Exercises - Hamiltonian mechanics - Hamilton's equations

4.1 Legendre transformation. Consider the function $A(x, y) = (1 + x^2)y^2$.

(a) Prove that its Legendre transformation of $A(x, y)$ gives

$$B(x, z) = \frac{z^2}{4(1 + x^2)} .$$

(b) Use the inverse Legendre transformation to find $y(x, z)$ and $A(x, y)$ from $B(x, z)$.

4.2 Particle on a parabolic wire. Consider a particle of mass m sliding, under the action of gravity and without friction, on a parabolic wire whose shape is given by the equation $y = x^2/2$ (Fig. 1).

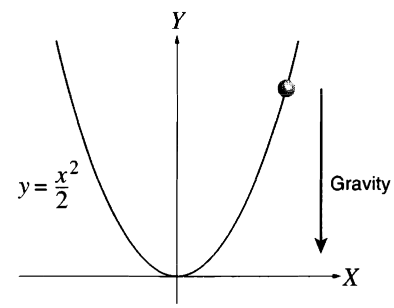

Figure 1: Spherical pendumum (Exercise 2).

(a) Write the Lagrangian $\mathcal{L}(x, x')$ of the system (in terms of the x coordinate and its velocity x').

(b) Obtain the generalized conjugate momentum p_x.

(c) Obtain the Hamiltonian of the system as a function of x and p_x.

(d) Obtain the equations of motion of x and p_x using Hamilton's equations.

(e) Considering that $H = E$ is the total energy and is conserved,[1] use your favorite software to plot the trajectories of the system in phase space (**Hint**: Given that $H = E =$ constant, you do not need to integrate the equations of motion to get the trajectory in phase space!).

[1] Why?

4.3 Pendulum hanging from a tilted wire. A pendulum is comprised of a mass m and a rigid rod of length l. The support of the pendulum can slide, without friction, on a straight wire described by the equation $y = ax$ (Fig. 2).

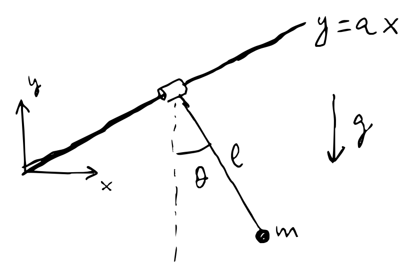

Figure 2: Pendumum hanging from a slope (Exercise 3).

(a) Write the Lagrangian of the system in terms of the horizontal position X of the support of the pendulum and the angle θ.

(b) Obtain the generalized momenta p_X and p_θ, and the expressions of the velocities X' and θ' as a function of p_X and p_θ. **Hint**: Use the following shorthands to write less:

$$A \equiv ml^2 \quad B \equiv m(1 + a^2) \quad C \equiv ml(\cos\theta + a \sin\theta) .$$

(c) Proof that the Hamiltonian of the system is

$$H = \frac{1}{AB - C^2}\left[\frac{A}{2}p_X^2 + \frac{B}{2}p_\theta^2 - Cp_X p_\theta\right] + mg(aX - l\cos\theta)\,.$$

4.4 Spherical pendulum. A pendulum of length R is connected to the center of a sphere as shown in Fig. 3; it is subject to gravity in the z direction as depicted.

(a) Write the Lagrangian $\mathcal{L}(\theta, \phi)$ of the system.

(b) Obtain the generalized conjugate momenta p_θ and p_ϕ.

(c) Obtain the Hamiltonian of the system.

(d) Obtain the equations of motion using Hamilton's equations.

(e) Differentiate the equation for θ' and substitute into the equation for p'_θ to obtain a 2nd-order differential equation for θ.

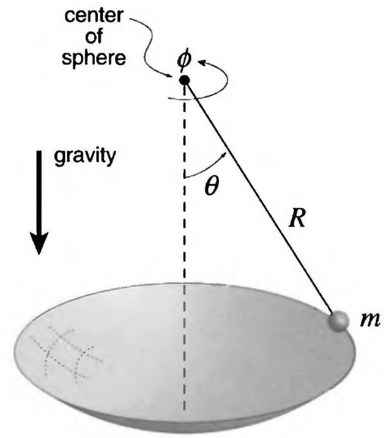

Figure 3: Spherical pendumum (Exercise 4).

4.5 Hamiltonian dynamics in an accelerated system. Consider a mass m sitting on a turntable that rotates at constant angular velocity ω (Fig. 4). The mass is subject to a potential $V(x, y)$, where x and y are the coordinates in the rotating frame. Obtain the Lagrangian and the generalized momenta from the rotating reference frame. Prove that the Hamiltonian in the rotating frame is[2]

$$H_\omega = \frac{p_x^2 + p_y^2}{2m} + V(x, y) + \omega(yp_x - xp_y)\,.$$

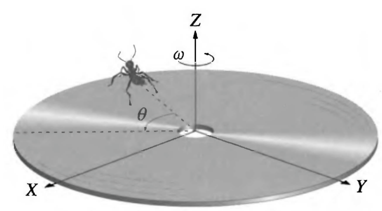

Figure 4: Bug of mass m on a rotating frame (Exercise 5).

[2] Notice that $H_\omega = H_{\omega=0} - \omega L_z$, where $L_z = (xp_y - yp_x)$ is the z component of the angular momentum. This means that, to get the Hamiltonian from the perspective of the rotating frame, we just need to substract ωL_z from the Hamiltonian we would have had if there was no rotation. This is one scenario in which the Hamiltonian approach does simplify things!

4.6 Momentum space Lagrangian. Consider the momentum space Lagrangian $\mathcal{K}$, which is defined as the double Legendre transform of the (coordinate space) Lagrangian

$$\mathcal{K}(p, p', t) = \mathcal{L}(q, q', t) - pq' - qp'\,.$$

(a) Prove that $\mathcal{K}$ and $\mathcal{L}$ are dynamically equivalent, that is, that the Euler-Lagrange equations for $\mathcal{K}$ are the same as for $\mathcal{L}$ (**Hint**: $d(pq)/dt = pq' + qp'$).

(b) Consider a harmonic oscillator with Hamiltonian

$$H = \frac{1}{2}\left(p^2 + \omega^2 q^2\right)$$

and obtain the corresponding Lagrangian $\mathcal{L}(q, q')$.

(c) Obtain $p = \partial\mathcal{L}/\partial q'$ and $p' = -\partial H/\partial q$.

(d) Obtain the momentum space Lagrangian $\mathcal{K}$ for this system, and show that it has the same functional form as the coordinate space Lagrangian $\mathcal{L}$.