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Vector Calculus

Fifth Edition

Chapter 6:
The Change of Variables Formula and Applications of Integration

6.3 Applications

6.3 Applications of Double and Triple Integrals

Key Points in this Section.

1. The ***average value*** of a function $f : [a, b] \to \mathbb{R}$ is

$$[f]_{\text{av}} = \frac{1}{b-a} \int_a^b f(x)dx,$$

of $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ is

$$[f]_{\text{av}} = \frac{1}{\text{Area}(D)} \iint_D f(x, y)\, dx\, dy$$

where $\text{Area}(D) = \iint_D dx\, dy$ and of $f : W \subset \mathbb{R}^3 \to \mathbb{R}$ is

$$[f]_{\text{av}} = \frac{1}{\text{Volume}(W)} \iiint_W f(x, y, z)\, dx\, dy\, dz,$$

where $\text{Volume}(W) = \iiint_W dx\, dy\, dz$.

2. The ***center of mass*** of a distribution of masses $m_1, \ldots, m_n$ at points $x_1, \ldots, x_n$ on $\mathbb{R}$ is

$$\bar{x} = \frac{1}{m_1 + \cdots + m_n} \left(x_1 m_1 + \cdots + x_n m_n \right),$$

of material with a mass density $\delta(x)$ on $[a, b]$ is

$$\bar{x} = \frac{1}{\int_a^b \delta(x) dx} \int_a^b x \delta(x) dx,$$

and of material with mass density $\delta(x, y)$ on $D \subset \mathbb{R}^2$ is $(\bar{x}, \bar{y})$, where

$$\bar{x} = \frac{1}{\iint_D \delta(x, y) dx dy} \iint_D x \delta(x, y) dx dy$$

$$\bar{y} = \frac{1}{\iint_D \delta(x, y) dx dy} \iint_D y \delta(x, y) dx dy,$$

and of a distribution of material with mass density $\delta(x, y, z)$ on a region $W \subset \mathbb{R}^3$ is $(\bar{x}, \bar{y}, \bar{z})$, where

$$\bar{x} = \frac{1}{\iiint_W \delta(x, y, z) dx dy dz} \iiint_W x \delta(x, y, z) dx dy dz$$

with similar formulas for $\bar{y}$ and $\bar{z}$. In each of these formulas, the denominator is the ***total mass***.

3. The ***moments of inertia*** of a solid body occupying a region $W \subset \mathbb{R}^3$ with mass density $\delta(x,y,z)$ about the x,y,and z-axes are

$$I_x = \iiint_W (y^2 + z^2)\delta(x,y,z)dxdydz,$$
$$I_y = \iiint_W (x^2 + z^2)\delta(x,y,z)dxdydz,$$
$$I_z = \iiint_W (x^2 + y^2)\delta(x,y,z)dxdydz.$$

4. The gravitational potential of a particle with mass m due to matter occupying a region W with mass density $\delta(x,y,z)$ at a point (X,Y,Z) outside the body is

$$V(X,Y,Z) = -Gm \iiint_W \frac{\delta(x,y,z)dxdydz}{\sqrt{(x-X)^2 + (y-Y)^2 + (z-Z)^2}}$$

Average Values The ***average value*** of a function of one variable on the interval $[a, b]$ is defined by

$$[f]_{\text{av}} = \frac{\int_a^b f(x)\,dx}{b-a}.$$

Likewise, for functions of two variables, the ratio of the integral to the area of D,

$$[f]_{\text{av}} = \frac{\iint_D f(x, y)\,dx\,dy}{\iint_D dx\,dy}, \tag{1}$$

is called the ***average value*** of f over D. Similarly, the ***average value*** of a function f on a region W in three space is defined by

$$[f]_{\text{av}} = \frac{\iiint_W f(x, y, z)\,dx\,dy\,dz}{\iiint_W dx\,dy\,dz}.$$

The Center of Mass of Two-Dimensional Plates

$$\bar{x} = \frac{\iint_D x\delta(x, y)\,dx\,dy}{\iint_D \delta(x, y)\,dx\,dy} \qquad \text{and} \qquad \bar{y} = \frac{\iint_D y\delta(x, y)\,dx\,dy}{\iint_D \delta(x, y)\,dx\,dy}, \tag{4}$$

where again $\delta(x, y)$ is the mass density (see Figure 6.3.2).

Coordinates for the Center of Mass of Three-Dimensional Regions

$$\bar{x} = \frac{\iiint_W x\delta(x, y, z)\, dx\, dy\, dz}{\text{mass}},$$

$$\bar{y} = \frac{\iiint_W y\delta(x, y, z)\, dx\, dy\, dz}{\text{mass}}, \tag{7}$$

$$\bar{z} = \frac{\iiint_W z\delta(x, y, z)\, dx\, dy\, dz}{\text{mass}}.$$

$$\text{mass} = \iiint_W \delta(x, y, z)\, dx\, dy\, dz.$$

Moments of Inertia About the Coordinate Axes

$$I_x = \text{moment of inertia about the } x\text{-axis} = \iiint_W (y^2 + z^2)\, \delta(x, y, z)\, dV;$$

$$I_y = \text{moment of inertia about the } y\text{-axis} = \iiint_W (x^2 + z^2)\, \delta(x, y, z)\, dV;$$

$$I_z = \text{moment of inertia about the } z\text{-axis} = \iiint_W (x^2 + y^2)\, \delta(x, y, z)\, dV.$$

Potencial gravitatori

$$V(x_1, y_1, z_1) = -Gm \iiint_W \frac{\delta(x, y, z)\, dx\, dy\, dz}{\sqrt{(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2}}$$