Exercises Quantum Physics (GEMF) – Mathematical concepts

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1. Given the complex number $z_\kappa = a_\kappa + b_\kappa i$ and its complex conjugate $z_\kappa^* = a_\kappa - b_\kappa i$, show that

   (a) zz^* is real

   (b) $z + z^*$ is real

   (c) $(z_1 + z_2)^* = z_1^* + z_2^*$

   (d) $(z_1 z_2)^* = z_1^* z_2^*$

   (e) $|z_1 z_2|^2 = |z_1|^2 |z_2|^2$

2. Use the state vectors $|\psi\rangle = \begin{pmatrix} 2 \\ -i \end{pmatrix}$ and $|\phi\rangle = \begin{pmatrix} i \\ 0 \end{pmatrix}$ to demonstrate that $\langle\psi|\phi\rangle = \big(\langle\phi|\psi\rangle\big)^*$

3. Show that the Pauli matrices are Hermitian.

4. The eigenvectors of the Pauli matrices $\hat{\sigma}_{x,y,z}$ are

$$|\psi_+\rangle_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad |\psi_+\rangle_y = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} \qquad |\psi_+\rangle_z = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$|\psi_-\rangle_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad |\psi_-\rangle_y = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix} \qquad |\psi_-\rangle_z = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

   Check that the eigenvalues of the Pauli matrices are indeed real as expected for Hermitian operators.

5. Demonstrate that $\big(\hat{\sigma}_y|\uparrow\rangle\big)^\dagger = \langle\uparrow|\hat{\sigma}_y^\dagger$

6. Consider the states $|\psi\rangle = 9i|\phi_1\rangle + 2|\phi_2\rangle$ and $|\chi\rangle = \frac{-i}{\sqrt{2}}|\phi_1\rangle + \frac{1}{\sqrt{2}}|\phi_2\rangle$. $|\phi_1\rangle$ and $|\phi_2\rangle$ form an orthogonal and complete basis of the Hilbert space.

   (a) Are the operators $|\psi\rangle\langle\chi|$ and $|\chi\rangle\langle\psi|$ equal?

   (b) Find the Hermitian conjugates of $|\psi\rangle$, $|\chi\rangle$, $|\psi\rangle\langle\chi|$ and $|\chi\rangle\langle\psi|$.

   (c) Calculate $|\psi\rangle\langle\psi|$, $|\chi\rangle\langle\chi|$.

   (d) Are $|\psi\rangle$ and $|\chi\rangle$ normalized?

(e) Check the validity of the Schwartz inequality for $|\psi\rangle$ and $|\chi\rangle$

7. Which of these operators is Hermitian: $\hat{X} = x$, d/dx, and id/dx?

8. Show that $\hat{P}_x = -\hat{P}_x^*$ and $\hat{P}_x = \hat{P}_x^\dagger$

9. Find the Hermitian conjugate of $\hat{X}\frac{d}{dx}$ (Remember $(\hat{A}\hat{B})^\dagger = \hat{B}^\dagger\hat{A}^\dagger$)

10. Demonstrate that $\hat{L}_x = -i\hbar(\hat{Y}\frac{\partial}{\partial z} - \hat{Z}\frac{\partial}{\partial y}) = \hat{L}_x^\dagger$

11. Calculate the commutator of $\hat{X}$ and $\frac{\partial^2}{\partial x^2}$

12. Given the operator $\hat{A} = -\mathcal{E}\frac{d^2}{dx^2} + 16\mathcal{E}\hat{X}^2$ and the wave function $\psi(x) = Ae^{-2x^2}$,

   (a) calculate the normalization constant A of $\psi(x)$

   (b) confirm that $\psi(x)$ is an eigenfunction of $\hat{A}$ and determine the eigenvalue

   (c) calculate the probability to measure the position of the particle between $-\infty$ and 0.

   (d) find the eigenvalue of the function $\phi(x) = 2x\psi(x)$

   Use the following integrals: $\displaystyle\int_{-\infty}^{\infty} e^{-ax^2}dx = \sqrt{\frac{\pi}{a}}$ $\qquad$ $\displaystyle\int_0^{\infty} e^{-ax^2}dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}$