Ex2.

a) $\langle N_0 \rangle = N = \dfrac{1}{e^{\beta(\varepsilon_0-\mu)}-1}$

$$N = \frac{1}{e^{-\beta\mu}-1} \Rightarrow e^{-\beta\mu}-1 = \frac{1}{N}$$

$$e^{-\beta\mu} = \frac{1}{N}+1 \qquad \log e^{-\beta\mu} = -\beta\mu = \log\left(\frac{1}{N}+1\right) \approx \frac{1}{N}$$

$$\boxed{\mu = -\frac{k_B T}{N}}$$

$e^{-\beta\mu} \approx \frac{1}{N}+1 \approx 1$ for large N

b) $g(\varepsilon) = a\varepsilon^2 \quad \varepsilon \geq 0$

$$\langle N \rangle_{ext} = \int_0^\infty g(\varepsilon) \frac{1}{e^{\beta(\varepsilon-\mu)}-1}\, d\varepsilon$$

$$\langle N \rangle_{ext} = \int_0^\infty a\varepsilon^2 \frac{1}{e^{\beta\varepsilon}-1}\, d\varepsilon$$

$$\boxed{\langle N_{ext} \rangle = \frac{a}{\beta^3} \cdot \int_0^\infty \frac{(\beta\varepsilon)^2\, d(\beta\varepsilon)}{e^{\beta\varepsilon}-1} = \frac{2a}{\beta^3}\zeta(3)}$$

c) $\langle N_{ext} \rangle \approx N = 2a k_B^3 T_c^3 \zeta(3)$

$$T_c = \left(\frac{N}{2a k_B^3} \frac{1}{\zeta(3)}\right)^{1/3}.$$

Ex. 1

$\omega \sim \lambda^{-s} \sim k^{+s}$

We redo al calculations for the expected energy using Bose-Einstein statistics + the Debye approximation.

We will have that

$$\langle E \rangle = \sum_{\varepsilon_i} \varepsilon_i \frac{1}{e^{\beta\varepsilon_i}-1} \rightarrow \int_0^\infty \frac{\varepsilon}{e^{\beta\varepsilon}-1} g(\varepsilon)\, d\varepsilon$$

We obtain $g(\varepsilon)$ as for the 3D case:

$$2 \times \int d\vec{n} = \frac{1}{4} \times 2 \times \int dn\, n\, 2\pi = 4\pi \int n\, dn = \frac{4\pi L^2}{\pi^2}\int k\, dk$$

↑ polarization modes

$$k = \sqrt{\left(\frac{n_x\pi}{L}\right)^2 + \left(\frac{n_y\pi}{L}\right)^2} = n\frac{\pi}{L}$$

in terms of the energy

$$\varepsilon_n = \hbar\omega_n = \hbar a k_n^s$$

$$k = \left(\frac{\varepsilon}{\hbar a}\right)^{1/s} \qquad dk = \left(\frac{\varepsilon}{\hbar a}\right)^{1/s-1} \frac{1}{s}\, d\varepsilon$$

$$\rightarrow \frac{4L^2}{4\pi}\int k\, dk = \frac{L^2}{\pi}\int \left(\frac{\varepsilon}{\hbar a}\right)^{1/s-1} \frac{1}{s} \left(\frac{\varepsilon}{\hbar a}\right)^{1/s} d\varepsilon$$

$$g(\varepsilon) = \frac{L^2}{\pi}\left(\frac{\varepsilon}{\hbar a}\right)^{2/s-1} \frac{1}{s}$$

$$\langle E \rangle = \int_0^{\hbar\omega_D} \varepsilon^{2/s} \frac{1}{s} \frac{L^2}{\pi} \left(\frac{1}{\hbar a}\right)^{2/s-1} \frac{1}{e^{\beta\varepsilon}-1}\, d\varepsilon$$

$$\langle E \rangle = \left\{ \begin{array}{l} x = \beta\varepsilon \\ dx = \beta d\varepsilon \end{array} \right\} = \frac{L^2}{s\pi}\left(\frac{1}{\hbar a}\right)^{2/s-1} \left(\frac{1}{\beta}\right)^{2/s+1} \int_0^{\beta\hbar\omega_D} \frac{x^{2/s}}{e^x-1}\, dx$$

At low T $\beta \rightarrow \infty$ so that

$$\langle E \rangle \approx \frac{L^2}{s\pi}\left(\frac{1}{\hbar a}\right)^{2/s-1} (k_B T)^{2/s+1} \underbrace{\int^{\hbar\omega_D} x^{2/s} e^{-x}\, dx}_{I(\hbar\omega_D)}$$

$$C_V = \left.\frac{\partial \langle E \rangle}{\partial T}\right)_V = k_B^{2/s+1} \frac{L^2}{s\pi}\left(\frac{1}{\hbar a}\right)^{2/s-1} \left(\frac{2}{s}+1\right) T^{2/s} I(\hbar\omega_D)$$

$C_V \sim T^{2/s}$ at low temp.

Ex. 3

$$(2s+1)\int d\vec{n}\; f(n) = \frac{1}{8}\left(\frac{L}{\pi}\right)^3 \int d\vec{k}\, f(k) = \frac{4\pi}{8}\left(\frac{L}{\pi}\right)^3 (2s+1) \int k^2\, dk\, f(k)$$

positive i negative

$$k = \sqrt{\frac{2m\varepsilon}{\hbar^2}} \qquad dk = \frac{1}{2}\frac{1}{\hbar}\sqrt{\frac{2m}{\varepsilon}}\, d\varepsilon$$

a) $$g(\varepsilon) = \frac{(2s+1)}{8}\frac{4V}{\pi^2}\frac{2m\varepsilon}{\hbar^2}\frac{1}{2\hbar}\sqrt{\frac{2m}{\varepsilon}}$$

$$= \frac{(2s+1)}{\hbar^3}\frac{V}{\pi^2} m^{3/2}\sqrt{\frac{\varepsilon}{2}}$$

b) $$\langle N_0 \rangle = \frac{1}{e^{-\beta\mu}-1}\cdot(2s+1) = (2s+1)\frac{e^{\beta\mu}}{1-e^{\beta\mu}}$$

$\varepsilon_0 = 0$

$$= (2s+1)\frac{z}{1-z}$$

c) $\phi = -PV \Rightarrow -PV = -k_B T \ln \mathcal{Z}$ grand free energy.

$$\beta PV = \ln \mathcal{Z} = (2s+1)\frac{V}{\hbar^3\pi^2} m^{3/2} \int d\varepsilon \sqrt{\frac{\varepsilon}{2}} \ln\left(1 - z e^{-\beta\varepsilon}\right)$$

$$+ (2s+1)\ln(1-z)$$

we need to consider specifically the contribution of the ground state.!

Note that in the large V, N limit the contribution of the ground state is vanishingly small.

Therefore, $$\lambda = \sqrt{\frac{2\pi\beta\hbar^2}{m}}$$

$$\beta PV \approx (2s+1)\frac{V}{\hbar^3\pi^2} m^{3/2} \int_0^\infty \frac{dx}{\beta}\frac{1}{\sqrt{2}}\sqrt{x}\frac{1}{\sqrt{\beta}} \ln\left(1 - z e^{-x}\right)$$

$$\beta PV = (2s+1)V\left(\frac{m}{\beta\hbar^2 2\pi}\right)^{3/2}\cdot\frac{2}{\sqrt{\pi}}\int dx\, x^{1/2}\ln(1 - z e^{-x})$$

$$\beta PV = (2s+1)V\frac{1}{\lambda^3}\frac{2}{\sqrt{\pi}} \boxed{\int dx\, x^{1/2} \ln(1 - \underbrace{z e^{-x}}_{})}$$

$$\int_0^\infty dx\, x^{1/2}\left(-\sum_{n=1}^{\infty}\frac{(ze^{-x})^n}{n}\right) = \sum_{n}\int_0^\infty dx\, \frac{x^{1/2}}{n} z^n e^{-xn} =$$

$$\sum_n \int_0^\infty dx\, x^{1/2}\frac{z^n e^{-xn}}{n} \overset{x=y^2}{=} \sum_n \frac{z^n}{n}\int_0^\infty dy\, 2y^2 e^{-y^2 n}$$

$$= \sum_n \frac{z^n}{n} I_n$$

$$I_n = \int_0^\infty dy\, 2y^2 e^{-y^2 n} = \cancel{-\frac{y}{n} e^{-y^2 n}\Big|_0^\infty} + \int_0^\infty \frac{e^{-y^2 n}}{n}$$

$$I_n = \frac{\sqrt{\pi}}{n} \qquad = \sqrt{\frac{\pi}{n}}\frac{1}{2}$$

$$\sum_n \frac{z^n}{n^2}\sqrt{\frac{\pi}{n}}\frac{1}{2} = \frac{\sqrt{\pi}}{2}\sum_n \frac{z^n}{n^{5/2}} = \frac{\sqrt{\pi}}{2} g_{5/2}(z)$$

Using this result

$$\beta P\cancel{V} = (2s+1)\cancel{V}\frac{1}{\lambda^3}\frac{2}{\sqrt{\pi}}\frac{\sqrt{\pi}}{2} g_{5/2}(z) = \frac{(2s+1)}{\lambda^3} V g_{5/2}(z)$$

$$\boxed{\beta P = (2s+1)\frac{1}{\lambda^3} g_{5/2}(z)}$$

equation of state.

EX 5.

$$g(\varepsilon) = D$$

$$N = \int_0^\infty g(\epsilon) f(\epsilon)\, d\varepsilon = \int_0^\infty D \frac{1}{1+e^{+\beta(\epsilon-\mu)}} d\varepsilon$$

$$= \int_0^\infty D \frac{e^{-\beta(\epsilon-\mu)}}{1+e^{-\beta(\epsilon-\mu)}} d\varepsilon = -\int_0^\infty \frac{D}{\beta} \frac{d}{d\varepsilon} \ln\left[1+e^{-\beta(\epsilon-\mu)}\right] d\varepsilon$$

$$= -\frac{D}{\beta} \ln\left[1+e^{-\beta(\epsilon-\mu)}\right]_0^\infty = +\frac{D}{\beta} \ln\left[1+e^{\beta\mu}\right]$$

$$\frac{N\beta}{D} = \ln\left[1+e^{\beta\mu}\right]$$

$$e^{N\beta/D} = 1+e^{\beta\mu} \qquad e^{\beta\mu} = e^{N\beta/D} - 1$$

R: $\boxed{\mu = k_B T \ln\left[e^{N\beta/D} - 1\right]}$.

at $T=0$ levels up to $\varepsilon_F = \mu(T=0)$ are occupied and the others are empty so that

$$N = \int_0^\infty g(\epsilon) f(\epsilon)\, d\varepsilon = \left\{ f(\epsilon) \begin{cases} 1 & \epsilon < \epsilon_F \\ 0 & \epsilon > \epsilon_F \end{cases} \right.$$

$$= \int_0^{\epsilon_F} D\, d\varepsilon = D\varepsilon_F \rightarrow \varepsilon_F = \mu(T=0) = N/D$$

$$\lim_{T\to 0} \mu = \lim_{T \to 0} k_B T \ln\left[e^{\beta N/D} - 1\right]$$

$$= \cancel{k_B T} \frac{N}{D \cancel{k_B T}} = \frac{N}{D}$$

Ex 6.

N $s=\frac{1}{2}$ fermions on a surface of area A

In this case we have 2 states per energy level

$$\mathcal{H}=\frac{\hbar^2k^2}{2m} \qquad \vec{k}=\left(\frac{\pi}{L}n_x, \frac{\pi}{L}n_y\right) \qquad k=\frac{\pi}{L}n$$

$$2\int d\vec{n} = \frac{2}{4}\int_{-\infty}^{\infty}dn_x\int_{-\infty}^{\infty}dn_y = 2\cdot 2\pi\int_0^\infty n\,dn$$

$$\frac{2\cdot 2\pi}{4}\left(\frac{L}{\pi}\right)^2\int_0^\infty k\,dk$$

To obtain the chemical potential we impose that:

$$N=\int_0^\infty \rho(\varepsilon)f(\varepsilon)\,d\varepsilon = N = \frac{A}{\pi}\int_0 dk\,k\,\frac{1}{1+e^{\beta\left(\frac{\hbar^2k^2}{2m}-\mu\right)}}$$

$$N=\frac{A}{\pi}\int_0^\infty dk\,k\,\frac{e^{-\beta\left(\frac{\hbar^2k^2}{2m}-\mu\right)}}{1+e^{-\beta\left(\frac{\hbar^2k^2}{2m}-\mu\right)}}$$

$$N=\frac{A}{\pi}\int_0^\infty dk\left(-\frac{1}{\frac{\beta\hbar^2}{2m}}\right)\cdot\frac{d}{dk}\left(\ln\left[1+e^{-\beta\left(\frac{\hbar^2k^2}{2m}-\mu\right)}\right]\right)$$

$$N=-\frac{A}{\pi}\frac{2m}{\beta\hbar^2}\ln\left[1+e^{-\beta\left(\frac{\hbar^2k^2}{2m}-\mu\right)}\right]\Bigg|_0^\infty$$

$$N=\frac{A}{\pi}\frac{2m}{\beta\hbar^2}\ln\left[1+e^{\beta\mu}\right]$$

$$e^{\frac{N\pi\beta\hbar^2}{2mA}} = 1+e^{\beta\mu}$$

$$\beta\mu=\ln\left(e^{\frac{N\pi\beta\hbar^2}{2mA}}-1\right)$$

at $T=0$ $\cancel{\beta}\mu=\frac{N\pi\cancel{\beta}\hbar^2}{2mA}$ $\qquad \epsilon_F=\frac{N\pi\hbar^2}{2mA}$

$$\boxed{\mu=k_BT\ln\left(e^{\epsilon_F\beta}-1\right)}$$

Ex 4.

a) $g(\varepsilon)$ will be the same as in exercise 3 with $s=\frac{1}{2}$

$$g(\epsilon) = \frac{2 V m^{3/2}\sqrt{\epsilon}}{\sqrt{2}\,\hbar^3\pi^2} = \frac{\sqrt{2}\, V\, m^{3/2}\sqrt{\epsilon}}{\hbar^3\pi^2}$$

b) Defining $g_0 = \frac{\sqrt{2}\,m^{3/2}}{\hbar^3\pi^2}$ $g(\epsilon) = g_0\sqrt{\varepsilon}$

we have that at $T=0$ only states up to $\epsilon=\epsilon_F$ are occupied

$$N = \int_0^{\infty} d\varepsilon\, g(\varepsilon) f(\varepsilon) = \int_0^{\epsilon_F} d\varepsilon\, g_0 \sqrt{\varepsilon}$$

$$f(\varepsilon) = \frac{1}{1+e^{-\beta(\epsilon-\mu)}} \begin{cases} 0 & \epsilon > \mu(T=0) = \epsilon_F \\ 1 & \epsilon < \mu(T=0) \end{cases}$$

$$N = \frac{2}{3}\varepsilon_F^{3/2} g_0 . V \rightarrow \frac{N}{V} = n_e$$

$$\varepsilon_F = \left(\frac{3}{2}\frac{n_e}{g_0}\right)^{2/3}$$

$T_F \Rightarrow$ temperature such that $k_B T_F = \varepsilon_F$

$$\boxed{T_F = \varepsilon_F / k_B}$$