Exercicis

9. Can there exist a C^2 function $f(x, y)$ with $f_x = 2x - 5y$ and $f_y = 4x + y$?

11. Show that the following functions satisfy the one-dimensional wave equation

$$\frac{\partial^2 f}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2}.$$

(a) $f(x, t) = \sin(x - ct)$

(b) $f(x, t) = \sin(x)\sin(ct)$

(c) $f(x, t) = (x - ct)^6 + (x + ct)^6$

12. (a) Show that $T(x, t) = e^{-kt}\cos x$ satisfies the one-dimensional heat equation

$$k\frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}.$$

(b) Show that $T(x, y, t) = e^{-kt}(\cos x + \cos y)$ satisfies the two-dimensional heat equation

$$k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) = \frac{\partial T}{\partial t}.$$

(c) Show that $T(x, y, z, t) = e^{-kt}(\cos x + \cos y + \cos z)$ satisfies the three-dimensional heat equation

$$k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) = \frac{\partial T}{\partial t}.$$

22. Let $w = f(x, y)$ be a function of two variables and let $x = u + v$, $y = u - v$. Show that

$$\frac{\partial^2 w}{\partial u\,\partial v} = \frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2}.$$

25. A function $u = f(x, y)$ with continuous second partial derivatives satisfying Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

is called a ***harmonic function***. Show that the function $u(x, y) = x^3 - 3xy^2$ is harmonic.

27. (a) Is the function $f(x, y, z) = x^2 - 2y^2 + z^2$ harmonic? What about $f(x, y, z) = x^2 + y^2 - z^2$?

(b) Laplace's equation for functions of n variables is

$$\frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2} = 0.$$

Find an example of a function of n variables that is harmonic, and show that your example is harmonic.

31. Show that Newton's potential $V = -GmM/r$ satisfies Laplace's equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \quad \text{for} \quad (x, y, z) \neq (0, 0, 0).$$

32. Let

$$f(x, y) = \begin{cases} xy(x^2 - y^2)/(x^2 + y^2), & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

(see Figure 3.1.4).

(a) If $(x, y) \neq (0, 0)$, calculate $\partial f/\partial x$ and $\partial f/\partial y$.

(b) Show that $(\partial f/\partial x)(0, 0) = 0 = (\partial f/\partial y)(0, 0)$.

(c) Show that $(\partial^2 f/\partial x\, \partial y)(0, 0) = 1$, $(\partial^2 f/\partial y\, \partial x)(0, 0) = -1$.

(d) What went wrong? Why are the mixed partials not equal?