Special relativity - Dynamics

Last updated: October 8, 2022

Special relativity posits that: (i) the laws of physics are the same in all inertial reference frames; (ii) the speed of light is the same in all reference frames. So far, we have been concerned with the implications of these postulates for how we measure space and time. Our goal, now, is to extend Newton's laws of mechanics to the (moderately weird) world of special relativity.

Proper time and invariant intervals

Proper time

Consider a rotation in ordinary 3-dimensional space, that takes us from a coordinate system S to a rotated coordinate system S'. The coordinates assigned to identify the same point are different in both coordinate systems, (x, y, z) and (x', y', z'), respectively (Fig. 1). However, if we take any two points in space and calculate the distance (or the square of distance) between them $d^2 = \Delta x^2 + \Delta y^2 + \Delta z^2$, this distance will be the same in any coordinate system, namely,

$$d^2 = \Delta x^2 + \Delta y^2 + \Delta z^2 = \Delta x'^2 + \Delta y'^2 + \Delta z'^2 = d'^2 \,. \tag{1}$$

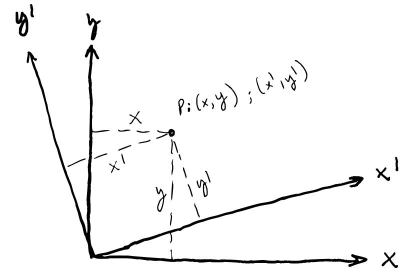

Figure 1: Rotation in the $x - y$ plane. Points are assigned different coordinates in each coordinate system, but distances between points (in particular, the distance between any point and the origin) are the same in any system.

Is there anything analogous to this *distance* in spacetime? That is, some quantity that is invariant under the Lorentz transformation. Since space and time mix in special relativity, and indeed play roles that are much more *similar* than in pre-relativity physics, we may be tempted to propose a *distance* $\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2$. However, we can see that this quantity is not invariant; let's do it ignoring y and z, as usual:[1]

$$\begin{aligned} \Delta t'^2 &= \gamma^2 \left(\Delta t - v\Delta x\right)^2 \\ \Delta x'^2 &= \gamma^2 \left(\Delta x - v\Delta t\right)^2 \\ \Delta t'^2 + \Delta x'^2 &= \gamma^2 \left(\Delta t^2 - 2v\Delta t\Delta x + v^2\Delta x^2 + \Delta x^2 - 2v\Delta t\Delta x + v^2\Delta t^2\right) \end{aligned} \tag{2}$$

and we can see immediately that this cannot be equal to $\Delta t^2 + \Delta x^2$ because the terms with $\Delta x \Delta t$ do *not* cancel out. Indeed, for these terms to cancel out, we would need to have a $-$ sign instead of a $+$, so let's try $\Delta t^2 - \Delta x^2$:[2]

[1] So far, we have been able to choose the origin of our reference frames, and therefore worked always with intervals from the origin to a certain point in spacetime. Thus, we have ignored the Δ and worked always with, e.g., x. Now, when we deal with velocities, accelerations, etc. it is imperative to bring back the increments. Note, however, that, because of its linearity, the Lorentz transformation remains the same, just changing x by Δx, and so forth. Prove it!

[2] There is at least another obvious reason to try this. For a particle moving at the speed of light, we have that $\Delta t^2 - \Delta x^2 = 0$ from any reference frame, so this looks promising!

$$\begin{aligned}
\Delta t'^2 - \Delta x'^2 &= \gamma^2 \left(\Delta t^2 - 2v\Delta t \Delta x + v^2 \Delta x^2 \right) - \\
&\quad \gamma^2 \left(\Delta x^2 - 2v\Delta t \Delta x + v^2 \Delta t^2 \right) \\
&= \gamma^2 \left(\Delta t^2 - v^2 \Delta t^2 \right) - \gamma^2 \left(\Delta x^2 - v^2 \Delta x^2 \right) \\
&= \Delta t^2 - \Delta x^2 \, . \qquad (3)
\end{aligned}$$

Alright! The quantity $\Delta\tau^2 = \Delta t^2 - \Delta \vec{x}^2$ *is* an invariant—its value is the same in any inertial reference frame.[3] The invariant τ is called the **proper time** and it plays a key role in formulating relativistic laws of motion that extend their Newtonian counterparts. But before we go there, let's see why we call it proper time—this is illuminating in itself.

[3] τ is also invariant to spatial rotations, because spatial rotations leave $\Delta\vec{x}^2$ and Δt^2 invariant separately.

Imagine a particle that travels between two points, so that an observer at rest with S measures a displacement ΔX (Fig. 2). The proper time for this motion as measured by the observer is $\Delta\tau^2 = \Delta t^2 - \Delta\vec{x}^2$. Now, from the reference frame S' that moves with the particle,[4] we have that $\Delta\vec{x'}^2 = 0$ (because the particle is at rest in S'), so that $\Delta\tau^2 = \Delta t'^2$. Therefore, proper time is simply the time measured by a clock that travels with the particle. Indeed, the Δt measured by different observers will always be different and there is no reason to prefer one over the other—thus, you can see why proper time, the only *privileged* time in the system, must play an important role in relativistic mechanics.

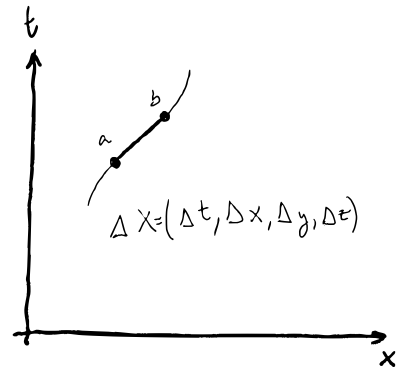

Figure 2: World-line of a moving particle.

[4] If the points are close enough, we can assume that the trajectory between them is rectilinear and at constant velocity, so that S' is, at least instantaneously, an inertial reference frame.

For completeness note that, reintroducing c, proper time is usually defined as

$$\Delta\tau^2 = \Delta t^2 - \frac{\Delta\vec{x}^2}{c^2} \qquad (4)$$

so that it has units of time.

Invariant spacetime interval

If $\Delta\tau^2 = \Delta t^2 - \Delta\vec{x}^2$ is invariant, it is trivial that $\Delta s^2 = \Delta\vec{x}^2 - \Delta t^2$ is also invariant. The quantity Δs is called the **spacetime interval**. But—why bother defining the spacetime interval when we have defined proper time? Why do we need an alter ego for proper time? The answer is that proper time is not always well defined because $\Delta\tau^2$ can be positive, but also negative or zero. To describe each of these situations, we choose to do it in terms of the spacetime interval Δs^2.

Timelike intervals are those for which $\Delta s^2 < 0$. For two events separated by a timelike interval, the space separation between them is smaller than the time separation, that is, than $c\Delta t$. Therefore, two points along the trajectory of a particle with $m > 0$ must be separated by timelike intervals; otherwise, the particle would have to move with velocity larger than c. It is for timelike intervals that it makes sense to define proper time as we've seen it in the previous subsection, that is, as the time measured in the reference frame in which the two events happen in the same place. Indeed, one can prove that, for events separated by a timelike interval, there is always a frame in which the events happen in the same place, and that the time measured in that frame (the proper time) is lower than in any other frame. The temporal order of events (causality) is always preserved in timelike intervals.[5]

[5] EXERCISE: Prove all the statements in the last two sentences.

SPACELIKE INTERVALS are those for which $\Delta s^2 > 0$. For two events separated by a spacelike interval, the space separation between them is larger than the time separation, that is, than $c\Delta t$. Because we know that nothing moves faster than c, events separated by spacelike intervals cannot be causally connected. Then, are these intervals physically meaningful? Sure! Consider, for example, the measurement of the length of an object, which consists in recording, simultaneously in a given reference frame, the spatial coordinates of both the head an the tail of the object. These two events (the determination of the location of the head of the object and the determination of the location of the tail of the object, both at some time $t = 0$) are separated by a spacelike interval. By analogy to proper time, we sometimes refer to the *proper length* of an object as the space interval measured in the reference frame in which the object is at rest.

FINALLY, LIGHTLIKE INTERVALS are those for which $\Delta s^2 = 0$. Two points in the trajectory of a light ray (more precisely, a photon), are separated by a lightlike interval.[6]

[6] To go back to the question of why we need to define spacetime intervals, note that proper time can only be defined for timelike intervals since a lightlike or spacelike world-line aren't possible for a material clock.

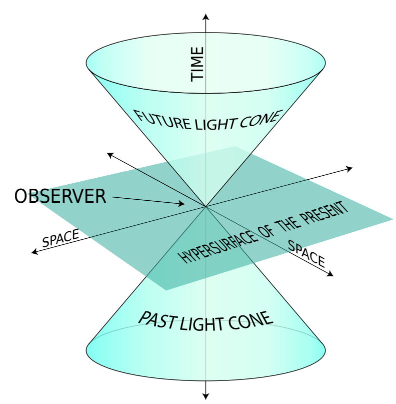

Figure 3: Minkowski light cone. The surface of the cone is defined by all possible lightlike intervals. Points inside the cone (past and future) are separated from the origin by timelike intervals.

Based on the separation between any event and a reference point (say, the origin), spacetime can be divided into the regions depicted in Fig. 3.

Minkowski spaces

We now formalize the mathematical properties of spacetime, which is characterized by the fact that "distances" are measured in terms of the spacetime interval, which, with all the c's, is $\Delta s^2 = \Delta \vec{x}^2 - c^2 \Delta t^2$.

Lorentz scalars

A Lorentz scalar is a quantity that remains invariant under both spatial rotations and Lorentz boosts. Mass or electric charge are

Lorentz scalars. As we have seen above, so is proper time. Even more generally, affine parameters, which exist independent of any metric at all, are scalars. As a trivial example, if τ is a particular object's proper time, then τ is a valid affine parameter, but so are 3 and $2\tau + 7$.

4-vectors

By definition, a 4-vector A is a 4-tuple whose elements A^μ that transform according to the Lorentz transformation (Lorentz boosts and spatial rotations).

So far, we have seen one particular type of 4-vectors,[7] namely, the spacetime displacement $\Delta X = (\Delta t, \Delta x, \Delta y, \Delta z)$. This may lead to the false impression that any 4-tuple is a 4-vector—that is not the case. Let's see what are valid 4-vectors and what not.

[7] Our basic 4-vector, if you wish.

1. Given a 4-vector, multiplying it by a Lorentz scalar results in another 4-vector. For example, $A = m\Delta X = (m\Delta t, m\Delta x, m\Delta y, m\Delta z)$ is a 4-vector. Indeed, for a boost along the x axis we have

$$A'^0 = \gamma\left(A^0 - vA^1\right) = \gamma\left(m\Delta t - vm\Delta x\right) = m\Delta t' \tag{5}$$

$$A'^1 = \gamma\left(A^1 - vA^0\right) = \gamma\left(m\Delta x - vm\Delta t\right) = m\Delta x' \tag{6}$$

   Spatial rotations also trivially leave $m\Delta X$ invariant. Therefore, $A = m\Delta X$ is a 4-vector because it transforms as $A' = m\Delta X'$.

2. The addition of 4-vectors or, more generally, any linear combination of 4-vectors (with scalar coefficients) is also a 4-vector.[8] This, again, is a consequence of the linearity of the Lorentz transformation.

[8] Exercise: Prove it.

3. The gradient of a Lorentz scalar is a 4-vector.

4. However, *not all operations* with a 4-vector result in 4-vectors. For example, given the spacetime displacement $(\Delta t, \Delta x, \Delta y, \Delta z)$, one can see that $A = (\Delta t, 2\Delta x, \Delta y, \Delta z)$ is *not* a 4-vector, because $A' \neq (\Delta t', 2\Delta x', \Delta y', \Delta z')$. For example,

$$A'^0 = \gamma\left(A^0 - vA^1\right) = \gamma\left(\Delta t - 2v\Delta x\right) \neq \Delta t' = \gamma\left(\Delta t - v\Delta x\right)\ . \tag{7}$$

Minkowski's metric

We now tie together some of the concepts that we have discussed so far (invariant interval and proper time, Lorentz scalars, and 4-vectors) by introducing Minkowski's metric, which allows us to define an inner product in Minkowski spaces that is analogous to the scalar product in Euclidean geometry.

We have seen that, for any 4-vector A, the spacetime interval $-(A^0)^2 + (A^1)^2 + (A^2)^2 + (A^3)^2$ is invariant under the Lorentz transformation. Now, we can introduce some notation that will help us to think more clearly about these objects. In particular, we define the *Minkowski metric* η as the following matrix

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{8}$$

Similarly, let's write the 4-vector as a column matrix

$$A^\mu = \begin{pmatrix} A^0 \\ A^1 \\ A^2 \\ A^3 \end{pmatrix} .$$

With this, the spacetime interval can be written as

$$A^2 = \sum_{\mu\nu} \eta_{\mu\nu} A^\mu A^\nu \equiv \eta_{\mu\nu} A^\mu A^\nu \tag{9}$$

where we have introduced *Einstein's sum convention*, which prescribes that, every time we encounter a repeated index, once as a superindex and once as a subindex, we have to sum over that index. If we go step by step, and first add over one of the indices (or *contract*) we have a new object that we are going to call A_μ[9]

$$A_\mu = \eta_{\mu\nu} A^\nu = \begin{pmatrix} -A^0 \\ A^1 \\ A^2 \\ A^3 \end{pmatrix} . \tag{10}$$

[9] It might be better to express A_μ as a row matrix. However, the summation convention minimizes the need to worry about thinking of contractions as matrix products.

With this, again, the spacetime interval is

$$A^2 = A_\mu A^\mu . \tag{11}$$

Note that the position of the indices is important. We call "upstairs" indices *contravariant* and "downstairs" indices *covariant*. A^μ is a 1-contravariant *tensor*, whereas A_μ is 1-covariant, and the metric $\eta_{\mu\nu}$ is 2-covariant.

It turns out that, just like $A_\mu A^\mu$ is invariant, $A_\mu B^\mu$ is also invariant for any two 4-vectors A and B,[10] just like in ordinary space the dot product between any two vectors is invariant. Indeed, you can think of $A^\mu B_\mu$ as the Minkowski-space version of the dot product. The only real difference is the change of sign for the time component.

[10] EXERCISE: Prove it using the Lorentz transformation for each of the components of A and B.

Although in this class we will not get to use the concept of covariant and contravariant tensors in all its depth, this formalism should be useful in the Electromagnetism class. Indeed, Maxwell's equations take a much simpler form when written using 4-vectors. This formalism will play a key role in the differential geometry class.

Relativistic laws of motion

We are finally ready to extend Newton's laws of mechanics to relativistic spacetime. As in Newtonian mechanics, we are interested in describing the trajectory of a particle from the inertial frame of an observer.

4-velocity

The velocity of a particle in Newtonian mechanics tells us about changes of position along time, which is the same in any inertial reference frame. 4-velocities in special relativity should, analogously, inform about changes ΔX^μ in spacetime (from the perspective of an observer) position along some time. But since time is not universal anymore, which time interval should we choose?

We could choose the time interval Δt in the observer's frame but, since Δt is not invariant, $\Delta X/\Delta t$ is not a 4-vector (i.e., it does not transform according to the Lorentz transformation).

Instead, we can use the proper time interval $\Delta\tau$, which has two very desirable properties: (i) it is a scalar (i.e. it is invariant), so that $\Delta X/\Delta\tau$ is a true 4-vector; (ii) it is the time measured by a clock that moves with the particle (which is the only *privileged* reference frame when studying the particle). Thus, we define the 4-velocity $V = (V^0, V^1, V^2, V^3)$ as the ratio[11]

$$V^\mu = \frac{dX^\mu}{d\tau} \tag{12}$$

[11] We now take infinitely small intervals $\Delta X^\mu \to dX^\mu$.

To see the relationship between this 4-velocity V and the ordinary velocity[12] $\vec{v} = d\vec{X}/dt$, we note that the proper time interval is $d\tau$ in the particle frame and $\sqrt{dt^2 - d\vec{X}^2}$ in the observer's frame; because the proper time interval is invariant, we have that[13]

$$d\tau = \sqrt{dt^2 - d\vec{X}^2} = \frac{dt}{\gamma} \ . \tag{13}$$

[12] We use V for the 4-velocity and reserve lowercase $\vec{v}$ (or v^i) for the standard Newtonian/spatial velocity. As in previous lectures, we reserve Greek letters for spacetime indices, and Latin letters for spatial indices.

[13] Yes, of course—the good old time dilation formula.

Therefore, we can write the 4-velocity in terms of the ordinary velocity as

$$V = \left(\frac{dt}{d\tau}, \frac{d\vec{X}}{d\tau}\right) = (\gamma, \gamma\vec{v}) \tag{14}$$

With the c's needed for units to be right we have

$$V = (\gamma c, \gamma\vec{v}) \ . \tag{15}$$

All in all, the temporal component of the 4-velocity gives, essentially, the Jacobian of the transformation between proper time in the frame of the moving particle and the time of the observer (that is, γ). In other words, this ubiquitous term in relativity appears because of the need to describe movement in terms of the only privileged time, the time that is measured by the moving particle itself. The spatial part gives the ordinary velocity from the observer's reference frame, scaled by γ.

There is one more interesting and useful fact about the 4-velocity. As for any other 4-vector, the magnitude $V^2 = V^\mu V_\mu$ of the 4-velocity in Minkowski space is invariant. What is this magnitude? It is easy to see[14] that

$$\boxed{V^\mu V_\mu = -V^{t2} + V^{x2} + V^{y2} + V^{z2} = -c^2} \,. \tag{16}$$

[14] EXERCISE: Prove it.

Energy-momentum 4-vector

The natural way to extend ordinary spatial momentum $\vec{p} = m\vec{v}$ (which we need to reformulate Newton's laws) to spacetime 4-momentum P is by doing $P = mV$, where V is the 4-velocity.[15] Note that this is a valid 4-vector because V is a 4-vector and m, the mass of the particle, is a scalar independent of the reference frame.

[15] Later in the course, when we learn about the Lagrangian and Hamiltonian formalisms, we will give more elegant arguments for why momentum, energy, etc should be defined in the way we introduce here.

The components of the 4-momentum are

$$P = (\gamma mc, \gamma m\vec{v}) \,. \tag{17}$$

Again, the spatial components correspond to the good old linear momentum, simply scaled by the factor γ that is needed to measure variations with respect to the proper time, rather than the observer's time.

Despite its harmless appearance, the temporal component $P^0 = \gamma mc$ of the 4-momentum turns out to be extremely important in special relativity. To interpret its meaning, let's see to what it corresponds in the limit of small, non-relativistic velocities[16]

$$P^0 = \frac{mc}{\sqrt{1 - v^2/c^2}} \approx mc + \frac{mv^2}{2c} \,. \tag{18}$$

[16] Here and in other places, we will often use the approximation $(1+x)^n = 1 + nx + O(x^2)$, which holds for small x. Prove it.

The second term is the non-relativistic kinetic energy except for a factor c. Indeed, $E = cP^0$ has dimensions of energy and is given by

$$E = cP^0 = \gamma mc^2 \approx mc^2 + \frac{mv^2}{2} \,. \tag{19}$$

Therefore, at low velocities, this *relativistic energy* of the particle corresponds to the Newtonian kinetic energy plus a term mc^2 (looks familiar?) that is an energy that results purely from having mass.

It is just natural to associate E with the energy of the particle, and $T = E - mc^2$ is the relativistic kinetic energy of the particle.[17] Then, the 4-momentum is[18]

$$P = (E/c, \gamma m\vec{v}) = (E/c, \vec{P}) \ , \tag{20}$$

and, for obvious reasons, it is sometimes referred to as the energy-momentum 4-vector.

[17] See footnote 15.

[18] As before, we reserve the capital letter for the spatial part of the 4-vector, $\vec{P} = \gamma m\vec{v}$

The fact that energy and linear momentum are, respectively, the temporal and spatial parts of a 4-vector has the important implication that the energy and momentum get mixed up under a Lorentz transformation. For example, an object at rest in one frame has energy but no momentum. In another frame, the same object has both energy and momentum. Indeed, given that P is a 4-vector, we know immediately that it transforms, under a Lorentz boost in the x direction, as any other 4-vector, that is (ignoring c)

$$\begin{aligned} E' &= \gamma\left(E - vP_x\right) \\ P_x' &= \gamma\left(P_x - vE\right) \ . \end{aligned} \tag{21}$$

Additionally, as for any other 4-vector, the invariant interval is preserved, which means that we have a new conserved quantity that takes the exact same value across reference frames, namely

$$\frac{E^2}{c^2} - \vec{P}^2 = \frac{E'^2}{c^2} - \vec{P}'^2 \ . \tag{22}$$

Indeed, we can do some algebra to get this invariant quantity

$$E^2 - \vec{P}^2c^2 = \gamma^2 m^2 c^4 - \gamma^2 m^2 v^2 c^2 = m^2c^4 \ , \tag{23}$$

or

$$\boxed{E^2 = m^2c^4 + \vec{P}^2c^2} \tag{24}$$

which is the relationship between mass, momentum, and energy.

Collisions and decays

Because a linear combination of 4-vectors is also a 4-vector, everything we have said about the energy and momentum of a particle (transformation, invariant interval, etc.) applies equally to the energy and momentum of a *system* with several particles. The total 4-momentum of a system is just $P = \sum_a P_a$, where P_a is the 4-momentum of particle a (not to be confused with a covariant component of the momentum).

Now consider collisions (or decays), in which one or more incoming particles with total 4-momentum P_{before} give rise to some potentially different particles with total 4-momentum P_{after}. In pre-relativity physics, one deals with such collisions by noting that, since all forces are internal to the system, total linear momentum must be conserved. Mass is also conserved but kinetic energy can be lost or not, in which cases we talk about *inelastic* or *elastic* collisions, respectively.

Relativistic collisions are a bit more constrained in that, imposing conservation of the momentum implies, automatically, the **conservation of total energy**. Let's see why. We have

$$\left(\sum_b \vec{P}_b\right)_{\text{before}} - \left(\sum_a \vec{P}_a\right)_{\text{after}} = 0 \tag{25}$$

and want this conservation law to hold in any inertial frame, so that

$$\left(\sum_b \vec{P'}_b\right)_{\text{before}} - \left(\sum_a \vec{P'}_a\right)_{\text{after}} = 0\ . \tag{26}$$

But using the Lorentz transformation the conservation law in the S' frame can be written as[19]

$$\sum_b P'^x_b - \sum_a P'^x_a = \gamma\left(\sum_b P^x_b - \sum_a P^x_a\right) - \gamma v\left(\sum_b E_b - \sum_a E_a\right)\ , \tag{27}$$

which implies

$$\left(\sum_b E_b\right)_{\text{before}} - \left(\sum_a E_a\right)_{\text{after}} = 0\ . \tag{28}$$

As usual, this results from the linearity of the Lorentz transformation.[20]

All in all, in a collision we have energy and momentum conservation, which can be written more concisely as the conservation of the whole energy-momentum 4-vector:

$$\boxed{P_{\text{before}} = P_{\text{after}}}\ . \tag{29}$$

But—what about elastic and inelastic collisions in this relativistic setting? Since total energy must be conserved, are all collisions elastic? The answer is *no*. For example, when the incoming particles in a collision (or decay) are different from the outgoing particles, then their total mass is, in general, different, and the mass difference needs to be compensated by a change in total kinetic energy so as to conserve total energy. We say that **a relativistic collision is elastic** when **kinetic energy** (or, equivalently, **mass**) is conserved, and inelastic otherwise.

Force and laws of motion

We are finally in a position to generalize Newton's second law to the relativistic case. At this point, it should be intuitive that the most

[19] In an abuse of notation, we use index b for "before" particles and a for "after" particles, and do not specify this anymore. Additionally, as usual, we assume that everything happens in the x direction.

[20] So if you are not convinced about the assumption of everything happening in the x axis when you have multiple particles, you can prove energy conservation using a matrix or tensor formalism for the transformation.

reasonable way to do it is by defining a force 4-vector F as

$$F^\mu = \frac{dP^\mu}{d\tau} \ . \tag{30}$$

The spatial part of this 4-vector is

$$\vec{F} = \frac{d\vec{P}}{d\tau} = \gamma \frac{d\vec{P}}{dt} = \gamma \vec{f} \ , \tag{31}$$

where $\vec{f} = d\vec{P}/dt = d(\gamma m \vec{v})/dt$ is the ordinary Newtonian force. Similarly, the temporal component is

$$F^0 = \frac{dP^0}{d\tau} = \gamma \frac{dP^0}{dt} = \frac{\gamma}{c}\frac{dE}{dt} \ , \tag{32}$$

so that

$$F = \left(\frac{\gamma}{c}\frac{dE}{dt}, \gamma \frac{d\vec{P}}{dt} \right) = \left(\frac{\gamma}{c}\frac{dE}{dt}, \gamma \frac{d(\gamma m \vec{v})}{dt} \right) \ . \tag{33}$$

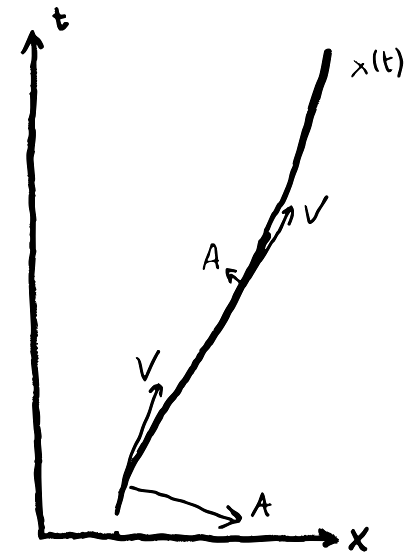

Figure 4: Spacetime diagram showing accelerated motion. The position four-vector gives the collection of points in spacetime that the world-line passes through. The velocity 4-vector is the tangent to that line and always has the same magnitude $V^\mu V_\mu = -1$. Thus, what determines how "fast" the particle moves is not the magnitude of the 4-velocity, but its "slope". Then, the acceleration 4-vector is always perpendicular to the 4-velocity ($A^\mu V_\mu = 0$) because it does not change its magnitude, but its "direction". The acceleration 4-vector is the curvature of the trajectory, so it is small when the trajectory is almost straight (constant velocity) and large when the trajectory changes abruptly in spacetime.

We can of course also define an acceleration 4-vector, by taking the derivative of the four-velocity with respect to the proper time

$$A^\mu = \frac{dV^\mu}{d\tau} = \gamma \frac{d}{dt}\left[\gamma \left(c, \vec{v} \right) \right] \ . \tag{34}$$

As with the forces, we are in for some surprises, this time because the proper time derivative acts on the γ factor in the velocity as well as on the components. So let's look, first, at this derivative

$$\frac{d\gamma}{dt} = \frac{d}{dt}\left(1 - \vec{v}^2/c^2 \right)^{-1/2} = \gamma^3 \frac{\vec{v}\cdot\vec{a}}{c^2} \ , \tag{35}$$

where $\vec{a} = d\vec{v}/dt$. Therefore

$$A^\mu = \left(\gamma^4 \frac{\vec{v}\cdot\vec{a}}{c}, \gamma^2 \vec{a} + \gamma^4 \frac{\vec{v}\cdot\vec{a}}{c^2}\vec{v} \right) \ . \tag{36}$$

This is some ugly beast, but it has an insightful interpretation. Indeed, note that $V^\mu A_\mu = 0$,[21] so that 4-velocity and 4-acceleration are always *perpendicular* in spacetime (Fig. 4).

We can also use this to understand why a particle cannot be accelerated beyond the speed of light. Indeed, consider a Newtonian force $\vec{f}$, which in Newtonian mechanics is proportional to the 3-acceleration $\vec{a}$. By contrast, in the relativistic setting we have[22]

$$f = \frac{d}{dt}(m\gamma v) = \cdots = m\gamma^3 a \ .$$

Therefore, inertia grows as γ^3 and becomes infinite when $v = c$, so at this point the particle cannot be accelerated anymore.

[21] Exercise: Prove it by using $V^\mu V_\mu = -1$, and then verify it by calculating the inner product term by term.

[22] We again assume a one-dimensional setting where 3-force, 3-velocity and 3-acceleration are all aligned in the x direction.