Exercises

1. a) Find the domain of the scalar function:

$$f(x,y)=\sum_{k=0}^{\infty}(xy)^k$$

b) Find and classify the critical points of $f(x,y)$

2. Assume that y is a differentiable function of x which satisfies the equation. Obtain dy/dx by implicit differentiation.

$x^2 - 2xy + y^4 = 4.$
$x e^y + y e^x - 2x^2 y = 0.$
$x^{2/3} + y^{2/3} = a^{2/3}.$ (a *is* a constant)
$x \cos xy + y \cos x = 2.$

3. Let $u = u(x,y)$, where $x = x(s)$ and $y = y(s)$, and assume that these functions have continuous second derivatives. Find the expression for $\frac{d^2u}{ds^2}$.

4. a) Show that the expression

$$x^6y + y^2\int_0^x \frac{1}{1+\sin^6 t}\,dt + y^5 - 1 = 0$$

defines y as an implicit differentiable function $y = f(x)$ around the point (0,1).

b) Write the equation of the tangent line to $y = f(x)$ at the point (0,1).

5. Assume that $u = u(x,y)$ is differentiable.

a) Show that the change of variables to polar coordinates $x = r\cos\theta, y = r\sin\theta$ gives

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta$$
$$\frac{\partial u}{\partial \theta} = -\frac{\partial u}{\partial x}\mathrm{r}\sin\theta + \frac{\partial u}{\partial y}r\cos\theta$$

b) Express in terms of $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ the following expression:

$$\left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r}\left(\frac{\partial u}{\partial \theta}\right)^2$$

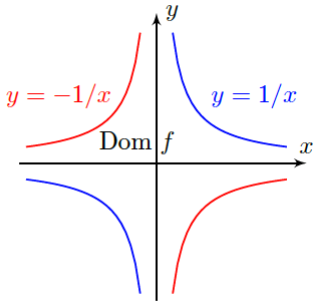
Solutions

1.

(a) La serie dada es una serie geométrica de razón xy, en consecuencia es convergente si y sólo si se verifica $|xy| < 1$ o equivalentemente, si y sólo si $-1 < xy < 1$. El dominio de f es por tanto:

$$\text{Dom } f = \{(x, y) \in \mathbb{R}^2 : -1 < xy < 1\}$$

Corresponde pues al conjunto abierto del plano que contiene al origen y cuya frontera son las hipérbolas equiláteras $y = 1/x$ e $y = -1/x$.



(b) Aplicando la conocida fórmula de la suma de la serie geométrica:

$$f(x, y) = 1 + xy + (xy)^2 + (xy)^3 + \ldots = \frac{1}{1 - xy} \quad (|xy| < 1).$$

Hallemos los puntos críticos de f :

$$\begin{cases} \dfrac{\partial f}{\partial x} = \dfrac{y}{(1 - xy)^2} = 0 \\ \dfrac{\partial f}{\partial y} = \dfrac{x}{(1 - xy)^2} = 0 \end{cases} \Leftrightarrow (x, y) = (0, 0).$$

Hallemos las parciales de orden dos de f en el punto crítico $(0, 0)$:

$$\frac{\partial^2 f}{\partial x^2} = \frac{-2(1 - xy)(-y)y}{(1 - xy)^4} = \frac{2y^2}{(1 - xy)^3},$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{1(1 - xy)^2 - 2(1 - xy)(-x)y}{(1 - xy)^4} = \frac{xy + 1}{(1 - xy)^3},$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{-2(1 - xy)(-x)x}{(1 - xy)^4} = \frac{2x^2}{(1 - xy)^3}.$$

La matriz hessiana de f en el punto crítico $(0, 0)$ es:

$$H(f, (0, 0)) = \begin{bmatrix} \dfrac{\partial^2 f}{\partial x^2}(0, 0) & \dfrac{\partial^2 f}{\partial x \partial y}(0, 0) \\ \dfrac{\partial^2 f}{\partial x \partial y}(0, 0) & \dfrac{\partial^2 f}{\partial x^2}(0, 0) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Dado que $\det H(f, (0, 0)) = -1 < 0$, en el punto $(0, 0)$ no hay extremo (punto de ensilladura).

2. Trivial

3.

$$\frac{d^2u}{ds^2} = \frac{\partial^2 u}{\partial x^2}\left(\frac{dx}{ds}\right)^2 + 2\frac{\partial^2 u}{\partial x \partial y}\frac{dx}{ds}\frac{dy}{ds} + \frac{\partial^2 u}{\partial y^2}\left(\frac{dy}{ds}\right)^2 + \frac{\partial u}{\partial x}\frac{d^2x}{ds^2} + \frac{\partial u}{\partial y}\frac{d^2y}{ds^2}$$

4.

Solución. (a) Denominemos

$$F(x,y) = x^6 y + y^2 \int_0^x \frac{1}{1+\sin^6 t}dt + y^5 - 1.$$

Aplicando conocidas propiedades, es claro que F está definida en $\mathbb{R}\times\mathbb{R}$ y que $F(0,1) = 0$. Por otra parte y usando el teorema fundamental del Cálculo

$$\frac{\partial F}{\partial x} = 6x^5 y + y^2 \cdot \frac{1}{1+\sin^6 x} \quad , \quad \frac{\partial F}{\partial y} = x^6 + 2y\int_0^x \frac{1}{1+\sin^6 t}dt + 5y^4.$$

Estas parciales son continuas en el abierto $A = \mathbb{R}\times\mathbb{R}$ y además $\frac{\partial F}{\partial y}(0,1) = 5 \neq 0$. Es decir, la ecuación dada determina una función diferenciable $y = f(x)$ en un entorno del punto $(0,1)$.

(b) La ecuación de la recta tangente a la curva $y = f(x)$ en el punto $(0,1)$ viene dada por $y - 1 = f'(0)(x-0)$. Pero

$$f'(0) = -\frac{\frac{\partial F}{\partial x}(0,1)}{\frac{\partial F}{\partial y}(0,1)} = -1/5,$$

con lo cual la ecuación de la recta pedida es $x + 5y - 5 = 0$.

5. Trivial