5.1 $F(\vec{q},\vec{p})$, $G(\vec{q},\vec{p})$ constants of motion $\Rightarrow$

$\Rightarrow [F,H]=0$, $[G,H]=0$

$$\frac{d}{dt}[F,G] \overset{?}{=} 0$$

Jacobi's identity

$$\frac{d}{dt}[F,G] = [[F,G],H] = -[H,[F,G]] =$$

$$= [F,\cancel{[G,H]}^{\,0}] + [G,\cancel{[H,F]}^{\,0}] = 0 \quad \square$$

5.2 1

$$H = p_1^2 + \frac{p_2^2}{q_1} + \frac{(p_4 - q_4)^2}{q_1 \cos q_2} + \frac{p_3^2}{q_1 q_4 \cos q_2} - \frac{\sin^2 q_2}{q_1} + q_1^2$$

1.1 q_3 cyclic $\Rightarrow p_3 = \text{constant} \equiv a_3$

$$H = p_1^2 + q_1^2 + \underbrace{\frac{1}{q_1}\left[p_2^2 - \sin^2 q_2 + \underbrace{\frac{1}{\cos q_2}\left(\underbrace{(p_4 - q_4)^2 + \frac{a_3^2}{q_4}}_{a_4}\right)}_{a_2}\right]}_{a_1}$$

- $[a_4, a_3] = \sum_k \left(\frac{\partial a_4}{\partial q_k}\frac{\partial a_3}{\partial p_k} - \frac{\partial a_3}{\partial q_k}\frac{\partial a_4}{\partial p_k}\right) =$ 0 for $k \neq 3$

$$= \frac{\cancel{\partial q_4}^{\,0}}{\cancel{\partial q_3}}\frac{\partial a_3}{\partial p_3} - \frac{\cancel{\partial q_3}^{\,0}}{\cancel{\partial q_3}}\frac{\partial a_4}{\partial p_3} = 0 \quad \text{Trivial}$$

- $[a_2, a_3] = 0$ trivial
- $[a_1, a_3] = 0$ trivial

⋮

We cannot find any new non-trivial constants

(1.2) $$H = \frac{1}{2}\underbrace{\left(\frac{p_1}{q_1^2} + k q_1\right)}_{a_1 = \text{constant}}^2 + \left(\frac{p_1}{q_1^2} + k q_1\right) q_3 t + \frac{p_3^2}{2} + (p_2 + B q_3)^2$$

$$H = \frac{1}{2} a_1^2 + a_1 q_3 t + \frac{p_3^2}{2} + (p_2 + B q_3)^2$$

q_2 is cyclic $\Rightarrow$ $p_2 = \text{constant} \equiv a_2$

$$H = \frac{a_1^2}{2} + a_1 q_3 t + \frac{p_3^2}{2} + (a_2 + B q_3)^2$$

Let's try to use Poisson's theorem:

$$[a_1, a_2] = \sum_k \left(\frac{\partial a_1}{\partial q_k}\frac{\partial a_2}{\partial p_k} - \frac{\partial a_2}{\partial q_k}\frac{\partial a_1}{\partial p_k}\right) = 0 \quad \underline{\text{trivial}}$$

No new constants.

5.3

$$T = \sum_{i=1}^{n} f_i(q_i)\, \dot{q}_i^2$$

$$V = \sum_{i=1}^{n} V_i(q_i)$$

a) $\mathcal{L}$?

$$\mathcal{L} = \sum_i \left(f_i(q_i)\dot{q}_i^2 - V_i(q_i) \right)$$

b) H?

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i} = 2 f_i(q_i)\, \dot{q}_i \Rightarrow \dot{q}_i = \frac{p_i}{2 f_i(q_i)}$$

$$H = \sum_i p_i \dot{q}_i - \mathcal{L} = \sum_i \left[\frac{p_i^2}{2 f_i(q_i)} - f_i\cancel{(q_i)} \frac{p_i^2}{4 f_i^{\cancel{2}}(q_i)} + V_i(q_i) \right] =$$

$$= \sum_i \left[\frac{p_i^2}{4 f_i(q_i)} + V_i(q_i) \right] = \sum_i H_i$$

$$H_i = \frac{p_i^2}{4 f_i(q_i)} + V_i(q_i)$$

c) Each H_i is conserved separately. H is also conserved.

5.4

$$u(q,p,t) = \ln(p + im\omega q) - i\omega t \quad , \quad \omega = \sqrt{\frac{k}{m}}$$

$$\parallel$$

$$k = \omega^2 m$$

$$H = \frac{p^2}{2m} + \frac{1}{2} k q^2$$

$$[u, H] = \frac{\partial u}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial H}{\partial q}\frac{\partial u}{\partial p} =$$

$$= \frac{i\cancel{m}\omega}{p + im\omega q} \cdot \frac{p}{\cancel{m}} - kq \frac{1}{p + im\omega q} =$$

$$= \frac{i\omega p - kq}{p + im\omega q} = \frac{i\omega p - \omega^2 m q}{p + im\omega q} =$$

$$= i\omega \frac{p - \frac{\omega m q}{i}}{p + im\omega q} = i\omega \frac{p + im\omega q}{p + im\omega q} =$$

$$= i\omega$$

$$\frac{du}{dt} = [u, H] + \frac{\partial u}{\partial t} = i\omega - i\omega = 0 \quad \square .$$

5.5

$$Q = p + iaq \quad ; \quad P = \frac{p - iaq}{2ia}$$

a) $[Q, P] = \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial P}{\partial q}\frac{\partial Q}{\partial p} = \frac{ia}{2ia} - \left(\frac{-1}{2}\cdot 1\right) = 1$ □.

b) $F_2(q, P, t)$?

$p = \frac{\partial F_2}{\partial q} \qquad Q = \frac{\partial F_2}{\partial P}$ Implicit transformation eqs.

We write p & Q as a function of q & P.

(1) $\frac{\partial F_2}{\partial q} = p = 2iaP + iaq = ia(2P + q)$

(2) $\frac{\partial F_2}{\partial P} = Q = p + iaq = 2iaP + iaq + iaq = 2ia(P + q)$

(1) $\Rightarrow F_2 = ia\left(2Pq + \frac{q^2}{2}\right) + f(P) \Rightarrow$

$\Rightarrow \frac{\partial F_2}{\partial P} = ia(2q) + \frac{d f(P)}{dP} \overset{(2)}{=} 2iaP + 2iaq$

$\Rightarrow \frac{df}{dP} = 2iaP \Rightarrow f = iaP^2$

$\Rightarrow \boxed{F_2 = ia\left(\frac{q^2}{2} + P^2 + 2Pq\right)}$

c) $H = \frac{p^2}{2m} + \frac{1}{2} k q^2$

We get Q & P as functions of q, p.

① $Q = p + iaq \Rightarrow Q = p + iaq$

② $P = \frac{p - iaq}{2ia} \Rightarrow 2iaP = p - iaq$

①+② $\Rightarrow p = \frac{Q}{2} + iaP$

①−② $\Rightarrow Q - 2iaP = 2iaq \Rightarrow q = \frac{Q}{2ia} - P$

$$p^2 = \frac{Q^2}{4} - a^2 P^2 + iaQP$$

$$q^2 = -\frac{Q^2}{4a^2} + P^2 - \frac{QP}{ia}$$

$$\boxed{\tilde{H} = \frac{1}{2m}\left(\frac{Q^2}{4} - a^2 P^2 + iaQP\right) + \frac{k}{2}\left(P^2 - \frac{Q^2}{2a^2} - \frac{QP}{ia}\right)}$$

d) $a = \sqrt{mk}$

$$\boxed{\tilde{H}} = \cancel{\frac{Q^2}{8m}} - \cancel{\frac{mkP^2}{2m}} + \frac{i\sqrt{mk}\,QP}{2m} + \cancel{\frac{kP^2}{2}} - \cancel{\frac{kQ^2}{8mk}} - \frac{k}{2}\frac{QP}{i\sqrt{mk}}$$

$$= i\sqrt{\frac{k}{m}}\frac{QP}{2} + i\sqrt{\frac{k}{m}}\frac{QP}{2} = \boxed{i\sqrt{\frac{k}{m}}\,QP}$$

$$\left.\begin{aligned}\dot{P} &= -\frac{\partial \tilde{H}}{\partial Q} = -i\sqrt{\frac{k}{m}}\,P \\ \dot{Q} &= \frac{\partial \tilde{H}}{\partial P} = i\sqrt{\frac{k}{m}}\,Q\end{aligned}\right\} \quad \sqrt{\frac{k}{m}} \equiv \omega$$

$$\frac{dP}{dt} = -i\omega P \Rightarrow \int \frac{dP}{P} = \int -i\omega\, dt \Rightarrow \log \frac{P}{P_0} = -i\omega(t-t_0)$$

$$\Rightarrow P = P_0\, e^{-i\omega(t-t_0)}$$

$$\ldots \Rightarrow Q = Q_0\, e^{i\omega(t-t_0)}$$

5.G

$$Q = \alpha \frac{p}{x} \qquad P = \beta x^2$$

necessary & sufficient

When is it canonical? We impose $[Q, P] = 1$

$$[Q,P] = \frac{\partial Q}{\partial x}\frac{\partial P}{\partial p} - \frac{\partial P}{\partial x}\frac{\partial Q}{\partial p} = -\frac{\alpha\beta}{x^2}\cdot 0 - 2\beta\cancel{x}\,\frac{\alpha}{\cancel{x}} = 1$$

$$\Rightarrow \boxed{\alpha\beta = -\frac{1}{2}} \Leftrightarrow \beta = -\frac{1}{2\alpha} \Rightarrow$$

$$\Rightarrow \boxed{Q = \alpha\frac{p}{x}, \quad P = -\frac{x^2}{2\alpha}}$$

Generating function:

$\rightarrow F_1(q, Q)$? $\quad p = \frac{Qx}{\alpha}, \quad P = -\frac{x^2}{2\alpha}$

$$p = \frac{\partial F_1}{\partial x} = \frac{Qx}{\alpha} \Rightarrow F_1 = \frac{Qx^2}{2\alpha} + f(Q)$$

$$P = -\frac{\partial F_1}{\partial Q} = -\frac{x^2}{2\alpha} - \frac{\partial f}{\partial Q} \stackrel{!}{=} -\frac{x^2}{2\alpha} \Rightarrow f = 0 \text{ is } \underline{OK}$$

$$\boxed{F_1 = \frac{Qx^2}{2\alpha}}$$

5.6

$$H = \frac{p_\phi^2}{4ma^2(1-\cos\phi)} - mag(1-\cos\phi)$$

a) $s = -4a\cos(\phi/2)$ $\quad p_s = \dfrac{p_\phi}{2a\sin(\phi/2)}$

$$[s, p_s] = \frac{\partial s}{\partial \phi}\frac{\partial p_s}{\partial p_\phi} - \frac{\partial p_s}{\partial \phi}\frac{\partial s}{\partial p_\phi} =$$

$$= \cancel{4}\cancel{a}\cancel{\sin\frac{\phi}{2}} \cdot \frac{1}{\cancel{2}} \frac{1}{\cancel{2}\cancel{a}\cancel{\sin\frac{\phi}{2}}} - 0 =$$

$$= 1 \quad \square.$$

b) $F_2(\phi, p_s)$?

$$p_\phi = \frac{\partial F_2}{\partial \phi} \qquad s = \frac{\partial F_2}{\partial p_s}$$

$$\frac{\partial F_2}{\partial \phi} = p_\phi = 2a\, p_s \sin\frac{\phi}{2} \Rightarrow F_2 = \int d\phi\; 2a p_s \sin\frac{\phi}{2} + f(p_s)$$

$$= -4a p_s \cos\frac{\phi}{2} + f(p_s)$$

$$s = \frac{\partial F_2}{\partial p_s} = -4a p_s \cos\frac{\phi}{2} + \frac{df}{dp_s} = -4a\cos\frac{\phi}{2} \Rightarrow$$

$\Rightarrow f(p_s) = \text{constant (or 0)}.$

$$\boxed{F_2 = -4a p_s \cos\frac{\phi}{2}}$$

b2) $F_1(s, \phi)$

$$p_\phi = \frac{\partial F_1}{\partial \phi} \qquad p_s = -\frac{\partial F_1}{\partial s}$$

$$p_\phi = 2a \underline{\underline{p_s}} \sin\frac{\phi}{2}$$ but this not expresed as a function of s!!

c) $H(s, p_s)$?

$$p_\phi = 2a\, p_s \sin\frac{\phi}{2} \Rightarrow p_\phi^2 = 4a^2 p_s^2 \sin^2\frac{\phi}{2}$$

Remember! $\sin\frac{\phi}{2} = \pm\sqrt{\frac{1-\cos\phi}{2}} \Rightarrow 1-\cos\phi = 2\sin^2\frac{\phi}{2}$

$$H = \frac{p_\phi^2}{4ma^2(1-\cos\phi)} - mag(1-\cos\phi) =$$

$$= \frac{\cancel{4}\cancel{a^2} p_s^2 \cancel{\sin^2\frac{\phi}{2}}}{\cancel{4}m\cancel{a^2}\, 2\cancel{\sin^2\frac{\phi}{2}}} - mag\, 2\sin^2\frac{\phi}{2} =$$

$$= \frac{p_s^2}{2m} - mag\, 2\left(1-\frac{s^2}{4^2a^2}\right) = \frac{p_s^2}{2m} + \frac{mg}{8a}s^2 + C \qquad \square.$$

$$\cos\frac{\phi}{2} = \frac{s}{-4a} \Rightarrow \cos^2\frac{\phi}{2} = \frac{s^2}{4^2a^2} \Rightarrow \sin^2\frac{\phi}{2} = 1 - \frac{s^2}{4^2a^2}$$

↳ from the given transformation