# Lesson 10- Master Equations (Bonus Track)

# Markov Processes

A **Markov process** is a stochastic process where the future state depends only on the current state and not on the history (memoryless property).

$$P(X_{t+1} | X_t, X_{t-1}, \ldots, X_0) = P(X_{t+1} | X_t)$$

Key Characteristics:

**States**: A finite or countable set of possible states.
**Transitions**: Probabilities of moving between states.
**Markov Property**: The probability of transitioning to the next state depends only on the current state.

# Markov Processes

**Define probabilities** $P_i(t)$: the probability of being in state i at time t.

**Recurrence relations describe how these probabilities evolve**:

$$P_i(t+1) = \sum_j P_j(t)\, \omega_{j \to i},$$

where $\omega_{j \to i}$ is the transition probability from j to i .

# Markov Processes

**Example**: Weather Prediction

Imagine a simple weather model with three states: Sunny (S), Cloudy (C), and Rainy (R). The weather tomorrow depends only on today's weather, not on the weather from previous days.

Transition Probabilities: The probabilities of transitioning from one state to another are captured in a transition matrix:

$$\omega = \begin{bmatrix} 0.8 & 0.15 & 0.05 \\ 0.2 & 0.6 & 0.2 \\ 0.25 & 0.25 & 0.5 \end{bmatrix}$$

Each row represents the current state, and each column represents the next state.
If today is Sunny, tomorrow's weather is determined by the probabilities in the first row: $\omega_{S \to S}$ (80%), $\omega_{S \to C}$ (15%), $\omega_{S \to R}$ (5%)

## What Are Master Equations?

- **Continuous-time** analog of recurrence relations for **Markov chains**.
- Differential equations describing the **evolution of probabilities** in Markov processes.
- Fundamental for understanding the dynamics of systems **out of equilibrium**.
- Connect **statistical mechanics** with the **theory of stochastic processes**.

## Applications

- **Quantum mechanics**: e.g. quantum optics, condensed matter, atomic physics, quantum information.
- **Birth and death processes**: e.g. bacteria population growth, demography, queueing theory, performance engineering.
- **Quantitative finance:** e.g. option Pricing, risk management.
- **Network dynamics:** e.g. information propagation, epidemics.
- **Social dynamics:** e.g. language competition, opinion dynamics.

## Two-State System Example

Consider a particle that can be in either state 1 or 2.
Let $P_1(t)$ and $P_2(t)$ represent the probabilities of being in state 1 or 2 at time t, respectively. Obviously, $P_1(t) + P_2(t) = 1$

Transition rates:

$\omega_{1\to 2}$: Rate of transitioning from state 1 to 2.
$\omega_{2\to 1}$: Rate of transitioning from state 2 to 1.

At $(t + dt)$, $P_1(t + dt)$ can be expressed as:

$$P_1(t+dt) = P_1(t)\text{Prob(staying in 1)} + P_2(t)\text{Prob(jumping from 2 to 1)}$$

By the definition of the rate, the probability of jumping from 2 to 1 in the time interval $(t, t+dt)$ is $\omega_{2\to 1}dt$ whereas the probability of staying in state 1 is one minus the probability of leaving state 1 in the same time interval, or $1 - \omega_{1\to 2}dt$. This leads to:

$$P_1(t+dt) = P_1(t)[1 - \omega_{1\to 2}dt] + P_2(t)\omega_{2\to 1}dt + O(dt^2)\,.$$

## Master equations for $P_1(t)$ and $P_2(t)$

In the limit $dt \to 0$:

$$\frac{P_1(t)}{dt} = -\,\omega_{1\to 2}P_1(t) + \omega_{2\to 1}P_2(t)$$

equivalent reasoning for $P_2(t)$

$$\frac{P_2(t)}{dt} = -\,\omega_{2\to 1}P_2(t) + \omega_{1\to 2}P_1(t)$$

These equations have to be solved for some **initial conditions** $P_1(t_0)$ and $P_2(t_0)$.

Note that $\dfrac{d}{dt}[P_1(t) + P_2(t)] = 0$, has it should be, given that $P_1(t) + P_2(t) = 1$ for all times t.

## Master equations for $P_1(t)$ and $P_2(t)$

For constant rates, we can solve these equations explicitly

$$\frac{P_1(t)}{dt} = -\omega_{1\to 2}P_1(t) + \omega_{2\to 1}P_2(t)$$

equivalent reasoning for $P_2(t)$

$$\frac{P_2(t)}{dt} = -\omega_{2\to 1}P_2(t) + \omega_{1\to 2}P_1(t)$$

These equations have to be solved for some **initial conditions** $P_1(t_0)$ and $P_2(t_0)$.

Note that $\frac{d}{dt}[P_1(t) + P_2(t)] = 0$, has it should be, given that $P_1(t) + P_2(t) = 1$ for all times t. We can use this to substitute $P_2(t) = 1 - P_1(t)$ in the first equation:

$$\frac{P_1(t)}{dt} = -\omega_{1\to 2}P_1(t) + \omega_{2\to 1}(1 - P_1(t)) = \omega_{2\to 1} - P_1(t)(\omega_{1\to 2} + \omega_{2\to 1})$$

## Reminder: Solving a First-Order Linear ODE

Consider the general equation:

$$\frac{dP(t)}{dt} = -aP(t) + b, \text{ where } a \text{ and } b \text{ are constants.}$$

**Step 1**: Solve the Homogeneous Equation:

$$\frac{dP_h(t)}{dt} = -aP_h(t) \implies P_h(t) = Ce^{-at},$$ where C is a constant determined by initial conditions.

**Step 2**: Find a Particular Solution:

$$P_p(t) = \frac{b}{a}.$$

General Solution:

$$P(t) = P_h(t) + P_p(t) = Ce^{-at} + \frac{b}{a}.$$

Apply Initial Condition $P(t_0) = P_0$: $\quad C = (P_0 - \frac{b}{a})e^{at_0}.$

# Explicit solution Master equations for $P_1(t)$ and $P_2(t)$

For constant rates, we can solve these equations explicitly

$$\frac{P_1(t)}{dt} = \omega_{2\to1} - P_1(t)(\omega_{1\to2} + \omega_{2\to1})$$

According to the previous general solution:

$$\frac{dP(t)}{dt} = -aP(t) + b \implies P(t) = (P_0 - \frac{b}{a})e^{-a(t-t_0)} + \frac{b}{a}.$$

Then: $a = (\omega_{1\to2} + \omega_{2\to1})$ and $b = \omega_{2\to1}$

$$P_1(t) = \left(P_1(t_0) - \frac{\omega_{2\to1}}{\omega_{1\to2} + \omega_{2\to1}}\right) e^{-(\omega_{1\to2}+\omega_{2\to1})(t-t_0)} + \frac{\omega_{2\to1}}{\omega_{1\to2} + \omega_{2\to1}}$$

## Explicit solution Master equations for $P_1(t)$ and $P_2(t)$

$$P_1(t) = \left(P_1(t_0) - \frac{\omega_{2\to1}}{\omega_{1\to2} + \omega_{2\to1}}\right) e^{-(\omega_{1\to2}+\omega_{2\to1})(t-t_0)} + \frac{\omega_{2\to1}}{\omega_{1\to2} + \omega_{2\to1}}$$
$$P_2(t) = \left(P_2(t_0) - \frac{\omega_{1\to2}}{\omega_{1\to2} + \omega_{2\to1}}\right) e^{-(\omega_{1\to2}+\omega_{2\to1})(t-t_0)} + \frac{\omega_{1\to2}}{\omega_{1\to2} + \omega_{2\to1}}$$

From here, we can obtain the **stationary** distribution as the limit $t \to \infty$:

$$P_1^{\mathrm{st}} = \frac{\omega_{2\to1}}{\omega_{1\to2} + \omega_{2\to1}}, \quad P_2^{\mathrm{st}} = \frac{\omega_{1\to2}}{\omega_{1\to2} + \omega_{2\to1}}.$$

Note that in this case, the stationary distribution satisfies: $\omega_{1\to2}P_1^{\mathrm{st}} = \omega_{2\to1}P_2^{\mathrm{st}}$, which is know as **detailed balance**.

Physical Meaning: It ensures no net flux of probability between states, preserving equilibrium.

## Conditional probabilities $P(i, t \mid j, t_0)$

The probability that the particle is in state $i$ at time $t$ given that it was in state $j$ at time $t_0$.

Challenges in Direct Computation:

- The particle may undergo multiple jumps $1 \to 2 \to 1$, or $2 \to 1 \to 2$, any number of times. Must sum over all possible paths, making direct computation difficult.

Alternative Approach:

- Use the master equation solutions. Set initial conditions:

$$P_1(t_0) = 1, \quad P_2(t_0) = 0 \quad \text{or} \quad P_1(t_0) = 0, \quad P_2(t_0) = 1.$$

Start with the master equation solution for $P_1(t)$ and $P_2(t)$:

$$P_1(t) = \left(P_1(t_0) - \frac{\omega_{2\to1}}{\omega_{1\to2} + \omega_{2\to1}}\right) e^{-(\omega_{1\to2}+\omega_{2\to1})(t-t_0)} + \frac{\omega_{2\to1}}{\omega_{1\to2} + \omega_{2\to1}}$$

We can obtain $P(1,t\,|\,1,t_0)$ from $P_1(t)$ setting as a initial condition $P_1(t_0) = 1, \quad P_2(t_0) = 0$, as we know (with probability 1) that the particle is in state 1 at time $t_0$.

$$P(1,t \mid 1,t_0) = \frac{\omega_{2\to1} + \omega_{1\to2} e^{-(\omega_{1\to2}+\omega_{2\to1})(t-t_0)}}{\omega_{1\to2} + \omega_{2\to1}},$$

$$P(2,t \mid 1,t_0) = \frac{\omega_{1\to2}(1 - e^{-(\omega_{1\to2}+\omega_{2\to1})(t-t_0)})}{\omega_{1\to2} + \omega_{2\to1}}.$$

equivalent for initial state 2:

$$P(1,t \mid 2,t_0) = \frac{\omega_{2\to1}(1 - e^{-(\omega_{1\to2}+\omega_{2\to1})(t-t_0)})}{\omega_{1\to2} + \omega_{2\to1}},$$

$$P(2,t \mid 2,t_0) = \frac{\omega_{1\to2} + \omega_{2\to1} e^{-(\omega_{1\to2}+\omega_{2\to1})(t-t_0)}}{\omega_{1\to2} + \omega_{2\to1}}.$$

## Conditional probabilities

In terms of these conditional probabilities, we can reason that the probability that the particle is in state 1 at time $t$ is the probability that it was in state 1 at time $t_0$ times the probability $P(1,t \mid 1,t_0)$ plus the probability that it was in state 2 at $t_0$ times the probability $P(1,t \mid 2,t_0)$:

$$P_1(t) = P_1(t_0)P(1,t \mid 1,t_0) + P_2(t_0)P(1,t \mid 2,t_0),$$

and similarly:

$$P_2(t) = P_1(t_0)P(2,t \mid 1,t_0) + P_2(t_0)P(2,t \mid 2,t_0)\,.$$

If we consider an intermediate time $t_0 < t_1 < t$, the probability that the particle is in state 1 at time $t$ provided it was in state 2 at time $t_0$ can be computed using the probabilities that the particle was at the intermediate time $t_1$ in one of the two states:

$$P(1,t \mid 2,t_0) = P(1,t \mid 1,t_1)P(1,t_1 \mid 2,t_0) + P(1,t \mid 2,t_1)P(2,t_1 \mid 1,t_0)$$

and similar for $P(1,t \mid 1,t_0)$, $P(2,t \mid 2,t_0)$, $P(2,t \mid 1,t_0)$, these relations are known as **Chapman-Kolmogorov equations**.

## Occupation numbers

So far a particle that can jump between two states and we asked for the probability for that particle to be in state 1 or in state 2. We now consider a **system which is composed by N of such particles**, each one of them making the random jumps from state 1 to 2 or vice-versa.

We introduce the **occupation number** $n$ of state 1 and ask now for the probability $p(n;t)$ that at a given time $t$ exactly $n$ of the $N$ particles are in that state.

If the $N$ particles are independent of each other and they all start in the same state, such that $P_1(t)$ is the same for all particles, then $p(n;t)$ is given by a binomial distribution:

$$p(n;t) = \binom{N}{n} P_1(t)^n \big(1 - P_1(t)\big)^{N-n}.$$

We will now find the equations governing the evolution of $p(n;t)$.

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We will now find the equations governing the evolution of $p(n; t)$. As before, we will relate the probabilities at times $t$ and $t + dt$.

## Occupation numbers

How can we have n particles in state 1 at time $t + dt$? There are three possibilities:

(1) There were $n$ particles in state 1 and $N - n$ in state 2 at time $t$ and **no one left the state it was in**.

(2) There were $n - 1$ particles in state 1 and one of the $N - (n - 1)$ particles that were in state 2 **jumped from 2 to 1 during that interval.**

(3) There were $n + 1$ particles in state 1 and one particle **jumped from 1 to 2 during that interval.**

$$
\begin{aligned}
p(n; t + dt) = \; & p(n; t) \cdot \textbf{Prob(no particle jumped)} \\
& + p(n - 1; t) \cdot \textbf{Prob(any of the N} - \textbf{n} + \textbf{1 particles jumps from 2} \rightarrow \textbf{1)} \\
& + p(n + 1; t) \cdot \textbf{Prob(any of the n} + \textbf{1 particles jumps from 1} \rightarrow \textbf{2)} \\
& + O(dt^2)\,.
\end{aligned}
$$

## Occupation numbers

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(1)**The probability that no particle jumped** is the product of probabilities that none of the $n$ particles in state 1 jumped to 2 and none of the $N-n$ particles in state 2 jumped to 1. The probability that one particle does not jump from 1 to 2 is $1-\omega_{1\rightarrow 2}dt$, hence, the probability that none of the $n$ particles jumps from 1 to 2 is the product for all particles of these probabilities, or $[1-\omega_{1\rightarrow 2}dt]^n$. Expanding to first order in $dt$, this is equal to $1-n\omega_{1\rightarrow 2}dt+O(dt^2)$. Similarly, the probability that none of the $N-n$ particles in 2 jumps to 1 is $1-(N-n)\omega_{2\rightarrow 1}dt+O(dt^2)$. Finally, the probability that no jump occurs whatsoever is the product of these two quantities

$$1-\left(n\omega_{1\rightarrow 2}+(N-n)\omega_{2\rightarrow 1}\right)dt+O(dt^2)$$

## Occupation numbers

$$
\begin{aligned}
p(n;t+dt) = \; & p(n;t)\cdot \textbf{Prob(no particle jumped)} \\
& + p(n-1;t)\cdot \textbf{Prob(any of the N}-\textbf{n}+\textbf{1 particles jumps from 2}\rightarrow\textbf{1)} \\
& + p(n+1;t)\cdot \textbf{Prob(any of the n}+\textbf{1 particles jumps from 1}\rightarrow\textbf{2)} \\
& + O(dt^2)\,.
\end{aligned}
$$

(2) **The probability that any of the $N-n+1$ jumped from $2\rightarrow 1$.** The probability that one particle jumps from $2\rightarrow 1$ is $\omega_{1\rightarrow 2}dt$, hence, the probability that any of the $N-n+1$ particles jumps from $2\rightarrow 1$ is the sum of these probabilities, $(N-n+1)\omega_{2\rightarrow 1}dt$.

(3) **The probability that any of the $n+1$ jumped from $1\rightarrow 2$.** The probability that one particle jumps from $1\rightarrow 2$ is $\omega_{2\rightarrow 1}dt$, hence, the probability that any of the $n+1$ particles jumps from $1\rightarrow 2$ is the sum of these probabilities, $(n+1)\omega_{1\rightarrow 2}dt$.

## Occupation numbers

$$
\begin{aligned}
p(n;t+dt) = \; & p(n;t) \cdot \textbf{Prob(no particle jumped)} \\
& + p(n-1;t) \cdot \textbf{Prob(any of the N} - \textbf{n} + \textbf{1 particles jumps from 2} \to \textbf{1)} \\
& + p(n+1;t) \cdot \textbf{Prob(any of the n} + \textbf{1 particles jumps from 1} \to \textbf{2)} \\
& + O(dt^2)\,.
\end{aligned}
$$

Putting all these terms together, we obtain:

$$
\begin{aligned}
p(n;t+dt) = \; & p(n;t) \cdot \Big(1 - \big(n\omega_{1\to 2} + (N-n)\omega_{2\to 1}\big)\Big)\, dt \\
& + p(n-1;t) \cdot (N-n+1)\omega_{2\to 1} dt \\
& + p(n+1;t) \cdot (n+1)\omega_{1\to 2} dt \\
& + O(dt^2)\,.
\end{aligned}
$$

Rearranging and taking the limit $dt \to 0$ we obtain the differential equations:

$$
\begin{aligned}
\frac{\partial p(n;t)}{\partial t} = & -(n \cdot \omega_{1\to 2} + (N-n) \cdot \omega_{2\to 1}) p(n;t) + (N-n+1) \cdot \omega_{2\to 1} \cdot p(n-1;t) \\
& + (n+1) \cdot \omega_{1\to 2} \cdot p(n+1;t)\,.
\end{aligned}
$$

## Occupation numbers

$$
\begin{aligned}
\frac{\partial p(n;t)}{\partial t} = &-(n \cdot \omega_{1\to 2} + (N-n)\cdot \omega_{2\to 1})p(n;t) + (N-n+1)\cdot \omega_{2\to 1} \cdot p(n-1;t) \\
&+(n+1)\cdot \omega_{1\to 2} \cdot p(n+1;t)\,.
\end{aligned}
$$

Now we have a system of $N+1$ equations for each $n = 0,1,\cdots,N$.

## Example

*Consider a closed economy with $N$ agents, where wealth $\omega$ takes discrete values $\omega = 0,1,2,\cdots, W_{\max}$. During each time step, wealth is transferred between two agents randomly. Agents can move between discrete wealth states $\omega - 1, \quad \omega + 1, \quad \omega$, depending on whether they lose, gain, or retain wealth during a transaction.*

The master equation describes the probability distribution of wealth across the entire population. Specifically $p(\omega, t)$: The probability that a randomly chosen agent has wealth $\omega$ at time $t$.

This is a population-level description that tracks how the fraction of agents with wealth $\omega$ evolves over time due to stochastic transitions.

$$
\begin{aligned}
\frac{\partial p(\omega, t)}{\partial t} = -\left[P_{\text{gain}}(\omega) + P_{\text{lose}}(\omega)\right] p(\omega, t) \\
+P_{\text{gain}}(\omega - 1)p(\omega - 1,t) \\
+P_{\text{lose}}(\omega + 1)p(\omega + 1,t)\,.
\end{aligned}
$$

## Example

$$\frac{\partial p(\omega,t)}{\partial t} = -\left[P_{\text{gain}}(\omega) + P_{\text{lose}}(\omega)\right] p(\omega,t)$$
$$+P_{\text{gain}}(\omega-1)p(\omega-1,t)$$
$$+P_{\text{lose}}(\omega+1)p(\omega+1,t)\,.$$

Let us assume that $P_{\text{gain}} = \alpha$, and $P_{\text{lose}} = \beta$, both constants.

At the **stationary state**: $\dfrac{\partial p(\omega,t)}{\partial t} = 0$, hence we obtain:

$$0 = -(\alpha+\beta)p(\omega) + \alpha p(\omega-1) + \beta p(\omega+1)$$

This is a **second order-recurrence**, the way of finding a solution is through the characteristic equation. Assume a solution of the form $p(\omega) = r^{\omega}$ where $r$ is a constant to be determined. Substituting:

$$0 = -(\alpha+\beta)r^{\omega} + \alpha r^{\omega-1} + \beta r^{\omega+1}$$

## Example

$$0 = -(\alpha+\beta)r^{\omega} + \alpha r^{\omega-1} + \beta r^{\omega+1}$$

factorizing:

$$0 = -(\alpha+\beta) + \alpha r^{-1} + \beta r^{1}$$

rearranging terms:

$$\beta r^2 - (\alpha+\beta)r + \alpha = 0$$

solutions:

$$r = \frac{-(\alpha+\beta) \pm \sqrt{(\alpha+\beta)^2 - 4\alpha\beta}}{2\beta}$$

simplifying:

$$r = \frac{(\alpha+\beta) \pm |\alpha-\beta|}{2\beta}\,.$$

Two solutions: $r_1 = 1$ (corresponding to uniform behaviour), and $r_2 = \alpha/\beta$ (corresponding to a geometric decay or growth solution)

## Example

**General solution**: $p(\omega) = C_1 1^\omega + C_2(\alpha/\beta)^\omega = C_1 + C_2(\alpha/\beta)^\omega$

**Boundary conditions and normalization**:

- there is a lower boundary at $\omega = 0$, we might require

$$p(\omega) \geq 0 \quad \text{and} \quad p(\omega) \to 0 \text{ as } \omega \to \infty\,.$$

- the total probability must sum 1

$$\sum_{\omega=0}^{\infty} p(\omega) = 1$$

$$\Longrightarrow C_1 = 0 \text{ and } p(\omega) = C_2(\alpha/\beta)^\omega$$

## Example

then:

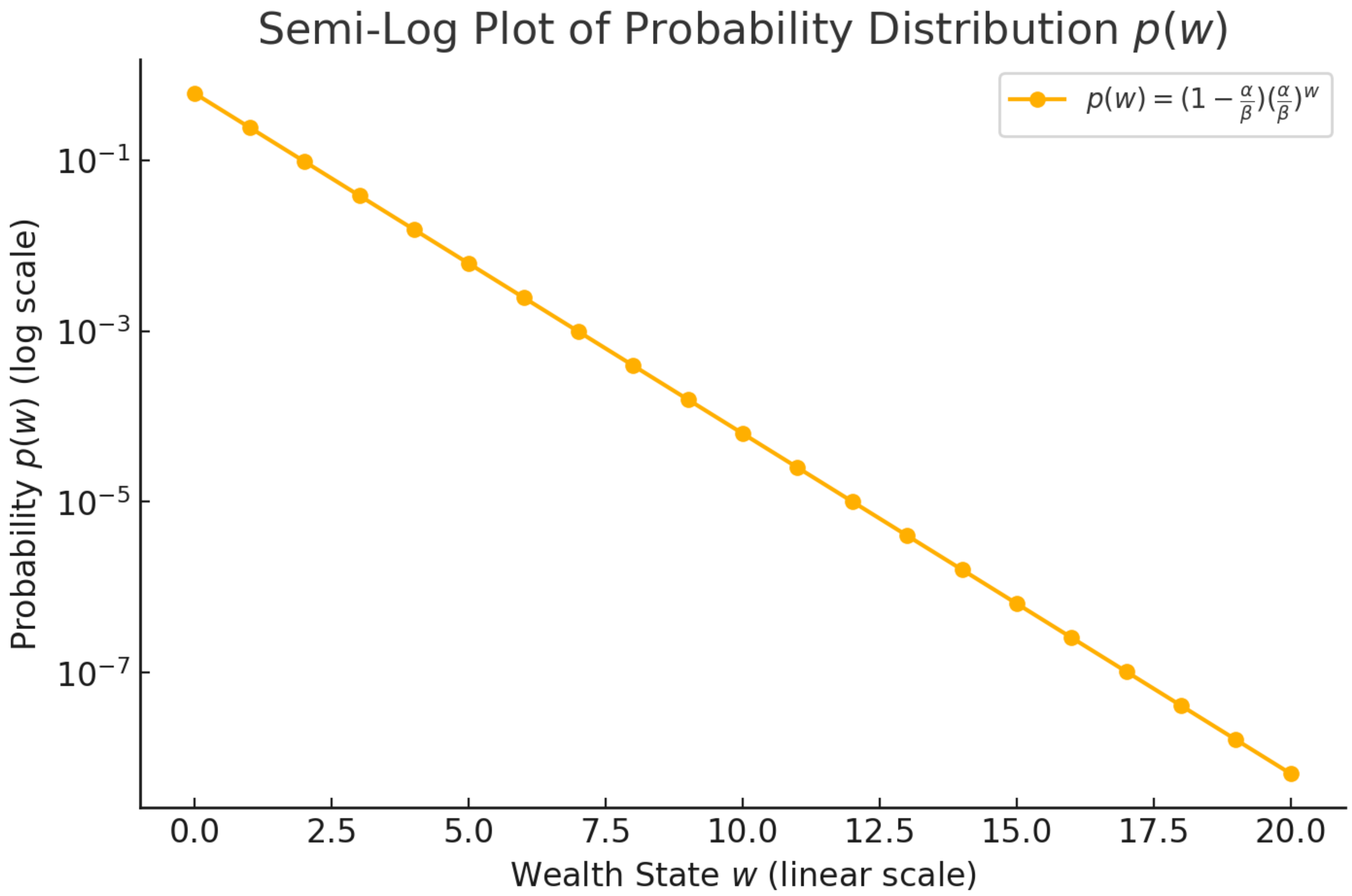
$$\sum_{\omega=0}^{\infty} C_2(\alpha/\beta)^\omega = 1 \text{ (only solution if } \alpha/\beta < 1)$$

$$C_2 \frac{1}{1-(\alpha/\beta)} = 1 \implies C_2 = 1 - \frac{\alpha}{\beta}$$

Back to the general solution in the stationary state:

$$p(\omega) = \left(1 - \frac{\alpha}{\beta}\right)\left(\frac{\alpha}{\beta}\right)^\omega$$

**Interpretation**: the probabilities decrease exponentially as $\omega$ increases. Most of the probability mass is concentrated at smaller values of $\omega$, meaning **low wealth states dominate**.



Semi-Log Plot of Probability Distribution $p(w)$
$p(w) = (1 - \frac{\alpha}{\beta})(\frac{\alpha}{\beta})^w$
$10^{-1}$
$10^{-3}$
$10^{-5}$
$10^{-7}$
Probability $p(w)$ (log scale)
0.0
2.5
5.0
7.5
10.0
12.5
15.0
17.5
20.0
Wealth State $w$ (linear scale)