COMBINATORICS AND PROBABILITY

PART III: GENERATING FUNCTIONS

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Ordinary generating function associated with a sequence

Given a sequence $\langle a_n \rangle_{n=0}^{\infty} = \langle a_0, a_1, \ldots \rangle$, the *(ordinary) generating function* (OGF) associated with $\langle a_n \rangle$ is

$$A(z) = \sum_{n=0}^{\infty} a_n z^n$$

Note: We use the variable z because we are working in the domain of complex numbers, in general.

Note: The generating function is a formal power series that is defined regardless of whether it converges or not. When it does converge we can perform additional operations with it.

Sum and product

Sum:

$$\sum_{n=0}^{\infty} a_n z^n \pm \sum_{n=0}^{\infty} b_n z^n = \sum_{n=0}^{\infty} (a_n \pm b_n) z^n$$

Product:

$$\sum_{n=0}^{\infty} a_n z^n \sum_{n=0}^{\infty} b_n z^n = \sum_{n=0}^{\infty} c_n z^n, \text{ where}$$

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}$$

The sequence $\langle c_n \rangle_{n=0}^{\infty}$ is called the **convolution** of $\langle a_n \rangle_{n=0}^{\infty}$ and $\langle b_n \rangle_{n=0}^{\infty}$.

Product and power

The product of generating functions arises when we have a Cartesian product of combinatorial classes.

From the product formula we get

$$(A(z))^k = \left(\sum_{n=0}^{\infty} a_n z^n\right)^k = \sum_{n=0}^{\infty} c_n z^n, \text{ where}$$

$$c_n = \sum_{n_1+n_2+\cdots+n_k=n} a_{n_1} a_{n_2} \cdots a_{n_k}$$

Reciprocal

Let $\langle a_n \rangle_{n=0}^{\infty}$ and $\langle b_n \rangle_{n=0}^{\infty}$ be two sequences. If

$$\left(\sum_{n=0}^{\infty} a_n z^n \right) \left(\sum_{n=0}^{\infty} b_n z^n \right) = 1,$$

then we say that $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=0}^{\infty} b_n z^n$ are *reciprocals*.

Proposition

A formal power series $\sum_{n=0}^{\infty} a_n z^n$ has a reciprocal if and only if $a_0 \neq 0$. In that case the reciprocal is unique.

Reciprocal

Computing the reciprocal:

$$b_0 = \frac{1}{a_0}$$

$$b_n = -\frac{1}{a_0}\sum_{k=1}^{n} a_k b_{n-k} \quad \text{for } n \geq 1$$

Division:

Division of $F(z)$ by $G(z)$ can be accomplished by multiplying $F(z)$ by the reciprocal of $G(z)$ (assuming that $G(z)$ has a reciprocal).

Derivative

Let $\langle a_n \rangle_{n=0}^{\infty}$ a sequence, and $A(z) = \sum_{n=0}^{\infty} a_n z^n$ its ordinary generating function. Then the (formal) derivative of $A(z)$ is

$$A'(z) = \sum_{n=0}^{\infty} (n+1) a_{n+1} z^n.$$

Note: The formal derivative is defined regardless of whether $A(z)$ converges or not. When it does converge then this *formal* derivative corresponds to the usual derivative of functions defined in Calculus.

Formal antiderivative (integral)

Let $\langle a_n \rangle_{n=0}^{\infty}$ a sequence, and $A(z) = \sum_{n=0}^{\infty} a_n z^n$ its ordinary generating function. Then the formal antiderivative (integral) of $A(z)$ is

$$\int_0^z A(t)dt = \sum_{n=1}^{\infty} \frac{a_{n-1}}{n} z^n.$$

Note: Again, the integral is defined regardless of whether $A(z)$ converges or not. When it does converge then this *formal* integral corresponds to the usual indefinite integral of functions defined in Calculus.

Shifting

If $A(z) = \sum_{n=0}^{\infty} a_n z^n$, then $\dfrac{A(z) - A(0)}{z} = \dfrac{A(z) - a_0}{z}$ is the OGF of the sequence $\langle a_{n+1} \rangle_{n=0}^{\infty}$.

$\dfrac{A(z) - a_0 - a_1 z}{z^2}$ is the OGF of the sequence $\langle a_{n+2} \rangle_{n=0}^{\infty}$.

In general,

$$\frac{A(z) - a_0 - a_1 z - \cdots - a_{h-1} z^{h-1}}{z^h}.$$

is the OGF of the sequence $\langle a_{n+h} \rangle_{n=0}^{\infty}$

Subsequence of alternating terms

If $A(z) = \sum_{n=0}^{\infty} a_n z^n$, then

$$\sum_{n=0}^{\infty} a_{2n} z^{2n} = \frac{1}{2}\left(A(z) + A(-z)\right)$$

$$\sum_{n=0}^{\infty} a_{2n+1} z^{2n+1} = \frac{1}{2}\left(A(z) - A(-z)\right)$$

Summary of the main operations for OGFs

Ordinary generating function	n-th element of the sequence
cA	ca_n
$A+B$	$a_n + b_n$
AB	$\sum_{k=0}^{n} a_k b_{n-k}$
$z^k A(z)$	if $n < k$ then 0 else a_{n-k}
$\frac{A(z)}{1-z}$	$\sum_{i=0}^{n} a_i$
$zA'(z)$	na_n
$\int_0^z A(t)dt$	if $n = 0$ then 0 else $\frac{a_{n-1}}{n}$

Extracting coefficients

Notation: Let $A(z)$ be a power series in the variable z. Then $[z^n]A(z)$ denotes the coefficient of z^n in the series $A(z)$.

A simple property of this symbol is

$$[z^n]\{z^a A(z)\} = [z^{n-a}]A(z)$$

Example 1 (extracting coefficients):

$$[z^n]e^z = \frac{1}{n!}, \quad [z^n]\left\{\frac{1}{1-3z}\right\} = 3^n, \quad [z^n](1+z)^p = \binom{p}{n}$$

Coefficients and Taylor series

If a power series converges to a function $A(z)$ that is complex-differentiable at the origin, then the coefficients $[z^n]$ are precisely the coefficients of the **Taylor-McLaurin series**:

$$A(z) = A(0) + \frac{A'(0)}{1!}z + \frac{A''(0)}{2!}z^2 + \frac{A'''(0)}{3!}z^3 + \cdots$$

That is,

$$[z^n]A(z) = \frac{A^{(n)}(0)}{n!}.$$

The Taylor-McLaurin series converges to $A(z)$ in an open disc $|z| < r$, where $r \geq 0$ is the **radius of convergence.**

Convergence and absolute convergence

A series $\sum_{i \geq 0} a_i z^i$ is **convergent** at a point z_0 if the sequence of partial sums:

$$S_n = \sum_{i=0}^{n} a_n z^n$$

converges to a finite limit. The same series is said to be **absolutely convergent** if $\sum_{i \geq 0} |a_i| z^i$ is convergent.

If the series is convergent at $z = r$, for some positive real r, then it is absolutely convergent for all $z \in \mathbb{C}$ satisfying $|z| < r$.

The **radius of convergence** of the series is the radius of the largest open disc in which the series converges.

Singularities

We can also consider series developments centered at other points. The series $\sum_{i \geq 0} a_i (z - z_0)^i$ is said to be centered at z_0. It is absolutely convergent in some disc $\left\{ z \in \mathbb{C} : |z - z_0| \leq R \right\}$.

The series $\sum_{i \geq 0} a_i (z - z_0)^i$ is called **analytic** at z_0 if its radius of convergence is positive, i.e. if it is convergent in some nontrivial open disc around z_0. The limit function $f(z)$ is also said to be **analytic** at z_0. If the function is analytic in the whole complex plane, then it is called **entire**.

The function $f(z)$ is **singular at** z_0 if it is not analytic at z_0, but is analytic in some neighborhood of z_0. The point z_0 is then called a **singularity** of $f(z)$.

Classification of singularities

A singularity z_0 of a function $f(z)$ is **isolated** if $f(z)$ is analytic in an open set around z_0, with z_0 removed. For example, if $f(z)$ is analytic in the **punctured disc** $\left\{z \in \mathbb{C} : 0 < |z - z_0| < R\right\}$.

An isolated singularity z_0 is called **removable** if the function $f(z)$ is not defined in z_0, but it is possible to define $f(z)$ in z_0 in such a way that the function becomes analytic in a disc centered at z_0.

Example 2 (removable singularities): The origin is a removable singularity of the functions $f(z) = \dfrac{\sin z}{z}$ and $g(z) = \dfrac{z}{e^z - 1}$.

An isolated singularity z_0 is called a **pole** (of order m) if the function $f(z)$ can be written as $f(z) = \dfrac{g(z)}{(z - z_0)^m}$ where $g(z)$ is analytic and nonzero in z_0, and m is a positive integer.

Poles

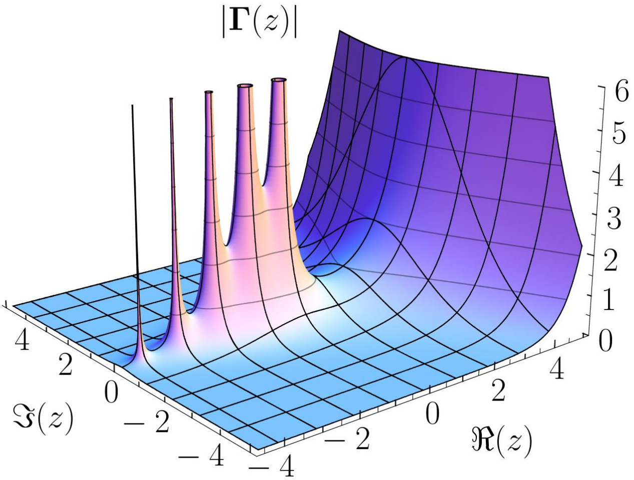

Figure: The poles of the gamma function

Other isolated singularities – Essential singularities

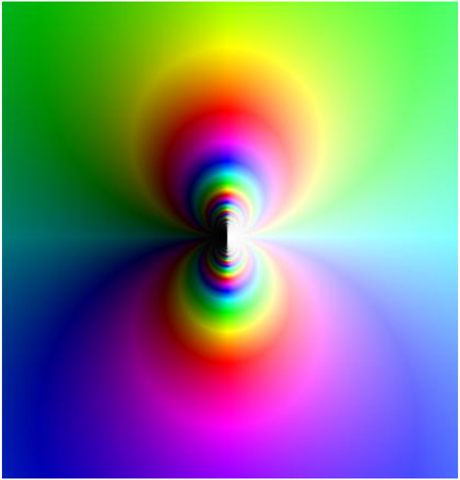

Figure: Essential singularity of the function $e^{1/z}$

Non-isolated singularities

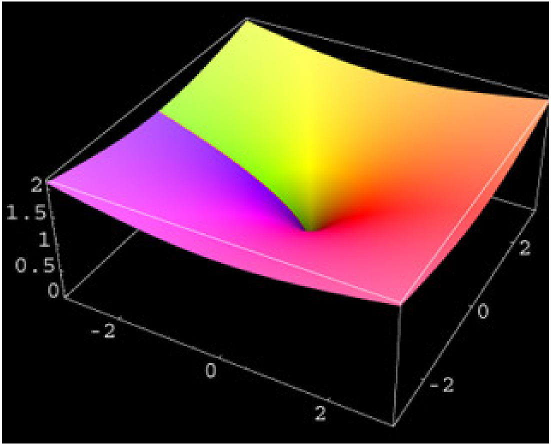

Figure: Non-isolated singularity of the function $\sqrt{z}$

Dominant singularities

Given a function $f(z)$ that is analytic at the origin, a **dominant singularity** of $f(z)$ is any non-removable singularity that is closest to the origin.

If $f(z)$ is analytic at the origin, then the radius of convergence r of $f(z)$ is the modulus of a dominant singularity.

Lemma (Pringsheim's lemma)

Suppose that $f(z)$ is analytic with series expansion $\sum_{i \geq 0} a_i z^i$, such that $a_i \geq 0$ for all $i \in \mathbb{N}$. If the series has radius of convergence $r > 0$, then the point $z = r$ is a singularity of $f(z)$ (hence a dominant singularity).

Many combinatorial sequences have this property.

Examples radius of convergence

Example 3: The functions e^z, $\sin z$ and $\cos z$ are **entire**, so their radius of convergence is ∞.

Example 4: The function $f(z) = \dfrac{1}{z^2+1}$ is analytic at $z = 0$ and has two isolated singularities at the points $z = \pm i$, hence its radius of convergence is 1. Its Taylor series about $z = 0$ is $\displaystyle\sum_{n=0}^{\infty}(-1)^n z^{2n}$.

Example 5: As we saw **before**, the function $g(z) = \dfrac{z}{e^z - 1}$ has a removable singularity at $z = 0$, hence by defining $g(0)$ properly we get a function $\hat{g}(z)$ that is analytic at $z = 0$. In other words, this singularity does not affect the radius of convergence. Apart from that, we have isolated singularities at $z = 2k\pi i$, for $k \in \mathbb{Z} - \{0\}$ (**verify !**). The singularities that are closest to the origin are $z = \pm 2\pi i$, hence the radius of convergence is 2π.

Some ordinary generating functions

Function	Sequence	Series
$\dfrac{1}{1-z}$	$\langle 1, 1, 1, \ldots \rangle$	$\sum_{n=0}^{\infty} z^n$
$\dfrac{1}{1+z}$	$\langle 1, -1, 1, -1, \ldots \rangle$	$\sum_{n=0}^{\infty} (-1)^n z^n$
$\dfrac{1}{1-az}$	$\langle 1, a, a^2, a^3, \ldots \rangle$	$\sum_{n=0}^{\infty} a^n z^n$
$\dfrac{1}{1-z^2}$	$\langle 1, 0, 1, 0, 1, \ldots \rangle$	$\sum_{n=0}^{\infty} z^{2n}$
$\dfrac{1}{(1-z)^2}$	$\langle 1, 2, 3, 4, 5, \ldots \rangle$	$\sum_{n=0}^{\infty} (n+1) z^n$

Generating functions involving binomial coefficients

$$\sum_{n=0}^{\infty} \binom{n+k}{n} a^n z^n = \frac{1}{(1-az)^{k+1}} \quad \text{(extension of binomial theorem)}$$

$$= \sum_{n=0}^{\infty} \binom{n+k}{k} a^n z^n$$

$$\sum_{n=0}^{\infty} \binom{2n}{n} z^n = \frac{1}{\sqrt{1-4z}} \quad \text{(central binomial coefficients)}$$

$$\sum_{n=0}^{\infty} \frac{1}{n+1} \binom{2n}{n} z^n = \frac{1}{2z} \left(1 - \sqrt{1-4z}\right) \quad \textbf{(Catalan numbers)}$$

$$\sum_{n=0}^{\infty} \binom{2n+k}{n} z^n = \frac{1}{\sqrt{1-4z}} \left(\frac{1-\sqrt{1-4z}}{2z}\right)^k$$

Generalized binomial coefficients

The above extension of the binomial theorem can be better understood if we extend the definition of the binomial coefficient $\binom{n}{k}$ to an arbitrary integer n, negative or non-negative (and not just for $0 \leq k \leq n$):

$$\binom{n}{k} = \begin{cases} 0 \quad \text{if} \quad k < 0, \\ \dfrac{n(n-1)(n-2)\cdots(n-k+1)}{k!} & \text{otherwise.} \end{cases}$$

Remark 1: If $0 \leq k \leq n$ this definition of the binomial coefficients is equivalent to the previous one.

Remark 2: We can also apply this extended definition to arbitrary $n \in \mathbb{R}$ and $k \in \mathbb{Z}$, or even when $n \in \mathbb{C}$, although here we will limit ourselves to the real case.

Generalized binomial coefficients

Remark 3: The expression

$$n^{\underline{k}} = \overbrace{n(n-1)(n-2)\cdots(n-k+1)}^{k \text{ factors}}$$

is called a **falling factorial** of n. Sometimes it is also denoted $(n)_k$.

Proposition

For any $n \in \mathbb{R}$ and $k \in \mathbb{Z}$,

$$\binom{-n}{k} = (-1)^k \binom{n+k-1}{k}$$

Prove it !!

Generalized binomial theorem

With the above extension of binomial coefficients we can generalize the binomial theorem as follows:

Theorem (Generalized binomial theorem)

Let $a, b \in \mathbb{R} - \{0\}$ *such that* $a + b \neq 0$, *and* $n \in \mathbb{Z}$

$$(a+b)^n = \sum_{k=0}^{\infty} \binom{n}{k} a^{n-k} b^k$$

Remark: This formula is also valid when the exponent n is an arbitrary real or complex number.

Exponential generating function associated with a sequence

Given a sequence $\langle a_n \rangle_{n=0}^{\infty} = \langle a_0, a_1, \ldots \rangle$, the *exponential generating function* (EGF) associated with $\langle a_n \rangle$ is

$$A(z) = \sum_{n=0}^{\infty} \frac{a_n}{n!} z^n$$

Note: We use the variable z because we are working in the domain of complex numbers, in general.

Note: The generating function is a formal power series that is defined regardless of whether it converges or not. When it does converge we can perform additional operations with it.

Some exponential generating functions

Function	Sequence	Series
e^z	$\langle 1, 1, 1, \dots \rangle$	$\sum_{n=0}^{\infty} \frac{z^n}{n!}$
e^{az}	$\langle 1, a, a^2, a^3, \dots \rangle$	$\sum_{n=0}^{\infty} \frac{a^n z^n}{n!}$
ze^z	$\langle 0, 1, 2, 3, 4, \dots \rangle$	$\sum_{n=0}^{\infty} \frac{nz^n}{n!}$
$z(z+1)e^z$	$\langle 0, 1, 4, 9, \dots \rangle$	$\sum_{n=0}^{\infty} \frac{n^2 z^n}{n!}$
$\frac{1}{1-z}$	$\langle 1, 1, 2, 6, 24, \dots \rangle$	$\sum_{n=0}^{\infty} \frac{n! z^n}{n!}$

Operations with exponential generating functions

Sum:

$$\sum_{n=0}^{\infty} a_n \frac{z^n}{n!} \pm \sum_{n=0}^{\infty} b_n \frac{z^n}{n!} = \sum_{n=0}^{\infty} (a_n \pm b_n) \frac{z^n}{n!}$$

Product:

$$\left(\sum_{n=0}^{\infty} a_n \frac{z^n}{n!}\right) \left(\sum_{n=0}^{\infty} b_n \frac{z^n}{n!}\right) = \sum_{n=0}^{\infty} c_n \frac{z^n}{n!}, \text{ where}$$

$$c_n = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}$$

Operations with exponential generating functions

Derivative:

Let $\langle a_n \rangle_{n=0}^{\infty}$ a sequence, and $A(z) = \sum_{n=0}^{\infty} a_n \frac{z^n}{n!}$ its exponential generating function. Then the derivative of $A(z)$ is

$$A'(z) = \sum_{n=0}^{\infty} a_{n+1} \frac{z^n}{n!}.$$

More generally,

$$A^{(k)}(z) = \sum_{n=0}^{\infty} a_{n+k} \frac{z^n}{n!}.$$

Summary of operations for EGFs

Exp. generating function	n-th element of the sequence
cA	ca_n
$A+B$	a_n+b_n
AB	$\sum_{k=0}^{n}\binom{n}{k}a_kb_{n-k}$
$A'(z)$	a_{n+1}
$\int_0^z A(t)dt$	if $n=0$ then 0 else a_{n-1}

Homogeneous linear recurrences - Fibonacci numbers

Example 6 – Fibonacci numbers:

The Fibonacci numbers are defined by the recurrence

$$F_{n+2} = F_{n+1} + F_n, \text{ with } n \geq 0 \text{ and } F_0 = 0, F_1 = 1$$

We multiply both sides of the equation by z^n and sum over $n \geq 0$:

$$\sum_{n=0}^{\infty} F_{n+2} z^n = \sum_{n=0}^{\infty} F_{n+1} z^n + \sum_{n=0}^{\infty} F_n z^n$$

$$\frac{F(z) - F_0 - F_1 z}{z^2} = \frac{F(z) - F_0}{z} + F(z)$$

$$F(z) = \frac{z}{1 - z - z^2}$$

Homogeneous linear recurrences - Fibonacci numbers

We can now decompose $F(z)$ in partial fractions. The roots of the denominator $1 - z - z^2$ are $-\varphi = -\frac{1+\sqrt{5}}{2} \approx -1.618033\dots$ (were φ is the golden ratio) and $-\hat{\varphi} = \varphi - 1 = \frac{1}{\varphi} = \frac{\sqrt{5}-1}{2} \approx 0.618033\dots$

Note also that $\varphi = -\frac{1}{\hat{\varphi}}$, and $\varphi\hat{\varphi} = -1$.

The traditional partial fraction decomposition proceeds as follows:

$$\frac{z}{1-z-z^2} = \frac{-z}{z^2+z-1} = \frac{-z}{(z-(-\varphi))\,(z-(-\hat{\varphi}))}$$
$$= \frac{A}{z+\varphi} + \frac{B}{z+\hat{\varphi}} = \frac{A\,(z+\hat{\varphi}) + B\,(z+\varphi)}{(z+\varphi)\,(z+\hat{\varphi})}$$

Hence $A\,(z+\hat{\varphi}) + B\,(z+\varphi) = -z$.

Homogeneous linear recurrences - Fibonacci numbers

By letting $z = -\varphi$ we get $A(\hat{\varphi} - \varphi) = \varphi$, hence

$$A = \frac{\varphi}{\hat{\varphi} - \varphi} = -\frac{1 + \sqrt{5}}{2\sqrt{5}} = -\frac{\varphi}{\sqrt{5}}.$$

On the other hand, if we let $z = -\hat{\varphi}$ we get $B(\varphi - \hat{\varphi}) = \hat{\varphi}$, hence

$$B = \frac{\hat{\varphi}}{\varphi - \hat{\varphi}} = \frac{1 - \sqrt{5}}{2\sqrt{5}} = \frac{\hat{\varphi}}{\sqrt{5}}.$$

All together we get

$$F(z) = \frac{\hat{\varphi}}{\sqrt{5}(z + \hat{\varphi})} - \frac{\varphi}{\sqrt{5}(z + \varphi)} = \frac{1}{\sqrt{5}} \left(\frac{\hat{\varphi}}{z + \hat{\varphi}} - \frac{\varphi}{z + \varphi} \right).$$

Homogeneous linear recurrences - Fibonacci numbers

We could already expand this function as a power series, but it is easier if we transform it with the aid of a small trick. We will multiply each numerator and denominator by φ and $\hat{\varphi}$, respectively, and then we will use the identity $\varphi\hat{\varphi} = -1$:

$$\begin{aligned} F(z) &= \frac{1}{\sqrt{5}}\left(\frac{\hat{\varphi}}{z+\hat{\varphi}} - \frac{\varphi}{z+\varphi}\right) = \frac{1}{\sqrt{5}}\left(\frac{\varphi\hat{\varphi}}{\varphi z+\varphi\hat{\varphi}} - \frac{\varphi\hat{\varphi}}{\hat{\varphi} z+\varphi\hat{\varphi}}\right) \\ &= \frac{1}{\sqrt{5}}\left(\frac{-1}{\varphi z-1} - \frac{-1}{\hat{\varphi} z-1}\right) = \frac{1}{\sqrt{5}}\left(\frac{1}{1-\varphi z} - \frac{1}{1-\hat{\varphi} z}\right) \end{aligned}$$

We can now proceed with the power series expansion, since we have two terms of the form $\dfrac{1}{1-az} = 1 + az + a^2z^2 + a^3z^3 + \cdots$

Homogeneous linear recurrences - Fibonacci numbers

We get the expansion

$$F(z) = \frac{1}{\sqrt{5}} \left(1 + \varphi z + \varphi^2 z^2 + \varphi^3 z^3 + \cdots - 1 - \hat{\varphi} z - \hat{\varphi}^2 z^2 - \hat{\varphi}^3 z^3 - \cdots \right)$$

Hence, the n-th Fibonacci number is

$$F_n = \frac{1}{\sqrt{5}} \left[\varphi^n - \hat{\varphi}^n \right] = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right].$$

Note that this formula is valid for all $n \geq 0$, and $F_0 = 0$, as defined.

Homogeneous linear recurrences - Fibonacci numbers

We can also solve the recurrence

$$F_{n+2} = F_{n+1} + F_n, \text{ with } n \geq 0 \text{ and } F_0 = 0, F_1 = 1$$

by exponential generating functions. Let $E(z) = \sum_{n=0}^{\infty} F_n \frac{z^n}{n!}$. Then we get the differential equation

$$E''(z) = E'(z) + E(z),$$

whose solution is

$$E(z) = c_1 e^{\varphi z} + c_2 e^{\hat{\varphi} z},$$

where c_1 and c_2 can be determined from the initial conditions $E(0) = 0$ and $E'(0) = 1$.

Homogeneous linear recurrences - Fibonacci numbers

We have

$$E'(z) = c_1 \varphi e^{\varphi z} + c_2 \hat{\varphi} e^{\hat{\varphi} z}.$$

That leads to the linear system

$$\begin{aligned} E(0) &= c_1 + c_2 = 0 \\ E'(0) &= c_1 \varphi + c_2 \hat{\varphi} = 1, \end{aligned}$$

with solutions $c_1 = \dfrac{1}{\sqrt{5}}$ and $c_2 = -\dfrac{1}{\sqrt{5}}$. Hence

$$E(z) = \frac{1}{\sqrt{5}} e^{\varphi z} - \frac{1}{\sqrt{5}} e^{\hat{\varphi} z}.$$

We must now expand $E(z)$ as a power series.

Homogeneous linear recurrences - Fibonacci numbers

$$
\begin{aligned}
E(z) &= \frac{1}{\sqrt{5}} e^{\varphi z} - \frac{1}{\sqrt{5}} e^{\hat{\varphi} z} \\
&= \frac{1}{\sqrt{5}} \sum_{n=0}^{\infty} \frac{\varphi^n z^n}{n!} - \frac{1}{\sqrt{5}} \sum_{n=0}^{\infty} \frac{\hat{\varphi}^n z^n}{n!} \\
&= \frac{1}{\sqrt{5}} \left(\sum_{n=0}^{\infty} (\varphi^n - \hat{\varphi}^n) \frac{z^n}{n!} \right)
\end{aligned}
$$

Therefore, the n-th Fibonacci number is

$$
F_n = \frac{1}{\sqrt{5}} \left[\varphi^n - \hat{\varphi}^n \right] = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right],
$$

as before.

Non-homogeneous linear recurrences

Example 7 – A non-homogeneous recurrence: (Example 9.25 of *Applied Combinatorics*, by M.T. Keller and W.T. Trotter):

Consider now the non-homogeneous recurrence

$$a_{n+2} = a_{n+1} + 2a_n + 2^{n+2}, \text{ with } n \geq 0 \text{ and } a_0 = 2, a_1 = 1$$

Let us define $A(z) = \sum_{n=0}^{\infty} a_n z^n$. Again, we multiply both sides of the equation by z^n and sum over $n \geq 0$:

$$\sum_{n=0}^{\infty} a_{n+2} z^n = \sum_{n=0}^{\infty} a_{n+1} z^n + 2 \sum_{n=0}^{\infty} a_n z^n + \sum_{n=0}^{\infty} 2^{n+2} z^n$$

$$\frac{A(z) - a_0 - a_1 z}{z^2} = \frac{A(z) - a_0}{z} + 2A(z) + \frac{4}{1 - 2z}$$

Non-homogeneous linear recurrences

$$\frac{A(z)-2-z}{z^2} - \frac{A(z)-2}{z} - 2A(z) - \frac{4}{1-2z} = 0$$

$$\frac{4A(z)z^3 - 3A(z)z + A(z) - 6z^2 + 5z - 2}{z^2(1-2z)} = 0$$

$$\frac{A(z)(4z^3 - 3z + 1) - 6z^2 + 5z - 2}{z^2(1-2z)} = 0$$

Solving for $A(z)$ we get

$$A(z) = \frac{6z^2 - 5z + 2}{4z^3 - 3z + 1} = \frac{6z^2 - 5z + 2}{(z+1)(2z-1)^2}$$

Decomposing $A(z)$ into partial fractions we get

$$A(z) = \frac{13}{9(1+z)} - \frac{1}{9(1-2z)} + \frac{2}{3(1-2z)^2}$$

Non-homogeneous linear recurrences

Develop $A(z)$ into power series:

$$\begin{aligned} A(z) &= \frac{13}{9(1+z)} - \frac{1}{9(1-2z)} + \frac{2}{3(1-2z)^2} \\ &= \frac{13}{9}\sum_{n=0}^{\infty}(-1)^n z^n - \frac{1}{9}\sum_{n=0}^{\infty} 2^n z^n + \frac{2}{3}\sum_{n=0}^{\infty}\binom{n+2-1}{1} 2^n z^n \end{aligned}$$

From the series we can read off the general term a_n:

$$a_n = \frac{13}{9}(-1)^n - \frac{1}{9}2^n + \frac{2(n+1)}{3}2^n = \frac{5}{9}2^n + \frac{2}{3}n2^n + \frac{13}{9}(-1)^n$$

Catalan numbers

- Named after the Belgian mathematician Eugène Charles Catalan (1814 – 1894)
- Student of Joseph Liouville
- These numbers arise in many combinatorial problems, for instance
  - Dyck words
  - Lattice paths
  - Triangulations of a convex polygon

Back to list of functions

Figure: Eugène Charles Catalan

Dyck words

The n-th Catalan number, C_n, represents the number of **Dyck words** of length $2n$.

Let Σ be an alphabet consisting of two letters, say X and Y. A Dyck word on Σ is a string S having n copies of X and n copies of Y, such that for every prefix of S the number of X's is greater than or equal to the number of Y's. Here we show the Dyck words for $n = 1, 2, 3$:

1. XY
2. $XXYY$, $XYXY$
3. $XXXYYY$, $XYXXYY$, $XYXYXY$, $XXYYXY$, $XXYXYY$

If we interpret X and Y as open and closed parenthesis, respectively, the Dyck words of length $2n$ corresponds to the expressions that are correctly parenthesized. For $n = 3$ they are:

$$((())) \quad (()()) \quad (())() \quad ()(()) \quad ()()()$$

Lattice paths

We can also interpret X as a step to the right, and Y as an upward step. In that case C_n represents the number of **lattice paths** that go from $(0;0)$ to $(n;n)$ and do not cross the diagonal.

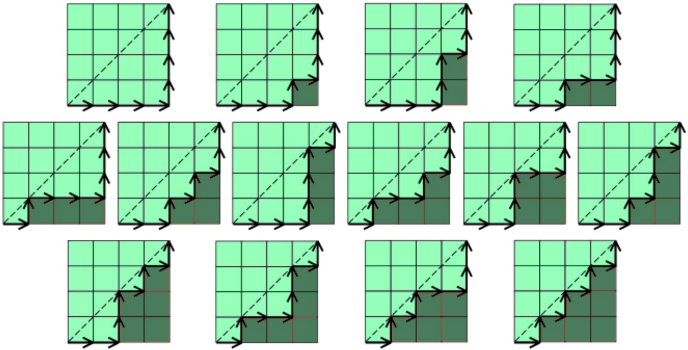

Figure: The 14 legal lattice paths for $n = 4$

Nonlinear recurrences – Catalan numbers

Example 8 – A more complex recurrence – Catalan numbers:

Let C_n denote the number of triangulations of a convex $(n+2)$-gon. By convention let $C_0 = 1$, and C_1 is also equal to 1, since it's the number of triangulations of a triangle.

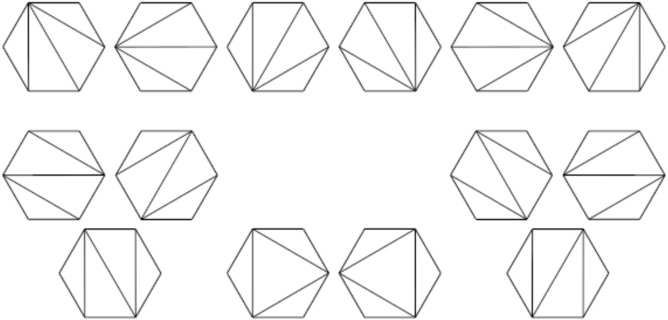

Figure: The 14 hexagon triangulations

Nonlinear recurrences – Catalan numbers

The numbers C_n obey the following recurrence:

$$C_{n+1} = \begin{cases} 1 \ \text{ if } \ n+1=0 \ \text{ or } \ n+1=1 \\ \sum_{k=0}^{n} C_k C_{n-k} \ \text{ otherwise} \end{cases}$$

As before, we multiply both sides of the recurrence by z^n and sum over $n \geq 0$:

$$\sum_{n=0}^{\infty} C_{n+1} z^n = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} C_k C_{n-k} \right) z^n, \quad \text{whence}$$

$$\frac{C(z)-1}{z} = C(z)^2$$

$$zC(z)^2 - C(z) + 1 = 0$$

Nonlinear recurrences – Catalan numbers

The solutions of this equation are:

$$C(z) = \frac{1 \pm \sqrt{1-4z}}{2z},$$

where the correct choice is

$$C(z) = \frac{1 - \sqrt{1-4z}}{2z}.$$

As we saw before, the series expansion of this function is

$$\sum_{n=0}^{\infty} \frac{1}{n+1}\binom{2n}{n} z^n,$$

therefore

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

Series involving Fibonacci numbers

Example 9 – Identity involving Fibonacci numbers: Going back to the generating function associated with Fibonacci numbers, it has two isolated singularities (poles) on the real axis, with $-\hat{\varphi} \approx 0.618033...$ being the closest one to the origin. Hence, the series converges to the function $F(z) = \dfrac{z}{1 - z - z^2}$ for $|z| < |\hat{\varphi}|$, i.e. for $|z| < 0.618033...$

Therefore, we can evaluate the function and the series at any value of z with $|z| < |\hat{\varphi}|$, and equate both results. For instance, we could take $z = \dfrac{1}{2}$, which leads us to the (otherwise elusive) formula

$$\sum_{n=0}^{\infty} F_n \left(\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} F_n \left(\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} \frac{F_n}{2^n} = \frac{\frac{1}{2}}{1 - \frac{1}{2} - \frac{1}{4}} = 2$$

Applications III – Counting

Example 10 (introductory example): Find the number of integer solutions of the equation $a + b = 5$, where $2 \leq a, b \leq 4$.

Solution: We can assign the (polynomial) generating function $f(z) = z^2 + z^3 + z^4$ to each variable, meaning that the variable can take the values 2, 3 or 4. Now we have to find the coefficient $[z^5]g(z)$, where $g(z) = (f(z))^2$, since there are two variables with identical conditions. Thus,

$$g(z) = (z^2 + z^3 + z^4)^2 = z^8 + 2z^7 + 3z^6 + 2z^5 + z^4$$

Hence the number of solutions is 2. Indeed, there are exactly two combinations that yield z^5, namely z^2z^3 and z^3z^2.

Applications III – Counting

Example 11 (adapted from *Matemática Discreta*, by Juan A. Rodríguez Velázquez): Find the number of integer solutions of the equation $a + b + c = 10$, where $2 \leq a, b, c \leq 4$.

Solution: We can assign the (polynomial) generating function $f(z) = z^2 + z^3 + z^4$ to each variable, meaning that the variable can take the values 2, 3 or 4. Now we have to find the coefficient $[z^{10}]g(z)$, where $g(z) = (f(z))^3$, since there are three variables with identical conditions. Thus,

$$g(z) = z^{12} + 3z^{11} + 6z^{10} + 7z^9 + 6z^8 + 3z^7 + z^6$$

Hence the number of solutions is 6.

Applications III – Counting

Example 12: Given two fair dice, find the number of combinations that yield a value k, where $2 \leq k \leq 12$.

Solution: To a single die we can associate the (polynomial) generating function $f(z) = z + z^2 + \cdots + z^6$. The combinations of two dice can be counted by the generating function

$$\begin{aligned}[f(z)]^2 &= z^2(z+1)^2(z^2-z+1)^2(z^2+z+1)^2 \\ &= z^{12} + 2z^{11} + 3z^{10} + 4z^9 + 5z^8 + 6z^7 + 5z^6 \\ &\quad + 4z^5 + 3z^4 + 2z^3 + z^2,\end{aligned}$$

where we can read the number of combinations for each value of k.

Alternatively, we are looking for the number of solutions of the equation $x + y = k$, where $1 \leq x, y \leq 6$ and $2 \leq k \leq 12$.

Applications IV – Probability

Probability generating functions:

Let X be a random variable which assumes nonnegative integer values, and let p_n the probability that $X = n$. Then the *probability generating function* of X is the generating function $P(z)$ for the sequence $\langle p_n \rangle_{n=0}^{\infty}$. This series converges in a radius greater than or equal to 1.

The *expected value* of X is

$$E(X) = \sum_{n=0}^{\infty} np_n = P'(1).$$

We will see more of that later !

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