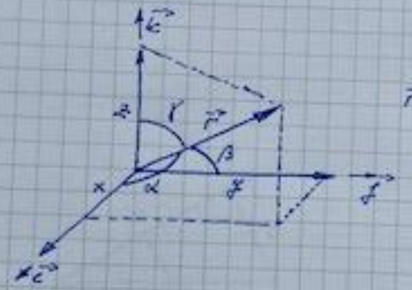
1) Demostreu que si els vectors $\boldsymbol{a}$ i $\boldsymbol{b}$ no tenen la mateixa direcció, la igualtat vectorial $x\boldsymbol{a} + y\boldsymbol{b} = 0$ implica que $x = y = 0$.

Si $A \Rightarrow B$ és equivalent a demostrar no $B \Rightarrow$ no A

Suposem $x \neq 0$ $\Rightarrow$

$\vec{a} = (-y/x)\vec{b}$ $\Rightarrow$ voldria dir que $\vec{a}$ i $\vec{b}$ serien paral·lels, això seria negar la hipòtesi inicial, llavors no pot ser $x \neq 0 \Rightarrow x = 0$ i donat que $x \cdot \vec{a} = -y\vec{b}$, si $x = 0 \Rightarrow y = 0$ c.e.v.d.

2) Determineu els angles α, β i γ que el vector $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$ forma amb els sentits positius dels eixos de coordenades, i demostreu que $cos^2\alpha + cos^2\beta + cos^2\gamma = 1$



$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$\alpha = \arccos x/|\vec{r}|$

$$\alpha = \arccos \frac{x}{\sqrt{x^2+y^2+z^2}} = \frac{x}{|\vec{r}|}$$

$$\beta = \arccos \frac{y}{\sqrt{x^2+y^2+z^2}} = \frac{y}{|\vec{r}|}$$

$$\gamma = \arccos \frac{z}{\sqrt{x^2+y^2+z^2}} = \frac{z}{|\vec{r}|}$$

$$\cos^2\alpha = \frac{x^2}{|\vec{r}|^2}$$

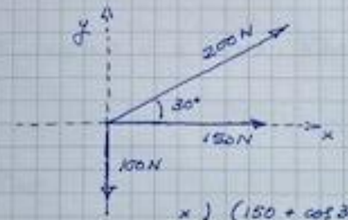
$$\cos^2\beta = \frac{y^2}{|\vec{r}|^2}$$

$$\cos^2\gamma = \frac{z^2}{|\vec{r}|^2}$$

$$\Sigma = \Sigma$$

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{x^2+y^2+z^2}{|\vec{r}|^2} = \frac{x^2+y^2+z^2}{x^2+y^2+z^2} = 1 \quad \text{c.e.v.d.}$$

3) Sobre un sòlid puntual en P actuen les tres forces coplanàries que mostra la figura. Trobeu la força que és necessari aplicar en P per a mantenir el sòlid donat en repòs.



x) $(150 + \cos 30° \cdot 200)$ N

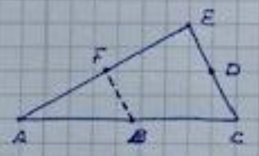
$R_x = 323$ N

y) $(200 \cdot \sin 30° - 100)$ N

$R_y = 0$ N

Per tant per mantenir el repòs necessitem $\vec{F} = -323\,\vec{i}$

4) Demostreu que la recta que uneix dos punts mitjans de dos costats d'un triangle, és paral·lela al tercer costat i igual a la seva meitat (paral·lela mitja).



$$\vec{AC} + \vec{CE} + \vec{EA} = 0$$

$$\Downarrow$$

$$\frac{\vec{AC}+\vec{CE}+\vec{EA}}{2} = \frac{\vec{AC}}{2} + \frac{\vec{CE}}{2} + \frac{\vec{EA}}{2} = 0$$

$$\frac{\vec{AC}}{2} = \vec{AB}$$

$$\frac{\vec{EF}}{2} = \vec{FA}$$

Per tant $\vec{AB} + \frac{\vec{CE}}{2} + \vec{FA} = (\vec{AB} + \vec{FA}) + \frac{\vec{CE}}{2} = 0$

$\vec{AB} + \vec{FA} = -\vec{BF}$

$\left.\right\}\ \vec{BF} = \frac{\vec{CE}}{2}$

5) Si $\boldsymbol{A} \cdot \boldsymbol{B} = 0$ i $\boldsymbol{A}$ *i* $\boldsymbol{B}$ són diferents a zero, demostreu que $\boldsymbol{A}$ és perpendicular a $\boldsymbol{B}$.

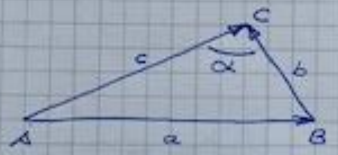
Si $\vec{A} \cdot \vec{B} = 0 \Rightarrow A \cdot B \cdot \cos\alpha = 0$ i si $A \neq 0$ i $\vec{B} \neq 0 \Rightarrow \cos\alpha = 0$

$\Rightarrow \alpha = \pi/2$

I recíprocament

Si $\alpha = \pi/2 \Rightarrow \vec{A} \cdot \vec{B} = 0$

6) Demostreu el teorema del cosinus d'un triangle qualsevol.



$$\vec{AB} + \vec{BC} = \vec{AC}$$

$$\vec{AB} = \vec{AC} - \vec{BC}$$

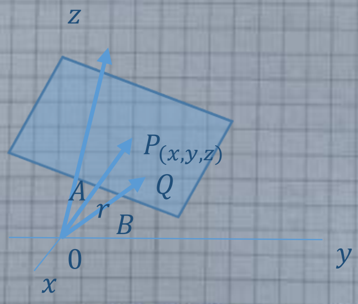
$$\Downarrow$$

$$\vec{AB} \cdot \vec{AB} = (\vec{AC} - \vec{BC})(\vec{AC} - \vec{BC})$$

$$AB^2 = AC^2 + BC^2 - 2|\vec{BC}| \cdot |\vec{AC}| \cos\alpha$$

$$a^2 = b^2 + c^2 - 2bc\cos\alpha$$

7) Trobeu l'equació del pla perpendicular al vector $\boldsymbol{A} = 2\boldsymbol{i} + 3\boldsymbol{j} + 6\boldsymbol{k}$ i que passa per l'extrem del vector $\boldsymbol{B} = \boldsymbol{i} + 5\boldsymbol{j} + 3\boldsymbol{k}$.



Sigui P un punt genèric del pla que busquem i extrem del vector $\vec{r}(P) = \vec{r}(x,y,z) = x\vec{i} + y\vec{j} + z\vec{k}$

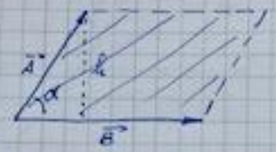
$\vec{PQ}$, estarà sobre el pla: $\vec{PQ} = (\vec{B} - \vec{r}) \perp \vec{A}$

$\Rightarrow (\vec{B} - \vec{r})\vec{A} = 0 \Rightarrow \vec{B} \cdot \vec{A} = \vec{r} \cdot \vec{A} \Rightarrow (\vec{i} + 5\vec{j} + 3\vec{k})(2\vec{i} + 3\vec{j} + 6\vec{k}) = 35$

$$\Downarrow$$

$$2x + 3y + 6z = 35$$

8) Demostreu que l'àrea d'un paral·lelogram de costats $\boldsymbol{A}$ i $\boldsymbol{B}$ és $|\boldsymbol{A} \times \boldsymbol{B}|$.

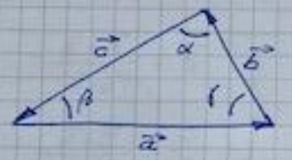


Àrea $\Rightarrow S = h\,|\vec{B}|$

$S = |\vec{A}| \cdot \sin\alpha \cdot |\vec{B}|$

$S = |\vec{A} \times \vec{B}|$

9) Deduïu el teorema dels sinus en un triangle pla.



$\vec{a} + \vec{b} + \vec{c} = 0$

$\vec{a}\times\vec{a} + \vec{a}\times\vec{b} + \vec{a}\times\vec{c} = 0 \Rightarrow \vec{a}\times\vec{b} = \vec{c}\times\vec{a}$

$\vec{b}\times\vec{a} + \vec{b}\times\vec{b} + \vec{b}\times\vec{c} = 0 \Rightarrow \vec{b}\times\vec{a} = \vec{c}\times\vec{b}$

$\vec{c}\times\vec{a} + \vec{c}\times\vec{b} + \vec{c}\times\vec{c} = 0 \Rightarrow \vec{c}\times\vec{a} = \vec{b}\times\vec{c}$

Per tant $\vec{a}\times\vec{b} = \vec{b}\times\vec{c} = \vec{c}\times\vec{a}$

$$a \cdot b \sin(\pi-\gamma) = bc \sin(\pi-\alpha) = ca \sin(\pi-\beta)$$

$$a\,b \sin\gamma = bc \sin\alpha = ca \sin\beta$$

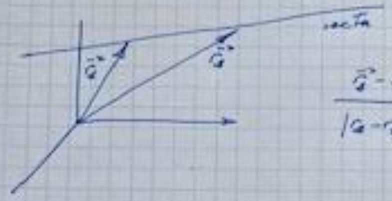
$$a\cancel{b}\sin\gamma = \cancel{b}c \sin\alpha \qquad b\cancel{c}\sin\alpha = \cancel{c}a \sin\beta$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\frac{\sin\gamma}{c} = \frac{\sin\alpha}{a} \qquad \frac{\sin\alpha}{a} = \frac{\sin\beta}{b}$$

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$$

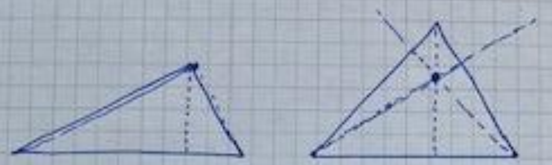
10) Trobeu els cosinus directors de la recta que passa pels punts $(3, 2, -4)$ i $(1, -1, 2)$.



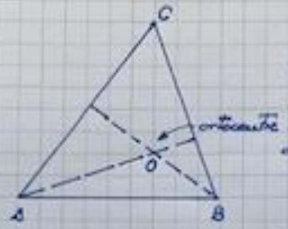
$$\frac{\vec{r_2} - \vec{r_1}}{|r_2 - r_1|} = \cos\alpha\,\vec{i} + \cos\beta\,\vec{j} + \cos\gamma\,\vec{k}$$

11) Trobeu la constant $\boldsymbol{a}$ de manera que els vectors $2\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}, \boldsymbol{i} + 2\boldsymbol{j} - 3\boldsymbol{k}$ i $3\boldsymbol{i} + a\boldsymbol{j} + 5\boldsymbol{k}$ siguin coplanars.

$$\left.\begin{array}{l} \dfrac{\vec{A} \times \vec{B}}{|\vec{A} \times \vec{B}|} = \vec{v_1} \\[2ex] \dfrac{\vec{A} \times \vec{C}}{|\vec{A} \times \vec{C}|} = \vec{v_1} \end{array}\right\} \quad \frac{\vec{A} \times \vec{B}}{|\vec{A} \times \vec{B}|} = \frac{\vec{A} \times \vec{C}}{|\vec{A} \times \vec{C}|} \Rightarrow a = -4$$



12) Demostreu que les alçades d'un triangle es tallen en un punt (ortocentre).



Si $\vec{AO} \perp \vec{BC}$ i $\vec{BO} \perp \vec{AC}$ $\overset{?}{\Longrightarrow} CO \perp \vec{AB}$

dem:

$\vec{AO} \perp \vec{CB} \Rightarrow \vec{AO} \cdot (\vec{CO} + \vec{OB}) = 0 \Rightarrow \vec{AO} \cdot \vec{CO} + \vec{AO} \cdot \vec{OB} = 0$

$\Updownarrow$

(1) $\vec{AO} \cdot \vec{CO} = -\vec{AO} \cdot \vec{OB}$

$\vec{BO} \perp \vec{AC} \Rightarrow \vec{BO} \cdot (\vec{AO} + \vec{OC}) = 0 \Rightarrow \vec{BO} \cdot \vec{AO} + \vec{BO} \cdot \vec{OC} = 0$

$\Updownarrow$

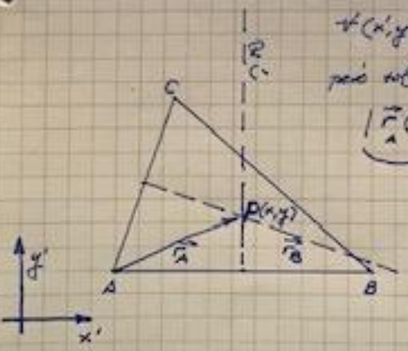
(2) $\vec{BO} \cdot \vec{AO} = -\vec{BO} \cdot \vec{OC}$

Per tant de (1) i (2) podem dir que: $\vec{AO} \cdot \vec{CO} = -\vec{BO} \cdot \vec{OC} \Rightarrow$

$0 = \vec{AO} \cdot \vec{CO} + \vec{BO} \cdot \vec{OC} = \vec{CO}(\vec{AO} - \vec{BO}) = \vec{CO}\,(\underbrace{\vec{AO} + \vec{OB}}_{\vec{AB}}) = 0 \Rightarrow$

$\vec{CO} \perp \vec{AB}$ c.e.v.d.

13) Demostreu que les mediatrius d'un triangle es tallen en un punt (circumcentre).

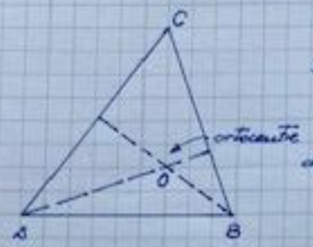


$\forall (x', y') \in R \Rightarrow |\vec{r}_A(x',y')| = |\vec{r}_B(x',y')| \Rightarrow$ recta mediatriu

però solament $\exists$ un $P(x,y)$ /

$|\vec{r}_A(P)| = |\vec{r}_B(P)| = \vec{r}_C(P)$

aquest sistema equacions $\Rightarrow x, y$ del circumcentre



Si $\vec{AO} \perp \vec{BC}$ i $\vec{BO} \perp \vec{AC}$ $\overset{?}{\Longrightarrow}$ $CO \perp \vec{AB}$

dem:

$$\vec{AO} \perp \vec{CB} \Rightarrow \vec{AO} \cdot (\vec{CO} + \vec{OB}) = 0 \Rightarrow \vec{AO} \cdot \vec{CO} + \vec{AO} \cdot \vec{OB} = 0$$

$$\Downarrow$$

$$(1) \quad \vec{AO} \cdot \vec{CO} = -\vec{AO} \cdot \vec{OB}$$

$$\vec{BO} \perp \vec{AC} \Rightarrow \vec{BO} \cdot (\vec{AO} + \vec{OC}) = 0 \Rightarrow \vec{BO} \cdot \vec{AO} + \vec{BO} \cdot \vec{OC} = 0$$

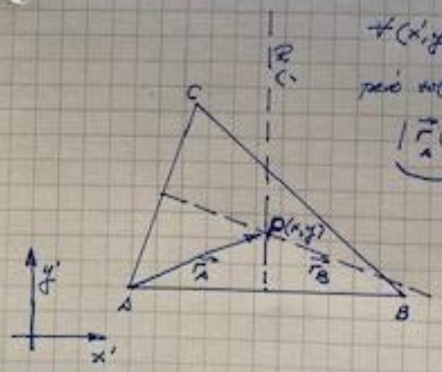
$$\Downarrow$$

$$(2) \quad \vec{BO} \cdot \vec{AO} = -\vec{BO} \cdot \vec{OC}$$

Per tant de (1) i (2) podem dir que: $\vec{AO} \cdot \vec{CO} = -\vec{BO} \cdot \vec{OC} \Rightarrow$

$$0 = \vec{AO} \cdot \vec{CO} + \vec{BO} \cdot \vec{OC} = \vec{CO}(\vec{AO} - \vec{BO}) = \vec{CO}(\underbrace{\vec{AO} + \vec{OB}}_{\vec{AB}}) = 0 \Rightarrow$$

$\vec{CO} \perp \vec{AB}$ c.q.v.d.



$\forall (x', y') \in R \Rightarrow |\vec{r}_A(x'y')| = |\vec{r}_B(x'y')| \Rightarrow$ recta mediatriz

però solament $\exists$ un $P(x,y)$ /

$$\underbrace{|\vec{r}_A(P)| = \overbrace{|\vec{r}_B(P)|}}\ \underbrace{= \vec{r}_C(P)}$$

aquest sistema equacions $\Rightarrow x, y$ del circumcentre

14) Trobeu $\nabla\emptyset$ essent $(a)\emptyset = 1n|r|, (b)\emptyset = \frac{1}{r}$.

$$\phi = \ln|\vec{r}|$$

a) $$\vec{\nabla}\phi = \vec{grad}\,\phi = \frac{\partial\left(\ln\left(\sqrt{x^2+y^2+z^2}\right)\right)}{\partial x}\vec{i} + \frac{\partial\left(\ln\sqrt{x^2+y^2+z^2}\right)}{\partial y}\vec{j} + \frac{\partial\left(\ln\sqrt{x^2+y^2+z^2}\right)}{\partial z}\vec{k}$$

$$\vec{\nabla}\ln|\vec{r}| = \frac{\vec{r}}{r^2}$$

b) $$\phi = \frac{1}{r} = \frac{1}{\sqrt{x^2+y^2+z^2}}$$

$$\vec{\nabla}\phi = \frac{1}{(x^2+y^2+z^2)^{3/2}}\,\vec{r} = \frac{1}{r^{3/2}}\,\vec{r}$$

$$\vec{\nabla}\times\vec{\nabla}\phi = \vec{rot}(\vec{grad}\,\phi) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix} =$$

$$= \vec{i}\left[\frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial y}\right)\right] + \vec{j}\left[\frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial x}\right) - \frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial z}\right)\right] + \vec{k}\left[\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial x}\right)\right]$$

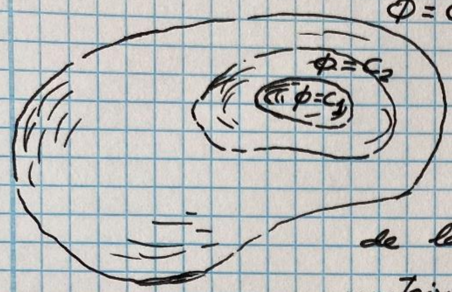
Si les derivacions parcials commuten (teorema Schwartz)

$$\frac{\partial^2\phi}{\partial y\partial z} = \frac{\partial^2\phi}{\partial z\partial y}; \ \ldots \Rightarrow \vec{\nabla}\times\vec{\nabla}\phi = 0$$ c.e.v.d.

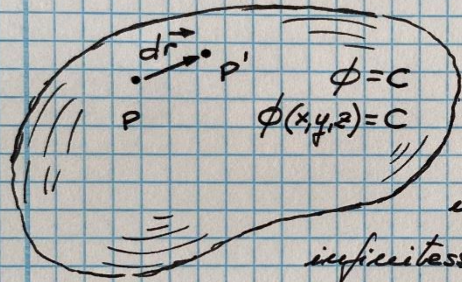
15) Demostreu que $\nabla\emptyset$ és un vector perpendicular a la superfície $\emptyset(x,y,z)=c$, essent c una constant.

Resolució:

$\phi = C$ (en genèric)



superfici equi-escalar: en tots el seus punts de la superfici el valor de la magnitud ϕ es el mateix es constant.



Per tant quan ens movem a sobre la superfici i fem un desplaçament infinitessim, $d\vec{r}$, no experimentem cap variació en el valor de ϕ.

En general: $d\phi = \dfrac{\partial\phi}{\partial x}dx + \dfrac{\partial\phi}{\partial y}dy + \dfrac{\partial\phi}{\partial z}dz$

$$d\phi = \vec{\nabla}\phi \cdot d\vec{r}$$

Entre P i P' $\Rightarrow$ $d\vec{r}$ i $d\phi = 0$, per tant:

$$0 = \vec{\nabla}\phi \cdot d\vec{r} \Rightarrow \vec{\nabla}\phi \perp d\vec{r}$$

I donat que $d\vec{r}$ és tangent a la superfici equi-escalar $\Rightarrow$ $\vec{\nabla}\phi$ (gradient del camp escalar) serà $\perp$ a la superfi equi-escalar. c.e.v.d.

16) Demostreu que:

a. $\nabla \times (\nabla \emptyset) = 0$ (rot grad $\emptyset = 0$).

b. $\nabla . (\nabla \times \boldsymbol{A}) = 0$ (div rot $\boldsymbol{A} = 0$).

a) $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = \text{div}\,(\text{rot}\,\vec{A}) =$

$$\vec{\nabla} \times \vec{A} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \vec{i}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) +$$

$$+\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\vec{j} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\vec{k} \Rightarrow$$

$$\vec{\nabla}(\vec{\nabla} \times \vec{A}) = \frac{\partial}{\partial x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) +$$

$$+ \frac{\partial}{\partial z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \Rightarrow$$

$$\vec{\nabla}(\vec{\nabla} \times \vec{A}) = \frac{\partial^2 A_z}{\partial x \partial y} - \frac{\partial^2 A_y}{\partial x \partial z} + \frac{\partial^2 A_x}{\partial y \partial z} - \frac{\partial^2 A_z}{\partial y \partial x}$$

$$+ \frac{\partial^2 A_y}{\partial z \partial x} - \frac{\partial^2 A_x}{\partial z \partial y}$$

si les derivades parcials podem commutar:

$$\frac{\partial^2 A_z}{\partial x \partial y} = \frac{\partial^2 A_z}{\partial y \partial x} \; ; \text{ etc, etc.}$$

Per tant $\vec{\nabla}(\vec{\nabla} \times \vec{A}) = 0$

podem dir:

"un camp rotacional es solenoidal"

b) $$\vec{\nabla}\times\vec{\nabla}\phi = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix} =$$

$$= \vec{i}\left(\frac{\partial}{\partial y}\frac{\partial\phi}{\partial z} - \frac{\partial}{\partial z}\frac{\partial\phi}{\partial y}\right) + \vec{j}\left(\frac{\partial}{\partial z}\frac{\partial\phi}{\partial x} - \frac{\partial}{\partial x}\frac{\partial\phi}{\partial z}\right) +$$

$$+ \vec{k}\left(\frac{\partial}{\partial x}\frac{\partial\phi}{\partial y} - \frac{\partial}{\partial y}\frac{\partial\phi}{\partial x}\right) =$$

$$= \vec{i}\left(\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial^2\phi}{\partial z\partial y}\right) + \vec{j}\left(\frac{\partial^2\phi}{\partial z\partial x} - \frac{\partial^2\phi}{\partial x\partial z}\right) + \vec{k}\left(\frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial y\partial x}\right) =$$

= { si ϕ compleix les condicions del teorema Schwartz }
(l'ordre de derivació parcial no canvia el resultat)

(els camps escalars FISICS tots compleixen aquestes condicions)

Per tant: $$\frac{\partial^2\phi}{\partial x_i\partial x_j} = \frac{\partial^2\phi}{\partial x_j\partial x_i}$$

i conseqüentment

$$\vec{\nabla}\times\vec{\nabla}\phi = 0.$$

Tot camp vectorial generat per un gradient es IRROTACIONAL !!

A més a més tenint present el teorema de Stokes

$$\int_S (\vec{\nabla}\times\vec{\nabla}\phi)\, d\vec{S} = \oint_C \vec{\nabla}\phi\cdot d\vec{l} \Rightarrow \oint_C \vec{\nabla}\phi\cdot d\vec{l} = 0$$

$$\Rightarrow \oint_C \vec{\nabla}\phi\cdot d\vec{l} = \int_{L_1(P_1)}^{P_2}\vec{\nabla}\phi\, d\vec{l} + \int_{L_2(P_2)}^{P_1}\vec{\nabla}\phi\, d\vec{l} = 0 \Rightarrow \int_{L_1(P_1)}^{P_2}\vec{\nabla}\phi\, d\vec{l} = \int_{L_2(P_1)}^{P_2}\vec{\nabla}\phi\, d\vec{l}$$

$C = L_1 + L_2$

Per tant $\vec{\nabla}\phi$ serà conservatiu !!