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Vector Calculus

Fifth Edition

Chapter 2:
Differentiation

2.4 Introduction to Paths and Curves

2.4 Introduction to Paths

Key Points in this Section.

1. A ***path*** in $\mathbb{R}^3$ is a map $\mathbf{c}$ of an interval $[a, b]$ to $\mathbb{R}^3$. The ***endpoints*** of the path are the points $\mathbf{c}(a)$ and $\mathbf{c}(b)$. The associated geometric curve C is the set of image points $\mathbf{c}(t)$ as t ranges from a to b. We say $\mathbf{c}$ is a ***parametrization*** of C. Paths in the plane are similar (leave off the last component).

2. A particle on the rim of a rolling circle of radius 1 traces out a path called a ***cycloid***:

$$\mathbf{c}(t) = (t - \sin t, 1 - \cos t).$$

3. If a path $\mathbf{c}$ is differentiable, its ***velocity*** is defined to be

$$\mathbf{c}'(t) = \lim_{h \to 0} \frac{\mathbf{c}(t+h) - \mathbf{c}(t)}{h} = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k},$$

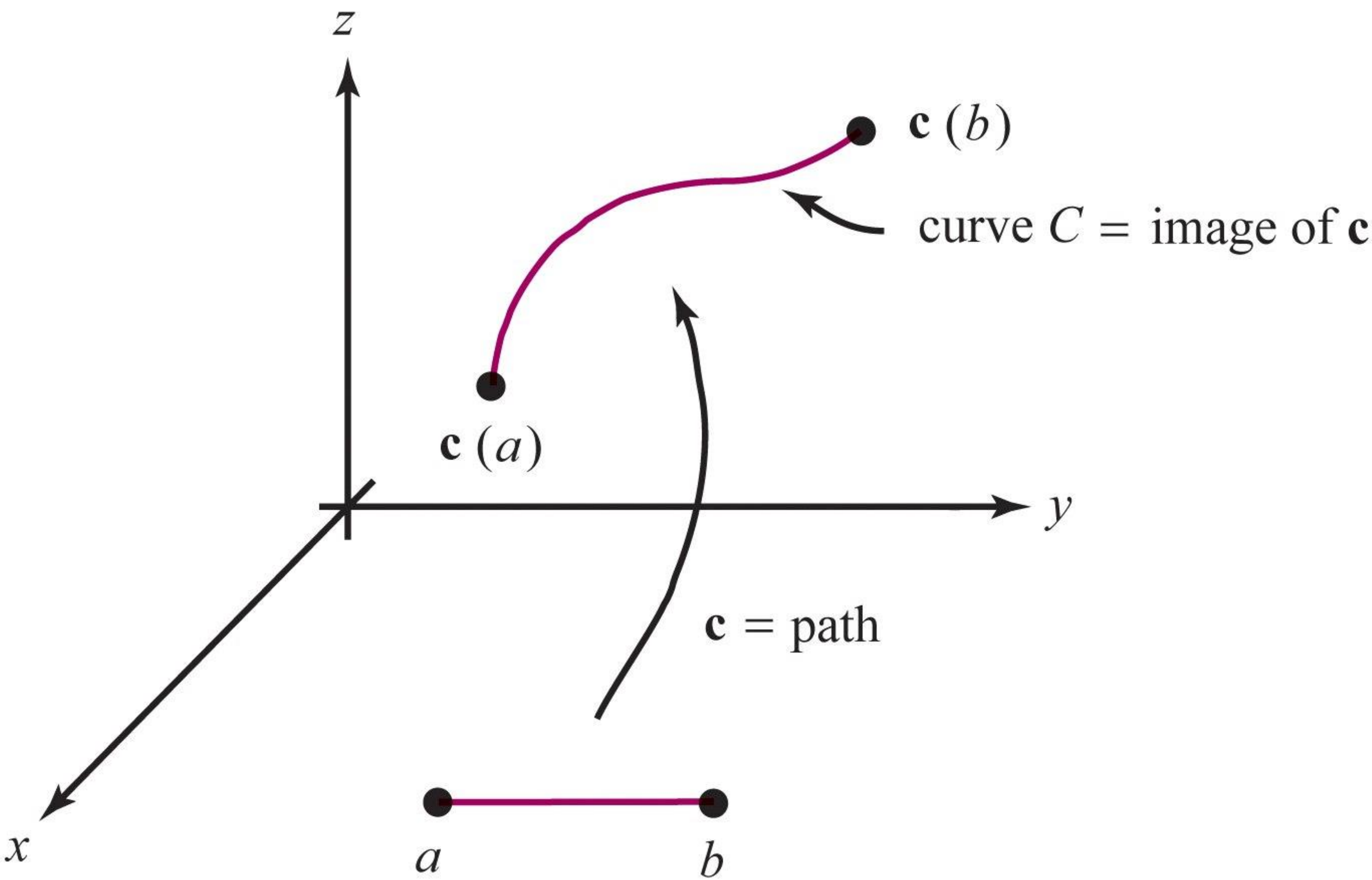
where $\mathbf{c}(t)$ has components $(x(t), y(t), z(t))$.

4. The vector $\mathbf{c}'(t_0)$ is tangent to the path at the point $\mathbf{c}(t_0)$. The ***tangent line*** at this point is

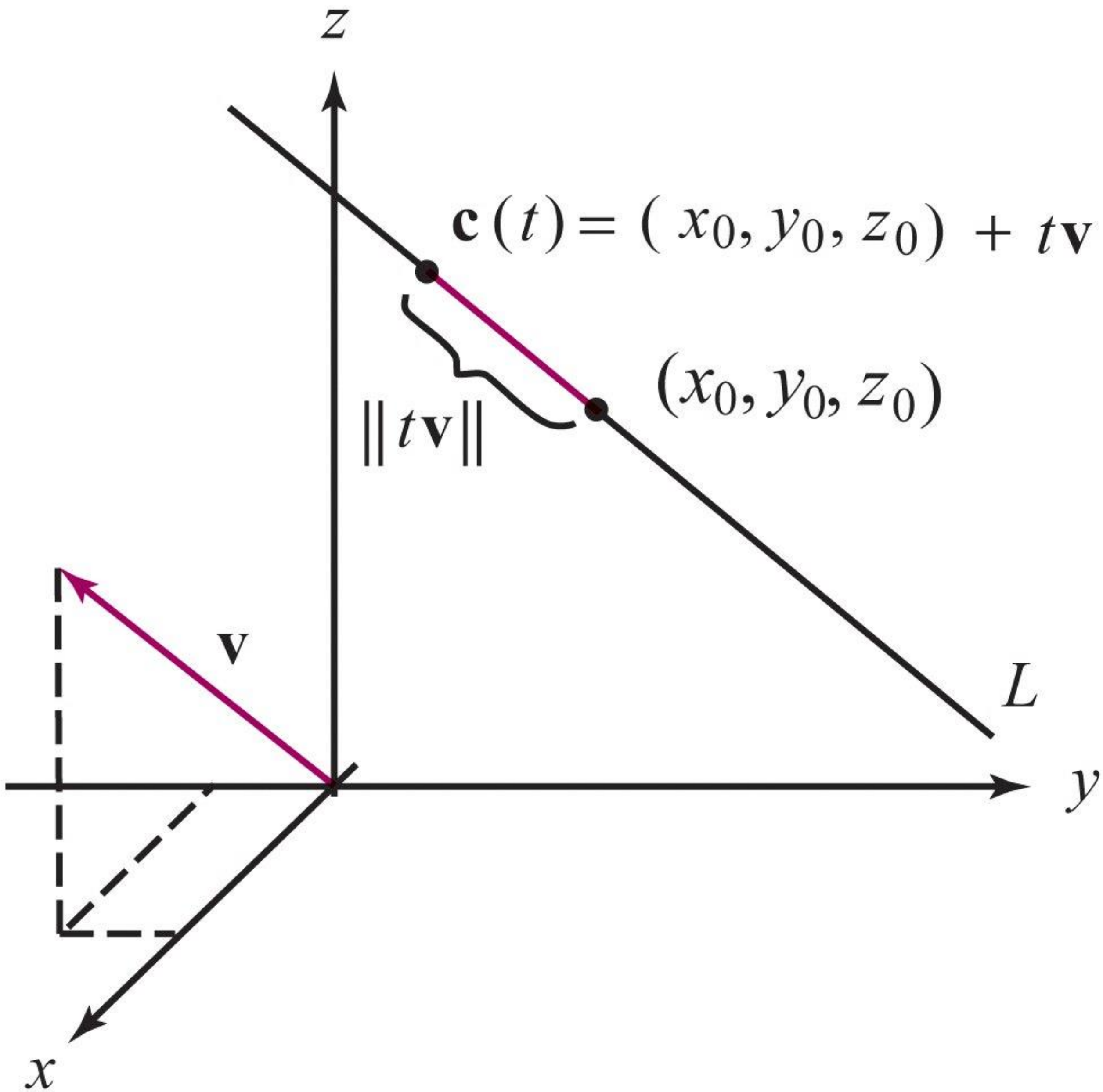
$$\ell(t) = \mathbf{c}(t_0) + (t - t_0)\mathbf{c}'(t_0).$$

Paths and Curves A ***path*** in $\mathbb{R}^n$ is a map $\mathbf{c}\colon [a, b] \to \mathbb{R}^n$; it is a ***path in the plane*** if $n = 2$ and a ***path in space*** if $n = 3$. The collection C of points $\mathbf{c}(t)$ as t varies in $[a, b]$ is called a ***curve***, and $\mathbf{c}(a)$ and $\mathbf{c}(b)$ are its ***endpoints***. The path $\mathbf{c}$ is said to ***parametrize*** the curve C. We also say $\mathbf{c}(t)$ ***traces out*** C as t varies.

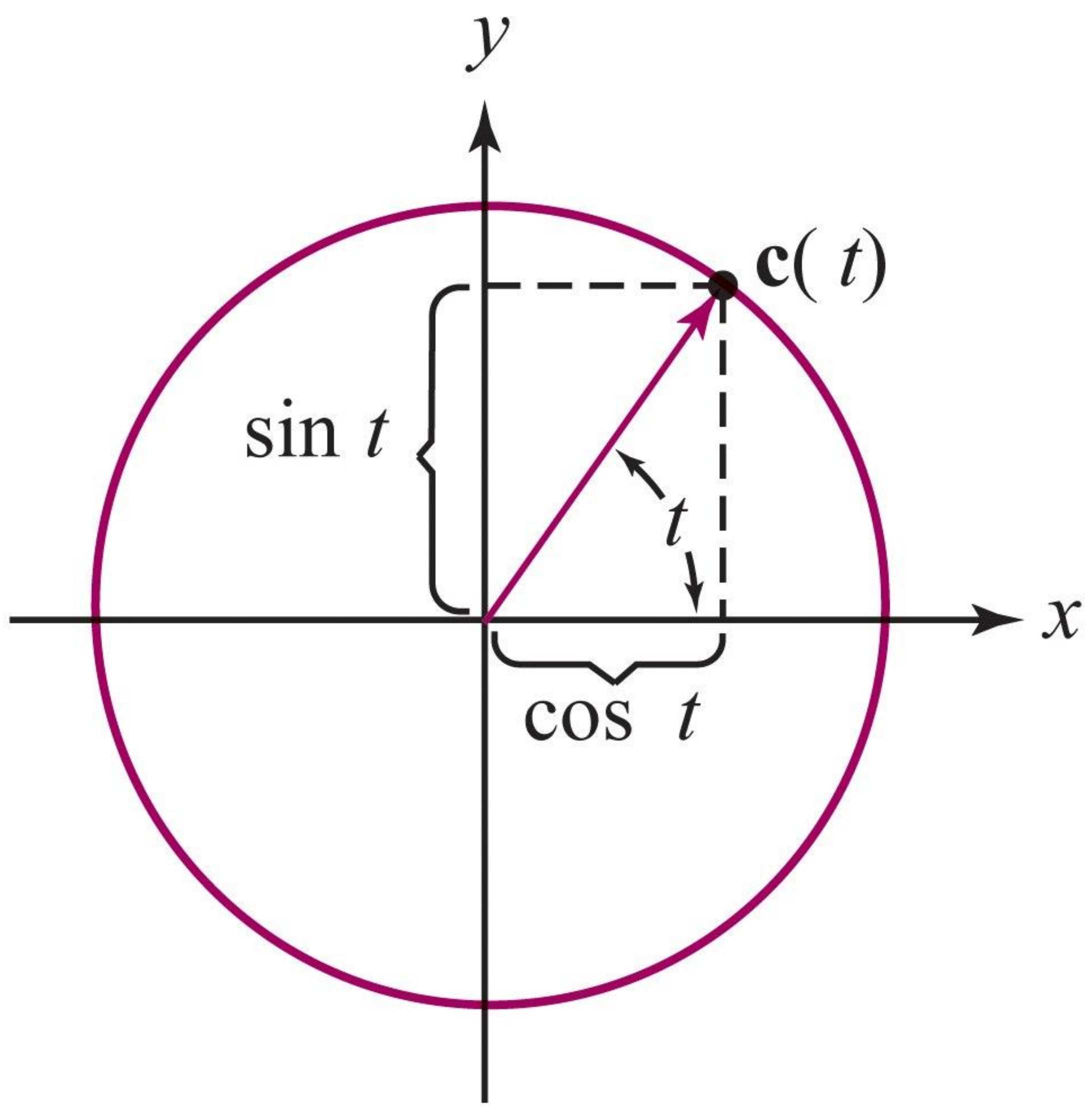
If $\mathbf{c}$ is a path in $\mathbb{R}^3$, we can write $\mathbf{c}(t) = (x(t), y(t), z(t))$ and we call $x(t)$, $y(t)$, and $z(t)$ the ***component functions*** of $\mathbf{c}$. We form component functions similarly in $\mathbb{R}^2$ or, generally, in $\mathbb{R}^n$.



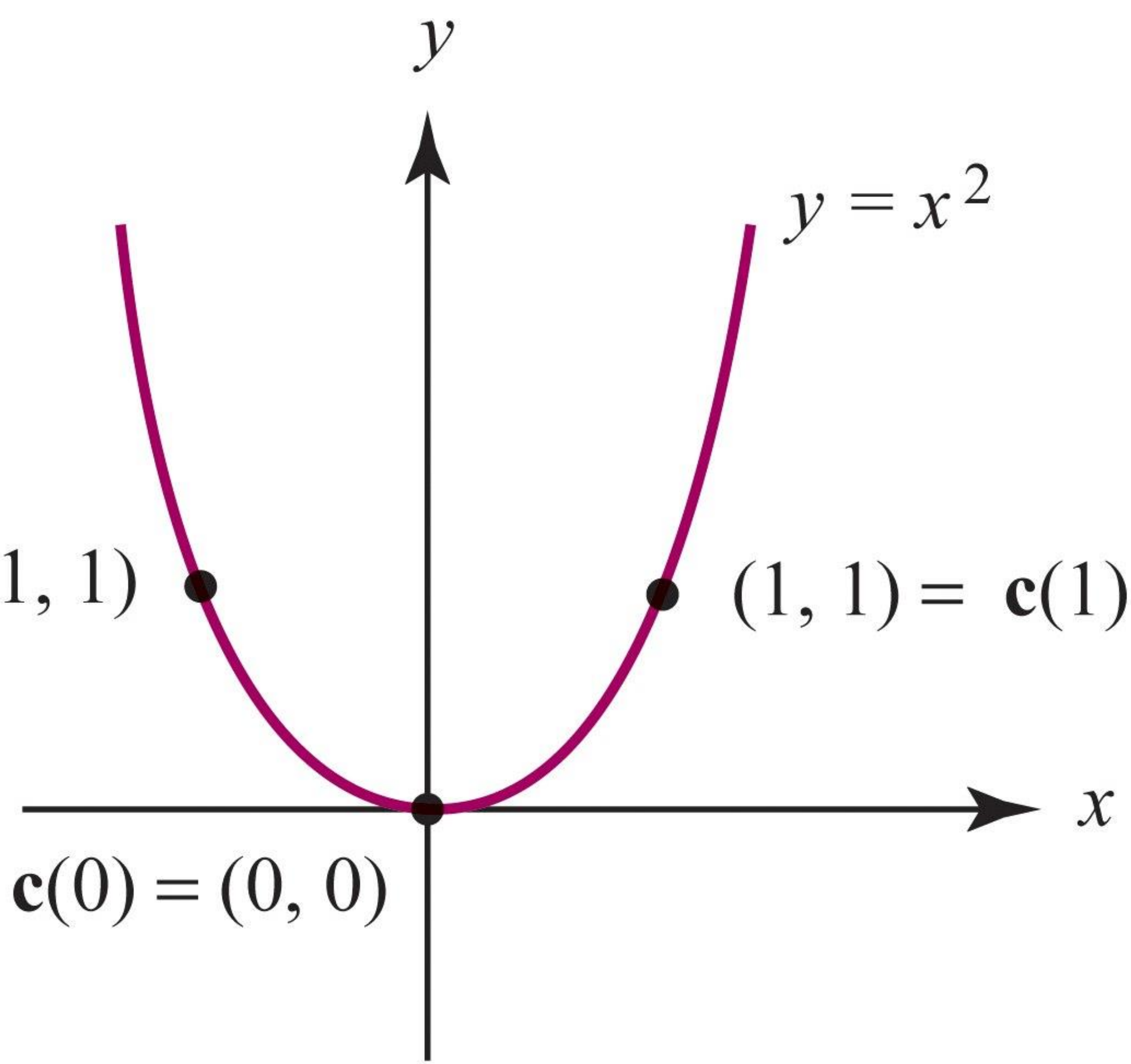
z
c (b)
curve C = image of c
c (a)
y
c = path
x
a
b

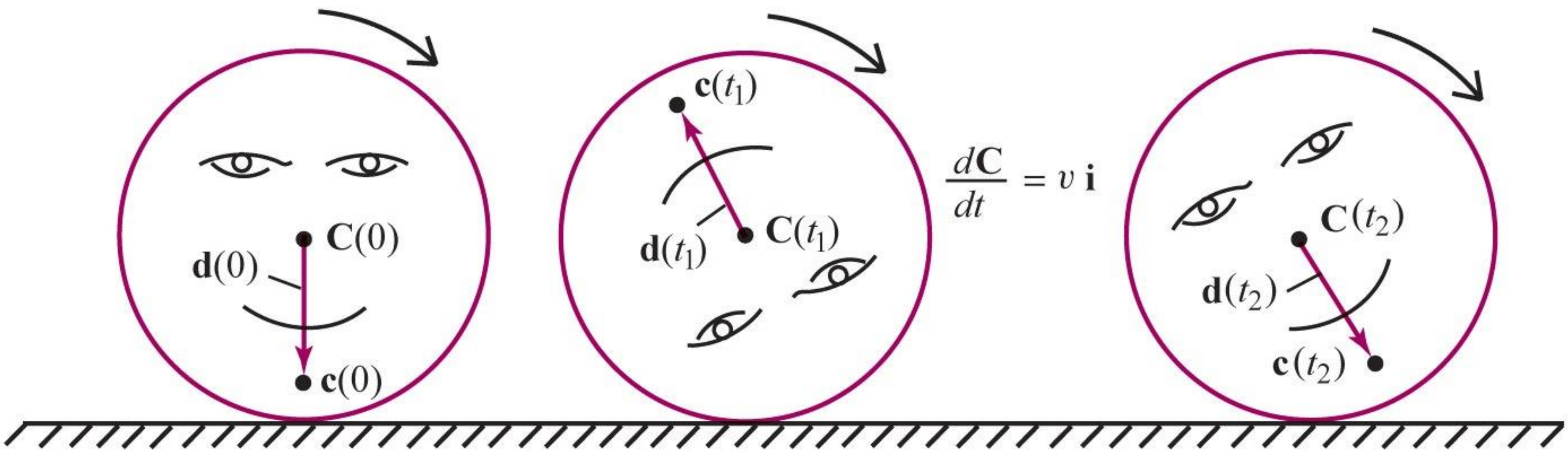

z
c(t) = (x0, y0, z0) + tv
(x0, y0, z0)
||tv||
v
L
y
x

$$\mathbf{c}(t) = (\cos t, \sin t), \qquad 0 \le t \le 2\pi$$



$$\mathbf{c}(t) = (t, t^2) \qquad t \in \mathbb{R}$$



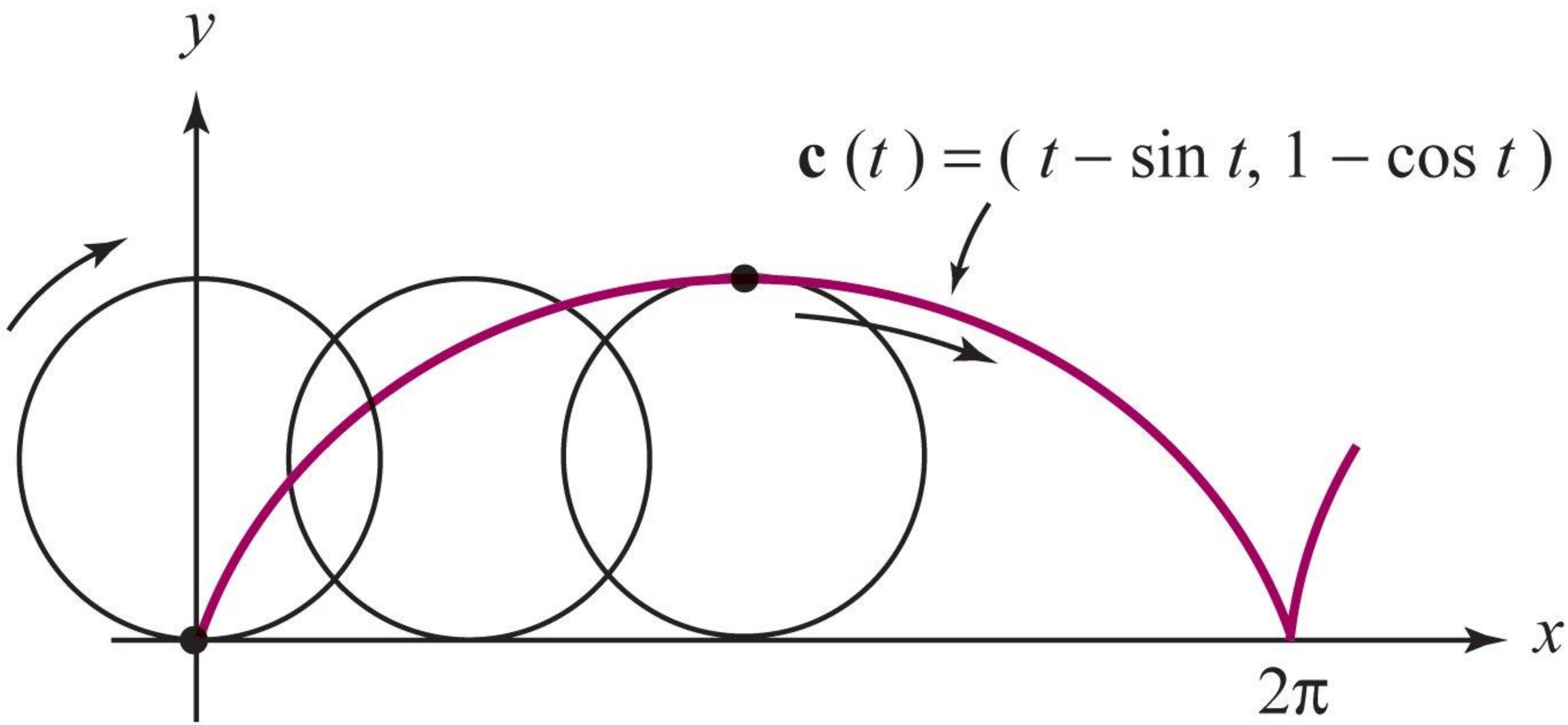


$$\mathbf{C}(t) = (vt,\ R) \qquad \mathbf{d}(t) = r\left(\cos\left[-\frac{v}{R}\,t - \frac{\pi}{2}\right]\mathbf{i} + \sin\left[-\frac{v}{R}\,t - \frac{\pi}{2}\right]\mathbf{j}\right)$$

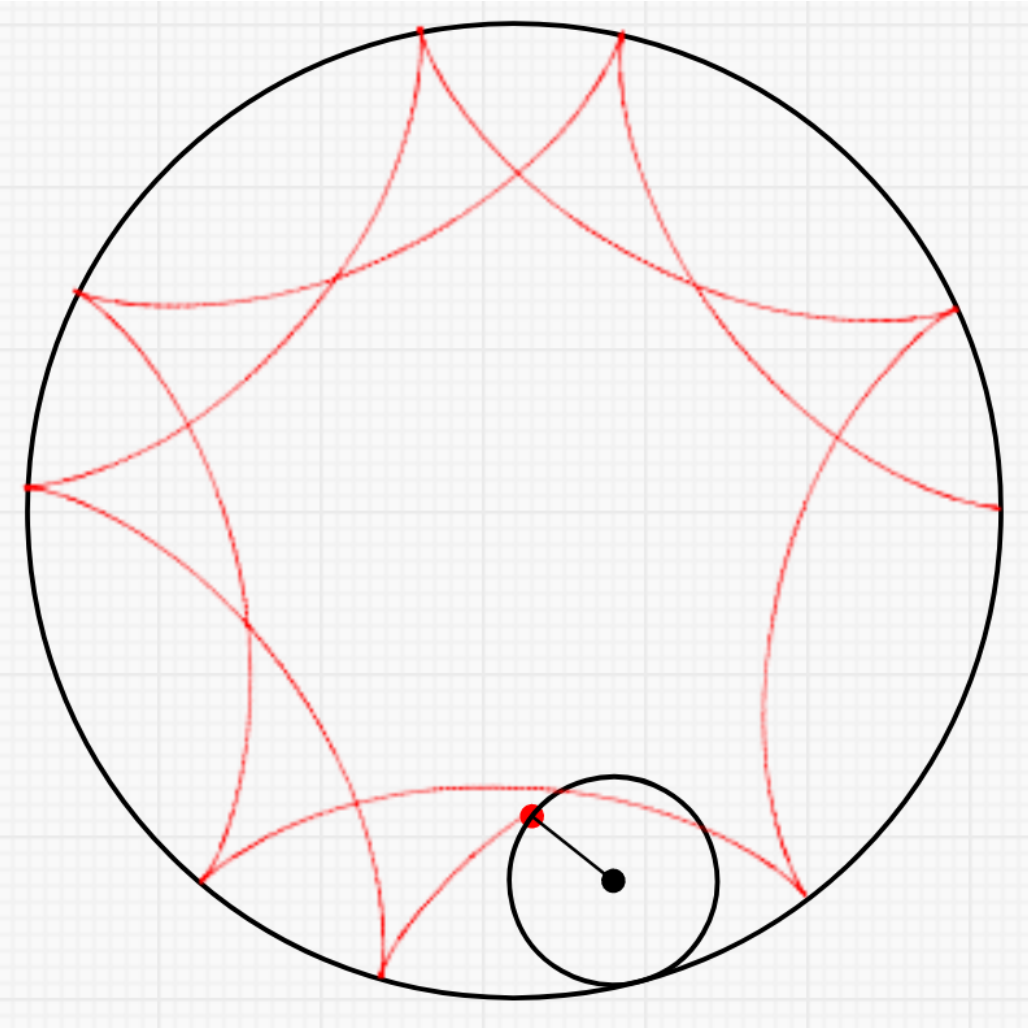
$$\mathbf{d}(t) = r\left(-\sin\frac{vt}{R}\,\mathbf{i} - \cos\frac{vt}{R}\,\mathbf{j}\right)$$

$$\mathbf{c}(t) = \left(vt - r\sin\frac{vt}{R},\ R - r\cos\frac{vt}{R}\right)$$

Cycloid



Hypocycloid



DEFINITION: Velocity Vector If $\mathbf{c}$ is a path and it is differentiable, we say $\mathbf{c}$ is a ***differentiable path***. The ***velocity*** of $\mathbf{c}$ at time t is defined by[3]

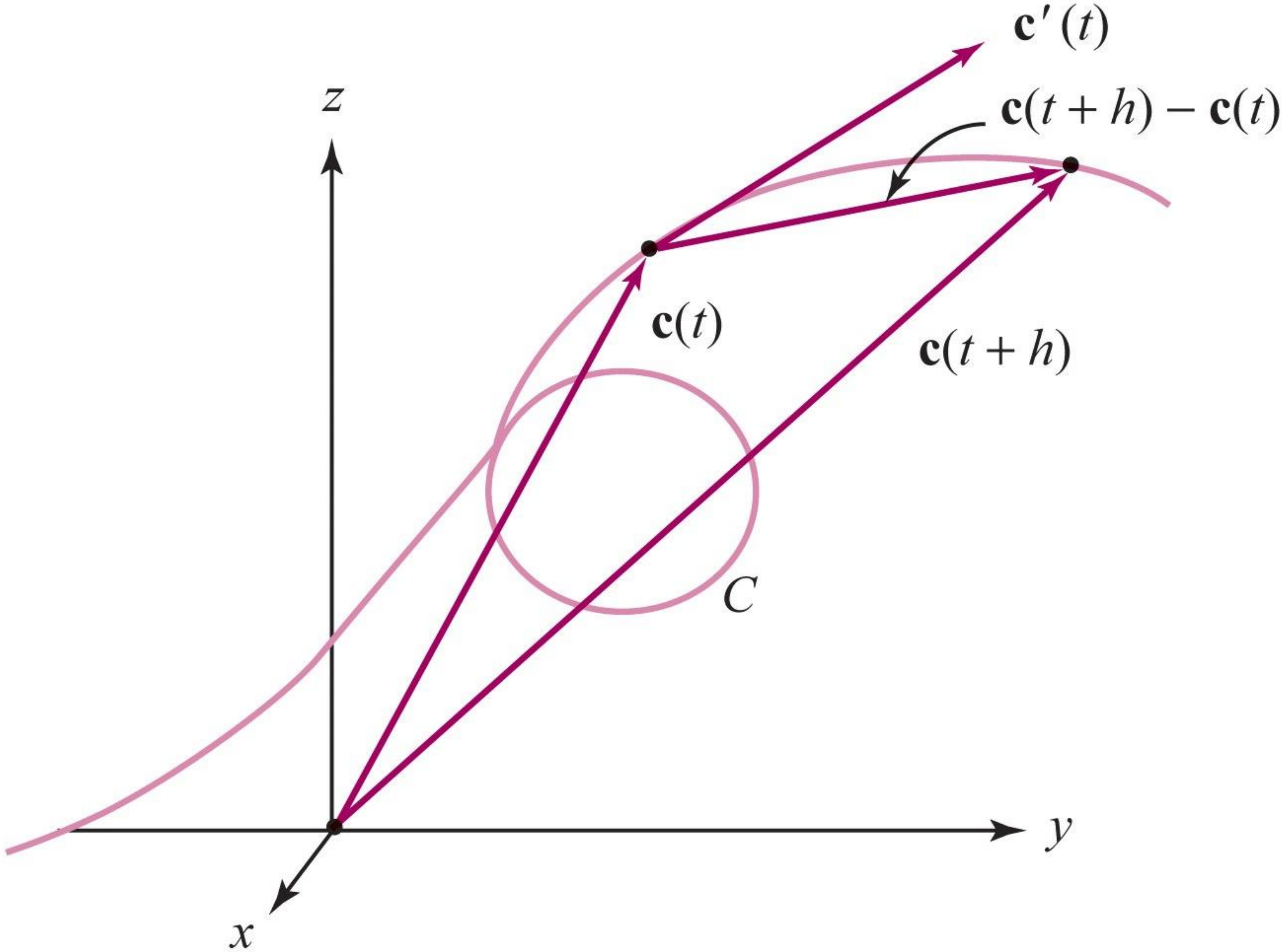
$$\mathbf{c}'(t) = \lim_{h \to 0} \frac{\mathbf{c}(t+h) - \mathbf{c}(t)}{h}.$$

We normally draw the vector $\mathbf{c}'(t)$ with its tail at the point $\mathbf{c}(t)$. The ***speed*** of the path $\mathbf{c}(t)$ is $s = \|\mathbf{c}'(t)\|$, the length of the velocity vector. If $\mathbf{c}(t) = (x(t), y(t))$ in $\mathbb{R}^2$, then

$$\mathbf{c}'(t) = (x'(t), y'(t)) = x'(t)\mathbf{i} + y'(t)\mathbf{j}$$

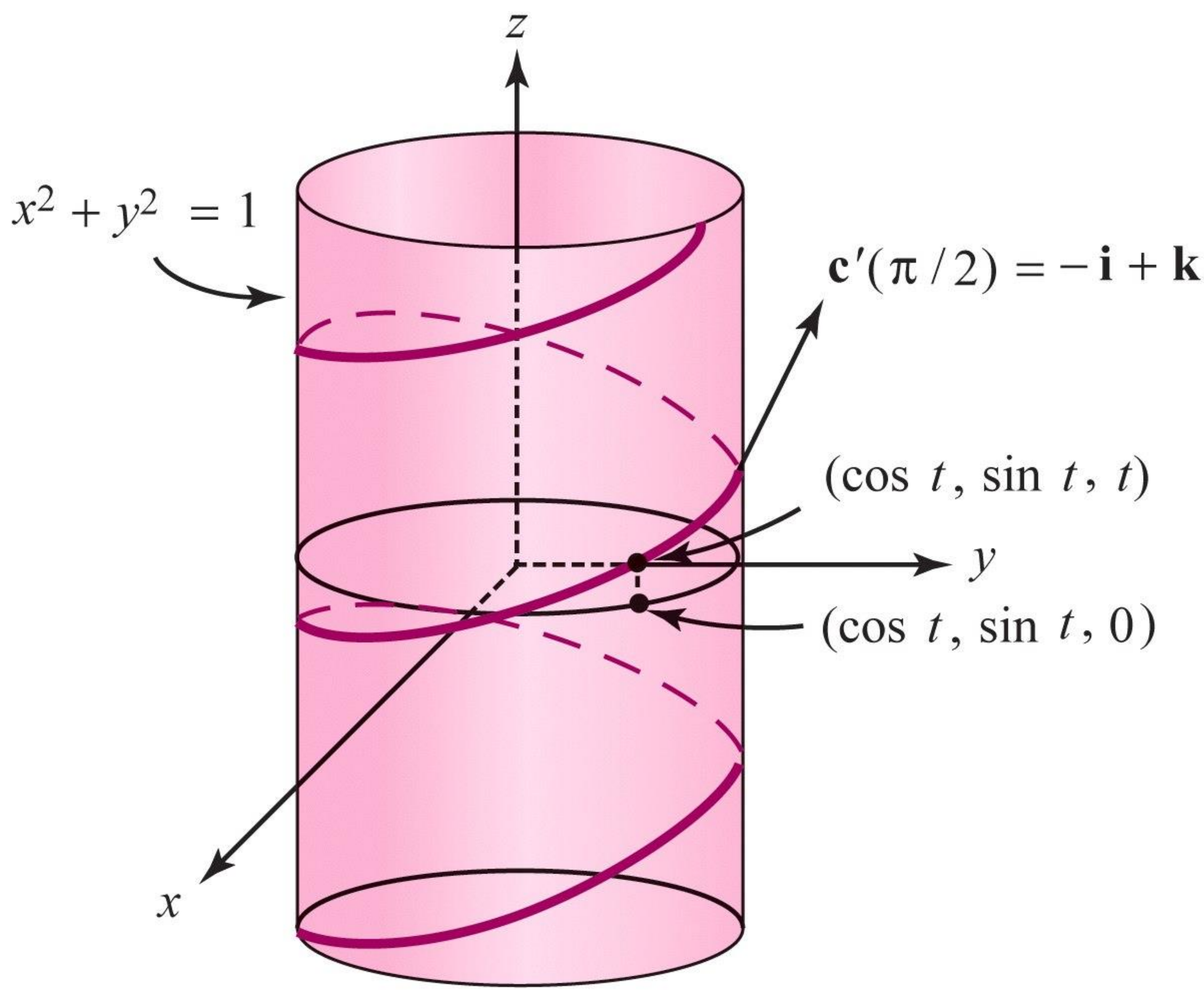
and if $\mathbf{c}(t) = (x(t), y(t), z(t))$ in $\mathbb{R}^3$, then

$$\mathbf{c}'(t) = (x'(t), y'(t), z'(t)) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}.$$

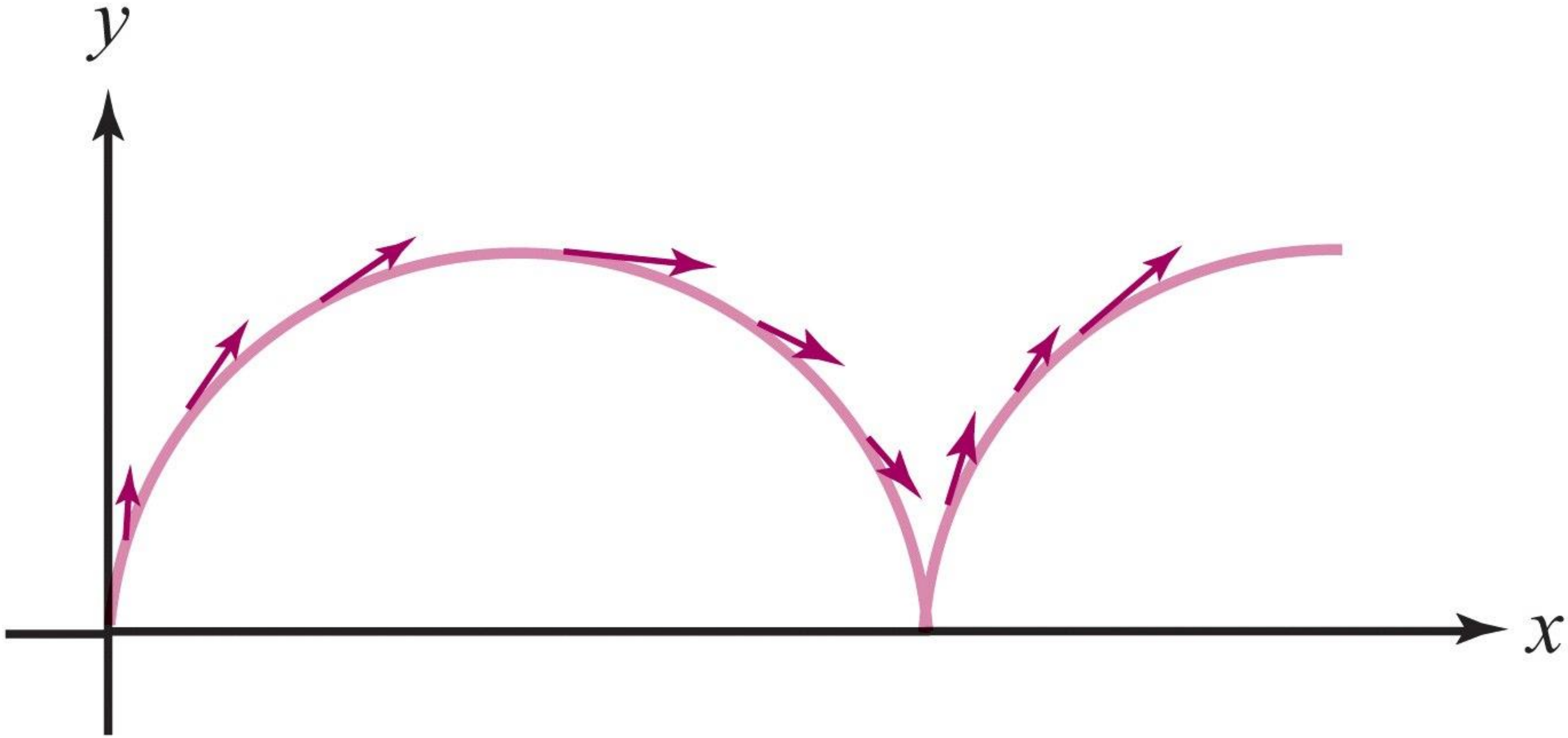


$\mathbf{c}'(t)$
z
$\mathbf{c}(t+h) - \mathbf{c}(t)$
$\mathbf{c}(t)$
$\mathbf{c}(t+h)$
C
y
x

Tangent Vector The velocity $\mathbf{c}'(t)$ is a vector ***tangent*** to the path $\mathbf{c}(t)$ at time t. If C is a curve traced out by $\mathbf{c}$ and if $\mathbf{c}'(t)$ is not equal to $\mathbf{0}$, then $\mathbf{c}'(t)$ is a vector tangent to the curve C at the point $\mathbf{c}(t)$.



z
$x^2 + y^2 = 1$
$\mathbf{c}'(\pi/2) = -\mathbf{i} + \mathbf{k}$
$(\cos t, \sin t, t)$
y
$(\cos t, \sin t, 0)$
x

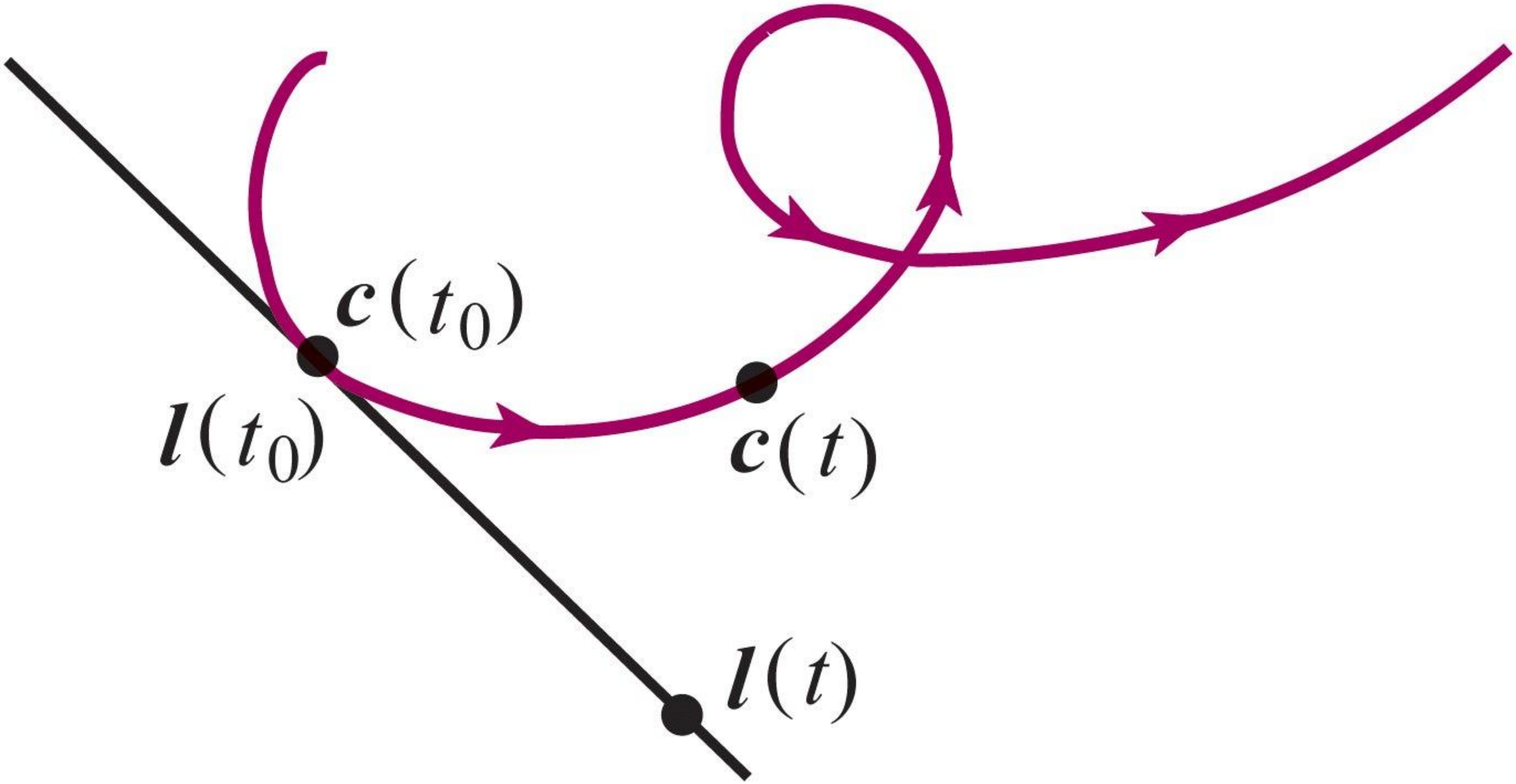


y
x

Tangent Line to a Path If $\mathbf{c}(t)$ is a path, and if $\mathbf{c}'(t_0) \neq \mathbf{0}$, the equation of its ***tangent line*** at the point $\mathbf{c}(t_0)$ is

$$\boldsymbol{l}(t) = \mathbf{c}(t_0) + (t - t_0)\mathbf{c}'(t_0).$$

If C is the curve traced out by $\mathbf{c}$, then the line traced out by $\boldsymbol{l}$ is the tangent line to the curve C at $\mathbf{c}(t_0)$.



$\boldsymbol{c}(t_0)$
$\boldsymbol{l}(t_0)$
$\boldsymbol{c}(t)$
$\boldsymbol{l}(t)$