6.1

$V(r)$ that leads to $r = r_0 e^{a\theta}$, $E = 0$

$$\frac{m}{2}\dot{r}^2 + \left(\frac{L^2}{2mr^2} + V(r)\right) = E = 0$$

$$\frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \frac{L}{mr^2}\frac{dr}{d\theta} = \left(\frac{L}{mr^2}\right) ar = \frac{La}{mr}$$

r(θ)

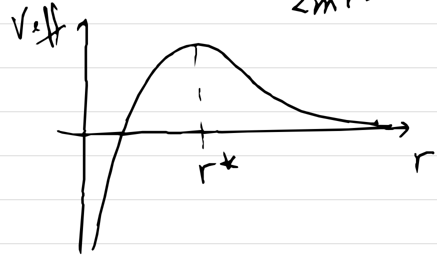
$$\frac{dr}{d\theta} = r_0 a e^{a\theta} = ar$$

$$\Rightarrow \frac{\cancel{m}}{2}\frac{L^2 a^2}{m^{\cancel{2}} r^2} + \frac{L^2}{2mr^2} + V(r) = 0 \Rightarrow \boxed{V(r) = -\frac{L^2}{2mr^2}(a^2+1)}$$

6.2

Potential $V(r) = -\dfrac{C}{3r^3}$

a) Draw $V_{eff} = \dfrac{L^2}{2mr^2} - \dfrac{C}{3r^3}$



b) L, V_{eff} max?

$$V_{eff} = \frac{L^2}{2mr^2} - \frac{C}{3r^3}$$

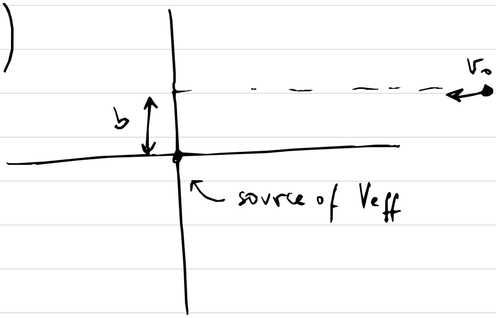
$$\frac{dV_{eff}}{dr} = \frac{-2L^2}{2mr^3} - \frac{-3C}{3r^4} = \frac{C}{r^4} - \frac{L^2}{mr^3} = \frac{1}{r^3}\left(\frac{C}{r} - \frac{L^2}{m}\right)$$

$$= 0 \Rightarrow \frac{C}{r^*} = \frac{L^2}{m} \Rightarrow r^* = \frac{Cm}{L^2}$$

$$\boxed{V_{eff}(r^*)} = \frac{L^2}{2m\frac{C^2m^2}{L^4}} - \frac{C}{3\frac{C^3m^3}{L^6}} = \frac{L^6}{2m^3C^2} - \frac{L^6}{3C^2m^3} =$$

$$= \boxed{\frac{L^6}{6C^2m^3}}$$

c)



$\vec{L} = \vec{r} \times \vec{p}$; $L_z = x p_y - y p_x$
$= b m v_0$

At infinity : $E = E_\infty = \frac{m}{2} v_0^2$, $L = L_\infty = m b v_0$

The condition for capture is $E > V_{eff}(r^*)$, that is (*)

$$\frac{(b m v_0)^6}{6 G^2 m^3} < \frac{m v_0^2}{2} \Rightarrow \frac{b^6 m^3 v_0^6}{6 G^2} < \frac{m v_0^2}{2}$$

$$\Rightarrow b^6 < \frac{3 G^2}{m^2 v_0^4} \Rightarrow \boxed{b_{max} = \left(\frac{3 G^2}{m^2 v_0^4}\right)^{1/6}}$$

d) The cross section for capture is

$$\boxed{\sigma = \pi b_{max}^2 = \pi \left(\frac{3 G^2}{m^2 v_0^4}\right)^{1/3}}.$$

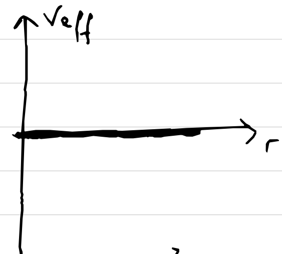
(*) Otherwise, the particle "bounces" against the potential.

6.3

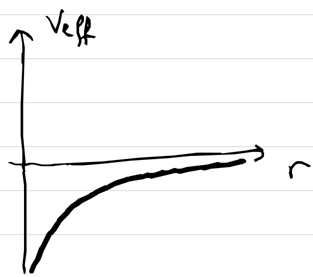
$$V(r) = \frac{\beta}{r^2}$$

a) V_{eff}? $V_{eff} = \frac{L^2}{2mr^2} + \frac{\beta}{r^2} = \frac{1}{r^2}\left(\frac{L^2}{2m} + \beta\right)$

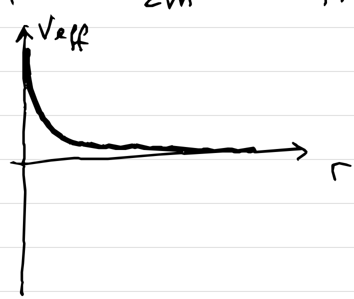
- $\beta = -\frac{L^2}{2m} \Rightarrow V_{eff} = 0$



- $\beta < -\frac{L^2}{2m} \Rightarrow V_{eff} = -\frac{a^2}{r^2}$



- $\beta > -\frac{L^2}{2m} \Rightarrow V_{eff} = \frac{a^2}{r^2}$



b) $r(\theta)$?

$A = \frac{L^2}{2m} + \beta$

Conservation of energy:

$$\frac{m}{2}\dot{r}^2 + V_{eff}(r) = E \Rightarrow \frac{m}{2}\dot{r}^2 + \frac{A}{r^2} = E$$

$$\frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \frac{dr}{d\theta}\frac{L}{mr^2} \qquad \Rightarrow$$

$$\dot{\theta} = \frac{L}{mr^2}$$

$$\Rightarrow \frac{m}{2}\frac{L^2}{m^2}\left(\frac{1}{r^2}\frac{dr}{d\theta}\right)^2 + \frac{A}{r^2} = E$$

Now, change to $u = \frac{1}{r} \Rightarrow \frac{du}{d\theta} = \frac{du}{dr}\frac{dr}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta}$, so that we have

$$\frac{L^2}{2m}\left(\frac{du}{d\theta}\right)^2 + Au^2 = E$$

Now, taking $\frac{d}{d\theta}$ of the previous expression, we have

$$\frac{L^2}{m}\left(\frac{du}{d\theta}\right)\frac{d^2u}{d\theta^2} + 2Au\left(\frac{du}{d\theta}\right) = 0$$ which, in general, requires:

$$\frac{L^2}{m}\frac{d^2u}{d\theta^2} + 2Au = 0 \Rightarrow \boxed{\frac{d^2u}{d\theta^2} + \frac{2Am}{L^2}u = 0}$$

c) • Case $A=0$

$$\frac{d^2u}{d\theta^2} = 0 \Rightarrow u = C_1\theta + C_2 \Rightarrow \frac{1}{r} = C_1\theta + C_2$$

By choosing the origin wisely we can make $C_2 = 0$

Then $\frac{du}{d\theta} = C_1 \Rightarrow \frac{L^2}{2m} C_1^2 = E \Rightarrow C_1 = \sqrt{\frac{2mE}{L^2}}$

$$\Rightarrow \boxed{r = \frac{1}{\theta}\frac{L}{\sqrt{2mE}}}$$

• Case $A>0$

choosing initial conditions wisely, again

$$\frac{d^2u}{d\theta} + \frac{2Am}{L^2}u^2 = 0 \Rightarrow u = \frac{1}{r} = C_1 \sin a\theta \longrightarrow a = \sqrt{\frac{2Am}{L^2}}$$

• Case $A<0$

$$\frac{d^2u}{d\theta} - a^2u^2 = 0 \Rightarrow u = \frac{1}{r} = \begin{cases} c_1 \operatorname{sh} a\theta \\ \text{or} \\ c_1 \operatorname{ch} a\theta \end{cases}$$