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Vector Calculus

Fifth Edition

Chapter 2:
Differentiation

2.1 The Geometry of Real-Value Functions

2.1 Functions, Graphs, and Level Surfaces

Key Points in this Section.

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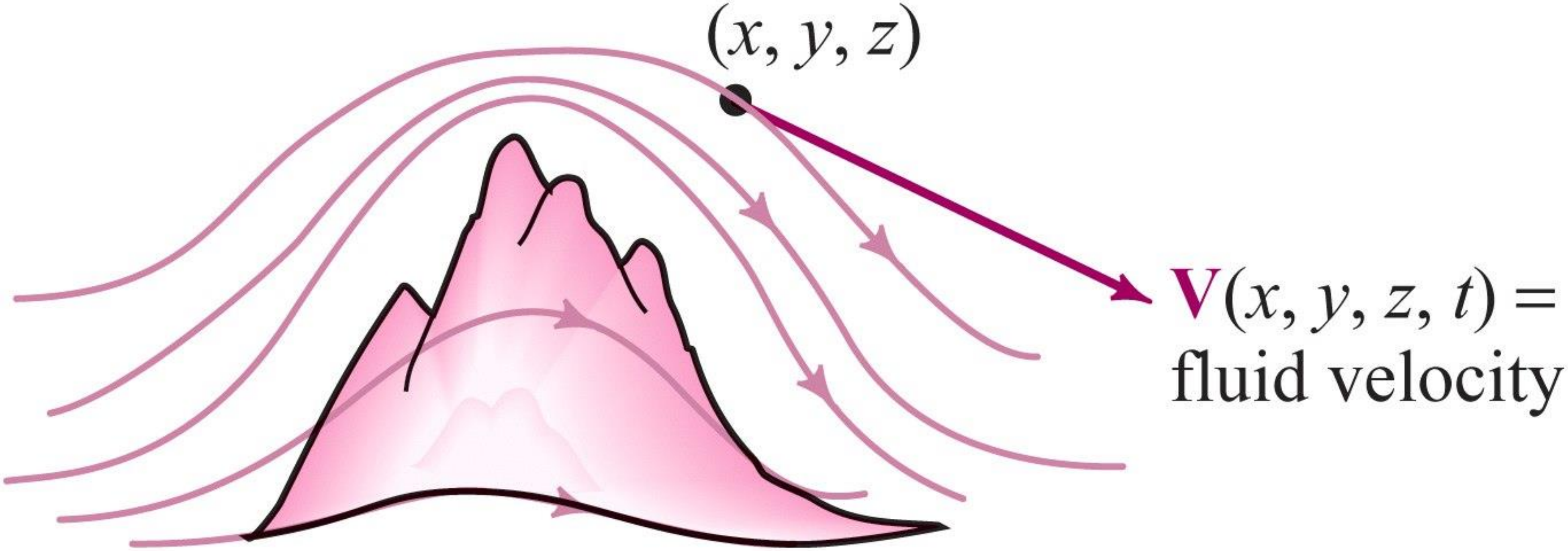
1. A ***mapping*** or ***function*** $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$ sends each point $\mathbf{x} \in A$ (the domain of f) to a specific point $f(\mathbf{x}) \in \mathbb{R}^m$. If $m = 1$, we call f a ***real valued function***.

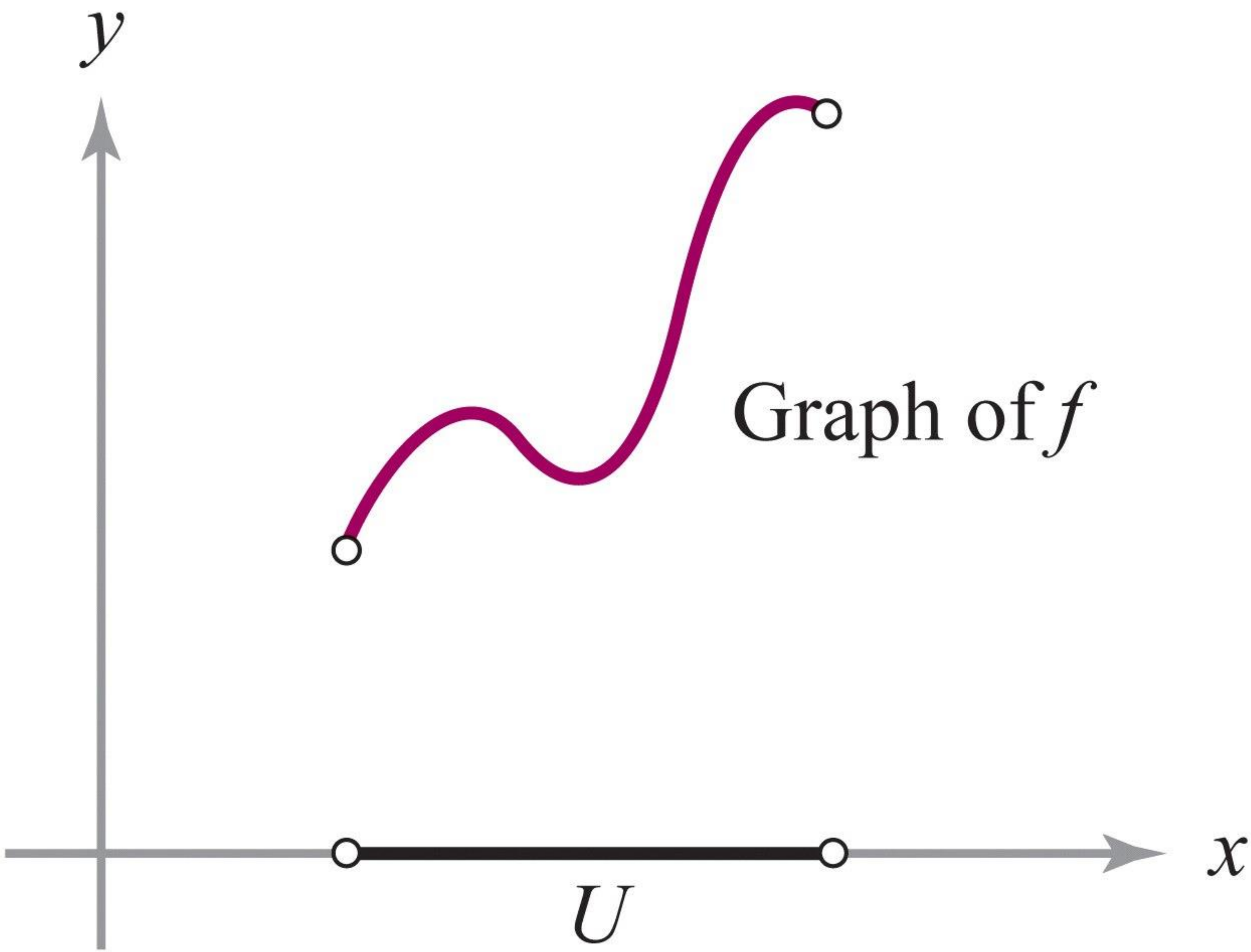
2. The ***graph*** of $f : U \subset \mathbb{R}^2 \to \mathbb{R}$ is the set of all points of the form (x, y, z) where $(x, y) \in U$ and $z = f(x, y)$. More generally, for $f : U \subset \mathbb{R}^n \to \mathbb{R}$ the graph is the subset of $\mathbb{R}^{n+1}$ consisting of points of the form $(x_1, \dots, x_n, z)$, where $(x_1, \dots, x_n) \in U$ (the domain of f) and $z = f(x_1, \dots, x_n)$.

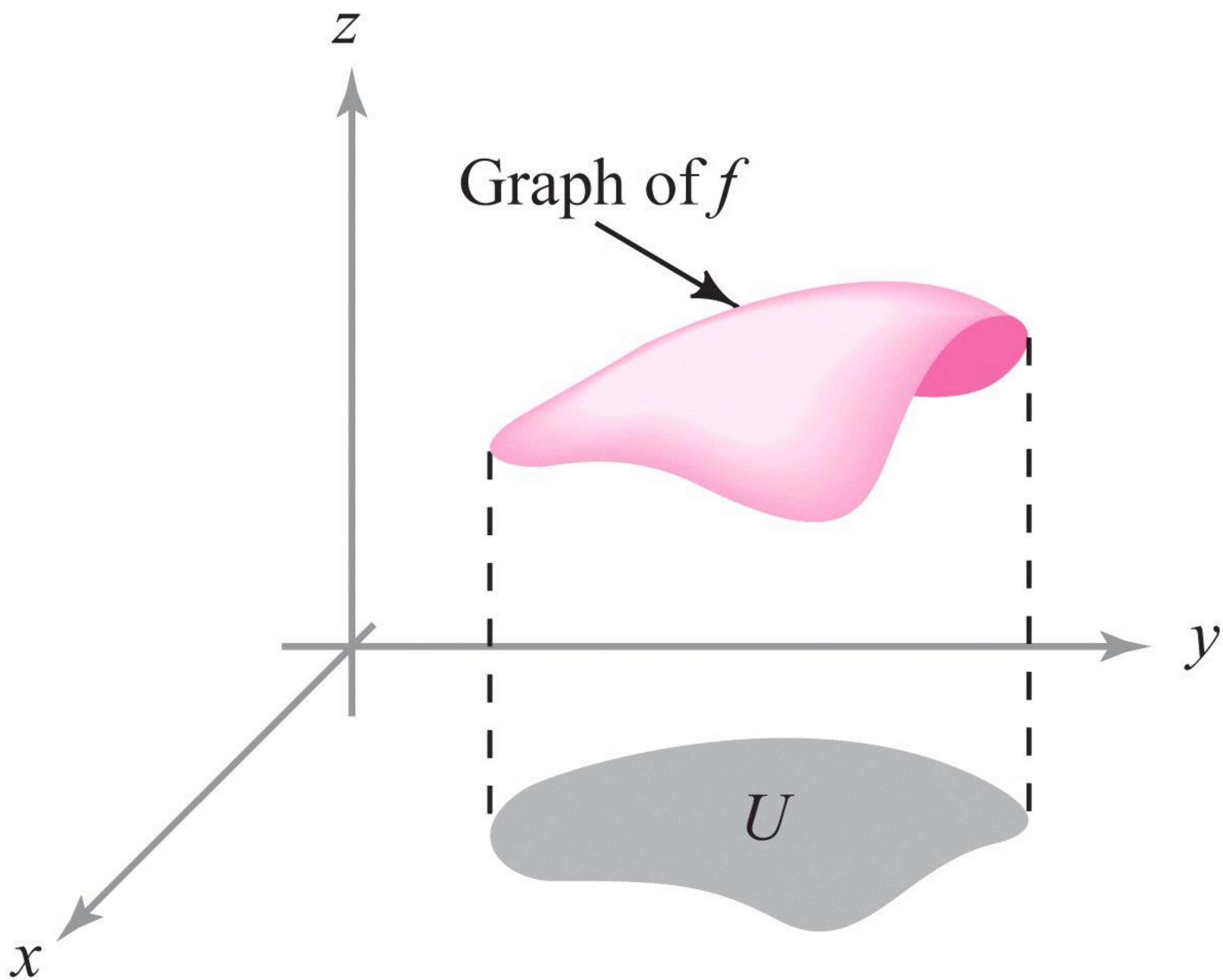
3. A ***level set*** of a real valued function $f : U \subset \mathbb{R}^n \to \mathbb{R}$ obtained by picking a constant c and forming the set of points $(x_1, \dots, x_n)$ in U such that $f(x_1, \dots, x_n) = c$. For $n = 3$ we speak of them as ***level surfaces*** and for $n = 2$, ***level curves***.

4. A ***section*** of a graph is obtained by intersecting the graph with a vertical plane. For instance, for $z = f(x, y)$, setting $y = 0$ produces the section $z = f(x, 0)$ which is the graph of one function of one variable.

5. Level sets and sections are useful tools in constructing and visualizing graphs.


(x, y, z)
V(x, y, z, t) =
fluid velocity


y
Graph of f
x
U

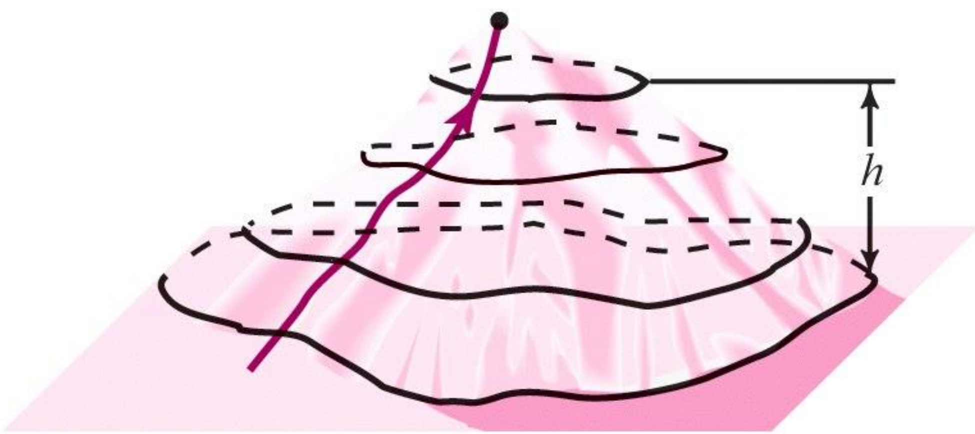

z
Graph of f
y
U
x

DEFINITION: Graph of a Function Let $f\colon U \subset \mathbb{R}^n \to \mathbb{R}$. Define the ***graph*** of f to be the subset of $\mathbb{R}^{n+1}$ consisting of all the points

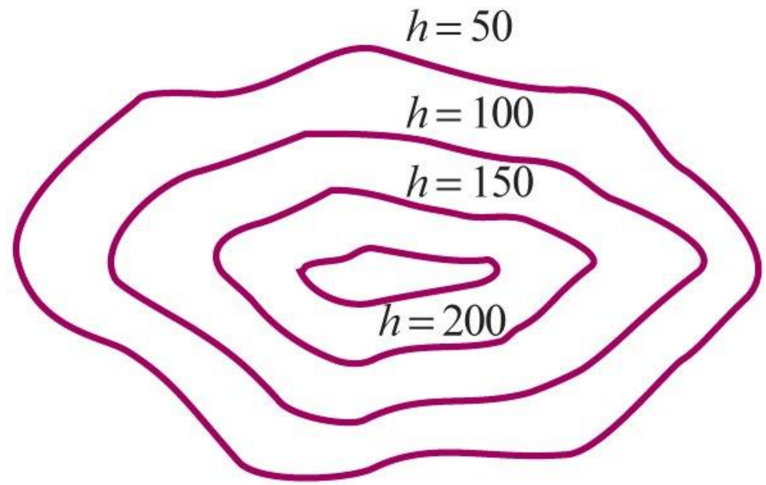
$$(x_1, \ldots, x_n, f(x_1, \ldots, x_n))$$

in $\mathbb{R}^{n+1}$ for $(x_1, \ldots, x_n)$ in U. In symbols,

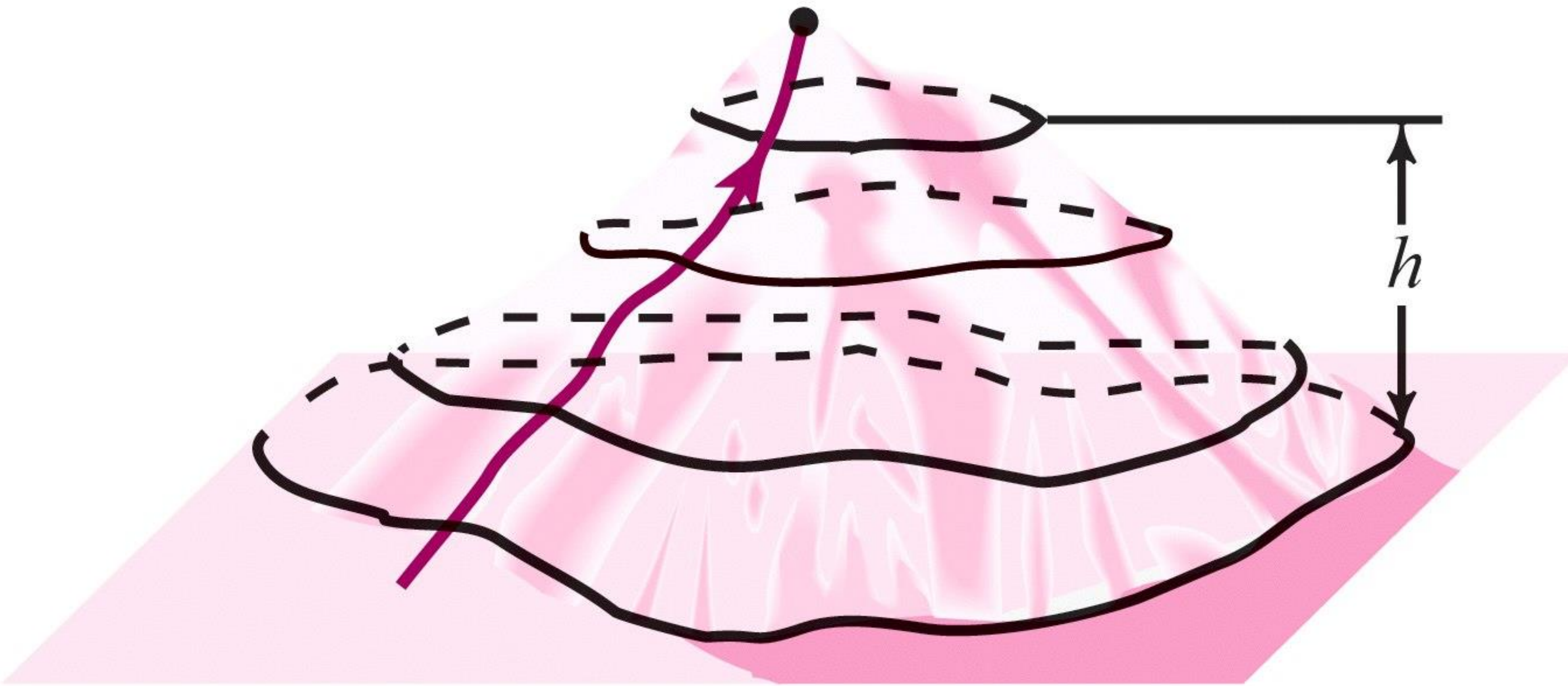
$$\text{graph } f = \{(x_1, \ldots, x_n, f(x_1, \ldots, x_n)) \in \mathbb{R}^{n+1} \mid (x_1, \ldots, x_n) \in U\}.$$

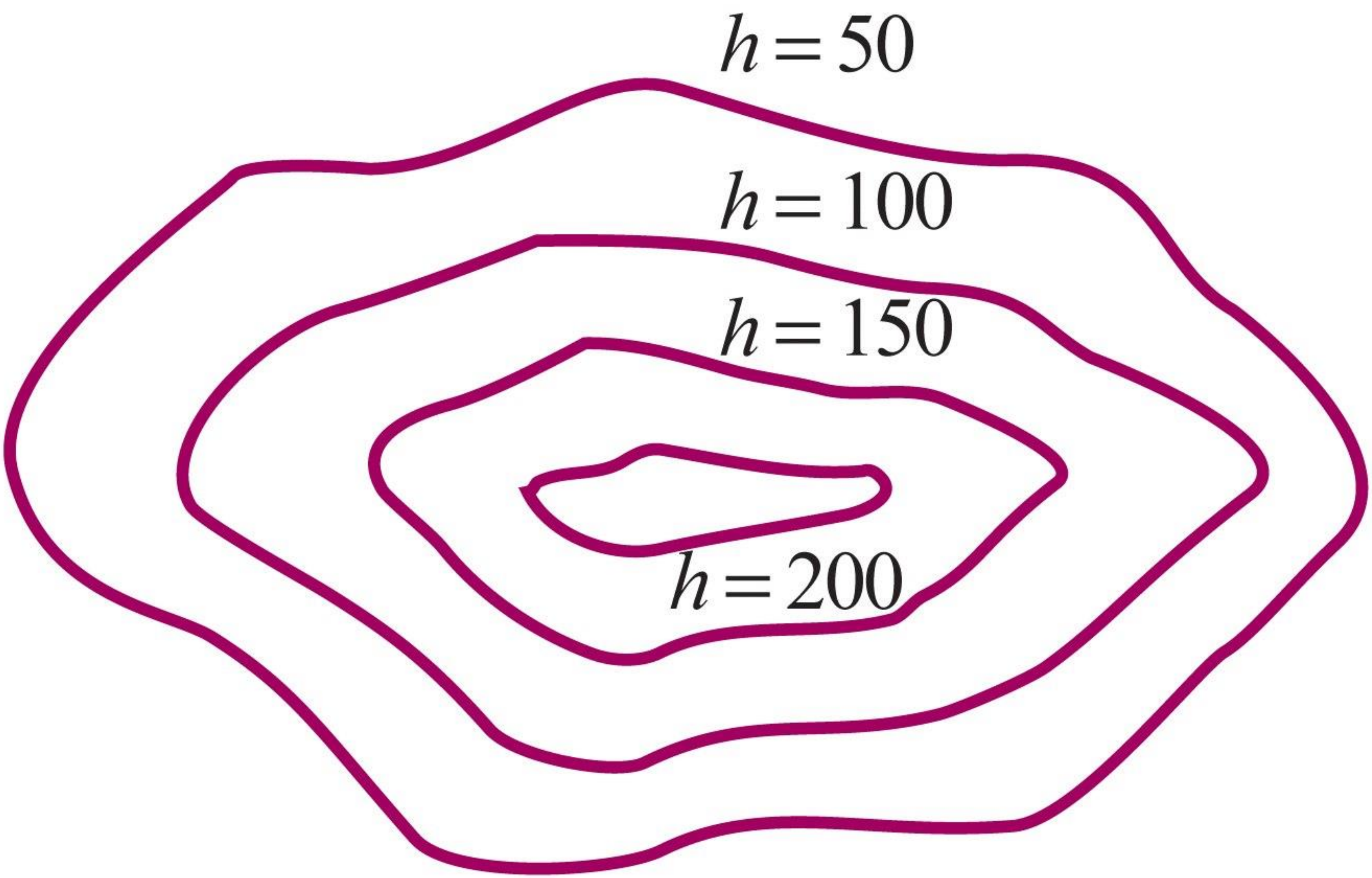


(a)

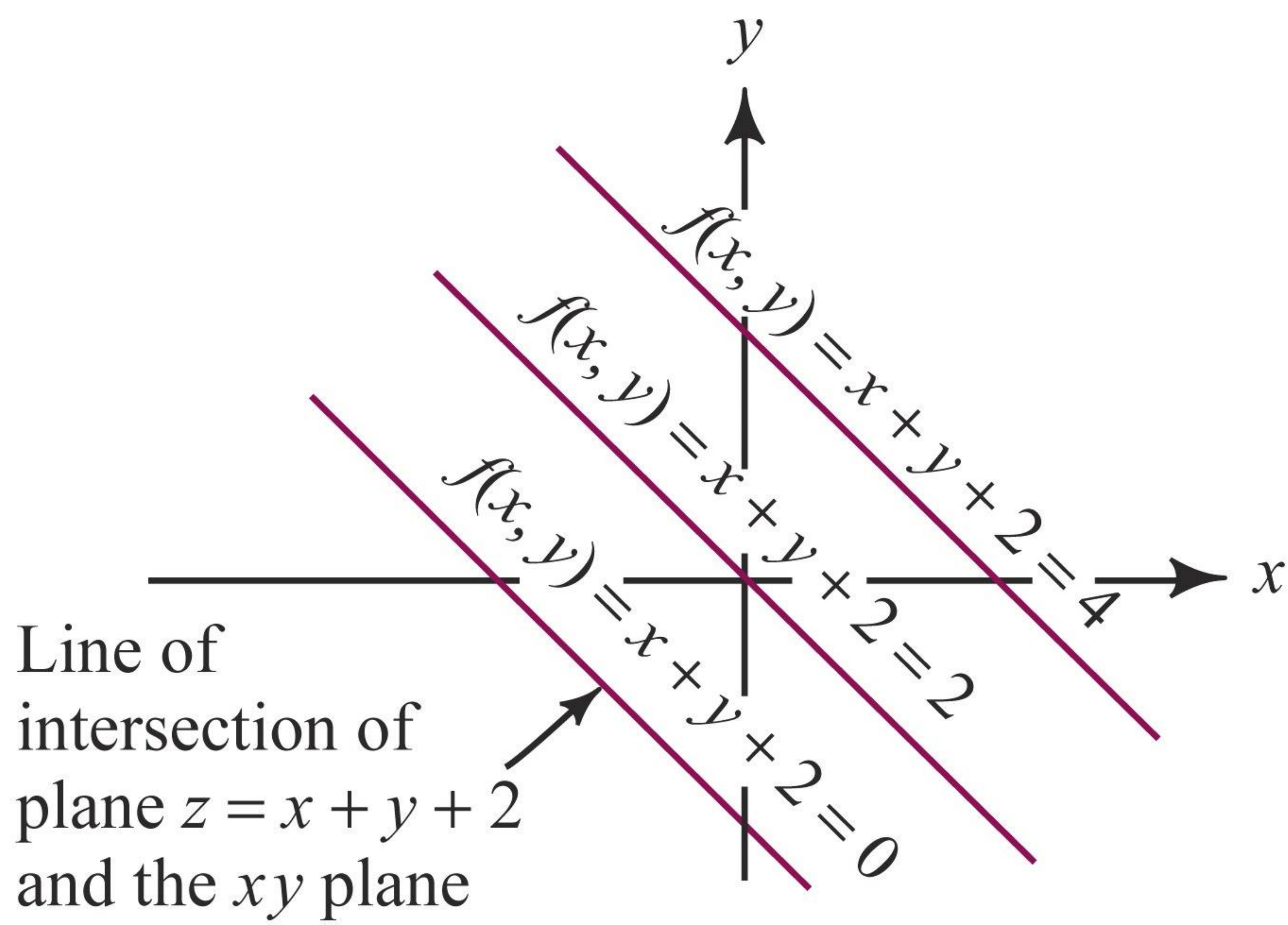


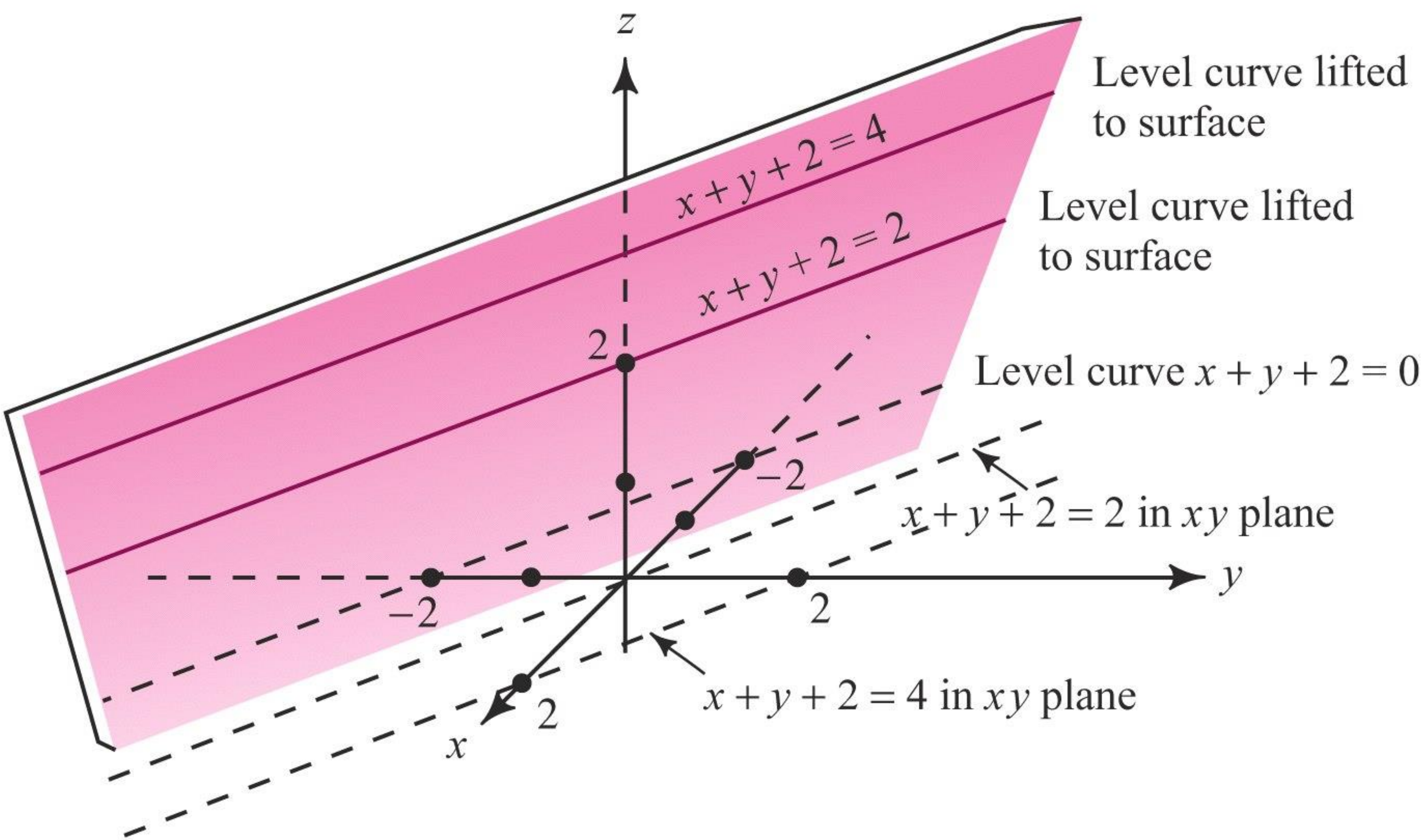
(b)


h



$h = 50$
$h = 100$
$h = 150$
$h = 200$


y
x
f(x, y) = x + y + 2 = 4
f(x, y) = x + y + 2 = 2
f(x, y) = x + y + 2 = 0
Line of intersection of plane z = x + y + 2 and the xy plane

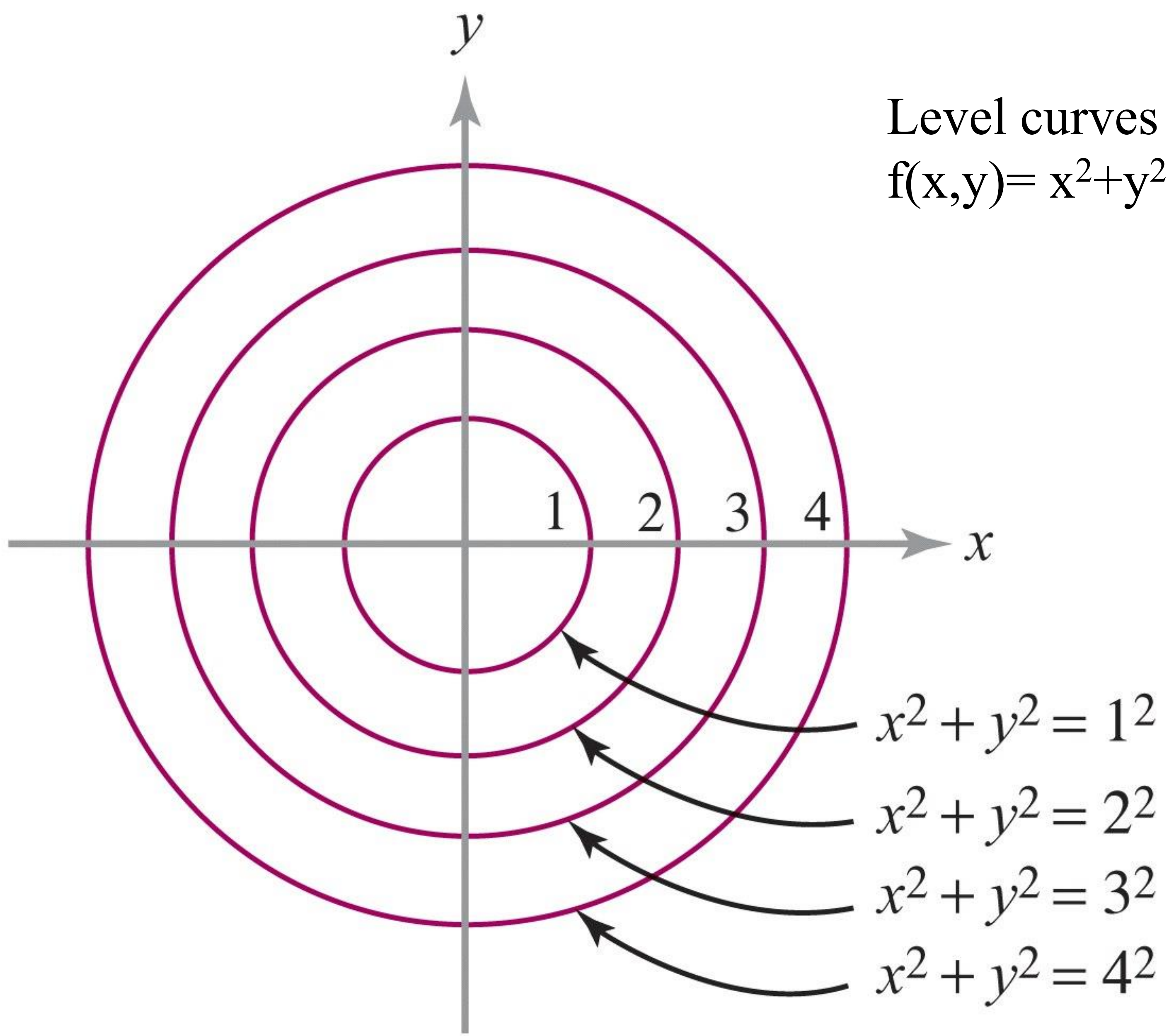

z
Level curve lifted to surface
$x + y + 2 = 4$
Level curve lifted to surface
$x + y + 2 = 2$
2
Level curve $x + y + 2 = 0$
−2
$x + y + 2 = 2$ in xy plane
y
−2
2
2
x
$x + y + 2 = 4$ in xy plane

DEFINITION: Level Curves and Surfaces Let $f\colon U \subset \mathbb{R}^n \to \mathbb{R}$ and let $c \in \mathbb{R}$. Then the ***level set of value*** c is defined to be the set of those points $\mathbf{x} \in U$ at which $f(\mathbf{x}) = c$. If $n = 2$, we speak of a ***level curve*** (of value c); and if $n = 3$, we speak of a ***level surface***. In symbols, the level set of value c is written

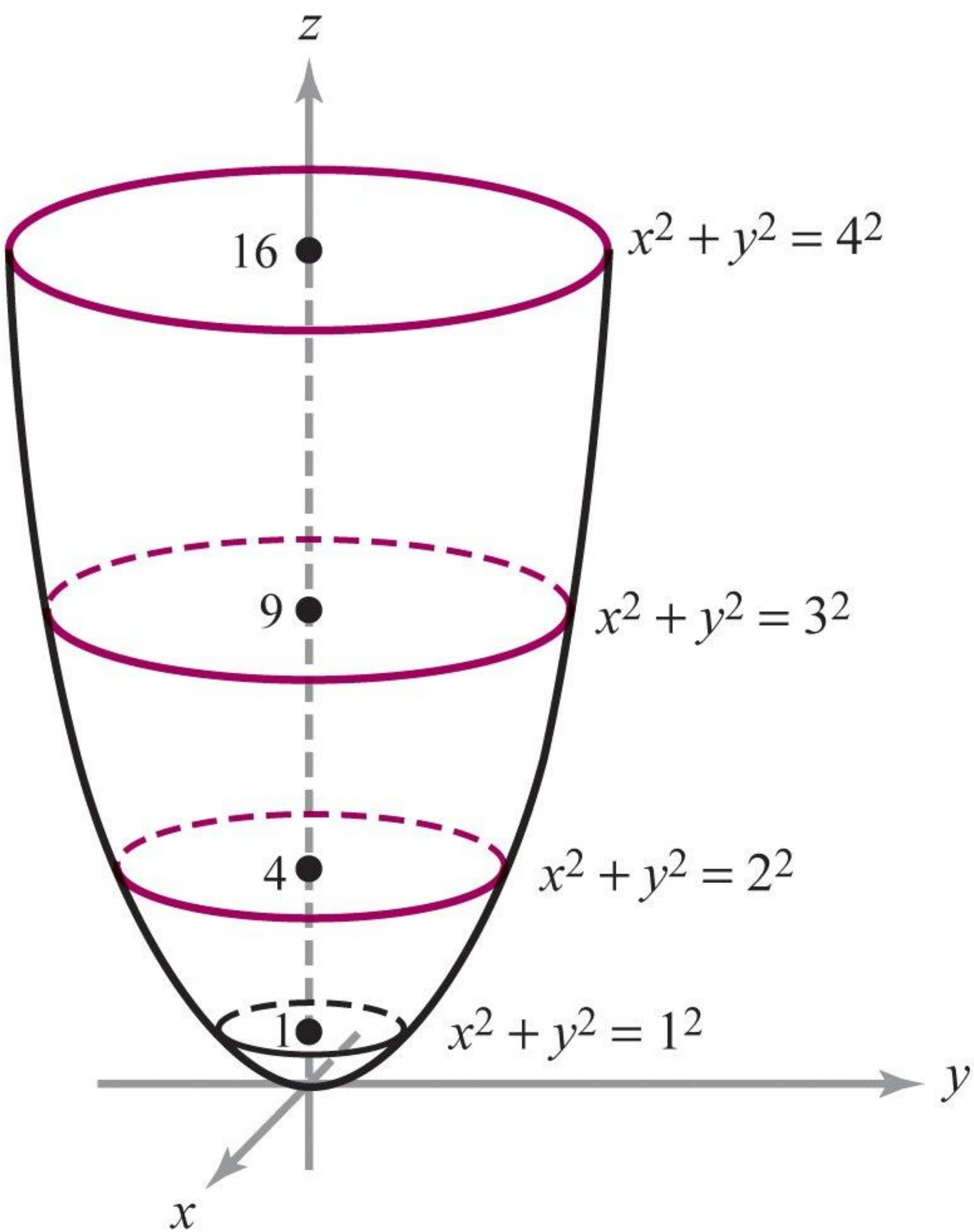
$$\{\mathbf{x} \in U \mid f(\mathbf{x}) = c\} \subset \mathbb{R}^n.$$

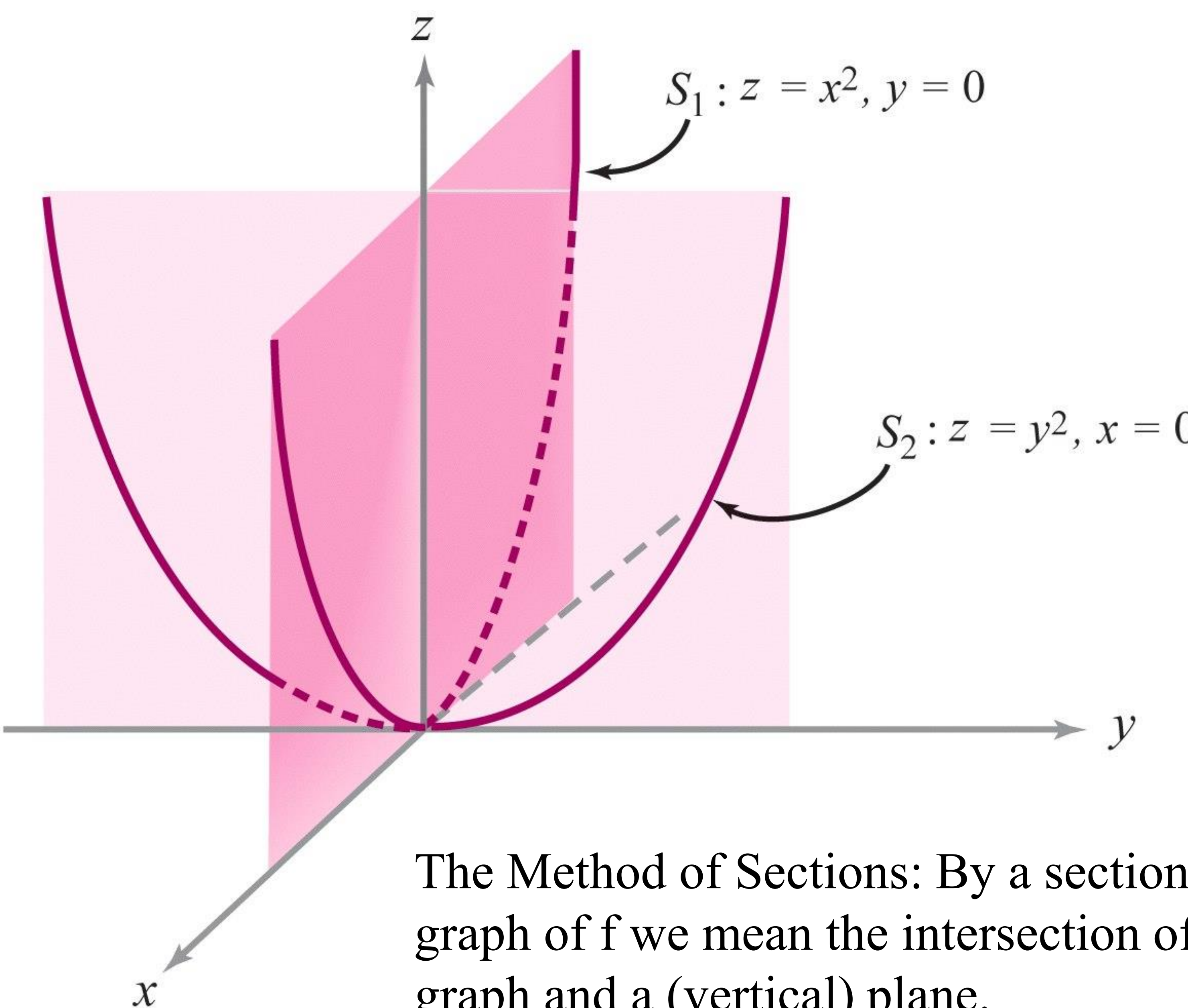
Note that the level set is always in the domain space.

Level curves of
$f(x,y)= x^2+y^2$



Level curves of
f(x,y)= x^2+y^2
3D view





The Method of Sections: By a section of the graph of f we mean the intersection of the graph and a (vertical) plane.

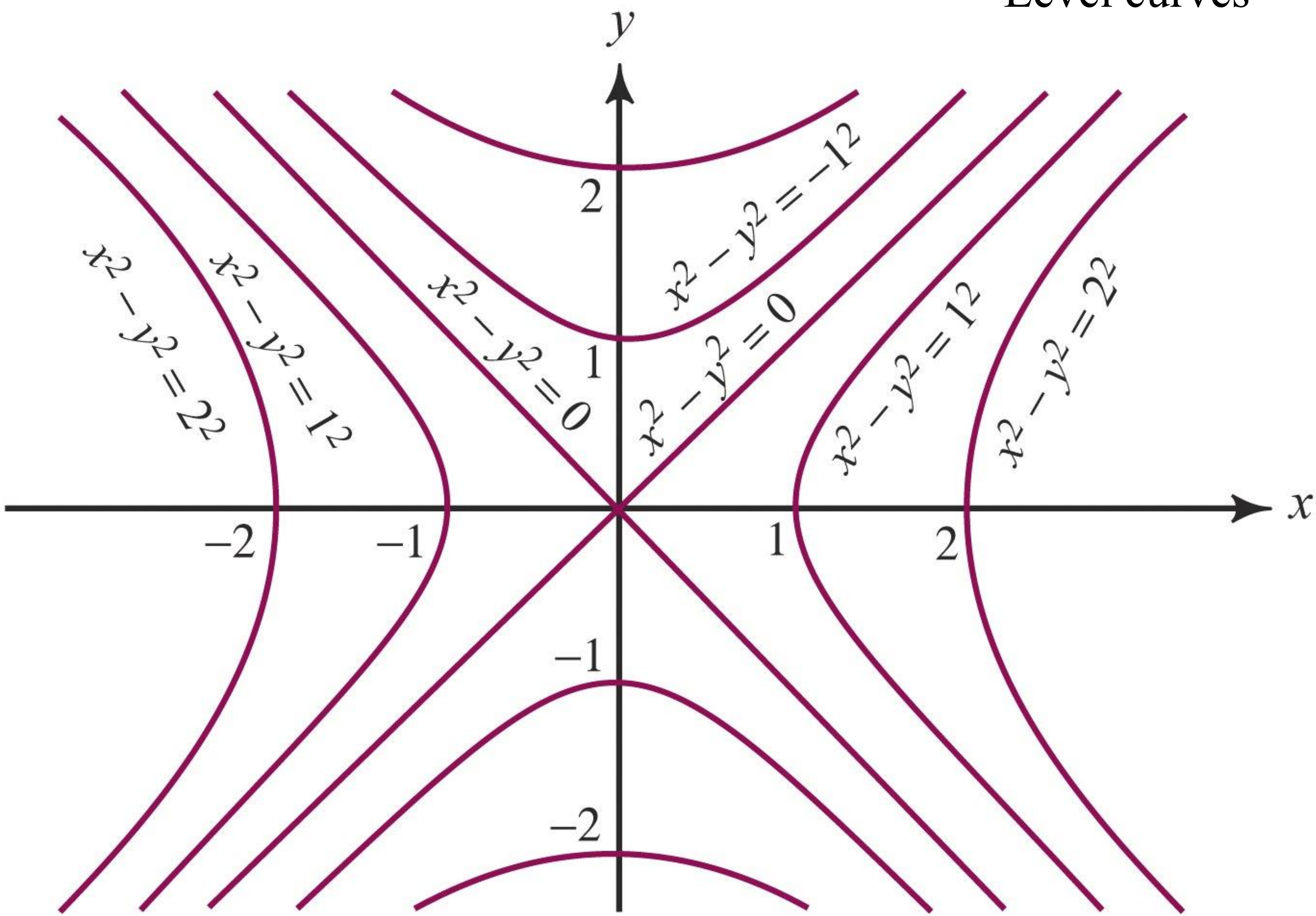
Exercise:

The graph of the quadratic function

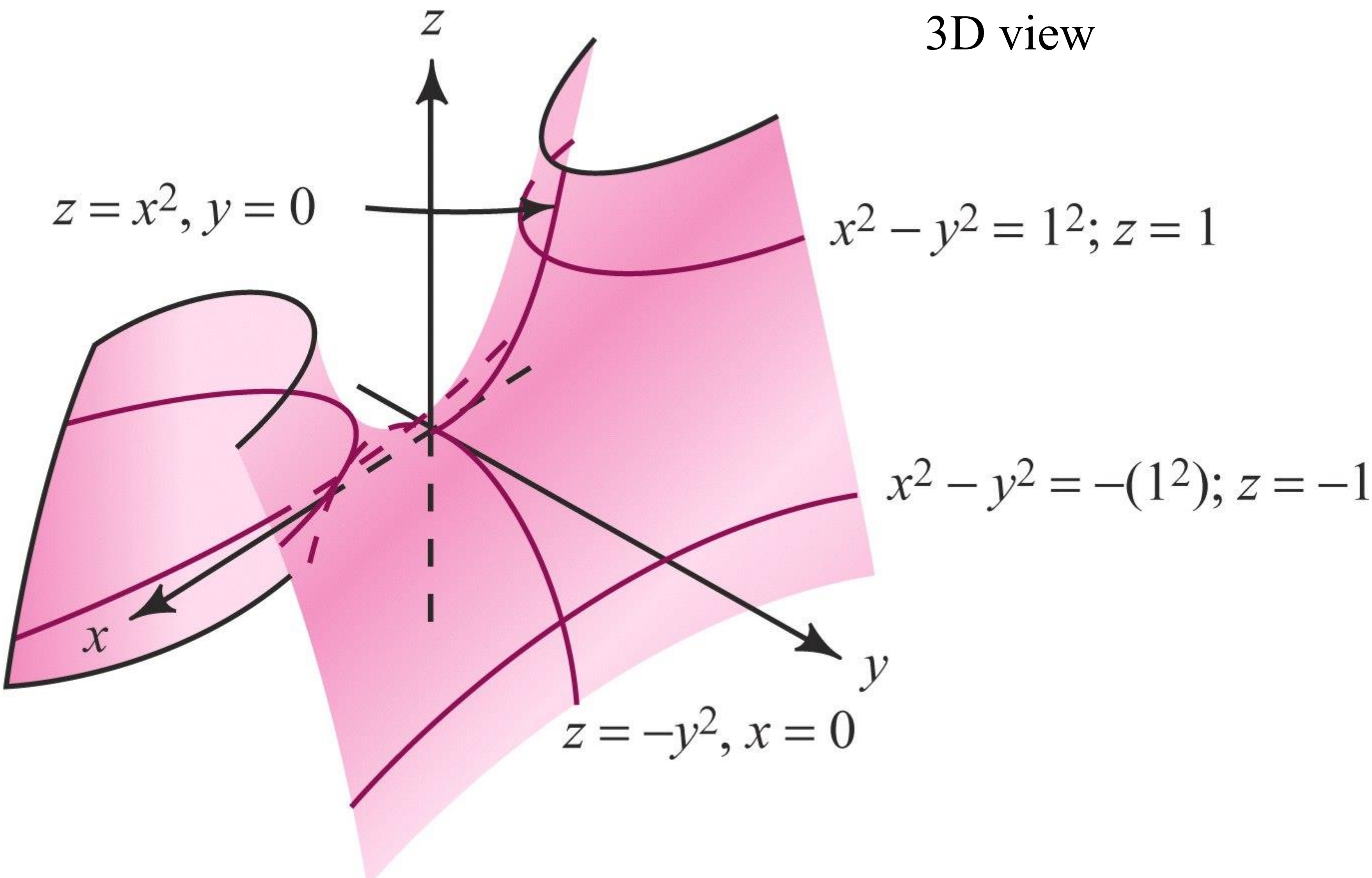
$$\begin{aligned} f : R^2 &\rightarrow R \\ (x, y) &\rightarrow x^2 - y^2 \end{aligned}$$

is called a hyperbolic paraboloid, or saddle, centered at the origin. Sketch the graph. (Hint: Consider the values $c = 0, \pm 1, \pm 4$)

Level curves

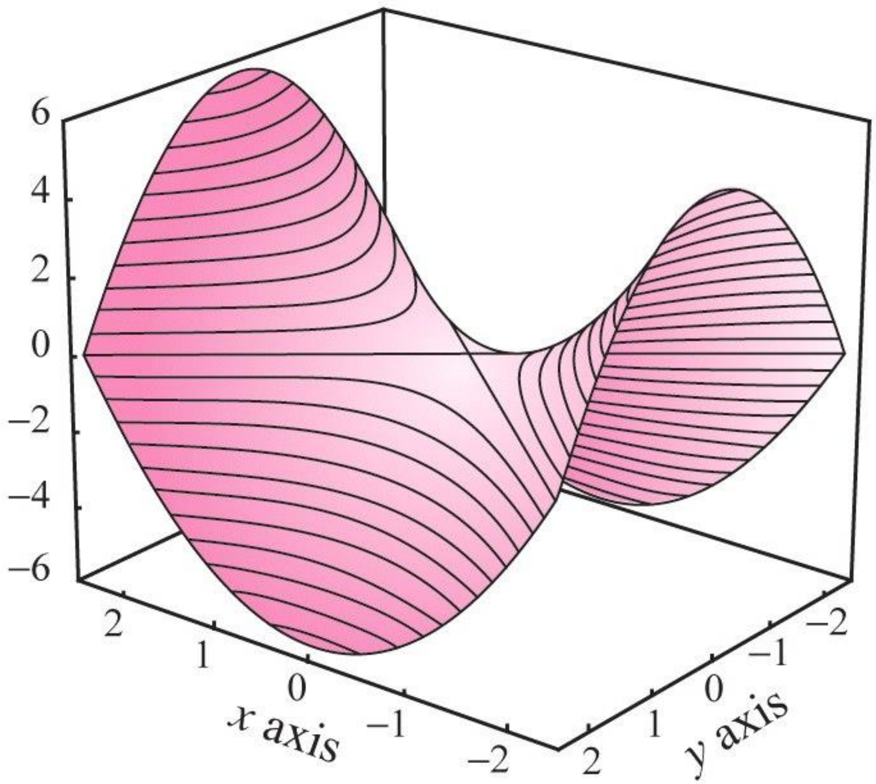


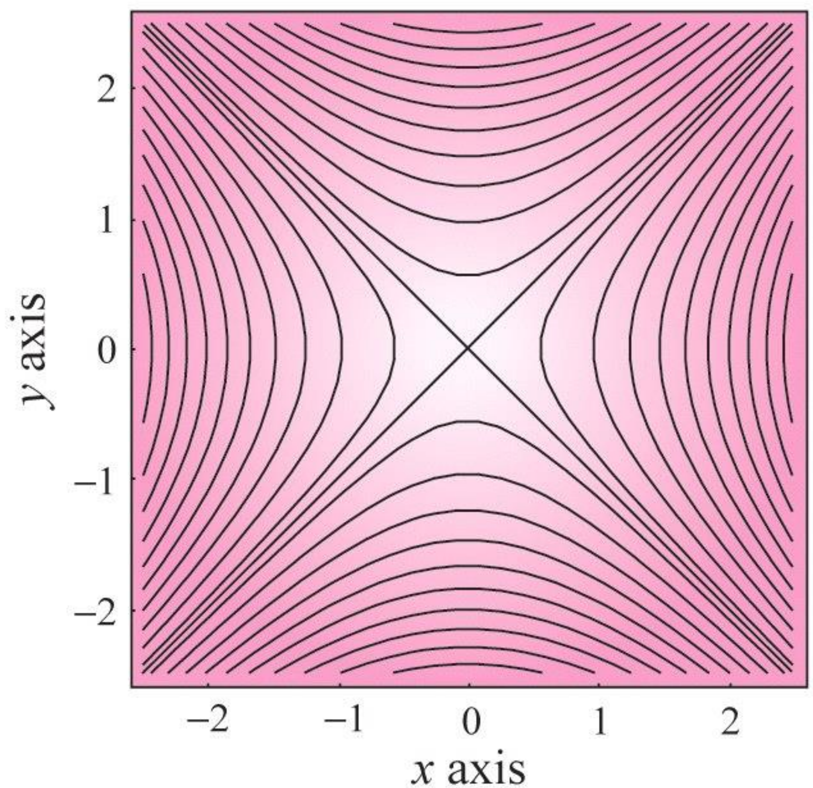
3D view



Level curves of $\quad f : \mathbb{R}^2 \to \mathbb{R}$

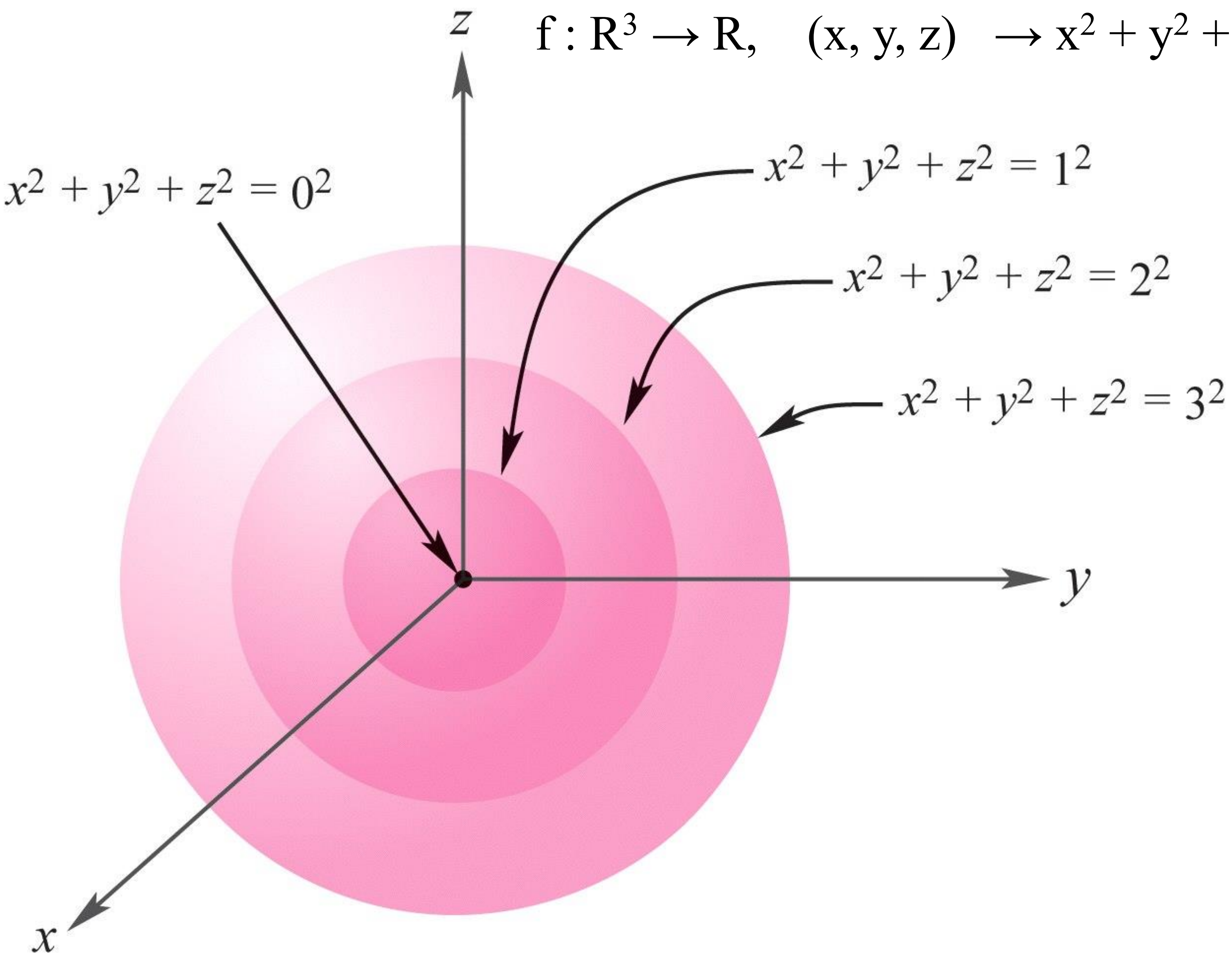
$(x, y) \to x^2 - y^2$

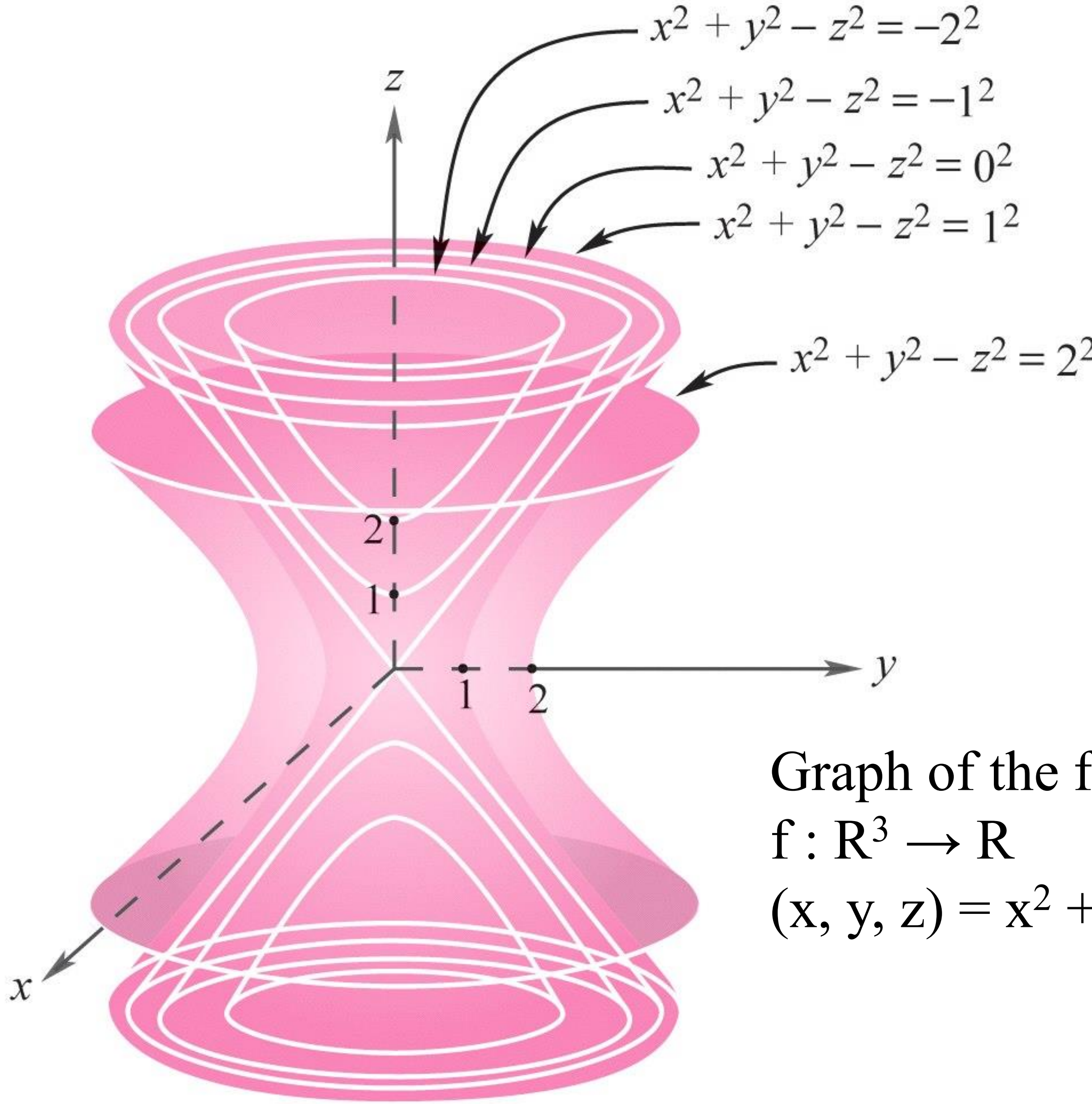




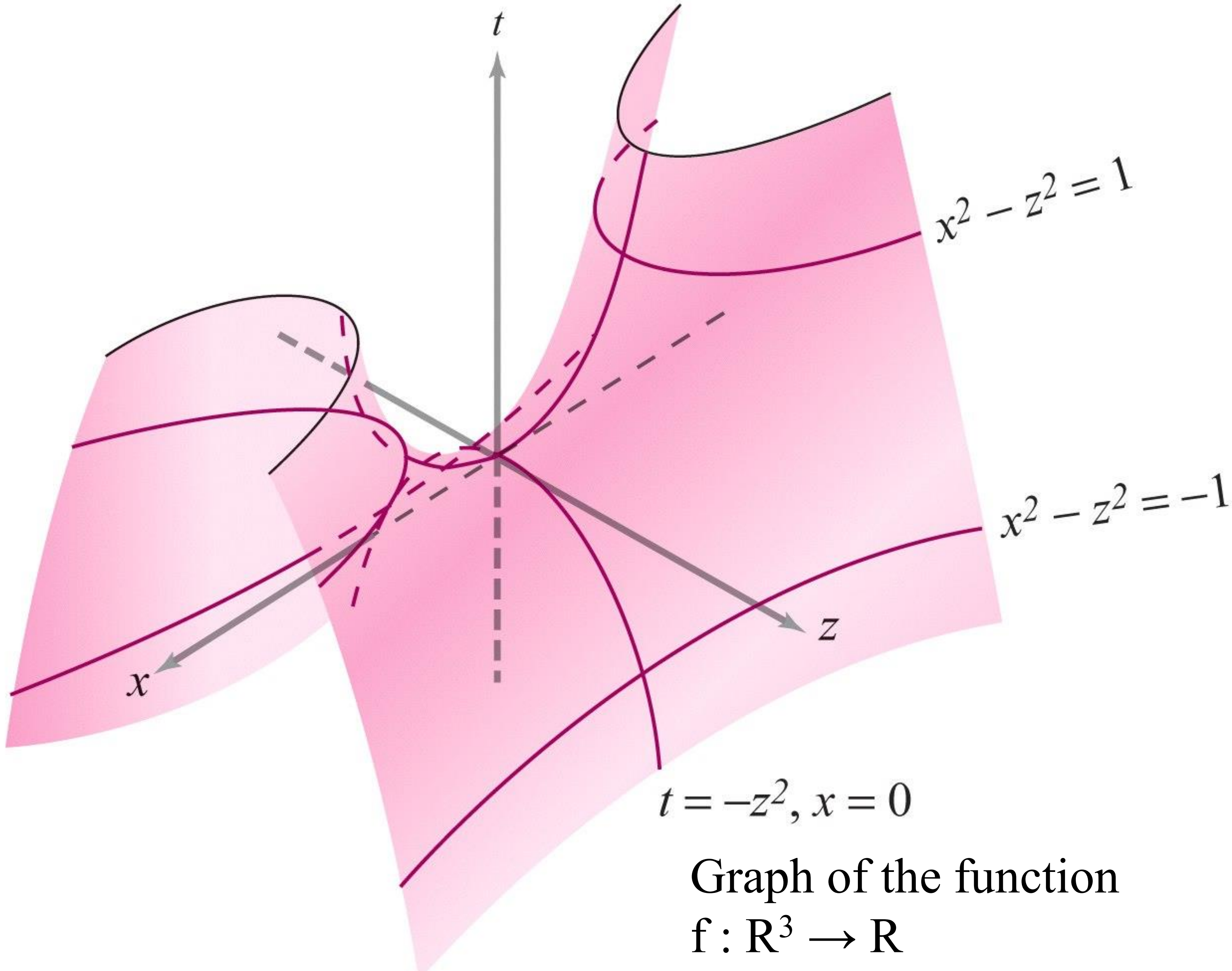
the level sets of the function

$f : R^3 \longrightarrow R, \quad (x, y, z) \quad \longrightarrow x^2 + y^2 + z^2.$



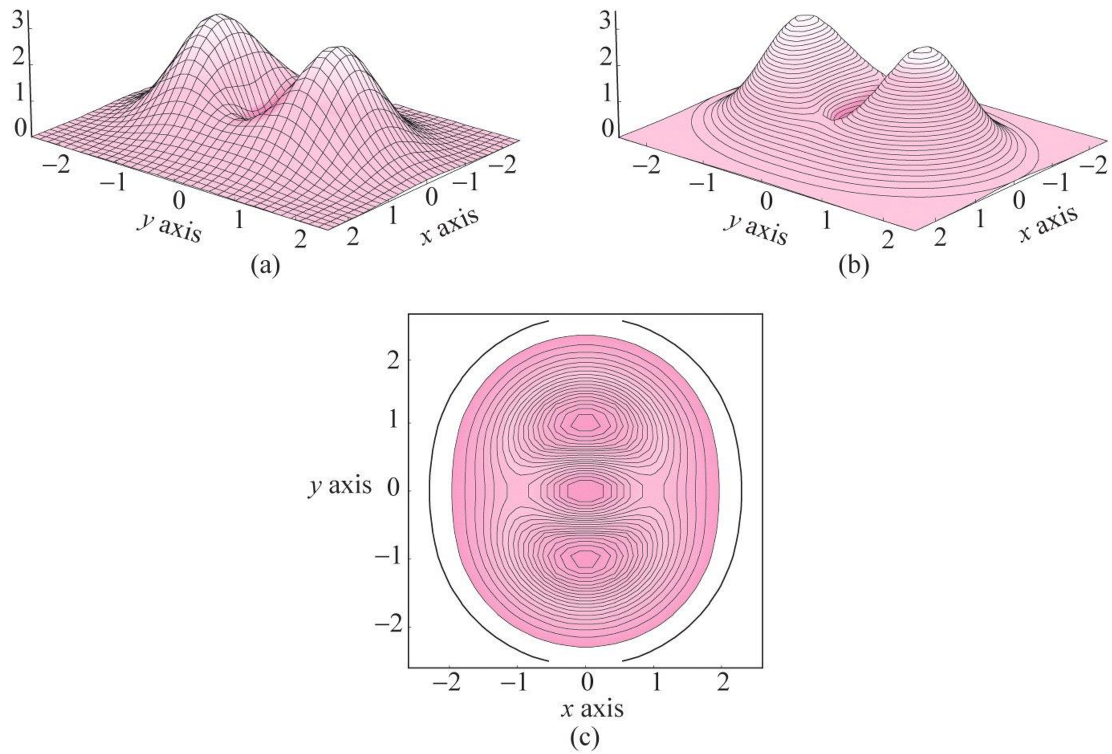


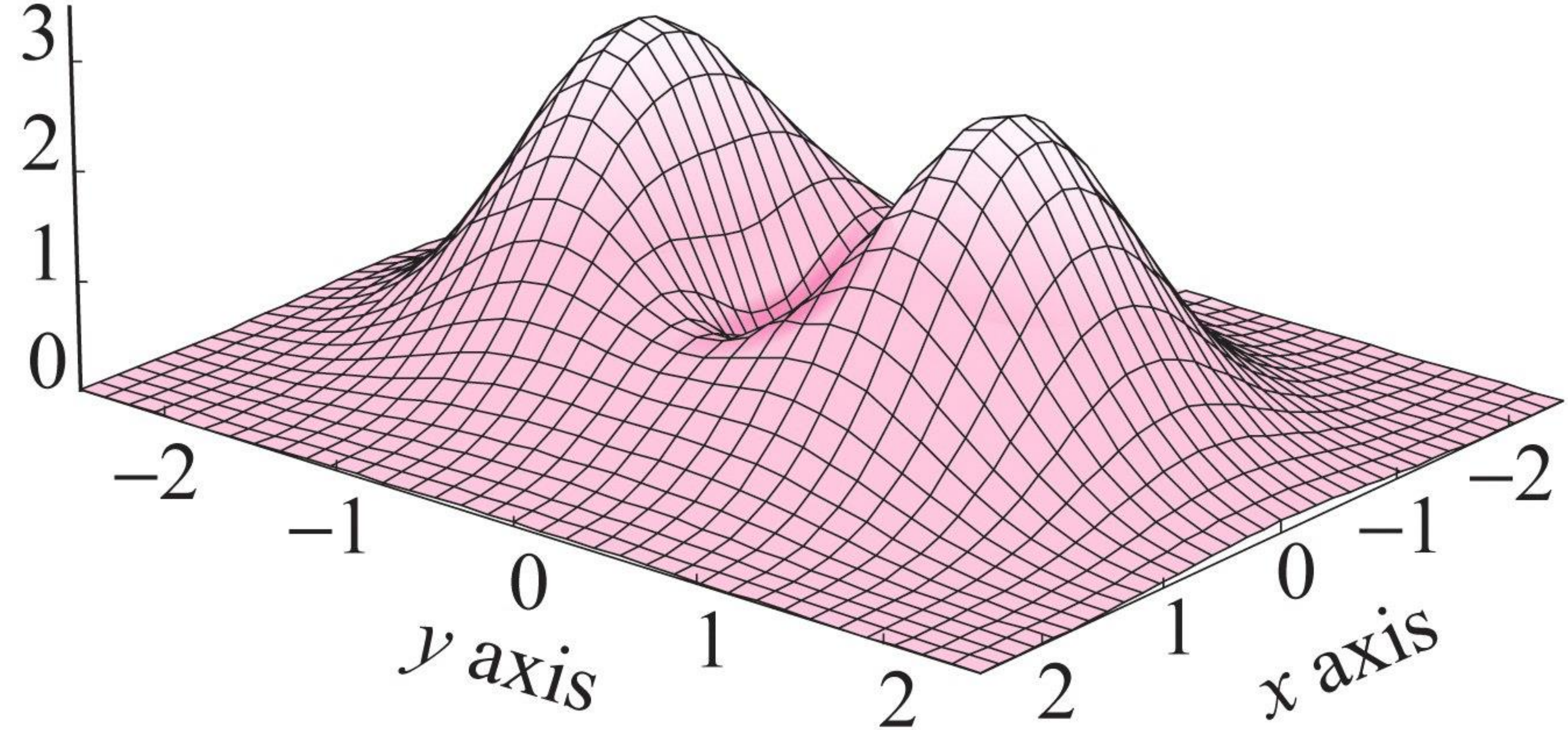
Graph of the function
$f : R^3 \longrightarrow R$
$(x, y, z) = x^2 + y^2 - z^2,$



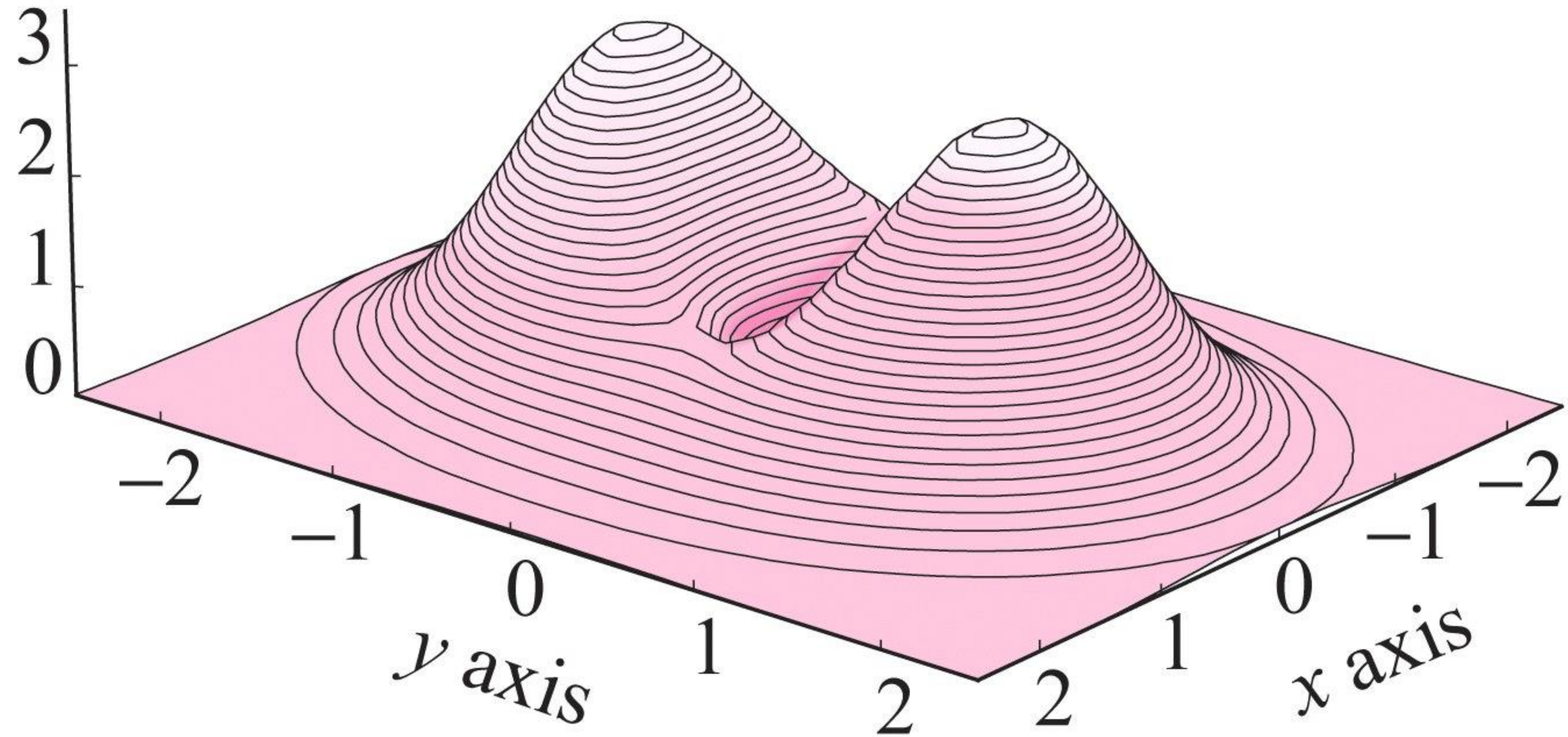
Graph of the function
$f : R^3 \rightarrow R$
$(x, y, z) = x^2 + y^2 - z^2$, for y=0

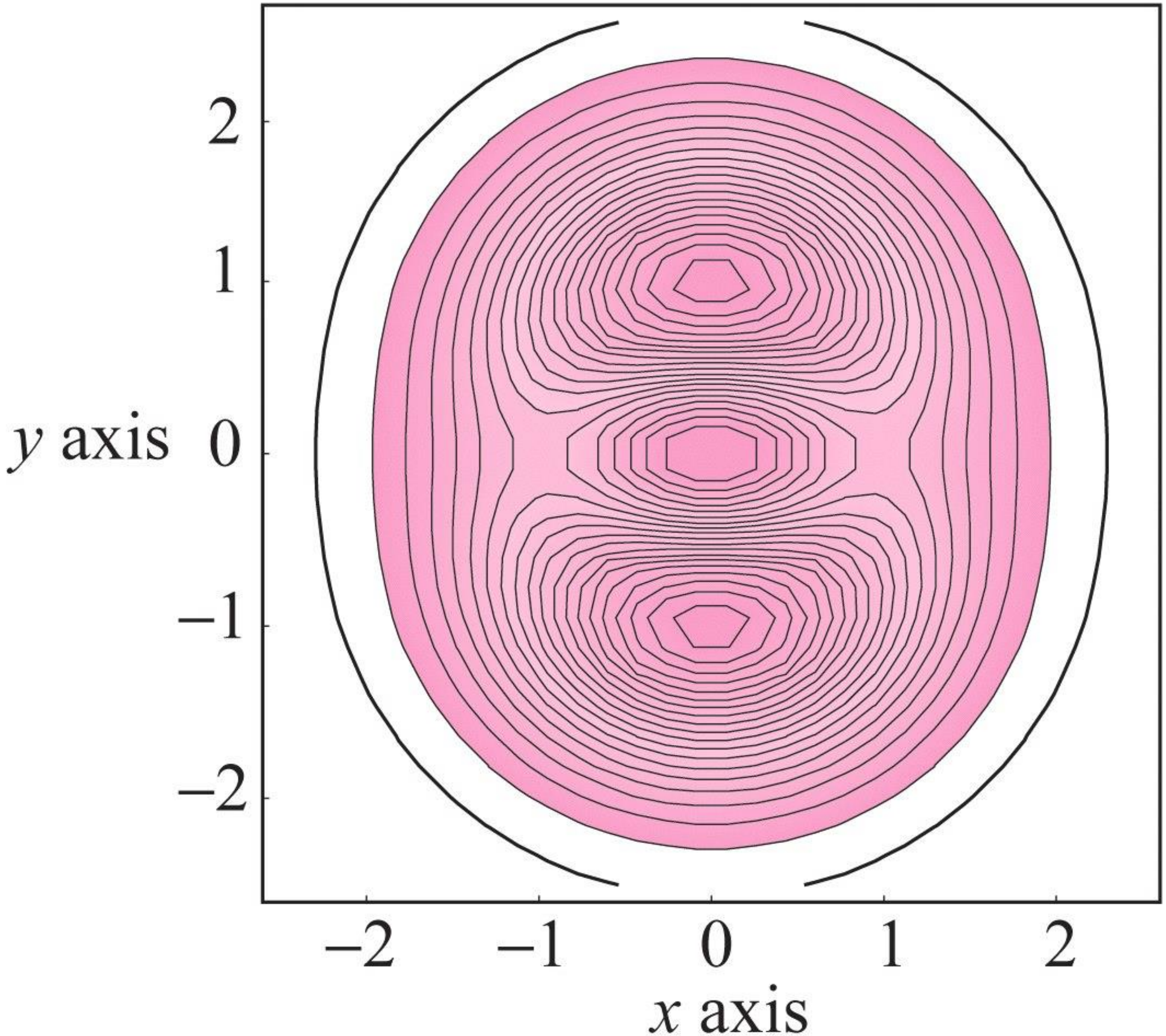
Computer-generated graph of $z = (x^2 + 3y^2) \exp (1 - x^2 - y^2)$ represented in three ways: (a) by sections, (b) by level curves on a graph, and (c) by level curves in the xy plane.





3
2
1
0
−2
−1
0
1
2
y axis
−2
−1
0
1
2
x axis


3
2
1
0
−2
−1
0
1
2
y axis
−2
−1
0
1
2
x axis


y axis
x axis
2
1
0
−1
−2
−2
−1
0
1
2