Integral definida

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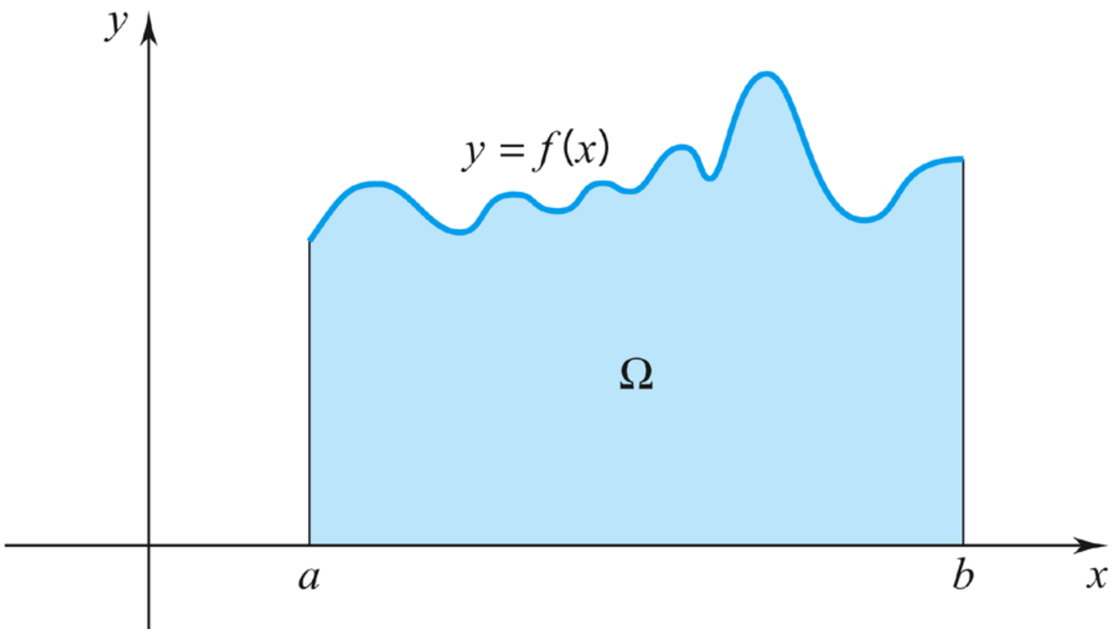
An Area Problem

Region Ω bounded above by the graph of a continuous function *f*, bounded below by the *x*-axis, bounded on the left by the line $x = a$, and bounded on the right by the line $x = b$. The question before us is this:

What number, if any, should be called the **area of Ω**?



An Area Problem

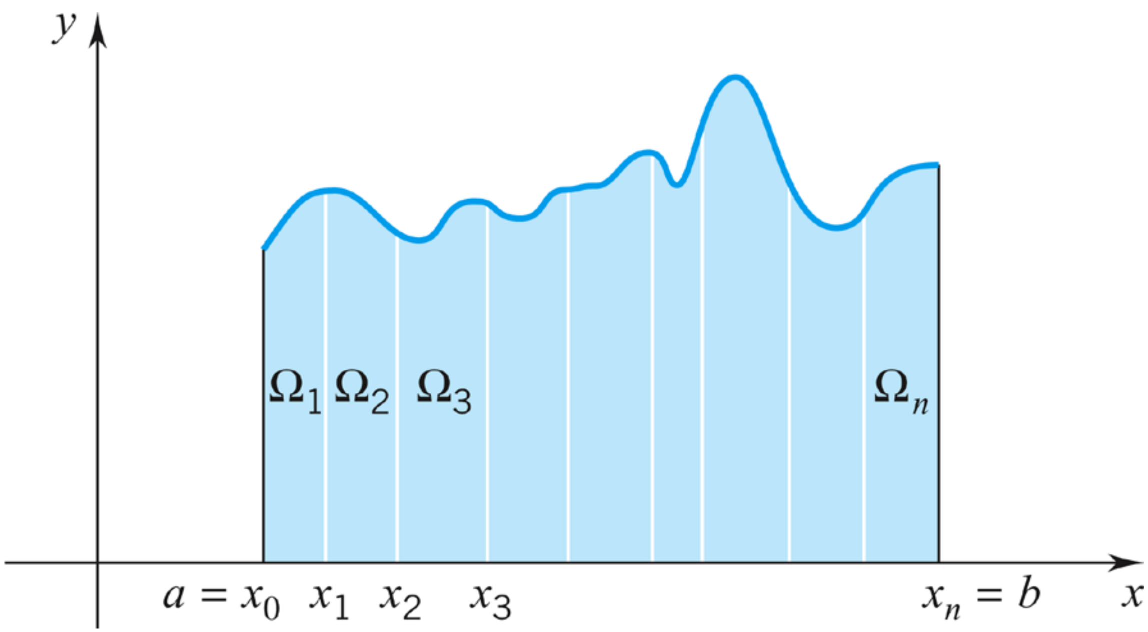
To begin to answer this question, we split up the interval $[a, b]$ into a finite number of subintervals

$$[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n] \quad \text{with} \quad a = x_0 < x_1 < \cdots < x_n = b.$$

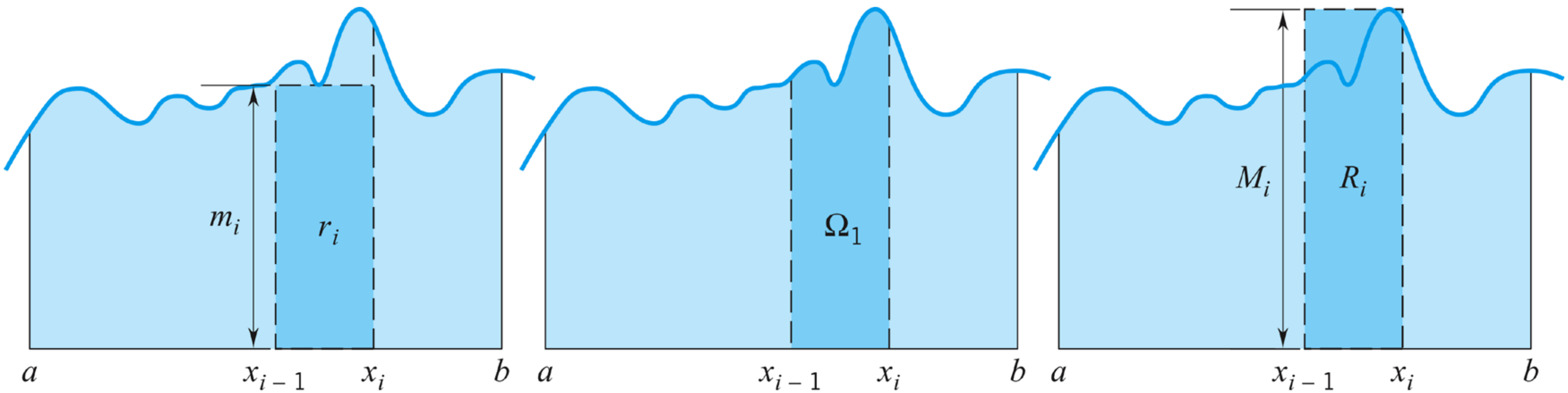
This breaks up the region Ω into n subregions:

$$\Omega_1, \Omega_2, \ldots, \Omega_n$$

We can estimate the total area of Ω by estimating the area of each subregion Ω_i and adding up the results.



An Area Problem



Adding up these inequalities, and defining $\Delta x_i = x_i - x_{i-1}$, we get on the one hand

$$\boxed{m_1 \Delta x_1 + m_2 \Delta x_2 + \cdots + m_n \Delta x_n \leq \text{area of } \Omega,}$$

and on the other hand

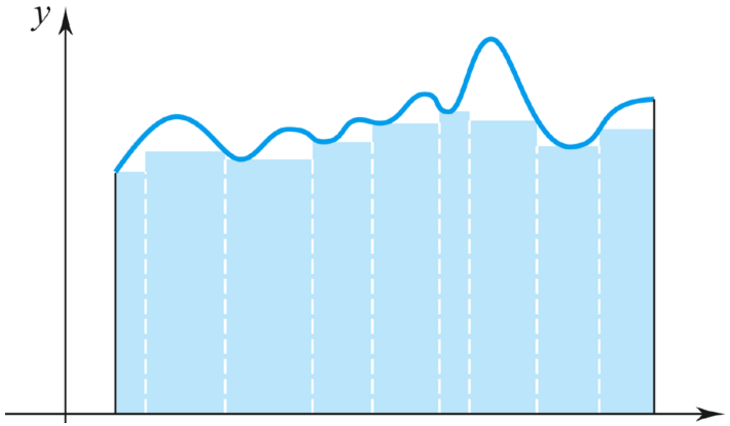
$$\boxed{\text{area of } \Omega \leq M_1 \Delta x_1 + M_2 \Delta x_2 + \cdots + M_n \Delta x_n.}$$

An Area Problem

A sum of the form

$$m_1\Delta x_1 + m_2\Delta x_2 + \cdots + m_n\Delta x_n$$

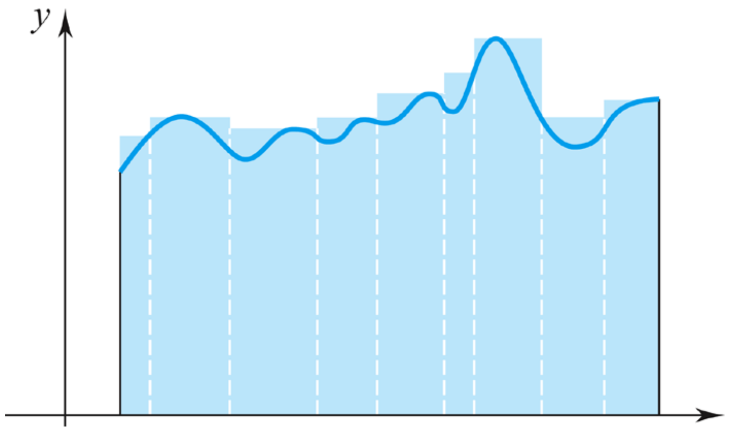
is called a ***lower sum for f***.



A sum of the form

$$M_1\Delta x_1 + M_2\Delta x_2 + \cdots + M_n\Delta x_n$$

is called an ***upper sum for f***.



For a number to be a candidate for the title "area of Ω," it must be greater than or equal to every lower sum for *f* and it must be less than or equal to every upper sum. It can be proven that with *f* continuous on $[a, b]$ there is one and only one such number. This number we call ***the area of* Ω**.

A Speed-Distance Problem

If an object moves at a constant speed for a given period of time, then the total distance traveled is given by the familiar formula

$$\text{distance} = \text{speed} \times \text{time}.$$

Suppose now that during the course of the motion the speed v does not remain constant; suppose that it varies continuously. How can we calculate the distance traveled in that case?

To answer this question, we suppose that the motion begins at time a, ends at time b, and during the time interval $[a, b]$ the speed varies continuously.
As in the case of the area problem, we begin by breaking up the interval $[a, b]$ into a finite number of subintervals:

$$[t_0, t_1], [t_1, t_2], \ldots, [t_{n-1}, t_n] \text{ with } a = t_0 < t_1 < \cdots < t_n = b.$$

On each subinterval $[t_{i-1}, t_i]$ the object attains a certain maximum speed M_i and a certain minimum speed m_i.

The total distance traveled during the full time interval $[a, b]$, call it s, must be the sum of the distances traveled during the subintervals $[t_{i-1}, t_i]$; thus, we must have

$$s = s_1 + s_2 + \cdots + s_n.$$

Similar to the area problem it can be shown that s must be greater than or equal to every lower sum for the speed function, and it must be less than or equal to every upper sum. It turns out that there is one and only one such number, and this is the total distance traveled.

The Definite Integral of a Continuous Function

By a *partition* of the closed interval $[a, b]$, we mean a finite subset of $[a, b]$ which contains the points a and b.

Example

The sets

$$\{0, 1\}, \ \{0, \tfrac{1}{2}, 1\}, \{0, \tfrac{1}{4}, \tfrac{1}{2}, 1\}, \{0, \tfrac{1}{4}, \tfrac{1}{3}, \tfrac{1}{2}, \tfrac{5}{8}, 1\}$$

are all partitions of the interval [0, 1].

The Definite Integral of a Continuous Function

If $P = \{x_0, x_1, x_2, \ldots, x_{n-1}, x_n\}$ is a partition of $[a, b]$, with $x_0 \leq x_1, \leq x_2 \leq \cdots \leq x_{n-1} \leq x_n = b$, then P breaks up $[a, b]$ into n subintervals

$$[x_0, x_1], [x_0, x_1], \ldots, [x_{n-1}, x_n] \text{ of lengths } \quad \Delta x_1, \Delta x_2, \ldots, \Delta x_n.$$

Suppose now that f is continuous on $[a, b]$. Then on each subinterval $[x_{i-1}, x_i]$ the function f takes on a maximum value, M_i, and a minimum value, m_i.

The number

$$U_f(P) = M_1 \Delta x_1 + M_2 \Delta x_2 + \cdots + M_n \Delta x_n$$

is called the P *upper sum for* f, and the number

$$L_f(P) = m_1 \Delta x_1 + m_2 \Delta x_2 + \cdots + m_n \Delta x_n$$

is called *the* P *lower sum for* f.

The Definite Integral of a Continuous Function

Example

The function $f(x) = 1 + x^2$ is continuous on [0, 1]. The partition $P = \{0, ½, ¾, 1\}$ breaks up [0, 1] into three subintervals

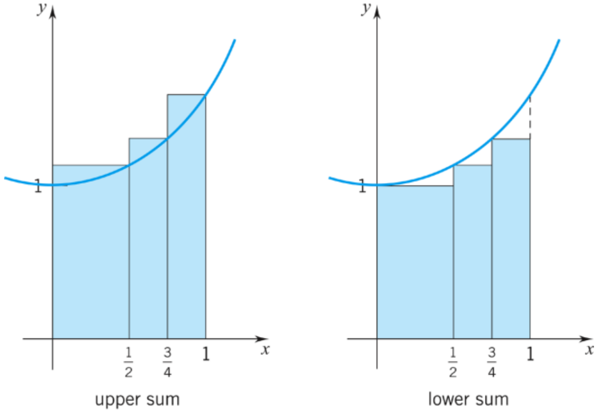
$$[x_0, x_1] = [0, ½], \qquad [x_1, x_2] = [½, ¾], \qquad [x_2, x_3] = [¾, 1]$$

of lengths

$$\Delta x_1 = ½ - 0 = ½, \qquad \Delta x_2 = ¾ - ½ = ¼, \qquad \Delta x_3 = 1 - ¾ = ¼.$$

Since f increases on [0, 1], it takes on its maximum value at the right endpoint of each subinterval:

$$M_1 = f\left(\frac{1}{2}\right) = \frac{5}{4}, \qquad M_2 = f\left(\frac{3}{4}\right) = \frac{25}{16}, \qquad M_3 = f(1) = 2$$



The minimum values are taken on at the left endpoints:

$$m_1 = f(0) = 1, \qquad m_2 = f\left(\frac{1}{2}\right) = \frac{5}{4}, \qquad m_3 = f\left(\frac{3}{4}\right) = \frac{25}{16}$$

Thus

$$U_f(P) = M_1\Delta x_1 + M_2\Delta x_2 + M_3\Delta x_3 = \frac{5}{4}\left(\frac{1}{2}\right) + \frac{25}{16}\left(\frac{1}{4}\right) + 2\left(\frac{1}{4}\right) = \frac{97}{64} \cong 1.52$$

and

$$L_f(P) = m_1\Delta x_1 + m_2\Delta x_2 + m_3\Delta x_3 = 1\left(\frac{1}{2}\right) + \frac{5}{4}\left(\frac{1}{4}\right) + \frac{25}{16}\left(\frac{1}{4}\right) = \frac{77}{64} \cong 1.20$$

The Definite Integral of a Continuous Function

Definició de Darboux de la Integral definida

DEFINITION 5.2.3 THE DEFINITE INTEGRAL OF A CONTINUOUS FUNCTION

Let f be continuous on $[a, b]$. The unique number I that satisfies the inequality

$$L_f(P) \le I \le U_f(P) \qquad \text{for all partitions } P \text{ of } [a, b]$$

is called the *definite integral* (or more simply *the integral*) of f from a to b and is denoted by

$$\int_a^b f(x)\, dx.$$

The Definite Integral of a Continuous Function

In the expression

$$\int_a^b f(x)\,dx$$

the letter *x* is a “dummy variable”; in other words, it can be replaced by any letter not already in use. Thus, for example,

$$\int_a^b f(x)\,d(x), \qquad \int_a^b f(t)\,dt, \qquad \int_a^b f(z)\,dz$$

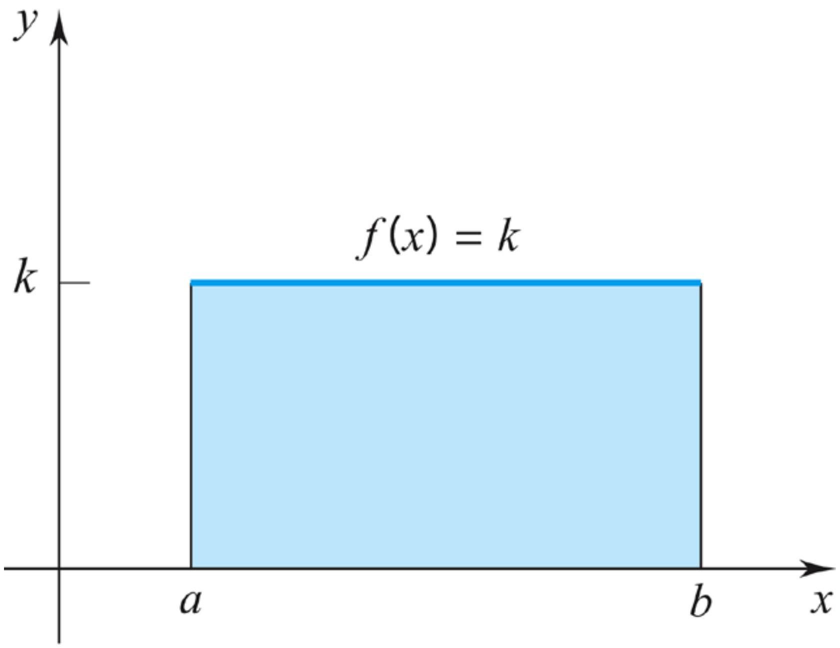
all denote exactly the same quantity, the definite integral of *f* from *a* to *b*.

From the introduction to this chapter, you know that if *f* is nonnegative and continuous on [*a*, *b*], then the integral of *f* from $x = a$ to $x = b$ gives the area below the graph of *f* from $x = a$ to $x = b$:

$$A = \int_a^b f(x)\,dx.$$

The Definite Integral of a Continuous Function

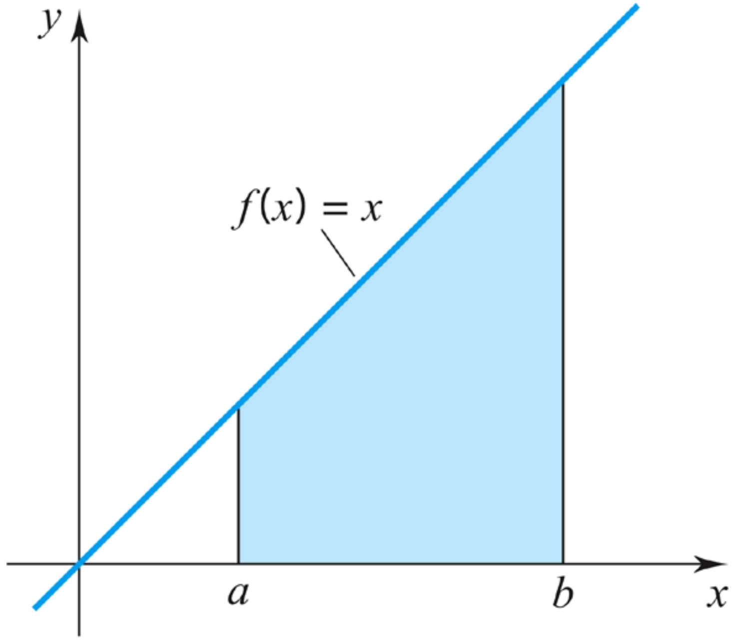
The integral of a constant function:



$$\boxed{\int_a^b k\,dx = k(b-a).}$$

The Definite Integral of a Continuous Function

The integral of the identity function:



$$\boxed{\int_a^b x\,dx = \tfrac{1}{2}(b^2 - a^2).}$$

The Integral as the Limit of Riemann Sums

The definite integral of a continuous function is the limit of Riemann sums.
The base interval is broken up into subintervals.
The point x_1^* is chosen from $[x_0, x_1]$, x_2^* from $[x_1, x_2]$, and so on: $x_i^* \in [x_{i-1}, x_i]$
Let $\|P\| = \max_i \Delta x_i$

Definició de Riemann de la Integral definida

$$\boxed{\int_a^b f(x)\,dx = \lim_{\|P\|\to 0} S^*(P),}$$

In expanded form
$$\int_a^b f(x)\,dx = \lim_{\|P\|\to 0} \left[f\left(x_1^*\right)\Delta x_1 + f\left(x_2^*\right)\Delta x_2 + \cdots + f\left(x_n^*\right)\Delta x_n\right]$$