Numerical Methods

Splines

- Parametric Curves

2022-2023

[DΣIM]

Cubic Splines

- Given a data set formed by *n* points, $\mathbf{P_1}$,.., $\mathbf{P_n}$, we could find an infinite number of curves passing through all these points. In order to build our curve, we would like to have a somewhat interactive algorithm that could allow the user to modify the shape of the curve until a final satisfactory shape is achieved.

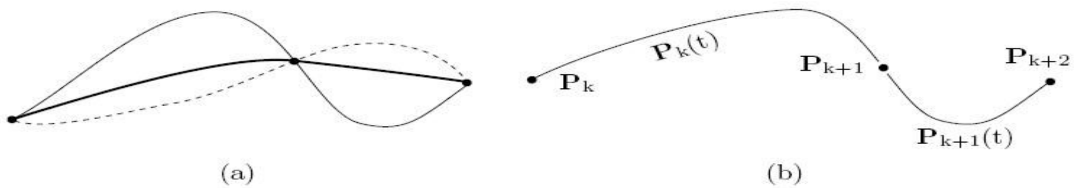


Figure 5.1: (a) Three Different Curves. (b) Two Segments.

Cubic Splines

- With the cubic spline algorithm we can build a curve that goes through the n data points, consisting in $n-1$ Hermite segments which are smoothly joined in the $n-2$ inner points
- The first condition is that the tangent vectors to the joining curves in the inner data points must be equal.
- In the cubic splines we add the additional condition that even the second derivatives must be equal.

Cubic Splines

- Thus, given the data points $\mathbf{P_1}$,.., $\mathbf{P_n}$, we look for $n-1$ parametric curves formed by cubic polynomials $\mathbf{S_1}(t), \dots, \mathbf{S_{n-1}}(t)$ in a way that $\mathbf{S_k}(t)$ is the curve segment that goes from $\mathbf{P_k}$ to the point $\mathbf{P_{k+1}}$
- The curves must join smoothly in the $n-2$ inner points, with equal first (tangent) and second (curvature) derivatives.

Cubic Splines

- We start by dividing the data set in pairs of consecutive data points ($\mathbf{P}_1$, $\mathbf{P}_2$), ($\mathbf{P}_2$, $\mathbf{P}_3$), until we reach the pair ($\mathbf{P}_{n-1}$, $\mathbf{P}_n$)
- The Hermite curves are determined by two points and the tangents at these points. The points are given, and we have to determine the tangents.
- As these must be equal in the contact points, we have only *n* tangents to be determined

Cubic Splines

- The unknown tangent are determined solving a linear system of equations. These equations are derived with the additional condition that the second derivatives must be equal in the contact points
- As we have only n-2 contact points, we have 2 ***undetermined tangents*** which give us two degrees of freedom to determine the final shape of the curve

Cubic Splines

- We start by selecting three consecutive points $\mathbf{P_k}$, $\mathbf{P_{k+1}}$ and $\mathbf{P_{k+2}}$. The point $\mathbf{P_{k+1}}$ must be necessarily an inner point. The other two can be an inner or an extreme point. The index k can vary then, from 1 up to n-2
- If $\mathbf{S_k}(t)$ is the cubic spline from $\mathbf{P_k}$ up to $\mathbf{P_{k+1}}$, then it must satisfy the two conditions:

$$\mathbf{S_k}(0) = \mathbf{P_k} \quad \text{and} \quad \mathbf{S_k}(1) = \mathbf{P_{k+1}}$$

Cubic Splines

- The tangent vectors in the extremes are unknown, they will be denoted by $\mathbf{v_k}$ and $\mathbf{v_{k+1}}$
- The Hermite cubic curve passing through the points $\mathbf{P_k}$ and $\mathbf{P_{k+1}}$ with tangents $\mathbf{v_k}$ and $\mathbf{v_{k+1}}$ at the extremes has the equation:

$$\mathbf{S_k}(t) = \mathbf{P_k} + \mathbf{v_k} t + \left[3\left(\mathbf{P_{k+1}} - \mathbf{P_k}\right) - 2\mathbf{v_k} - \mathbf{v_{k+1}}\right] t^2 + \left[2\left(\mathbf{P_k} - \mathbf{P_{k+1}}\right) + \mathbf{v_k} + \mathbf{v_{k+1}}\right] t^3$$

- This expression can be written also as:

$$\mathbf{S_k}(t) = \mathbf{A_k} t^3 + \mathbf{B_k} t^2 + \mathbf{C_k} t + \mathbf{D_k}$$

- The next Hermite curve, coming from the point $\mathbf{P_{k+1}}$ has a similar expression:

$$\mathbf{S_{k+1}}(t) = \mathbf{P_{k+1}} + \mathbf{v_{k+1}} t + \left[3\left(\mathbf{P_{k+2}} - \mathbf{P_{k+1}}\right) - 2\mathbf{v_{k+1}} - \mathbf{v_{k+2}}\right] t^2 + \left[2\left(\mathbf{P_{k+1}} - \mathbf{P_{k+2}}\right) + \mathbf{v_{k+1}} + \mathbf{v_{k+2}}\right] t^3$$

Cubic Splines

- Note that we have been using the same tangent $\mathbf{v}_{\mathbf{k+1}}$ to the point $\mathbf{P}_{\mathbf{k+1}}$ for both expressions $\mathbf{S}_{\mathbf{k}}(t)$ and $\mathbf{S}_{\mathbf{k+1}}(t)$
- In this way we include the condition that both tangents must be equal in the joining inner points.

Cubic Splines

- Now, deriving two times both expressions and imposing the conditions that both must be equal, we obtain the following expressions:

$$\ddot{\mathbf{S}}_{\mathbf{k}}(t) = 6\mathbf{A}_{\mathbf{k}}t + 2\mathbf{B}_{\mathbf{k}}$$

$$\ddot{\mathbf{S}}_{\mathbf{k+1}}(t) = 6\mathbf{A}_{\mathbf{k+1}}t + 2\mathbf{B}_{\mathbf{k+1}}$$

$$\ddot{\mathbf{S}}_{\mathbf{k}}(1) = \ddot{\mathbf{S}}_{\mathbf{k+1}}(0) \Leftrightarrow 6\mathbf{A}_{\mathbf{k}} + 2\mathbf{B}_{\mathbf{k}} = 6\mathbf{A}_{\mathbf{k+1}}$$

- Substituting for the constants we obtain the equation:

$$6\left[2\left(\mathbf{P}_k-\mathbf{P}_{\mathbf{k+1}}\right)+\mathbf{v}_{\mathbf{k}}+\mathbf{v}_{\mathbf{k+1}}\right]+2\left[3\left(\mathbf{P}_{\mathbf{k+1}}-\mathbf{P}_{\mathbf{k}}\right)-2\mathbf{v}_{\mathbf{k}}-\mathbf{v}_{\mathbf{k+1}}\right]$$
$$=2\left[3\left(\mathbf{P}_{\mathbf{k+2}}-\mathbf{P}_{\mathbf{k+1}}\right)-2\mathbf{v}_{\mathbf{k+1}}-\mathbf{v}_{\mathbf{k+2}}\right]$$

- And after some algebra, we obtain the expression which relates the data points and the tangents to be determined:

$$\boxed{\mathbf{v}_{\mathbf{k}}+4\mathbf{v}_{\mathbf{k+1}}+\mathbf{v}_{\mathbf{k+2}}=3(\mathbf{P}_{\mathbf{k+2}}-\mathbf{P}_{\mathbf{k}})}$$

Cubic Splines

- The three vectors $\mathbf{v}_k$, $\mathbf{v}_{k+1}$ and $\mathbf{v}_{k+2}$ are the unknowns, while the value of the data points $\mathbf{P}_{k+2}$ and $\mathbf{P}_k$, are known magnitudes
- We have n-2 equations as this one, one for each inner point. We have however, n unknowns, the tangents $\mathbf{v}_1, \ldots, \mathbf{v}_n$
- We have then two degrees of freedom to be determined for the user, normally $\mathbf{v}_1$ and $\mathbf{v}_n$

Cubic Splines

- The first and last equations are then:

$$4\mathbf{v}_2 + \mathbf{v}_3 = 3(\mathbf{P}_3 - \mathbf{P}_1) - \mathbf{v}_1$$
$$\mathbf{v}_{n-2} + 4\mathbf{v}_{n-1} = 3(\mathbf{P}_n - \mathbf{P}_{n-2}) - \mathbf{v}_n$$

- The problem of building the cubic spline curves is transformed into a linear algebra problem, the solution of a linear system of equations.

Cubic Splines

- These equations can be written in matrix form as follows:

$$\begin{pmatrix} 4 & 1 & 0 & \cdots & 0 \\ 1 & 4 & 1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & 1 & 4 & 1 \\ 0 & \cdots & 0 & 1 & 4 \end{pmatrix} \begin{pmatrix} \mathbf{v}_2 \\ \mathbf{v}_3 \\ \vdots \\ \mathbf{v}_{n-2} \\ \mathbf{v}_{n-1} \end{pmatrix} = \begin{pmatrix} 3(\mathbf{P}_3 - \mathbf{P}_1) - \mathbf{v}_1 \\ 3(\mathbf{P}_4 - \mathbf{P}_2) \\ \vdots \\ 3(\mathbf{P}_{n-1} - \mathbf{P}_{n-3}) \\ 3(\mathbf{P}_n - \mathbf{P}_{n-2}) - \mathbf{v}_n \end{pmatrix}$$

Cubic Splines

- Note that the system matrix is tridiagonal, (only the elements in the three principal diagonals are not zero), and it also has a dominant diagonal:

$$a_{nn} > a_{nn-1} + a_{nn+1}$$

- This last fact guarantees that the matrix is not singular, and then it is invertible and the system has a unique solution.
- Moreover, being diagonal dominant, the iterative methods of solution are convergent.

- We can write alternatively the system as follows, using a *n* x *n*-2 system matrix:

$$\begin{pmatrix} 1 & 4 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 4 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 4 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{v_1} \\ \mathbf{v_2} \\ \vdots \\ \mathbf{v_n} \end{pmatrix} = \begin{pmatrix} 3(\mathbf{P_3} - \mathbf{P_1}) \\ 3(\mathbf{P_4} - \mathbf{P_2}) \\ \vdots \\ 3(\mathbf{P_n} - \mathbf{P_{n-2}}) \end{pmatrix}$$

- But then, the system will not be computer oriented

Cubic Splines

- The splines method is able to generate smooth curves, but it has some problems that must be known
- We do not have local control on the shape of the curve, changing any of the free tangents we change the global shape of the curve
- A linear system of n unknowns must be solved, and if n big, we will need a good deal of CPU time.

Cubic Splines

- Consider the following example. Suppose that our data points are four points in the plane, namely: $\mathbf{P_1}=(0,0)$, $\mathbf{P_2}=(1,0)$, $\mathbf{P_3}=(1,1)$ i $\mathbf{P_4}=(0,1)$ and we want a smooth curve through these points.
- We have to choose the value of the tangents at the extremes, so we take the vectors: $\mathbf{v_1}=(1,-1)$ and $\mathbf{v_4}=(-1,-1)$

- The resulting system of equations is then:

$$\begin{pmatrix} 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \end{pmatrix} \begin{pmatrix} (1,-1) \\ \mathbf{v_2} \\ \mathbf{v_3} \\ (-1,-1) \end{pmatrix} = \begin{pmatrix} 3\left[(1,1)-(0,0)\right] \\ 3\left[(0,1)-(1,0)\right] \end{pmatrix}$$

$$= \begin{pmatrix} (3,3) \\ (-3,3) \end{pmatrix}$$

- After the necessary algebra we obtain:

$$\left(1,-1\right)+4\mathbf{v_2}+\mathbf{v_3}=(3,3)$$

$$\mathbf{v_2}+4\mathbf{v_3}+(-1,-1)=(-3,3)$$

- Equation which is solved by the vectors:

$$\mathbf{v_2}=\begin{pmatrix}\frac{2}{3} & \frac{4}{5}\end{pmatrix} \quad \text{and} \quad \mathbf{v_3}=\begin{pmatrix}-\frac{2}{3} & \frac{4}{5}\end{pmatrix}$$

Cubic Splines

- Each segment will be an Hermite curve, that can be computed using the Hermite matrix:

$$\mathbf{S}_1(t) = (t^3, t^2, t, 1)\begin{pmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} (0,0) \\ (1,0) \\ (1,-1) \\ \left(\frac{2}{3},\frac{4}{5}\right) \end{pmatrix}$$

$$= \left(-\frac{1}{3}, -\frac{1}{5}\right)t^3 + \left(\frac{1}{3}, \frac{6}{5}\right)t^2 + (1,-1)\,t$$

- Following the same procedure for the two additional curves we obtain:

$$\mathbf{S_2}(t) = \left(0, -\frac{2}{5}\right)t^3 + \left(-\frac{2}{3}, \frac{3}{5}\right)t^2 + \left(\frac{2}{3}, \frac{4}{5}\right)t + (1, 0)$$

- And:

$$\mathbf{S_3}(t) = \left(\frac{1}{3}, -\frac{1}{5}\right)t^3 - \left(\frac{2}{3}, \frac{3}{5}\right)t^2 + \left(-\frac{2}{3}, \frac{4}{5}\right)t + (1, 1)$$

Natural Cubic Splines

- Until now, the additional free conditions are fixed the value of the two tangent vectors in the extremes, $\mathbf{v_1}$ and $\mathbf{v_n}$. This kind of free condition is called "*clamped end condition*" and the resulting splines are called ***natural splines.***
- The are other ways to fix these two free conditions resulting in spline curves of different shape.

Relaxed Cubic Splines

- We could also fix the second derivatives at the extremes. This means to fix the value of the vectors:

$$\ddot{\mathbf{S}}_{\mathbf{1}}(0) \quad \text{and} \quad \ddot{\mathbf{S}}_{\mathbf{n-1}}(1)$$

- This second derivative are associated to the curvature of the curve. This condition is called "*relaxed condition*" and the resulting spline curves ***relaxed cubic splines***

Relaxed Cubic Splines

- In order to calculate this kind of splines, note that the second derivative of each of the polynomial curves can be written as:

$$\ddot{\mathbf{S}}_{\mathbf{i}}(t) = 6\mathbf{A}_{\mathbf{i}}t + 2\mathbf{B}_{\mathbf{i}}$$

$$\mathbf{A}_{\mathbf{i}} = 2\left(\mathbf{P}_{\mathbf{i}} - \mathbf{P}_{\mathbf{i+1}}\right) + \mathbf{v}_{\mathbf{i}} + \mathbf{v}_{\mathbf{i+1}}$$

$$\mathbf{B}_{\mathbf{i}} = 3\left(\mathbf{P}_{\mathbf{i+1}} - \mathbf{P}_{i}\right) - 2\mathbf{v}_{\mathbf{i}} - \mathbf{v}_{\mathbf{i+1}}$$

Relaxed Cubic Splines

- The relaxed conditions impose the equations:

$$-3\mathbf{P_1} + 3\mathbf{P_2} - 2\mathbf{v_1} - \mathbf{v_2} = 0 \Leftrightarrow$$

$$\mathbf{v_1} = \frac{3}{2}\left(\mathbf{P_2} - \mathbf{P_1}\right) - \frac{1}{2}\mathbf{v_2} 6\left(2\mathbf{P_{n\text{-}1}} - 2\mathbf{P_n} + \mathbf{v_{n\text{-}1}} + \mathbf{v_n}\right)$$

$$+2\left(-3\mathbf{P_{n\text{-}1}} + 3\mathbf{P_n} - 2\mathbf{v_{n\text{-}1}} - \mathbf{v_n}\right) = 0$$

$$\Leftrightarrow \mathbf{v_n} = \frac{3}{2}\left(\mathbf{P_n} - \mathbf{P_{n\text{-}1}}\right) - \frac{1}{2}\mathbf{v_{n\text{-}1}}$$

Relaxed Cubic Splines

- While the system matrix is now:

$$
\begin{pmatrix}
1 & 4 & 1 & 0 & \cdots & 0 \\
0 & 1 & 4 & 1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 1 & 4 & 1 & 0 \\
0 & \cdots & 0 & 1 & 4 & 1
\end{pmatrix}
\begin{pmatrix}
\mathbf{v}_2 \\ \vdots \\ \vdots \\ \vdots \\ \mathbf{v}_{n-1}
\end{pmatrix}
=
\begin{pmatrix}
3\left(\mathbf{P}_3 - \mathbf{P}_1\right) \\
3\left(\mathbf{P}_4 - \mathbf{P}_2\right) \\
\vdots \\
3\left(\mathbf{P}_{n-1} - \mathbf{P}_{n-3}\right) \\
3(\mathbf{P}_n - \mathbf{P}_{n-2})
\end{pmatrix}
$$

Relaxed Cubic Splines

- In the former system we have *n-2* rows and *n-2* columns.
- In order to compute the spline curve we use first the system of equations to calculate the tangent vectors $\mathbf{v}_2,\ldots,\mathbf{v}_{n-1}$, then, with the values of $\mathbf{v}_2$ and $\mathbf{v}_{n-1}$ we calculate $\mathbf{v}_1$ and $\mathbf{v}_n$ using:

$$\mathbf{v}_1 = \frac{3}{2}\left(\mathbf{P}_2 - \mathbf{P}_1\right) - \frac{1}{2}\mathbf{v}_2$$

$$\mathbf{v}_n = \frac{3}{2}\left(\mathbf{P}_n - \mathbf{P}_{n-1}\right) - \frac{1}{2}\mathbf{v}_{n-1}$$

Relaxed Cubic Splines

- We write now the Hermite curve of each piece. Using again the same points as in the former example: $\mathbf{P_1}$=(0,0), $\mathbf{P_2}$=(1,0), $\mathbf{P_3}$=(1,1) i $\mathbf{P_4}$=(0,1) , we obtain the system:

$$\begin{pmatrix} 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \end{pmatrix} \begin{pmatrix} \left(\frac{3}{2},0\right)-\frac{1}{2}\mathbf{v_2} \\ \mathbf{v_2} \\ \mathbf{v}_3 \\ \left(-\frac{3}{2},0\right)-\frac{1}{2}\mathbf{v}_3 \end{pmatrix} = \begin{pmatrix} (3,3) \\ (-3,3) \end{pmatrix}$$

Relaxed Cubic Splines

- The solutions are:

$$\mathbf{v}_2 = \left(\frac{3}{5}, \frac{2}{3}\right), \quad \mathbf{v}_3 = \left(-\frac{3}{5}, \frac{2}{3}\right)$$

- And then:

$$\mathbf{v}_1 = \frac{3}{2}(\mathbf{P}_2 - \mathbf{P}_1) - \frac{1}{2}\mathbf{v}_2 = \left(\frac{3}{2}, 0\right) - \frac{1}{2}\left(\frac{3}{5}, \frac{2}{3}\right) = \left(\frac{6}{5}, -\frac{1}{3}\right)$$

$$\mathbf{v}_4 = \frac{3}{2}(\mathbf{P}_4 - \mathbf{P}_3) - \frac{1}{2}\mathbf{v}_3 = \left(-\frac{3}{2}, 0\right) - \frac{1}{2}\left(-\frac{3}{5}, \frac{2}{3}\right) = \left(-\frac{6}{5}, -\frac{1}{3}\right)$$

Relaxed Cubic Splines

- We have only to build the Hermite curves. Thus the first segment is given by:

$$\mathbf{S}_1(t) = \left(t^3, t^2, t, 1\right)\begin{pmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} (0,0) \\ (1,0) \\ \left(\frac{6}{5}, -\frac{1}{3}\right) \\ \left(\frac{3}{5}, \frac{2}{3}\right) \end{pmatrix}$$

$$= \left(-\frac{1}{5}, \frac{1}{3}\right)t^3 + \left(\frac{6}{5}, -\frac{1}{3}\right)t$$

Cyclic Cubic Splines

- ***Cyclic Cubic Splines***

$$\mathbf{v_1} = \mathbf{v_n} \quad \text{i} \quad \ddot{\mathbf{S}}_1(0) = \dot{\mathbf{v}}_1 = \ddot{\mathbf{v}}_n = \ddot{\mathbf{S}}_{n-1}(1)$$

- In this case we proceed in a similar way than in the relaxed cubic splines. We first solve for the *n-2* inner tangents:

$$\begin{pmatrix} 1 & 4 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 4 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 4 & 1 & 0 \\ 0 & \cdots & 0 & 1 & 4 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{v}_2 \\ \vdots \\ \vdots \\ \vdots \\ \mathbf{v}_{n-1} \end{pmatrix} = \begin{pmatrix} 3(\mathbf{P}_3 - \mathbf{P}_1) \\ 3(\mathbf{P}_4 - \mathbf{P}_2) \\ \vdots \\ 3(\mathbf{P}_{n-1} - \mathbf{P}_{n-3}) \\ 3(\mathbf{P}_n - \mathbf{P}_{n-2}) \end{pmatrix}$$

Cyclic Cubic Splines

- Then the to additional tangents at the extremes are obtained from:

$$\mathbf{v}_1 = \mathbf{v}_n = \frac{3}{4}\left(\mathbf{P}_2 - \mathbf{P}_1 + \mathbf{P}_n - \mathbf{P}_{n\text{-}1}\right) - \frac{1}{4}\left(\mathbf{v}_2 - \mathbf{v}_{n\text{-}1}\right)$$

- Note that this curve does not need to be closed as we do not require a segment from $\mathbf{P_n}$ to $\mathbf{P_1}$

More Cubic Splines

- ***Periodic Cubic Splines***. In this case, the last coordinates of $\mathbf{P_1}$ and $\mathbf{P_n}$ must be the same:

$$\mathbf{v_1} = \mathbf{v_n} \quad \text{i} \quad \mathbf{P_0} = (x, y), \quad \mathbf{P_n}(x + p, y)$$

- where p is the period of the curve.
- This spline can be obtained as in the case of cyclic splines just imposing the additional condition that the y coordinates of the initial and last points be equal

$$\mathbf{P_1}(:,2) = \mathbf{P_n}(:,2)$$

- In 3D, however, this condition is more relaxed and it is enough to impose the equality of the tangents at the ending points

Closed Cubic Splines

- There is a somewhat different variant that we use to build a closed curve. To build these curves we simply add to new artificial points, equal to the first and second points.

$$\mathbf{P_{n+1} = P_1} \quad \text{and} \quad \mathbf{P_{n+2} = P_2}$$

Closed Spline Curves

- The problem is now the same as for the building the natural splines. The system matrix is now:

$$
\begin{pmatrix}
1 & 4 & 1 & 0 & \cdots & \cdots & 0 \\
0 & 1 & 4 & 1 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & \ddots & \ddots & \ddots & \ddots & 4 & 1 \\
1 & \ddots & \ddots & \ddots & \ddots & 1 & 4 \\
4 & 1 & 0 & \cdots & \cdots & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \vdots \\ \vdots \\ \mathbf{v}_{n-1} \\ \mathbf{v}_n
\end{pmatrix}
=
\begin{pmatrix}
\mathbf{3(P_3 - P_1)} \\ \mathbf{3(P_4 - P_2)} \\ \vdots \\ \vdots \\ \vdots \\ \mathbf{3(P_{n+1} - P_{n-1})} \\ \mathbf{3(P_{n+2} - P_n)}
\end{pmatrix}
$$

Closed Spline Curves

- The matrix of the system has now n x n dimensions, and we have to determine n tangent vectors
- The matrix is no more dominant diagonal dominant and we will need different algorithms in order to solve this kind of system, like the Gauss method.
- The iterative algorithms can also be used.

Closed Spline Curves

- We can build now the closed spline curve through the four points $\mathbf{P_1}$=(0,0), $\mathbf{P_2}$=(1,0), $\mathbf{P_3}$=(1,1) i $\mathbf{P_4}$=(0,1). The linear system is:

$$\begin{pmatrix} 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 1 & 0 & 1 & 4 \\ 4 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \\ \mathbf{v}_4 \end{pmatrix} = \begin{pmatrix} 3(\mathbf{P}_3 - \mathbf{P}_1) \\ 3(\mathbf{P}_4 - \mathbf{P}_2) \\ 3(\mathbf{P}_1 - \mathbf{P}_3) \\ 3(\mathbf{P}_2 - \mathbf{P}_4) \end{pmatrix}$$

Closed Spline Curves

- The solution of this system are the vectors:

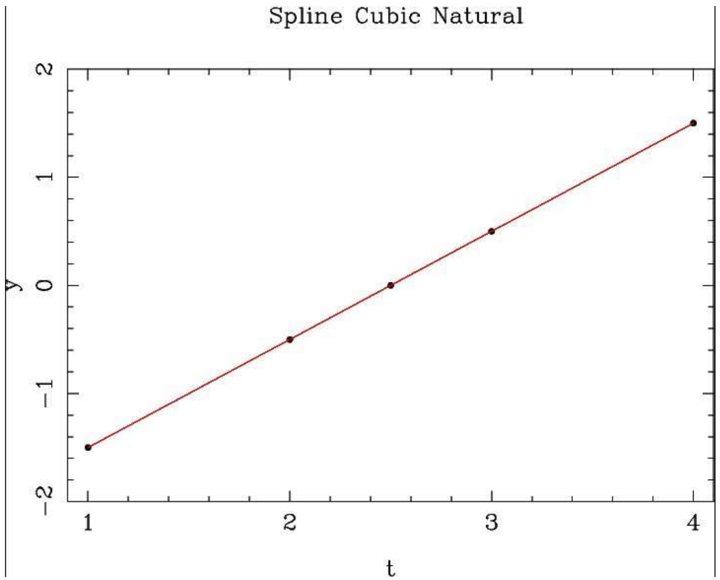
$$\mathbf{v}_1 = \left(\frac{3}{4}, -\frac{3}{4}\right), \quad \mathbf{v}_2 = \left(\frac{3}{4}, \frac{3}{4}\right)$$

$$\mathbf{v}_3 = \left(-\frac{3}{4}, \frac{3}{4}\right), \quad \mathbf{v}_4 = \left(-\frac{3}{4}, -\frac{3}{4}\right)$$

- From which we could eventually build the Hermite curves.

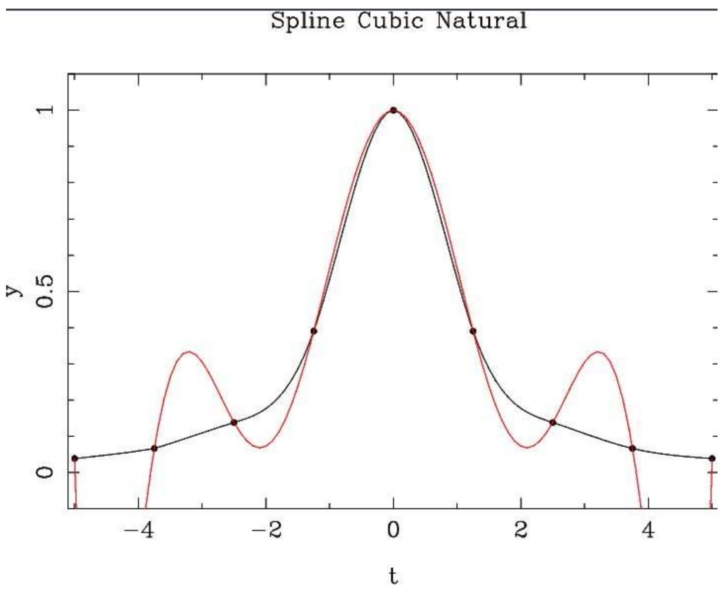
Cubic Splines

- With the spline curves we will recover exactly the polynomic functions of degree lower or equal to three. As in the case of the classic interpolating polynomials



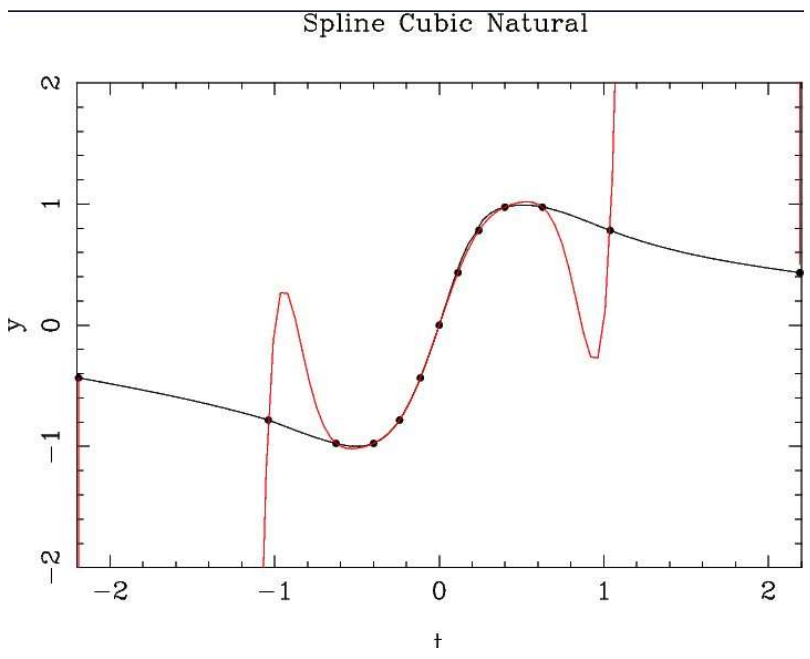
Cubic Splines

- With splines, however, we can get better representations of rational functions and we do not have to cope with the Runge phenomenon



Cubic Splines

- Splines can deal also with functions with asymptotes:



Cardinal Splines

- The cardinal splines are a variant of the cubic splines
- With the cubic splines we do not control the global shape of the curve and we have to solve a linear system of equations, which could be big is our data set is so
- With the cardinal splines we do not have these problems but we loose the continuity of the second derivatives

Cardinal Splines

- In the following figure we can se a curve passing through 7 points. The curve is continuous and made of pieces, two of them in thick black line

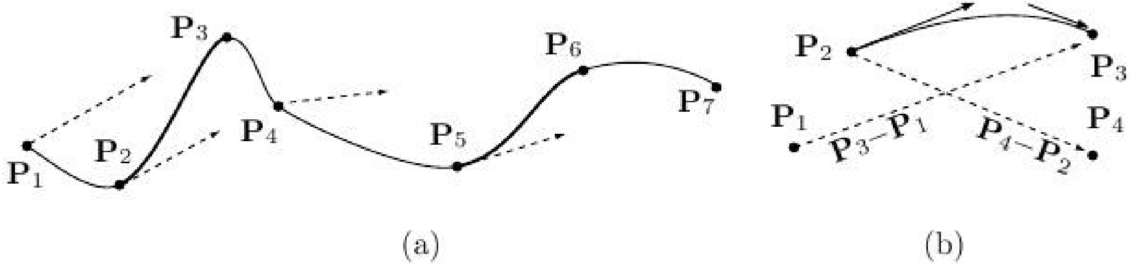


Figure 5.7: Tangent Vectors in a Cardinal Spline.

Cardinal Splines

- Following the piece of curve from $\mathbf{P_2}$ to $\mathbf{P_3}$ we observe that it starts in the same direction as the piece going from $\mathbf{P_1}$ to $\mathbf{P_3}$ while it ends with a tangent as the curve going from $\mathbf{P_2}$ to $\mathbf{P_4}$ This is true also for the piece going from $\mathbf{P_5}$ to $\mathbf{P_6}$.

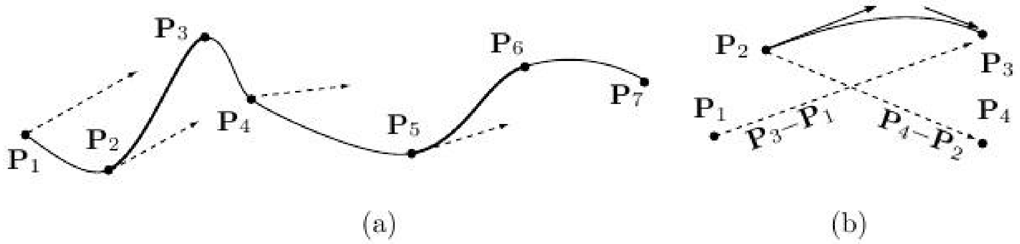


Figure 5.7: Tangent Vectors in a Cardinal Spline.

Cardinal Splines

- The cardinal splines for n points is calculated and draw in segments, each one depends on four points. On the other hand, each point participates in four segments at the most.
- This gives a certain local control on the curve. Varying one of the points we can modify up to four segments of the total curve

Cardinal Splines

- The segments are connected softly as the first derivatives are equal in the contact points. The cardinal splines will have continuity of first order. We cannot guarantee the second order continuity, as the second derivatives do not need to be equal
- To build the curve we organize the points in groups of 4 points.

$$\left[\mathbf{P}_1,\mathbf{P}_2,\mathbf{P}_3,\mathbf{P}_4\right],\left[\mathbf{P}_2,\mathbf{P}_3,\mathbf{P}_4,\mathbf{P}_5\right],\ldots,\left[\mathbf{P}_{n-3},\mathbf{P}_{n-2},\mathbf{P}_{n-1},\mathbf{P}_n\right]$$

Cardinal Splines

- Now we can use the Hermite interpolation to build a curve segment for each group, **P**(t). The two inner points give the start and ending points, while the outer points allow to build the tangents.
- Thus, in the case of $[\mathbf{P_1},\mathbf{P_2},\mathbf{P_3},\mathbf{P_4}]$ the curve goes from $\mathbf{P_2}$ to $\mathbf{P_3}$ while the tangents are given by $s(\mathbf{P_3}-\mathbf{P_1})$ and $s(\mathbf{P_4}-\mathbf{P_2})$, where *s* is a real parameter associated to the curve tension.

Cardinal Splines

- To define the segments going from $\mathbf{P}_1$ to $\mathbf{P}_2$ and from $\mathbf{P}_n$ to $\mathbf{P}_{n-1}$, we add to additional points, $\mathbf{P}_0$ and $\mathbf{P}_{n+1}$ which will control the start and ending of the curve interactively.
- This construction guarantees that the segments will join softly, as the last tangent vector of a segment is the same as the starting one of the following segment.

Cardinal Splines

- Thus in our case, the curve segment starting in $\mathbf{P}_2$ and ending in $\mathbf{P}_3$ will have the expression:

$$\mathbf{P}(t)=\left(t^3,t^2,t,1\right)\begin{pmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \mathbf{P}_2 \\ \mathbf{P}_3 \\ s(\mathbf{P}_3-\mathbf{P}_1) \\ s(\mathbf{P}_4-\mathbf{P}_2) \end{pmatrix}$$

$$=\left(t^3,t^2,t,1\right)\begin{pmatrix} -s & 2-s & s-2 & s \\ 2s & s-3 & 3-2s & -s \\ -s & 0 & s & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} \mathbf{P}_1 \\ \mathbf{P}_2 \\ \mathbf{P}_3 \\ \mathbf{P}_4 \end{pmatrix}$$

Cardinal Splines

- The tension (shape) of the curve can be controlled by the real parameter *s*. The larger this parameter, the larger the modulus of the initial tangent and the curve will follow this direction during more time. Lowering *s*, we have the contrary effect. If s is zero, we get an straight line between consecutive points. Normally s varies within $[0,1]$.

Cardinal Splines

- We can define also the parameter T as:

$$s = \frac{(1-T)}{2}, \quad \text{or} \quad T = 1-2s$$

- Zero tension T=0 is equivalent to s=1/2. This particular case is know as the ***Catmull-Rom spline***
- Varying from T=0 to T=1 we add tension until we get straight line, while varying from T=0 to T=-1 we obtain looser curves

Cardinal Splines

- We may consider the points $\mathbf{P_1}$=(1,0), $\mathbf{P_2}$=(3,1), $\mathbf{P_3}$=(6,2) and $\mathbf{P_4}$=(6,2) the cardinal spline joining $\mathbf{P_2}$ and $\mathbf{P_3}$ is:

$$\mathbf{P}(t) = (t^3, t^2, t, 1)\begin{pmatrix} -s & 2-s & s-2 & s \\ 2s & s-3 & 3-2s & -s \\ -s & 0 & s & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\begin{bmatrix} (1,0) \\ (3,1) \\ (6,2) \\ (2,3) \end{bmatrix}$$

$$= t^3(4s-6, 4s-2) + t^2(-9s+9, -6s+3) + t(5s, 2s) + (3,1)$$

Cardinal Splines

- If we use the maximum tension (T=1 or s=0) we obtain the straight line:

$$\begin{aligned}\mathbf{P}(t) &= (-6,2)t^3 + (9,3)t^2 + (3,1) = (3,1)(-2t^3 + 3t^2) + (3,1) \\ &= (3,1)u + (3,1)\end{aligned}$$

- While if we use T=0 or s=1/2 we obtain the Catmull-Rom curve

$$\mathbf{P}(t) = (-4,0)t^3 + (4.5,0)t^2 + (2.5,1)t + (3,1)$$