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Vector Calculus

Fifth Edition

Chapter 3:
High-Order Derivatives: Maxima and Minima

3.5 The Implicit Function Theorem

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Key Points in this Section.

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1. **One-Variable Version.** If $f : (a, b) \to \mathbb{R}$ is C^1 and if $f'(x_0) \neq 0$, then locally near x_0, f has a C^1 inverse function $x = f^{-1}(y)$. If $f'(x) > 0$ on all of (a, b) and is continuous on $[a, b]$, then f has an inverse defined on $[f(a), f(b)]$. This result is used in one-variable calculus to define, for example, the log function as the inverse of $f(x) = e^x$ and $\sin^{-1}$ as the inverse of $f(x) = \sin x$.

2. **Special n-variable Version.** If $F : \mathbb{R}^{n+1} \to \mathbb{R}$ is C^1 and at a point $(\mathbf{x}_0, z) \in \mathbb{R}^{n+1}$, $F(\mathbf{x}_0, z) = 0$ and $\frac{\partial F}{\partial z}(\mathbf{x}_0, z_0) \neq 0$, then locally near $(\mathbf{x}_0, z_0)$ there is a unique solution $z = g(\mathbf{x})$ of the equation $F(\mathbf{x}, z) = 0$. We say that $F(\mathbf{x}, z) = 0$ ***implicitly defines*** z as a function of $\mathbf{x} = (x_1, \ldots, x_n)$.

3. The partial derivatives are computed by ***implicit differentiation***:

$$\frac{\partial F}{\partial x_i} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x_i} = 0,$$

so

$$\frac{\partial z}{\partial x_i} = -\frac{\partial F/\partial x_i}{\partial F/\partial z}$$

4. The special implicit function theorem guarantees that if $\nabla g(\mathbf{x}_0) \neq \mathbf{0}$, then the level set $g = c$ is a smooth surface near $\mathbf{x}_0$, a fact needed in the proof of the Lagrange multiplier theorem.

5. The general implicit function theorem deals with solving m equations

$$\begin{aligned} F_1(x_1, \ldots, x_n, z_1, \ldots, z_m) &= 0 \\ &\vdots \\ F_m(x_1, \ldots, x_n, z_1, \ldots, z_m) &= 0 \end{aligned}$$

for m unknowns $\mathbf{z} = (z_1, \ldots, z_m)$. If

$$\begin{vmatrix} \dfrac{\partial F_1}{\partial z_1} & \cdots & \dfrac{\partial F_1}{\partial z_m} \\ \vdots & & \vdots \\ \dfrac{\partial F_m}{\partial z_1} & \cdots & \dfrac{\partial F_m}{\partial z_m} \end{vmatrix} \neq 0$$

at $(\mathbf{x}_0, \mathbf{z}_0)$, then these equations define $(z_1, \ldots, z_m)$ as functions of $(x_1, \ldots, x_n)$. The partial derivatives $\partial z_i / \partial x_j$ may again be computed by using implicit differentiation.

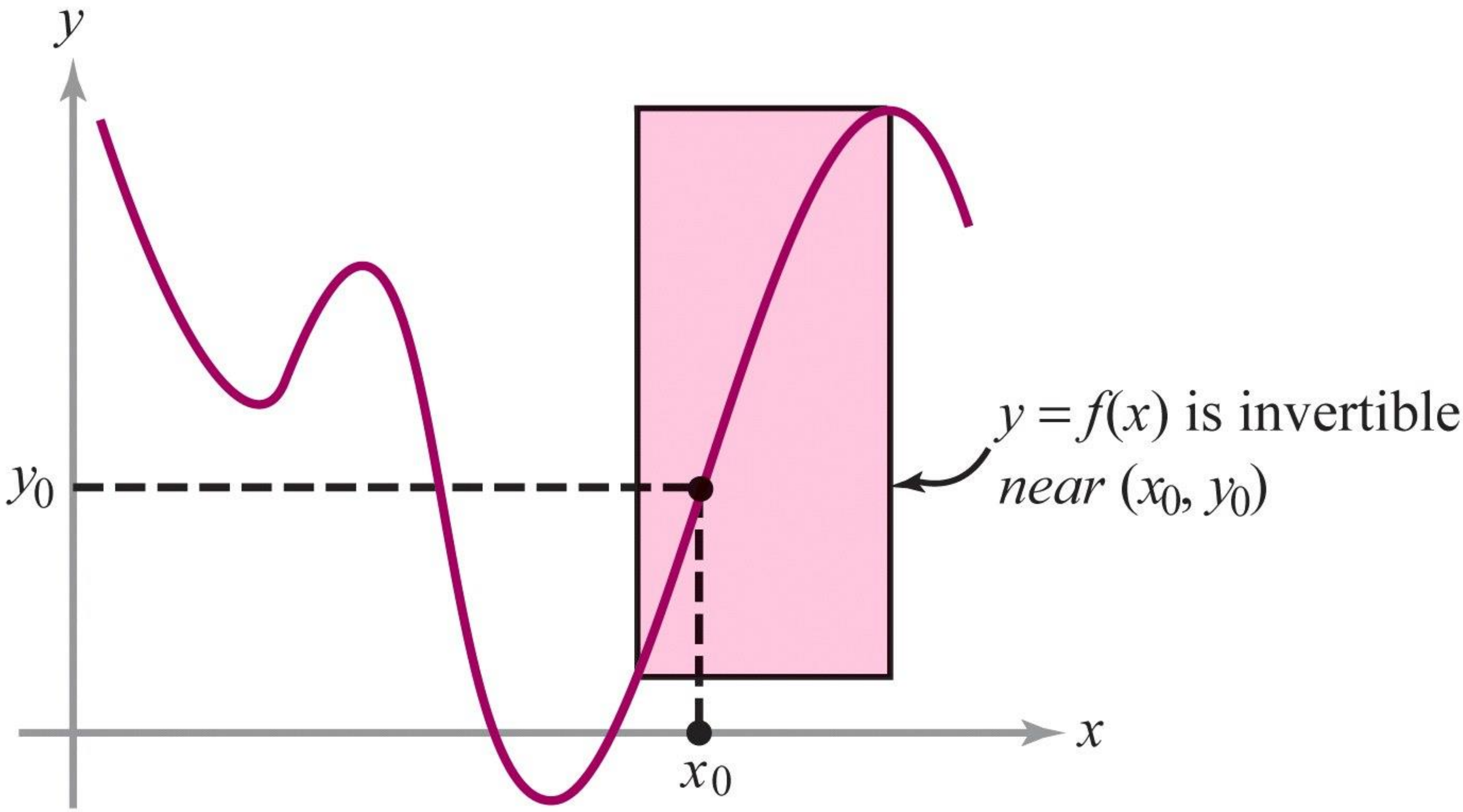
6. The ***Inverse Function Theorem***, which is a special case of the general implicit function theorem, states that a system

$$\begin{aligned} f_1(x_1,\ldots,x_n) &= y_1 \\ &\vdots \\ f_n(x_1,\ldots,x_n) &= y_n \end{aligned}$$

where $f=(f_1,\ldots,f_n)$ is a C^1 mapping, can be solved for the x_i's as functions of $(y_1,\ldots,y_n)$ near a given point $\mathbf{x}_0$, $\mathbf{y}_0 = f(\mathbf{x}_0)$ provided the ***Jacobian determinant***

$$\left.\frac{\partial(f_1,\ldots,f_n)}{\partial(x_1,\ldots,x_n)}\right|_{\mathbf{x}=\mathbf{x}_0} = J(f)(\mathbf{x}_0) = \begin{vmatrix} \dfrac{\partial f_1}{\partial x_1} & \cdots & \dfrac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \dfrac{\partial f_n}{\partial x_1} & \cdots & \dfrac{\partial f_n}{\partial x_n} \end{vmatrix}$$

(where partials are evaluated at $\mathbf{x}_0$) is non-zero. Again the partial derivatives $\partial x_i/\partial y_j$ can be determined by implicit differentiation.



y
y_0
$y = f(x)$ is invertible
near (x_0, y_0)
x
x_0

THEOREM 11: Special Implicit Function Theorem Suppose that $F\colon \mathbb{R}^{n+1} \to \mathbb{R}$ has continuous partial derivatives. Denoting points in $\mathbb{R}^{n+1}$ by $(\mathbf{x}, z)$, where $\mathbf{x} \in \mathbb{R}^n$ and $z \in \mathbb{R}$, assume that $(\mathbf{x}_0, z_0)$ satisfies

$$F(\mathbf{x}_0, z_0) = 0 \qquad \text{and} \qquad \frac{\partial F}{\partial z}(\mathbf{x}_0, z_0) \neq 0.$$

Then there is a ball U containing $\mathbf{x}_0$ in $\mathbb{R}^n$ and a neighborhood V of z_0 in $\mathbb{R}$ such that there is a unique function $z = g(\mathbf{x})$ defined for $\mathbf{x}$ in U and z in V that satisfies

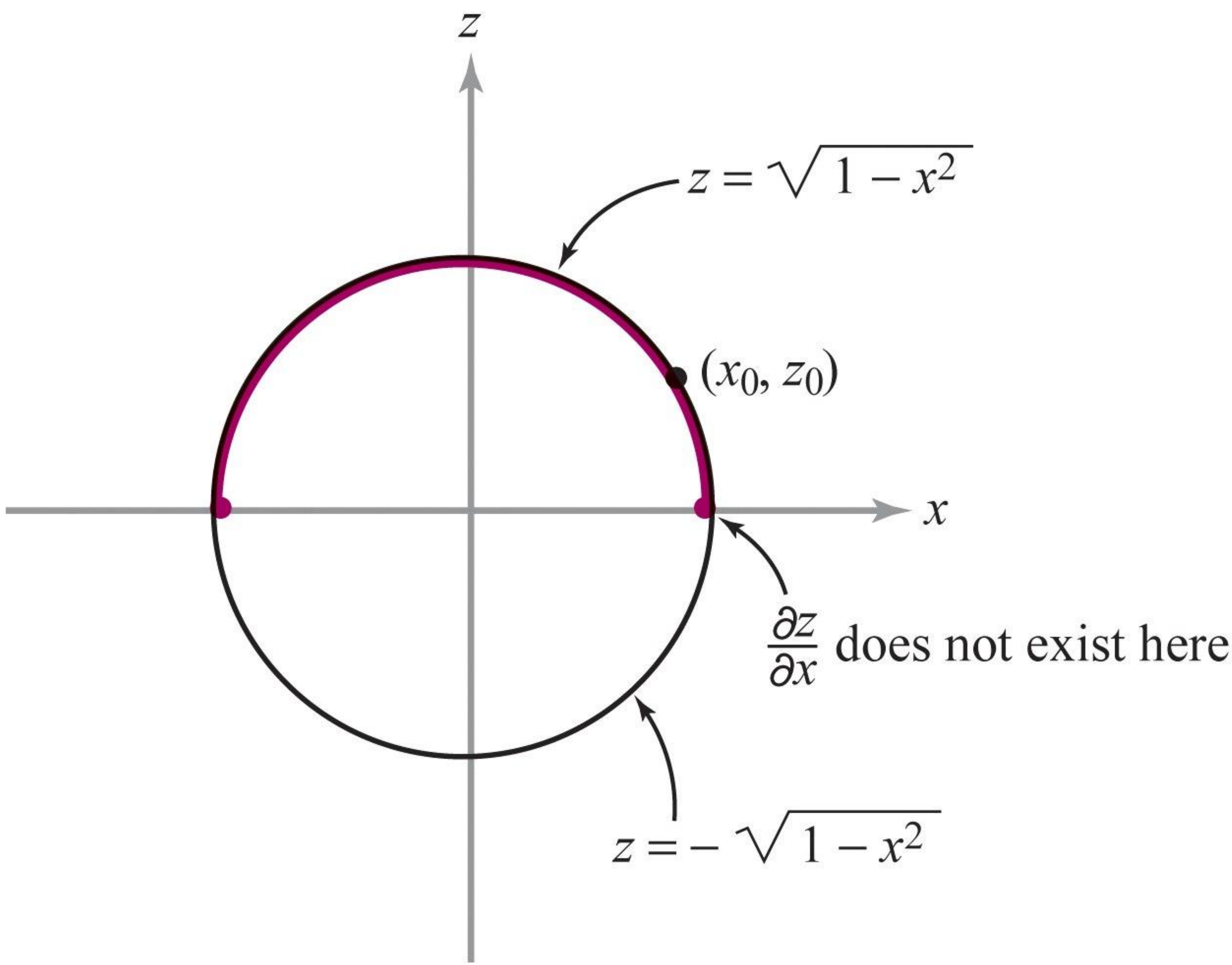
$$F(\mathbf{x}, g(\mathbf{x})) = 0.$$

Moreover, if $\mathbf{x}$ in U and z in V satisfy $F(\mathbf{x}, z) = 0$, then $z = g(\mathbf{x})$. Finally, $z = g(\mathbf{x})$ is continuously differentiable, with the derivative given by

$$\mathbf{D}g(\mathbf{x}) = -\frac{1}{\dfrac{\partial F}{\partial z}(\mathbf{x}, z)}\, \mathbf{D}_{\mathbf{x}} F(\mathbf{x}, z)\bigg|_{z=g(\mathbf{x})},$$

where $\mathbf{D}_{\mathbf{x}} F$ denotes the (partial) derivative of F with respect to the variable $\mathbf{x}$, that is, we have $\mathbf{D}_{\mathbf{x}} F = [\partial F/\partial x_1, \dots, \partial F/\partial x_n]$; in other words,

$$\frac{\partial g}{\partial x_i} = -\frac{\partial F/\partial x_i}{\partial F/\partial z}, \qquad i = 1, \dots, n. \tag{1}$$



z
$z = \sqrt{1 - x^2}$
(x_0, z_0)
x
$\frac{\partial z}{\partial x}$ does not exist here
$z = -\sqrt{1 - x^2}$

THEOREM 12: General Implicit Function Theorem

The general implicit function theorem deals with solving m equations

$$\begin{aligned} F_1(x_1, \ldots, x_n, z_1, \ldots, z_m) &= 0 \\ \vdots \quad & \quad \vdots \\ F_m(x_1, \ldots, x_n, z_1, \ldots, z_m) &= 0 \end{aligned}$$

for m unknowns $\mathbf{z} = (z_1, \ldots, z_m)$. If

$$\begin{vmatrix} \dfrac{\partial F_1}{\partial z_1} & \cdots & \dfrac{\partial F_1}{\partial z_m} \\ \vdots & & \vdots \\ \dfrac{\partial F_m}{\partial z_1} & \cdots & \dfrac{\partial F_m}{\partial z_m} \end{vmatrix} \neq 0$$

at $(\mathbf{x}_0, \mathbf{z}_0)$, then these equations define $(z_1, \ldots, z_m)$ as functions of $(x_1, \ldots, x_n)$. The partial derivatives $\partial z_i / \partial x_j$ may again be computed by using implicit differentiation.

THEOREM 13: Inverse Function Theorem

The ***Inverse Function Theorem***, which is a special case of the general implicit function theorem, states that a system

$$\begin{aligned} f_1(x_1, \ldots, x_n) &= y_1 \\ \vdots \quad & \quad \vdots \\ f_n(x_1, \ldots, x_n) &= y_n \end{aligned}$$

where $f = (f_1, \ldots, f_n)$ is a C^1 mapping, can be solved for the x_i's as functions of $(y_1, \ldots, y_n)$ near a given point $\mathbf{x}_0$, $\mathbf{y}_0 = f(\mathbf{x}_0)$ provided the ***Jacobian determinant***

$$\left.\frac{\partial(f_1, \ldots, f_n)}{\partial(x_1, \ldots, x_n)}\right|_{\mathbf{x}=\mathbf{x}_0} = J(f)(\mathbf{x}_0) = \begin{vmatrix} \dfrac{\partial f_1}{\partial x_1} & \cdots & \dfrac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \dfrac{\partial f_n}{\partial x_1} & \cdots & \dfrac{\partial f_n}{\partial x_n} \end{vmatrix}$$

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