Mètodes d'integració

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Anàlisi Matemàtica 1

Integral Tables

1. $\int u^n\,du = \dfrac{u^{n+1}}{n+1} + C,\ n \neq -1$

2. $\int \dfrac{du}{u} = \ln|u| + C$

3. $\int e^u\,du = e^u + C$

4. $\int a^u\,du = \dfrac{a^u}{\ln a} + C$

5. $\int \sin u\,du = -\cos u + C$

6. $\int \cos u\,du = \sin u + C$

7. $\int \sec^2 u\,du = \tan u + C$

8. $\int \csc^2 u\,du = -\cot u + C$

9. $\int \sec u \tan u\,du = \sec u + C$

10. $\int \csc u \cot u\,du = -\csc u + C$

11. $\int \tan u\,du = \ln|\sec u| + C$

12. $\int \cot u\,du = \ln|\sin u| + C$

13. $\int \sec u\,du = \ln|\sec u + \tan u| + C$

14. $\int \csc u\,du = \ln|\csc u - \cot u| + C$

15. $\int \dfrac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\dfrac{u}{a} + C$

16. $\int \dfrac{du}{a^2+u^2} = \dfrac{1}{a}\tan^{-1}\dfrac{u}{a} + C$

17. $\int \dfrac{du}{u\sqrt{u^2-a^2}} = \dfrac{1}{a}\sec^{-1}\dfrac{u}{a} + C$

42. $\int ue^{au}\,du = \dfrac{1}{a^2}(au-1)e^{au} + C$

46. $\int \ln u\,du = u\ln u - u + C$

59. $\int \sin^{-1} u\,du = u\sin^{-1} u + \sqrt{1-u^2} + C$

60. $\int \cos^{-1} u\,du = u\cos^{-1} u - \sqrt{1-u^2} + C$

61. $\int \tan^{-1} u\,du = u\tan^{-1} u - \dfrac{1}{2}\ln\left(1+u^2\right) + C$

34. $\int \sin au \sin bu\,du = \dfrac{\sin(a-b)u}{2(a-b)} - \dfrac{\sin(a+b)u}{2(a+b)} + C$

35. $\int \cos au \cos bu\,du = \dfrac{\sin(a-b)u}{2(a-b)} + \dfrac{\sin(a+b)u}{2(a+b)} + C$

36. $\int \sin au \cos bu\,du = -\dfrac{\cos(a-b)u}{2(a-b)} - \dfrac{\cos(a+b)u}{2(a+b)} + C$

49. $\int \sinh u\,du = \cosh u + C$

50. $\int \cosh u\,du = \sinh u + C$

51. $\int \tanh u\,du = \ln\cosh u + C$

18. $\int \sin^2 u\,du = \dfrac{1}{2}u - \dfrac{1}{4}\sin 2u + C$

19. $\int \cos^2 u\,du = \dfrac{1}{2}u + \dfrac{1}{4}\sin 2u + C$

20. $\int \tan^2 u\,du = \tan u - u + C$

21. $\int \cot^2 u\,du = -\cot u - u + C$

Integral Tables

Using a Table of Integrals

- *Gradshteyn & Ryzhik: Table of integrals, series and products*
- *Schaum: Fórmulas y tablas de matemática aplicada*
- *CRC Standard Mathematical Tables and Formulae*
- *Burington: Handbook of Mathematical Tables and Formulas*

Example

We use the table to calculate $$\int \frac{\sqrt{9-4x^2}}{x^2}\,dx.$$

Solution

Closest to what we need is this formula: $$\int \frac{\sqrt{a^2-u^2}}{u^2}\,du = -\frac{1}{u}\sqrt{a^2-u^2} - \arcsin\frac{u}{a} + C.$$

We can write our integral to fit the formula by setting

$$u = 2x,\ du = 2\,dx.$$

Doing this, we have

$$\int \frac{\sqrt{9-4x^2}}{x^2}\,dx = 2\int \frac{\sqrt{9-u^2}}{u^2}\,du = 2\left[-\frac{1}{u}\sqrt{9-u^2} - \arcsin\frac{u}{3}\right] + C$$

Check this out.

$$= 2\left[-\frac{1}{2x}\sqrt{9-4x^2} - \arcsin\frac{2x}{3}\right] + C.$$

Integration by Substitution

Integration by substitution or change of variable

THEOREM 5.7.1

If f is a continuous function and $F' = f$, then

$$\int f(u(x))u'(x)\,dx = F(u(x)) + C$$

for all functions $u = u(x)$ which have values in the domain of f and continuous derivative u'

In other words:

$$\int f\big(u(x)\big)u'(x)dx = \int f(u)du$$

where the second integral, after being calculated for variable u, must be expressed in terms of $u = u(x)$

Integration by Substitution

Integration by substitution or change of variable

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$$\int f(u(x))u'(x)\,dx = F(u(x)) + C$$

for all functions $u = u(x)$ which have values in the domain of f and continuous derivative u'

It is a direct consequence of the chain rule of the derivative: if $F'(x) = f(x)$

$$\int f(u(x))u'(x)\,dx = \int F'(u(x))u'(x)\,dx \underset{\text{by the chain rule}}{=} \int \frac{d}{dx}[F(u(x))]\,dx = F(u(x)) + C$$

Integration by Substitution

Example

Calculate $\displaystyle\int \frac{1}{(3+5x)^2}\,dx$

Solution

Set $u = 3 + 5x$, $du = 5\,dx$. Then

$$\frac{1}{(3+5x)^2}\,dx = \frac{1}{u^2}\left(\frac{1}{5}\,du\right) = \frac{1}{5}u^{-2}\,du$$

and

$$\int \frac{1}{(3+5x)^2}\,dx = \frac{1}{5}\int u^{-2}\,du = -\frac{1}{5}u^{-1} + C = -\frac{1}{5(3+5x)} + C$$

Integration by Substitution

Integration by substitution or change of variable

$$\boxed{\int_a^b f(u(x))u'(x)\,dx = \int_{u(a)}^{u(b)} f(u)\,du.}$$

This formula is called the *change-of-variables formula*. The formula can be used to evaluate $\int_a^b f\left(u(x)\right)u'(x)\,dx$ provided that u' is continuous on $[a, b]$ and f is continuous on the set of values taken on by u on $[a, b]$. Since u is continuous, this set is an interval that contains a and b.

Integration by Substitution

Example

Calculate

$$\int_0^1 \frac{e^x}{e^x+2}\,dx$$

Solution

Set $u = e^x + 2$, $du = e^x dx$. At $x = 0$, $u = 3$; *at* $x = 1$, $u = e + 2$.

Thus

$$\int_0^1 \frac{e^x}{e^x+2}\,dx = \int_3^{e+2} \frac{du}{u} = \Big[\ln|u|\Big]_3^{e+2}$$

Formula 3

$$= \ln(e+2) - \ln 3 = \ln\left[\tfrac{1}{3}(e+2)\right] \cong 0.45.$$

Integration by Parts

Integration by parts

$$\boxed{\int u(x)v'(x)\,dx = u(x)v(x) - \int v(x)u'(x)\,dx.}$$

It is a direct consequence of the derivative of the product:

$$\big(u(x)v(x)\big)' = u'(x)v(x) + u(x)v'(x)$$

Integrating both sides:

$$\int \big(u(x)v(x)\big)' dx = u(x)v(x) = \int u'(x)v(x)dx + \int u(x)v'(x)dx$$

Integration by Parts

Integration by parts

$$\boxed{\int u(x)v'(x)\,dx = u(x)v(x) - \int v(x)u'(x)\,dx.}$$

Usually, we write

$$\begin{aligned} u &= u(x), \qquad & dv &= v'(x)\,dx \\ du &= u'(x)\,dx, \qquad & v &= v(x). \end{aligned}$$

Then the formula for integration by parts reads

$$\boxed{\int u\,dv = uv - \int v\,du.}$$

Integration by Parts

Example
Calculate $\int x^2 e^{-x} dx$

Solution
Setting $u = x^2$ and $v' = e^{-x}$, we have $u' = 2x$ and $v = -e^{-x}$
This gives

$$\int x^2 e^{-x} dx = -x^2 e^{-x} - \int (-2x) e^{-x} dx = -x^2 e^{-x} + \int 2x e^{-x} dx$$

We now calculate the integral on the right, again by parts. This time we set $u = 2x$ and $v' = e^{-x}$, we have $u' = 2$ and $v = -e^{-x}$
and thus

$$\int 2x e^{-x} dx = -2x e^{-x} + \int 2 e^{-x} dx = -2x e^{-x} - 2e^{-x}$$

Combining this with our earlier calculations, we have

$$\int x^2 e^{-x} dx = -x^2 e^{-x} - 2x e^{-x} - 2e^{-x} = -\left(x^2 + 2x + 2\right) e^{-x} + C$$

Integration by Parts

Example
Calculate $\int x^2 e^{-x} dx$

Solution
Setting $u = x^2$ and $dv = e^{-x}\,dx$, we have $du = 2x\,dx$ and $v = -e^{-x}$.
This gives

$$\int x^2 e^{-x} dx = \int u\ dv = uv - \int v\ du = -x^2 e^{-x} - \int -2xe^{-x} dx = -x^2 e^{-x} + \int 2xe^{-x} dx$$

We now calculate the integral on the right, again by parts. This time we set
$u = 2x$ and $dv = e^{-x}\,dx$ which gives $du = 2\,dx$ and $v = -e^{-x}$
and thus

$$\int 2xe^{-x} dx = \int u\ dv = uv - \int v\ du = -2xe^{-x} - \int -2e^{-x} dx = -2xe^{-x} + \int 2e^{-x} dx = -2xe^{-x} - 2e^{-x} + C$$

Combining this with our earlier calculations, we have

$$\int x^2 e^{-x} dx = -x^2 e^{-x} - 2xe^{-x} - 2e^{-x} + C = -\left(x^2 + 2x + 2\right)e^{-x} + C$$

Integration by Parts

Integration by parts for definite integrals

$$\boxed{\int_a^b u(x)v'(x)\,dx = \Big[u(x)v(x)\Big]_a^b - \int_a^b v(x)u'(x)\,dx.}$$

Integration by Parts

Through integration by parts, we construct an antiderivative for the logarithm, for the arc sine, and for the arc tangent.

$$\boxed{\int \ln x \, dx = x \ln x - x + C.}$$

$$\boxed{\int \arcsin x \, dx = x \arcsin x + \sqrt{1 - x^2} + C.}$$

$$\boxed{\int \arctan x \, dx = x \arctan x - \tfrac{1}{2} \ln(1 + x^2) + C.}$$

Integration by Parts

To find the integral of the arc sine, we set

$$u = \arcsin x, \qquad dv = dx$$
$$du = \frac{1}{\sqrt{1-x^2}}\,dx, \qquad v = x$$

This gives

$$\int \arcsin x\; dx = x \arcsin x - \int \frac{x}{\sqrt{1-x^2}}\; dx = x \arcsin x + \sqrt{1-x^2} + C$$

Powers and Products of Trigonometric Functions

Integrals of trigonometric powers and products can usually be reduced to elementary integrals by the imaginative use of the basic trigonometric identities and, here and there, some integration by parts. These are the identities that we'll rely on:

Sign relations

$$\sin(-\alpha) = -\sin\alpha \qquad \cos(-\alpha) = \cos\alpha \qquad \tan(-\alpha) = -\tan\alpha$$

Unit circle relations

$$\sin^2\alpha + \cos^2\alpha = 1 \qquad \tan^2\alpha + 1 = \sec^2\alpha \qquad \cot^2\alpha + 1 = \csc^2\alpha$$

Addition formulas

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\tan(\alpha\pm\beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$$

Double-angle formulas

$$\sin 2\alpha = 2\sin\alpha\cos\alpha$$
$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$$
$$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}$$

Half-angle formulas

$$2\sin^2\alpha = 1 - \cos 2\alpha$$
$$2\cos^2\alpha = 1 + \cos 2\alpha$$
$$\tan^2\alpha = \frac{1-\cos 2\alpha}{1+\cos 2\alpha}$$

Product-to-sum formulas

$$2\sin\alpha\cos\beta = \sin(\alpha+\beta) + \sin(\alpha-\beta)$$
$$2\cos\alpha\cos\beta = \cos(\alpha+\beta) + \cos(\alpha-\beta)$$
$$2\sin\alpha\sin\beta = \cos(\alpha-\beta) - \cos(\alpha+\beta)$$

Powers and Products of Trigonometric Functions

Sines and Cosines

Example
Calculate

$$\int \sin^2 x \cos^5 x \, dx$$

Solution
The relation $\cos^2 x = 1 - \sin^2 x$ enables us to express $\cos^4 x$ in terms of $\sin x$. The integrand then becomes (a polynomial in $\sin x$) $\cos x$, an expression that we can integrate by the chain rule.

$$\begin{aligned}
\int \sin^2 x \cos^5 x \, dx &= \int \sin^2 x \cos^4 x \cos x \, dx \\
&= \int \sin^2 x \left(1 - \sin^2 x\right)^2 \cos x \, dx \\
&= \int \left(\sin^2 x - 2\sin^4 x + \sin^6 x\right) \cos x \, dx \\
&= \frac{1}{3}\sin^3 x - \frac{2}{5}\sin^5 x + \frac{1}{7}\sin^7 x + C
\end{aligned}$$

Powers and Products of Trigonometric Functions

Example

Calculate

$$\int \sin^2 x\, dx \quad \text{and} \quad \int \cos^2 x\, dx$$

Solution

Since $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$ and $\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x$

$$\int \sin^2 x\, dx = \int \left(\frac{1}{2} - \frac{1}{2}\cos 2x\right) dx = \frac{1}{2}x - \frac{1}{4}\sin 2x + C$$

and

$$\int \cos^2 x\, dx = \int \left(\frac{1}{2} + \frac{1}{2}\cos 2x\right) dx = \frac{1}{2}x + \frac{1}{4}\sin 2x + C$$

Integration by Substitution (2)

Another change of variables

$$\text{if } F' = f, \qquad \text{then} \qquad \int f(x(u))x'(u)\,du = F(x(u)) + C$$

and

$$\int_{x(a)}^{x(b)} f(x)\,dx = \int_a^b f(x(u))x'(u)\,du.$$

Note that instead of $u = u(x)$ we have now $x = x(u)$

Integrals involving $\sqrt{a^2 - x^2}, \sqrt{a^2 + x^2}, \sqrt{x^2 - a^2}$

Integrals that feature $\sqrt{a^2 - x^2}, \sqrt{a^2 + x^2}$ or $\sqrt{x^2 - a^2}$ can often be calculated by a *trigonometric substitution*. Taking $a > 0$, we proceed as follows:

For $\sqrt{a^2 - x^2}$ we use

$$x = a \sin u, \quad dx = a \cos u \,\mathrm{du}, \quad \sqrt{a^2 - x^2} = a\sqrt{1 - \sin^2 u} = a \cos u$$

For $\sqrt{a^2 + x^2}$ we use

$$x = a \tan u, \quad dx = \frac{a}{\cos^2 u} du, \quad \sqrt{a^2 + x^2} = a\sqrt{1 + \tan^2 u} = \frac{a}{\cos u} = a \sec u$$

For $\sqrt{x^2 - a^2}$ we use

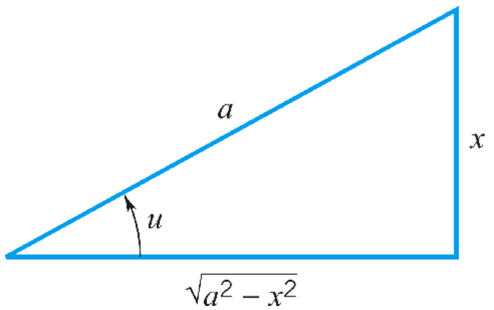
$$x = a \sec u, \quad dx = \frac{a \sin u}{\cos^2 u} du, \quad \sqrt{x^2 - a^2} = a\sqrt{\sec^2 u - 1} = a \tan u$$

In making such substitutions, we must make clear exactly what values of u we are using.

Integrals involving $\sqrt{a^2 - x^2}, \sqrt{a^2 + x^2}, \sqrt{x^2 - a^2}$

Example

To calculate $\displaystyle\int \frac{dx}{\left(a^2 - x^2\right)^{3/2}}\, dx$



we note that the integral can be written

$$\int \left(\frac{1}{\sqrt{a^2 - x^2}} \right)^3 dx$$

This integral features $\sqrt{a^2 - x^2}$. For each x between $-a$ and a, we set

$$x = a \sin u, \quad dx = a \cos u \, du,$$

taking u between $-\tfrac{1}{2}\pi$ and $\tfrac{1}{2}\pi$. For such u, $\cos u > 0$ and $\sqrt{a^2 - x^2} = a \cos u$

Therefore
$$\begin{aligned}
\int \frac{dx}{(a^2 - x^2)^{3/2}} &= \int \frac{a \cos u}{(a \cos u)^3}\, du \\
&= \frac{1}{a^2} \int \frac{1}{\cos^2 u}\, du \\
&= \frac{1}{a^2} \int \sec^2 u \, du \\
&= \frac{1}{a^2} \tan u + C = \frac{x}{a^2 \sqrt{a^2 - x^2}} + C.
\end{aligned}$$

$$\tan u = \frac{\sin u}{\cos u}$$

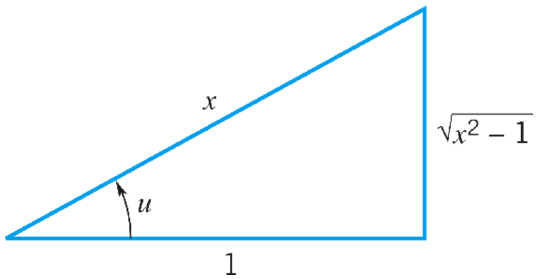
Integrals involving $\sqrt{a^2 - x^2}, \sqrt{a^2 + x^2}, \sqrt{x^2 - a^2}$

$$\boxed{\int \sqrt{a^2 + x^2}\, dx = \tfrac{1}{2} x \sqrt{a^2 + x^2} + \tfrac{1}{2} a^2 \ln\left(x + \sqrt{a^2 + x^2}\right) + C.}$$

Integrals involving $\sqrt{a^2 - x^2}, \sqrt{a^2 + x^2}, \sqrt{x^2 - a^2}$

Example

We calculate $\displaystyle\int \frac{dx}{\sqrt{x^2 - 1}}$



The domain of the integrand consists of two separated sets: all $x > 1$ and all $x < -1$. Both for $x > 1$ and $x < -1$, we set

$$x = \sec u, \quad dx = \sec u \tan u \, du.$$

For $x > 1$ we take u between 0 and $\frac{1}{2}\pi$; for $x < -1$ we take u between π and $\frac{3}{2}\pi$. For such u, $\tan u > 0$ and

$$\sqrt{x^2 - 1} = \tan u$$

Therefore,

$$\int \frac{dx}{\sqrt{x^2 - 1}} = \int \frac{\sec u \tan u}{\tan u} \, du = \int \sec u \, du$$

$$= \ln\left|\sec u + \tan u\right| + C$$

$$= \ln\left|x + \sqrt{x^2 - 1}\right| + C$$

Integrals involving $\sqrt{a^2 - x^2}, \sqrt{a^2 + x^2}, \sqrt{x^2 - a^2}$

Trigonometric substitutions can be effective in cases where the quadratic is not under a radical sign. In particular, the reduction formula

$$\boxed{\int \frac{dx}{(x^2 + a^2)^n}\, dx = \frac{1}{a^{2n-1}} \int \cos^{2(n-1)} u \, du}$$

(a very useful little formula) can be obtained by setting $x = a \tan u$, taking u between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$.

Rational Functions

A rational function $R(x) = P(x)/Q(x)$ is said to be *proper* if the degree of the numerator is less than the degree of the denominator. If the degree of the numerator is greater than or equal to the degree of the denominator, then the rational function is called *improper*. We will focus our attention on *proper rational functions* because any improper rational function can be written as a sum of a polynomial and a proper rational function:

$$\frac{P(x)}{Q(x)} = p(x) + \frac{r(x)}{Q(x)}$$

This is obtained by division of polynomials

$$\begin{array}{r|l} P(x) & Q(x) \\ \cline{2-2} r(x) & p(x) \end{array}$$

which means

$$P(x) = p(x)Q(x) + r(x)$$

and where $\deg r(x) < \deg Q(x)$

Rational Functions

Example

$$\frac{x^3+1}{x^2-4} = \frac{x^3-4x+4x+1}{x^2-4} = \frac{x(x^2-4)+4x+1}{x^2-4} = x + \frac{4x+1}{x^2-4}$$

Rational Functions

Example

$$\frac{6x^4 + 5x^3 + 4x - 4}{2x^2 + x - 1} = 3x^2 + x + 1 + \frac{4x - 3}{2x^2 + x - 1}$$

since

$$\begin{array}{rrrrr|l}
6x^4 & +\,5x^3 & & +\,4x & -\,4 & \;2x^2 + x - 1 \\ \cline{6-6}
-(6x^4 & +\,3x^3 & -\,3x^2 & &) & \;3x^2 + x + 1 \\ \cline{1-5}
 & 2x^3 & +\,3x^2 & +\,4x & -\,4 & \\
 & -(2x^3 & +\,x^2 & -\,x &) & \\ \cline{2-5}
 & & 2x^2 & +\,5x & -\,4 & \\
 & & -(2x^2 & +\,x & -\,1) & \\ \cline{3-5}
 & & & 4x & -\,3 &
\end{array}$$

Rational Functions

Every polynomial

$$q(x) = x^m + b_{m-1}x^{m-1} + \cdots + b_0$$

can be decomposed as a product of polynomials of degree 1 and/or 2

$$q(x) = (x-\alpha_1)^{r_1} \cdot \ldots \cdot (x-\alpha_k)^{r_k}(x^2+\beta_1 x+\gamma_1)^{s_1} \cdot \ldots \cdot (x^2+\beta_l x+\gamma_l)^{s_l}$$

with

$$r_1 + \cdots + r_k + 2(s_1 + \cdots + s_l) = m$$

$${\beta_i}^2 - 4\gamma_i < 0$$

Rational Functions

Every proper rational function can be written as the sum of **partial fractions**, fractions of the form

$$\boxed{\frac{A}{(x-\alpha)^k} \quad \text{and} \quad \frac{Bx+C}{(x^2+\beta x+\gamma)^k}}$$

Rational Functions

The denominator splits into distinct linear factors

In general, each distinct linear factor $x - \alpha$ in the denominator gives rise to a term of the form

$$\frac{A}{x-\alpha}.$$

Rational Functions

The denominator has a repeated linear factor

In general, each factor of the form $(x - a)^k$ in the denominator gives rise to an expression of the form

$$\frac{A_1}{x-\alpha} + \frac{A_2}{(x-\alpha)^2} + \cdots + \frac{A_k}{(x-\alpha)^k}.$$

Rational Functions

The denominator has an irreducible quadratic factor

In general, each irreducible quadratic factor $x^2 + \beta x + \gamma$ in the denominator gives rise to a term of the form

$$\frac{Ax + B}{x^2 + \beta x + \gamma}.$$

Rational Functions

The denominator has a repeated irreducible quadratic factor

In general, each multiple irreducible quadratic factor $(x^2 + \beta x + \gamma)^k$ in the denominator gives rise to an expression of the form

$$\frac{A_1 x + B_1}{x^2 + \beta x + \gamma} + \frac{A_2 x + B_2}{(x^2 + \beta x + \gamma)^2} + \cdots + \frac{A_k x + B_k}{(x^2 + \beta x + \gamma)^k}.$$

Rational Functions

Example

$$\frac{2x}{x^2 - x - 2} = \frac{2x}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1}$$

To find A and B

$$\frac{A}{x-2} + \frac{B}{x+1} = \frac{A(x+1) + B(x-2)}{(x-2)(x+1)}$$

thus

$$A(x+1) + B(x-2) = 2x$$

Method 1: substitute values for x

If $x = 2$ then $3A = 4 \quad \Rightarrow \quad A = 4/3$

If $x = -1$ then $-3B = -2 \quad \Rightarrow \quad B = 2/3$

Method 2: two polynomials are equal iff all the coefficients are equal

$(A + B)x + (A - 2B) = 2x$

$$\begin{cases} A + B = 2 \\ A - 2B = 0 \end{cases} \Rightarrow \begin{cases} A = 4/3 \\ B = 2/3 \end{cases}$$

Rational Functions

Example

$$\frac{2x^7+8x^6+13x^5+20x^4+15x^3+16x^2+7x+10}{(x^2+x+1)^2(x^2+2x+2)(x-1)^2}$$

$$=\frac{a}{x-1}+\frac{b}{(x-1)^2}+\frac{cx+d}{x^2+2x+2}+\frac{ex+f}{x^2+x+1}+\frac{gx+h}{(x^2+x+1)^2}$$

$$\begin{aligned}
&2x^7+8x^6+13x^5+20x^4+15x^3+16x^2+7x+10\\
&= a(x-1)(x^2+2x+2)(x^2+x+1)^2+b(x^2+2x+2)(x^2+x+1)^2\\
&\quad +(cx+d)(x-1)^2(x^2+x+1)^2+(ex+f)(x-1)^2(x^2+2x+2)(x^2+x+1)\\
&\quad +(gx+h)(x-1)^2(x^2+2x+2).
\end{aligned}$$

$$\begin{aligned}
&a=1, \quad b=2, \quad c=1, \quad d=3,\\
&e=0, \quad f=0, \quad g=0, \quad h=1.
\end{aligned}$$

Rational Functions

Solution of integrals of rational functions

$$\int \frac{P(x)}{Q(x)}dx$$

1. Divide the polynomials if $\deg P(x) \geq \deg Q(x)$

$$\int \frac{P(x)}{Q(x)}dx = \int p(x)dx + \int \frac{r(x)}{Q(x)}dx$$

2. Decompose the integral of the proper rational function into sum of integrals of the form

$$\int \frac{A}{(x-\alpha)^k}dx$$

$$\int \frac{Ax+B}{(x^2+\beta x+\gamma)^k}dx$$

Rational Functions

Solution of integrals of rational functions

3. Solve the remaining integrals using

- Case 1

$$\int \frac{A}{x-\alpha}\,dx = A\ln(x-\alpha) + C$$

- Case 2

$$\int \frac{A}{(x-\alpha)^k}\,dx = A\int (x-\alpha)^{-k}dx = \frac{A}{(1-k)(x-\alpha)^{k-1}} + C, \qquad k > 1$$

Rational Functions

Solution of integrals of rational functions

3. Solve the remaining integrals using

- Case 3

$$\int \frac{Ax+B}{x^2+\beta x+\gamma}\,dx = A\int \frac{x}{x^2+\beta x+\gamma}\,dx + B\int \frac{1}{x^2+\beta x+\gamma}\,dx$$

$$= \frac{A}{2}\int \frac{2x+\beta-\beta}{x^2+\beta x+\gamma}\,dx + B\int \frac{1}{\left(x+\frac{\beta}{2}\right)^2+\frac{4\gamma-\beta^2}{4}}\,dx$$

$$= \frac{A}{2}\int \frac{2x+\beta}{x^2+\beta x+\gamma}\,dx + \left(B-\frac{\beta A}{2}\right)\int \frac{1}{\left(x+\frac{\beta}{2}\right)^2+\frac{4\gamma-\beta^2}{4}}\,dx$$

$$= \frac{A}{2}\ln\left|x^2+\beta x+\gamma\right| + \frac{2B-\beta A}{\sqrt{4\gamma-\beta^2}}\arctan\left(\frac{2x+\beta}{\sqrt{4\gamma-\beta^2}}\right)+C$$

Rational Functions

Solution of integrals of rational functions

3. Solve the remaining integrals using

- Case 4

$$\int \frac{Ax+B}{(x^2+\beta x+\gamma)^k}\,dx = \frac{A}{2}\int \frac{2x+\beta}{(x^2+\beta x+\gamma)^k}\,dx + \left(B-\frac{\beta A}{2}\right)\int \frac{1}{(x^2+\beta x+\gamma)^k}\,dx$$

$$= \frac{A}{2(1-k)(x^2+\beta x+\gamma)^{k-1}} + \left(B-\frac{\beta A}{2}\right)\int \frac{1}{\left[\left(x+\frac{\beta}{2}\right)^2+\frac{4\gamma-\beta^2}{4}\right]^k}\,dx$$

$$I_k = \int \frac{dt}{(t^2+m^2)^k} = \frac{1}{m^2}\int \frac{m^2+t^2-t^2}{(t^2+m^2)^k}\,dt = \frac{1}{m^2}I_{k-1} - \frac{1}{m^2}\int \frac{t^2\,dt}{(t^2+m^2)^k}$$

By parts: $\displaystyle \int \frac{t^2\,dt}{(t^2+m^2)^k} = \frac{-1}{2(k-1)}\left[\frac{t}{(t^2+m^2)^{k-1}} - I_{k-1}\right]$

$$I_k = \frac{t}{2m^2(k-1)(t^2+m^2)^{k-1}} + \frac{2k-3}{2m^2(k-1)}I_{k-1}, \qquad t = x+\frac{\beta}{2},\ m^2 = \gamma-\frac{\beta^2}{4}$$

Rational Functions

Example 1

$$\int \frac{6x^4 + 5x^3 + 4x - 4}{2x^2 + x - 1}\,dx$$

Solution

First, divide to obtain a proper fraction, as already done a few slides before

$$\frac{6x^4 + 5x^3 + 4x - 4}{2x^2 + x - 1} = 3x^2 + x + 1 + \frac{4x - 3}{2x^2 + x - 1}$$

Thus

$$\int \frac{6x^4 + 5x^3 + 4x - 4}{2x^2 + x - 1}\,dx = x^3 + \frac{x^2}{2} + x + \int \frac{4x - 3}{2x^2 + x - 1}\,dx$$

To calculate the remaining integral, first decompose it in sum of partial fractions. Since $x = -1$ is a zero of $2x^2 + x - 1$, then $2x^2 + x - 1 = (x + 1)(2\mathrm{x} - 1)$ and

$$\frac{4x - 3}{2x^2 + x - 1} = \frac{4x - 3}{(x + 1)(2\mathrm{x} - 1)} = \frac{A}{x + 1} + \frac{B}{2x - 1} = \frac{A(2x - 1) + B(x + 1)}{(x + 1)(2\mathrm{x} - 1)}$$

Rational Functions

Example 1

$$\int \frac{6x^4 + 5x^3 + 4x - 4}{2x^2 + x - 1}\, dx$$

Solution

$$\frac{4x - 3}{(x+1)(2\mathrm{x} - 1)} = \frac{A(2x-1) + B(x+1)}{(x+1)(2\mathrm{x} - 1)}$$

We have to find A and B such that $4x - 3 = A(2x - 1) + B(x + 1)$

Setting $x = -1 \quad \Rightarrow \quad -7 = -3A \quad \Rightarrow \quad A = \frac{7}{3}$

Setting $x = \frac{1}{2} \quad \Rightarrow \quad -1 = \frac{3}{2}B \quad \Rightarrow \quad B = -\frac{2}{3}$

Therefore

$$\int \frac{4x - 3}{2x^2 + x - 1}\, dx = \frac{7}{3}\int \frac{1}{x+1}\, dx - \frac{2}{3}\int \frac{1}{2x - 1}\, dx = \frac{7}{3}\ln|x + 1| - \frac{1}{3}\ln|2x - 1|$$

Finally

$$\int \frac{6x^4 + 5x^3 + 4x - 4}{2x^2 + x - 1}\, dx = x^3 + \frac{x^2}{2} + x + \frac{7}{3}\ln|x + 1| - \frac{1}{3}\ln|2x - 1| + C$$

Rational Functions

Example 2

$$\int \frac{50}{(x^2+4)(x-1)^2}\,dx$$

Solution

$$\frac{50}{(x^2+4)(x-1)^2} = \frac{Ax+B}{x^2+4} + \frac{C}{x-1} + \frac{D}{(x-1)^2}$$

$$= \frac{(Ax+B)(x-1)^2 + C\big(x^2+4\big)(x-1) + D(x^2+4)}{(x^2+4)(x-1)^2}$$

We have to find the coefficients such that

$$(Ax+B)(x-1)^2 + C\big(x^2+4\big)(x-1) + D\big(x^2+4\big) = 50$$

Setting $x = 1 \quad \Rightarrow \quad 5D = 50 \quad \Rightarrow \quad D = 10$

Setting $x = 0 \quad \Rightarrow \quad B - 4C + 40 = 50 \quad \Rightarrow \quad B = 10 + 4C$

The coefficient of x^3 is $A + C \quad \Rightarrow \quad A + C = 0 \quad \Rightarrow \quad A = -C$

The numerator becomes $5\big((C+4)x^2 - (C+4)x + 10\big) = 50 \quad \Rightarrow \quad C = -4$

Thus, $A = 4,\ B = -6,\ C = -4,\ D = 10$

$$\frac{50}{(x^2+4)(x-1)^2} = \frac{4x-6}{x^2+4} - \frac{4}{x-1} + \frac{10}{(x-1)^2}$$

Rational Functions

Example 2

$$\int \frac{50}{(x^2+4)(x-1)^2}\,dx$$

Solution

$$\int \frac{50}{(x^2+4)(x-1)^2}\,dx = \int \frac{4x-6}{x^2+4}\,dx - 4\int \frac{1}{x-1}\,dx + 10\int \frac{1}{(x-1)^2}\,dx$$

$$= \int \frac{4x}{x^2+4}\,dx - 6\int \frac{1}{x^2+4}\,dx - 4\ln|x-1| - \frac{10}{x-1}$$

$$= 2\ln\left|x^2+4\right| - \frac{3}{2}\int \frac{1}{\left(\frac{x}{2}\right)^2+1}\,dx - 4\ln|x-1| - \frac{10}{x-1}$$

$$= 2\ln\left|x^2+4\right| - 3\arctan\left(\frac{x}{2}\right) - 4\ln|x-1| - \frac{10}{x-1} + C$$

$$= 2\ln\frac{x^2+4}{(x-1)^2} - 3\arctan\left(\frac{x}{2}\right) - \frac{10}{x-1} + C$$

Some Rationalizing Substitutions

There are integrands which are not rational functions but can be transformed into rational functions by a suitable substitution. Such substitutions are known as *rationalizing substitutions.*

Example

Find $\int \frac{dx}{1+\sqrt{x}}$

Solution

To rationalize the integrand, we set

$$u^2 = x, \quad 2u\,du = dx,$$

taking $u \geq 0$. Then $u = \sqrt{x}$ and

$$\int \frac{dx}{1+\sqrt{x}} = \int \frac{2u}{1+u}du \underset{\text{divide}}{=} \int \left(2 - \frac{2}{1+u}\right) du$$

$$= 2u - 2\ln(1+u) + C$$

$$\underset{1+u>0}{=} 2\sqrt{x} - 2\ln(1+\sqrt{x}) + C.$$

Some Rationalizing Substitutions

Example

Find $\int \sqrt{1-e^x}\, dx$

Solution

To rationalize the integrand, we set $u = \sqrt{1-e^x}$

Then $0 \le u < 1$. To express dx in terms of u and du, we solve the equation for x:

$$u^2 = 1 - e^x, \qquad 1 - u^2 = e^x, \qquad \ln(1-u^2) = x, \qquad -\frac{2u}{1-u^2}\, du = dx.$$

$$\begin{aligned}
\int \sqrt{1-e^x}\, dx &= \int u\left(-\frac{2u}{1-u^2}\right) du \\
&= \int \frac{2u^2}{u^2-1}\, du = \int \left(2 + \frac{1}{u-1} - \frac{1}{u+1}\right) du
\end{aligned}$$

divide; then use partial fractions

$$\begin{aligned}
&= 2u + \ln|u-1| - \ln|u+1| + C \\
&= 2u + \ln\left|\frac{u-1}{u+1}\right| + C \\
&= 2\sqrt{1-e^x} + \ln\left|\frac{\sqrt{1-e^x}-1}{\sqrt{1-e^x}+1}\right| + C. \quad \square
\end{aligned}$$

Some Rationalizing Substitutions

Integració de binomis diferencials

$$\int x^m (a + bx^n)^p dx$$

on m, n i p son nombres **racionals** i els coeficients a i b son nombres reals, $a\, b \neq 0$. Aquestes integrals són un cas particular d'integrals irracionals.

Pel *Teorema de Chevyshev*, aquestes integrals es poden expressar en termes de funcions elementals **només** en els següents casos:

a) Si p enter: fent la substitució $x = t^s$, amb s el mínim comú múltiple dels denominadors de m i n
b) Si $\frac{m+1}{n}$ enter: fent la substitució $a + bx^n = t^s$, amb s el denominador de p
c) Si $\frac{m+1}{n} + p$ enter: fent la substitució $\frac{a}{x^n} + b = t^s$, amb s el denominador de p

Amb els canvis de variable esmentats la integral es converteix en racional.

Some Rationalizing Substitutions

Exemple

$$\int \sqrt[3]{x}\left(1+\sqrt{x}\right)^2 dx$$

Solució

És binomi diferencial amb $m=\frac{1}{3}, n=\frac{1}{2}, p=2$. Com p enter fem $x=t^6$, $dx=6t^5dt$

$$\int \sqrt[3]{x}\left(1+\sqrt{x}\right)^2 dx = \int t^{6/3}\left(1+t^{6/2}\right)^2 6t^5 dt = 6\int t^7\left(1+t^3\right)^2 dt$$

$$= 6\int t^7(1+2t^3+t^6)dt = 6\frac{t^8}{8} + 12\frac{t^{11}}{11} + 6\frac{t^{14}}{14}$$

$$= \frac{3}{4}x^{8/6} + \frac{12}{11}x^{11/6} + \frac{3}{7}x^{14/6}$$

$$= x^{4/3}\left(\frac{3}{4} + \frac{12}{11}\sqrt{x} + \frac{3}{7}x\right)$$

$$= \frac{3}{308}x\sqrt[3]{x}\left(77 + 112\sqrt{x} + 44x\right) + C$$

Some Rationalizing Substitutions

Exemple

$$\int \sqrt[3]{1-\sqrt{x}}\,dx$$

Solució

$$\int \sqrt[3]{1-\sqrt{x}}\,dx = \int \left(1-x^{1/2}\right)^{1/3} dx$$

És binomi diferencial amb $m = 0, n = \frac{1}{2}, p = \frac{1}{3}$. Com $\frac{m+1}{n} = 2$ enter fem

$1-\sqrt{x} = t^3, -\frac{1}{2\sqrt{x}}dx = 3t^2 dt, dx = -6t^2\left(1-t^3\right)dt$

$$\int \sqrt[3]{1-\sqrt{x}}\,dx = -6\int \sqrt[3]{t^3}\; t^2\left(1-t^3\right)dt = -6\int \left(t^3-t^6\right)dt$$

$$= -6\frac{t^4}{4} + 6\frac{t^7}{7} = \frac{6}{28}t^4\left(4t^3-7\right)$$

$$= -\frac{3}{14}\left(1-\sqrt{x}\right)^{\frac{4}{3}}\left(4\sqrt{x}+3\right) + C$$

Some Rationalizing Substitutions

Exemple

$$\int \frac{1}{x^{11}\sqrt{2+3x^4}}\,dx$$

Solució

$$\int \frac{1}{x^{11}\sqrt{2+3x^4}}\,dx = \int x^{-11}\left(2+3x^4\right)^{-1/2}dx$$

És binomi diferencial amb $m=-11, n=4, p=-\frac{1}{2}$. Com $\frac{m+1}{n}+p=-3$ enter fem $2x^{-4}+3=t^2$, $-8x^{-5}dx=2tdt$, $dx=-\frac{1}{4}x^5tdt$

$$\int x^{-11}\left(2+3x^4\right)^{-1/2}dx = -\frac{1}{4}\int x^{-11}\left(t^2x^4\right)^{-\frac{1}{2}}x^5t\,dt = -\frac{1}{4}\int x^{-8}dt$$

$$= -\frac{1}{4}\int\left(\frac{t^2-3}{2}\right)^2 dt = -\frac{1}{16}\int\left(t^4-6t^2+9\right)dt = -\frac{1}{16}\left(\frac{t^5}{5}-2t^3+9t\right)$$

$$= -\frac{t}{80}\left(t^4-10t^2+45\right) = -\frac{1}{80}\frac{\sqrt{2+3x^4}}{x^2}\left(\frac{\left(2+3x^4\right)^2}{x^8}-10\frac{2+3x^4}{x^4}+45\right)$$

$$= -\frac{1}{20}\frac{\sqrt{2+3x^4}}{x^{10}}\left(6x^8-2x^4+1\right)+C$$

Some Rationalizing Substitutions

Rational functions of terms $\sin(x)$, $\cos(x)$ **and** $\tan(x)$

General solution

They are rationalized using

$$t = \tan\frac{x}{2}$$

since

$$x = 2\arctan t\,, \qquad dx = \frac{2dt}{1+t^2}$$

$$\sin x = \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\sin^2\frac{x}{2} + \cos^2\frac{x}{2}} = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \frac{2t}{1+t^2}$$

$$\cos x = \frac{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}}{\cos^2\frac{x}{2} + \sin^2\frac{x}{2}} = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \frac{1-t^2}{1+t^2}$$

$$\tan x = \frac{2t}{1-t^2}$$

Some Rationalizing Substitutions

Rational functions of terms $\sin(x)$, $\cos(x)$ **and** $\tan(x)$

Example

Using the change of variable $t = \tan\frac{x}{2}$ we can solve

$$\int \frac{1}{\sin x}\,dx = \int \frac{1}{\frac{2t}{1+t^2}}\,\frac{2dt}{1+t^2} = \int \frac{dt}{t} = \ln|t| = \ln\left|\tan\frac{x}{2}\right| + C$$

Some Rationalizing Substitutions

Rational functions of terms $\tan(x)$**,** $\sin^2(x)$ **and** $\cos^2(x)$

General solution

They are rationalized using

$$t = \tan x$$

since

$$x = \arctan t\,, \qquad dx = \frac{dt}{1+t^2}$$

$$\cos^2 x = \frac{1}{1+\tan^2 x} = \frac{1}{1+t^2}$$

$$\sin^2 x = \frac{\tan^2 x}{1+\tan^2 x} = \frac{t^2}{1+t^2}$$

$$\tan x = t$$

Some Rationalizing Substitutions

Rational functions of terms $\tan(x)$**,** $\sin^2(x)$ **and** $\cos^2(x)$

Examples

Using the change of variable $t = \tan x$ we can solve

$$\int \frac{1}{\sin^4 x}dx = \int \frac{1}{\left(\frac{t^2}{1+t^2}\right)^2}\,\frac{dt}{1+t^2} = \int \frac{1+t^2}{t^4}dt = \frac{t^{-3}}{-3} + \frac{t^{-1}}{-1}$$

$$= -\left(\frac{1}{\tan x} + \frac{1}{3\tan^3 x}\right) + C = -\operatorname{ctg} x - \frac{1}{3}\operatorname{ctg}^3\ x + C$$

$$\int \frac{\tan^2 x - 1}{\cos^8 x}dx = \int \frac{t^2-1}{\left(\frac{1}{1+t^2}\right)^4}\,\frac{dt}{1+t^2} = \int (t^2-1)\left(1+t^2\right)^3 dt$$

$$= \int (t^8 + 2t^6 - 2t^2 - 1)dt = \frac{1}{9}\tan^9 x + \frac{2}{7}\tan^7 x - \frac{2}{3}\tan^3 x - \tan x + C$$

Nonelementary integrals

Integrals that cannot be expressed in terms of elementary functions (i.e., a finite number of quotients of constant, algebraic, exponential, trigonometric or logarithmic functions)

$$\int e^{-x^2}dx$$

$$\int \ln(\ln x)\,dx$$

$$\int \frac{1}{\ln x}dx$$

$$\int \frac{e^{-x}}{x}dx$$

$$\int \sin(x^2)dx$$

$$\int \frac{\sin x}{x}dx$$

$$\int x^{c-1}e^{-x}dx\,, \qquad c \notin \mathbb{Z}$$