Exercicis integral de superfície

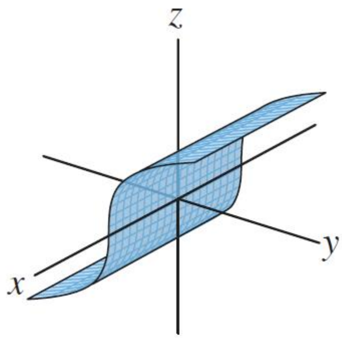
1. Emparelleu parametrització amb superfície:

(a) $(u, \cos v, \sin v)$ **(b)** $(u, u + v, v)$

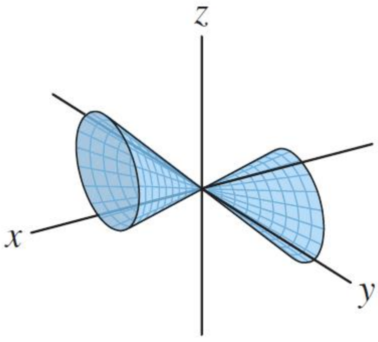
(c) (u, v^3, v)

(d) $(\cos u \sin v, 3\cos u \sin v, \cos v)$

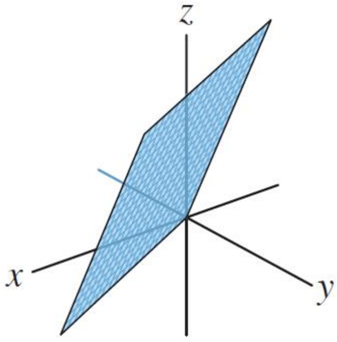
(e) $(u, u(2 + \cos v), u(2 + \sin v))$



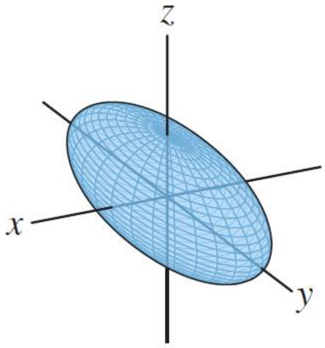
(i)



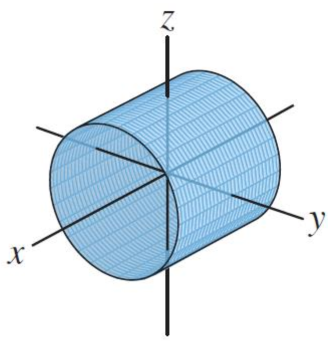
(ii)



(iii)



(iv)



(v)

Solució

SOLUTION (a) = (v), because the y and z coordinates describe a circle with fixed radius.
(b) = (iii), because the coordinates are all linear in u and v.
(c) = (i), because the parametrization gives $y = z^3$.
(d) = (iv), an ellipsoid.
(e) = (ii), because the y and z coordinates describe a circle with varying radius.

2. Siguin les següents parametritzacions de superfícies:

$$\mathbf{X}(s,t) = (s\cos t, s\sin t, 3s^2), \qquad 0 \le s \le 2, 0 \le t \le 2\pi.$$

$$\mathbf{Y}(s,t) = (2s\cos t, 2s\sin t, 12s^2), \qquad 0 \le s \le 1, 0 \le t \le 4\pi.$$

a) Demostra que les imatges de $\boldsymbol{X}$ i $\boldsymbol{Y}$ són iguals. [Pista: troba l'equació de la superfície en funció de x, y, z].

b) Calcula la integral de superfície del camp $\boldsymbol{F} = y\boldsymbol{i} - x\boldsymbol{j} + z^2\boldsymbol{k}$ per a les dues parametritzacions. Reconcilia els resultats.

Solució

(a) You can easily verify that both $\mathbf{X}$ and $\mathbf{Y}$ parametrize the surface $z = 3x^2 + 3y^2$ for $0 \le x^2 + y^2 \le 4$. The major difference is that $\mathbf{X}$ covers the surface once while $\mathbf{Y}$ covers the surface twice.

(b) For $\mathbf{X}$, the standard normal $\mathbf{N}$ is

$$(\cos t, \sin t, 6s) \times (-s\sin t, s\cos t, 0) = (-6s^2\cos t, -6s^2\sin t, s)$$

so

$$\iint_{\mathbf{X}} (y\mathbf{i} - x\mathbf{j} + z^2\mathbf{k}) \cdot d\mathbf{S} = \int_0^{2\pi}\int_0^2 (s\sin t, -s\cos t, 9s^4) \cdot (-6s^2\cos t, -6s^2\sin t, s)\, ds\, dt$$
$$= \int_0^{2\pi}\int_0^2 9s^5\, ds\, dt = \int_0^{2\pi} \left.\frac{9s^6}{6}\right|_0^2 dt = \int_0^{2\pi} 96\, dt = 192\pi.$$

For $\mathbf{Y}$, the standard normal $\mathbf{N}$ is

$$(2\cos t, 2\sin t, 24s) \times (-2s\sin t, 2s\cos t, 0) = (-48s^2\cos t, -48s^2\sin t, 4s)$$

so

$$\iint_{\mathbf{Y}} (y\mathbf{i} - x\mathbf{j} + z^2\mathbf{k}) \cdot d\mathbf{S} = \int_0^{4\pi}\int_0^1 (2s\sin t, -2s\cos t, 144s^4) \cdot (-48s^2\cos t, -48s^2\sin t, 4s)\, ds\, dt$$
$$= \int_0^{4\pi}\int_0^1 576s^5\, ds\, dt = \int_0^{4\pi} \left.\frac{576s^6}{6}\right|_0^1 dt = \int_0^{4\pi} 96\, dt = 384\pi.$$

As noted in part (a), the integral over $\mathbf{Y}$ should be twice the integral over $\mathbf{X}$ since they both parametrize the same space but $\mathbf{Y}$ covers the space twice.

3. Sigui $\boldsymbol{\phi}(x,y) = (x,y,xy)$.

   a) Calcula $\boldsymbol{T}_x, \boldsymbol{T}_y$ i $\boldsymbol{n}(x,y)$.
   b) Sigui S la part de la superfície amb domini de paràmetres $D = \{(x,y): x^2 + y^2 \leq 1, x \geq 0, y \geq 0\}$. Verifica la següent fórmula i avalua-la utilitzant coordenades polar:

$$\iint_S 1\, dS = \iint_{\mathcal{D}} \sqrt{1 + x^2 + y^2}\, dx\, dy$$

   c) Verifica la següent fórmula i avalua-la:

$$\iint_S z\, dS = \int_0^{\pi/2} \int_0^1 (\sin\theta \cos\theta) r^3 \sqrt{1 + r^2}\, dr\, d\theta$$

Solució

(a) The tangent vectors are:

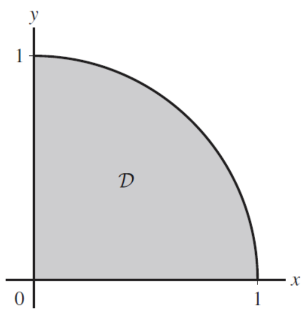
$$\mathbf{T}_x = \frac{\partial \phi}{\partial x} = \frac{\partial}{\partial x}(x, y, xy) = \langle 1, 0, y \rangle$$
$$\mathbf{T}_y = \frac{\partial \phi}{\partial y} = \frac{\partial}{\partial y}(x, y, xy) = \langle 0, 1, x \rangle$$

The normal vector is the cross product:

$$\mathbf{N}(x, y) = \mathbf{T}_x \times \mathbf{T}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & y \\ 0 & 1 & x \end{vmatrix} = \begin{vmatrix} 0 & y \\ 1 & x \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & y \\ 0 & x \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \mathbf{k}$$
$$= -y\mathbf{i} - x\mathbf{j} + \mathbf{k} = \langle -y, -x, 1 \rangle$$

(b) Using the Theorem on evaluating surface integrals we have:

$$\iint_S 1\, dS = \iint_{\mathcal{D}} \|\mathbf{N}(x, y)\|\, dx\, dy = \iint_{\mathcal{D}} \|\langle -y, -x, 1 \rangle\|\, dx\, dy = \iint_{\mathcal{D}} \sqrt{y^2 + x^2 + 1}\, dx\, dy$$



We convert the integral to polar coordinates $x = r\cos\theta$, $y = r\sin\theta$. The new region of integration is:

$$0 \le r \le 1, \quad 0 \le \theta \le \frac{\pi}{2}.$$

We get:

$$\iint_S 1\, dS = \int_0^{\pi/2}\int_0^1 \sqrt{r^2+1}\cdot r\, dr\, d\theta = \int_0^{\pi/2}\left(\int_0^1 \sqrt{r^2+1}\cdot r\, dr\right) d\theta$$

$$= \int_0^{\pi/2}\left(\int_1^2 \frac{\sqrt{u}}{2}\, du\right) d\theta = \int_0^{\pi/2} \frac{2\sqrt{2}-1}{3}\, d\theta = \frac{\left(2\sqrt{2}-1\right)\pi}{6}$$

(c) The function z expressed in terms of the parameters x, y is $f(\Phi(x, y)) = xy$. Therefore,

$$\iint_S z\, dS = \iint_{\mathcal{D}} xy \cdot \|\mathbf{N}(x, y)\|\, dx\, dy = \iint_{\mathcal{D}} xy\sqrt{1+x^2+y^2}\, dx\, dy$$

We compute the double integral by converting it to polar coordinates. We get:

$$\iint_S z\, dS = \int_0^{\pi/2}\int_0^1 (r\cos\theta)(r\sin\theta)\sqrt{1+r^2}\cdot r\, dr\, d\theta = \int_0^{\pi/2}\int_0^1 (\sin\theta\cos\theta) r^3\sqrt{1+r^2}\, dr\, d\theta$$

$$= \left(\int_0^{\pi/2} (\sin\theta\cos\theta)\, d\theta\right)\left(\int_0^1 r^3\sqrt{1+r^2}\, dr\right) \tag{1}$$

We compute each integral in (1). Using the substitution $u = 1 + r^2$, $du = 2r\, dr$ we get:

$$\int_0^1 r^3\sqrt{1+r^2}\, dr = \int_0^1 r^2\sqrt{1+r^2}\cdot r\, dr = \int_1^2 \left(u^{3/2} - u^{1/2}\right)\frac{du}{2} = \frac{u^{5/2}}{5} - \frac{u^{3/2}}{3}\Bigg|_1^2 = \frac{2\left(\sqrt{2}+1\right)}{15}$$

Also,

$$\int_0^{\pi/2} \sin\theta\cos\theta\, d\theta = \int_0^{\pi/2} \frac{\sin 2\theta}{2}\, d\theta = -\frac{\cos 2\theta}{4}\Bigg|_0^{\pi/2} = \frac{1}{2}$$

We substitute the integrals in (1) to obtain the following solution:

$$\iint_S z\, dS = \frac{1}{2}\cdot\frac{2\left(\sqrt{2}+1\right)}{15} = \frac{\sqrt{2}+1}{15}$$

4. Calcula $\boldsymbol{T}_u$, $\boldsymbol{T}_v$ i $\boldsymbol{n}(u,v)$ per a les superfícies parametritzades següents, i calcula el pla tangent en el punt indicat:

a)

$$\Phi(u, v) = (2u + v, u - 4v, 3u); \qquad u = 1, \quad v = 4$$

b)

$$\Phi(\theta, \phi) = (\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi); \qquad \theta = \frac{\pi}{2}, \quad \phi = \frac{\pi}{4}$$

Solució

a)

SOLUTION The tangent vectors are the following vectors,

$$\mathbf{T}_u = \frac{\partial \Phi}{\partial u} = \frac{\partial}{\partial u}(2u + v, u - 4v, 3u) = \langle 2, 1, 3 \rangle$$
$$\mathbf{T}_v = \frac{\partial \Phi}{\partial v} = \frac{\partial}{\partial v}(2u + v, u - 4v, 3u) = \langle 1, -4, 0 \rangle$$

The normal is the cross product:

$$\mathbf{N}(u, v) = \mathbf{T}_u \times \mathbf{T}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 3 \\ 1 & -4 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ -4 & 0 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & 3 \\ 1 & 0 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & 1 \\ 1 & -4 \end{vmatrix} \mathbf{k}$$
$$= 12\mathbf{i} + 3\mathbf{j} - 9\mathbf{k} = 3\langle 4, 1, -3 \rangle$$

The equation of the plane passing through the point $P : \Phi(1, 4) = (6, -15, 3)$ with the normal vector $\langle 4, 1, -3 \rangle$ is:

$$\langle x - 6, y + 15, z - 3 \rangle \cdot \langle 4, 1, -3 \rangle = 0$$

or

$$4(x - 6) + y + 15 - 3(z - 3) = 0$$
$$4x + y - 3z = 0$$

b)

SOLUTION We compute the tangent vectors:

$$\mathbf{T}_\theta = \frac{\partial \Phi}{\partial \theta} = \frac{\partial}{\partial \theta}(\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi) = \langle -\sin\theta \sin\phi, \cos\theta \sin\phi, 0\rangle$$
$$\mathbf{T}_\phi = \frac{\partial \Phi}{\partial \phi} = \frac{\partial}{\partial \phi}(\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi) = \langle \cos\theta \cos\phi, \sin\theta \cos\phi, -\sin\phi\rangle$$

The normal vector is the cross product:

$$\mathbf{N}(\theta, \phi) = \mathbf{T}_\theta \times \mathbf{T}_\phi = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin\theta \sin\phi & \cos\theta \sin\phi & 0 \\ \cos\theta \cos\phi & \sin\theta \cos\phi & -\sin\phi \end{vmatrix}$$
$$= \left(-\cos\theta\ \sin^2\phi\right)\mathbf{i} - \left(\sin\theta\ \sin^2\phi\right)\mathbf{j} + \left(-\sin^2\theta \sin\phi \cos\phi - \cos^2\theta \cos\phi \sin\phi\right)\mathbf{k}$$
$$= -\left(\cos\theta\ \sin^2\phi\right)\mathbf{i} - \left(\sin\theta\ \sin^2\phi\right)\mathbf{j} - (\sin\phi \cos\phi)\mathbf{k}$$

The tangency point and the normal at this point are,

$$P = \Phi\left(\frac{\pi}{2}, \frac{\pi}{4}\right) = \left(\cos\frac{\pi}{2} \sin\frac{\pi}{4}, \sin\frac{\pi}{2} \sin\frac{\pi}{4}, \cos\frac{\pi}{4}\right) = \left(0, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$$
$$\mathbf{N}\left(\frac{\pi}{2}, \frac{\pi}{4}\right) = -\frac{1}{2}\mathbf{j} - \frac{1}{2}\mathbf{k} = -\frac{1}{2}(\mathbf{j} + \mathbf{k}) = -\frac{1}{2}\langle 0, 1, 1\rangle$$

The equation of the plane orthogonal to the vector $\langle 0, 1, 1\rangle$ and passing through $P = \left(0, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ is:

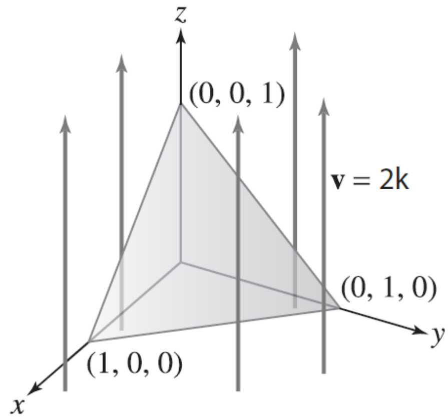
$$\left\langle x, y - \frac{\sqrt{2}}{2}, z - \frac{\sqrt{2}}{2}\right\rangle \cdot \langle 0, 1, 1\rangle = 0$$

or

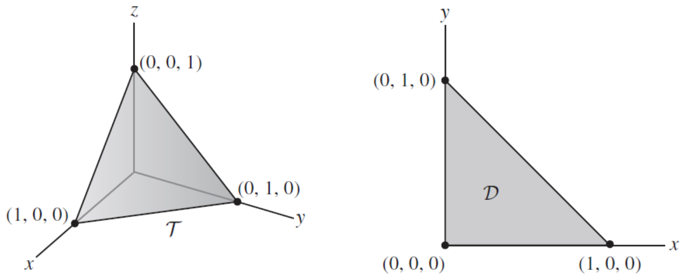
$$y - \frac{\sqrt{2}}{2} + z - \frac{\sqrt{2}}{2} = 0$$
$$y + z = \sqrt{2}$$

5. Un fluid flueix amb un camp de velocitats constant $v = 2\boldsymbol{k}$ (m/s). Calcula:

   a) El flux a través del triangle T.
   b) El flux a través de la projecció del triangle T sobre el pla xy.



Solució



The equation of the plane through the three vertices is $x + y + z = 1$, hence the upward pointing normal vector is:

$$\mathbf{N} = \langle 1, 1, 1 \rangle$$

and the unit normal is:

$$\mathbf{e_n} = \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle$$

We compute the dot product $\mathbf{v} \cdot \mathbf{e_n}$:

$$\mathbf{v} \cdot \mathbf{e_n} = \langle 0, 0, 2 \rangle \cdot \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle = \frac{2}{\sqrt{3}}$$

The flow rate through $\mathcal{T}$ is equal to the flux of $\mathbf{v}$ through $\mathcal{T}$. That is,

$$\iint_S \mathbf{v} \cdot d\mathbf{S} = \iint_S (\mathbf{v} \cdot \mathbf{e_n})\, dS = \iint_S \frac{2}{\sqrt{3}}\, dS = \frac{2}{\sqrt{3}} \iint_S 1\, dS = \frac{2}{\sqrt{3}} \cdot \text{Area}(S)$$

The area of the equilateral triangle $\mathcal{T}$ is $\frac{(\sqrt{2})^2 \cdot \sqrt{3}}{4} = \frac{\sqrt{3}}{2}$. Therefore,

$$\iint_S \mathbf{v} \cdot d\mathbf{S} = \frac{2}{\sqrt{3}} \cdot \frac{\sqrt{3}}{2} = 1$$

Let $\mathcal{D}$ denote the projection of $\mathcal{T}$ onto the xy-plane. Then the upward pointing normal is $\mathbf{N} = \langle 0, 0, 1 \rangle$. We compute the dot product $\mathbf{v} \cdot \mathbf{N}$:

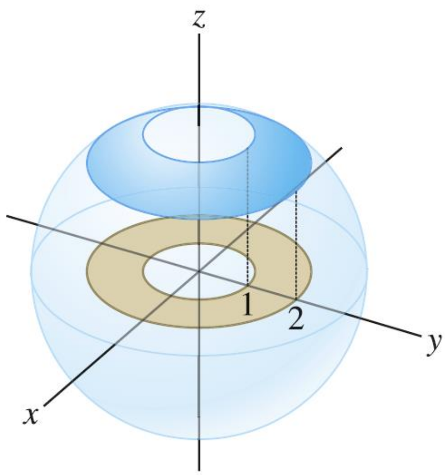
$$\mathbf{v} \cdot \mathbf{N} = \langle 0, 0, 2 \rangle \cdot \langle 0, 0, 1 \rangle = 2$$

The flow rate through $\mathcal{D}$ is equal to the flux of $\mathbf{v}$ through $\mathcal{D}$. That is,

$$\iint_{\mathcal{D}} \mathbf{v} \cdot d\mathbf{S} = \iint_{\mathcal{D}} (\mathbf{v} \cdot \mathbf{N}) \, dS = \iint_{\mathcal{D}} 2 \, dS = 2 \iint_{\mathcal{D}} 1 \, dS = 2 \cdot \text{Area}(\mathcal{D}) = 2 \cdot \frac{1 \cdot 1}{2} = 1$$

6. Sigui S la porció d'una esfera $x^2 + y^2 + z^2 = 9$ amb $1 \le x^2 + y^2 \le 4$ and $z \ge 0$. Troba una parametrització de S en coordenades esfèriques i utilitza-la per a calcular:

a) L'àrea de S.
b) $\int\int_S z^{-1}\,dS$.



Solució

Parametrització

$$\mathbf{\Phi}(\phi,\theta) = (3\sin\phi\cos\theta\,, 3\sin\phi\sin\theta\,, 3\cos\phi),\ \ \phi_1 \le \phi \le \phi_2,\ \ 0 \le \theta < 2\pi$$

amb

$$\sin\phi_1 = \frac{1}{3} \quad \Rightarrow \quad \cos\phi_1 = \frac{\sqrt{8}}{3}$$
$$\sin\phi_2 = \frac{2}{3} \quad \Rightarrow \quad \cos\phi_2 = \frac{\sqrt{5}}{3}$$

Tenim

$$T_\phi = (3\cos\phi\cos\theta\,, 3\cos\phi\sin\theta\,, -3\sin\phi)$$
$$T_\theta = (-3\sin\phi\sin\theta\,, 3\sin\phi\cos\theta\,, 0)$$
$$T_\phi \times T_\theta = 9(\sin^2\phi\cos\theta\,, \sin^2\phi\sin\theta\,, \sin\phi\cos\phi)$$

El resultat final és conegut ($R^2\sin\phi$):

$$\|T_\phi \times T_\theta\| = 9\sin\phi$$

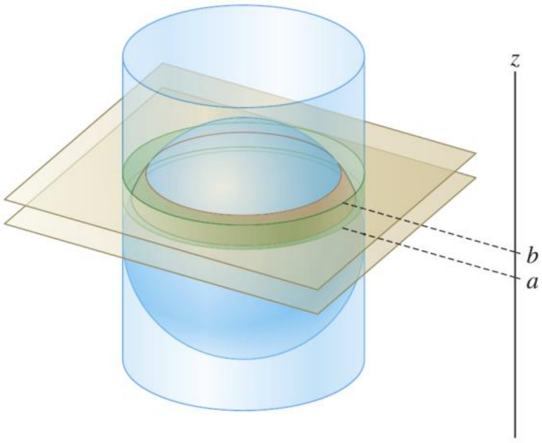
a) L'àrea de S:

$$\int\int_S dS = 9\int_0^{2\pi}\int_{\phi_1}^{\phi_2}\sin\phi\,d\phi\,d\theta = 18\pi(\cos\phi_1 - \cos\phi_2) = 6\pi(\sqrt{8} - \sqrt{5})$$

b)

$$\int\int_S z^{-1}dS = 9\int_0^{2\pi}\int_{\phi_1}^{\phi_2}\frac{\sin\phi}{3\cos\phi}d\phi\,d\theta = 6\pi[-\ln\cos\phi]_{\phi_1}^{\phi_2} = 6\pi\ln\frac{\phi_1}{\phi_2} = 3\pi\ln\frac{8}{5}$$

7. Demostra el famós resultat d'Arquímedes: l'àrea de la porció de superfície d'una esfera de radi R entre dos plans horitzontals $z = a$ i $z = b$ és igual a la corresponent porció de superfície del cilindre circumscrit.



Solució

L'àrea de la porció de superfície del cilindre és senzillament:

$$S_c = 2\pi R(b - a)$$

Podem aprofitar el resultat de l'exercici anterior per a l'àrea de la porció de superfície esfèrica:

$$S_s = 2\pi R^2(\cos\phi_1 - \cos\phi_2)$$

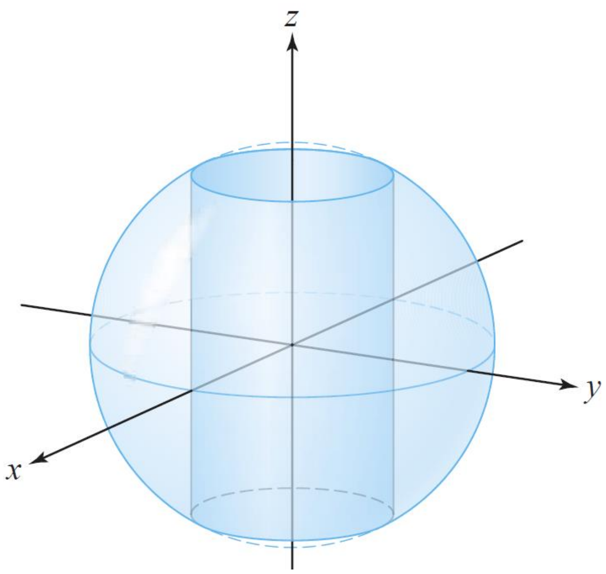
En aquest cas

$$\cos\phi_1 = \frac{b}{R}, \qquad \cos\phi_2 = \frac{a}{R}$$

Per tant:

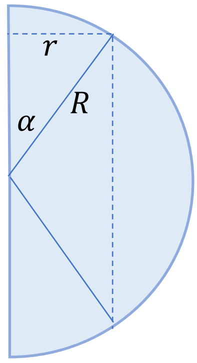
$$S_s = 2\pi R(b - a) = S_c \quad \blacksquare$$

8. Calcula la superfície exterior i el volum d'una esfera de radi R, centrada a l'origen, a la qual se li ha fet un forat cilíndric de radi r i d'eix del cilindre igual a l'eix z.



Solució

Calculem l'àrea com una integral de superfície, parametritzada amb les coordenades esfèriques θ i ϕ.



Tenim

$$\sin\alpha = \frac{r}{R} \quad \Rightarrow \quad \cos\alpha = \frac{1}{R}\sqrt{R^2 - r^2}$$

Ja sabem que

$$\|T_\phi \times T_\theta\| = R^2 \sin\phi$$

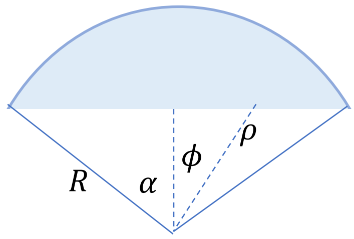
Per tant

$$S = \int\int_S dS = \int_0^{2\pi}\int_\alpha^{\pi-\alpha} R^2 \sin\phi \, d\phi \, d\theta = 2\pi R^2[-\cos\phi]_\alpha^{\pi-\alpha} = 2\pi R^2[\cos\alpha - \cos(\pi - \alpha)]$$
$$= 4\pi R^2 \cos\alpha = 4\pi R\sqrt{R^2 - r^2}$$

Pel volum, ho calculem com el volum de l'esfera menys el volum del cilindre menys el volum dels dos casquets:

$$V = \frac{4}{3}\pi R^3 - 2\pi r^2 R\cos\alpha - 2V_{\text{casquet}}$$

on s'ha utilitzat que l'alçada del cilindre és $2R\cos\alpha = 2\sqrt{R^2 - r^2}$. Pel casquet, el parametritzem utilitzant coordenades esfèriques. La part complicada és expressar la base plana:



$$\cos\phi = \frac{R\cos\alpha}{\rho} \quad \Rightarrow \quad \rho = \frac{R\cos\alpha}{\cos\phi}$$

$$V_{\text{casquet}} = \int_0^{2\pi}\int_0^{\alpha}\int_{\frac{R\cos\alpha}{\cos\phi}}^{R} \rho^2 \sin\phi \, d\rho \;\, d\phi \, d\theta = 2\pi \int_0^{\alpha} \frac{1}{3}\left[R^3 - \left(\frac{R\cos\alpha}{\cos\phi}\right)^3\right] \sin\phi \, d\phi$$

$$= \frac{2}{3}\pi R^3 \int_0^{\alpha} \sin\phi \, d\phi - \frac{2}{3}\pi R^3 \cos^3\alpha \int_0^{\alpha} \frac{\sin\phi}{\cos^3\phi} d\phi$$

$$= \frac{2}{3}\pi R^3 [-\cos\phi]_0^{\alpha} - \frac{2}{3}\pi R^3 \cos^3\alpha \left[\frac{1}{2\cos^2\phi}\right]_0^{\alpha}$$

$$= \frac{2}{3}\pi R^3 (1-\cos\alpha) - \frac{1}{3}\pi R^3 \cos^3\alpha \left(\frac{1}{\cos^2\alpha} - 1\right)$$

$$= \frac{2}{3}\pi R^3 (1-\cos\alpha) - \frac{1}{3}\pi R^3 \cos\alpha\,(1-\cos^2\alpha)$$

$$= \frac{1}{3}\pi R^3 (2+\cos\alpha)(1-\cos\alpha)^2$$

Per tant, el volum total és:

$$V = \frac{4}{3}\pi R^3 - 2\pi r^2 R\cos\alpha - \frac{2}{3}\pi R^3 (2+\cos\alpha)(1-\cos\alpha)^2$$

$$V = \frac{4}{3}\pi R^3 - 2\pi r^2 \sqrt{R^2 - r^2} - \frac{2}{3}\pi \left(2R + \sqrt{R^2 - r^2}\right)\left(R - \sqrt{R^2 - r^2}\right)^2$$