Numerical Methods

Boundary Value Problems

Ordinary Differential Equations

Course 2022-2023

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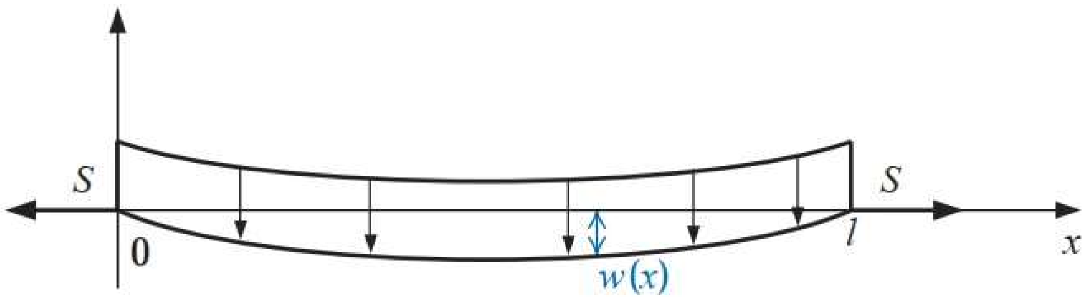
Boundary Value Problems

- A common problem in civil engineering concerns the deflection of a beam of rectangular cross section subject to uniform loading, while the ends of the beam are supported so that they undergo no deflection



Boundary Value Problems

- Suppose that l, q, E, S and I represent, respectively, the length of the beam, the intensity of the uniform load, the modulus of elasticity, the stress at the endpoints, and the central moment of inertia. The differential equation approximating the physical situation is:

$$\frac{d^2 w(x)}{dx^2} = \frac{S}{EI} w(x) + \frac{qx}{2EI}(x - l),$$

- Where $w(x)$ is the deflection a distance x from the left end of the beam. Since no deflection occurs at the ends, we have also

$$w(0) = 0 \quad \text{and} \quad w(l) = 0.$$

Boundary Value Problems

- This is an example of a Boundary Value Problem (BVP). We need to find approximate solutions to differential equations where the conditions are imposed at different points.
- In first-order differential equations, only one condition is specified, so there is no distinction between initial-value and boundary value problems. So, for this kind of problems, we will need to deal with ***second order differential equations.***

Boundary Value Problems

- Physical problems that are position-dependent rather that time-dependent are often described in terms of differential equations with conditions imposed at more than one point, but BVP's can also be associated with time dependent problems, particularly in the case of partial differential equations.
- From now on, we will discuss solutions to problems of the form

$$y'' = f(x, y, y'), \qquad a \le x \le b$$

$$y(a) = \alpha \quad \text{and} \quad y(b) = \beta$$

Existence and Unicity

- ***Theorem.*** Suppose the function f in the BVP problem:

$$y'' = f(x, y, y'), \qquad a \le x \le b$$
$$y(a) = \alpha \quad \text{and} \quad y(b) = \beta$$

- Is continuous on the set

$$D = \{(x, y, y') : a \le x \le b, -\infty < y < \infty, -\infty < y' < \infty\}$$

- And the partial derivatives f_y and $f_{y'}$ are also continuous on D. If:
  - $f_y(x, y, y') > 0$, for all $(x, y, y') \in D$, and
  - $|f_y(x, y, y')| \le M$, for all $(x, y, y') \in D$ and some constant M,
- Then the BVP has ***unique solution***.

Linear Boundary Value Problems

- A ***linear problem*** can be written as:

$$y'' = f(x, y, y') = p(x)y' + q(x)y + r(x)$$

- For this kind of problems, the existence and unicity of the solution is more relaxed. So, if
  - $p(x), q(x)$, and $r(x)$ are continuous on $[a, b]$
  - $q(x) > 0$ on $[a, b]$
- Then, the linear boundary value problem has a unique solution.

Linear Shooting

- To approximate the unique solution to this linear problem, we first consider the initial value problems:

$$(1)\ \ y'' = p(x)y' + q(x)y + r(x),\ \ a \le x \le b,\ \ y(a) = \alpha,\ \ y'(a) = 0$$

- And

$$(2)\ \ y'' = p(x)y' + q(x)y,\ \ a \le x \le b,\ \ y(a) = 0,\ \ y'(a) = 1$$

- If $y_1(x)$ is the solution to the first problem and $y_2(x)$ the solution of the second, with the condition that $y_2(b) \neq 0$, then the solution of the BVP is:

$$\boxed{y(x) = y_1(x) + \frac{\beta - y_1(b)}{y_2(b)} y_2(x).}$$

Linear Shooting

- We have:

$$y'(x) = y'_1(x) + \frac{\beta - y_1(b)}{y_2(b)} y'_2(x) \quad \text{and} \quad y''(x) = y''_1(x) + \frac{\beta - y_1(b)}{y_2(b)} y''_2(x)$$

- Substituting in the differential equation gives

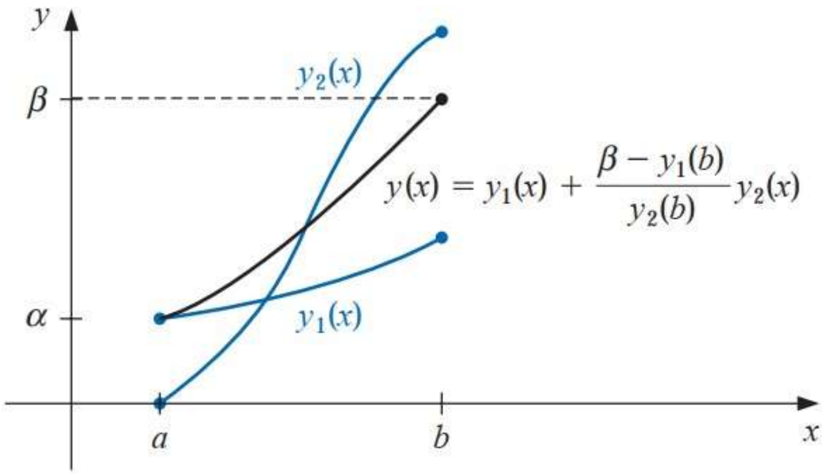
$$y'' = p(x)y_1' + q(x)y_1 + r(x) + \frac{\beta - y_1(b)}{y_2(b)}\left(p(x)y'_2 + q(x)y_2\right)$$

$$= p(x)\left(y'_1 + \frac{\beta - y_1(b)}{y_2(b)} y'_2\right) + q(x)\left(y_1 + \frac{\beta - y_1(b)}{y_2(b)} y_2\right) + r(x)$$

$$= p(x)y'(x) + q(x)y(x) + r(x)$$

Linear Shooting

- We can solve numerically the IVP's (1) and (2) and obtain the solutions $y_1(x)$ and $y_2(x)$. Once these approximations are available, we can build the solution $y(x)$ as a linear combination of these solutions.



Nonlinear Shooting

- The shooting technique for the nonlinear second order BVP:

$$y'' = f(x, y, y'), \quad a \le x \le b, \quad y(a) = \alpha, \quad y(b) = \beta$$

- Works like the linear technique, except that the solution to a nonlinear problem cannot be expressed as a linear combination of the solutions to two initial-value problems.
- Instead, we approximate the solution by using the solutions of a sequence of IVPs of the form:

$$y'' = f(x, y, y'), \quad a \le x \le b, \quad y(a) = \alpha, \quad y(b) = t$$

Nonlinear Shooting

- We do this by choosing the parameters $t = t_k$ ensuring that:

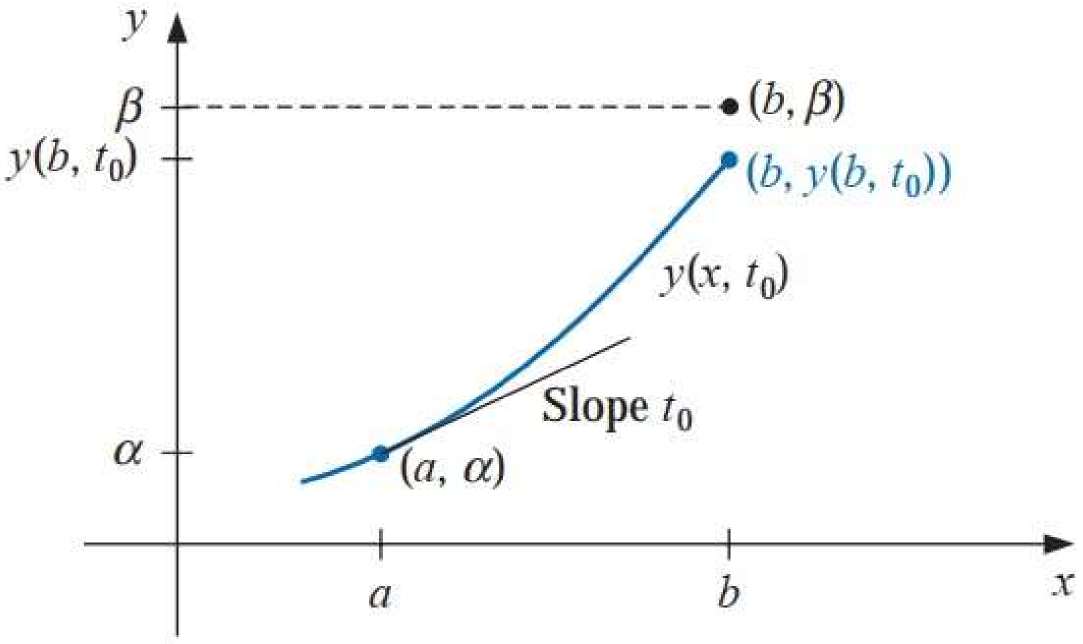
$$\lim_{k \to \infty} y(b, t_k) = y(b) = \beta$$

- Where $y(x, t_k)$ denotes the solution to the IVP with $t = t_k$ and $y(x)$ denotes the solution of the BVP.
- We start with a parameter t_0 that determines the initial elevation at which the object is fired from the point (a, α) along the IVP

$$y'' = f(x, y, y'), \quad a \le x \le b, \quad y(a) = \alpha, \quad y'(\alpha) = t_0$$

Numerical Methods

Nonlinear Shooting



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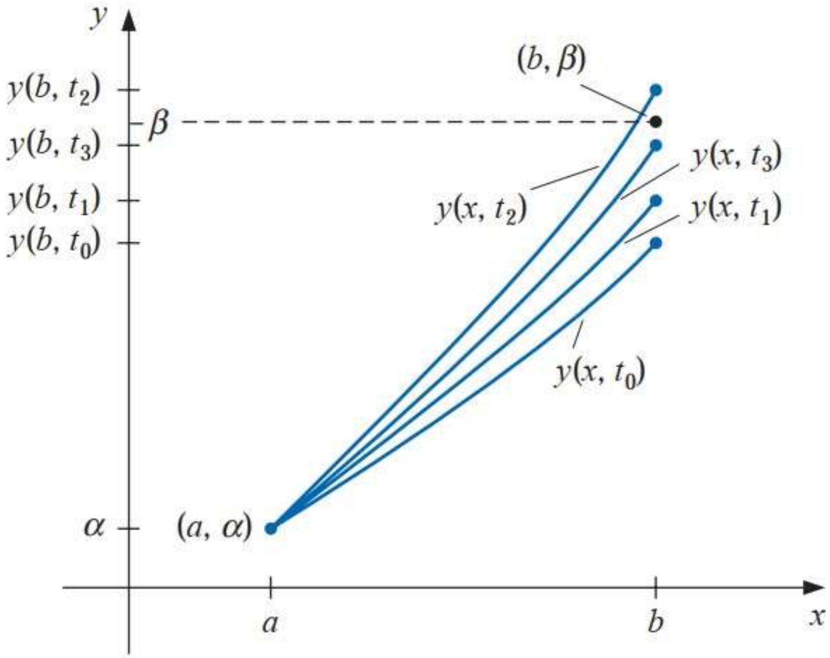
Nonlinear Shooting

- If $y(b, t_0)$ is not sufficiently close to β, we correct our approximation by choosing elevations t_1, t_2 and so on, until $y(b, t_k)$ is sufficiently close to β.



Nonlinear Shooting

- To determine the parameters t_k, suppose that a BVP satisfies the conditions of unicity and $y(x,t)$ denotes this unique solution. We next determine t solving the equation:

$$y(b,t)-\beta=0$$

- This is a nonlinear equation in the variable t that must be solved numerically. We may use the Secant method. Selecting the initial values t_0, t_1, we then generate the sequence:

$$t_k = t_{k-1} - \frac{\left(y(b,t_{k-1})-\beta\right)\left(t_{k-1}-t_{k-2}\right)}{y(b,t_{k-1})-y(b,t_{k-2})}, \quad k=2,3,\ldots$$

- Until we get the desired precision.

Linear Finite-Difference

- Consider again the linear problem:

$$y'' = p(x)y' + q(x)y + r(x), \quad y(a) = \alpha, \quad y(b) = \beta$$

- We can approximate the functions $y''(x)$ and $y'(x)$ using finite differences as follows:

$$y''(x_i) = \frac{1}{h^2}\left[y(x_{i+1}) - 2y(x_i) + y(x_{i-1})\right] - \frac{h^2}{12} y^{(4)}(\xi_i)$$

$$y'(x_i) = \frac{1}{2h}\left[y(x_{i+1}) - y(x_{i-1})\right] - \frac{h^2}{6} y^{(3)}(\eta_i)$$

Linear Finite-Difference

- Then the BVP can be approximated by

$$\frac{y(x_{i+1})-2y(x_i)+y(x_{i-1})}{h^2}=p(x_i)\left[\frac{y(x_{i+1})-y(x_{i-1})}{2h}\right]+q(x_i)y(x_i)$$
$$+r(x_i)-\frac{h^2}{12}\left[2p(x_i)y^{(3)}(\eta_i)-y^{(4)}(\xi_i)\right].$$

- Which must be complemented by the boundary conditions $y(a)=\alpha$, and $y(b)=\beta$. If we use the notation $w_i=y(x_i)$ and divide $[a,b]$ using N steps, we have the approximation:

$$\frac{-w_{i+1}+2w_i-w_{i-1}}{h^2}+p(x_i)\left[\frac{w_{i+1}-w_{i-1}}{2h}\right]+q(x_i)w_i=-r(x_i),\quad i=1,2,\ldots,N$$

Linear Finite-Difference

- This relation can be rewritten as a system of N linear equations

$$-\left(1+\frac{h}{2}p(x_i)\right)w_{i-1}+\left(2+h^2q(x_i)\right)w_i-\left(1-\frac{h}{2}p(x_i)\right)w_{i+1}=-h^2r(x_i),$$

- Of the form

$$\mathbf{Aw}=\mathbf{b}$$

- where

$$w_0=\alpha,\quad w_{N+1}=\beta$$

$$\mathbf{w}=(w_1,\ldots\ldots,w_N)^T$$

Linear Finite-Difference

- And

$$\mathbf{A}=\begin{pmatrix} 2+h^2q(x_1) & -1+\frac{h}{2}p(x_1) & 0 & \cdots & 0 \\ -1-\frac{h}{2}p(x_2) & 2+h^2q(x_2) & -1+\frac{h}{2}p(x_2) & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1+\frac{h}{2}p(x_{N-1}) \\ 0 & \cdots & 0 & -1-\frac{h}{2}p(x_{N-1}) & 2+h^2q(x_N) \end{pmatrix},$$

$$\mathbf{b}=\begin{pmatrix} -h^2r(x_1)+\left(1+\frac{h}{2}p(x_1)\right)w_0 \\ -h^2r(x_2) \\ \vdots \\ -h^2r(x_{N-1}) \\ -h^2r(x_N)+\left(1-\frac{h}{2}p(x_N)\right)w_{N+1} \end{pmatrix}$$

Linear Finite-Difference

- This linear system can be solved using any of the standard methods used to solve linear systems. We must guarantee, however, that the matrix $\mathbf{A}$ is non-singular. For a linear system, the conditions of the following theorem must be satisfied.
- Theorem. Suppose that p, q and r are continuous on $[a, b]$. If $q(x) \geq 0$ on $[a, b]$, then the tridiagonal linear system has a unique solution provided that:

$$h < \frac{L}{2}, \quad \text{where} \quad L = \max_{a \leq x \leq b} \left| p(x) \right|.$$

Linear Finite-Difference

- This method, is going to be less precise than the linear shooting. It is based on finite approximations of $O(h^2)$ for the derivatives while the shooting method computes the solution using a RK4 method to integrate the IVP, which is of the order of $O(h^4)$.
- We could use of course, better approximations for the derivatives but this will result in a banded linear system of equations which more than 3 diagonals. This would imply longer CPU times for the solution.

Linear Finite-Difference

- It is normally much simpler to use the fact that the error of this method is some power of the step h. This means that we can use the Richardson Extrapolation to refine our computations.
- We could compute two solutions using different values of the step, h and $h/2$ and then use the Richardson recursion:

$$w_{3_i} = \frac{4w_{2_i} - w_i}{3}$$

- Where $w_i(h), w_{2i}(h/2)$. The new approximation w_{3i} would be two orders of magnitude more precise.

Nonlinear Finite-Difference

- For the general nonlinear BVP problem:

$$y'' = f(x, y, y'), \quad a \le x \le b, \quad y(a) = \alpha, \quad y(b) = \beta$$

- We can proceed in a similar way. As in the linear case we divide $[a, b]$ into $N + 1$ equal subintervals whose endpoints are at $x_i = a + ih, i = 0,1, \dots, N + 1$. If the exact solution has a bounded fourth derivative, this allows us to replace $y''(x_i)$ and $y'(x_i)$ in each of the equations:

$$y''(x_i) = f(x_i, y(x_i), y'(x_i)), \quad i = 1, 2, \dots, N$$

Nonlinear Finite-Difference

- Then we have

$$\frac{y(x_{i+1})-2y(x_i)+y(x_{i+1})}{h^2}=f\left(x_i,y(x_i),\frac{y(x_{i+1})-y(x_{i-1})}{2h}-\frac{h^2}{6}y^{(3)}(\eta_i)\right)+\frac{h^2}{12}y^{(4)}(\xi_i)$$

- For some η_i,ξ_i in the interval (x_{i-1},x_{i+1}). Now, deleting the error terms, and using the values of the boundary conditions $w_0=\alpha$, $w_{N+1}=\beta$, we obtain the system of nonlinear equations

$$-\frac{w_{i+1}-2w_i+w_{i-1}}{h^2}+f\left(x_i,w_i,\frac{w_{i+1}-w_{i-1}}{2h}\right)=0,\quad i=1,2,..,N.$$

Nonlinear Finite-Difference

- This nonlinear system will have a unique solution provided that $h < L/2$ where $L = \max_{a \le x \le b} |p(x)|$.
- To solve this nonlinear system, we will use Newton's method to generate a sequence of iterates $\left\{\left(w_1^{(k)}, w_2^{(k)}, \dots, w_N^{(k)}\right)^T\right\}$ that will converge to the solution of the system, provided that the initial approximation $\left\{\left(w_1^{(0)}, w_2^{(0)}, \dots, w_N^{(0)}\right)^T\right\}$ is sufficiently close to the solution $\{(w_1, w_2, \dots, w_N)^T\}$.

Nonlinear Finite-Difference

- The nonlinear system of equations is linearized using a first order approximation of $f(x,y,y')$. The Jacobian can be written as:

$$J(w_1,w_2,\dots,w_N)=\begin{cases}-1+\dfrac{h}{2}f_{y'}\left(x_i,w_i,\dfrac{w_{i+1}-w_{i-1}}{2h}\right), & \text{for } i=j-1 \text{ and } j=2,\dots,N\\ 2+h^2 f_y\left(x_i,w_i,\dfrac{w_{i+1}-w_{i-1}}{2h}\right), & \text{for } i=j \text{ and } j=1,\dots,N\\ -1-\dfrac{h}{2}f_{y'}\left(x_i,w_i,\dfrac{w_{i+1}-w_{i-1}}{2h}\right), & \text{for } i=j+1 \text{ and } j=1,\dots,N-1\end{cases}$$

- Where $w_0=\alpha$, $w_{N+1}=\beta$.

Nonlinear Finite-Differences

- At each iteration we need to solve the $N \times N$ linear system:

$$J(w_1, w_2, \ldots, w_N)(v_1, \ldots, v_N)^T =$$

$$-\begin{pmatrix} 2w_1 - w_2 - \alpha + h^2 f\left(x_1, w_1, \dfrac{w_2 - \alpha}{2h}\right), \\ -w_1 + 2w_2 - w_3 + h^2 f\left(x_2, w_2, \dfrac{w_3 - w_1}{2h}\right), \ldots\ldots, \\ -w_{N-2} + 2w_{N-1} - w_N + h^2 f\left(x_{N-1}, w_{N-1}, \dfrac{w_N - w_{N-2}}{2h}\right), \\ -w_{N-1} + 2w_N + h^2 f\left(x_N, w_N, \dfrac{\beta - w_{N-1}}{2h}\right) - \beta \end{pmatrix}^T,$$

Nonlinear Finite-Differences

- Then, the new approximation is obtained as:

$$w_i^{(k)} = w_i^{(k-1)} + v_i, \quad i = 1,2,\ldots,N.$$

- The Jacobian is tridiagonal, and the problem can be solved with any linear solver, particularly any factorization algorithm. This nonlinear finite-difference method if of order of $O(h^2)$.
- We need a good initial approximation. We normally use the solution of the linear system. Richardson approximation could be used also to improve the accuracy.