > 0$ there is a $\delta > 0$ such that $0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta$ and $\mathbf{x} \in A$ implies $\|f(\mathbf{x}) - \mathbf{b}\| < \varepsilon$. Let N be a neighborhood of $\mathbf{b}$. We have to show that f is eventually in N as $\mathbf{x} \to \mathbf{x}_0$; that is, we must find an open set $U \subset \mathbb{R}^n$ such that $\mathbf{x} \in U, \mathbf{x} \in A$, and $\mathbf{x} \neq \mathbf{x}_0$ implies $f(\mathbf{x}) \in N$. Now since N is open, there is an $\varepsilon > 0$ such that $D_\varepsilon(\mathbf{b}) \subset N$. If we choose $U = D_\delta(\mathbf{x})$ (according to our assumption), then $\mathbf{x} \in U$, $\mathbf{x} \in A$ and $\mathbf{x} \neq \mathbf{x}_0$ means $\|f(\mathbf{x}) - \mathbf{b}\| < \varepsilon$, that is $f(\mathbf{x}) \in D_\varepsilon(\mathbf{b}) \subset N$. ■

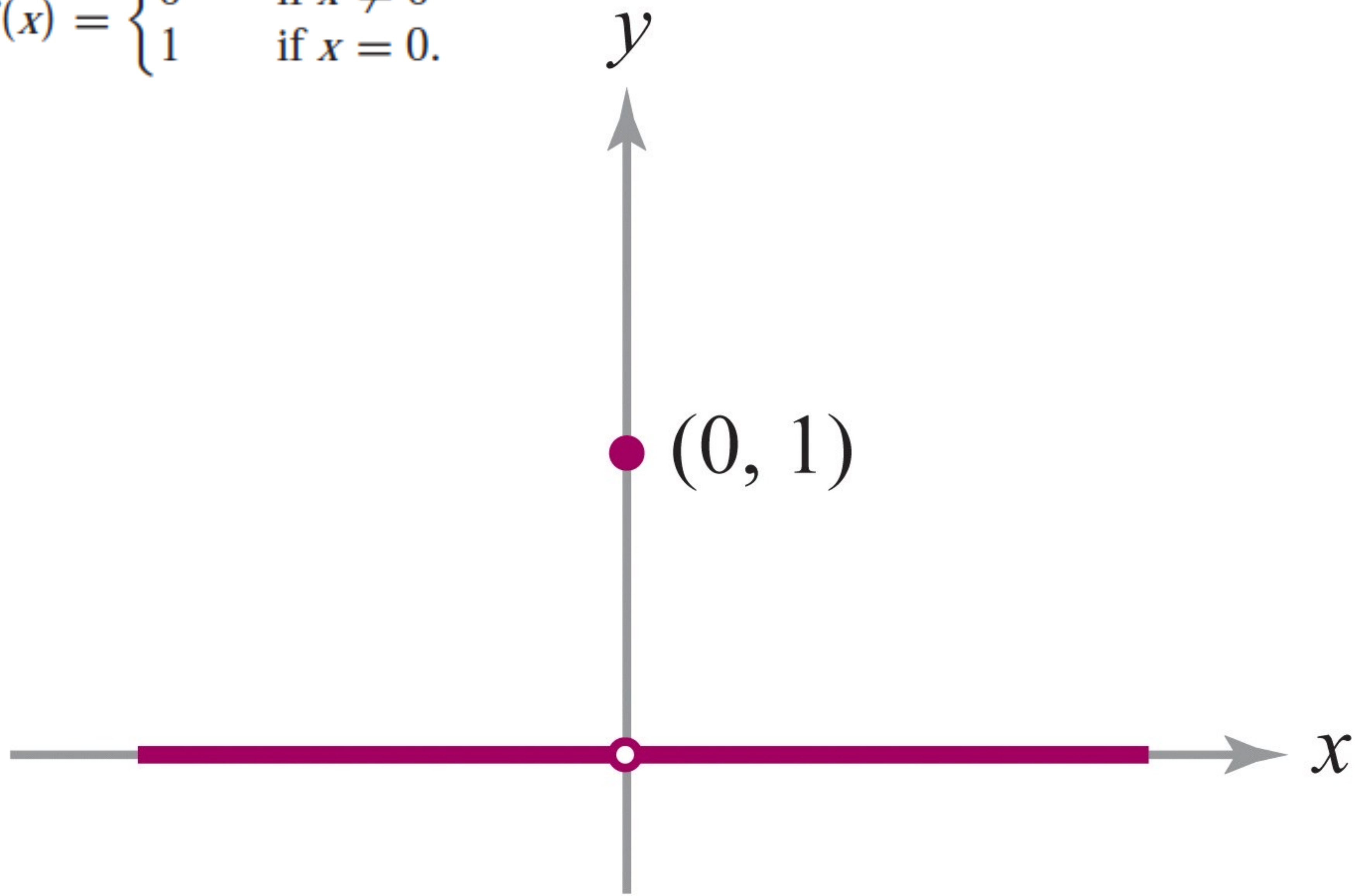


z
N
b
(x, f(x))
x
y
x_0
A
U
x



y
$f(x_1) = 1$
x_2
x
x_1
$f(x_2) = -1$
$f(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x \leq 0. \end{cases}$

$$f(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0. \end{cases}$$



PARENTESI

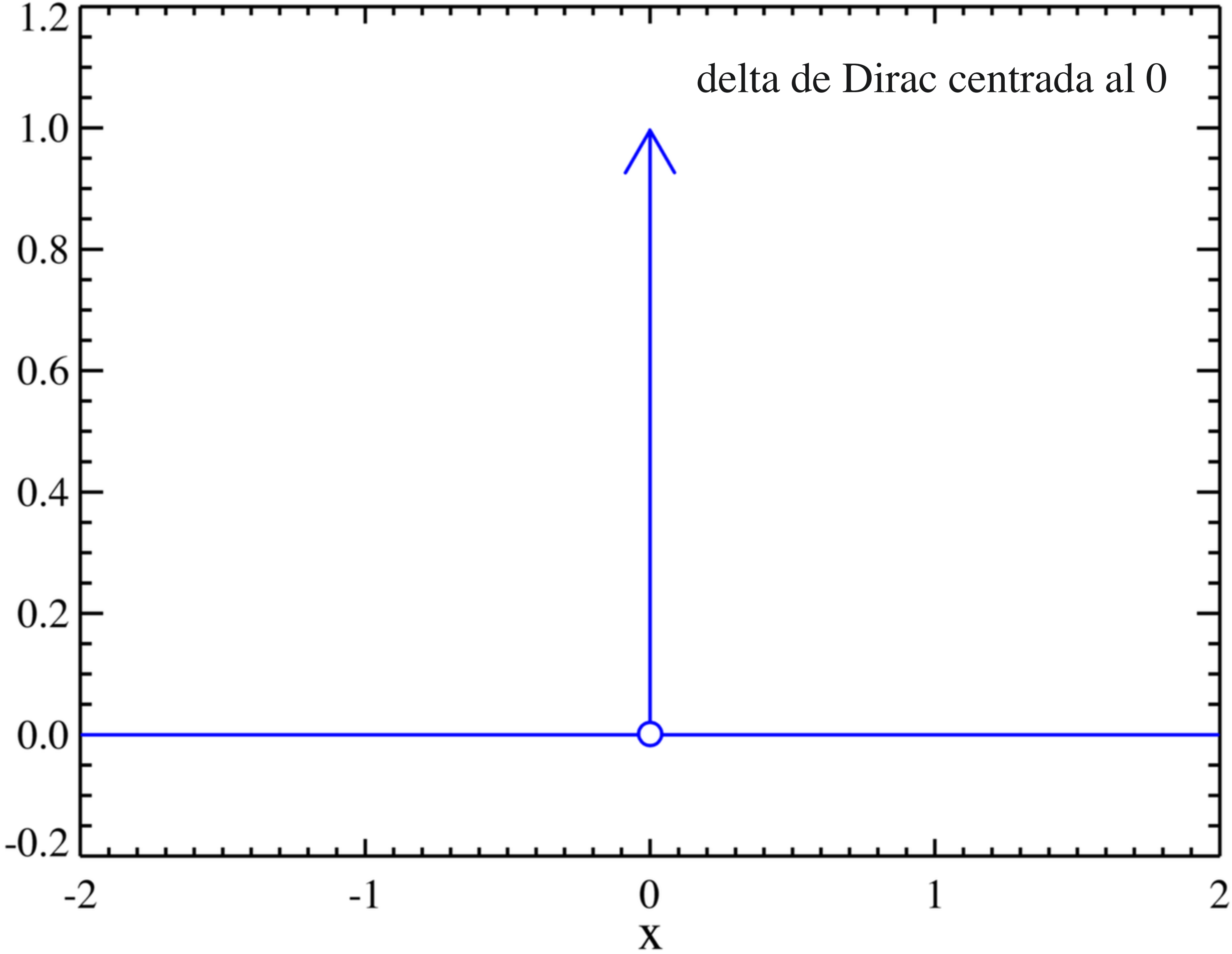
La delta de Dirac es una funció genealitzada definida com:

$$\int_{-\infty}^{\infty} \delta(x-a) f(x)\, dx = f(a) \qquad \left[\text{e.g.} \int_{-\infty}^{\infty} \delta(x-a)\, dx = 1\right]$$

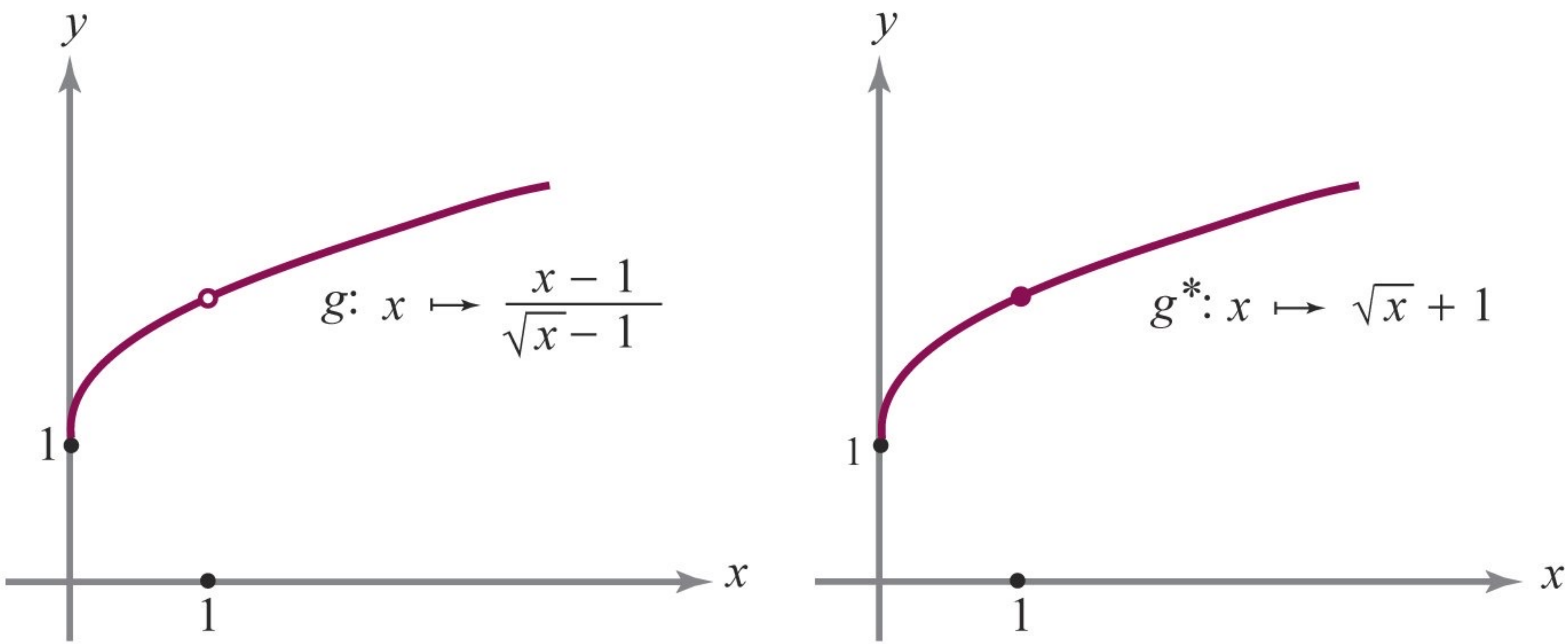
La delta de Dirac no es una funció estrictament parlant, sinò una distribució (o funció generalitzada).

De vegades, informalment, s'expressa la delta de Dirac com el límit d'una successió de funcions que tendeix a zero a tot punt de l'espai excepte en un punt per al qual divergiria cap a infinit; per això la "expressió informal" com funció definida a trams:

$$\delta(x) = \begin{cases} \infty, & x = 0 \\ 0, & x \neq 0 \end{cases};$$



delta de Dirac centrada al 0
1.2
1.0
0.8
0.6
0.4
0.2
0.0
-0.2
-2
-1
0
1
2
x



y
1
x
1
$g\colon x \longmapsto \dfrac{x-1}{\sqrt{x}-1}$
y
1
x
1
$g^{*}\colon x \longmapsto \sqrt{x}+1$

(a) (b)

THEOREM 2: Uniqueness of Limits

$$\text{If} \qquad \operatorname*{limit}_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = \mathbf{b}_1 \qquad \text{and} \qquad \operatorname*{limit}_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = \mathbf{b}_2, \qquad \text{then} \qquad \mathbf{b}_1 = \mathbf{b}_2.$$

THEOREM 3: Properties of Limits Let $f\colon A \subset \mathbb{R}^n \to \mathbb{R}^m$, $g\colon A \subset \mathbb{R}^n \to \mathbb{R}^m$, $\mathbf{x}_0$ be in A or be a boundary point of A, $\mathbf{b} \in \mathbb{R}^m$, and $c \in \mathbb{R}$; then

(i) If $\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = \mathbf{b}$, then $\lim_{\mathbf{x}\to\mathbf{x}_0} cf(\mathbf{x}) = c\mathbf{b}$, where $cf\colon A \to \mathbb{R}^m$ is defined by $\mathbf{x} \mapsto c(f(\mathbf{x}))$.

(ii) If $\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = \mathbf{b}_1$ and $\lim_{\mathbf{x}\to\mathbf{x}_0} g(\mathbf{x}) = \mathbf{b}_2$, then $\lim_{\mathbf{x}\to\mathbf{x}_0} (f+g)(\mathbf{x}) = \mathbf{b}_1 + \mathbf{b}_2$, where $(f+g)\colon A \to \mathbb{R}^m$ is defined by $\mathbf{x} \mapsto f(\mathbf{x}) + g(\mathbf{x})$.

(iii) If $m = 1$, $\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = b_1$, and $\lim_{\mathbf{x}\to\mathbf{x}_0} g(\mathbf{x}) = b_2$, then $\lim_{\mathbf{x}\to\mathbf{x}_0} (fg)(\mathbf{x}) = b_1 b_2$, where $(fg)\colon A \to \mathbb{R}$ is defined by $\mathbf{x} \mapsto f(\mathbf{x})g(\mathbf{x})$.

(iv) If $m = 1$, $\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = b \neq 0$, and $f(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in A$, then $\lim_{\mathbf{x}\to\mathbf{x}_0} 1/f(\mathbf{x}) = 1/b$, where $1/f\colon A \to \mathbb{R}$ is defined by $\mathbf{x} \mapsto 1/f(\mathbf{x})$.

(v) If $f(\mathbf{x}) = (f_1(\mathbf{x}), \ldots, f_m(\mathbf{x}))$ where $f_i\colon A \to \mathbb{R}$, $i = 1, \ldots, m$, are the component functions of f, then $\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = \mathbf{b} = (b_1, \ldots, b_m)$ if and only if $\lim_{\mathbf{x}\to\mathbf{x}_0} f_i(\mathbf{x}) = b_i$ for each $i = 1, \ldots, m$.

2.2 Limits and Continuity

Key Points in this Section.

1. A set $U \subset \mathbb{R}^n$ is ***open*** when, for every point $\mathbf{x}_0 \in U$, there is an $r > 0$ such that $D_r(\mathbf{x}_0) \subset U$. Here, $D_r(\mathbf{x}_0)$ is the ***open disk***, consisting of all points $\mathbf{x} \in \mathbb{R}^n$ such that $\| \mathbf{x} - \mathbf{x}_0 \| < r$. Open disks themselves are open sets.

2. A ***neighborhood*** of a point $\mathbf{x} \in \mathbb{R}^n$ is an open set containing $\mathbf{x}$.

3. A ***boundary point*** of a set $A \subset \mathbb{R}^n$ is a point $\mathbf{x} \in \mathbb{R}^n$ such that every neighborhood of $\mathbf{x}$ contains a point in A and a point not in A.

4. **Limits**. Let $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$ and $\mathbf{x}_0$ be in A or be a boundary point of A and let $\mathbf{b} \in \mathbb{R}^m$. When we write

$$\lim_{\mathbf{x} \to \mathbf{x}_0} f(\mathbf{x}) = \mathbf{b}$$

we mean that for any neighborhood N of $\mathbf{b}$, there is a neighborhood U of $\mathbf{x}_0$ such that if $\mathbf{x} \in A \cap U$, then $f(\mathbf{x}) \in N$.

5. Limits, if they exist, are unique. Also, the properties of limits from one-variable calculus (such as: the limit of a sum is the sum of the limits) also hold for functions of several variables.