Propietats de la integral

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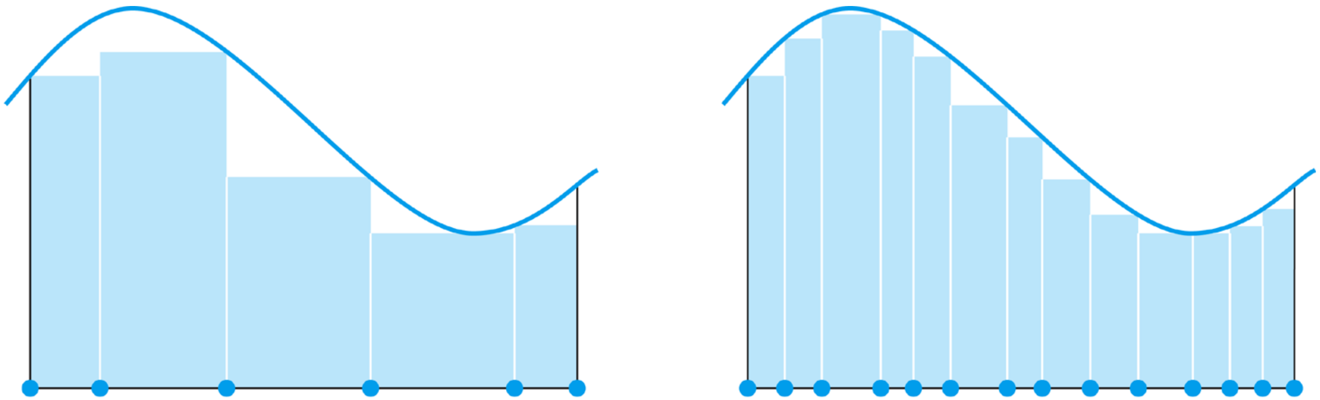
The Fundamental Theorem of Integral Calculus

THEOREM 5.3.1

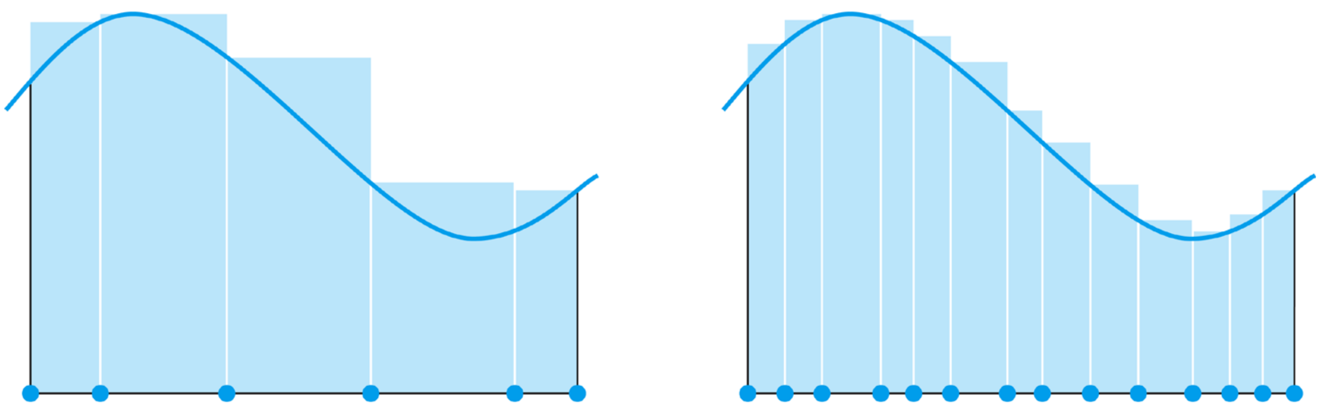
Suppose that f is continuous on $[a, b]$, and P and Q are partitions of $[a, b]$. If $Q \supseteq P$, then

$$L_f(P) \leq L_f(Q) \qquad \text{and} \qquad U_f(Q) \leq U_f(P).$$

The Fundamental Theorem of Integral Calculus



as points are added to a partition, the lower sums tend to get bigger



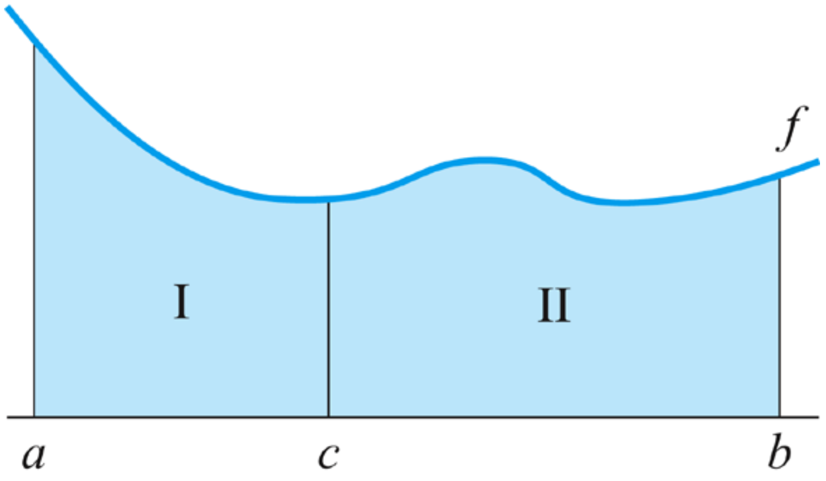
as points are added to a partition, the upper sums tend to get smaller

The Fundamental Theorem of Integral Calculus

THEOREM 5.3.2

If f is continuous on $[a, b]$ and $a < c < b$, then

$$\int_a^c f(t)\,dt + \int_c^b f(t)\,dt = \int_a^b f(t)\,dt.$$



The Fundamental Theorem of Integral Calculus

Until now we have integrated only from left to right: from a number *a* to a number *b* greater than *a*. We integrate in the other direction by defining

$$\boxed{\int_b^a f(t)\,dt = -\int_a^b f(t)\,dt.}$$

The integral from any number to itself is defined to be zero:

$$\boxed{\int_c^c f(t)\,dt = 0.}$$

The Fundamental Theorem of Integral Calculus

Primer Teorema Fonamental del Càlcul

THEOREM 5.3.5

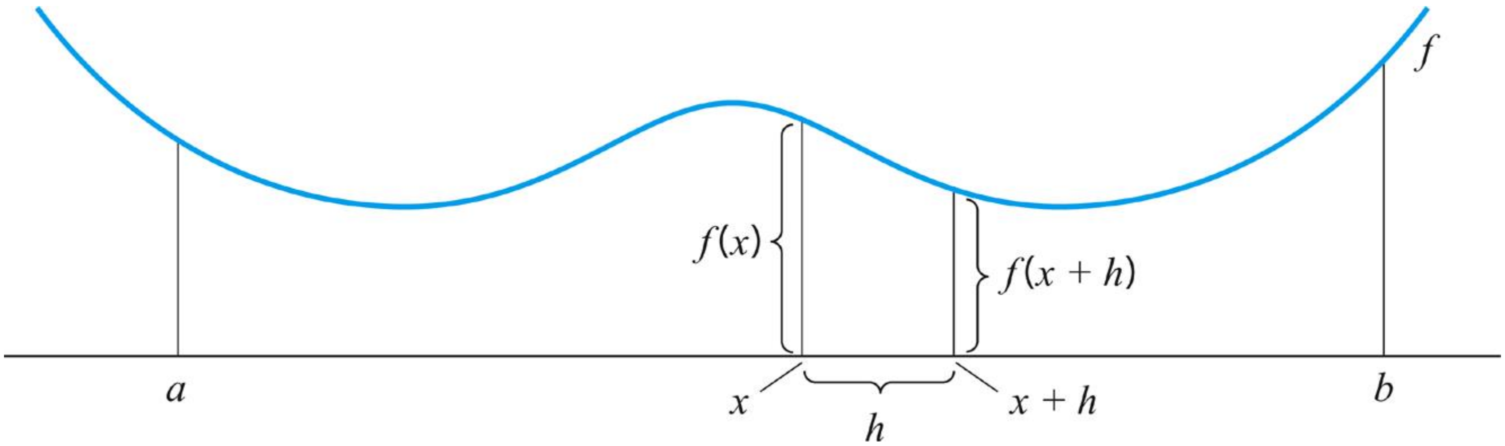
Let f be continuous on $[a, b]$ and let c be any number in $[a, b]$. The function F defined on $[a, b]$ by setting

$$F(x) = \int_c^x f(t)\,dt$$

is continuous on $[a, b]$, differentiable on (a, b), and has derivative

$$F'(x) = f(x) \quad \text{for all } x \text{ in } (a, b).$$

The Fundamental Theorem of Integral Calculus



Idea: $F(x)$ = area from a to x and $F(x + h)$ = area from a to $x + h$. Therefore $F(x + h) - F(x)$ = area from x to $x + h$. For small h this is approximately $f(x)\,h$. Thus

$$\frac{F(x+h)-F(x)}{h} \quad \text{is approximately} \quad \frac{f(x)h}{h} = f(x)$$

In the limit

$$F'(x) = \lim_{h\to 0^+} \frac{F(x+h)-F(x)}{h} = f(x)$$

The Fundamental Theorem of Integral Calculus

Proof: Let us start with $a \le c \le x < x + h < b$. We know

$$\int_c^{x+h} f(t)dt = \int_c^x f(t)dt + \int_x^{x+h} f(t)dt$$

Thus

$$F(x+h) - F(x) = \int_x^{x+h} f(t)dt$$

Since f is continuous in $[x, x+h]$, it attains a maximum value M_h and a minimum value m_h inside the interval (Weierstrass theorem), and thus

$$m_h\, h \le \int_x^{x+h} f(t)dt \le M_h\, h \quad \Longrightarrow \quad m_h \le \frac{F(x+h) - F(x)}{h} \le M_h$$

The Fundamental Theorem of Integral Calculus

Proof: Let us start with $a \leq c \leq x < x + h < b$. We know

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$$m_h\, h \leq \int_x^{x+h} f(t)dt \leq M_h\, h \quad \Longrightarrow \quad m_h \leq \frac{F(x+h) - F(x)}{h} \leq M_h$$

Since f is continuous in $[x, x+h]$,

$$\lim_{h \to 0^+} m_h = f(x) = \lim_{h \to 0^+} M_h$$

Therefore, by the sandwich theorem,

$$f(x) = \lim_{h \to 0^+} \frac{F(x+h) - F(x)}{h}$$

Similarly, we could prove

$$f(x) = \lim_{h \to 0^-} \frac{F(x+h) - F(x)}{h}$$

which lead to the desired

$$f(x) = F'(x)$$

The Fundamental Theorem of Integral Calculus

Antiderivative for *f* = Primitive function of *f*

DEFINITION 5.4.1 ANTIDERIVATIVE ON (a, b)

Let f be continuous on $[a, b]$. A function G is called an *antiderivative for* f *on* $[a, b]$ if

$$G \text{ is continuous on } [a, b] \qquad \text{and} \qquad G'(x) = f(x) \qquad \text{for all } x \in (a, b).$$

Segon Teorema Fonamental del Càlcul (Regla de Barrow)

THEOREM 5.4.2 THE FUNDAMENTAL THEOREM OF INTEGRAL CALCULUS

Let f be continuous on $[a, b]$. If G is any antiderivative for f on $[a, b]$, then

$$\int_a^b f(t)\,dt = G(b) - G(a).$$

The Fundamental Theorem of Integral Calculus

Proof: We know

$$F(x) = \int_a^x f(t)dt$$

is a primitive function of f, i.e., $F'(x) = f(x)$. If G is any other primitive of f, then it must satisfy $G'(x) = f(x) = F'(x)$. Thus, there exists a constant C such that

$$F(x) = G(x) + C$$

Since

$$F(a) = \int_a^a f(t)dt = 0 = G(a) + C$$

then

$$C = -G(a)$$

Therefore,

$$F(b) = \int_a^b f(t)dt = G(b) + C = G(b) - \mathrm{G}(a)$$

The Fundamental Theorem of Integral Calculus

Example

Evaluate

$$\int_1^4 x^2 dx$$

Solution

As an antiderivative for $f(x) = x^2$, we can use the function

$$G(x) = \frac{1}{3}x^3.$$

By the fundamental theorem,

$$\int_1^4 x^2 dx = G(4) - G(1) = \frac{1}{3}(4)^3 - \frac{1}{3}(1)^3 = \frac{64}{3} - \frac{1}{3} = 21$$

NOTE: Any other antiderivative of $f(x) = x^2$ has the form $H(x) = \frac{1}{3}x^3 + C$ for some constant C. Had we chosen such an H instead of G, then we would have had

$$\int_1^4 x^2 dx = H(4) - H(1) = \left[\frac{1}{3}(4)^3 + C\right] - \left[\frac{1}{3}(1)^3 + C\right] = \frac{64}{3} + C - \frac{1}{3} - C = 21$$

the C's would have canceled out.

The Fundamental Theorem of Integral Calculus

Function	Antiderivative	Function	Antiderivative
$\sin x$	$-\cos x$	$\cos x$	$\sin x$
$\sec^2 x$	$\tan x$	$\csc^2 x$	$-\cot x$
$\sec x \tan x$	$\sec x$	$\csc x \cot x$	$-\csc x$

Some examples:

$$\int_1^2 \frac{dx}{x^3} = \int_1^2 x^{-3} dx = \left[\frac{x^{-2}}{-2}\right]_1^2 = \left[-\frac{1}{2x^2}\right]_1^2 = -\tfrac{1}{8} - \left(-\tfrac{1}{2}\right) = \tfrac{3}{8},$$

$$\int_0^1 t^{5/3}\, dt = \left[\tfrac{3}{8} t^{8/3}\right]_0^1 = \tfrac{3}{8}(1)^{8/3} - \tfrac{3}{8}(0)^{8/3} = \tfrac{3}{8}.$$

$$\int_{-\pi/4}^{\pi/3} \sec^2 t\, dt = \left[\tan t\right]_{-\pi/4}^{\pi/3} = \tan\frac{\pi}{3} - \tan\frac{-\pi}{4} = \sqrt{3} - (-1) = \sqrt{3} + 1.$$

$$\int_{\pi/6}^{\pi/2} \csc x \cot x\, dx = \left[-\csc x\right]_{\pi/6}^{\pi/2} = -\csc\frac{\pi}{2} - \left[-\csc\frac{\pi}{6}\right] = -1 - (-2) = 1.$$

The Linearity of the Integral

I. Constants may be factored through the integral sign:

$$\int_a^b \alpha f(x)\,dx = \alpha \int_a^b f(x)\,dx.$$

II. The integral of a sum is the sum of the integrals:

$$\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

III. The integral of a linear combination is the linear combination of the integrals:

$$\int_a^b [\alpha f(x) + \beta g(x)]\,dx = \alpha \int_a^b f(x)\,dx + \beta \int_a^b g(x)\,dx.$$

The Linearity of the Integral

Example

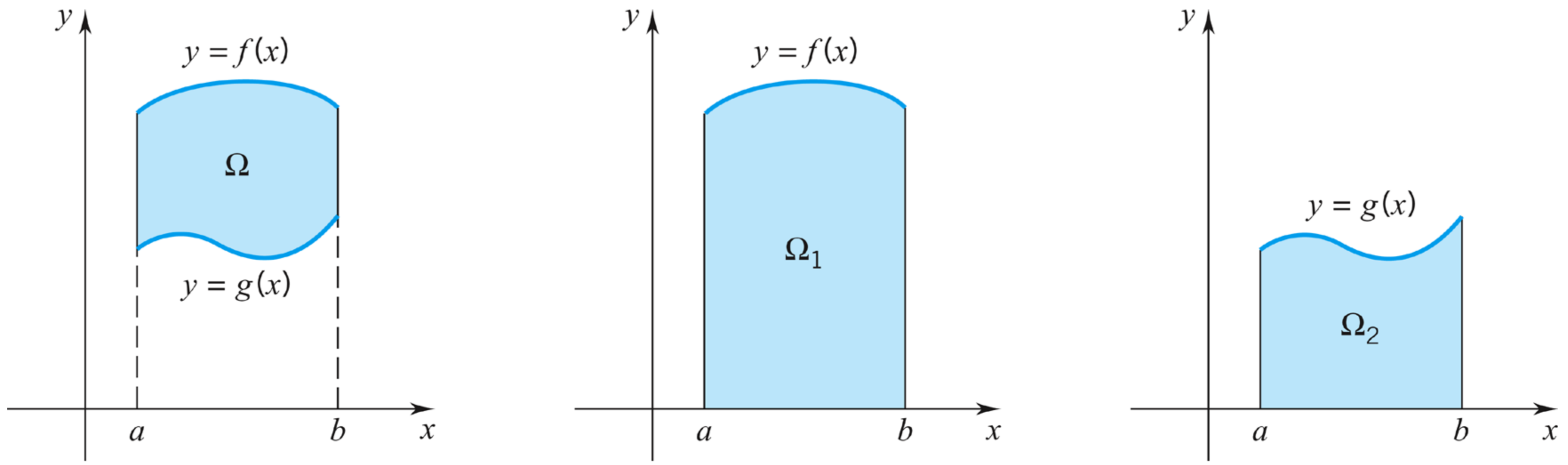
Evaluate

$$\int_0^{\pi/4} \sec x\left[2\tan x - 5\sec x\right]dx$$

Solution

$$\begin{aligned}
\int_0^{\pi/4} \sec x\left[2\tan x - 5\sec x\right]dx &= \int_0^{\pi/4}\left[2\sec x\tan x - 5\sec^2 x\right]dx \\
&= 2\int_0^{\pi/4} \sec x\tan x\,dx - 5\int_0^{\pi/4} \sec^2 x\,dx \\
&= 2\left[\sec x\right]_0^{\pi/4} - 5\left[\tan x\right]_0^{\pi/4} \\
&= 2\left[\sec\frac{\pi}{4} - \sec 0\right] - 5\left[\tan\frac{\pi}{4} - \tan 0\right] \\
&= 2\left[\sqrt{2} - 1\right] = 5\left[1 - 0\right] = 2\sqrt{2} - 7
\end{aligned}$$

Some Area Problems



area of Ω = area of Ω_1 – area of Ω_2

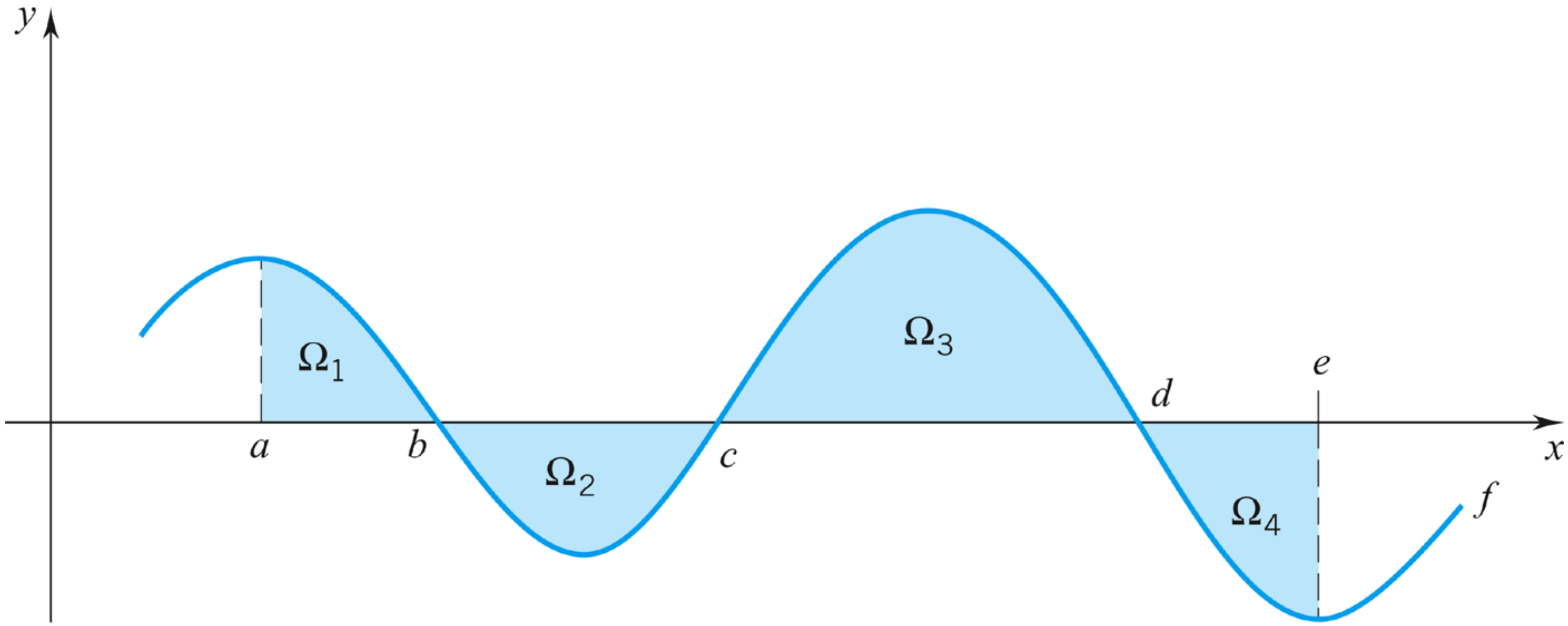
The upper boundary of Ω is the graph of a nonnegative function f and the lower boundary is the graph of a nonnegative function g. We can obtain the area of Ω by calculating the area of Ω_1 and subtracting off the area of Ω_2. Since

$$\text{area of } \Omega_1 = \int_a^b f(x)\,dx \quad \text{and} \quad \text{area of } \Omega_2 = \int_a^b g(x)\,dx$$

we have

$$\boxed{\text{area of } \Omega = \int_a^b [f(x) - g(x)]\,dx.}$$

Some Area Problems



The area between the graph of *f* and the *x*-axis from $x = a$ to $x = e$ is the sum

area of Ω_1 + area of Ω_2 + area of Ω_3 + area of Ω_4

This area is

$$\int_a^b f(x)\,dx - \int_b^c f(x)\,dx + \int_c^d f(x)\,dx - \int_d^e f(x)\,dx.$$

Indefinite Integrals

Consider a continuous function f. If F is an antiderivative for f on $[a, b]$, then

$$\int_a^b f(x)\,dx = \left[F(x)\right]_a^b$$

If C is a constant, then

$$\left[F(x)+C\right]_a^b = \left[F(b)+C\right] - \left[F(a)+C\right] = F(b) - F(a) = \left[F(x)\right]_a^b$$

Thus we can replace (1) by writing

$$\int_a^b f(x)\,dx = \left[F(x)+C\right]_a^b.$$

If we have no particular interest in the interval $[a, b]$ but wish instead to emphasize that F is an antiderivative for f, which on open intervals simply means that $F' = f$, then we omit the a and the b and simply write

$$\int f(x)\,dx = F(x) + C$$

Antiderivatives expressed in this manner are called *indefinite integrals*. The constant C is called the *constant of integration*; it is an *arbitrary* constant and we can assign to it any value we choose. Each value of C gives a particular antiderivative, and each antiderivative is obtained from a particular value of C.

Indefinite Integrals

$$\int \sin x\, dx = -\cos x + C \qquad \int \cos x\, dx = \sin x + C$$

$$\int \sec^2 x\, dx = \tan x + C \qquad \int \csc^2 x\, dx = -\cot x + C$$

$$\int \sec x \tan x\, dx = \sec x + C \qquad \int \csc x \cot x\, dx = -\csc x + C$$

Indefinite Integrals

The linearity properties of definite integrals also hold for indefinite integrals.

$$\boxed{\int [\alpha f(x) + \beta g(x)]\, dx = \alpha \int f(x)\, dx + \beta \int g(x)\, dx.^{\dagger}}$$

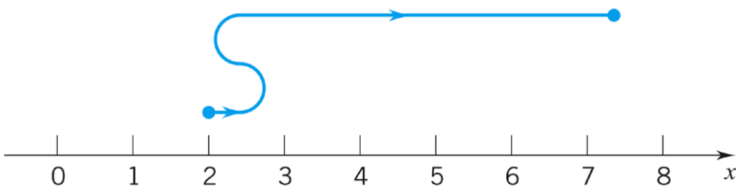
Example
Calculate $\int \left[5x^{3/2} - 2\csc^2 x\right] dx$

Solution

$$\int \left[5x^{3/2} - 2\csc^2 x\right] dx = 5\int x^{3/2} dx - 2\int \csc^2 x dx$$
$$= 5\left(\frac{2}{5}\right) x^{5/2} + C_1 - 2\left(-\cot x\right) + C_2$$
$$= 2x^{5/2} + 2\cot x + C \qquad \text{(writing } C \text{ for } C_1 + C_2\text{)}$$

Indefinite Integrals

Application to Motion



Example
An object moves along a coordinate line with velocity

$$v(t) = 2 - 3t + t^2 \text{ units per second.}$$

Its initial position (position at time $t = 0$) is 2 units to the right of the origin. Find the position of the object 4 seconds later.

Solution
Let $x(t)$ be the position (coordinate) of the object at time t. We are given that $x(0) = 2$. Since $x'(t) = v(t)$,

$$x(t) = \int v(t)\,dt = \int \left(2 - 3t + t^2\right)dt = 2t - \frac{3}{2}t^2 + \frac{1}{3}t^3 + C$$

Since $x(0) = 2$ and $x(0) = 2(0) - \tfrac{3}{2}(0)^2 + \tfrac{1}{3}(0)^3 + C = C$, we have $C = 2$ and

$$x(t) = 2t - \tfrac{3}{2}t^2 + \tfrac{1}{3}t^3 + 2.$$

The position of the object at time $t = 4$ is the value of this function at $t = 4$:

$$x(4) = 2(4) - \tfrac{3}{2}(4)^2 + \tfrac{1}{3}(4)^3 + 2 = 7\tfrac{1}{3}$$

At the end of 4 seconds the object is $7\tfrac{1}{3}$ units to the right of the origin.

Additional Properties of the Definite Integral

I. The integral of a nonnegative continuous function is nonnegative:

$$\boxed{\text{if} \quad f(x) \geq 0 \quad \text{for all } x \in [a, b], \quad \text{then} \quad \int_a^b f(x)\,dx \geq 0.}$$

The integral of a positive continuous function is positive:

$$\boxed{\text{if} \quad f(x) > 0 \quad \text{for all } x \in [a, b], \quad \text{then} \quad \int_a^b f(x)\,dx > 0.}$$

II. The integral is order-preserving: for continuous functions *f* and *g*,

$$\boxed{\text{if} \quad f(x) \leq g(x) \text{ for all } x \in [a, b], \quad \text{then} \quad \int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx}$$

and

$$\boxed{\text{if} \quad f(x) < g(x) \quad \text{for all} \quad x \in [a, b], \quad \text{then} \quad \int_a^b f(x)\,dx < \int_a^b g(x)\,dx.}$$

Additional Properties of the Definite Integral

III. Just as the absolute value of a sum of numbers is less than or equal to the sum of the absolute values of those numbers,

$$|x_1 + x_2 + \cdots + x_n| \leq |x_1| + |x_2| + \cdots + |x_n|,$$

the absolute value of an integral of a continuous function is less than or equal to the integral of the absolute value of that function:

$$\boxed{\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx.}$$

IV. If f is continuous on $[a, b]$, then

$$\boxed{m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)}$$

where m is the minimum value of f on $[a, b]$ and M is the maximum.
Reasoning: $m(b-a)$ is a lower sum for f and $M(b-a)$ is an upper sum.

Additional Properties of the Definite Integral

V. If f is continuous on $[a, b]$ and u is a differentiable function of x with values in $[a, b]$, then for all $u(x) \in (a, b)$

$$\boxed{\frac{d}{dx}\left(\int_a^{u(x)} f(t)\,dt\right) = f(u(x))u'(x).}$$

Example

Find $\quad \dfrac{d}{dx}\left(\displaystyle\int_0^{x^3} \frac{1}{1+t}dt\right)$

Solution

$$\frac{d}{dx}\left(\int_0^{x^3} \frac{1}{1+t}\,dt\right) = \frac{1}{1+x^3}3x^2 = \frac{3x^2}{1+x^3}$$

Additional Properties of the Definite Integral

V. If f is continuous on $[a, b]$ and u is a differentiable function of x with values in $[a, b]$, then for all $u(x) \in (a, b)$

$$\boxed{\frac{d}{dx}\left(\int_a^{u(x)} f(t)\,dt\right) = f(u(x))u'(x).}$$

PROOF OF (5.8.7) Since f is continuous on $[a, b]$, the function

$$F(u) = \int_a^u f(t)\,dt$$

is differentiable on (a, b) and

$$F'(u) = f(u).$$

This we know from Theorem 5.3.5. The result that we are trying to prove follows from noting that

$$\int_a^{u(x)} f(t)\,dt = F(u(x))$$

and applying the chain rule:

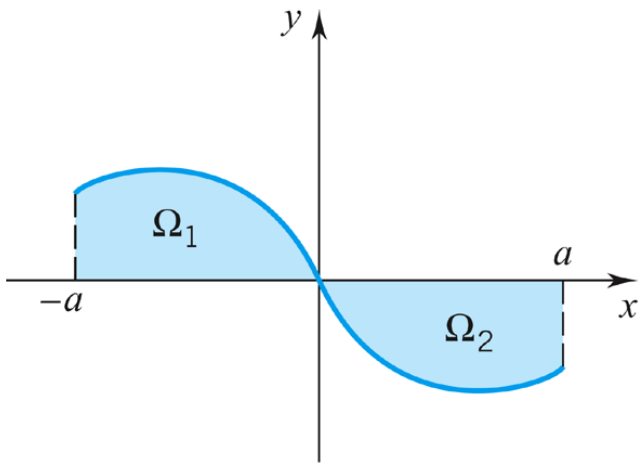
$$\frac{d}{dx}\left(\int_a^{u(x)} f(t)\,dt\right) = \frac{d}{dx}[F(u(x))] = F'(u(x))u'(x) = f(u(x))u'(x). \quad \square$$

Additional Properties of the Definite Integral

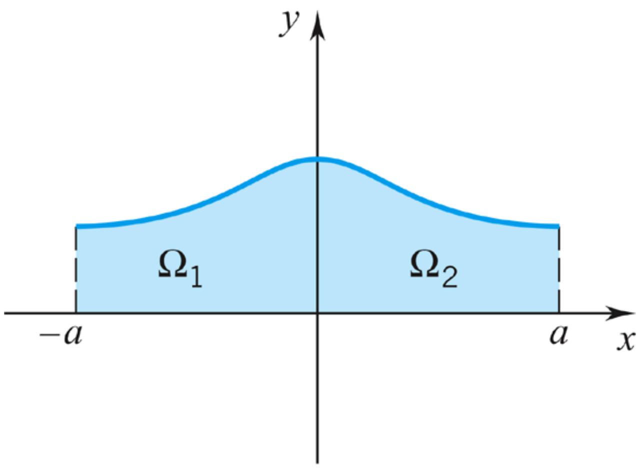
VI. Now a few words about the role of symmetry in integration. Suppose that f is continuous on an interval of the form $[-a, a]$, a closed interval symmetric about the origin.

(a) if f is odd on $[-a, a]$, then $\displaystyle\int_{-a}^{a} f(x)\,dx = 0.$

(b) if f is even on $[-a, a]$, then $\displaystyle\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx.$



f odd



f even

Mean-Value Theorems for Integrals

THEOREM 5.9.1 THE FIRST MEAN-VALUE THEOREM FOR INTEGRALS

If f is continuous on $[a, b]$, then there is at least one number c in (a, b) for which

$$\int_a^b f(x)\,dx = f(c)(b-a).$$

This number $f(c)$ is called *the average value* (or *mean value*) *of f on* $[a, b]$.

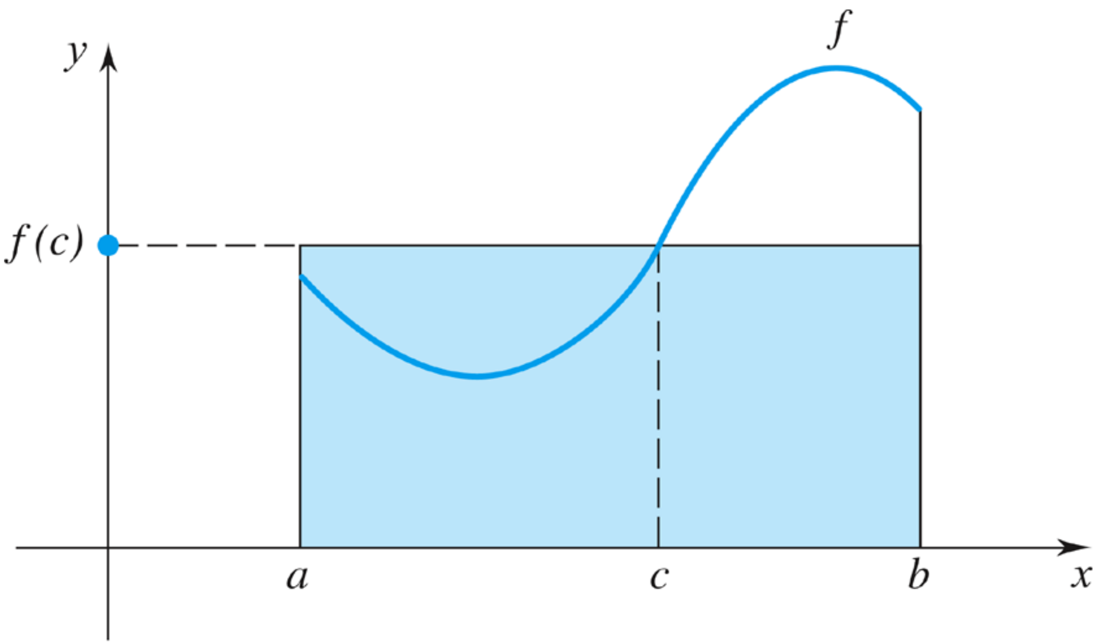
This theorem is proved using the Intermediate Value theorem of continuous functions

In practice, point c is not so useful, what emerges is a **definition for the mean value of a function in an interval** $[a, b]$

$$f_{\text{avg}} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Mean-Value Theorems for Integrals

$$\text{area of } \Omega = (\textit{the average value of } f \text{ on } [a, b]) \cdot (b - a).$$



Mean-Value Theorems for Integrals

THEOREM 5.9.3 THE SECOND MEAN-VALUE THEOREM FOR INTEGRALS

If f and g are continuous on $[a, b]$ and g is nonnegative, then there is a number c in (a, b) for which

$$\int_a^b f(x)g(x)\,dx = f(c)\int_a^b g(x)\,dx.$$

This number $f(c)$ is called the *g-weighted average of f on* $[a, b]$.

In this case, we get a definition for the **weighted mean value of a function in an interval** $[a, b]$

$$f_{\text{avg}}^{w(x)} = \frac{\int_a^b f(x)w(x)dx}{\int_a^b w(x)dx}, \qquad w(x) \geq 0, \forall x \in [a, b]$$

The First Mean-Value Theorem is a particular case of the Second Mean-Value Theorem in which $w(x) = 1, \forall x \in [a, b], \int_a^b w(x)dx = \int_a^b dx = b - a$

Mean-Value Theorems for Integrals

Proof: Since f is continuous in $[a, b]$, by the Weierstrass theorem, it attains its maximum (M) and minimum (m) values in the interval, $m \leq f(x) \leq M, \forall x \in [a, b]$. And since $w(x) \geq 0, \forall x \in [a, b]$

$$m\, w(x) \leq f(x)w(x) \leq M\, w(x)$$

Integration leads to

$$m \int_a^b w(x)dx \leq \int_a^b f(x)w(x)dx \leq M \int_a^b w(x)dx$$

Let us call

$$I = \int_a^b w(x)dx$$

Since $w(x) \geq 0, \forall x \in [a, b]$, we know $I \geq 0$

If $I = 0$ then

$$0 \leq \int_a^b f(x)w(x)dx \leq 0 \implies \int_a^b f(x)w(x)dx = 0 = f(c)\, I, \qquad \forall c \in [a, b]$$

If $I > 0$ then

$$m \leq \frac{1}{I} \int_a^b f(x)w(x)dx \leq M$$

Since f is continuous in $[a, b]$, by the mean-value theorem, $\exists c \in [a, b]$ such that

$$f(c) = \frac{1}{I} \int_a^b f(x)w(x)dx \implies \int_a^b f(x)w(x)dx = f(c)\, I$$