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Try to answer these questions as if it were an exam, that is, without the aid of any electronic devices (calculators, mobile phones, etc.).
All answers must include the supporting deductions and calculations (if the calculations are too heavy then they can be left implied).

BASIC PRINCIPLES OF COMBINATORICS

1. Find the number of
   (a) two-digit even numbers
   (b) two-digit odd numbers
   (c) two-digit odd numbers with distinct digits
   (d) two-digit even numbers with distinct digits
   (e) positive integers with distinct digits
2. Prove the following statements by mathematical induction:
   (a) $11^n - 6$ is divisible by 5 for every positive integer n.
   (b) $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ for every positive integer n.
   (c) $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ for every positive integer n.
   (d) $2^n > 2n$ for every positive integer $n > 2$.
3. Consider a tournament with n players, where every player plays against every other player. Suppose that each player wins at least one game. Show that there are at least two players having the same number of wins.
4. Let T be an equilateral triangle with sides of length 2. Show that if five points are chosen at random inside T, then at least a pair of points will have a separation of less than 1 unit.
5. (1st partial exam, 2022–2023) An employee is attending a meeting of a special committee, and he spends his time playing around with the word ‘COMMITTEE’. He’s trying to make a list of all the distinct words that can be constructed with the letters of the word ‘COMMITTEE’, possibly with some additional conditions, but he is not good at Math. Help him find:
   (a) The number of all distinct words where all the vowels come before all the consonants.
   (b) The number of all distinct words where equal letters are always together.

(c) The number of all distinct words where the two Ts are always apart and the two Es are always apart.

6. (1st partial exam, 2022–2023) An idle bookkeeper is waiting for clients, and in the meantime he spends his time playing around with the word 'BOOKKEEPER'. He's trying to make a list of all the distinct words that can be constructed with the letters of the word 'BOOKKEEPER', possibly with some additional conditions, but he is not good at Math. Help him find:

   (a) The number of all distinct words where all the vowels come before all the consonants.

   (b) The number of all distinct words where equal letters are always together.

   (c) The number of all distinct words where the two Os are always apart and the two Ks are always apart.

7. (1st partial exam, 2022–2023) Prove the following version of the hockey-stick identity:

$$\sum_{j=k}^{n} \binom{j}{k} = \binom{n+1}{k+1}$$

   Hint: Use induction on n while keeping k fixed.

8. (1st partial exam, 2022–2023) Prove the following version of the hockey-stick identity:

$$\sum_{i=0}^{k} \binom{n+i}{i} = \binom{n+k+1}{k}$$

   Hint: Use induction on k while keeping n fixed.

9. Prove the following identity of binomial coefficients:

$$\sum_{k=0}^{n} \binom{n}{k} k = n2^{n-1}$$

   Hint: Differentiate Newton's binomial formula $(x+y)^n$ with respect to x and substitute suitable values for x and y.

10. Prove the following identity of binomial coefficients:

$$\sum_{k=0}^{n} \binom{n}{k} \frac{1}{k+1} = \frac{2^{n+1}-1}{n+1}$$

   Hint: Use the following version of the absortion identity: $\dfrac{1}{n+1}\dbinom{n+1}{k+1} = \dfrac{1}{k+1}\dbinom{n}{k}$.

11. Prove the inclusion-exclusion formula for three sets, A, B and C:

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

   Hint: Assume that the formula is valid for two sets, and use the properties of the set operations (union, intersection, etc.).

12. Show that the number of terms in the sum

$$\sum_{k_1+k_2+\cdots+k_m=n} \binom{n}{k_1, k_2, \ldots, k_m}$$

is $\dbinom{n+m-1}{m-1}$.

Hint: What is the number of nonnegative integer solutions of the equation $k_1 + k_2 + \cdots + k_m = n$?

13. The Stirling number of the second kind, written $\left\{ {n \atop k} \right\}$, counts the number of distinct partitions of the set $S_n = \{1, 2, \ldots, n\}$ into k (nonempty) parts. For example, $\left\{ {3 \atop 2} \right\} = 3$ because the set S_3 can be partitioned into two parts in 3 ways, as follows:

$$\{\{1\}, \{2, 3\}\}, \quad \{\{2\}, \{1, 3\}\}, \quad \{\{3\}, \{1, 2\}\}.$$

$\left\{ {n \atop k} \right\}$ can be calculated by the recursive formula

$$\left\{ {n \atop k} \right\} = \begin{cases} 0 \quad \text{if} \quad k \leq 0 \quad \text{or} \quad k > n, \\ 1 \quad \text{if} \quad k = 1 \quad \text{or} \quad k = n, \\ \left\{ {n-1 \atop k-1} \right\} + k\left\{ {n-1 \atop k} \right\} \text{ otherwise.} \end{cases}$$

(a) Prove the recursive formula given above. **Hint**: Fix an element $x \in S_n$, and consider two kinds of partitions of S_n: those where $\{x\}$ is a part by itself, and those partitions where x belongs to some part, together with other elements of S_n.

(b) Construct a triangle, similar to Pascal's triangle, to calculate $\left\{ {n \atop k} \right\}$ for all $1 \leq k \leq n$.

14. The Ministry of Education wants to distribute five supercomputers among three universities. It has been agreed that each university must get at least one supercomputer. Find the number of ways to carry out this distribution (adapted from *Matemática Discreta*, by Juan Alberto Rodríguez Velázquez).

Hint: The solution can be computed with the aid of Stirling numbers of the second kind, and also with the aid of multinomial coefficients.

RECURRENCES

1. (1st partial exam, 2022–2023) Let a_n be the number of n-letter words from the alphabet $\{A, B, C\}$ such that any nonterminal A has to be immediately followed by a B.
   (a) Find a recurrence relation for $\langle a_n \rangle$ (including its initial values).
   (b) Prove that $\langle a_n \rangle$ is monotonically increasing. **Hint**: Use induction on n.
   (c) Solve the recurrence.
2. (1st partial exam, 2022–2023) Let a_n be the number of n-letter words from the alphabet $\{A, B, C\}$ such that any nonterminal A or B has to be immediately followed by a C.
   (a) Find a recurrence relation for $\langle a_n \rangle$ (including its initial values).
   (b) Prove that $\langle a_n \rangle$ is monotonically increasing. **Hint**: Use induction on n.
   (c) Solve the recurrence.
3. Solve the following homogeneous linear recurrences (adapted from *Matemática Discreta*, by Juan Alberto Rodríguez Velázquez):
   (a) $a_n - 5a_{n-1} + 6a_{n-2} = 0$, with $a_0 = 1$ and $a_1 = 2$.
   (b) $a_n - 2a_{n-1} - a_{n-2} + 2a_{n-3} = 0$, with $a_0 = 0$ and $a_1 = a_2 = 1$.
   (c) $a_n + 4a_{n-1} + 5a_{n-2} + 2a_{n-3} = 0$, with $a_0 = 1$, $a_1 = 2$ and $a_2 = 3$.
   (d) $a_n - 9a_{n-1} + 15a_{n-2} + 25a_{n-3} = 0$, with $a_0 = 0$, $a_1 = 4$ and $a_2 = 76$.
4. Find a closed-form formula for the following homogeneous linear recurrences in terms of the roots of the characteristic equation and, if possible, also in terms of trigonometric functions:
   (a) $a_n = 2a_{n-1} - 2a_{n-2}$, with $a_0 = 0$ and $a_1 = 1$.
   (b) $a_n = 8a_{n-3}$, with $a_0 = 0$ and $a_1 = a_2 = 1$.
5. Tribonacci numbers are a generalization of Fibonacci numbers. There are several versions of Tribonacci numbers; one of them is defined by the recurrence

$$T_n = \begin{cases} 0 \text{ if } n = 0, \\ 1 \text{ if } n = 1 \text{ or } n = 2, \\ T_{n-1} + T_{n-2} + T_{n-3} \text{ if } n > 2. \end{cases}$$

   (a) Find an approximate closed-form formula for T_n (i.e. solve the recurrence approximately). **Remark**: The roots of the characteristic equation are quite complicated. As an alternative we could work with numerical approximations of the roots, and thus we will get numerical approximations for the terms T_n.
   (b) Compute the first few terms (a dozen or so) of your approximate sequence, and compare them with the exact terms T_n.

6. Lucas numbers are named after the French mathematician François Édouard Anatole Lucas (1842–1891), who studied this number sequence, together with the closely related Fibonacci sequence. Lucas numbers are defined by the recurrence

$$L_n = \begin{cases} 2 \text{ if } n = 0, \\ 1 \text{ if } n = 1, \\ L_{n-1} + L_{n-2} \text{ if } n > 1. \end{cases}$$

   (a) Find a closed-form formula for L_n (i.e. solve the recurrence).

   (b) Formulate the recurrence in matrix form and find a Cassini-type identity for the sequence $\langle L_n \rangle$.

7. The 'Tower of Hanoi' puzzle (also know as 'Tower of Brahma') is a game that was also invented by Édouard Lucas in 1883. It consists of three rods (denoted A, B and C) and n disks of various diameters. The puzzle begins with the disks stacked on rod A in order of decreasing size, the smallest at the top, thus approximating a conical shape. The goal is to move the entire stack from rod A to rod C, using the intermediate rod (i.e. rod B) as an auxiliary station, and obeying the following rules:

   i Only one disk may be moved at a time.

   ii Each move consists of taking the upper disk from one of the stacks and placing it on top of another stack or on an empty rod.

   iii No disk may be placed on top of a disk that is smaller than it.

   Now,

   (a) Formulate a recursive algorithm (pseudocode) that solves the puzzle.

   (b) Formulate the recurrence that describes the (minimal) number of moves performed by the previous recursive algorithm.

   (c) Solve the recurrence.

8. Solve the following non-homogeneous linear recurrences (adapted from *Matemática Discreta*, by Juan Alberto Rodríguez Velázquez):

   (a) $a_n = a_{n-1} + 2a_{n-2} - 1$, with $a_0 = 0$ and $a_1 = 1$.

   (b) $a_n - 5a_{n-1} + 6a_{n-2} = 2^n$, with $a_0 = 4$ and $a_1 = 5$.

   (c) $a_n - 4a_{n-1} + 4a_{n-2} = 2^n$, with $a_0 = 1$, $a_1 = 2$ and $a_2 = 8$.

9. Use recurrence relations to derive closed-form formulas for the sums:

   (a) $S_n = \sum_{i=1}^{n} i$

   (b) $S_n = \sum_{i=1}^{n} i^2$

   (c) $S_n = \sum_{i=1}^{n} i^3$

   (d) $S_n = \sum_{i=1}^{n} i2^i$

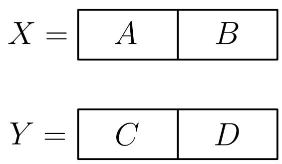
10. (1st partial exam, 2022–2023) Solve the divide-and-conquer recurrence

$$T_n = \begin{cases} 0 \text{ if } n = 1, \\ 4T_{\frac{n}{2}} + n^2 \text{ for } n = 2^k,\ k \geq 1. \end{cases}$$

11. (1st partial exam, 2022–2023) Solve the divide-and-conquer recurrence

$$T_n = \begin{cases} 0 \text{ if } n = 1, \\ 4T_{\frac{n}{2}} + n \text{ for } n = 2^k,\ k \geq 1. \end{cases}$$

12. A divide-and-conquer algorithm for multiplying integers was devised by the Soviet computer scientists A. Karatsuba and Yu. Ofman in 1962. For two large integers X and Y this algorithm is more efficient than the conventional pencil-and-paper algorithm. For the sake of simplicity we will assume that X and Y are represented in binary notation, with a number of digits n that is a power of 2, i.e. $n = 2^k$, and X and Y are both positive. The algorithm starts by decomposing each number into two pieces of size $\dfrac{n}{2} = 2^{k-1}$, as shown in Figure 1:



$X = A^{\frac{n}{2}} + B$

$Y = C^{\frac{n}{2}} + D$

Figure 1: Breaking n-bit integers into $\frac{n}{2}$-bit pieces

Now the product XY can be calculated by the formula

$$XY = AC2^n + \Big[(A - B)(D - C) + AC + BD\Big]2^{\frac{n}{2}} + BD$$

In the above formula we have to compute three products that involve $\frac{n}{2}$-bit integers, which can be accomplished by recursively calling the same algorithm. Additionally, there are two products that involve powers of two, which can be accomplished by shifting, as well as some addition operations. Note that the latter two types of operations do not involve bitwise multiplications. Algorithm 1 depicts this procedure in more detail.

(a) Formulate a recurrence equation that describes the number of bitwise multiplications performed by Algorithm 1.

(b) Solve the recurrence.

13. Let B_n denote the number of Boolean functions on n variables.

(a) Show that B_n satisfies the recurrence

$$B_n = \begin{cases} 4 \text{ if } n = 1, \\ B_{n-1}^2 \text{ if } n > 1. \end{cases}$$

(b) Find a closed-form formula for B_n

Algorithm 1: DIVIDE-AND-CONQUER MULTIPLICATION

```
Function Mult(X, Y, n):
    Input  : Two n-bit numbers, X and Y.
    Output: The product Z = XY.

Function Mult(X, Y, n):
    if n = 1 then
        if X = 1 and Y = 1 then
            return 1
        else
            return 0
        end
    else
        A:= left half of X
        B:= right half of X
        C:= left half of Y
        D:= right half of Y
        M_1:= Mult(A, C, n/2)
        M_2:= Mult(A − B, D − C, n/2)
        M_3:= Mult(B, D, n/2)
        Z:= M_1 2^n + (M_1 + M_2 + M_3) 2^(n/2) + M_3
    end
return Z
```

GENERATING FUNCTIONS

1. Find the coefficient of z^{21} in the expression $(z^3 + z^4 + \cdots + z^{10})^4$ (Adapted from *Matemática Discreta*, by Juan Alberto Rodríguez Velázquez).
2. Find the coefficient of z^{16} in the expression $(z^2 + z^3 + \cdots)^5$ (Adapted from *Matemática Discreta*, by Juan Alberto Rodríguez Velázquez).
3. Find the coefficient of z^{25} in the generating function $g(z) = (z^2 + \cdots + z^6)^7$ (Adapted from *Matemática Discreta*, by Juan Alberto Rodríguez Velázquez).
4. Find the ordinary generating functions associated with the following sequences:
   (a) $\langle 0, 1, 2, 3, \ldots \rangle$.
   (b) $\langle 1, -2, 3, -4, \ldots \rangle$.
5. Let $\langle a_n \rangle_{n=0}^{\infty}$ be a number sequence, and $A(z)$ its ordinary generating function. Suppose that $A(z)$ is known. Find the general term a_n when
   (a) $A(z) = \dfrac{3x}{x^2 + x - 2}$.
   (b) $A(z) = \dfrac{3x + 5}{2x^2 - 5x - 3}$.
   (c) $A(z) = \dfrac{x^2 + 1}{(x-1)^2(x+1)}$.

   Hint: Decompose $A(z)$ into partial fractions, and then use the known formulas.
6. Calculate the first three coefficients of the reciprocals of the Taylor-McLaurin series of the functions
   (a) $\cos z$
   (b) $(1+z)^m$
   (c) $1 + z^2 + z^3 + z^5 + z^7 + z^{11} + \cdots$
7. (2nd partial exam, 2022–2023) Solve the following recurrence equation by means of (ordinary) generating functions
$$a_n = \begin{cases} 1 \text{ if } n = 0, \\ 0 \text{ if } n = 1, \\ a_{n-1} + 2a_{n-2} \text{ for } n \geq 2. \end{cases}$$
8. (2nd partial exam, 2022–2023) Solve the following recurrence equation by means of (ordinary) generating functions
$$a_n = \begin{cases} 0 \text{ if } n = 0, \\ 1 \text{ if } n = 1, \\ 3a_{n-1} - 2a_{n-2} \text{ for } n \geq 2. \end{cases}$$
9. Find the radius of convergence of each of the following power series:
   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{z^n}{n^2}$

(b) $1 + z^3 + z^6 + z^9 + z^{12} + \cdots$

(c) $1 + 5z^2 + 25z^4 + 125z^6 + \cdots$

10. Show that the radius of convergence of the Taylor-McLaurin series of the function $\dfrac{z}{e^z - 1}$ is 2π.
Hint: Find the singularities that are closest to the origin, other than the origin itself. Use the trigonometric representation of e^z.

11. Find the radius of convergence of the Taylor-McLaurin series of the function $\dfrac{1}{z^2 - 2z + 2}$.

12. Find the generating function for the number of integer solutions to the equation $x_1 + x_2 + x_3 + x_4 = k$, where $0 \leq x_1 \leq \infty$, $3 \leq x_2 \leq \infty$, $2 \leq x_3 \leq 5$ and $1 \leq x_4 \leq 5$.

13. Find the exponential generating functions associated with the following sequences:

    (a) $\langle 0, 1, 8, 27, 64, \ldots \rangle$.

    (b) $\langle 1, 2, 4, 8, 16, \ldots \rangle$.

14. Let $\langle a_n \rangle_{n=0}^{\infty}$ and $\langle b_n \rangle_{n=0}^{\infty}$ be two number sequences. Prove that the following statements are equivalent:

    (a) $a_n = \displaystyle\sum_{k=0}^{n} \binom{n}{k} b_{n-k}$ for all $n \geq 0$.

    (b) $b_n = \displaystyle\sum_{k=0}^{n} (-1)^k \binom{n}{k} a_{n-k}$ for all $n \geq 0$.

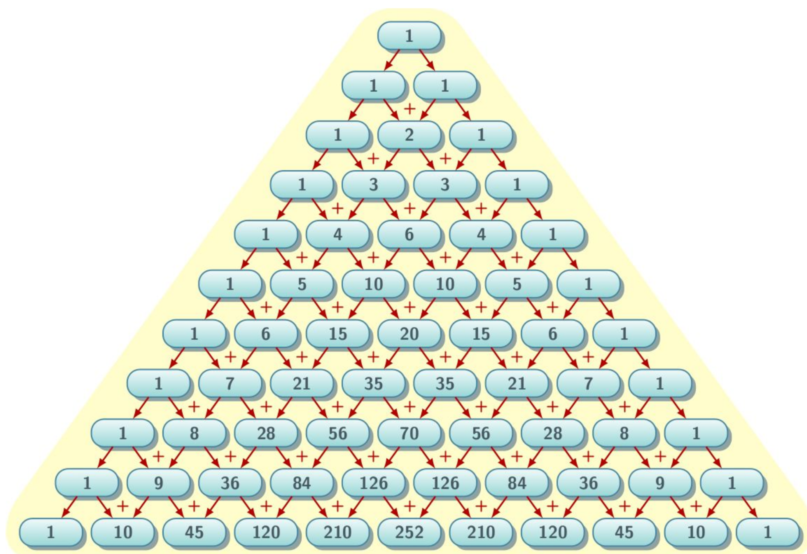
    Hint: Construct the exponential generating functions $A(z)$ and $B(z)$ for $\langle a_n \rangle_{n=0}^{\infty}$ and $\langle b_n \rangle_{n=0}^{\infty}$, respectively. Then observe the relationship between both generating functions. This result is called the Binomial Inversion Formula.

SOME USEFUL TABLES AND FORMULAS

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}, \quad \text{for } 0 \le k \le n$$

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$$

Pascal's triangle:



Symmetry:

$$\binom{n}{k} = \binom{n}{n-k}, \quad n \ge k \ge 0$$

Absorption:

$$\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}, \quad n \ge k \ge 1$$

Trinomial revision:

$$\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}, \quad n \ge m \ge k \ge 0$$

Newton's binomial formula:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

$$(a-b)^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} a^{n-k} b^k$$

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n \qquad \sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

Table of factorials

n	$n!$	n	$n!$	n	$n!$
0	1	1	1	2	2
3	6	4	24	5	120
6	720	7	5 040	8	40 320
9	362 880	10	3 628 800	11	39 916 800

Table of subfactorials

n	$!n$	n	$!n$	n	$!n$
1	0	2	1	3	2
4	9	5	44	6	265
7	1 854	8	14 833	9	133 496
10	1 334 961	11	14 684 570	12	176 214 841

Table of ordinary generating functions:

Function	Sequence	Series
$\frac{1}{1-z}$	$\langle 1, 1, 1, \ldots \rangle$	$\sum_{n=0}^{\infty} z^n$
$\frac{1}{1+z}$	$\langle 1, -1, 1, -1, \ldots \rangle$	$\sum_{n=0}^{\infty} (-1)^n z^n$
$\frac{1}{1-az}$	$\langle 1, a, a^2, a^3, \ldots \rangle$	$\sum_{n=0}^{\infty} a^n z^n$
$\frac{1}{1-z^2}$	$\langle 1, 0, 1, 0, 1, \ldots \rangle$	$\sum_{n=0}^{\infty} z^{2n}$
$\frac{1}{(1-z)^2}$	$\langle 1, 2, 3, 4, 5, \ldots \rangle$	$\sum_{n=0}^{\infty} (n+1) z^n$

Shifting of ordinary generating functions:

If $A(z) = \sum_{n=0}^{\infty} a_n z^n$, then $\frac{A(z) - A(0)}{z} = \frac{A(z) - a_0}{z}$ is the OGF of the sequence $\langle a_{n+1} \rangle_{n=0}^{\infty}$. $\frac{A(z) - a_0 - a_1 z}{z^2}$ is the OGF of the sequence $\langle a_{n+2} \rangle_{n=0}^{\infty}$.

Operations with ordinary generating functions:

Ordinary generating function	n-th element of the sequence
cA	ca_n
$A + B$	$a_n + b_n$
AB	$\sum_{k=0}^{n} a_k b_{n-k}$
$z^k A(z)$	if $n < k$ then 0 else a_{n-k}
$\frac{A(z)}{1-z}$	$\sum_{i=0}^{n} a_i$
$zA'(z)$	na_n
$\int_0^z A(t)dt$	if $n = 0$ then 0 else $\frac{a_{n-1}}{n}$

Table of exponential generating functions:

Function	Sequence	Series
e^z	$\langle 1, 1, 1, \ldots \rangle$	$\sum_{n=0}^{\infty} \frac{z^n}{n!}$
e^{az}	$\langle 1, a, a^2, a^3, \ldots \rangle$	$\sum_{n=0}^{\infty} \frac{a^n z^n}{n!}$
ze^z	$\langle 0, 1, 2, 3, 4, \ldots \rangle$	$\sum_{n=0}^{\infty} \frac{nz^n}{n!}$
$z(z+1)e^z$	$\langle 0, 1, 4, 9, \ldots \rangle$	$\sum_{n=0}^{\infty} \frac{n^2 z^n}{n!}$
$\frac{1}{1-z}$	$\langle 1, 1, 2, 6, 24, \ldots \rangle$	$\sum_{n=0}^{\infty} \frac{n! z^n}{n!}$

Operations with exponential generating functions:

Exp. generating function	n-th element of the sequence
cA	ca_n
$A + B$	$a_n + b_n$
AB	$\sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}$
$A'(z)$	a_{n+1}
$\int_0^z A(t)dt$	if $n = 0$ then 0 else a_{n-1}