2. Fourier Transform

I have drawn the content for this lecture mostly from the book *Mathematical Methods for the Physical Sciences* by K. F. Riley

In the last lecture we showed that we could represent a periodic function by a sum of sine and cosine terms of alternately by complex exponentials.

$$y(t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{2\pi nt}{T}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{2\pi nt}{T}\right) = \sum_{n=-\infty}^{\infty} C_n \exp\left(\frac{i2\pi nt}{T}\right) \tag{2-1}$$

Furthermore we showed that we could evaluate the coefficients in the first representation

$$\begin{aligned} A_n &= \frac{2}{T}\int_{-T/2}^{T/2} y(t)\cos\left(\frac{2\pi nt}{T}\right)dt \\ B_n &= \frac{2}{T}\int_{-T/2}^{T/2} y(t)\sin\left(\frac{2\pi nt}{T}\right)dt \end{aligned} \tag{2-2}$$

For the second representation the coefficients are obtained by

$$C_n = \frac{1}{T}\int_{-T/2}^{T/2} y(t)\exp\left(-\frac{i2\pi nt}{T}\right)dt \tag{2-3}$$

(Note that the negative sign in the exponential, which arises because orthogonality for complex functions is defined with the complex conjugate - see equation 1.9 of the previous lecture)

Fourier Transform

We can extend the Fourier series to non-periodic functions that are defined over an infinite interval. To do this we formally let the interval T of the Fourier series tend to infinity. If we substitute $\omega = 2\pi n/T$ into equation (2-3) and replace C_n by $C(\omega)$ to indicate the dependency of C on ω, we get

$$C(\omega) = \frac{1}{T}\int_{-T/2}^{T/2} y(t)\exp(-i\omega t)\,dt \tag{2-4}$$

We can write $y(t)$ as a sum of its components

$$\begin{aligned} y(t) &= \sum_{m=-\infty}^{\infty} C(\omega)\exp(i\omega t) \\ &= \frac{T}{2\pi}\sum_{m=-\infty}^{\infty} \frac{2\pi}{T} C(\omega)\exp(i\omega t) \\ &= \frac{T}{2\pi}\sum_{m=-\infty}^{\infty} C(\omega)\exp(i\omega t)\delta\omega \end{aligned} \tag{2-5}$$

where $\delta\omega = 2\pi/T$ is the spacing of adjacent $C(\omega)$ (or C_n) terms. Now we can see that as $\delta\omega$ becomes small the above sum can be represented as an integral

$$y(t) = \frac{T}{2\pi}\int_{-\infty}^{\infty} C(\omega)\exp(i\omega t)\,d\omega \tag{2-6}$$

Now for equations (2-4) and (2-6) we set $Y(\omega) = TC(\omega)$ and let $T \to \infty$. This yields

$$Y(\omega) = \int_{-\infty}^{\infty} y(t)\exp(-i\omega t)\,dt$$
$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(\omega)\exp(i\omega t)\,d\omega \tag{2-7}$$

Alternatively, we can set $Y(\omega) = TC(\omega)/(2\pi)^{1/2}$ and let $T \to \infty$. This yields

$$Y(\omega) = \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty} y(t)\exp(-i\omega t)\,dt$$
$$y(t) = \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty} Y(\omega)\exp(i\omega t)\,d\omega \tag{2-8}$$

Since $\omega = 2\pi f$, we can also write (2-7) as

$$Y(f) = \int_{-\infty}^{\infty} y(t)\exp(-i2\pi ft)\,dt$$
$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(f)\exp(i2\pi ft)\,2\pi df = \int_{-\infty}^{\infty} Y(f)\exp(i2\pi ft)\,df \tag{2-9}$$

Equations (2-7) to (2-9) are three alternate but equivalent representations of the Fourier transform pair (i.e., the Fourier transform and the inverse Fourier transform) and you need to be careful to make sure which version is being used. In this class, we will follow the convection adopted in Bob Crosson's class notes and use (2-9).

Properties of the Fourier Transform

Differentiation is very easy in the frequency domain – you just multiply by $i\omega$

$$y'(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} i\omega Y(\omega)\exp(i\omega t)\,d\omega$$
$$FT\left[y'(t)\right] = i\omega Y(\omega) \tag{2-10}$$

Similarly integration involves dividing by $i\omega$

$$\int_t y(s)ds = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{Y(\omega)}{i\omega}\exp(i\omega t)\,d\omega + \text{constant}$$
$$FT\left[\int_t y(s)ds\right] = \frac{Y(\omega)}{i\omega} + \text{constant} \tag{2-11}$$

Translation by $-a$ involves multiplying by $\exp(i\omega a)$

$$y(t+a) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(\omega)\exp\left[i\omega(t+a)\right]d\omega$$
$$FT\left[y(t+a)\right] = \exp(i\omega a)Y(\omega) \tag{2-12}$$

The δ-function

The δ-function is a very useful function that we will come across repeatedly when consider filters and the discrete (digital) version of the Fourier transform. It is defined mathematically by

$$\delta(x) = 0, \quad x \neq 0$$

$$\int_a^b \delta(x)\,dx = 1, \quad a < 0 < b \qquad (2\text{-}13)$$

$$\int_a^b f(y)\delta(y-x)\,dy = f(x), \quad a < y < b$$

It can be thought of an infinitely sharp narrow pulse – it not physically realizable but for practical purposes it just needs to be narrower than a system can resolve (i.e., shorter in duration than the time between samples in digital system).

The Fourier transform of a δ-function is

$$D(\omega) = \int_{-\infty}^{\infty} \delta(t)\exp(-i\omega t)\,dt = 1 \qquad (2\text{-}14)$$

All frequencies are equally represented. For a δ-function at time t_0, the Fourier transform is

$$D(\omega) = \int_{-\infty}^{\infty} \delta(t-t_0)\exp(-i\omega t)\,dt = \exp(-i\omega t_0) \qquad (2\text{-}15)$$

We get a similar result when we consider the inverse Fourier transform of a δ-function in the frequency domain.

$$d(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega-\omega_0)\exp(i\omega t)\,d\omega = \exp(i\omega_0 t) \qquad (2\text{-}16)$$

In the 2nd class exercise you will investigate at a form of "Uncertainty Principle" - it is impossible to produce a signal that is narrow in both the time and frequency domain.

The Meaning of complex coefficients

The complex coefficients carry information about the amplitude and phase of a signal. Consider a monochromatic signal of frequency *f*. For the frequency +*f*, we can write the Fourier coefficient as

$$Y(f) = a + ib \qquad (2\text{-}17)$$

In the time domain this gives

$$\begin{aligned}&(a+ib)(\cos 2\pi ft + i\sin 2\pi ft)\\ &= (a\cos 2\pi ft - b\sin 2\pi ft) + i(a\sin 2\pi ft + b\cos 2\pi ft)\end{aligned} \qquad (2\text{-}18)$$

For the frequency –*f*, we can write the Fourier coefficient as

$$Y(-f) = c + id \qquad (2\text{-}19)$$

In the time domain this gives

$$\begin{aligned}&(c+id)(\cos 2\pi ft - i\sin 2\pi ft)\\ &= (c\cos 2\pi ft + d\sin 2\pi ft) + i(-c\sin 2\pi ft + d\cos 2\pi ft)\end{aligned} \qquad (2\text{-}20)$$

The combined effects of $+f$ and $-f$ in the time domain are thus

$$\left[(a+c)\cos 2\pi ft+(d-b)\sin 2\pi ft\right]+i\left[(b+d)\cos 2\pi ft+(a-c)\sin 2\pi ft\right] \tag{2-21}$$

Clearly if the signal is real

$$\begin{aligned} d &= -b \\ a &= c \end{aligned} \tag{2-22}$$

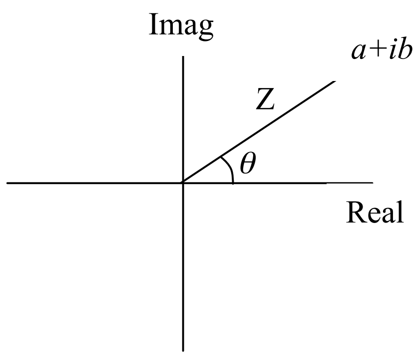
so that

$$\begin{aligned} Y(f) &= a+ib \\ Y(-f) &= a-ib \end{aligned} \tag{2-23}$$

In the time domain this yields

$$y(t)=2(a\cos 2\pi ft-b\sin 2\pi ft) \tag{2-24}$$

The relative values of the real and imaginary coefficients can be written in terms of a magnitude and phase



$$a+ib=Z\exp(i\theta) \tag{2-25}$$

where

$$\theta=\arctan(b/a) \tag{2-26}$$

while their magnitude gives the amplitude of the signal

$$Z=\sqrt{a^2+b^2} \tag{2-27}$$

From equation (2-24) it is clear that the amplitude of the $y(t)$ is $2Z$. By drawing the cosine function and negative sine function you can also see that the effect of a positive value of b is to move the signal earlier in time which would be described as a phase shift of $-\theta$ (by convention a positive phase shift moves the signal to the right on a plot or to a later time)

Pictorial Dictionary of Fourier Transforms

In this class we will soon be moving on to the discrete Fourier transform because that is what we deal with numerically. You should all be aware of the classic textbook for the continuous Fourier Transform - Bracewell, R. N., *The Fourier transform and its applications*, McGraw-Hill, 2nd edition, 1986. Chapter 21 is a pictorial dictionary of Fourier transforms.