1. Vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$

Definition 1.1. *A vector $\vec{v} \in \mathbb{R}^3$ is a 3-tuple of real numbers (v_1, v_2, v_3).*

Hopefully the reader can well imagine the definition of a vector in $\mathbb{R}^2$.

Example 1.2. $(1,1,0)$ *and* $(\sqrt{2}, \pi, 1/e)$ *are vectors in* $\mathbb{R}^3$.

Definition 1.3. *The **zero vector** in $\mathbb{R}^3$, denoted $\vec{0}$, is the vector $(0,0,0)$. If $\vec{v} = (v_1, v_2, v_3)$ and $\vec{w} = (w_1, w_2, w_3)$ are two vectors in $\mathbb{R}^3$, the **sum of** $\vec{v}$ **and** $\vec{w}$, denoted $\vec{v} + \vec{w}$, is the vector $(v_1 + w_1, v_2 + w_2, v_3 + w_3)$.*

*If $\vec{v} = (v_1, v_2, v_3) \in \mathbb{R}^3$ is a vector and $\lambda \in \mathbb{R}$ is a **scalar**, the **scalar product of** λ **and** v, denoted $\lambda \cdot \vec{v}$, is the vector $(\lambda v_1, \lambda v_2, \lambda v_3)$.*

Example 1.4. *If $\vec{v} = (2, -3, 1)$ and $\vec{w} = (1, -5, 3)$ then $\vec{v} + \vec{w} = (3, -8, 4)$. If $\lambda = -3$ then $\lambda \cdot \vec{v} = (-6, 9, -3)$.*

Lemma 1.5. *If λ and μ are scalars and $\vec{u}$, $\vec{v}$ and $\vec{w}$ are vectors in $\mathbb{R}^3$, then*

(1) $\vec{0} + \vec{v} = \vec{0}$.
(2) $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$.
(3) $\vec{u} + \vec{v} = \vec{v} + \vec{u}$.
(4) $\lambda \cdot (\mu \cdot \vec{v}) = (\lambda\mu) \cdot \vec{v}$.
(5) $(\lambda + \mu) \cdot \vec{v} = \lambda \cdot \vec{v} + \mu \cdot \vec{v}$.
(6) $\lambda \cdot (\vec{u} + \vec{v}) = \lambda \cdot \vec{u} + \lambda \cdot \vec{v}$.

Proof. We check (3). If $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$, then

$$\begin{aligned} \vec{u} + \vec{v} &= (u_1 + v_1, u_2 + v_2, u_3 + v_3) \\ &= (v_1 + u_1, v_2 + u_2, v_3 + u_3) \\ &= \vec{v} + \vec{u}. \end{aligned}$$

□

We can interpret vector addition and scalar multiplication geometrically. We can think of a vector as representing a displacement from the origin. Geometrically a vector $\vec{v}$ has a *magnitude* (or length) $|\vec{v}| = (v_1^2 + v_2^2 + v_3^2)^{1/2}$ and every non-zero vector has a *direction*

$$\vec{u} = \frac{\vec{v}}{|\vec{v}|}.$$

Multiplying by a scalar leaves the direction unchanged and rescales the magnitude. To add two vectors $\vec{v}$ and $\vec{w}$, think of transporting the tail of $\vec{w}$ to the endpoint of $\vec{v}$. The sum of $\vec{v}$ and $\vec{w}$ is the vector whose tail is the tail of the transported vector.

One way to think of this is in terms of directed line segments. Note that given a point P and a vector $\vec{v}$ we can add $\vec{v}$ to P to get another point Q. If $P = (p_1, p_2, p_3)$ and $\vec{v} = (v_1, v_2, v_3)$ then

$$Q = P + \vec{v} = (p_1 + v_1, p_2 + v_2, p_3 + v_3).$$

If $Q = (q_1, q_2, q_3)$, then there is a unique vector $\overrightarrow{PQ}$, such that $Q = P + \vec{v}$, namely

$$\overrightarrow{PQ} = (q_1 - p_1, q_2 - p_2, q_3 - p_3).$$

Lemma 1.6. *Let P, Q and R be three points in $\mathbb{R}^3$.*
Then $\overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR}$.

Proof. Let us consider the result of adding $\overrightarrow{PQ} + \overrightarrow{QR}$ to P,

$$\begin{aligned} P + (\overrightarrow{PQ} + \overrightarrow{QR}) &= (P + \overrightarrow{PQ}) + \overrightarrow{QR} \\ &= Q + \overrightarrow{QR} \\ &= R. \end{aligned}$$

On the other hand, there is at most one vector $\vec{v}$ such that when we add it P we get R, namely the vector $\overrightarrow{PR}$. So $\overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR}$. □

Note that (1.6) expresses the geometrically obvious statement that if one goes from P to Q and then from Q to R, this is the same as going from P to R.

Vectors arise quite naturally in nature. We can use vectors to represent forces; every force has both a magnitude and a direction. The combined effect of two forces is represented by the vector sum. Similarly we can use vectors to measure both velocity and acceleration. The equation

$$\vec{F} = m\vec{a},$$

is the vector form of Newton's famous equation.

Note that $\mathbb{R}^3$ comes with three standard unit vectors

$$\hat{\imath} = (1, 0, 0) \qquad \hat{\jmath} = (0, 1, 0) \qquad \text{and} \qquad \hat{k} = (0, 0, 1),$$

which are called the *standard basis*. Any vector can be written uniquely as a linear combination of these vectors,

$$\vec{v} = (v_1, v_2, v_3) = v_1\hat{\imath} + v_2\hat{\jmath} + v_3\hat{k}.$$

We can use vectors to parametrise lines in $\mathbb{R}^3$. Suppose we are given two different points P and Q of $\mathbb{R}^3$. Then there is a unique line l containing P and Q. Suppose that $R = (x, y, z)$ is a general point of the line. Note that the vector $\overrightarrow{PR}$ is parallel to the vector $\overrightarrow{PQ}$, so that $\overrightarrow{PR}$ is a scalar multiple of $\overrightarrow{PQ}$. Algebraically,

$$\overrightarrow{PR} = t\overrightarrow{PQ},$$

for some scalar $t \in \mathbb{R}$. If $P = (p_1, p_2, p_3)$ and $Q = (q_1, q_2, q_3)$, then

$$(x - p_1, y - p_2, z - p_3) = t(q_1 - p_1, q_2 - p_2, q_3 - p_3) = t(v_1, v_2, v_3),$$

where $(v_1, v_2, v_3) = (q_1 - p_1, q_2 - p_2, q_3 - p_3)$. We can always rewrite this as,

$$(x, y, z) = (p_1, p_2, p_3) + t(v_1, v_2, v_3) = (p_1 + tv_1, p_2 + tv_2, p_3 + tv_3).$$

Writing these equations out in coordinates, we get

$$x = p_1 + tv_1 \qquad y = p_2 + tv_2 \qquad \text{and} \qquad z = p_3 + tv_3.$$

Example 1.7. *If $P = (1, -2, 3)$ and $Q = (1, 0, -1)$, then $\vec{v} = (0, 2, -4)$ and a general point of the line containing P and Q is given parametrically by*

$$(x, y, z) = (1, -2, 3) + t(0, 2, -4) = (1, -2 + 2t, 3 - 4t).$$

Example 1.8. *Where do the two lines l_1 and l_2*

$$(x, y, z) = (1, -2+2t, 3-4t) \qquad \text{and} \qquad (x, y, z) = (2t-1, -3+t, 3t),$$

intersect?

We are looking for a point (x, y, z) common to both lines. So we have

$$(1, -2 + 2s, 3 - 4s) = (2t - 1, -3 + t, 3t).$$

Looking at the first component, we must have $t = 1$. Looking at the second component, we must have $-2 + 2s = -2$, so that $s = 0$. By inspection, the third component comes out equal to 3 in both cases. So the lines intersect at the point $(1, -2, 3)$.

Example 1.9. *Where does the line*

$$(x, y, z) = (1 - t, 2 - 3t, 2t + 1)$$

intersect the plane

$$2x - 3y + z = 6?$$

We must have

$$2(1 - t) - 3(2 - 3t) + (2t + 1) = 6.$$

Solving for t we get

$$9t - 3 = 6,$$

so that $t = 1$. The line intersects the plane at the point

$$(x, y, z) = (0, -1, 3).$$

Example 1.10. *A cycloid is the path traced in the plane, by a point on the circumference of a circle as the circle rolls along the ground.*

Let's find the parametric form of a cycloid. Let's suppose that the circle has radius a, the circle rolls along the x-axis and the point is at the origin at time $t = 0$. We suppose that the cylinder rotates through an angle of t radians in time t. So the circumference travels a distance of at. It follows that the centre of the circle at time t is at the point $P = (at, a)$. Call the point on the circumference $Q = (x, y)$ and let O be the centre of coordinates. We have

$$(x, y) = \overrightarrow{OQ} = \overrightarrow{OP} + \overrightarrow{PQ}.$$

Now relative to P, the point Q just goes around a circle of radius a. Note that the circle rotates backwards and at time $t = 0$, the angle $3\pi/2$. So we have

$$\overrightarrow{PQ} = (a\cos(3\pi/2 - t), a\sin(3\pi/2 - t)) = (-a\sin t, -a\cos t)$$

Putting all of this together, we have

$$(x, y) = (at - a\sin t, a - a\cos t).$$

2. Dot product

Definition 2.1. *Let $\vec{v} = (v_1, v_2, v_3)$ and $\vec{w} = (w_1, w_2, w_3)$ be two vectors in $\mathbb{R}^3$. The **dot product** of $\vec{v}$ and $\vec{w}$, denoted $\vec{v} \cdot \vec{w}$, is the scalar $v_1w_1 + v_2w_2 + v_3w_3$.*

Example 2.2. *The dot product of $\vec{v} = (1, -2, -1)$ and $\vec{w} = (2, 1, -3)$ is*

$$1 \cdot 2 + (-2) \cdot 1 + (-1) \cdot (-3) = 2 - 2 + 3 = 3.$$

Lemma 2.3. *Let $\vec{u}$, $\vec{v}$ and $\vec{w}$ be three vectors in $\mathbb{R}^3$ and let λ be a scalar.*

(1) $(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$.
(2) $\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$.
(3) $(\lambda\vec{v}) \cdot \vec{w} = \lambda(\vec{v} \cdot \vec{w})$.
(4) $\vec{v} \cdot \vec{v} = 0$ *if and only if* $\vec{v} = \vec{0}$.

Proof. (1–3) are straightforward.

To see (4), first note that one direction is clear. If $\vec{v} = \vec{0}$, then $\vec{v} \cdot \vec{v} = 0$. For the other direction, suppose that $\vec{v} \cdot \vec{v} = 0$. Then $v_1^2 + v_2^2 + v_3^2 = 0$. Now the square of a real number is non-negative and if a sum of non-negative numbers is zero, then each term must be zero. It follows that $v_1 = v_2 = v_3 = 0$ and so $\vec{v} = \vec{0}$. □

Definition 2.4. *If $\vec{v} \in \mathbb{R}^3$, then the **norm** or **length** of $\vec{v} = (v_1, v_2, v_3)$ is the scalar*

$$\|v\| = \sqrt{\vec{v} \cdot \vec{v}} = (v_1^2 + v_2^2 + v_3^2)^{1/2}.$$

It is interesting to note that if you know the norm, then you can calculate the dot product:

$$\begin{aligned}(\vec{v} + \vec{w}) \cdot (\vec{v} + \vec{w}) &= \vec{v} \cdot \vec{v} + 2\vec{v} \cdot \vec{w} + \vec{w} \cdot \vec{w} \\ (\vec{v} - \vec{w}) \cdot (\vec{v} - \vec{w}) &= \vec{v} \cdot \vec{v} - 2\vec{v} \cdot \vec{w} + \vec{w} \cdot \vec{w}.\end{aligned}$$

Subtracting and dividing by 4 we get

$$\begin{aligned}\vec{v} \cdot \vec{w} &= \frac{1}{4}\left((\vec{v} + \vec{w}) \cdot (\vec{v} + \vec{w}) - (\vec{v} - \vec{w}) \cdot (\vec{v} - \vec{w})\right) \\ &= \frac{1}{4}(\|\vec{v} + \vec{w}\|^2 - \|\vec{v} - \vec{w}\|^2).\end{aligned}$$

Given two non-zero vectors $\vec{v}$ and $\vec{w}$ in space, note that we can define the angle θ between $\vec{v}$ and $\vec{w}$. $\vec{v}$ and $\vec{w}$ lie in at least one plane (which is in fact unique, unless $\vec{v}$ and $\vec{w}$ are parallel). Now just measure the angle θ between the $\vec{v}$ and $\vec{w}$ in this plane. By convention we always take $0 \leq \theta \leq \pi$.

Theorem 2.5. *If $\vec{v}$ and $\vec{w}$ are any two vectors in $\mathbb{R}^3$, then*

$$\vec{v} \cdot \vec{w} = \|\vec{v}\| \ \|\vec{w}\| \cos\theta.$$

Proof. If $\vec{v}$ is the zero vector, then both sides are equal to zero, so that they are equal to each other and the formula holds (note though, that in this case the angle θ is not determined).

By symmetry, we may assume that $\vec{v}$ and $\vec{w}$ are both non-zero. Let $\vec{u} = \vec{w} - \vec{v}$ and apply the law of cosines to the triangle with sides parallel to $\vec{u}$, $\vec{v}$ and $\vec{w}$:

$$\|\vec{u}\|^2 = \|\vec{v}\|^2 + \|\vec{w}\|^2 - 2\|\vec{v}\|\|\vec{w}\| \cos\theta.$$

We have already seen that the LHS of this equation expands to

$$\vec{v} \cdot \vec{v} - 2\vec{v} \cdot \vec{w} + \vec{w} \cdot \vec{w} = \|\vec{v}\|^2 - 2\vec{v} \cdot \vec{w} + \|\vec{w}\|.$$

Cancelling the common terms $\|\vec{v}\|^2$ and $\|\vec{w}\|^2$ from both sides, and dividing by 2, we get the desired formula. $\square$

We can use (2.5) to find the angle between two vectors:

Example 2.6. *Let $\vec{v} = -\hat{\imath} + \hat{k}$ and $\vec{w} = \hat{\imath} + \hat{\jmath}$. Then*

$$-1 = \vec{v} \cdot \vec{w} = \|\vec{v}\|\|\vec{w}\| \cos\theta = 2\cos\theta.$$

Therefore $\cos\theta = -1/2$ and so $\theta = 2\pi/3$.

Definition 2.7. *We say that two vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^3$ are **orthogonal** if $\vec{v} \cdot \vec{w} = 0$.*

Remark 2.8. *If neither $\vec{v}$ nor $\vec{w}$ are the zero vector, and $\vec{v} \cdot \vec{w} = 0$ then the angle between $\vec{v}$ and $\vec{w}$ is a right angle. Our convention is that the zero vector is orthogonal to every vector.*

Example 2.9. *$\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ are pairwise orthogonal.*

Given two vectors $\vec{v}$ and $\vec{w}$, we can project $\vec{v}$ onto $\vec{w}$. The resulting vector is called the **projection** of $\vec{v}$ onto $\vec{w}$ and is denoted $\operatorname{proj}_{\vec{w}} \vec{v}$. For example, if $\vec{F}$ is a force and $\vec{w}$ is a direction, then the projection of $\vec{F}$ onto $\vec{w}$ is the force in the direction of $\vec{w}$.

As $\operatorname{proj}_{\vec{w}} \vec{v}$ is parallel to $\vec{w}$, we have

$$\operatorname{proj}_{\vec{w}} \vec{v} = \lambda \vec{w},$$

for some scalar λ. Let's determine λ. Let's deal with the case that $\lambda \geq 0$ (so that the angle θ between $\vec{v}$ and $\vec{w}$ is between 0 and $\pi/2$). If we take the norm of both sides, we get

$$\|\operatorname{proj}_{\vec{w}} \vec{v}\| = \|\lambda \vec{w}\| = \lambda \|\vec{w}\|,$$

(note that $\lambda \geq 0$), so that

$$\lambda = \frac{\| \operatorname{proj}_{\vec{w}} \vec{v}\|}{\|\vec{w}\|}.$$

But

$$\cos\theta = \frac{\| \operatorname{proj}_{\vec{w}} \vec{v}\|}{\|\vec{v}\|},$$

so that

$$\| \operatorname{proj}_{\vec{w}} \vec{v}\| = \|\vec{v}\| \cos\theta.$$

Putting all of this together we get

$$\begin{aligned} \lambda &= \frac{\|\vec{v}\| \cos\theta}{\|\vec{w}\|} \\ &= \frac{\|\vec{v}\|\|\vec{w}\| \cos\theta}{\|\vec{w}\|^2} \\ &= \frac{\vec{v} \cdot \vec{w}}{\|\vec{w}\|^2}. \end{aligned}$$

There are a number of ways to deal with the case when $\lambda < 0$ (so that $\theta > \pi/2$). One can carry out a similar analysis to the one given above. Here is another way. Note that the angle ϕ between $\vec{w}$ and $\vec{u} = -\vec{v}$ is equal to $\pi - \theta < \pi/2$. By what we already proved

$$\operatorname{proj}_{\vec{w}} \vec{u} = \frac{\vec{u} \cdot \vec{w}}{\|\vec{w}\|^2}\vec{w}.$$

But $\operatorname{proj}_{\vec{w}} \vec{u} = -\operatorname{proj}_{\vec{w}} \vec{v}$ and $\vec{u} \cdot \vec{w} = -\vec{v} \cdot \vec{w}$, so we get the same formula in the end. To summarise:

Theorem 2.10. *If $\vec{v}$ and $\vec{w}$ are two vectors in $\mathbb{R}^3$, where $\vec{w}$ is not zero, then*

$$\operatorname{proj}_{\vec{w}} \vec{v} = \left(\frac{\vec{v} \cdot \vec{w}}{\|\vec{w}\|^2} \right) \vec{w}.$$

Example 2.11. *Find the distance d between the line l containing the points $P_1 = (1, -1, 2)$ and $P_2 = (4, 1, 0)$ and the point $Q = (3, 2, 4)$.*

Suppose that R is the closest point on the line l to the point Q. Note that $\overrightarrow{QR}$ is orthogonal to the direction $\overrightarrow{P_1P_2}$ of the line. So we want the length of the vector $\overline{P_1}Q - \operatorname{proj}_{\overrightarrow{P_1P_2}} \overline{P_1}Q$, that is, we want

$$d = \|\overline{P_1}Q - \operatorname{proj}_{\overrightarrow{P_1P_2}} \overline{P_1}Q\|.$$

Now

$$\overrightarrow{P_1Q} = (2, 3, 2) \qquad \textit{and} \qquad \overrightarrow{P_1P_2} = (3, 2, -2).$$

We have

$\|\overrightarrow{P_1P_2}\|^2 = 3^2 + 2^2 + 2^2 = 17$ *and* $\overrightarrow{P_1P_2} \cdot \overrightarrow{P_1Q} = 6 + 6 - 4 = 8.$

It follows that

$$\mathrm{proj}_{\overrightarrow{P_1P_2}} \overrightarrow{P_1Q} = \frac{8}{17}(3, 2, -2).$$

Subtracting, we get

$$\overrightarrow{P_1Q} - \mathrm{proj}_{\overrightarrow{P_1P_2}} \overrightarrow{P_1Q} = (2, 3, 2) - \frac{8}{17}(3, 2, -2) = \frac{1}{17}(10, 35, 50) = \frac{5}{17}(2, 7, 10).$$

Taking the length, we get

$$\frac{5}{17}(2^2 + 7^2 + 10^2)^{1/2} \approx 3.64.$$

Theorem 2.12. *The angle subtended on the circumference of a circle by a diameter of the circle is always a right angle.*

Proof. Suppose that P and Q are the two endpoints of a diameter of the circle and that R is a point on the circumference. We want to show that the angle between $\overrightarrow{PR}$ and $\overrightarrow{QR}$ is a right angle.

Let O be the centre of the circle. Then

$$\overrightarrow{PR} = \overrightarrow{PO} + \overrightarrow{OR} \qquad \text{and} \qquad \overrightarrow{QR} = \overrightarrow{QO} + \overrightarrow{OR}.$$

Note that $\overrightarrow{QO} = -\overrightarrow{PO}$. Therefore

$$\begin{aligned}\overrightarrow{PR} \cdot \overrightarrow{QR} &= (\overrightarrow{PO} + \overrightarrow{OR}) \cdot (\overrightarrow{QO} + \overrightarrow{OR}) \\ &= (\overrightarrow{PO} + \overrightarrow{OR}) \cdot (\overrightarrow{OR} - \overrightarrow{PO}) \\ &= \|\overrightarrow{OR}\|^2 - \|\overrightarrow{PO}\|^2 \\ &= r^2 - r^2 = 0,\end{aligned}$$

where r is the radius of the circle. It follows that $\overrightarrow{PR}$ and $\overrightarrow{QR}$ are indeed orthogonal. □

3. Cross product

Definition 3.1. *Let $\vec{v}$ and $\vec{w}$ be two vectors in $\mathbb{R}^3$. The **cross product** of $\vec{v}$ and $\vec{w}$, denoted $\vec{v} \times \vec{w}$, is the vector defined as follows:*

- *the length of $\vec{v} \times \vec{w}$ is the area of the parallelogram with sides $\vec{v}$ and $\vec{w}$, that is, $\|\vec{v}\|\|\vec{w}\| \sin\theta$.*
- *$\vec{v} \times \vec{w}$ is orthogonal to both $\vec{v}$ and $\vec{w}$.*
- *the three vectors $\vec{v}$, $\vec{w}$ and $\vec{v} \times \vec{w}$ form a right-handed set of vectors.*

Remark 3.2. *The cross product only makes sense in $\mathbb{R}^3$.*

Example 3.3. *We have*

$$\hat{\imath} \times \hat{\jmath} = \hat{k}, \qquad \hat{\jmath} \times \hat{k} = \hat{\imath} \qquad \textit{and} \qquad \hat{k} \times \hat{\imath} = \hat{\jmath}.$$

By contrast

$$\hat{\jmath} \times \hat{\imath} = -\hat{k}, \qquad \hat{k} \times \hat{\jmath} = -\hat{\imath} \qquad \textit{and} \qquad \hat{\imath} \times \hat{k} = -\hat{\jmath}.$$

Theorem 3.4. *Let $\vec{u}$, $\vec{v}$ and $\vec{w}$ be three vectors in $\mathbb{R}^3$ and let λ be a scalar.*

(1) $\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}$.
(2) $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$.
(3) $(\vec{u} + \vec{v}) \times \vec{w} = \vec{u} \times \vec{w} + \vec{v} \times \vec{w}$.
(4) $\lambda(\vec{v} \times \vec{w}) = (\lambda\vec{v}) \times \vec{w} = \vec{v} \times (\lambda\vec{w})$.

Before we prove (3.4), let's draw some conclusions from these properties.

Remark 3.5. *Note that (1) of* (3.4) *is what really distinguishes the cross product (the cross product is skew commutative).*

Consider computing the cross product of $\hat{\imath}$, $\hat{\imath}$ and $\hat{\jmath}$. On the one hand,

$$(\hat{\imath} \times \hat{\imath}) \times \hat{\jmath} = \vec{0} \times \vec{j} = \vec{0}.$$

On the other hand,

$$\hat{\imath} \times (\hat{\imath} \times \hat{\jmath}) = \hat{\imath} \times \hat{k} = -\hat{\jmath}.$$

In other words, the order in which we compute the cross product is important (the cross product is not associative).

Note that if $\vec{v}$ and $\vec{w}$ are parallel, then the cross product is the zero vector. One can see this directly from the formula; the area of the parallelogram is zero and the only vector of zero length is the zero vector. On the other hand, we know that $\vec{w} = \lambda\vec{v}$. In this case,

$$\begin{aligned}\vec{v} \times \vec{w} &= \vec{v} \times (\lambda\vec{v}) \\ &= \lambda\vec{v} \times \vec{v} \\ &= -\lambda\vec{v} \times \vec{v}.\end{aligned}$$

To get from the second to the third line, we just switched the factors. But the only vector which is equal to its inverse is the zero vector.

Let's try to compute the cross product using (3.4). If $\vec{v} = (v_1, v_2, v_3)$ and $\vec{w} = (w_1, w_2, w_3)$, then

$$\begin{aligned}\vec{v} \times \vec{w} &= (v_1\hat{\imath} + v_2\hat{\jmath} + v_3\hat{k}) \times (w_1\hat{\imath} + w_2\hat{\jmath} + w_3\hat{k}) \\ &= v_1w_1(\hat{\imath} \times \hat{\imath}) + v_1w_2(\hat{\imath} \times \hat{\jmath}) + v_1w_3(\hat{\imath} \times \hat{k}) \\ &+ v_2w_1(\hat{\jmath} \times \hat{\imath}) + v_2w_2(\hat{\jmath} \times \hat{\jmath}) + v_2w_3(\hat{\jmath} \times \hat{k}) \\ &+ v_3w_1(\hat{k} \times \hat{\imath}) + v_3w_2(\hat{k} \times \hat{\jmath}) + v_3w_3(\hat{k} \times \hat{k}) \\ &= (v_2w_3 - v_3w_2)\hat{\imath} + (v_3w_1 - v_1w_3)\hat{\jmath} + (v_1w_2 - v_2w_1)\hat{k}.\end{aligned}$$

Definition 3.6. *A matrix $A = (a_{ij})$ is a rectangular array of numbers, where a_{ij} is in the i row and jth column. If A has m rows and n columns, then we say that A is a $m \times n$ matrix.*

Example 3.7.

$$A = \begin{pmatrix} -2 & 1 & -7 \\ 0 & 2 & -4 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix},$$

is a 2×3 matrix. $a_{23} = -4$.

Definition 3.8. *If*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

*is a 2×2 matrix, then the **determinant** of A is the scalar*

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

If

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

*is a 3×3 matrix, then the **determinant** of A is the scalar*

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}.$$

Note that the cross product of $\vec{v}$ and $\vec{w}$ is the (formal) determinant

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

Let's now turn to the proof of (3.4).

Definition 3.9. *Let $\vec{u}$, $\vec{v}$ and $\vec{w}$ be three vectors in $\mathbb{R}^3$. The **triple scalar product** is $(\vec{u} \times \vec{v}) \cdot \vec{w}$.*

The triple scalar product is the *signed* volume of the parallelepiped formed using the three vectors, $\vec{u}$, $\vec{v}$ and $\vec{w}$. Indeed, the volume of the parallelepiped is the area of the base times the height. For the base, we take the parallelogram with sides $\vec{u}$ and $\vec{v}$. The magnitude of $\vec{u} \times \vec{v}$ is the area of this parallelogram. The height of the parallelepiped, up to sign, is the length of $\vec{w}$ times the cosine of the angle, let's call this ϕ, between $\vec{u} \times \vec{v}$ and $\vec{w}$. The sign is positive, if $\vec{u}$, $\vec{v}$ and $\vec{w}$ form a *right-handed set* and negative if they form a *left-handed set*.

Lemma 3.10. *If $\vec{u} = (u_1, u_2, u_3)$, $\vec{v} = (v_1, v_2, v_3)$ and $\vec{w} = (w_1, w_2, w_3)$ are three vectors in $\mathbb{R}^3$, then*

$$(\vec{u} \times \vec{v}) \cdot \vec{w} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

Proof. We have already seen that

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

If one expands this determinant and dots with $\vec{w}$, this is the same as replacing the top row by (w_1, w_2, w_3),

$$(\vec{u} \times \vec{v}) \cdot \vec{w} = \begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

Finally, if we switch the first row and the second row, and then the second row and the third row, the sign changes twice (which makes no change at all):

$$(\vec{u} \times \vec{v}) \cdot \vec{w} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

□

Example 3.11. *The scalar triple product of $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ is one. One way to see this is geometrically; the parallelepiped determined by these three vectors is the unit cube, which has volume* 1*, and these vectors form a right-handed set, so that the sign is positive.*

Another way to see this is to compute directly

$$(\hat{\imath} \times \hat{\jmath}) \cdot \hat{k} = \hat{k} \cdot \hat{k} = 1.$$

Finally one can use determinants,

$$(\hat{\imath} \times \hat{\jmath}) \cdot \hat{k} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1.$$

Lemma 3.12. *Let $\vec{u}$, $\vec{v}$ and $\vec{w}$ be three vectors in $\mathbb{R}^3$.*

Then

$$(\vec{u} \times \vec{v}) \cdot \vec{w} = (\vec{v} \times \vec{w}) \cdot \vec{u} = (\vec{w} \times \vec{u}) \cdot \vec{v}.$$

Proof. In fact all three numbers have the same absolute value, namely the volume of the parallelepiped with sides $\vec{u}$, $\vec{v}$ and $\vec{w}$. On the other hand, if $\vec{u}$, $\vec{v}$, and $\vec{w}$ is a right-handed set, then so is $\vec{v}$, $\vec{w}$ and $\vec{u}$ and vice-versa, so all three numbers have the same sign as well. □

Lemma 3.13. *Let $\vec{v}$ and $\vec{w}$ be two vectors in $\mathbb{R}^3$.*

Then $\vec{v} = \vec{w}$ if and only if $\vec{v} \cdot \vec{x} = \vec{w} \cdot \vec{x}$, for every vector $\vec{x}$ in $\mathbb{R}^3$.

Proof. One direction is clear; if $\vec{v} = \vec{w}$, then $\vec{v} \cdot \vec{x} = \vec{w} \cdot \vec{x}$ for any vector $\vec{x}$.

So, suppose that we know that $\vec{v} \cdot \vec{x} = \vec{w} \cdot \vec{x}$, for every vector $\vec{x}$. Suppose that $\vec{v} = (v_1, v_2, v_3)$ and $\vec{w} = (w_1, w_2, w_3)$. If we take $\vec{x} = \hat{\imath}$, then we see that

$$v_1 = \vec{v} \cdot \hat{\imath} = \vec{w} \cdot \hat{\imath} = w_1.$$

Similarly, if we take $\vec{x} = \hat{\jmath}$ and $\vec{x} = \hat{k}$, then we also get

$$v_2 = \vec{v} \cdot \hat{\jmath} = \vec{w} \cdot \hat{\jmath} = w_2,$$

and

$$v_3 = \vec{v} \cdot \hat{k} = \vec{w} \cdot \hat{k} = w_3.$$

But then $\vec{v} = \vec{w}$ as they have the same components. □

Proof of (3.4). We first prove (1). Both sides have the same magnitude, namely the area of the parallelogram with sides $\vec{v}$ and $\vec{w}$. Further both sides are orthogonal to $\vec{v}$ and $\vec{w}$, so the only thing to check is the change in sign.

As $\vec{v}$, $\vec{w}$ and $\vec{v} \times \vec{w}$ form a right-handed triple, it follows that $\vec{w}$, $\vec{v}$ and $\vec{v} \times \vec{w}$ form a left-handed triple. But then $\vec{w}$, $\vec{v}$ and $-\vec{v} \times \vec{w}$ form a right-handed triple. It follows that

$$\vec{w} \times \vec{v} = -\vec{v} \times \vec{w}.$$

This is (1).

To check (2), we check that

$$(\vec{u} \times (\vec{v} + \vec{w})) \cdot \vec{x} = (\vec{u} \times \vec{v} + \vec{u} \times \vec{w}) \cdot \vec{x},$$

for an arbitrary vector $\vec{x}$. We first attack the LHS. By (3.12), we have

$$\begin{aligned}(\vec{u} \times (\vec{v} + \vec{w})) \cdot \vec{x} &= (\vec{x} \times \vec{u}) \cdot (\vec{v} + \vec{w}) \\ &= (\vec{x} \times \vec{u}) \cdot \vec{v} + (\vec{x} \times \vec{u}) \cdot \vec{w} \\ &= (\vec{u} \times \vec{v}) \cdot \vec{x} + (\vec{u} \times \vec{w}) \cdot \vec{x}.\end{aligned}$$

We now attack the RHS.

$$(\vec{u} \times \vec{v} + \vec{u} \times \vec{w}) \cdot \vec{x} = (\vec{u} \times \vec{v}) \cdot \vec{x} + (\vec{u} \times \vec{v}) \cdot \vec{x}.$$

It follows that both sides are equal. This is (2).

We could check (3) by a similar argument. Here is another way.

$$\begin{aligned}(\vec{u} + \vec{v}) \times \vec{w} &= -\vec{w} \times (\vec{u} + \vec{v}) \\ &= -\vec{w} \times \vec{u} - \vec{w} \times \vec{v} \\ &= \vec{u} \times \vec{w} + \vec{v} \times \vec{w}.\end{aligned}$$

This is (3).

To prove (4), it suffices to prove the first equality, since the fact that the first term is equal to the third term follows by a similar derivation. If $\lambda = 0$, then both sides are the zero vector, and there is nothing to prove. So we may assume that $\lambda \neq 0$. Note first that the magnitude of both sides is the area of the parallelogram with sides $\lambda\vec{v}$ and $\vec{w}$.

If $\lambda > 0$, then $\vec{v}$ and $\lambda\vec{v}$ point in the same direction. Similarly $\vec{v} \times \vec{w}$ and $\lambda(\vec{v} \times \vec{w})$ point in the same direction. As $\vec{v}$, $\vec{w}$ and $\vec{v} \times \vec{w}$ for a right-handed set, then so do $\lambda\vec{v}$, $\vec{w}$ and $\lambda(\vec{v} \times \vec{w})$. But then $\lambda(\vec{v} \times \vec{w})$ is the cross product of $\lambda\vec{v}$ and $\vec{w}$, that is,

$$(\lambda\vec{v}) \times \vec{w} = \lambda(\vec{v} \times \vec{w}).$$

If $\lambda < 0$, then $\vec{v}$ and $\lambda\vec{v}$ point in the opposite direction. Similarly $\vec{v} \times \vec{w}$ and $\lambda(\vec{v} \times \vec{w})$ point in the opposite direction. But then $\lambda\vec{v}$, $\vec{w}$ and $\lambda(\vec{v} \times \vec{w})$ still form a right-handed set. This is (4). $\square$

4. Planes and distances

How do we represent a plane Π in $\mathbb{R}^3$? In fact the best way to specify a plane is to give a normal vector $\vec{n}$ to the plane and a point P_0 on the plane. Then if we are given any point P on the plane, the vector $\overrightarrow{P_0P}$ is a vector in the plane, so that it must be orthogonal to the normal vector $\vec{n}$. Algebraically, we have

$$\overrightarrow{P_0P} \cdot \vec{n} = 0.$$

Let's write this out as an explicit equation. Suppose that the point $P_0 = (x_0, y_0, z_0)$, $P = (x, y, z)$ and $\vec{n} = (A, B, C)$. Then we have

$$(x - x_0, y - y_0, z - z_0) \cdot (A, B, C) = 0.$$

Expanding, we get

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0,$$

which is one common way to write down a plane. We can always rewrite this as

$$Ax + By + Cz = D.$$

Here

$$D = Ax_0 + By_0 + Cz_0 = (A, B, C) \cdot (x_0, y_0, z_0) = \vec{n} \cdot \overrightarrow{OP_0}.$$

This is perhaps the most common way to write down the equation of a plane.

Example 4.1.

$$3x - 4y + 2z = 6,$$

is the equation of a plane. A vector normal to the plane is $(3, -4, 2)$.

Example 4.2. *What is the equation of a plane passing through* $(1, -1, 2)$, *with normal vector* $\vec{n} = (2, 1, -1)$? *We have*

$$(x - 1, y + 1, z - 2) \cdot (2, 1, -1) = 0.$$

So

$$2(x - 1) + y + 1 - (z - 2) = 0,$$

so that in other words,

$$2x + y - z = -1.$$

A line is determined by two points; a plane is determined by three points, provided those points are not collinear (that is, provided they don't lie on the same line). So given three points P_0, P_1 and P_2, what is the equation of the plane Π containing P_0, P_1 and P_2? Well, we would like to find a vector $\vec{n}$ orthogonal to any vector in the plane. Note that $\overrightarrow{P_0P_1}$ and $\overrightarrow{P_0P_2}$ are two vectors in the plane, which by assumption are not parallel. The cross product is a vector which is orthogonal to both vectors,

$$\vec{n} = \overrightarrow{P_0P_1} \times \overrightarrow{P_0P_2}.$$

So the equation we want is

$$\overrightarrow{P_0P} \cdot (\overrightarrow{P_0P_1} \times \overrightarrow{P_0P_2}) = 0.$$

We can rewrite this a little. $\overrightarrow{P_0P} = \overrightarrow{OP} - \overrightarrow{OP_0}$. Expanding and rearranging gives

$$\overrightarrow{OP} \cdot (\overrightarrow{P_0P_1} \times \overrightarrow{P_0P_2}) = \overrightarrow{OP_0} \cdot (\overrightarrow{P_0P_1} \times \overrightarrow{P_0P_2}).$$

Note that both sides involve the triple scalar product.

Example 4.3. *What is the equation of the plane* Π *through the three points,* $P_0 = (1, 1, 1)$, $P_1 = (2, -1, 0)$ *and* $P_2 = (0, -1, -1)$*?*

$$\overrightarrow{P_0P_1} = (1, -2, -1) \qquad \textit{and} \qquad \overrightarrow{P_0P_2} = (-1, -2, -2).$$

Now a vector orthogonal to both of these vectors is given by the cross product:

$$\begin{aligned}
\vec{n} &= \overrightarrow{P_0P_1} \times \overrightarrow{P_0P_2} \\
&= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -2 & -1 \\ -1 & -2 & -2 \end{vmatrix} \\
&= \hat{\imath} \begin{vmatrix} -2 & -1 \\ -2 & -2 \end{vmatrix} - \hat{\jmath} \begin{vmatrix} 1 & -1 \\ -1 & -2 \end{vmatrix} + \hat{k} \begin{vmatrix} 1 & -2 \\ -1 & -2 \end{vmatrix} \\
&= 2\hat{\imath} + 3\hat{\jmath} - 4\hat{k}.
\end{aligned}$$

Note that

$$\vec{n} \cdot \overrightarrow{P_0P_1} = 2 - 6 + 4 = 0,$$

as expected. It follows that the equation of Π *is*

$$2(x - 1) + 3(y - 1) - 4(z - 1) = 0,$$

so that

$$2x + 3y - 4z = 1.$$

For example, if we plug in $P_2 = (0, -1, -1)$*, then*

$$2 \cdot 0 + 3 \cdot -1 + 4 = 1,$$

as expected.

Example 4.4. *What is the parametric equation for the line l given as the intersection of the two planes $2x - y + z = 1$ and $x + y - z = 2$?*

Well we need two points on the intersection of these two planes. If we set $z = 0$, then we get the intersection of two lines in the xy-plane,

$$\begin{aligned} 2x - y &= 1 \\ x + y &= 2. \end{aligned}$$

Adding these two equations we get $3x = 3$, so that $x = 1$. It follows that $y = 1$, so that $P_0 = (1, 1, 0)$ is a point on the line.

Now suppose that $y = 0$. Then we get

$$\begin{aligned} 2x + z &= 1 \\ x - z &= 2. \end{aligned}$$

As before this says $x = 1$ and so $z = -1$. So $P_1 = (1, 0, -1)$ is a point on l.

$$\overrightarrow{P_0P} = t\overrightarrow{P_0P_1},$$

for some parameter t. Expanding

$$(x - 1, y - 1, z) = t(0, -1, -1),$$

so that

$$(x, y, z) = (1, 1 - t, -t).$$

We can also calculate distances between planes and points, lines and points, and lines and lines.

Example 4.5. *What is the distance between the plane $x - 2y + 3z = 4$ and the point $P = (1, 2, 3)$?*

Call the closest point R. Then $\overrightarrow{PR}$ is orthogonal to every vector in the plane, that is, $\overrightarrow{PR}$ is normal to the plane. Note that $\vec{n} = (1, -2, 3)$ is normal to the plane, so that $\overrightarrow{PR}$ is parallel to the plane.

Pick any point Q belonging to the plane. Then the triangle PQR has a right angle at R, so that

$$\overrightarrow{PR} = \pm \operatorname{proj}_{\vec{n}} \overrightarrow{PQ}.$$

When $x = z = 0$, then $y = -2$, so that $Q = (0, -2, 0)$ is a point on the plane.

$$\overrightarrow{PQ} = (-1, -4, -3).$$

Now

$$\|\vec{n}\|^2 = \vec{n}. \cdot \vec{n}. = 1^2 + 2^2 + 3^2 = 14 \qquad \text{and} \qquad \vec{n} \cdot \overrightarrow{PQ} = 4.$$

So

$$\operatorname{proj}_{\vec{n}} \overrightarrow{PQ} = \frac{2}{7}(1, -2, 3).$$

So the distance is

$$\frac{2}{7}\sqrt{14}.$$

Here is another way to proceed. The line through P, pointing in the direction $\vec{n}$, will intersect the plane at the point R. Now this line is given parametrically as

$$(x-1, y-2, z-3) = t(1, -2, 3),$$

so that

$$(x, y, z) = (t+1, 2-2t, 3+3t).$$

The point R corresponds to

$$(t+1) - 2(2-2t) + 3(3+3t) = 4,$$

so that

$$14t = -2 \qquad \text{that is} \qquad t = \frac{2}{7}.$$

So the point R is

$$\frac{1}{7}(9, 10, 27).$$

It follows that

$$\overrightarrow{PR} = \frac{1}{7}(2, -4, 6) = \frac{2}{7}(1, -2, 3),$$

the same answer as before (phew!).

Example 4.6. *What is the distance between the two lines*

$$(x, y, z) = (t-2, 3t+1, 2-t) \qquad \text{and} \qquad (x, y, z) = (2t-1, 2-3t, t+1)?$$

If the two closest points are R and R' then $\overrightarrow{RR'}$ is orhogonal to the direction of both lines. Now the direction of the first line is $(1, 3, -1)$ and the direction of the second line is $(2, -3, 1)$. A vector orthogonal to both is given by the cross product:

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 3 & -1 \\ 2 & -3 & 1 \end{vmatrix} = -3\hat{\jmath} - 9\hat{k}.$$

To simplify some of the algebra, let's take

$$\vec{n} = \hat{\jmath} + 3\hat{k},$$

which is parallel to the vector above, so that it is still orthogonal to both lines.

It follows that $\overrightarrow{RR'}$ is parallel to $\vec{n}$. Pick any two points P and P' on the two lines. Note that the length of the vector

$$\operatorname{proj}_{\vec{n}} \overline{P'}P,$$

is the distance between the two lines.

Now if we plug in $t = 0$ to both lines we get

$$P' = (-2, 1, 2) \qquad \textit{and} \qquad P = (-1, 2, 1).$$

So

$$\overline{P'}P = (1, 1, -1).$$

Then

$$\|\vec{n}\|^2 = 1^2 + 3^2 = 10 \qquad \textit{and} \qquad \vec{n} \cdot \overline{P'}P = -2.$$

It follows that

$$\operatorname{proj}_{\vec{n}} \overline{P'}P = \frac{-2}{10}(0, 1, 3) = \frac{-1}{5}(0, 1, 3).$$

and so the distance between the two lines is

$$\frac{1}{5}\sqrt{10}.$$

5. n-DIMENSIONAL SPACE

Definition 5.1. *A **vector** in $\mathbb{R}^n$ is an n-tuple $\vec{v} = (v_1, v_2, \dots, v_n)$. The **zero vector** $\vec{0} = (0, 0, \dots, 0)$. Given two vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^n$, the sum $\vec{v}+\vec{w}$ is the vector $(v_1 + w_1, v_2 + w_2, \dots, v_n + w_n)$. If λ is a scalar, the **scalar product** $\lambda\vec{v}$ is the vector $(\lambda v_1, \lambda v_2, \dots, \lambda v_n)$.*

The sum and scalar product of vectors in $\mathbb{R}^n$ obey the same rules as the sum and scalar product in $\mathbb{R}^2$ and $\mathbb{R}^3$.

Definition 5.2. *Let $\vec{v}$ and $\vec{w} \in \mathbb{R}^n$. The **dot product** is the scalar*

$$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + \dots v_n w_n.$$

*The **norm** (or **length**) of $\vec{v}$ is the scalar*

$$\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}}.$$

The scalar product obeys the usual rules.

Example 5.3. *Suppose that $\vec{v} = (1, 2, 3, 4)$ and $\vec{w} = (2, -1, 1, -1)$ and $\lambda = -2$. Then*

$$\vec{v} + \vec{w} = (3, 1, 4, 3) \qquad \textit{and} \qquad \lambda\vec{w} = (-4, 2, -2, 2).$$

We have

$$\vec{v} \cdot \vec{w} = 2 - 2 + 3 - 4 = -1.$$

The standard basis of $\mathbb{R}^n$ is the set of vectors,

$$e_1 = (1, 0, \dots, 0), \quad e_2 = (0, 1, \dots, 0), \quad e_1 = (0, 0, 1, \dots, 0), \quad \dots e_n = (0, 0, \dots, 1).$$

Note that if $\vec{v} = (v_1, v_2, \dots, v_n)$, then

$$\vec{v} = v_1 e_1 + v_2 e_2 + \dots + v_n e_n.$$

Let's adopt the (somewhat ad hoc) convention that $\vec{v}$ and $\vec{w}$ are parallel if and only if either $\vec{v}$ is a scalar multiple of $\vec{w}$, or vice-versa. Note that if both $\vec{v}$ and $\vec{w}$ are non-zero vectors, then $\vec{v}$ is a scalar multiple of $\vec{w}$ if and only if $\vec{w}$ is a scalar multiple of $\vec{v}$.

Theorem 5.4 (Cauchy-Schwarz-Bunjakowski)**.** *If $\vec{v}$ and $\vec{w}$ are two vectors in $\mathbb{R}^n$, then*

$$|\vec{v} \cdot \vec{w}| \leq \|v\|\|w\|,$$

with equality if and only if $\vec{v}$ is parallel to $\vec{w}$.

Proof. If either $\vec{v}$ or $\vec{w}$ is the zero vector, then there is nothing to prove. So we may assume that neither vector is the zero vector.

Let $\vec{u} = x\vec{v} + \vec{w}$, where x is a scalar. Then

$$0 \leq \vec{u} \cdot \vec{u} = (\vec{v} \cdot \vec{v})x^2 + 2(\vec{v} \cdot \vec{w})x + \vec{w} \cdot \vec{w} = ax^2 + bx + c.$$

So the quadratic function $f(x) = ax^2 + bx + c$ has at most one root. It follows that the discriminant is less than or equal to zero, with equality if and only if $f(x)$ has a root. So

$$4(\vec{v} \cdot \vec{w})^2 - 4\|\vec{v}\|^2\|\vec{w}\|^2 = b^2 - 4ac \leq 0.$$

Rearranging, gives

$$(\vec{v} \cdot \vec{w})^2 \leq \|\vec{v}\|^2\|\vec{w}\|^2.$$

Taking square roots, gives

$$|\vec{v} \cdot \vec{w}| \leq \|v\|\|w\|.$$

Now if we have equality here, then the discriminant must be equal to zero, in which case we may find a scalar λ such that the vector $\lambda\vec{v} + \vec{w}$ has zero length. But the only vector of length zero is the zero vector, so that $\lambda\vec{v} + \vec{w} = \vec{0}$. In other words, $\vec{w} = -\lambda\vec{v}$ and $\vec{v}$ and $\vec{w}$ are parallel. □

Definition 5.5. *If $\vec{v}$ and $\vec{w} \in \mathbb{R}^n$ are non-zero vectors, then the angle between them is the unique angle $0 \leq \theta \leq \pi$ such that*

$$\cos\theta = \frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\|\|\vec{w}\|}.$$

Note that the fraction is between -1 and 1, by (5.4), so this does makes sense, and we showed in (5.4) that the angle is 0 or π if and only if $\vec{v}$ and $\vec{w}$ are parallel.

Definition 5.6. *If $A = (a_{ij})$ and $B = (b_{ij})$ are two $m \times n$ matrices, then the* ***sum*** *$A + B$ is the $m \times n$ matrix $(a_{ij} + b_{ij})$. If λ is a scalar, then the* ***scalar multiple*** *λA is the $m \times n$ matrix (λa_{ij}).*

Example 5.7. *If*

$$A = \begin{pmatrix} 1 & -1 \\ 3 & -4 \end{pmatrix}, \qquad \text{and} \qquad B = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix},$$

then

$$A + B = \begin{pmatrix} 2 & 0 \\ 5 & -5 \end{pmatrix},$$

and

$$3A = \begin{pmatrix} 3 & -3 \\ 9 & -12 \end{pmatrix}.$$

Note that if we *flattened* A and B to $(1, -1, 3, -4)$ and $(2, 0, 5, -5)$ then the sum corresponds to the usual vector sum $(3, -1, 8, -9)$. Ditto for scalar multiplication.

Definition 5.8. *Suppose that $A = (a_{ij})$ is an $m \times n$ matrix and $B = (b_{ij})$ is an $n \times p$ matrix. The* ***product*** *$C = AB = (c_{ij})$ is the $m \times p$ matrix where*

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdots + a_{in}b_{nj} = \sum_{l=1}^{n} a_{il}b_{lj}.$$

In other words, the entry in the ith row and jth column of C is the dot product of the ith row of A and the jth column of B. This only makes sense because the ith row and the jth column are both vectors in $\mathbb{R}^n$.

Example 5.9. *Let*

$$A = \begin{pmatrix} 1 & -2 & 1 \\ 1 & -1 & 5 \end{pmatrix},$$

and

$$B = \begin{pmatrix} 2 & 1 \\ 1 & -4 \\ -1 & 1 \end{pmatrix}.$$

Then $C = AB$ has shape 2×2, and in fact

$$C = AB = \begin{pmatrix} -1 & 10 \\ -4 & 10 \end{pmatrix}.$$

Theorem 5.10. *Let A, B and C be three matrices, and let λ and μ be scalars.*

(1) *If A, B and C have the same shape, then $(A + B) + C = A + (B + C)$.*
(2) *If A and B have the same shape, then $A + B = B + A$.*
(3) *If A and B have the same shape, then $\lambda(A + B) = \lambda A + \lambda B$.*
(4) *If Z is the zero matrix with the same shape as A, then $Z + A = A + Z$.*
(5) *$\lambda(\mu A) = (\lambda\mu)A$.*
(6) *$(\lambda + \mu)A = \lambda A + \mu A$.*
(7) *If I_n is the matrix with ones on the diagonal and zeroes everywhere else and A has shape $m \times n$, then $AI_n = A$ and $I_m A = A$.*
(8) *If A has shape $m \times n$ and B has shape $n \times p$ and C has shape $p \times q$, then $A(BC) = (AB)C$.*
(9) *If A has shape $m \times n$ and B and C have the same shape $n \times p$, then $A(B + C) = AB + AC$.*
(10) *If A and B have the same shape $m \times n$ and C has shape $n \times p$, then $(A + B)C = AC + BC$.*

Example 5.11. *Note however that $AB \neq BA$ in general. For example if A has shape 1×3 and B has shape 3×2, then it makes sense to multiply A and B but it does not make sense to multiply B and A.*

In fact even if it makes sense to multiply A and B and B and A, the two products might not even have the same shape. For example, if

$$A = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix},$$

and

$$B = \begin{pmatrix} 2 & -1 & 3 \end{pmatrix},$$

then AB has shape 3×3,

$$AB = \begin{pmatrix} 2 & -1 & 3 \\ 4 & -2 & 6 \\ -2 & 1 & -3 \end{pmatrix},$$

but BA has shape 1×1,

$$BA = (2 - 2 - 3) = (-3).$$

But even both products AB and BA make sense, and they have the same shape, the products still don't have to be equal. Suppose

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \text{and} \qquad B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Then AB and BA are both 2×2 matrices. But

$$AB = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \qquad \text{and} \qquad BA = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

One can also define determinants for $n \times n$ matrices. It is probably easiest to explain the general rule using an example:

$$\begin{vmatrix} 1 & 0 & 0 & 2 \\ 2 & 0 & 1 & -1 \\ 1 & -2 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 1 & -1 \\ -2 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} - 2 \begin{vmatrix} 2 & 0 & 1 \\ 1 & -2 & 1 \\ 0 & 1 & 0 \end{vmatrix}.$$

Notice that we as expand about the top row, the sign alternates $+-+-$, so that the last term comes with a minus sign.

Finally, we try to explain the real meaning of a matrix. Let

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Given A, we can construct a function

$$f \colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2,$$

by the rule

$$f(\vec{v}) = A\vec{v}.$$

If $\vec{v} = (x, y)$, then

$$A\vec{v} = (x + y, y).$$

Here I cheat a little, and write row vectors instead of column vectors. Geometrically this is called a *shear*; it leaves the y-axis alone but one goes further along the x-axis according to the value of y. If

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

the resulting function sends (x, y) to $(ax + by, cx + dy)$. In fact the functions one gets this way are always linear. If

$$A = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix},$$

then $f(x, y) = (2x - y)$, and this has the result of scaling by a factor of 2 in the x-direction and reflects in the y-direction.

In general if A is an $m \times n$ matrix, we get a function

$$f \colon \mathbb{R}^n \longrightarrow \mathbb{R}^m,$$

using the same rule, $f(\vec{v}) = A\vec{v}$. If B is an $n \times p$ matrix, then we get a function

$$g \colon \mathbb{R}^p \longrightarrow \mathbb{R}^n,$$

by the rule $g(\vec{w}) = B\vec{w}$. Note that we can compose the functions f and g, to get a function

$$f \circ g \colon \mathbb{R}^p \longrightarrow \mathbb{R}^m.$$

First we apply g to $\vec{w}$ to get a vector $\vec{v}$ in $\mathbb{R}^n$ and then we apply f to $\vec{v}$ to get a vector in $\mathbb{R}^m$. The composite function $f \circ g$ is given by the rule $(f \circ g)(\vec{w}) = (AB)\vec{w}$. In other words, matrix multiplication is chosen so that it represents composition of functions.

As soon as one realises this, many aspects of matrix multiplication become far less mysterious. For example, composition of functions is not commutative, for example

$$\sin 2x \neq 2 \sin x,$$

and this is why $AB \neq BA$ in general. Note that it is not hard to check that composition of functions is associative,

$$f \circ (g \circ h) = (f \circ g) \circ h.$$

This is the easiest way to check that matrix multiplication is associative, that is, (8) of (5.10).

Functions given by matrices are obviously very special. Note that if $f(\vec{v}) = A\vec{v}$, then

$$f(\vec{v} + \vec{w}) = A(\vec{v} + \vec{w}) = A\vec{v} + A\vec{w} = f(\vec{v}) + f(\vec{w}),$$

and

$$f(\lambda\vec{v}) = A(\lambda\vec{v}) = \lambda(A\vec{v}) = \lambda f(\vec{v}).$$

Any function which respects both addition of vectors and scalar multiplication is called **linear** and it is precisely the linear functions which are given by matrices. In fact if $e_1, e_2, \dots, e_n$ and $f_1, f_2, \dots, f_m$ are standard bases for $\mathbb{R}^n$ and $\mathbb{R}^m$, and f is linear, then

$$f(e_j) = \sum a_{ij} f_i,$$

for some scalars a_{ij}, since $f(e_j)$ is a vector in $\mathbb{R}^m$ and any vector in $\mathbb{R}^m$ is a linear combinaton of the standard basis vectors $f_1, f_2, \dots, f_m$. If we put $A = (a_{ij})$ then one can check that f is the function

$$f(\vec{v}) = A\vec{v}.$$

4. Planes and distances

How do we represent a plane Π in $\mathbb{R}^3$? In fact the best way to specify a plane is to give a normal vector $\vec{n}$ to the plane and a point P_0 on the plane. Then if we are given any point P on the plane, the vector $\overrightarrow{P_0P}$ is a vector in the plane, so that it must be orthogonal to the normal vector $\vec{n}$. Algebraically, we have

$$\overrightarrow{P_0P} \cdot \vec{n} = 0.$$

Let's write this out as an explicit equation. Suppose that the point $P_0 = (x_0, y_0, z_0)$, $P = (x, y, z)$ and $\vec{n} = (A, B, C)$. Then we have

$$(x - x_0, y - y_0, z - z_0) \cdot (A, B, C) = 0.$$

Expanding, we get

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0,$$

which is one common way to write down a plane. We can always rewrite this as

$$Ax + By + Cz = D.$$

Here

$$D = Ax_0 + By_0 + Cz_0 = (A, B, C) \cdot (x_0, y_0, z_0) = \vec{n} \cdot \overrightarrow{OP_0}.$$

This is perhaps the most common way to write down the equation of a plane.

Example 4.1.

$$3x - 4y + 2z = 6,$$

is the equation of a plane. A vector normal to the plane is $(3, -4, 2)$.

Example 4.2. *What is the equation of a plane passing through* $(1, -1, 2)$, *with normal vector* $\vec{n} = (2, 1, -1)$? *We have*

$$(x - 1, y + 1, z - 2) \cdot (2, 1, -1) = 0.$$

So

$$2(x - 1) + y + 1 - (z - 2) = 0,$$

so that in other words,

$$2x + y - z = -1.$$

A line is determined by two points; a plane is determined by three points, provided those points are not collinear (that is, provided they don't lie on the same line). So given three points P_0, P_1 and P_2, what is the equation of the plane Π containing P_0, P_1 and P_2? Well, we would like to find a vector $\vec{n}$ orthogonal to any vector in the plane. Note that $\overrightarrow{P_0P_1}$ and $\overrightarrow{P_0P_2}$ are two vectors in the plane, which by assumption are not parallel. The cross product is a vector which is orthogonal to both vectors,

$$\vec{n} = \overrightarrow{P_0P_1} \times \overrightarrow{P_0P_2}.$$

So the equation we want is

$$\overrightarrow{P_0P} \cdot (\overrightarrow{P_0P_1} \times \overrightarrow{P_0P_2}) = 0.$$

We can rewrite this a little. $\overrightarrow{P_0P} = \overrightarrow{OP} - \overrightarrow{OP_0}$. Expanding and rearranging gives

$$\overrightarrow{OP} \cdot (\overrightarrow{P_0P_1} \times \overrightarrow{P_0P_2}) = \overrightarrow{OP_0} \cdot (\overrightarrow{P_0P_1} \times \overrightarrow{P_0P_2}).$$

Note that both sides involve the triple scalar product.

Example 4.3. *What is the equation of the plane Π through the three points, $P_0 = (1, 1, 1)$, $P_1 = (2, -1, 0)$ and $P_2 = (0, -1, -1)$?*

$$\overrightarrow{P_0P_1} = (1, -2, -1) \qquad \textit{and} \qquad \overrightarrow{P_0P_2} = (-1, -2, -2).$$

Now a vector orthogonal to both of these vectors is given by the cross product:

$$\begin{aligned}
\vec{n} &= \overrightarrow{P_0P_1} \times \overrightarrow{P_0P_2} \\
&= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -2 & -1 \\ -1 & -2 & -2 \end{vmatrix} \\
&= \hat{\imath} \begin{vmatrix} -2 & -1 \\ -2 & -2 \end{vmatrix} - \hat{\jmath} \begin{vmatrix} 1 & -1 \\ -1 & -2 \end{vmatrix} + \hat{k} \begin{vmatrix} 1 & -2 \\ -1 & -2 \end{vmatrix} \\
&= 2\hat{\imath} + 3\hat{\jmath} - 4\hat{k}.
\end{aligned}$$

Note that

$$\vec{n} \cdot \overrightarrow{P_0P_1} = 2 - 6 + 4 = 0,$$

as expected. It follows that the equation of Π is

$$2(x - 1) + 3(y - 1) - 4(z - 1) = 0,$$

so that

$$2x + 3y - 4z = 1.$$

For example, if we plug in $P_2 = (0, -1, -1)$, then

$$2 \cdot 0 + 3 \cdot -1 + 4 = 1,$$

as expected.

Example 4.4. *What is the parametric equation for the line l given as the intersection of the two planes $2x - y + z = 1$ and $x + y - z = 2$?*

Well we need two points on the intersection of these two planes. If we set $z = 0$, then we get the intersection of two lines in the xy-plane,

$$\begin{aligned} 2x - y &= 1 \\ x + y &= 2. \end{aligned}$$

Adding these two equations we get $3x = 3$, so that $x = 1$. It follows that $y = 1$, so that $P_0 = (1, 1, 0)$ is a point on the line.

Now suppose that $y = 0$. Then we get

$$\begin{aligned} 2x + z &= 1 \\ x - z &= 2. \end{aligned}$$

As before this says $x = 1$ and so $z = -1$. So $P_1 = (1, 0, -1)$ is a point on l.

$$\overrightarrow{P_0P} = t\overrightarrow{P_0P_1},$$

for some parameter t. Expanding

$$(x - 1, y - 1, z) = t(0, -1, -1),$$

so that

$$(x, y, z) = (1, 1 - t, -t).$$

We can also calculate distances between planes and points, lines and points, and lines and lines.

Example 4.5. *What is the distance between the plane $x - 2y + 3z = 4$ and the point $P = (1, 2, 3)$?*

Call the closest point R. Then $\overrightarrow{PR}$ is orthogonal to every vector in the plane, that is, $\overrightarrow{PR}$ is normal to the plane. Note that $\vec{n} = (1, -2, 3)$ is normal to the plane, so that $\overrightarrow{PR}$ is parallel to the plane.

Pick any point Q belonging to the plane. Then the triangle PQR has a right angle at R, so that

$$\overrightarrow{PR} = \pm \operatorname{proj}_{\vec{n}} \overrightarrow{PQ}.$$

When $x = z = 0$, then $y = -2$, so that $Q = (0, -2, 0)$ is a point on the plane.

$$\overrightarrow{PQ} = (-1, -4, -3).$$

Now

$$\|\vec{n}\|^2 = \vec{n}. \cdot \vec{n}. = 1^2 + 2^2 + 3^2 = 14 \qquad \text{and} \qquad \vec{n} \cdot \overrightarrow{PQ} = 4.$$

So

$$\operatorname{proj}_{\vec{n}} \overrightarrow{PQ} = \frac{2}{7}(1, -2, 3).$$

So the distance is

$$\frac{2}{7}\sqrt{14}.$$

Here is another way to proceed. The line through P, pointing in the direction $\vec{n}$, will intersect the plane at the point R. Now this line is given parametrically as

$$(x-1, y-2, z-3) = t(1,-2,3),$$

so that

$$(x,y,z) = (t+1, 2-2t, 3+3t).$$

The point R corresponds to

$$(t+1) - 2(2-2t) + 3(3+3t) = 4,$$

so that

$$14t = -2 \qquad \text{that is} \qquad t = \frac{2}{7}.$$

So the point R is

$$\frac{1}{7}(9, 10, 27).$$

It follows that

$$\overrightarrow{PR} = \frac{1}{7}(2,-4,6) = \frac{2}{7}(1,-2,3),$$

the same answer as before (phew!).

Example 4.6. *What is the distance between the two lines*

$$(x,y,z) = (t-2, 3t+1, 2-t) \qquad \text{and} \qquad (x,y,z) = (2t-1, 2-3t, t+1)?$$

If the two closest points are R and R' then $\overrightarrow{RR'}$ is orhogonal to the direction of both lines. Now the direction of the first line is $(1,3,-1)$ and the direction of the second line is $(2,-3,1)$. A vector orthogonal to both is given by the cross product:

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 3 & -1 \\ 2 & -3 & 1 \end{vmatrix} = -3\hat{\jmath} - 9\hat{k}.$$

To simplify some of the algebra, let's take

$$\vec{n} = \hat{\jmath} + 3\hat{k},$$

which is parallel to the vector above, so that it is still orthogonal to both lines.

It follows that $\overrightarrow{RR'}$ is parallel to $\vec{n}$. Pick any two points P and P' on the two lines. Note that the length of the vector

$$\operatorname{proj}_{\vec{n}} \overline{P'}P,$$

is the distance between the two lines.

Now if we plug in $t = 0$ *to both lines we get*

$$P' = (-2, 1, 2) \qquad \textit{and} \qquad P = (-1, 2, 1).$$

So

$$\overline{P'}P = (1, 1, -1).$$

Then

$$\|\vec{n}\|^2 = 1^2 + 3^2 = 10 \qquad \textit{and} \qquad \vec{n} \cdot \overline{P'}P = -2.$$

It follows that

$$\operatorname{proj}_{\vec{n}} \overline{P'}P = \frac{-2}{10}(0, 1, 3) = \frac{-1}{5}(0, 1, 3).$$

and so the distance between the two lines is

$$\frac{1}{5}\sqrt{10}.$$

6. Cylindrical and spherical coordinates

Recall that in the plane one can use polar coordinates rather than Cartesian coordinates. In polar coordinates we specify a point using the distance r from the origin and the angle θ with the x-axis.

In polar coordinates, if a is a constant, then $r = a$ represents a circle of radius a, centred at the origin, and if α is a constant, then $\theta = \alpha$ represents a half ray, starting at the origin, making an angle α.

Suppose that $r = a\theta$, a a constant. This represents a spiral (in fact, the Archimedes spiral), starting at the origin. The smaller a, the 'tighter' the spiral.

By convention, if r is negative, we use this to mean that we point in the opposite direction to the direction given by θ. Also by convention, θ and $\theta + 2\pi$ represent the same point. We may require $r \geq 0$ and $0 \leq \theta < 2\pi$ and if we are not at the origin, this gives us unique polar coordinates.

It is straightforward to convert to and from polar coordinates:

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta, \end{aligned}$$

and

$$\begin{aligned} r^2 &= x^2 + y^2 \\ \tan\theta &= y/x. \end{aligned}$$

For example, what curve does the equation $r = 2a\cos\theta$ represent? Well if we multiply both sides by r, then we get

$$r^2 = 2ar\cos\theta.$$

So we get

$$x^2 + y^2 = 2ax.$$

Completing the square gives

$$(x-a)^2 + y^2 = a^2.$$

So this is a circle radius a, centred at $(a, 0)$. Polar coordinates can be very useful when we have circles or lines through the origin, or there is a lot of radially symmetry.

Instead of using the vectors $\hat{\imath}$ and $\hat{\jmath}$, in polar coordinates it makes sense to use orthogonal vectors of unit length, that move as the point moves (these are called **moving frames**). At a point P in the plane, with polar coordinates (r, θ), we use the vector $\hat{e}_r$ to denote the vector of unit length pointing in the radial direction:

$$\hat{e}_r = \cos\theta\hat{\imath} + \sin\theta\hat{\jmath}.$$

$\hat{e}_r$ points in the direction of increasing r. The vector $\hat{e}_\theta$ is a unit vector pointing in the direction of increasing θ. It is orthogonal to $\hat{e}_r$ and so in fact

$$\hat{e}_\theta = -\sin\theta\hat{\imath} + \cos\theta\hat{\jmath}.$$

We will call a set of unit vectors which are pairwise orthogonal, an **orthonormal** basis if we have two in the plane or three in space.

We want do something similar in space but now there are two choices beyond Cartesian coordinates. The first just takes polar coordinates in the xy-plane and throws in the extra variable z. So a point P is specified by three coordinates, (r, θ, z). r is the distance to the origin, of the projection P' of P down to the xy-plane, θ is the angle $\overline{OP'}$ makes with the x-axis, so that (r, θ) are just polar coordinates for the point P' in the xy-plane, and z is just the height of P from the xy-plane.

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z. \end{aligned}$$

Note that the locus $r = a$, specifies a cylinder in three space. For this reason we call (r, θ, z) cylindrical coordinates. The locus $\theta = \alpha$, specifies a half-plane which is vertical (if we allow $r < 0$ then we get the full vertical plane). The locus $z = a$ specifies a horizontal plane, parallel to the xy-plane.

The locus $z = ar$ specifies a half cone. At height one, the cone has radius a, so the larger a the more 'open' the cone.

The locus $z = a\theta$ is rather complicated. If we fix the angle, then we get a line of this height and this angle. The resulting surface is called a helicoid, and looks a little bit like a spiral staircase.

Again it is useful to write down an orthonormal coordinate frame. In this case there are three vectors, pointing in the direction of increasing r, increasing θ and increasing z:

$$\begin{aligned} \hat{e}_r &= \cos\theta\hat{\imath} + \sin\theta\hat{\jmath} \\ \hat{e}_\theta &= -\sin\theta\hat{\imath} + \cos\theta\hat{\jmath} \\ \hat{e}_z &= \hat{k}. \end{aligned}$$

The third coordinate system in space uses two angles and the distance to the origin, (ρ, θ, ϕ). ρ is the distance to the origin, θ is the angle made by the projection of P down to the xy-plane and ϕ is the angle the radius vector makes with the z-axis. Typically we use coordinates such that $0 \leq z \leq \infty$, $0 \leq \theta < 2\pi$ and $0 \leq \phi \leq \pi$. To get from spherical coordinates to Cartesian coordinates, we first convert to cylindrical coordinates,

$$\begin{aligned}
r &= \rho \sin \phi \\
\theta &= \theta \\
z &= \rho \cos \phi.
\end{aligned}$$

So, in Cartesian coordinates we get

$$\begin{aligned}
x &= \rho \sin \phi \cos \theta \\
y &= \rho \sin \phi \sin \theta \\
z &= \rho \cos \phi.
\end{aligned}$$

The locus $z = a$ represents a sphere of radius a, and for this reason we call (ρ, θ, ϕ) cylindrical coordinates. The locus $\phi = a$ represents a cone.

Example 6.1. *Describe the region*

$$x^2 + y^2 + z^2 \leq a^2 \qquad \text{and} \qquad x^2 + y^2 \geq z^2,$$

in spherical coordinates. The first region is the region inside the sphere of radius,

$$\rho \leq a.$$

The second is the region outside a cone. The surface of the cone is given by $z^2 = x^2 + y^2$. *Now one point on this cone is the point* $(1, 1, 1)$, *so that this a right-angled cone, and the region is given by*

$$\pi/4 \leq \phi \leq 3\pi/4.$$

So we can describe this region by the inequalities

$$\rho \leq a \qquad \text{and} \qquad \pi/4 \leq \phi \leq 3\pi/4.$$

Finally, let's write down the moving frame given by spherical coordinates, the one corresponding to increasing ρ, increasing θ and increasing ϕ.

$$\begin{aligned}
\hat{e}_\rho &= \frac{x\hat{\imath} + y\hat{\jmath} + z\hat{k}}{\sqrt{x^2 + y^2 + z^2}} \\
&= \sin \phi \cos \theta \hat{\imath} + \sin \phi \sin \theta \hat{\jmath} + \cos \phi \hat{k}. \\
\hat{e}_\theta &= -\sin \theta \hat{\imath} + \cos \theta \hat{\jmath}.
\end{aligned}$$

To calculate $\hat{e}_\phi$, we use the fact that it has unit length and it is orthogonal to both $\hat{e}_\rho$ and $\hat{e}_\theta$. We have

$$\begin{aligned}
\hat{e}_\phi &= \pm \hat{e}_\theta \times \hat{e}_\rho \\
&= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -\sin\theta & \cos\theta & 0 \\ \sin\phi\cos\theta & \sin\phi\sin\theta & \cos\phi \end{vmatrix} \\
&= \cos\phi\cos\theta\hat{\imath} + \sin\theta\cos\phi\hat{\jmath} - (sin^2\theta\sin\phi + \cos^2\theta\sin\phi)\hat{k} \\
&= \cos\phi\cos\theta\hat{\imath} + \cos\phi\sin\theta\hat{\jmath} - \sin\phi\hat{k}
\end{aligned}$$

Now when ϕ increases, z decreases. So we want the vector with negative z-component, which is exactly the last vector we wrote down.

5. n-DIMENSIONAL SPACE

Definition 5.1. *A* ***vector*** *in* $\mathbb{R}^n$ *is an* n*-tuple* $\vec{v} = (v_1, v_2, \dots, v_n)$*. The* ***zero vector*** $\vec{0} = (0, 0, \dots, 0)$*. Given two vectors* $\vec{v}$ *and* $\vec{w}$ *in* $\mathbb{R}^n$*, the sum* $\vec{v} + \vec{w}$ *is the vector* $(v_1 + w_1, v_2 + w_2, \dots, v_n + w_n)$*. If* λ *is a scalar, the* ***scalar product*** $\lambda\vec{v}$ *is the vector* $(\lambda v_1, \lambda v_2, \dots, \lambda v_n)$*.*

The sum and scalar product of vectors in $\mathbb{R}^n$ obey the same rules as the sum and scalar product in $\mathbb{R}^2$ and $\mathbb{R}^3$.

Definition 5.2. *Let* $\vec{v}$ *and* $\vec{w} \in \mathbb{R}^n$*. The* ***dot product*** *is the scalar*

$$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + \dots v_n w_n.$$

The ***norm*** *(or* ***length****) of* $\vec{v}$ *is the scalar*

$$\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}}.$$

The scalar product obeys the usual rules.

Example 5.3. *Suppose that* $\vec{v} = (1, 2, 3, 4)$ *and* $\vec{w} = (2, -1, 1, -1)$ *and* $\lambda = -2$*. Then*

$$\vec{v} + \vec{w} = (3, 1, 4, 3) \qquad \text{and} \qquad \lambda\vec{w} = (-4, 2, -2, 2).$$

We have

$$\vec{v} \cdot \vec{w} = 2 - 2 + 3 - 4 = -1.$$

The standard basis of $\mathbb{R}^n$ is the set of vectors,

$$e_1 = (1, 0, \dots, 0), \quad e_2 = (0, 1, \dots, 0), \quad e_1 = (0, 0, 1, \dots, 0), \quad \dots e_n = (0, 0, \dots, 1).$$

Note that if $\vec{v} = (v_1, v_2, \dots, v_n)$, then

$$\vec{v} = v_1 e_1 + v_2 e_2 + \dots + v_n e_n.$$

Let's adopt the (somewhat ad hoc) convention that $\vec{v}$ and $\vec{w}$ are parallel if and only if either $\vec{v}$ is a scalar multiple of $\vec{w}$, or vice-versa. Note that if both $\vec{v}$ and $\vec{w}$ are non-zero vectors, then $\vec{v}$ is a scalar multiple of $\vec{w}$ if and only if $\vec{w}$ is a scalar multiple of $\vec{v}$.

Theorem 5.4 (Cauchy-Schwarz-Bunjakowski)**.** *If* $\vec{v}$ *and* $\vec{w}$ *are two vectors in* $\mathbb{R}^n$*, then*

$$|\vec{v} \cdot \vec{w}| \leq \|v\|\|w\|,$$

with equality if and only if $\vec{v}$ *is parallel to* $\vec{w}$*.*

Proof. If either $\vec{v}$ or $\vec{w}$ is the zero vector, then there is nothing to prove. So we may assume that neither vector is the zero vector.

Let $\vec{u} = x\vec{v} + \vec{w}$, where x is a scalar. Then

$$0 \leq \vec{u} \cdot \vec{u} = (\vec{v} \cdot \vec{v})x^2 + 2(\vec{v} \cdot \vec{w})x + \vec{w} \cdot \vec{w} = ax^2 + bx + c.$$

So the quadratic function $f(x) = ax^2 + bx + c$ has at most one root. It follows that the discriminant is less than or equal to zero, with equality if and only if $f(x)$ has a root. So

$$4(\vec{v} \cdot \vec{w})^2 - 4\|\vec{v}\|^2\|\vec{w}\|^2 = b^2 - 4ac \leq 0.$$

Rearranging, gives

$$(\vec{v} \cdot \vec{w})^2 \leq \|\vec{v}\|^2\|\vec{w}\|^2.$$

Taking square roots, gives

$$|\vec{v} \cdot \vec{w}| \leq \|v\|\|w\|.$$

Now if we have equality here, then the discriminant must be equal to zero, in which case we may find a scalar λ such that the vector $\lambda\vec{v} + \vec{w}$ has zero length. But the only vector of length zero is the zero vector, so that $\lambda\vec{v} + \vec{w} = \vec{0}$. In other words, $\vec{w} = -\lambda\vec{v}$ and $\vec{v}$ and $\vec{w}$ are parallel. □

Definition 5.5. *If $\vec{v}$ and $\vec{w} \in \mathbb{R}^n$ are non-zero vectors, then the angle between them is the unique angle $0 \leq \theta \leq \pi$ such that*

$$\cos\theta = \frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\|\|\vec{w}\|}.$$

Note that the fraction is between -1 and 1, by (5.4), so this does makes sense, and we showed in (5.4) that the angle is 0 or π if and only if $\vec{v}$ and $\vec{w}$ are parallel.

Definition 5.6. *If $A = (a_{ij})$ and $B = (b_{ij})$ are two $m \times n$ matrices, then the **sum** $A + B$ is the $m \times n$ matrix $(a_{ij} + b_{ij})$. If λ is a scalar, then the **scalar multiple** λA is the $m \times n$ matrix (λa_{ij}).*

Example 5.7. *If*

$$A = \begin{pmatrix} 1 & -1 \\ 3 & -4 \end{pmatrix}, \qquad \text{and} \qquad B = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix},$$

then

$$A + B = \begin{pmatrix} 2 & 0 \\ 5 & -5 \end{pmatrix},$$

and

$$3A = \begin{pmatrix} 3 & -3 \\ 9 & -12 \end{pmatrix}.$$

Note that if we *flattened* A and B to $(1, -1, 3, -4)$ and $(2, 0, 5, -5)$ then the sum corresponds to the usual vector sum $(3, -1, 8, -9)$. Ditto for scalar multiplication.

Definition 5.8. *Suppose that $A = (a_{ij})$ is an $m \times n$ matrix and $B = (b_{ij})$ is an $n \times p$ matrix. The* ***product*** *$C = AB = (c_{ij})$ is the $m \times p$ matrix where*

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdots + a_{in}b_{nj} = \sum_{l=1}^{n} a_{il}b_{lj}.$$

In other words, the entry in the ith row and jth column of C is the dot product of the ith row of A and the jth column of B. This only makes sense because the ith row and the jth column are both vectors in $\mathbb{R}^n$.

Example 5.9. *Let*

$$A = \begin{pmatrix} 1 & -2 & 1 \\ 1 & -1 & 5 \end{pmatrix},$$

and

$$B = \begin{pmatrix} 2 & 1 \\ 1 & -4 \\ -1 & 1 \end{pmatrix}.$$

Then $C = AB$ has shape 2×2, and in fact

$$C = AB = \begin{pmatrix} -1 & 10 \\ -4 & 10 \end{pmatrix}.$$

Theorem 5.10. *Let A, B and C be three matrices, and let λ and μ be scalars.*

(1) *If A, B and C have the same shape, then $(A + B) + C = A + (B + C)$.*
(2) *If A and B have the same shape, then $A + B = B + A$.*
(3) *If A and B have the same shape, then $\lambda(A + B) = \lambda A + \lambda B$.*
(4) *If Z is the zero matrix with the same shape as A, then $Z + A = A + Z$.*
(5) *$\lambda(\mu A) = (\lambda\mu)A$.*
(6) *$(\lambda + \mu)A = \lambda A + \mu A$.*
(7) *If I_n is the matrix with ones on the diagonal and zeroes everywhere else and A has shape $m \times n$, then $AI_n = A$ and $I_m A = A$.*
(8) *If A has shape $m \times n$ and B has shape $n \times p$ and C has shape $p \times q$, then $A(BC) = (AB)C$.*
(9) *If A has shape $m \times n$ and B and C have the same shape $n \times p$, then $A(B + C) = AB + AC$.*
(10) *If A and B have the same shape $m \times n$ and C has shape $n \times p$, then $(A + B)C = AC + BC$.*

Example 5.11. *Note however that $AB \neq BA$ in general. For example if A has shape 1×3 and B has shape 3×2, then it makes sense to multiply A and B but it does not make sense to multiply B and A.*

In fact even if it makes sense to multiply A and B and B and A, the two products might not even have the same shape. For example, if

$$A = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix},$$

and

$$B = \begin{pmatrix} 2 & -1 & 3 \end{pmatrix},$$

then AB has shape 3×3,

$$AB = \begin{pmatrix} 2 & -1 & 3 \\ 4 & -2 & 6 \\ -2 & 1 & -3 \end{pmatrix},$$

but BA has shape 1×1,

$$BA = (2 - 2 - 3) = (-3).$$

But even both products AB and BA make sense, and they have the same shape, the products still don't have to be equal. Suppose

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \text{and} \qquad B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Then AB and BA are both 2×2 matrices. But

$$AB = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \qquad \text{and} \qquad BA = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

One can also define determinants for $n \times n$ matrices. It is probably easiest to explain the general rule using an example:

$$\begin{vmatrix} 1 & 0 & 0 & 2 \\ 2 & 0 & 1 & -1 \\ 1 & -2 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 1 & -1 \\ -2 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} - 2 \begin{vmatrix} 2 & 0 & 1 \\ 1 & -2 & 1 \\ 0 & 1 & 0 \end{vmatrix}.$$

Notice that we as expand about the top row, the sign alternates $+-+-$, so that the last term comes with a minus sign.

Finally, we try to explain the real meaning of a matrix. Let

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Given A, we can construct a function

$$f \colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2,$$

by the rule

$$f(\vec{v}) = A\vec{v}.$$

If $\vec{v} = (x, y)$, then

$$A\vec{v} = (x + y, y).$$

Here I cheat a little, and write row vectors instead of column vectors. Geometrically this is called a *shear*; it leaves the y-axis alone but one goes further along the x-axis according to the value of y. If

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

the resulting function sends (x, y) to $(ax + by, cx + dy)$. In fact the functions one gets this way are always linear. If

$$A = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix},$$

then $f(x, y) = (2x - y)$, and this has the result of scaling by a factor of 2 in the x-direction and reflects in the y-direction.

In general if A is an $m \times n$ matrix, we get a function

$$f\colon \mathbb{R}^n \longrightarrow \mathbb{R}^m,$$

using the same rule, $f(\vec{v}) = A\vec{v}$. If B is an $n \times p$ matrix, then we get a function

$$g\colon \mathbb{R}^p \longrightarrow \mathbb{R}^n,$$

by the rule $g(\vec{w}) = B\vec{w}$. Note that we can compose the functions f and g, to get a function

$$f \circ g\colon \mathbb{R}^p \longrightarrow \mathbb{R}^m.$$

First we apply g to $\vec{w}$ to get a vector $\vec{v}$ in $\mathbb{R}^n$ and then we apply f to $\vec{v}$ to get a vector in $\mathbb{R}^m$. The composite function $f \circ g$ is given by the rule $(f \circ g)(\vec{w}) = (AB)\vec{w}$. In other words, matrix multiplication is chosen so that it represents composition of functions.

As soon as one realises this, many aspects of matrix multiplication become far less mysterious. For example, composition of functions is not commutative, for example

$$\sin 2x \neq 2 \sin x,$$

and this is why $AB \neq BA$ in general. Note that it is not hard to check that composition of functions is associative,

$$f \circ (g \circ h) = (f \circ g) \circ h.$$

This is the easiest way to check that matrix multiplication is associative, that is, (8) of (5.10).

Functions given by matrices are obviously very special. Note that if $f(\vec{v}) = A\vec{v}$, then

$$f(\vec{v} + \vec{w}) = A(\vec{v} + \vec{w}) = A\vec{v} + A\vec{w} = f(\vec{v}) + f(\vec{w}),$$

and

$$f(\lambda\vec{v}) = A(\lambda\vec{v}) = \lambda(A\vec{v}) = \lambda f(\vec{v}).$$

Any function which respects both addition of vectors and scalar multiplication is called **linear** and it is precisely the linear functions which are given by matrices. In fact if $e_1, e_2, \dots, e_n$ and $f_1, f_2, \dots, f_m$ are standard bases for $\mathbb{R}^n$ and $\mathbb{R}^m$, and f is linear, then

$$f(e_j) = \sum a_{ij} f_i,$$

for some scalars a_{ij}, since $f(e_j)$ is a vector in $\mathbb{R}^m$ and any vector in $\mathbb{R}^m$ is a linear combinaton of the standard basis vectors $f_1, f_2, \dots, f_m$. If we put $A = (a_{ij})$ then one can check that f is the function

$$f(\vec{v}) = A\vec{v}.$$