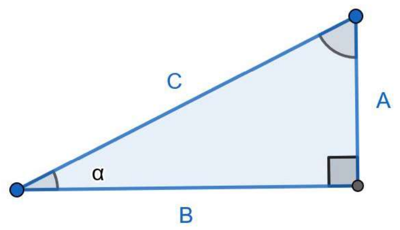
2.3.4 Trigonometry

We assume that students have a basic knowledge of trigonometry. We will therefore limit ourselves to prove some well known results.

Exercise 2.24. *Prove the Pythagorean theorem.*

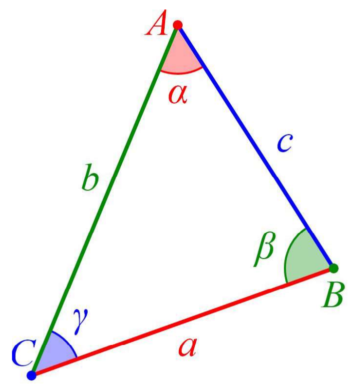
Solution: Consider the following triangle.



Since $\sin\alpha = \frac{A}{C}$, $\cos\alpha = \frac{B}{C}$ and $\sin^2\alpha + \cos^2\alpha = 1$, we have that $C^2 = A^2 + B^2$. □

Exercise 2.25 (Law of sinus)**.** *With the notation used in the following triangle, prove that*

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}.$$



Solution: If one angle is right, then the result follows by definition of sinus as a trigonometric ratio. Assume that no angle is right.

Let x be the altitude to the side c. If $\beta < \frac{\pi}{2}$ and $\alpha < \frac{\pi}{2}$, then we have $a\sin\beta = x = b\sin\alpha$. Therefore, $\dfrac{a}{\sin\alpha} = \dfrac{b}{\sin\beta}$.

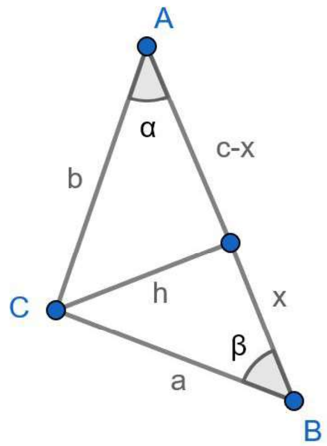
Notice that, if $\beta > \frac{\pi}{2}$, then $\sin\beta = \sin(\pi - \beta) = \frac{x}{a}$, and so $\dfrac{a}{\sin\alpha} = \dfrac{b}{\sin\beta}$.

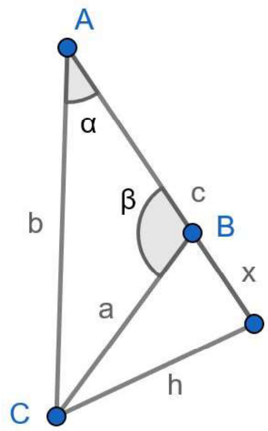
The case $\dfrac{b}{\sin\beta} = \dfrac{c}{\sin\gamma}$ is deduced by analogy. □

Exercise 2.26 (Law of cosine, or generalized Pythagorean theorem)**.** *With the notation of Exercise 2.25, prove that the following statements hold.*

- $a^2 = b^2 + c^2 - 2cb \cdot \cos\alpha$.
- $b^2 = a^2 + c^2 - 2ca \cdot \cos\beta$.
- $c^2 = a^2 + b^2 - 2ab \cdot \cos\gamma$.

Solution: By symmetry, we only need to prove one item. Let h be the altitude to side c, differentiating the two possible cases, according to the figures.





In the first case, the Pythagorean theorem and definition of cosine lead to

$$b^2 = h^2 + (c^2 - 2cx + x^2), \quad a^2 = x^2 + h^2, \quad \cos\beta = \frac{x}{a}.$$

Therefore, $b^2 = a^2 + c^2 - 2ca \cdot \cos\beta$.

Analogously, in the second case,

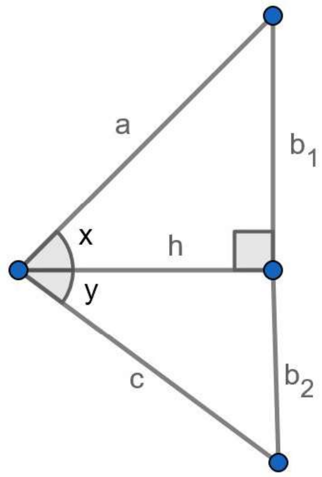
$$b^2 = h^2 + (c^2 + 2cx + x^2), \quad a^2 = x^2 + h^2, \quad -\cos\beta = \cos(\pi - \beta) = \frac{x}{a}.$$

Therefore, $b^2 = a^2 + c^2 - 2ca \cdot \cos\beta$. □

Exercise 2.27. *Prove that* $\cos(x + y) = \cos x \cdot \cos y - \sin x \cdot \sin y$.

Solution: By the law of cosine,

$$(b_1 + b_2)^2 = a^2 + c^2 - 2ac \cdot \cos(x + y).$$



Since $a^2 = h^2 + b_1^2$ and $c^2 = h^2 + b_2^2$, we have

$$2b_1 b_2 = 2h^2 - 2ac \cdot \cos(x+y).$$

Now, since $b_1 = a \cdot \sin x$, $b_2 = c \cdot \sin y$, $h = a \cdot \cos x$ and $h = c \cdot \cos y$, we have

$$2ac \cdot \sin x \cdot \sin y = 2ac \cdot \cos x \cdot \cos y - 2ac \cdot \cos(x+y).$$

Therefore, $\cos(x+y) = \cos x \cdot \cos y - \sin x \cdot \sin y$. □

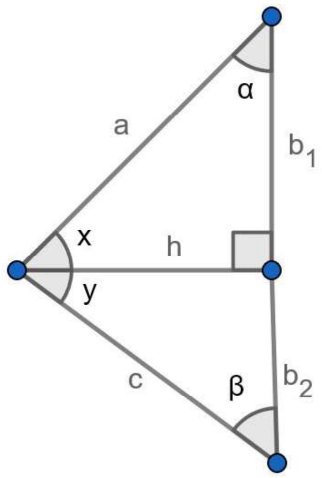
Notice that the following identity is a particular case of the previous statement.

$$\cos(2x) = \cos^2 x - \sin^2 x.$$

Exercise 2.28. *Prove that* $\sin(x+y) = \cos x \cdot \sin y + \sin x \cdot \cos y$.

Solution: By the law of sinus,

$$\frac{\sin(x+y)}{b_1 + b_2} = \frac{\sin\alpha}{c} = \frac{\sin\beta}{a}.$$



Therefore,

$$\begin{aligned}\sin(x+y) &= (b_1+b_2)\frac{\sin\alpha}{c}\\ &= (a\cdot\sin x + c\cdot\sin y)\frac{h}{ac}\\ &= \frac{h}{c}\cdot\sin x + \frac{h}{a}\cdot\sin y\\ &= \cos x\cdot\sin y + \sin x\cdot\cos y.\end{aligned}$$

□

Notice that the following identity is a particular case of the previous statement.

$$\sin(2x) = 2\sin x\cdot\cos x.$$

Exercise 2.29. *Find the length of a circle of radius r.*

Solution: We consider a circle C of radius r and center o. Let P_n be a regular polygon of n sides and center o whose sides are chords of C, i.e., P_n is inscribed over C. The central angle associated to any side x of P_n has measures $\frac{2\pi}{n}$. Thus, $\sin\left(\frac{\alpha}{2}\right) = \frac{x}{2r}$. Hence, the perimeter of P_n is

$$\begin{aligned}l(P_n) &= nx\\ &= 2nr\sin\left(\frac{\pi}{n}\right)\\ &= 2\pi r\frac{\sin\left(\frac{\pi}{n}\right)}{\frac{\pi}{n}}.\end{aligned}$$

Therefore, the length of C is

$$l(C) = \lim_{n\to\infty} l(P_n) = 2\pi r.$$

$\square$

Notice that the number π is well defined from any circle C as the ratio

$$\pi = \frac{l(C)}{D(C)},$$

where $D(C)$ denotes the diameter of C. In fact, in the previous proof, we do not need to use the constant π, as we can write the proof by using any symbol to denote the length of the flat angle. Note also that the proof of the trigonometric identities used in this exercise can also be done without knowing the value of the constant π. For this reason, this definition of the constant π is correct. A different problem is to find an approximate value of the number π, but this is only possible on a practical level and has been done for thousands of years.

Exercise 2.30. *Prove that if α is the angle defined by the pair of vectors $\overrightarrow{u}, \overrightarrow{v}$ in an Euclidean space, then*

$$\cos\alpha = \frac{\overrightarrow{u}\cdot\overrightarrow{v}}{\|\overrightarrow{u}\|\|\overrightarrow{v}\|}.$$

Solution: By the cosine law we have

$$\|\overrightarrow{u}-\overrightarrow{v}\|^2 = \|\overrightarrow{u}\|^2 + \|\overrightarrow{v}\|^2 - 2\|\overrightarrow{u}\|\cdot\|\overrightarrow{v}\|\cos\alpha.$$

Now, since

$$\|\overrightarrow{u}-\overrightarrow{v}\|^2 = (\overrightarrow{u}-\overrightarrow{v})\cdot(\overrightarrow{u}-\overrightarrow{v}) = \|\overrightarrow{u}\|^2 + \|\overrightarrow{v}\|^2 - 2\overrightarrow{u}\cdot\overrightarrow{v},$$

we conclude that $\overrightarrow{u}\cdot\overrightarrow{v} = \|\overrightarrow{u}\|\cdot\|\overrightarrow{v}\|\cos\alpha$, as required. $\square$

Notice that this exercise is a proof of the well-known fact that for non-null vectors, $\overrightarrow{u} \perp \overrightarrow{v}$ if and only if $\overrightarrow{u}\cdot\overrightarrow{v} = 0$.