22. Double integrals

Definition 22.1. *Let $R = [a, b] \times [c, d] \subset \mathbb{R}^2$ be a rectangle in the plane. A* ***partition*** *$\mathcal{P}$ of R is a pair of sequences:*

$$a = x_0 < x_1 < \cdots < x_n = b$$
$$c = y_0 < y_1 < \cdots < y_n = d.$$

The ***mesh*** *of $\mathcal{P}$ is*

$$m(\mathcal{P}) = \max\{\, x_i - x_{i-1}, y_i - y_{i-1} \,|\, 1 \leq i \leq k \,\}.$$

Now suppose we are given a function

$$f \colon R \longrightarrow \mathbb{R}$$

Pick

$$\vec{c}_{ij} \in R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j].$$

Definition 22.2. *The sum*

$$S = \sum_{i=1}^{n} \sum_{j=1}^{n} f(\vec{c}_{ij})(x_i - x_{i-1})(y_j - y_{j-1}),$$

is called a ***Riemann sum****.*

We will use the short hand notation

$$\Delta x_i = x_i - x_{i-1} \qquad \text{and} \qquad \Delta y_j = y_j - y_{j-1}.$$

Definition 22.3. *The function $f \colon R \longrightarrow \mathbb{R}$ is called* ***integrable****, with integral I, if for every $\epsilon > 0$, we may find a $\delta > 0$ such that for every mesh $\mathcal{P}$ whose mesh size is less than δ, we have*

$$|I - S| < \epsilon,$$

where S is any Riemann sum associated to $\mathcal{P}$.

We write

$$\iint_R f(x, y)\,\mathrm{d}x\,\mathrm{d}y = I,$$

to mean that f is integrable with integral I.

We use a sneaky trick to integrate over regions other than rectangles. Suppose that D is a bounded subset of the plane. Then we can find a rectangle R which completely contains D.

Definition 22.4. *The* ***indicator function*** *of $D \subset R$ is the function*

$$i_D \colon R \longrightarrow \mathbb{R},$$

given by

$$i_D(x) = \begin{cases} 1 & \text{if } x \in D \\ 0 & \text{if } x \notin D. \end{cases}$$

If i_D is integrable, then we say that the ***area of*** *D is the integral*

$$\iint_R i_D \,\mathrm{d}x\,\mathrm{d}y.$$

If i_D is not integrable, then D does not have an area.

Example 22.5. *Let*

$$D = \{\, (x, y) \in [0, 1] \times [0, 1] \mid x, y \in \mathbb{Q} \,\}.$$

Then D does not have an area.

Definition 22.6. *If $f\colon D \longrightarrow \mathbb{R}$ is a function and D is bounded, then pick $D \subset R \subset \mathbb{R}^2$ a rectangle. Define*

$$\tilde{f}\colon R \longrightarrow \mathbb{R},$$

by the rule

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in D \\ 0 & \text{otherwise.} \end{cases}$$

We say that f is ***integrable*** *over D if $\tilde{f}$ is integrable over R. In this case*

$$\iint_D f(x, y) \,\mathrm{d}x\,\mathrm{d}y = \iint_R \tilde{f}(x, y) \,\mathrm{d}x\,\mathrm{d}y.$$

Proposition 22.7. *Let $D \subset \mathbb{R}^2$ be a bounded subset and let $f\colon D \longrightarrow \mathbb{R}$ and $g\colon D \longrightarrow \mathbb{R}$ be two integrable functions. Let λ be a scalar.*

Then

(1) *$f + g$ is integrable over D and*

$$\iint_D f(x, y) + g(x, y) \,\mathrm{d}x\,\mathrm{d}y = \iint_D f(x, y) \,\mathrm{d}x\,\mathrm{d}y + \iint_D g(x, y) \,\mathrm{d}x\,\mathrm{d}y.$$

(2) *λf is integrable over D and*

$$\iint_D \lambda f(x, y) \,\mathrm{d}x\,\mathrm{d}y = \lambda \iint_D f(x, y) \,\mathrm{d}x\,\mathrm{d}y.$$

(3) *If $f(x, y) \leq g(x, y)$ for any $(x, y) \in D$, then*

$$\iint_D f(x, y) \,\mathrm{d}x\,\mathrm{d}y \leq \iint_D g(x, y) \,\mathrm{d}x\,\mathrm{d}y.$$

(4) $|f|$ *is integrable over* D *and*

$$|\iint_D f(x,y)\,\mathrm{d}x\,\mathrm{d}y| \le \iint_D |f(x,y)|\,\mathrm{d}x\,\mathrm{d}y.$$

It is straightforward to integrate continuous functions over regions of three special types:

Definition 22.8. *A bounded subset $D \subset \mathbb{R}^2$ is an* ***elementary region*** *if it is one of three types:*
Type 1:

$$D = \{\, (x,y) \in \mathbb{R}^2 \mid a \le x \le b, \gamma(x) \le y \le \delta(x) \,\},$$

where $\gamma\colon [a,b] \longrightarrow \mathbb{R}$ and $\delta\colon [a,b] \longrightarrow \mathbb{R}$ are continuous functions.
Type 2:

$$D = \{\, (x,y) \in \mathbb{R}^2 \mid c \le y \le d, \alpha(y) \le x \le \beta(y) \,\},$$

where $\alpha\colon [c,d] \longrightarrow \mathbb{R}$ and $\beta\colon [c,d] \longrightarrow \mathbb{R}$ are continuous functions.
Type 3: *D is both type 1 and 2.*

Theorem 22.9. *Let $D \subset \mathbb{R}^2$ be an elementary region and let $f\colon D \longrightarrow \mathbb{R}$ be a continuous function.*
Then

(1) *If D is of type 1, then*

$$\iint_D f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_a^b \left(\int_{\gamma(x)}^{\delta(x)} f(x,y)\,\mathrm{d}y \right) \mathrm{d}x.$$

(2) *If D if of type 2, then*

$$\iint_D f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_c^d \left(\int_{\alpha(y)}^{\beta(y)} f(x,y)\,\mathrm{d}x \right) \mathrm{d}y.$$

Example 22.10. *Let D be the region bounded by the lines $x = 0$, $y = 4$ and the parabola $y = x^2$. Let $f\colon D \longrightarrow \mathbb{R}$ be the function given by $f(x,y) = x^2 + y^2$.*

If we view D as a region of type 1, then we get

$$
\begin{aligned}
\iint_D f(x,y)\,\mathrm{d}x\,\mathrm{d}y &= \int_0^2 \left(\int_{x^2}^4 x^2 + y^2 \,\mathrm{d}y \right) \mathrm{d}x \\
&= \int_0^2 \left[x^2 y + \frac{y^3}{3} \right]_{x^2}^4 \mathrm{d}x \\
&= \int_0^2 4x^2 + \frac{2^6}{3} - x^4 - \frac{x^6}{3} \,\mathrm{d}x \\
&= \left[\frac{4x^3}{3} + \frac{2^6 x}{3} - \frac{x^5}{5} - \frac{x^7}{3\cdot 7} \right]_0^2 \\
&= \frac{2^5}{3} + \frac{2^7}{3} - \frac{2^5}{5} - \frac{2^7}{3\cdot 7} \\
&= \frac{2^6}{3\cdot 5} + \frac{2^8}{7} \\
&= 2^6 \left(\frac{1}{3\cdot 5} + \frac{2^2}{7} \right).
\end{aligned}
$$

On the other hand, if we view D as a region of type 2, then we get

$$
\begin{aligned}
\iint_D f(x,y)\,\mathrm{d}x\,\mathrm{d}y &= \int_0^4 \left(\int_0^{\sqrt{y}} x^2 + y^2 \,\mathrm{d}x \right) \mathrm{d}y \\
&= \int_0^4 \left[\frac{x^3}{3} + xy^2 \right]_0^{\sqrt{y}} \mathrm{d}y \\
&= \int_0^4 \frac{y^{3/2}}{3} + y^{5/2} \,\mathrm{d}y \\
&= \left[\frac{2y^{5/2}}{3\cdot 5} + \frac{2y^{7/2}}{7} \right]_0^4 \\
&= \frac{2^6}{3\cdot 5} + \frac{2^8}{7} \\
&= 2^6 \left(\frac{1}{3\cdot 5} + \frac{2^2}{7} \right).
\end{aligned}
$$

23. Inclusion-Exclusion

Proposition 23.1. *Let $D = D_1 \cup D_2$ be a bounded region and let $f \colon D \longrightarrow \mathbb{R}$ be a function.*

If f is integrable over D_1 and over D_2, then f is integrable over D and and $D_1 \cap D_2$, and we have

$$\iint_D f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \iint_{D_1} f(x,y)\,\mathrm{d}x\,\mathrm{d}y + \iint_{D_2} f(x,y)\,\mathrm{d}x\,\mathrm{d}y - \iint_{D_1\cap D_2} f(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

Example 23.2. *Let*

$$D = \{\, (x,y) \in \mathbb{R}^2 \mid 1 \leq x^2 + y^2 \leq 9 \,\}.$$

Then D is not an elementary region. Let

$$D_1 = \{\, (x,y) \in D \mid y \geq 0 \,\} \qquad \text{and} \qquad D_2 = \{\, (x,y) \in D \mid y \leq 0 \,\}.$$

Then D_1 and D_2 are both of type 1.

If f is continuous, then f is integrable over D and $D_1 \cap D_2$. In fact

$$\begin{aligned} D_1 \cap D_2 = L \cup R &= \{\, (x,y) \in \mathbb{R}^2 \mid -3 \leq x \leq -1, 0 \leq y \leq 0 \,\} \\ &\qquad \cup \{\, (x,y) \in \mathbb{R}^2 \mid 1 \leq x \leq 3, 0 \leq y \leq 0 \,\}. \end{aligned}$$

Now L and R are elementary regions. We have

$$\iint_R f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_1^3 \left(\int_0^0 f(x,y)\,\mathrm{d}y \right) \mathrm{d}x = 0.$$

Therefore, by symmetry,

$$\iint_L f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \iint_R f(x,y)\,\mathrm{d}x\,\mathrm{d}y = 0$$

and so

$$\iint_D f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \iint_{D_1} f(x,y)\,\mathrm{d}x\,\mathrm{d}y + \iint_{D_2} f(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

To integrate f over D_1, break D_1 into three parts.

$$
\begin{aligned}
\iint_{D_1} f(x,y)\,\mathrm{d}x\,\mathrm{d}y &= \int_{-3}^{3}\left(\int_{\gamma(x)}^{\delta(x)} f(x,y)\,\mathrm{d}y\right)\mathrm{d}x \\
&= \int_{-3}^{-1}\left(\int_{0}^{\sqrt{9-x^2}} f(x,y)\,\mathrm{d}y\right)\mathrm{d}x \\
&+ \int_{-1}^{1}\left(\int_{\sqrt{1-x^2}}^{\sqrt{9-x^2}} f(x,y)\,\mathrm{d}y\right)\mathrm{d}x \\
&+ \int_{1}^{3}\left(\int_{0}^{\sqrt{9-x^2}} f(x,y)\,\mathrm{d}y\right)\mathrm{d}x.
\end{aligned}
$$

One can do something similar for D_2.

Example 23.3. *Suppose we are given that*

$$\iint_D f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_0^1\left(\int_y^{2y} f(x,y)\,\mathrm{d}x\right)\mathrm{d}y.$$

What is the region D?

It is the region bounded by the two lines $y = x$ and $x = 2y$ and between the two lines $y = 0$ and $y = 1$.

Change order of integration:

$$\iint_D f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_0^1\left(\int_{x/2}^{x} f(x,y)\,\mathrm{d}x\right)\mathrm{d}y + \int_1^2\left(\int_{x/2}^{1} f(x,y)\,\mathrm{d}x\right)\mathrm{d}y.$$

Example 23.4. *Calculate the volume of a solid ball of radius a. Let*

$$B = \{\,(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2+z^2 \leq a^2\,\}.$$

We want the volume of B. Break into two pieces. Let

$$B^+ = \{\,(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2+z^2 \leq a^2, z \geq 0\,\}.$$

Let

$$D = \{\,(x,y)\in\mathbb{R}^2 \mid x^2+y^2 \leq a^2\,\}.$$

Then B^+ is bounded by the xy-plane and the graph of the function

$$f\colon D \longrightarrow \mathbb{R},$$

given by

$$f(x,y) = \sqrt{a^2-x^2-y^2}.$$

It follows that

$$
\begin{aligned}
\operatorname{vol}(B^+) &= \iint_D \sqrt{a^2 - x^2 - y^2}\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_{-a}^{a} \left(\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \sqrt{a^2 - x^2 - y^2}\,\mathrm{d}y \right) \mathrm{d}x \\
&= \int_{-a}^{a} \left(\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \sqrt{1 - \frac{y^2}{a^2 - x^2}} \sqrt{a^2 - x^2}\,\mathrm{d}y \right) \mathrm{d}x.
\end{aligned}
$$

Now let's make the substitution

$$
t = \frac{y}{\sqrt{a^2 - x^2}} \qquad \textit{so that} \qquad dt = \frac{dy}{\sqrt{a^2 - x^2}}.
$$

$$
\begin{aligned}
\operatorname{vol}(B^+) &= \int_{-a}^{a} \left(\int_{-1}^{1} \sqrt{1 - t^2}(a^2 - x^2)\,\mathrm{d}t \right) \mathrm{d}x \\
&= \int_{-a}^{a} (a^2 - x^2) \left(\int_{-1}^{1} \sqrt{1 - t^2}\,\mathrm{d}t \right) \mathrm{d}x
\end{aligned}
$$

Now let's make the substitution

$$
t = \sin u \qquad \textit{so that} \qquad dt = \cos u\, du.
$$

$$
\begin{aligned}
\operatorname{vol}(B^+) &= \int_{-a}^{a} (a^2 - x^2) \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 u\,\mathrm{d}u \right) \mathrm{d}x \\
&= \int_{-a}^{a} (a^2 - x^2) \frac{\pi}{2}\,\mathrm{d}x \\
&= \frac{\pi}{2} \left[a^2 x - \frac{x^3}{3} \right]_{-a}^{a} \\
&= \pi (a^3 - \frac{a^3}{3}) \\
&= \frac{2\pi a^3}{3}.
\end{aligned}
$$

Therefore, we get the expected answer

$$
\operatorname{vol}(B) = 2\operatorname{vol}(B^+) = \frac{4\pi a^3}{3}.
$$

Example 23.5. *Now consider the example of a cone whose base radius is a and whose height is b. Put the central axis along the x-axis and* *the base in the yz-plane. In the xy-plane we get an equilateral triangle of height b and base $2a$. If we view this as a region of type 1, we have*

$$\gamma(x) = -a\left(1 - \frac{x}{b}\right) \qquad \textit{and} \qquad \delta(x) = a\left(1 - \frac{x}{b}\right).$$

We want to integrate the function

$$f\colon D \longrightarrow \mathbb{R},$$

given by

$$f(x, y) = \sqrt{a^2\left(1 - \frac{x}{b}\right)^2 - y^2}.$$

So half of the volume of the cone is

$$\begin{aligned}
\int_0^b \left(\int_{-a(1-\frac{x}{b})}^{a(1-\frac{x}{b})} \sqrt{a^2\left(1 - \frac{x}{b}\right)^2 - y^2}\,\mathrm{d}y\right)\mathrm{d}x &= \frac{\pi}{2}\int_0^b a^2\left(1 - \frac{x}{b}\right)^2\,\mathrm{d}x \\
&= \frac{\pi a^2}{2}\int_0^b 1 - \frac{2x}{b} + \frac{x^2}{b^2}\,\mathrm{d}x \\
&= \frac{\pi a^2}{2}\left[x - \frac{x^2}{b} + \frac{x^3}{3b^2}\right]_0^b \\
&= \frac{1}{6}(\pi a^2 b).
\end{aligned}$$

Therefore the volume is

$$\frac{1}{3}(\pi a^2 b).$$

24. Triple integrals

Definition 24.1. *Let $B = [a, b] \times [c, d] \times [e, f] \subset \mathbb{R}^3$ be a box in space. A **partition** $\mathcal{P}$ of R is a triple of sequences:*

$$
\begin{aligned}
a = x_0 < x_1 < \cdots < x_n = b \\
c = y_0 < y_1 < \cdots < y_n = d \\
e = z_0 < z_1 < \cdots < z_n = f.
\end{aligned}
$$

*The **mesh** of $\mathcal{P}$ is*

$$m(\mathcal{P}) = \max\{\, x_i - x_{i-1}, y_i - y_{i-1}, z_i - z_{i-1} \,|\, 1 \leq i \leq k \,\}.$$

Now suppose we are given a function

$$f \colon B \longrightarrow \mathbb{R}$$

Pick

$$\vec{c}_{ijk} \in B_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_i, z_{i-1}].$$

Definition 24.2. *The sum*

$$S = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} f(\vec{c}_{ijk})(x_i - x_{i-1})(y_j - y_{j-1})(z_i - z_{i-1}),$$

*is called a **Riemann sum**.*

Definition 24.3. *The function $f \colon B \longrightarrow \mathbb{R}$ is called **integrable**, with integral I, if for every $\epsilon > 0$, we may find a $\delta > 0$ such that for every mesh $\mathcal{P}$ whose mesh size is less than δ, we have*

$$|I - S| < \epsilon,$$

where S is any Riemann sum associated to $\mathcal{P}$.

If $W \subset \mathbb{R}^3$ is a bounded subset and $f \colon W \longrightarrow \mathbb{R}$ is a bounded function, then pick a box B containing W and extend f by zero to a function $\tilde{f} \colon B \longrightarrow \mathbb{R}$,

$$\tilde{f}(x) = \begin{cases} x & \text{if } x \in W \\ 0 & \text{otherwise.} \end{cases}$$

If $\tilde{f}$ is integrable, then we write

$$\iiint_W f(x, y, z)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z = \iiint_B \tilde{f}(x, y, z)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z.$$

In particular

$$\mathrm{vol}(W) = \iiint_W \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z.$$

There are two pairs of results, which are much the same as the results for double integrals:

Proposition 24.4. *Let $W \subset \mathbb{R}^2$ be a bounded subset and let $f\colon W \longrightarrow \mathbb{R}$ and $g\colon W \longrightarrow \mathbb{R}$ be two integrable functions. Let λ be a scalar.*

Then

(1) *$f + g$ is integrable over W and*

$$\iiint_W f(x,y,z)+g(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iiint_W f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z + \iiint_W g(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

(2) *λf is integrable over W and*

$$\iiint_W \lambda f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \lambda \iiint_W f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

(3) *If $f(x,y,z) \leq g(x,y,z)$ for any $(x,y,z) \in W$, then*

$$\iiint_W f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \leq \iiint_W g(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

(4) *$|f|$ is integrable over W and*

$$|\iiint_W f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z| \leq \iiint_W |f(x,y,z)|\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

Proposition 24.5. *Let $W = W_1 \cup W_2 \subset \mathrm{R}^3$ be a bounded subset and let $f\colon W \longrightarrow \mathbb{R}$ be a bounded function.*

If f is integrable over W_1 and over W_2, then f is integrable over W and and $W_1 \cap W_2$, and we have

$$\iiint_W f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iiint_{W_1} f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z + \iiint_{W_2} f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$
$$- \iiint_{W_1 \cap W_2} f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

Definition 24.6. *Define three maps*

$$\pi_{ij}\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^2,$$

by projection onto the ith and jth coordinate.

In coordinates, we have

$$\pi_{12}(x,y,z) = (x,y), \qquad \pi_{23}(x,y,z) = (y,z), \qquad \text{and} \qquad \pi_{13}(x,y,z) = (x,z).$$

For example, if we start with a solid pyramid and project onto the xy-plane, the image is a square, but it project onto the xz-plane, the image is a triangle. Similarly onto the yz-plane.

Definition 24.7. *A bounded subset $W \subset \mathbb{R}^3$ is an* ***elementary subset*** *if it is one of four types:*

Type 1: *$D = \pi_{12}(W)$ is an elementary region and*

$$W = \{\, (x,y,z) \in \mathbb{R}^2 \mid (x,y) \in D, \epsilon(x,y) \leq z \leq \phi(x,y) \,\},$$

where $\epsilon \colon D \longrightarrow \mathbb{R}$ and $\phi \colon D \longrightarrow \mathbb{R}$ are continuous functions.

Type 2: *$D = \pi_{23}(W)$ is an elementary region and*

$$W = \{\, (x, y, z) \in \mathbb{R}^2 \,|\, (y, z) \in D, \alpha(y, z) \leq x \leq \beta(y, z) \,\},$$

where $\alpha \colon D \longrightarrow \mathbb{R}$ and $\beta \colon D \longrightarrow \mathbb{R}$ are continuous functions.

Type 3: *$D = \pi_{13}(W)$ is an elementary region and*

$$W = \{\, (x, y, z) \in \mathbb{R}^2 \,|\, (x, z) \in D, \gamma(x, z) \leq y \leq \delta(x, z) \,\},$$

where $\gamma \colon D \longrightarrow \mathbb{R}$ and $\delta \colon D \longrightarrow \mathbb{R}$ are continuous functions.

Type 4: *W is of type 1, 2 and 3.*

The solid pyramid is of type 4.

Theorem 24.8. *Let $W \subset \mathbb{R}^3$ be an elementary region and let $f \colon W \longrightarrow \mathbb{R}$ be a continuous function.*

Then

(1) *If W is of type 1, then*

$$\iiint_W f(x, y, z) \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z = \iint_{\pi_{12}(W)} \left(\int_{\epsilon(x,y)}^{\phi(x,y)} f(x, y, z) \, \mathrm{d}z \right) \mathrm{d}x \, \mathrm{d}y.$$

(2) *If W if of type 2, then*

$$\iiint_W f(x, y, z) \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z = \iint_{\pi_{23}(W)} \left(\int_{\alpha(y,z)}^{\beta(y,z)} f(x, y, z) \, \mathrm{d}x \right) \mathrm{d}y \, \mathrm{d}z.$$

(3) *If W if of type 3, then*

$$\iiint_W f(x, y, z) \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z = \iint_{\pi_{13}(W)} \left(\int_{\gamma(x,z)}^{\delta(x,z)} f(x, y, z) \, \mathrm{d}y \right) \mathrm{d}x \, \mathrm{d}z.$$

Let's figure out the volume of the solid ellipsoid:

$$W = \{\, (x, y, z) \in \mathbb{R}^3 \,|\, \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 \leq 1 \,\}.$$

This is an elementary region of type 4.

$$
\begin{aligned}
\operatorname{vol}(W) &= \iiint_W \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\
&= \int_{-a}^{a}\left(\int_{-b\sqrt{1-\left(\frac{x}{a}\right)^2}}^{b\sqrt{1-\left(\frac{x}{a}\right)^2}}\left(\int_{-c\sqrt{1-\left(\frac{x}{a}\right)^2-\left(\frac{y}{b}\right)^2}}^{c\sqrt{1-\left(\frac{x}{a}\right)^2-\left(\frac{y}{b}\right)^2}}\mathrm{d}z\right)\,\mathrm{d}y\right)\,\mathrm{d}x \\
&= \int_{-a}^{a}\left(\int_{-b\sqrt{1-\left(\frac{x}{a}\right)^2}}^{b\sqrt{1-\left(\frac{x}{a}\right)^2}}2c\sqrt{1-\left(\frac{x}{a}\right)^2-\left(\frac{y}{b}\right)^2}\,\mathrm{d}y\right)\mathrm{d}x \\
&= 2c\int_{-a}^{a}\left(\int_{-b\sqrt{1-\left(\frac{x}{a}\right)^2}}^{b\sqrt{1-\left(\frac{x}{a}\right)^2}}\sqrt{1-\left(\frac{x}{a}\right)^2-\left(\frac{y}{b}\right)^2}\,\mathrm{d}y\right)\mathrm{d}x \\
&= \frac{2c}{b}\int_{-a}^{a}\left(\int_{-b\sqrt{1-\left(\frac{x}{a}\right)^2}}^{b\sqrt{1-\left(\frac{x}{a}\right)^2}}\sqrt{b^2\left(1-\left(\frac{x}{a}\right)^2\right)-y^2\,\mathrm{d}y}\right)\mathrm{d}x \\
&= \frac{\pi c}{b}\int_{-a}^{a}b^2\left(1-\left(\frac{x}{a}\right)^2\right)\,\mathrm{d}x \\
&= \pi bc\int_{-a}^{a}1-\left(\frac{x}{a}\right)^2\,\mathrm{d}x \\
&= \pi bc\left[x-\frac{x^3}{3a^2}\right]_{-a}^{a} \\
&= \pi bc\left(2a-2\frac{a^3}{3a^2}\right) \\
&= \frac{4\pi}{3}abc.
\end{aligned}
$$