Combinatorics and Probability

Part V: Random variables

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Random variable

Informally, a **random variable** is quantity or object which depends on random events.

More formally, given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ a **random variable** is a *measurable function* χ from the sample space Ω to a measurable space (in our case, the real numbers).

Note: A measurable real function is a function $\chi : \Omega \longrightarrow \mathbb{R}$ such that the pre-image of a measurable subset $S \subseteq \mathbb{R}$ is a measurable subset of Ω, i.e. $\chi^{-1}(S) \in \mathcal{F}$.

If the range of χ is finite or countable then χ is a **discrete random variable**, otherwise it is a **continuous random variable**.

Examples of random variables

Example 1: The action of casting a (fair) die is a random experiment with sample space Ω_1, consisting of the faces of the die, represented in the Figure below

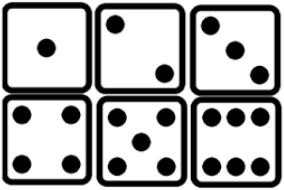

Figure: Faces of a die

The function $\chi_1 : \Omega_1 \longrightarrow \mathbb{R}$ that assigns a number in $\{1, \ldots, 6\}$ to each face of the die (in the obvious manner) is a (discrete) random variable.

Examples of random variables

Example 2: Let Ω_2 be the set of all students at the Sescelades Campus, URV. The function χ_2 that establishes the correspondence between one student and his/her height (in centimetres) is a discrete random variable. The same for the weight in kilograms, etc.

Example 3: Let Ω_3 be the set of all citizens of Tarragona, and let χ_3 be the same function as before, but given not in integer values (centimetres), but with 'infinite' accuracy. Then χ_3 can be regarded as a continuous random variable.

Example 4: In the dart game let $\Omega_4 = \mathbb{R}^2$, and let χ_4 the function that assigns to each measurable region $D \subseteq \mathbb{R}^2$ a real number, interpreted as the probability that a dart lands in D. Then χ_4 is a continuous random variable.

Discrete random variables – Probability mass function

Recall that a discrete real random variable is a function $\chi : \Omega \longrightarrow S$, where S is a discrete subset of $\mathbb{R}$.

Probability mass function (PMF):

$$\begin{aligned} p_\chi : S &\longrightarrow [0;1] \\ p_\chi(s) &= \mathbb{P}(\chi = s) \\ &= \mathbb{P}\Big(\{\omega \in \Omega : \chi(\omega) = s\}\Big) \end{aligned}$$

Example 1, revisited: In the experiment of rolling a fair die, all the numbers in $\{1, \ldots, 6\}$ are equally likely, hence

$$p_{\chi_1}(k) = \frac{1}{6} \text{ for all } k \in \{1, \ldots, 6\}.$$

Probability mass function - Another example

Example 2, revisited: Ω_2 is the set of all students at the Sescelades Campus, URV. Our random variable is the function χ_2 that establishes the correspondence between one student and his/her height in cm.

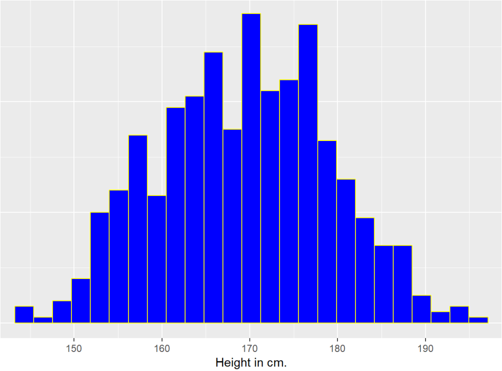

Figure: Histogram of height distribution at Sescelades

Probability mass function - Another example

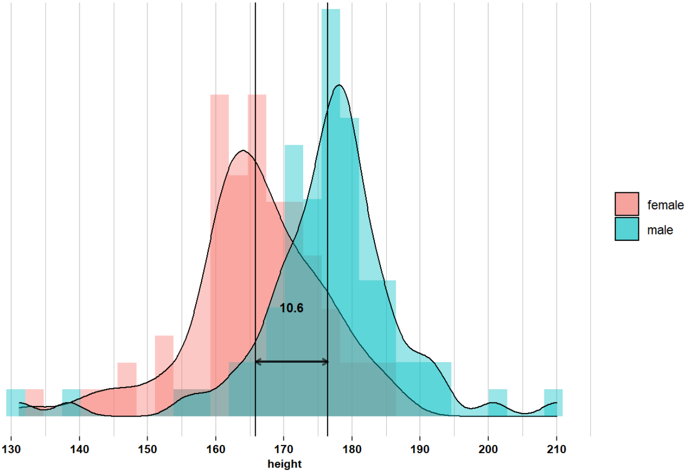

Figure: Histogram of height distribution at Sescelades by gender

Properties of probability mass functions

$$\forall s \in S \left(p_\chi(s) \geq 0 \right),$$
$$\sum_{s \in S} p_\chi(s) = 1$$

Cumulative probability distribution function (CDF):

$$F_\chi : S \longrightarrow [0;1]$$
$$F_\chi(s) = \mathbb{P}(\chi \leq s)$$
$$= \sum_{\substack{t \in S \\ t \leq s}} p_\chi(t)$$

Example discrete CDF

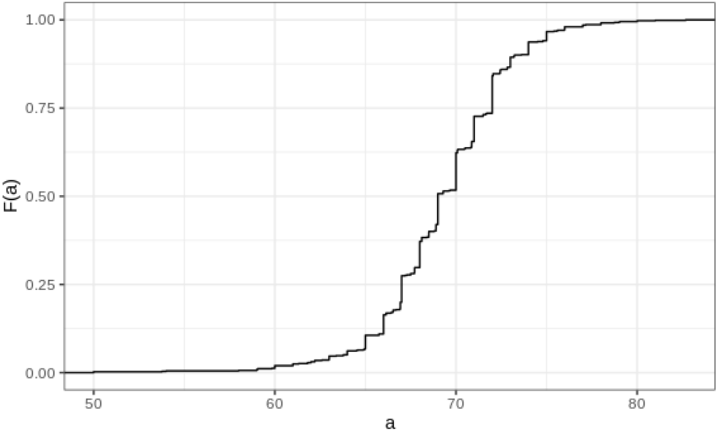

Figure: Cumulative distribution of heights at Sescelades (in inches)

Continuous random variables – Probability density function

- Recall that a discrete real random variable is a function $\chi : \Omega \longrightarrow S$, where S is a discrete subset of $\mathbb{R}$.
- If we consider subsets $S \in \mathbb{R}$ with ever increasing accuracy, our PMF resembles more and more a *continuous function*.
- In the limit, when S becomes a continuous subset of $\mathbb{R}$, our PMF becomes a continuous function called **probability density function**, or **PDF**.

Probability density function - Example

Example 3, revisited: Ω_3 is the set of all citizens of Tarragona, and χ_3 assignes every person his/her height in cm with infinite accuracy.

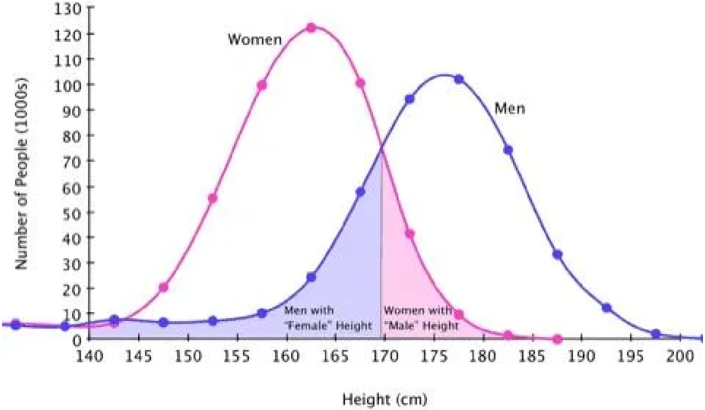

Figure: Height distribution in Tarragona by gender

Cumulative probability distribution function of heights

Example 3, revisited: Ω_3 is the set of all citizens of Tarragona, and χ_3 assignes every person his/her height in cm with infinite accuracy.

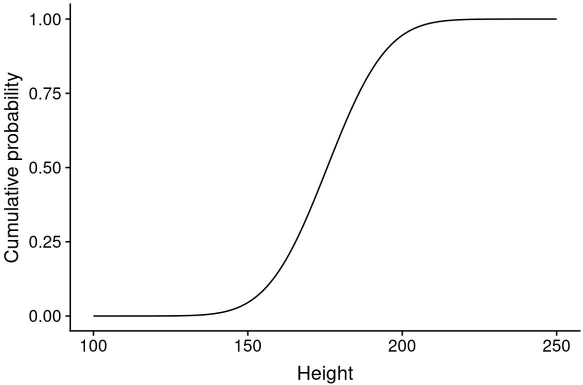

Figure: Cumulative distribution of heights in Tarragona

Probability density function - Another example

Example 4, revisited: In the dart game $\Omega_4 = \mathbb{R}^2$, and χ_4 is the function that assigns to each measurable region $D \subseteq \mathbb{R}^2$ the probability that a dart lands in D (bi-variate PDF).

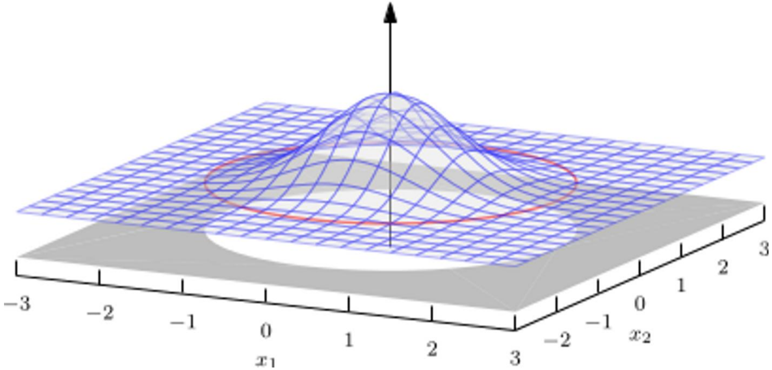

Figure: Probability density function of the dart game

Continuous random variables – Cumulative probability distribution and probability density function

Recall that a continuous real random variable is a function $\chi : \Omega \longrightarrow S$, where S is a continuous subset of $\mathbb{R}$.

Cumulative probability distribution function (CDF):

$$F_\chi : S \longrightarrow [0;1]$$
$$F_\chi(s) = \mathbb{P}(\chi \leq s)$$

Probability density function (PDF):

$$f_\chi : S \longrightarrow [0;1]$$
$$f_\chi(s) = F'_\chi(s)$$

Some properties of the PDF and the CDF

$$f_\chi(s) = F'_\chi(s)$$

$$F_\chi(s) = \int_{-\infty}^{s} f_\chi(t)\, dt$$

$$\int_{-\infty}^{\infty} f_\chi(t)\, dt = 1$$

$$\mathbb{P}(a \leq \chi \leq b) = \int_{a}^{b} f_\chi(t)\, dt$$

$$\mathbb{P}(\chi = a) = \int_{a}^{a} f_\chi(t)\, dt = 0$$

Remark: Note that the PDF cannot be interpreted exactly as the PMF.

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Expected value

Discrete random variable χ, with PMF $p_\chi : S \longrightarrow [0;1]$:

$$\mathbb{E}(\chi) = \sum_{s \in S} s\, p_\chi(s)$$

Continuous random variable χ, with PDF $f_\chi : S \longrightarrow [0;1]$:

$$\mathbb{E}(\chi) = \int_{-\infty}^{\infty} t\, f_\chi(t)\, dt$$

Also called *mean value*, *expectation* or *expectancy*.

Expected value – Examples

Example 1, revisited: In the experiment of rolling a fair die the expected value is

$$\mathbb{E}(\chi_1) = \sum_{n=1}^{6} n\,\frac{1}{6} = \frac{1}{6}\sum_{n=1}^{6} n = \frac{21}{6} = \frac{7}{2} = 3.5$$

Example 5: In the experiment of tossing a fair coin let χ_5 be the random variable that takes the value 1 if the result is HEADS, and 0 if the result is TAILS. Then $p_{\chi_5} : \{0,1\} \longrightarrow [0;1]$, with $p_{\chi_5}(0) = p_{\chi_5}(1) = \frac{1}{2}$.

$$\mathbb{E}(\chi_5) = 0 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = \frac{1}{2} = 0.5$$

Expected value – Some remarks

- The previous examples are examples of random variables with *uniform distribution*.
- The expectation is a measure of **central tendency.**
- From the previous examples we can see that the expectation does not always correspond to one of the possible values of the random variable, and it does not always provide an accurate picture of the variable's behaviour.
- In order to get a more accurate picture of the random variable the expectation has to be accompanied by other descriptors, such as the *variance*, that we will see later.

Expected value – Further examples

Example 6: Suppose we toss a fair coin three times, and let the random variable $\chi_6 : \{\text{HEADS}, \text{TAILS}\}^3 \longrightarrow \{0, \ldots, 3\}$ be defined as the number of heads. Then $p_{\chi_6} : \{0, \ldots, 3\} \longrightarrow [0; 1]$ is defined as follows

$$p_{\chi_6}(s) = \begin{cases} \frac{1}{8} & \text{if } s = 0 \\ \frac{3}{8} & \text{if } s = 1 \\ \frac{3}{8} & \text{if } s = 2 \\ \frac{1}{8} & \text{if } s = 3 \end{cases}$$

and the expectation of χ_6 is

$$\mathbb{E}(\chi_6) = 0 \cdot \frac{1}{8} + 1 \cdot \frac{3}{8} + 2 \cdot \frac{3}{8} + 3 \cdot \frac{1}{8} = \frac{12}{8} = \frac{3}{2} = 1.5$$

Expected value – Further examples

Example 7 (Generalization of Example 6): Now suppose we flip a (not necessarily fair) coin n consecutive times, and let the random variable $\chi_7 : \{\text{HEADS}, \text{TAILS}\}^n \longrightarrow \{0, \ldots, n\}$ be defined again as the total number of heads. We can assume that the probability of getting HEADS in one toss is p (and hence the probability of getting TAILS is $q = 1 - p$).
Then the probability of getting exactly k heads is

$$\mathbb{P}(\chi_7 = k) = p_{\chi_7}(k) = \binom{n}{k} p^k q^{n-k}$$

and the expectation of χ_7 is

Expectation – Another example

Example 8 (Generalization of Example 1): Now suppose we toss two fair dice. As we saw before, the number of combinations that yield a particular value k is given by the generating function

$$[f(z)]^2 = z^2(z+1)^2(z^2-z+1)^2(z^2+z+1)^2$$
$$= z^{12} + 2z^{11} + 3z^{10} + 4z^9 + 5z^8 + 6z^7 + 5z^6$$
$$+ 4z^5 + 3z^4 + 2z^3 + z^2,$$

Therefore, the probability of obtaining a particular value k is

$$\mathbb{P}(\chi_1 + \chi_2 = k) = \frac{1}{36}[z^k]\,(f(z))^2\,,$$

where χ_1 and χ_2 are the random variables associated with each die.

Example 8 (continued)

We define the new random variable $\chi = \chi_1 + \chi_2$. The PMF of χ is

$$\mathbb{P}(\chi = k) = \begin{cases} \frac{1}{36} & \text{if } k = 2 \text{ or } k = 12 \\ \frac{2}{36} = \frac{1}{18} & \text{if } k = 3 \text{ or } k = 11 \\ \frac{3}{36} = \frac{1}{12} & \text{if } k = 4 \text{ or } k = 10 \\ \frac{4}{36} = \frac{1}{9} & \text{if } k = 5 \text{ or } k = 9 \\ \frac{5}{36} & \text{if } k = 6 \text{ or } k = 8 \\ \frac{6}{36} = \frac{1}{6} & \text{if } k = 7 \\ 0 \text{ otherwise.} \end{cases}$$

In more compact form

$$\mathbb{P}(\chi = k) = \frac{1}{36} \begin{cases} (k-1) \text{ for } 2 \leq k \leq 7 \\ (13-k) \text{ for } 8 \leq k \leq 12 \\ 0 \text{ otherwise.} \end{cases}$$

Example 8 (continued)

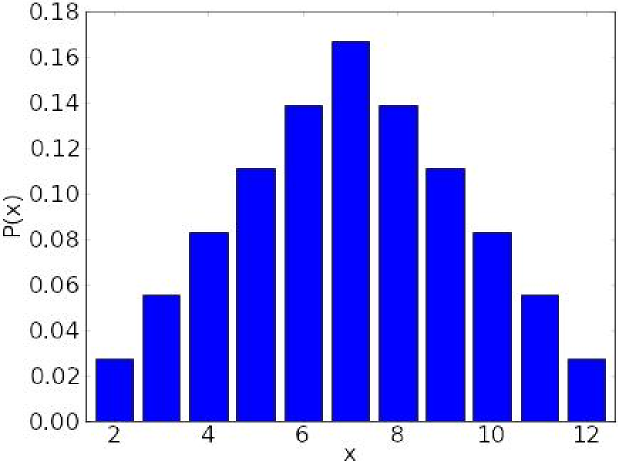

Figure: PMF of $\chi = \chi_1 + \chi_2$

Example 8 (continued)

Hence, the expectation of χ is

$$\begin{aligned}
\mathbb{E}(\chi) &= 2\cdot\frac{1}{36}+3\cdot\frac{2}{36}+4\cdot\frac{3}{36}+5\cdot\frac{4}{36}+6\cdot\frac{5}{36}\\
&\quad+7\cdot\frac{6}{36}+8\cdot\frac{5}{36}+9\cdot\frac{4}{36}+10\cdot\frac{3}{36}+11\cdot\frac{2}{36}+12\cdot\frac{1}{36}\\
&=\frac{2}{36}+\frac{6}{36}+\frac{12}{36}+\frac{20}{36}+\frac{30}{36}+\frac{42}{36}+\frac{40}{36}+\frac{36}{36}+\frac{30}{36}+\frac{22}{36}+\frac{12}{36}\\
&=\frac{252}{36}=7
\end{aligned}$$

This result is not surprising, given the symmetry of the PMF.

Properties of the expectation

Proposition

Let χ be a (real) random variable, and $\alpha, \beta \in \mathbb{R}$ two constants. Then

$$\mathbb{E}(\alpha\chi + \beta) = \alpha\mathbb{E}(\chi) + \beta$$

Remark: In general, if g is any real-valued function we have $\mathbb{E}(g(\chi)) \neq g(\mathbb{E}(\chi))$.

Expectation of a function of a discrete random variable

Proposition

Let χ be a discrete (real) random variable with PMF $p_\chi : S \longrightarrow [0;1]$, $g : S \longrightarrow T$ a real valued function, and $\psi = g(\chi)$ another random variable with PMF $p_\psi : T \longrightarrow [0;1]$. Then

$$\begin{aligned} \mathbb{E}(\psi) &= \sum_{t \in T} t p_\psi(t) \quad \text{(by definition)} \\ &= \sum_{s \in S} g(s) p_\chi(s) \quad \text{(Law of the Unconscious Statistician – LOTUS)} \end{aligned}$$

Functions of a continuous random variable

Proposition

Let χ be a continuous (real) random variable with PDF $f_\chi : S \longrightarrow [0;1]$, $g : S \longrightarrow T$ a real valued function, and $\psi = g(\chi)$ another random variable with PDF $f_\psi : T \longrightarrow [0;1]$. Then

$$\begin{aligned}\mathbb{E}(\psi) &= \int_{-\infty}^{\infty} y\ f_\psi(y)\ dy \quad \text{(by definition)} \\ &= \int_{-\infty}^{\infty} g(x)\ f_\chi(x)\ dx \ \ \text{(LOTUS)}\end{aligned}$$

Variance I

Definition (Variance)

$$\mathbb{V}(\chi) = \mathbb{E}\left((\chi - \mathbb{E}(\chi))^2\right)$$

Proposition

$$\mathbb{V}(\chi) = \mathbb{E}\left(\chi^2\right) - \mathbb{E}(\chi)^2$$

Proof:

$$\begin{aligned}
\mathbb{V}(\chi) &= \mathbb{E}\left((\chi - \mathbb{E}(\chi))^2\right) \\
&= \mathbb{E}\left(\chi^2 - 2\,\chi\mathbb{E}(\chi) + \mathbb{E}(\chi)^2\right) \\
&= \mathbb{E}\left(\chi^2\right) - 2\,\mathbb{E}(\chi)\,\mathbb{E}(\chi) + \mathbb{E}(\chi)^2 \\
&= \mathbb{E}\left(\chi^2\right) - \mathbb{E}(\chi)^2
\end{aligned}$$

Variance II

Discrete random variable χ, with PMF $p_\chi : S \longrightarrow [0;1]$, and expected value $\mathbb{E}(\chi) = \mu$:

$$\mathbb{V}(\chi) = \sum_{s \in S} (s - \mu)^2 \; p_\chi(s)$$

Continuous random variable χ, with PDF $f_\chi : S \longrightarrow [0;1]$, and expected value $\mathbb{E}(\chi) = \mu$:

$$\mathbb{V}(\chi) = \int_{-\infty}^{\infty} (t - \mu)^2 \; f_\chi(t) \; dt$$

Variance – Examples

Example 1, revisited: As we saw before, the expected value of a fair die is $\frac{7}{2} = 3.5$. The variance is

$$\begin{aligned}
\mathbb{V}(\chi_1) &= \sum_{n=1}^{6} \left(n - \frac{7}{2}\right)^2 \frac{1}{6} \\
&= \frac{1}{6}\left(\left(-\frac{5}{2}\right)^2 + \left(-\frac{3}{2}\right)^2 + \left(-\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{3}{2}\right)^2 + \left(\frac{5}{2}\right)^2\right) \\
&= \frac{1}{6}\left(\frac{25}{4} + \frac{9}{4} + \frac{1}{4} + \frac{1}{4} + \frac{9}{4} + \frac{25}{4}\right) = \frac{1}{6} \cdot \frac{70}{4} = \frac{70}{24} \approx 2.9166
\end{aligned}$$

Example 8, revisited

Example 8, revisited: As we saw before, the expected value of the sum of two fair dice is 7. Let's now calculate the variance by the definition:

$$\begin{aligned}\mathbb{V}(\chi) &= \frac{25}{36} + \frac{32}{36} + \frac{27}{36} + \frac{16}{36} + \frac{5}{36} + \frac{5}{36} + \frac{16}{36} + \frac{27}{36} + \frac{32}{36} + \frac{25}{36} \\ &= \frac{50}{36} + \frac{64}{36} + \frac{54}{36} + \frac{32}{36} + \frac{10}{36} = \frac{210}{36} \approx 5.833\end{aligned}$$

Alternatively,

$$\begin{aligned}\mathbb{E}\left(\chi^2\right) = 2^2 \cdot \frac{1}{36} + 3^2 \cdot \frac{2}{36} + 4^2 \cdot \frac{3}{36} + 5^2 \cdot \frac{4}{36} + 6^2 \cdot \frac{5}{36} + 7^2 \cdot \frac{6}{36} \\ + 8^2 \cdot \frac{5}{36} + 9^2 \cdot \frac{4}{36} + 10^2 \cdot \frac{3}{36} + 11^2 \cdot \frac{2}{36} + 12^2 \cdot \frac{1}{36} \approx 54.833\end{aligned}$$

Hence, $\mathbb{V}(\chi) = \mathbb{E}\left(\chi^2\right) - \mathbb{E}(\chi)^2 \approx 54.833 - 49 \approx 5.833$.

Properties of the variance

- The variance is a measure of **dispersion** or **spread**.
- $\mathbb{V}(\chi) \geq 0$ for any random variable χ.
- The square root $\sigma = \sqrt{\mathbb{V}(\chi)}$ is called the **standard deviation** of χ. It is another commonly used measure of dispersion.

Proposition

Let χ be a (real) random variable, and $\alpha, \beta \in \mathbb{R}$ two constants. Then

$$\mathbb{V}(\alpha\chi + \beta) = \alpha^2 \mathbb{V}(\chi)$$

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Probability generating function

Definition

Let χ be a discrete random variable which assumes nonnegative integer values, and let $p_\chi(k) = p_k$ denote the probability that $\chi = k$, for all $k \in \mathbb{N}$. Then the **probability generating function** of χ is the generating function $P_\chi(z)$ for the sequence $\langle p_k \rangle_{k=0}^{\infty}$, i.e.

$$P_\chi(z) = \sum_{k=0}^{\infty} p_k z^k$$

Remark 1: The above series has a radius of convergence greater than or equal to 1.

Remark 2: $P_\chi(1) = 1$.

Applications of probability generating functions (I)

Proposition

Let χ be a random variable and $P_\chi(z)$ its probability generating function. Then the *expected value* of χ is

$$\mathbb{E}(\chi) = \sum_{k=0}^{\infty} k\, p_k = P'_\chi(1),$$

and the *variance* of χ is

$$\mathbb{V}(\chi) = P''_\chi(1) + P'_\chi(1) - [P'_\chi(1)]^2.$$

Applications of probability generating functions (II)

Proposition

Let χ and ψ be two **independent** random variables, with probability generating functions $P_\chi(z)$ and $P_\psi(z)$, respectively. Then the probability generating function for the sum $\chi + \psi$ is

$$P_{\chi+\psi}(z) = P_\chi(z)P_\psi(z)$$

Proposition

Let χ and ψ be two **independent** random variables with expected values $\mathbb{E}(\chi)$ and $\mathbb{E}(\psi)$, respectively. Then the expected value of the sum $\chi + \psi$ is

$$\mathbb{E}(\chi + \psi) = \mathbb{E}(\chi) + \mathbb{E}(\psi)$$

Prove !!

Example 8, revisited

Example 8, revisited: Again, let χ be the random variable associated with the outcomes of a fair die. The corresponding probability generating function is $P_1(z) = \frac{1}{6}(z + z^2 + \cdots + z^6)$. The random variable associated to the sum of two dice is $\chi + \chi$, and its probability generating function is $P_2(z) = (P_1(z))^2 = \frac{1}{36}(z + z^2 + \cdots + z^6)^2$. Therefore,

$$P_2'(z) = \frac{1}{18}(z + z^2 + \cdots + z^6)(1 + 2z + \cdots + 6z^5), \text{ thus}$$

$$\mathbb{E}(\chi + \chi) = P_2'(1) = 7.$$

$$P_2''(z) = \frac{1}{18}\Big[(1 + 2z + \cdots + 6z^5)^2 + \\ + (z + z^2 + \cdots + z^6)(2 + 6z + 12z^2 + 20z^3 + 30z^4)\Big],$$

$$\mathbb{V}(\chi + \chi) = P_2''(1) + P_2'(1) - [P_2'(1)]^2 \approx 5.833. \quad \text{(Verify !!)}$$

Discrete uniform distribution

- **Probability mass function**: Let $n = b - a + 1$, then

$$\mathbb{P}(\chi = k) = \begin{cases} \frac{1}{n} \text{ if } k \in \mathbb{N},\ a \leq k \leq b \\ 0 \text{ otherwise} \end{cases}$$

- **Expected value**: $\dfrac{a+b}{2}$
- **Variance**: $\dfrac{n^2 - 1}{12}$
- **Probability generating function**: $\dfrac{z^a - z^{b+1}}{n(1-z)}$

Discrete uniform distribution for $n = 5$

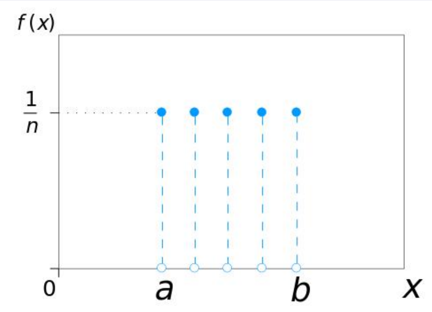

Figure: Discrete uniform PMF

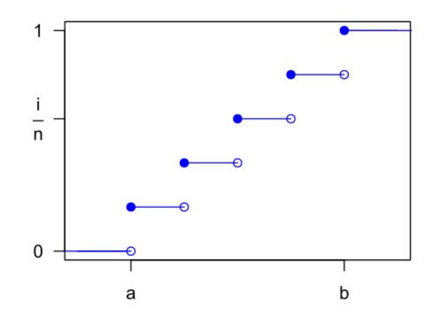

Figure: Discrete uniform CDF

Binomial distribution

- Measures the number of successes in a sequence of n independent **Bernoulli trials**, i.e. a sequence of n random experiments where there are only two possible outcomes: **success** (with probability p), or **failure** (with probability $q = 1 - p$).
- **Probability mass function**: $\mathbb{P}(\chi = k) = \binom{n}{k} p^k q^{n-k}$
- **Expected value**: np
- **Variance**: npq
- **Probability generating function**: $(q + pz)^n = (1 - p + pz)^n$

Binomial distribution for different values of p and n

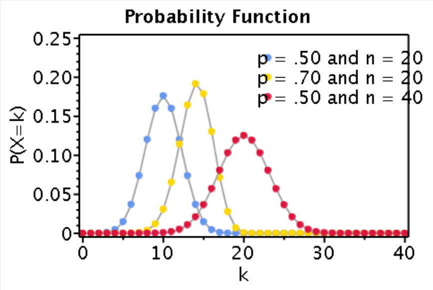

Figure: Binomial PMF

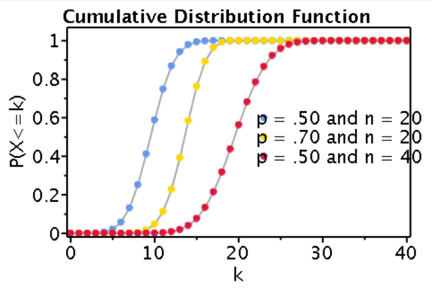

Figure: Binomial CDF

Geometric distribution

- Measures the number of **Bernoulli trials** that are necessary for getting the first success. Again, the probability of success at each trial is p, and the probability of failure is $q = 1 - p$.
- **Probability mass function**: $\mathbb{P}(\chi = k) = q^{k-1}p$
- **Expected value**: $\dfrac{1}{p}$
- **Variance**: $\dfrac{q}{p^2}$
- **Probability generating function**: $\dfrac{pz}{1 - qz}$

Geometric distribution

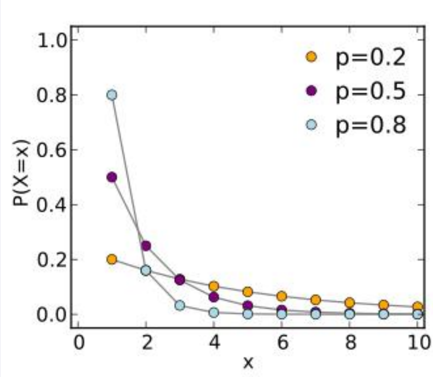

Figure: Geometric PMF

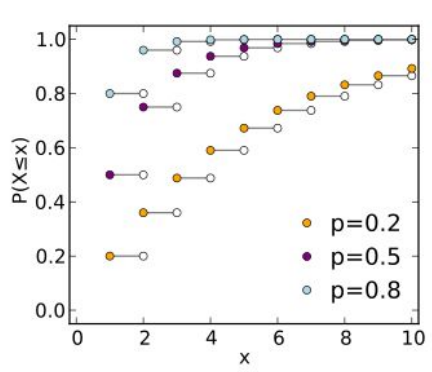

Figure: Geometric CDF

Poisson distribution

- Models the number of independent events that take place during a time interval of given length (e.g. arrivals at a queue in one minute, number of floods in a decade, number of goals per match in the World Cup).
- **Probability mass function**: $\mathbb{P}(\chi = k) = \dfrac{\lambda^k e^{-\lambda}}{k!}$
- **Expected value**: λ
- **Variance**: λ
- **Probability generating function**: $e^{\lambda(z-1)}$

Poisson distribution

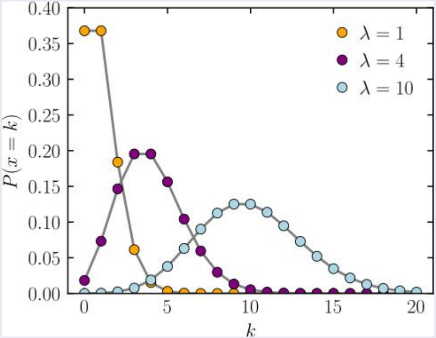

Figure: Poisson PMF

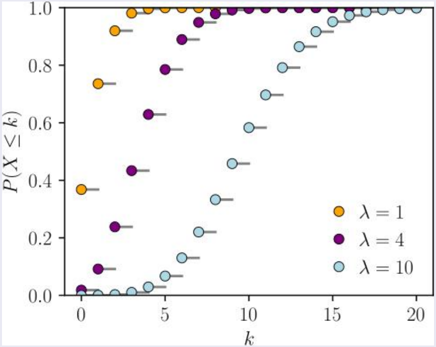

Figure: Poisson CDF

Summary of discrete distributions

Distribution	p_k (PMF)	Mean	Variance	Gen. Function
Discrete uniform	$\frac{1}{n}$	$\frac{a+b}{2}$	$\frac{n^2-1}{12}$	$\frac{z^a - z^{b+1}}{n(1-z)}$
Binomial	$\binom{n}{k} p^k q^{n-k}$	np	npq	$(q+pz)^n$
Geometric	pq^{k-1}	$\frac{1}{p}$	$\frac{1-p}{p^2}$	$\frac{pz}{1-qz}$
Poisson	$\frac{\lambda^k e^{-\lambda}}{k!}$	λ	λ	$e^{\lambda(z-1)}$

1. Random variables
   - Real random variables
   - Numerical properties of random variables
   - Probability generating functions

2. Probability Distributions
   - Discrete probability distributions
   - Continuous probability distributions

Continuous uniform distribution

- **Probability density function**:

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{for } x \in [a; b] \\ 0 & \text{otherwise} \end{cases}$$

- **Cumulative distribution function**:

$$F(x) = \begin{cases} 0 & \text{for } x < a \\ \frac{x-a}{b-a} & \text{for } x \in [a; b] \\ 1 & \text{for } x > b \end{cases}$$

- **Expected value**: $\dfrac{a+b}{2}$
- **Variance**: $\dfrac{(b-a)^2}{12}$

Continuous uniform distribution

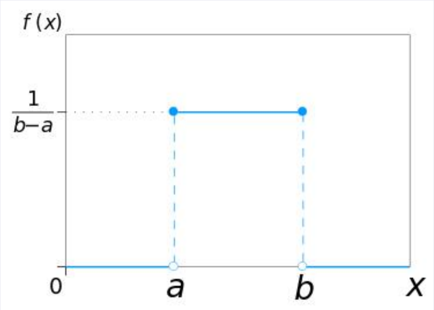

Figure: Uniform PDF

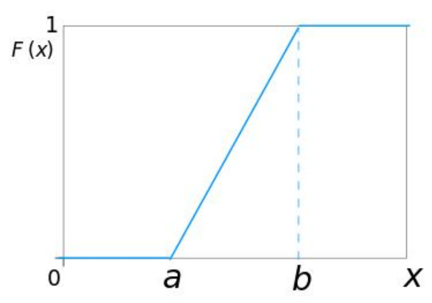

Figure: Uniform CDF

Exponential distribution

- Related to the Poisson distribution. Models the distribution of the time intervals between two events in a Poisson process (e.g. the time between two arrivals at a queue, the time between two goals at a World Cup match, etc.).

- **Probability density function**:

$$f(x) = \begin{cases} 0 & \text{for } x < 0 \\ \lambda e^{-\lambda x} & \text{for } x \geq 0 \end{cases}$$

- **Cumulative distribution function**:

$$f(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 - e^{-\lambda x} & \text{for } x \geq 0 \end{cases}$$

- **Expected value**: $\dfrac{1}{\lambda}$ **Variance**: $\dfrac{1}{\lambda^2}$

Exponential distribution

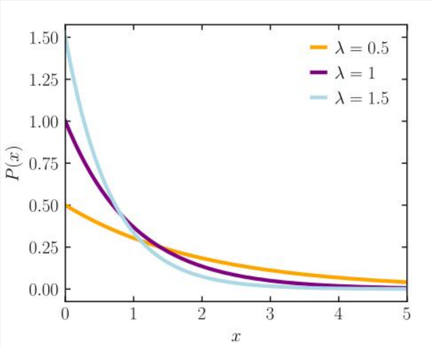

Figure: Exponential PDF

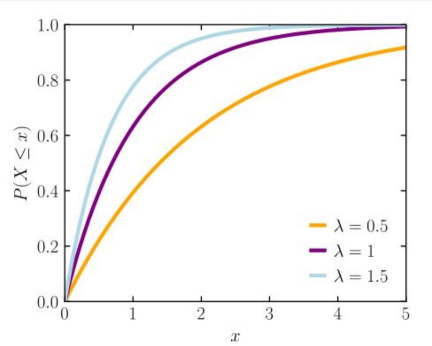

Figure: Exponential CDF

Normal distribution

- Limit distribution of a sum of identically distributed random observations (central limit theorem). Models the distribution of heights, weights, IQs, etc. in a human population.
- **Probability density function**:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

- **Expected value**: μ
- **Variance**: σ^2

Normal distribution

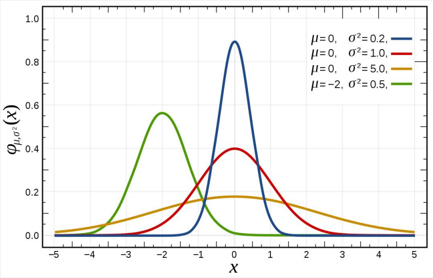

Figure: Normal PDF

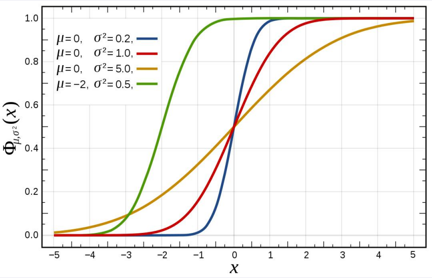

Figure: Normal CDF

Since the normal PDF has the shape of a bell, it is sometimes called “the bell curve” or “the Gaussian bell”.

Table of normal distribution

The function $f(x) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ cannot be integrated analytically, but it can be computed numerically and tabulated.

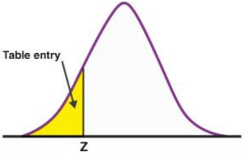

Table entry for z is the area under the standard normal curve to theleft of z.

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−3.4	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0002
−3.3	.0005	.0005	.0005	.0004	.0004	.0004	.0004	.0004	.0004	.0003
−3.2	.0007	.0007	.0006	.0006	.0006	.0006	.0006	.0005	.0005	.0005
−3.1	.0010	.0009	.0009	.0009	.0008	.0008	.0008	.0008	.0007	.0007
−3.0	.0013	.0013	.0013	.0012	.0012	.0011	.0011	.0011	.0010	.0010
−2.9	.0019	.0018	.0018	.0017	.0016	.0016	.0015	.0015	.0014	.0014
−2.8	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019
−2.7	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026
−2.6	.0047	.0045	.0044	.0043	.0041	.0040	.0039	.0038	.0037	.0036
−2.5	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051	.0049	.0048

Figure: Fragment of the table of the normal CDF

Central limit theorem

The normal distribution plays a distinguished role in Probability Theory.

Theorem (The Central Limit Theorem)

Let $\chi_1, \chi_2, \ldots, \chi_n$ *be independent identically distributed random variables with expected value* $\mathbb{E}(\chi_i) = \mu < \infty$*. Then the random variable*

$$\psi_n = \frac{\overline{\chi} - \mu}{\sigma/\sqrt{n}} = \frac{\chi_1 + \chi_2 + \ldots + \cdots + \chi_n - n\mu}{\sqrt{n}\sigma}$$

converges in distribution to the standard normal random variable as n goes to infinity, that is

$$\lim_{n \longrightarrow \infty} \mathbb{P}(\psi_n \leq x) = \Psi(x), \qquad \text{for all } x \in \mathbb{R},$$

where $\Psi(x)$ *is the standard normal CDF.*

See Introduction to Probability, Statistics and Random Processes

The Central Limit Theorem in action

The Galton board is a device invented by the British mathematician Sir Francis Galton (1822–1911), which provides a visual demonstration of the Central Limit Theorem.

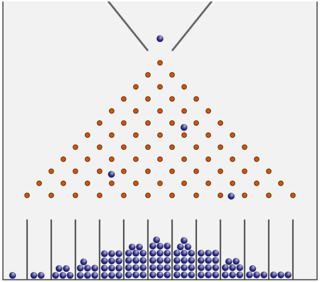

Figure: Galton board

This video shows a computer simulation of a Galton board at work.

Summary of continuous distributions

Distribution	PDF	Mean	Variance
Cont. uniform	$\frac{1}{b-a}$ for $x \in [a; b]$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
Exponential	$\lambda e^{-\lambda x}$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
Normal	$\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$	μ	σ^2

Bibliography Probability I

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