Jerrold E. Marsden and Anthony J. Tromba

Vector Calculus

Fifth Edition

Chapter 6:
The Change of Variables Formula and Applications of Integration

6.1 The Geometry of Maps…

6.1 The Geometry of Maps from $\mathbb{R}^2$ to $\mathbb{R}^2$

Key Points in this Section.

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1. A ***mapping*** T of a region D^* in $\mathbb{R}^2$ to $\mathbb{R}^2$ associates to each point (u, v) in D^* a point $(x, y) = T(u, v)$. The set of all such (x, y) is the ***image*** domain $D = T(D^*)$.

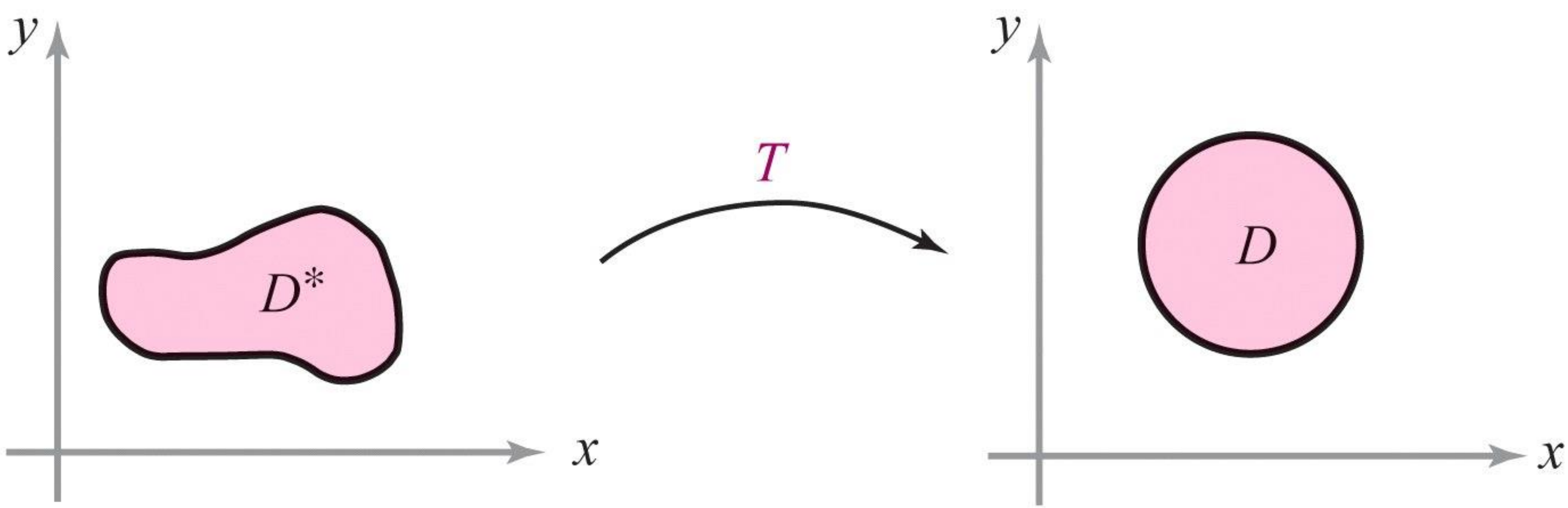
2. If T is ***linear***; that is if $T(u, v) = A \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]$, where A is a 2×2 matrix (and identifying points (u, v) with column vectors $\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]$), then T maps parallelograms to parallelograms, mapping the sides and vertices of the first, to those of the second.

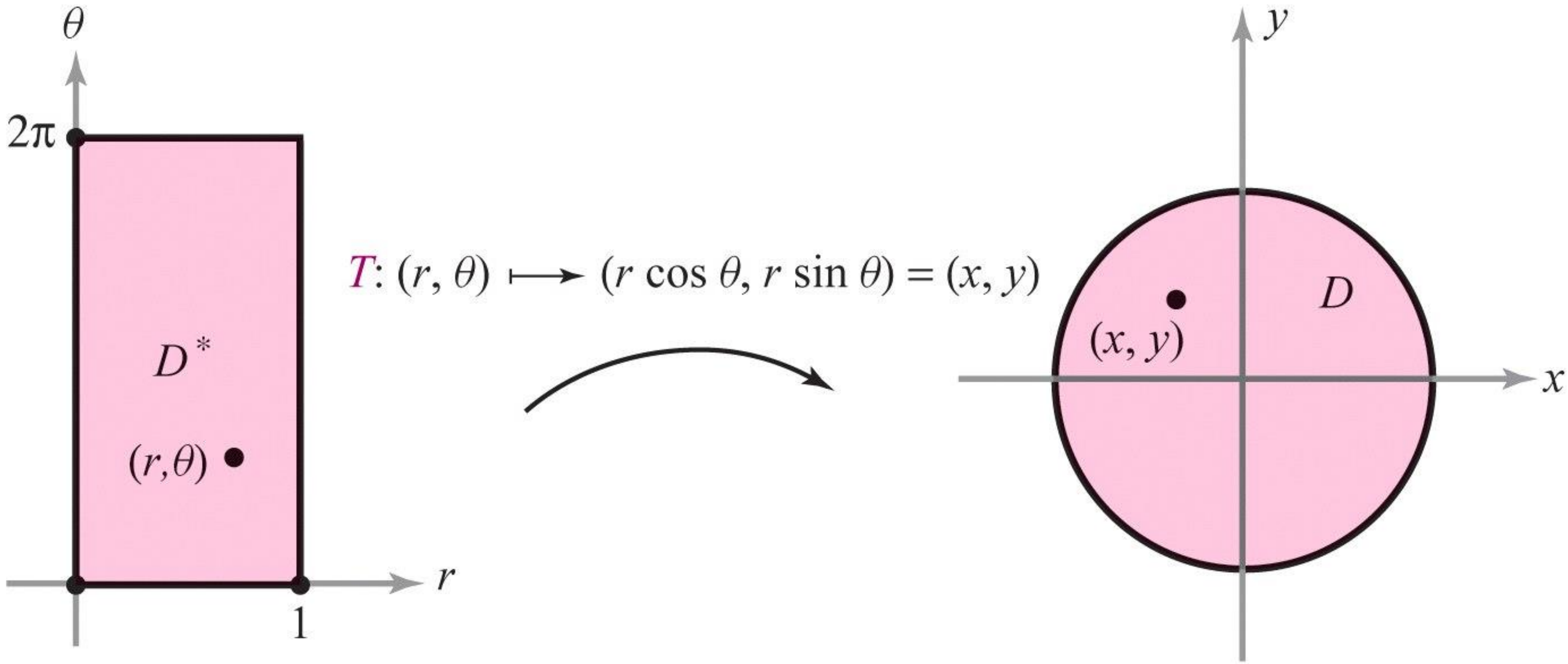
3. A map T is called ***one-to-one*** if different points (that is, $(u, v) \neq (u', v')$) get sent to different points (that is $T(u, v) \neq T(u', v')$).

4. If T is linear, determined by a 2×2 matrix A, then T is one-to-one when $\det A \neq 0$.

5. When D is the image of T; that is, $D = T(D^*)$, we say T maps D^* ***onto*** D.

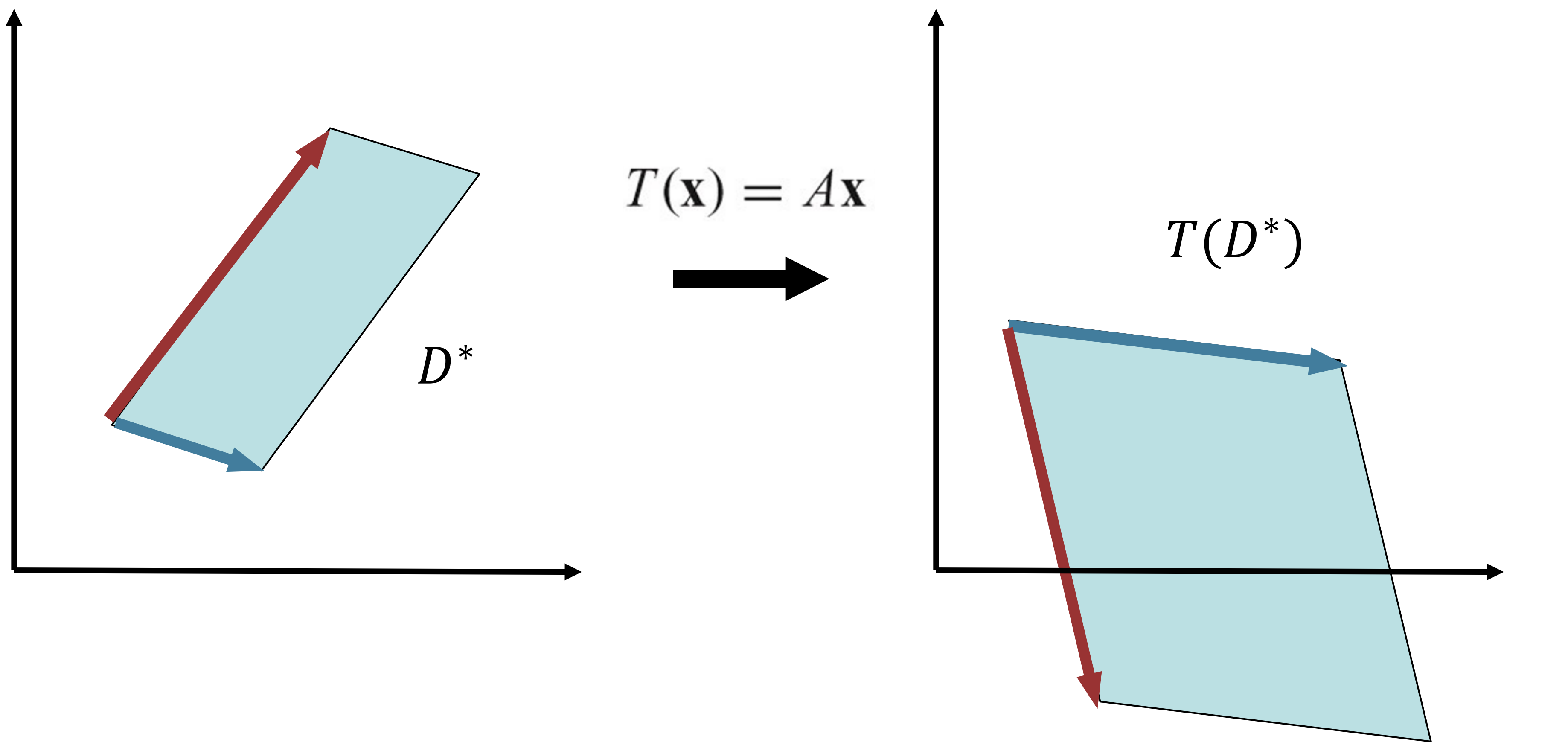
A ***mapping*** T of a region D^* in $\mathbb{R}^2$ to $\mathbb{R}^2$ associates to each point (u, v) in D^* a point $(x, y) = T(u, v)$. The set of all such (x, y) is the ***image*** domain $D = T(D^*)$.

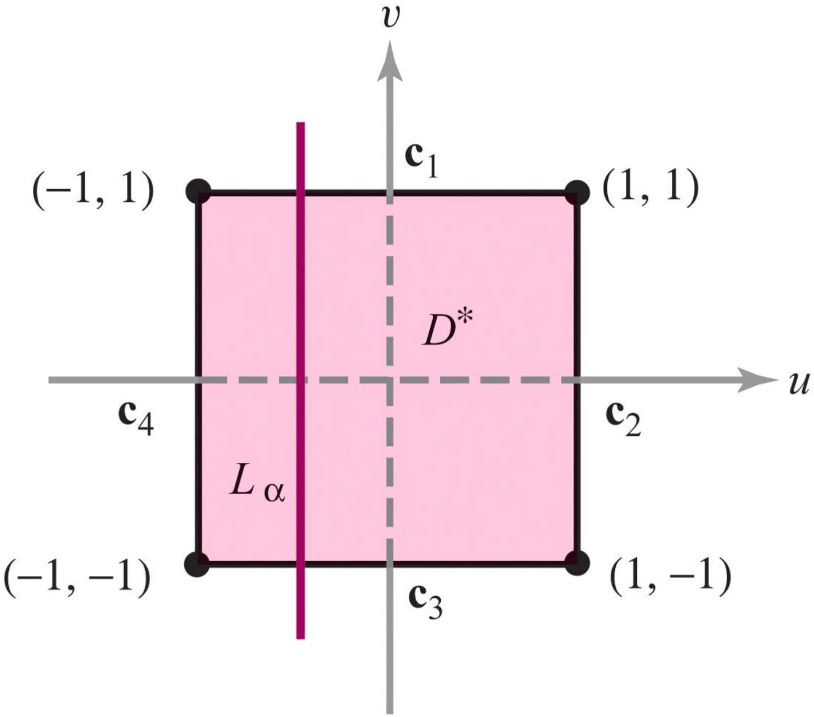




θ
2π
D^*
(r,θ)
r
1
$T\colon (r,\theta) \longmapsto (r\cos\theta, r\sin\theta) = (x, y)$
y
D
(x, y)
x

THEOREM 1 Let A be a 2×2 matrix with $\det A \neq 0$ and let T be the linear mapping of $\mathbb{R}^2$ to $\mathbb{R}^2$ given by $T(\mathbf{x}) = A\mathbf{x}$ (matrix multiplication). Then T transforms parallelograms into parallelograms and vertices into vertices. Moreover, if $T(D^*)$ is a parallelogram, D^* must be a parallelogram.

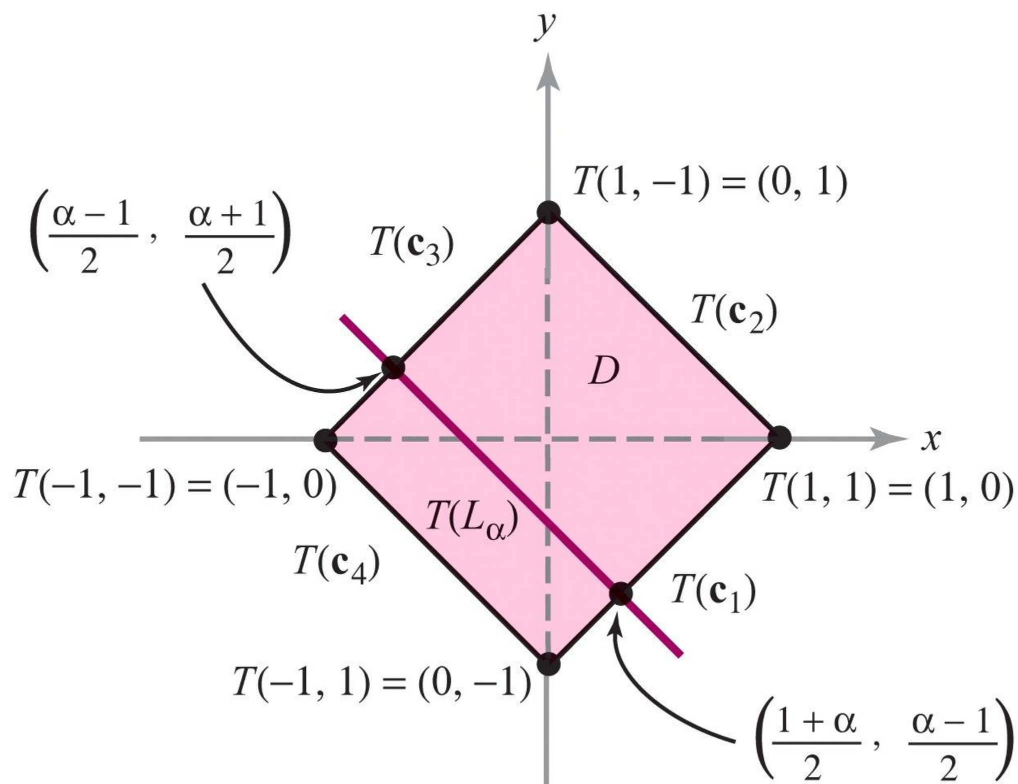




$$T(u, v) = \left(\frac{u+v}{2}, \frac{u-v}{2}\right)$$

$$T(\boldsymbol{u}) = A\boldsymbol{u}$$

$$A = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}$$



DEFINITION A mapping T is ***one-to-one*** on D^* if for (u, v) and $(u', v') \in D^*$, $T(u, v) = T(u', v')$ implies that $u = u'$ and $v = v'$.

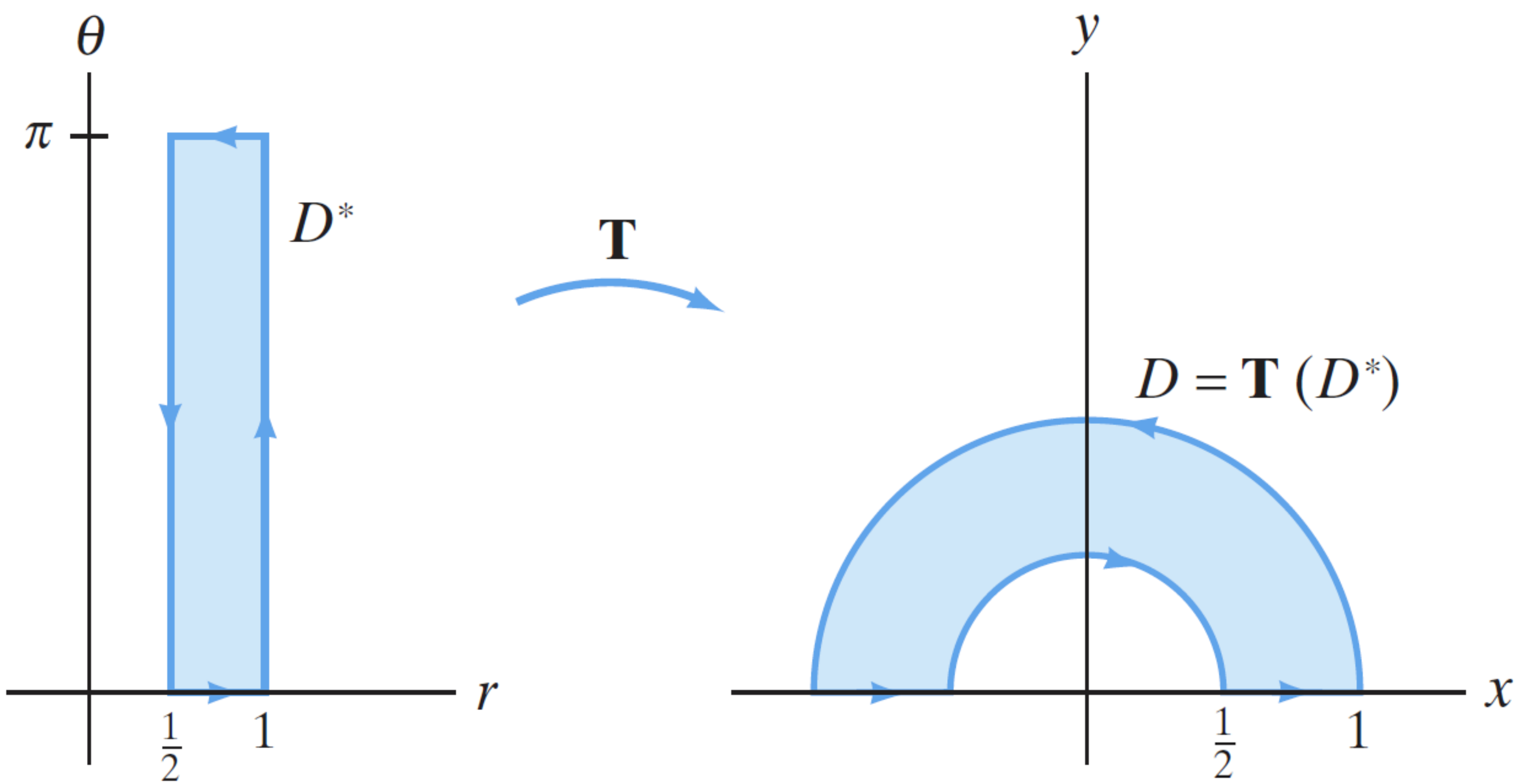
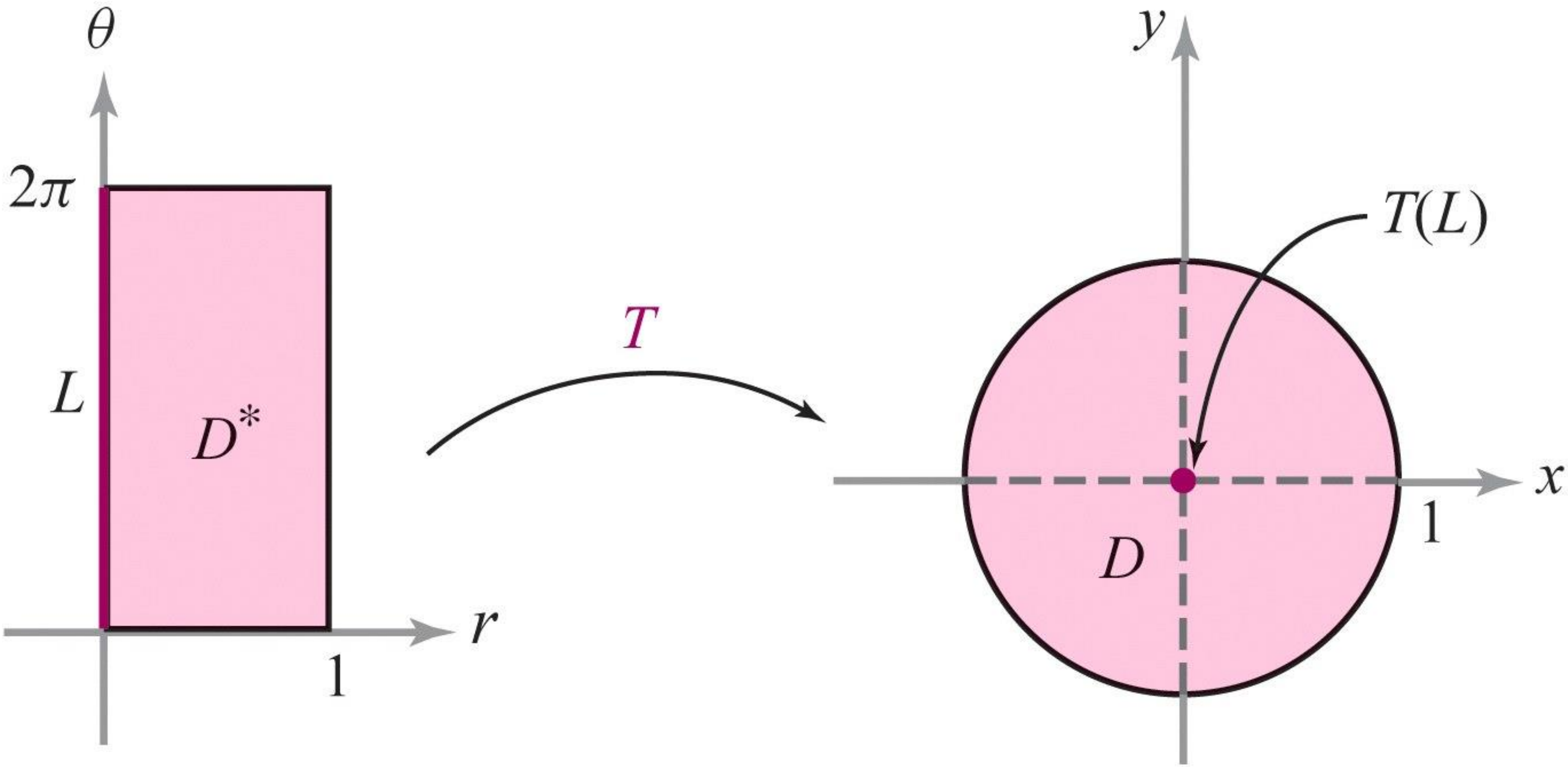


Figure 5.82 The image of the rectangle $D^* = [\frac{1}{2}, 1] \times [0, \pi]$ under $\mathbf{T}(r, \theta) = (r\cos\theta, r\sin\theta)$.

DEFINITION The mapping T is ***onto*** D if for every point $(x, y) \in D$ there exists at least one point (u, v) in the domain of T such that $T(u, v) = (x, y)$.



θ
2π
L
D^*
r
1
T
y
$T(L)$
x
1
D

One-to-One and Onto Mappings A mapping $T: D^* \to D$ is ***one-to-one*** when it maps distinct points to distinct points. It is ***onto*** when the image of D^* under T is all of D.

A *linear* transformation of $\mathbb{R}^n$ to $\mathbb{R}^n$ given by multiplication by a matrix A is one-to-one and onto when and only when $\det A \neq 0$.

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Chapter 6: The Change of Variables Formula and Applications of Integration

6.2 The Change of Variables Theorem

6.2 The Change of Variables Theorem

Key Points in this Section.

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1. The ***Jacobian determinant*** of a C^1 mapping $T : D^* \subset \mathbb{R}^2 \to \mathbb{R}$; $T(u, v) = (x(u, v), y(u, v))$ is defined by

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\[2ex] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}.$$

2. The ***singe variable change of variables formula***, which is an integrated version of the chain rule, states that for $u \mapsto x(u)$ a C^1 mapping and $f(x)$ continuous,

$$\int_{x(a)}^{x(b)} f(x)\, dx = \int_a^b f(x(u))\frac{dx}{du} du$$

3. The ***two-variable change of variables formula*** states that for a C^1 map $\tau : D^* \to D$ that is one-to-one and onto D, and an integrable function $f : D \to \mathbb{R}$,

$$\iint_D f(x,y)\,dx\,dy = \iint_{D^*} f(x(u,v),y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv.$$

4. The key idea in the proof is to put together these facts

   (a) the double integral is a limit of Riemann sums

   (b) the mapping T is nearly equal to its linear approximation on each term in the Riemann sum

   (c) the absolute value of the determinant of a linear map is the factor by which the map distorts area.

5. For polar coordinates $(r, \theta) \mapsto (x, y)$, where $x = r\cos\theta$ and $y = r\sin\theta$, the change of variables formula reads

$$\iint_D f(x, y)\, dx\, dy = \iint_{D^*} f(r\cos\theta, r\sin\theta) r\, dr\, d\theta$$

and we write the relation between the area elements as

$$dx\, dy = r\, dr\, d\theta$$

6. **Guassian Integral.** An interesting combination of reduction to iterated integrals and a change of variables to polar coordinates applied to the integral $\iint_{\mathbb{R}^2} e^{-x^2-y^2}\,dx\,dy$ shows that

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}.$$

7. The ***triple integral change of variables formula*** states that for a C^1 one-to-one map $T : W^* \to W$ that is onto W (except possibly on a finite union of curves), and an integrable function $f : W \to \mathbb{R}$,

$$\iiint_W f(x,y,z)\,dx\,dy\,dz$$
$$= \iiint_{W^*} f(x(u,v,w), y(u,v,w), z(u,v,w)) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw,$$

where $T(u,v,w) = (x(u,v,w), y(u,v,w), z(u,v,w))$ and where the ***Jacobian determinant***

$$\frac{\partial(x,y,z)}{\partial(u,v,w)}$$

is the determinant of $\mathbf{D}T$, the matrix of partial derivatives of T.

8. **Cylindrical Coordinates.** For $x = r\cos\theta$, $y = r\sin\theta$, $z = z$,

$$\iiint_W f(x, y, z)\, dx\, dy\, dz = \iiint_{W^*} f(r\cos\theta, r\sin\theta, z)\, r\, dr\, d\theta\, dz$$

and the volume elements are related by

$$dx\, dy\, dz = r\, dr\, d\theta\, dz$$

9. **Spherical Coordinates.** For $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, $z = \rho\cos\phi$,

$$\begin{aligned} &\iiint_W f(x, y, z)\, dx\, dy\, dz \\ =& \iiint_{W^*} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\, \rho^2\sin\phi\, d\rho\, d\theta\, d\phi \end{aligned}$$

and the volume elements are related by

$$dx\, dy\, dz = \rho^2\sin\phi\, d\rho\, d\theta\, d\phi.$$

Canvi de variables

Volem simplificar el càlcul de certes integrals múltiples

$$\iint_D f(x, y)\, dx\, dy$$

amb

$$f: D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$$

Per a fer-ho, volem utilitzar **canvi de variables**, com per integrals d'una variable

$$x = x(u, v), \quad y = y(u, v)$$

Aquest canvi de variables correspon a una aplicació

$$T: D^* \subset \mathbb{R}^2 \longrightarrow D \subset \mathbb{R}^2$$

Canvi de variables

Sabem que això **no** és correcte

$$\iint_D f(x, y)\, dx\, dy \stackrel{?}{=} \iint_{D^*} f(x(u, v), y(u, v))\, du\, dv$$

ja que en el cas unidimensional tenim

$$\int_{x(a)}^{x(b)} f(x)\, dx = \int_a^b f(x(u))x'(u)\, du$$

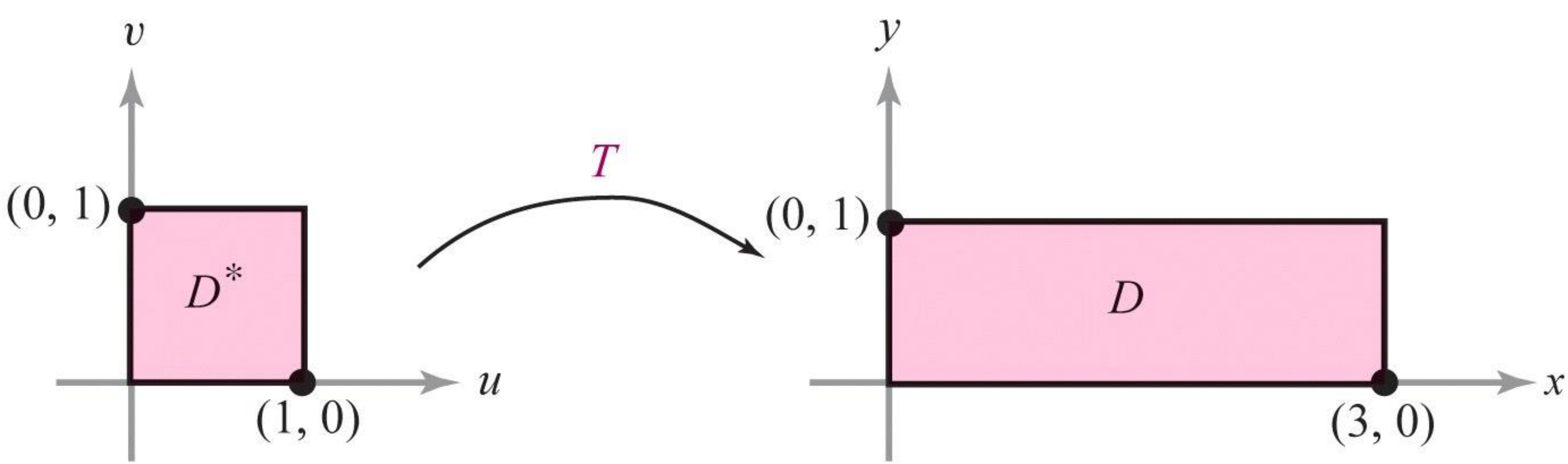
És dir, ens manca l'equivalent al factor $x'(u)$ per a integrals múltiples

Contraexemple

$$T(u,v) = \left(-u^2 + 4u, v\right), \qquad f(x,y) = 1 = (f \circ T)(u,v)$$

$$D^* = [0,1] \times [0,1] \longrightarrow D = [0,3] \times [0,1]$$

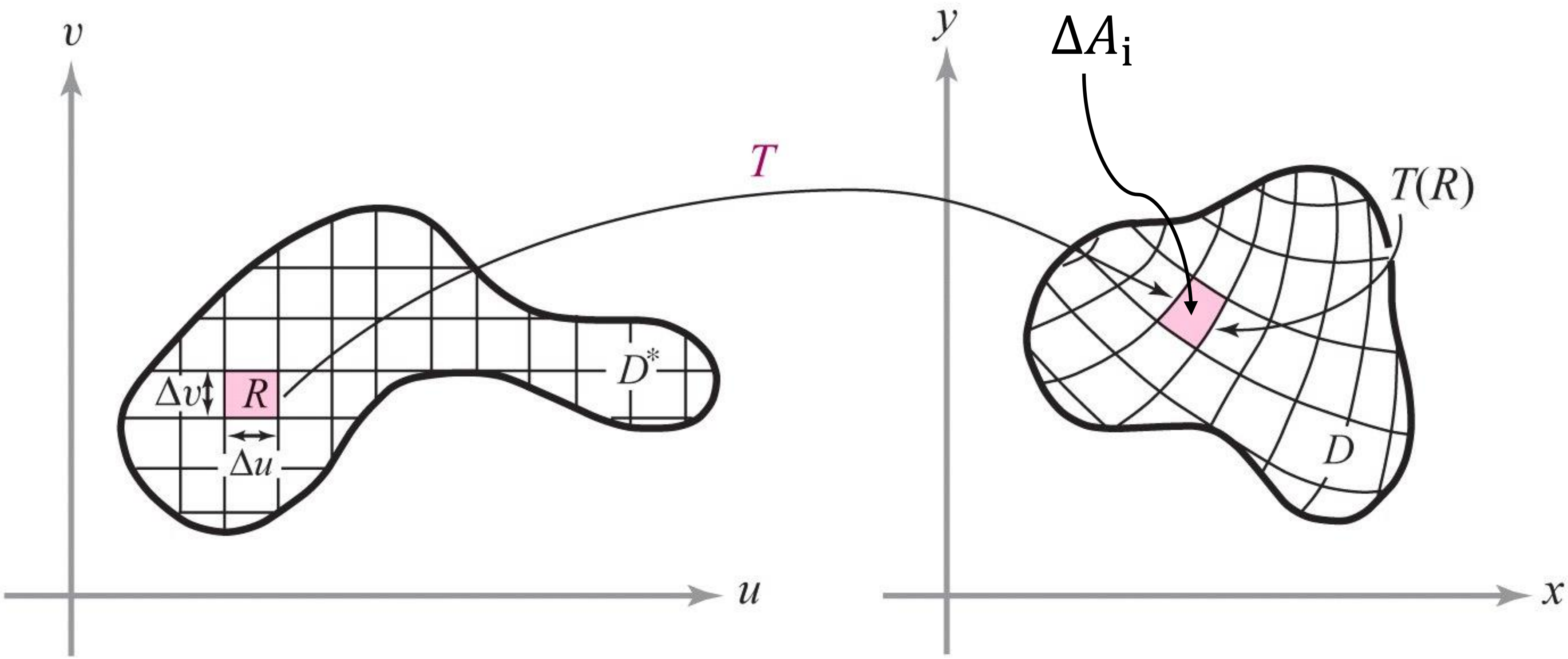
$$A(D^*) = \iint_{D^*} du\, dv = 1 \quad \neq \quad A(D) = \iint_{D} dx\, dy = 3$$



$$\iint_D f(x,y)\,dx\,dy \approx \sum_i f(T(\boldsymbol{d}_i))\Delta A_{\mathrm{i}}$$

Ara els ΔA_{i} no són rectangles

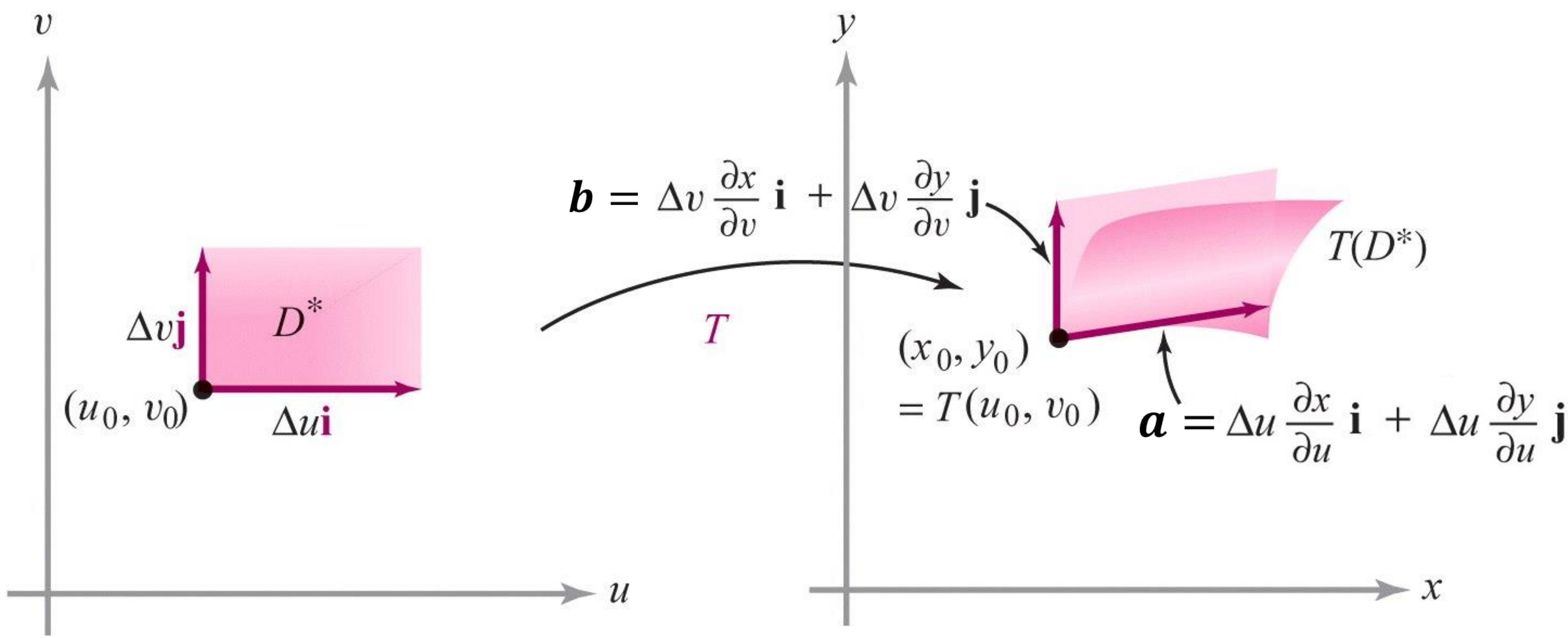
Com s'expressen els ΔA_{i} en funció de les variables $\boldsymbol{u} = (u, v)$?



$\Delta A_{\mathrm{i}} \approx$ àrea del paral·lelogram de costats els vectors

$$\boldsymbol{a} = \left(\frac{\partial x(u_0,v_0)}{\partial u}\Delta u, \frac{\partial y(u_0,v_0)}{\partial u}\Delta u\right) = \Delta \mathrm{u}\left(\frac{\partial x(u_0,v_0)}{\partial u}, \frac{\partial y(u_0,v_0)}{\partial u}\right)$$

$$\boldsymbol{b} = \left(\frac{\partial x(u_0,v_0)}{\partial v}\Delta v, \frac{\partial y(u_0,v_0)}{\partial v}\Delta v\right) = \Delta v\left(\frac{\partial x(u_0,v_0)}{\partial v}, \frac{\partial y(u_0,v_0)}{\partial v}\right)$$



S'ha utilitzat Taylor a primer ordre

$$\boldsymbol{x}(\boldsymbol{u}_0 + \Delta\boldsymbol{u}) = \boldsymbol{x}(\boldsymbol{u}_0) + DT(\boldsymbol{u}_0)\Delta\boldsymbol{u}$$

amb

$$DT(\boldsymbol{u}_0) = \begin{pmatrix} \dfrac{\partial x(\boldsymbol{u}_0)}{\partial u} & \dfrac{\partial x(\boldsymbol{u}_0)}{\partial v} \\ \dfrac{\partial y(\boldsymbol{u}_0)}{\partial u} & \dfrac{\partial y(\boldsymbol{u}_0)}{\partial v} \end{pmatrix}$$

aplicat a l'increment horitzontal

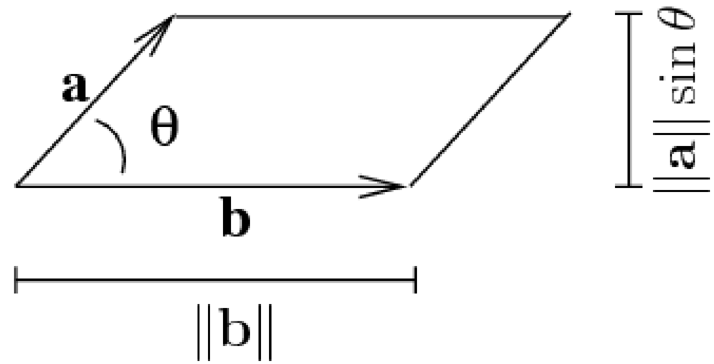
$$\Delta\boldsymbol{u} = \begin{pmatrix} \Delta u \\ 0 \end{pmatrix} \Rightarrow \boldsymbol{a} = \boldsymbol{x}(\boldsymbol{u}_0 + \Delta\boldsymbol{u}) - \boldsymbol{x}(\boldsymbol{u}_0) = \Delta u \begin{pmatrix} \dfrac{\partial x(\boldsymbol{u}_0)}{\partial u} \\ \dfrac{\partial y(\boldsymbol{u}_0)}{\partial u} \end{pmatrix}$$

i a l'increment vertical

$$\Delta\boldsymbol{u} = \begin{pmatrix} 0 \\ \Delta v \end{pmatrix} \Rightarrow \boldsymbol{b} = \boldsymbol{x}(\boldsymbol{u}_0 + \Delta\boldsymbol{u}) - \boldsymbol{x}(\boldsymbol{u}_0) = \Delta v \begin{pmatrix} \dfrac{\partial x(\boldsymbol{u}_0)}{\partial v} \\ \dfrac{\partial y(\boldsymbol{u}_0)}{\partial v} \end{pmatrix}$$

$\Delta A_{\mathrm{i}} \approx$ àrea del paral·lelogram de costats els vectors

$$\boldsymbol{a} = \Delta u \left(\frac{\partial x(u_0,v_0)}{\partial u}, \frac{\partial y(u_0,v_0)}{\partial u}\right)$$
$$\boldsymbol{b} = \Delta v \left(\frac{\partial x(u_0,v_0)}{\partial v}, \frac{\partial y(u_0,v_0)}{\partial v}\right)$$



$$\Delta A_{\mathrm{i}} \approx \|\boldsymbol{a}\|\,\|\boldsymbol{b}\| \sin\theta = \|\boldsymbol{a} \times \boldsymbol{b}\|$$

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \Delta u \frac{\partial x(u_0,v_0)}{\partial u} & \Delta u \frac{\partial x(u_0,v_0)}{\partial u} & 0 \\ \Delta v \frac{\partial x(u_0,v_0)}{\partial v} & \Delta v \frac{\partial x(u_0,v_0)}{\partial v} & 0 \end{vmatrix}$$

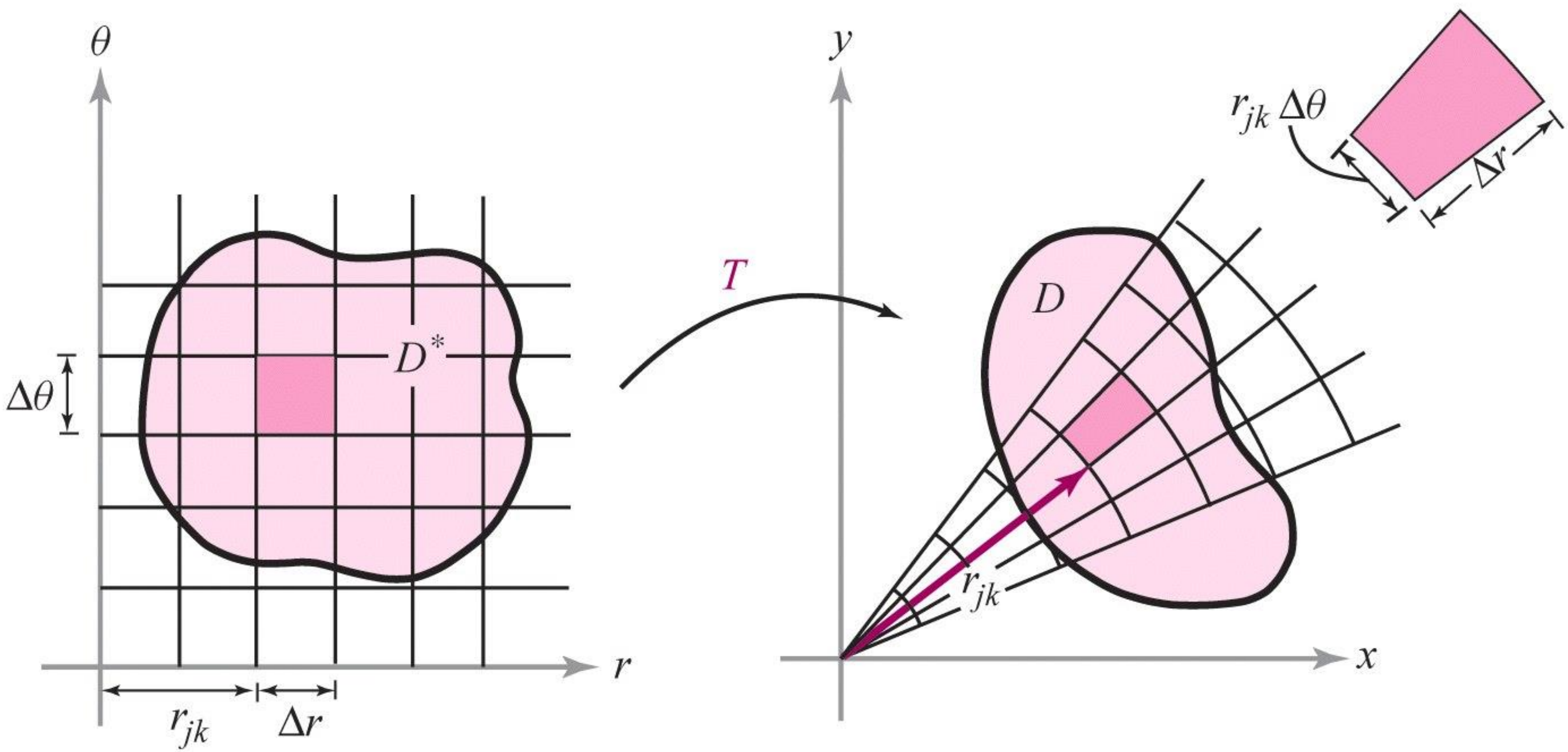
$$\Delta A_{\mathrm{i}} \approx \|\boldsymbol{a} \times \boldsymbol{b}\| = |\det(DT(\boldsymbol{u}_0))|\, \Delta u\, \Delta v$$

DEFINITION: Jacobian Determinant Let $T\colon D^* \subset \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 transformation given by $x = x(u, v)$ and $y = y(u, v)$. The ***Jacobian determinant*** of T, written $\partial(x, y)/\partial(u, v)$, is the determinant of the derivative matrix $\mathbf{D}T(u, v)$ of T:

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}.$$

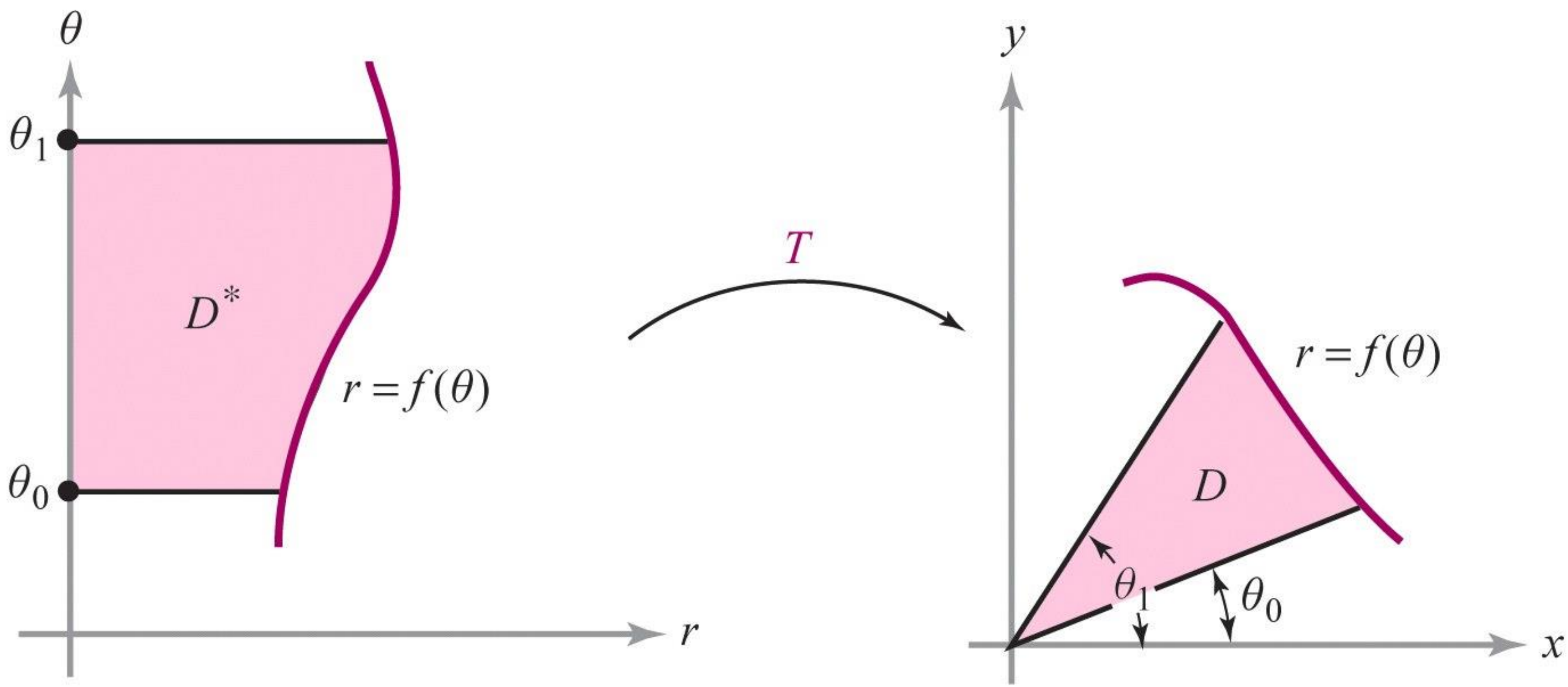
Coordenades polars

$$x = r\cos\theta \qquad\qquad \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r \qquad\qquad \Delta A = r\,\Delta r\,\Delta\theta$$
$$y = r\sin\theta$$



THEOREM 2: Change of Variables: Double Integrals Let D and D^* be elementary regions in the plane and let $T\colon D^* \to D$ be of class C^1; suppose that T is one-to-one on D^*. Furthermore, suppose that $D = T(D^*)$. Then for any integrable function $f\colon D \to \mathbb{R}$, we have

$$\iint_D f(x, y)\, dx\, dy = \iint_{D^*} f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv. \tag{6}$$



$$A(D) = \iint_D dx\,dy = \iint_{D^*} \left| \frac{\partial(x, y)}{\partial(r, \theta)} \right| dr\,d\theta$$

$$= \iint_{D^*} r\,dr\,d\theta = \int_{\theta_0}^{\theta_1} \left[\int_0^{f(\theta)} r\,dr \right] d\theta$$

$$= \int_{\theta_0}^{\theta_1} \left[\frac{r^2}{2} \right]_0^{f(\theta)} d\theta = \int_{\theta_0}^{\theta_1} \frac{[\,f(\theta)]^2}{2} d\theta$$

Let P be the parallelogram bounded by $y = 2x$, $y = 2x - 2$, $y = x$, and $y = x + 1$ (see Figure 6.2.6). Evaluate $\iint_P xy\, dx\, dy$ by making the change of variables

$$x = u - v, \qquad y = 2u - v,$$

that is, $T(u, v) = (u - v, 2u - v)$.

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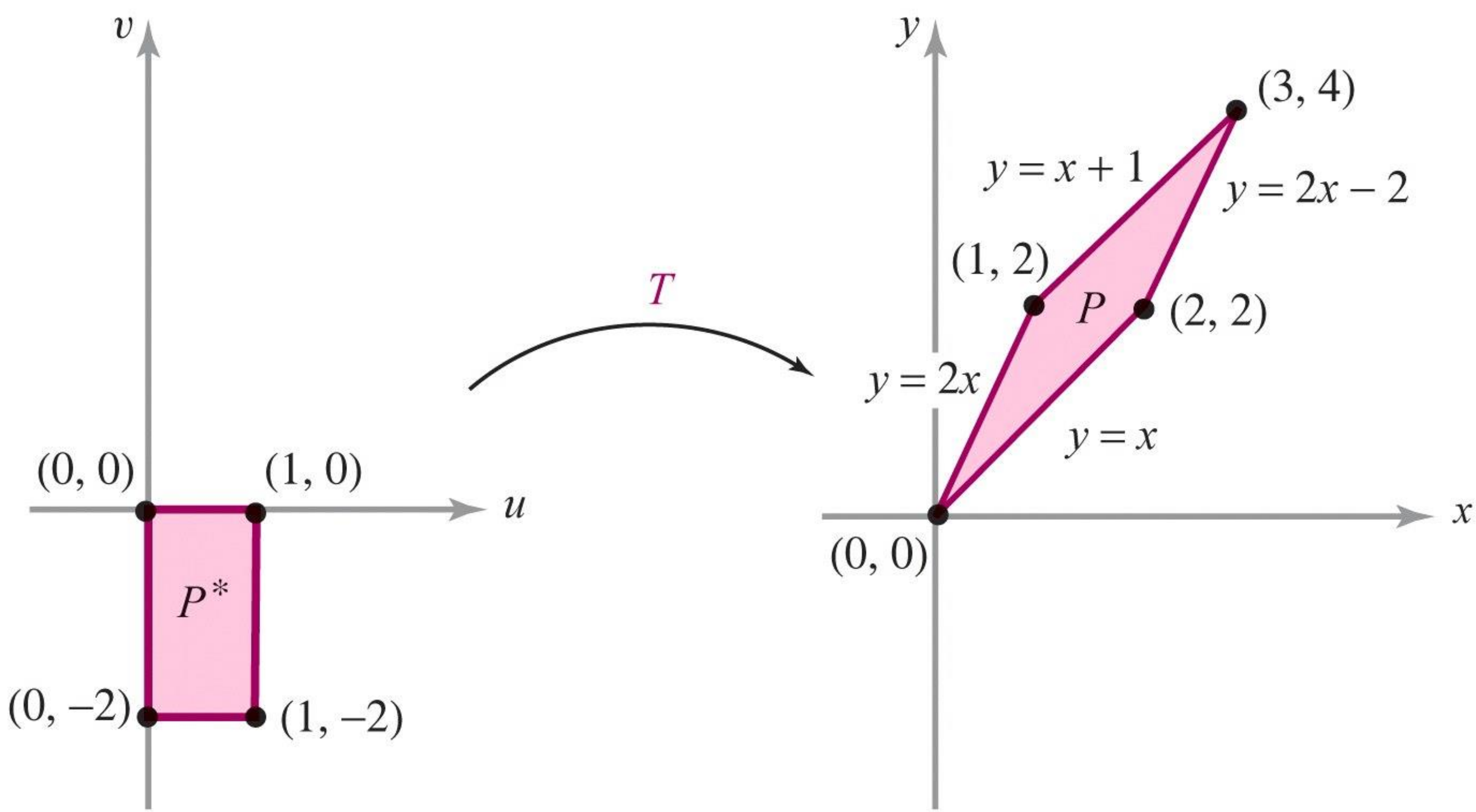


Let P be the parallelogram bounded by $y = 2x$, $y = 2x - 2$, $y = x$, and $y = x + 1$ (see Figure 6.2.6). Evaluate $\iint_P xy\,dx\,dy$ by making the change of variables

$$x = u - v, \qquad y = 2u - v,$$

that is, $T(u, v) = (u - v, 2u - v)$.

$$\left|\frac{\partial(x, y)}{\partial(u, v)}\right| = \left|\det\begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}\right| = 1$$

$$\begin{aligned}
\iint_P xy\,dx\,dy &= \iint_{P^*} (u - v)(2u - v)\,du\,dv = \int_{-2}^{0}\int_{0}^{1} (2u^2 - 3vu + v^2)\,du\,dv \\
&= \int_{-2}^{0} \left[\frac{2}{3}u^3 - \frac{3u^2 v}{2} + v^2 u\right]_0^1 dv = \int_{-2}^{0} \left[\frac{2}{3} - \frac{3}{2}v + v^2\right] dv \\
&= \left[\frac{2}{3}v - \frac{3}{4}v^2 + \frac{v^3}{3}\right]_{-2}^{0} = -\left[\frac{2}{3}(-2) - 3 - \frac{8}{3}\right] \\
&= -\left[-\frac{12}{3} - 3\right] = 7.
\end{aligned}$$

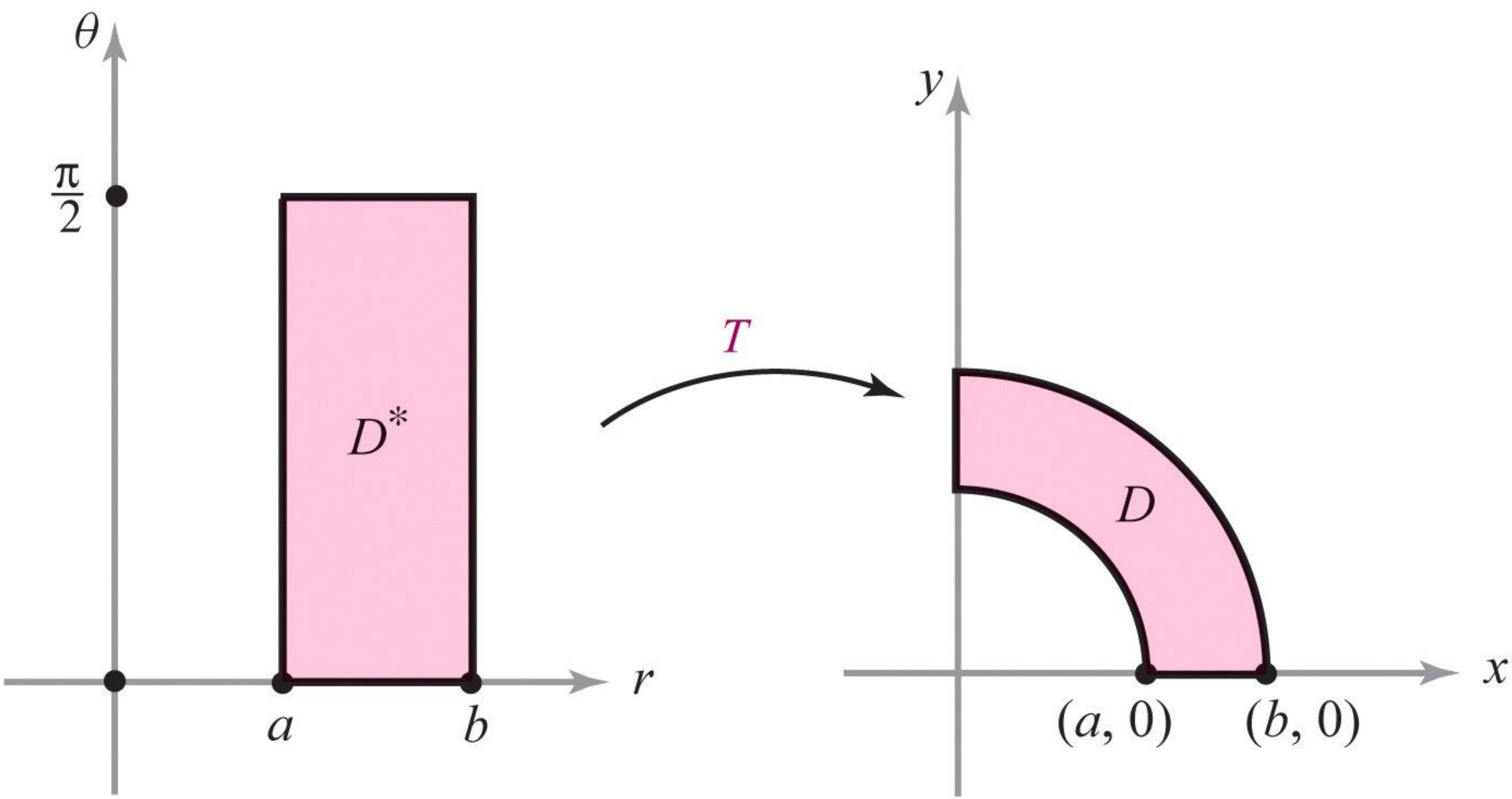
▲

Change of Variables---Polar Coordinates

$$\iint_D f(x, y)\, dx\, dy = \iint_{D^*} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta \tag{7}$$

Evaluate $\iint_D \log(x^2 + y^2)\, dx\, dy$, where D is the region in the first quadrant lying between the arcs of the circles $x^2 + y^2 = a^2$ and $x^2 + y^2 = b^2$, where $0 < a < b$

Evaluate $\iint_D \log(x^2 + y^2)\, dx\, dy$, where D is the region in the first quadrant lying between the arcs of the circles $x^2 + y^2 = a^2$ and $x^2 + y^2 = b^2$, where $0 < a < b$



Evaluate $\iint_D \log(x^2 + y^2)\, dx\, dy$, where D is the region in the first quadrant lying between the arcs of the circles $x^2 + y^2 = a^2$ and $x^2 + y^2 = b^2$, where $0 < a < b$

$$\iint_D \log(x^2 + y^2)\, dx\, dy = \int_a^b \int_0^{\pi/2} r \log r^2 \, d\theta \, dr$$

$$= \frac{\pi}{2} \int_a^b r \log r^2 \, dr = \frac{\pi}{2} \int_a^b 2r \log r \, dr.$$

Applying integration by parts, or using the formula

$$\int x \log x \, dx = \frac{x^2}{2} \log x - \frac{x^2}{4}$$

from the table of integrals at the back of the book, we obtain the result

$$\frac{\pi}{2} \int_a^b 2r \log r \, dr = \frac{\pi}{2} \left[b^2 \log b - a^2 \log a - \frac{1}{2}(b^2 - a^2) \right].$$

Observació

Per les propietats de la funció inversa

$$\boxed{\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}}}$$

Això és útil per a fer canvis de variable quan coneixem

$$u = u(x,y)$$
$$v = v(x,y)$$

en lloc de

$$x = x(u,v)$$
$$y = y(u,v)$$

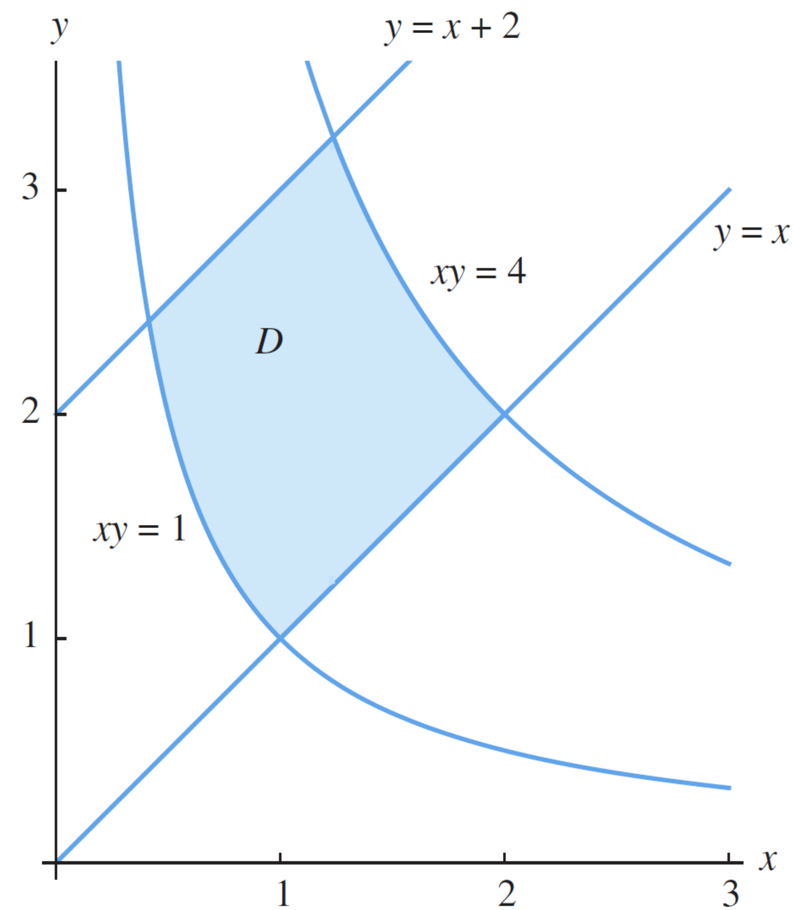
Demostra que

$$\iint_D (x^2 - y^2) e^{xy} \, dx \, dy = 2e(1 - e^3)$$

on D és la regió del primer quadrant compresa entre les hipèrboles $xy = 1$, $xy = 4$, i les línies $y = x$, $y = x + 2$

Utilitza el canvi de variables

$$u = x - y$$
$$v = xy$$



DEFINITION Let $T\colon W \subset \mathbb{R}^3 \to \mathbb{R}^3$ be a C^1 function defined by the equations $x = x(u, v, w)$, $y = y(u, v, w)$, $z = z(u, v, w)$. Then the ***Jacobian*** of T, which is denoted $\partial(x, y, z)/\partial(u, v, w)$, is the determinant

$$\begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix}.$$

Change of Variables Formula: Triple Integrals

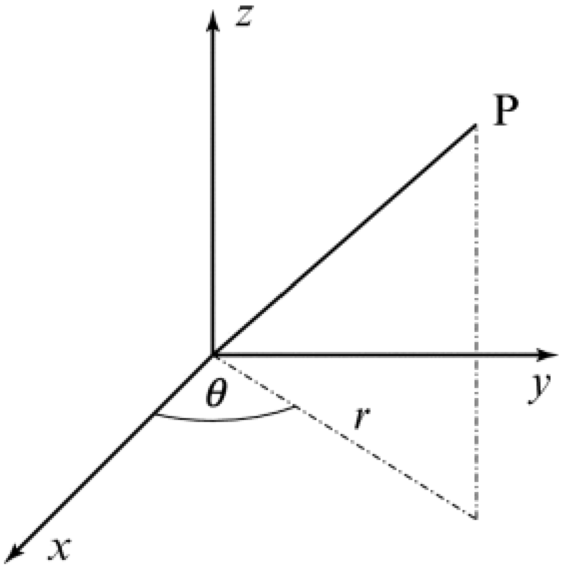
$$\iiint_W f(x, y, z)\, dx\, dy\, dz$$
$$= \iiint_{W^*} f(x(u, v, w), y(u, v, w), z(u, v, w)) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| du\, dv\, dw, \tag{8}$$

where W^* is an elementary region in uvw space corresponding to W in xyz space, under a mapping $T\colon (u, v, w) \mapsto (x(u, v, w), y(u, v, w), z(u, v, w))$, provided T is of class C^1 and is one-to-one, except possibly on a set that is the union of graphs of functions of two variables.

Coordenades cilíndriques

$$(x, y, z) \;\rightarrow\; (r, \theta, z)$$

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$



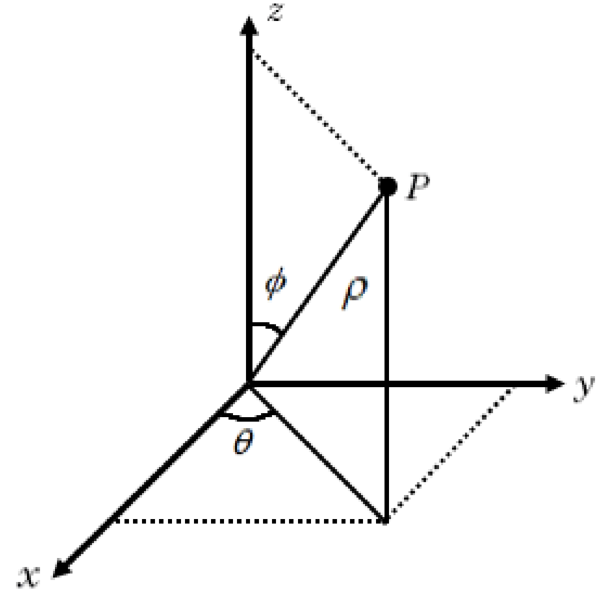
Change of Variables--Cylindrical Coordinates

$$\iiint_W f(x, y, z)\,dx\,dy\,dz = \iiint_{W^*} f(r\cos\theta, r\sin\theta, z)\,r\,dr\,d\theta\,dz. \quad (9)$$

Coordenades esfèriques

$$(x, y, z) \rightarrow (\rho, \theta, \phi)$$

$$x = \rho \cos\theta \sin\phi$$
$$y = \rho \sin\theta \sin\phi$$
$$z = \rho \cos\phi$$



Change of Variables---Spherical Coordinates

$$\iiint_W f(x, y, z)\, dx\, dy\, dz$$
$$= \iiint_{W*} f(\rho \sin\phi \cos\theta,\, \rho \sin\phi \sin\theta,\, \rho \cos\phi)\rho^2 \sin\phi\, d\rho\, d\theta\, d\phi. \tag{10}$$

Calcula el volum d'una esfera de radi R utilitzant integral triple i canvi de variables a coordenades esfèriques

Demostra que

$$\iiint_W e^{\sqrt{(x^2+y^2+x^2)^3}}\,dx\,dy\,dx = \frac{4}{3}\pi(e-1)$$

on W és una bola de radi unitat