# THE BOOK

ON

# THE STUDY OF ANALYSIS

## FOR THE FALL STUDY ON GRADUATE ANALYSIS


BY

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The following textbook is dedicated in favor of my respectable faculty teachers at Beaconhouse School System:

Faisal Jaber Abbasi
Gul Khatab

# PREFACE

This comprehensive textbook has been meticulously crafted as part of an initiative to provide graduate students with a robust foundation across various disciplines of Analysis. Its primary emphasis lies in elucidating fundamental concepts ranging from introductory to diversified analysis, ensuring a thorough understanding of the subject matter.

Great care was taken to present the material in a concise and accessible manner, placing a special focus on articulating theorems along with insightful remarks. This approach not only facilitates a clear comprehension of the theoretical underpinnings but also encourages students to delve into the nuances and implications of each concept.

Beyond the dissemination of knowledge, this textbook aspires to instill in students a deep conscientiousness towards research endeavors. By emphasizing the importance of critical thinking and intellectual exploration, the authors aim to nurture a mindset that goes beyond the surface-level understanding of mathematical principles. The inclusion of induction as a cornerstone further underscores the significance of cultivating logical reasoning skills, providing students with the tools necessary for engaging with the core tenets of various mathematical disciples.

In essence, this textbook serves as a catalyst for intellectual curiosity, pushing students to not only grasp the intricacies of analysis but also to actively contribute to the ongoing dialogue within the realm of mathematical research. Its thoughtful design and content promote a holistic approach, encouraging students to transcend rote learning and embrace a deeper, more profound understanding of the mathematical landscape. As a valuable resource for those navigating the complexities of graduate-level courses, this textbook stands as a testament to the commitment to excellence in mathematical education.

# Contents

# Chapter 1

# INTRODUCTORY ANALYSIS

Calculus, formulated between the 1630s and 1700, was independently developed by Isaac Newton (1642-1727) and Gottfried Wilhelm Leibniz (1646-1716). The mathematical foundation was significantly influenced by Euclid's "Elements" (300 BC), which employed axiomatic and rigorous development of Euclidean geometry. In 1821, Augustin-Louis Cauchy provided the first predominantly rigorous development of Calculus.

**Example 1.**—Define $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ as the set of integers.

**Proposition 1** (M5)**.**—For any nonzero $c \in \mathbb{Q}$, there exists an element $d \in \mathbb{Q}$ such that $c \cdot d = 1$.

**Example 2.**—Identify the smallest set containing $\mathbb{Z}$ where axiom (M5) holds. Answer: The set of rational numbers $\mathbb{Q} = \{a/b \mid a, b \in \mathbb{Z}, b \neq 0\}$. $\mathbb{Q}$ forms a field.

**Definition.**—A field is a set with at least two elements equipped with two defined operations.

In a field $\mathbb{F}$, the additive identity (0) is distinct from the multiplicative identity (1).

**Example 3.**—Consider $\mathbb{F}_2$, a field containing two elements $\bar{0}$ and $\bar{1}$.

**Example 4.**—Let $x$ be an irrational number. Prove the existence of an irrational number $y$ such that $xy$ is rational.

**Proof.** (M5): For any non-zero real number $a$, there exists an element $b \in \mathbb{R}$ such that $ab = 1$.

Assume there exists a non-zero number $y$ such that $xy = 1$. Since $x$ is irrational, it must be non-zero. Assume $y$ is rational. If $y$ is rational, so is $\frac{1}{y}$ since $x = \frac{1}{y}$. Consequently, $x$ is rational, contradicting the initial assumption that $x$ is irrational.

**Theorem.** *If $x \in \mathbb{R}$ and $\forall \epsilon > 0 (|x| < \epsilon)$, then $x = 0$ by (A12).*

*Proof (by Contradiction).* Assume $x \neq 0$. According to (Q2), $|x| \neq 0$, so $0 < |x|$. Let $\epsilon = \frac{|x|}{2}$, ensuring $\epsilon > 0$. By assumption, $|x| < \frac{|x|}{2}$ implies multiplication by $\frac{1}{|x|} > 0$. This results in $1 < \frac{1}{2}$, a contradiction. Therefore, $x = 0$.

Given: $\mathbb{R}$ is an "ordered field," and $\mathbb{Q}$ is an "ordered field." However, an additional axiom is needed to fully distinguish $\mathbb{R}$ from $\mathbb{Q}$ and to conduct calculus.

**Example 5.**—$\sqrt{2}$ is irrational. $\mathbb{R}$ denotes the set of real numbers.

While we are familiar with rational numbers $\mathbb{Q}$ and employ the "metric" $|x-y|$ to measure the distance between two rational numbers $x$ and $y$, irrational numbers, such as $\sqrt{2}$, $5^{1/3}$, $\pi$, $\ln 4$, $\sin 27\,\text{deg}$, $\tan 17\,\text{deg}$, etc., are more challenging to comprehend. Nonetheless, many numbers encountered in scientific applications are irrational, compelling us to grapple with these numbers.

For instance, how do we conceptualize $\pi$ as a number? We represent it as $\pi = 3.1415926\ldots$. Notice that an (infinite) sequence of rational numbers is implied:

$$\frac{3}{1}, \frac{31}{10}, \frac{314}{100}, \frac{3141}{1000}, \ldots .$$

We denote this sequence as $x_1, x_2, x_3, x_4, \ldots$, and with each rational number in this sequence, we get one step closer to $\pi$. We have $x_1 < x_2 < x_3 < \ldots$.

Since we understand $\mathbb{Q}$ well, we use it as our base field. Understanding irrational numbers in terms of rational numbers requires the use of limits, i.e., through infinite processes. The sequence $x_1, x_2, x_3, \ldots$ is a sequence of rational numbers that converges to an irrational number.

To completely characterize the system of real numbers $\mathbb{R}$, an additional axiom is needed to ensure that certain sequences of rational and real numbers converge within the real number system.

$\sim$ 1870: The set of real numbers can be constructed from the set of rational numbers alone (Weierstrass, Cantor, Dedekind).

Consider the prime numbers: $2, 3, 5, 7, \ldots$ Completing $\mathbb{Q}$ with respect to the usual metric results in $\mathbb{R}$. Completing $\mathbb{Q}$ with respect to the $p$-adic metric yields $\mathbb{Q}_p$.

Completeness can be proven as a theorem; however, for our purposes, we assume it as an axiom.

**Definition** (Function).—A function $f : A \to B$ is a rule of correspondence that assigns to each element of the set $A$ exactly one element in the set $B$ such that $a \in A \mapsto f(a) \in B$. The domain is $A$, the codomain is $B$, and the range is $\{f(a) \mid a \in A\}$.

**Definition** (Infinite sequence of real numbers).—A sequence of real numbers is a function $f : \mathbb{Z}^+ \to \mathbb{R}$ with domain

$\mathbb{Z}^+$ and codomain $\mathbb{R}$. A refined definition is when the domain is $\{n \in \mathbb{Z} \mid n \geq m\}$, where $m$ is a fixed integer. If $m = 1$, we recover the book definition, but sometimes we may want $m = 0$ or even $m = 37$.

$f(1) = x_1, f(2) = x_2, f(3) = x_3, \ldots$ can be represented as $\{x_n\}_{n=1}^{\infty} = \{x_n\}$.

**Definition.**—If $\{x_n\}$ is an infinite sequence of real numbers, we say that $x_1, x_2, x_3, \ldots$ converges to $x \in \mathbb{R}$ if, for any $\epsilon > 0$, there exists a positive integer $N \in \mathbb{Z}$ such that $|x_n - x| < \epsilon$ for all $n \geq N$.

**Definition.**—A sequence $x_1, x_2, x_3, \ldots$ of real numbers is said to converge to a number $L \in \mathbb{R}$ if for each $\epsilon > 0$ or $\forall \epsilon > 0$, there exists $N \in \mathbb{Z}^+$ such that

$$\forall n \geq N(|x_n - L| < \epsilon),$$

where $\epsilon$ is an arbitrary positive real number.

The objective is always: given some $\epsilon$, find $N$.

**Note 1.**—$(|x_n - L| < \epsilon) \implies (L - \epsilon < x_n < L + \epsilon)$.

**Remark 1.**—Infinitesimals are ghosts of departed quantities; we do not need them here.

**Definition.**—The sequence $\{x_n\}$ is convergent (or converges) if there exists a number $L \in \mathbb{R}$ such that $\{x_n\}$ converges to $L$. The sequence $\{x_n\}$ is divergent (or diverges) if $\{x_n\}$ does not converge.

**Example 6.**—$x_1 = 1, x_2 = -1, x_3 = 1, x_4 = -1, \ldots$ does not converge.

**Theorem.** *If a sequence $\{x_n\}$ converges, then the value $L$ referred to in equation 1 is uniquely determined. This value $L$ is denoted as the limit of the sequence $\{x_n\}$.*

**Proof.** Consider a convergent sequence $\{x_n\}$ and assume it converges to real numbers $a$ and $b$. Let $\epsilon > 0$. According to the definition, there exists $N_1 \in \mathbb{Z}^+$ such that for all $n \geq N_1$, $|x_n - a| < \frac{\epsilon}{2}$. Similarly, there exists $N_2 \in \mathbb{Z}^+$ such that for all $n \geq N_2$, $|x_n - b| < \frac{\epsilon}{2}$. Set $M = \max\{N_1, N_2\}$. Then, using the triangle inequality, we have

$$|a - b| = |a - x_M + x_M - b| \leq |a - x_M| + |x_M - b| = |x_M - a| + |x_M - b|.$$

Since $|x_M - a| < \frac{\epsilon}{2}$ and $|x_M - b| < \frac{\epsilon}{2}$, it follows that $|x_M - a| + |x_M - b| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. Consequently, $|a - b| < \epsilon$ for every $\epsilon > 0$. By (A12), $a - b = 0$ or $a = b$, establishing the unique definition of the limit $L$.

**Definition.**—A sequence $\{x_n\}$ of real numbers is said to converge to a number $L \in \mathbb{R}$ if, for every $\epsilon > 0$, there exists $N \in \mathbb{Z}^+$ such that $|x_n - L| < \epsilon$ for all $n \geq N$.

**Definition.**—Let $\{x_n\}$ be a sequence of real numbers.

a) The sequence $\{x_n\}$ is said to be bounded above if there exists $M \in \mathbb{R}$ such that $x_n \leq M$ for all $n \geq 1$. The number $M$ is referred to as an upper bound for $\{x_n\}$.

d) The sequence $\{x_n\}$ is said to be increasing if $x_n \leq x_{n+1}$ for all $n \geq 1$. The sequence $\{x_n\}$ is strictly increasing if $x_n < x_{n+1}$ for all $n \geq 1$.

**Example 7.**—The sequence $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots$ is strictly decreasing and bounded below (0 is a lower bound).

**Example 8.**—The sequence $1, 1, 1, 1, \ldots$ is an increasing sequence (and also decreasing).

**Definition** (Completeness Axiom (version I))**.**—An infinite sequence $x_1, x_2, \ldots$ of real numbers that is increasing and bounded above always converges to a unique real number $L \in \mathbb{R}$. This also holds when the sequence is decreasing and bounded below.

**Note 2.**—The set of real numbers $\mathbb{R}$ is a complete ordered field.

**Example 9.**—Consider the sequence $x_n = (1+\frac{1}{n})^n$, where $x_1 = 2$, $x_2 = \frac{9}{4}$, $x_3 = \frac{64}{27}$. It can be proven that this sequence is strictly increasing and bounded above by 3. By the completeness axiom, this sequence converges to a unique real number, which we denote as $e$.

For any positive real number $t > 0$, it can be proved that there exists a positive real number denoted by $\sqrt{t}$ such that $(\sqrt{t})^2 = t$, and there is no other positive real number whose square is $t$. The completeness axiom is essential for this proof. We define: $\sqrt{0} = 0$.

- s1. $|t| = \sqrt{t^2}$ for all $t \in \mathbb{R}$.
- s2. If $0 \leq s \leq t$, then $\sqrt{s} \leq \sqrt{t}$.
- s3. $\sqrt{s} \cdot \sqrt{t} = \sqrt{s \cdot t}$ for all $s, t \geq 0 \in \mathbb{R}$.

**Note 3.**—The expression $||(x_1, y_1) - (x_2, y_2)|| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ represents the definition of the distance between points in $\mathbb{R}^2$.

**Remark 2.**—A sequence $(x_1, y_1), (x_2, y_2), \ldots$ of points in $\mathbb{R}^2$ is said to converge to a point $(L, M) \in \mathbb{R}^2$ if, for every $\epsilon > 0$, there exists $N \in \mathbb{Z}^+$ such that $||(x_n, y_n) - (L, M)|| < \epsilon$ for all $n \geq N$.

**Proposition 2.**—Archimedean Property of $\mathbb{R}$: For any $x \in \mathbb{R}$, there exists $N \in \mathbb{Z}^+$ such that $x < N$.

**Proof.** Let's assume the opposite (proof by contradiction)! Consider a fixed real number $x$. We aim to demonstrate two assertions:

1. There exists a positive integer $N$ such that $x < N$.
2. The negation of (1) states: $n \leq x$ for all positive integers $n$.

Utilizing the completeness axiom, we will illustrate that (2) leads to a contradiction. Notice that the positive integers $1, 2, 3, \ldots$ form an infinitely increasing sequence of real numbers. According to assumption (2), this sequence is bounded above by the real number $x$. Therefore, by the completeness axiom, there exists a unique real number $L \in \mathbb{R}$ such that $\lim_{n\to\infty} n = L$. We will now show that this is impossible, leading to a contradiction. Set $\epsilon = \frac{1}{2}$. There is at most one positive integer in the open interval $(L-\frac{1}{2}, L+\frac{1}{2})$. If $1, 2, 3, \ldots$ truly converged to $L$, we would need more than one positive integer in $(L-\frac{1}{2}, L+\frac{1}{2})$—in fact, we would need an infinite number.

**Note 4.**—When negating an existential quantifier, we switch to a universal quantifier. If we negate a universal quantifier, we switch to an existential quantifier.

**Proposition 3.**—$\{\frac{7n}{4n+5}\} \underset{n\to\infty}{\longrightarrow} \frac{7}{4}$. We need to show that for every $\epsilon > 0$, there exists $N \in \mathbb{Z}^+$ such that $|x_n - L| < \epsilon$ for all $n \geq N$. Given $\epsilon$, the goal is to find $N \in \mathbb{Z}^+$ such that everything works out.

**Proof.** $|\frac{7n}{4n+5} - \frac{7}{4}| = |\frac{28n-7(4n+5)}{4(4n+5)}| = |\frac{-35}{4(4n+5)}| = \frac{35}{4(4n+5)} < \frac{35}{4(4n)} < \frac{3}{n} \leq \frac{3}{N} < \epsilon$ because $N \leq n \implies \frac{1}{n} \leq \frac{1}{N}$. Choose $\epsilon > 0$. Let $x = \frac{3}{\epsilon}$. By the Archimedean principle of $\mathbb{R}$, there exists $N \in \mathbb{Z}^+$ such that $x < N$ or $\frac{3}{\epsilon} < N$ or $\frac{3}{N} < \epsilon$.

For a sequence not to converge to $L$, there must be at least one positive real number $\epsilon(> 0)$ such that for every positive integer $N$, there exists an integer $n \geq N$ such that $|x_n - L| \geq \epsilon$.

$x_1, x_2, x_3, \ldots$ converges to $L \in \mathbb{R}$ if $\forall \epsilon > 0$, there exists $N \in \mathbb{Z}^+$ such that $|x_n - L| < \epsilon$ for all $n \geq N$.

**Theorem.** *If $\{x_n\}$ is convergent (say $\lim_{n\to\infty} x_n = L$), then $\{x_n\}$ is bounded (there exists $M$, a positive real number, such that $|x_n| \leq M$ for all $n \geq 1$).*

**Proof.** Set $\epsilon = 1$. By assumption, there exists $N \in \mathbb{Z}^+$ such that $|x_n - L| < 1$ for all $n > N$. Claim: The positive real number $M = \max\{|x_1|, |x_2|, \ldots, |x_N|, |L| + 1\}$ is a bound for our sequence.

**Note 5.**—$|x_j| \leq M$ for $j = 1, \ldots, N$.

Assume $n > N$. Then $|x_n| = |x_n - L + L| \leq |x_n - L| + |L| < 1 + |L| \leq M$.

If $A$, then $B$. Contrapositive is: If not $B$, then not $A$. Contrapositive of Theorem 3: If $\{x_n\}$ is not bounded, then $\{x_n\}$ does not converge. Converse to If $A$, then $B$: If $B$, then $A$. Converse of Theorem 3: If $\{x_n\}$ is bounded, then $\{x_n\}$ converges. This is not true. Counter example: $\{(-1)^n\} = -1, 1, -1, 1, \ldots$.

**Theorem.** *Suppose that $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = M$. Then:*

*a) $\{ca_n\}$ converges to $c \cdot L$ for any given fixed constant $c \in \mathbb{R}$.*

*b) $\{a_n + b_n\}$ converges to $L + M$.*

*c) $\{a_n - b_n\}$ converges to $L - M$.*

*d) $\{a_n \cdot b_n\}$ converges to $L \cdot M$.*

*e) Assume furthermore that $b_j \neq 0$ for all $j \geq 1$ and that $M \neq 0$. Then $\{\frac{a_n}{b_n}\}$ converges to $\frac{L}{M}$.*

Bounded above $(\forall n \geq 1(x_n \leq M))$ & bounded below $(\forall n \geq 1(m \leq x_n)) \implies$ bounded $(\forall n \geq 1(|x_n| \leq B)$, where $B \in \mathbb{R}^+)$. Since $-B \leq x_n \leq B$ so $B = \max\{|m|, |M|\}$.

**Example 10.**—$0 \leq x_n$ for all $n \geq 1$. $\{x_n\}$ converges to $L$. We want to prove $\underline{0 \leq L}$.

**Proof.** Proof by contradiction: Assume $L < 0$. Given any $\epsilon > 0$ there exists $N \in \mathbb{Z}^+$ such that $|x_n - L| < \epsilon$ for all $n \geq N$, so $L - \epsilon < x_n < L + \epsilon$. Pick $\epsilon$ such that $L + \epsilon = 0$ by choosing $\epsilon = -L > 0$. For this $\epsilon$ there exists $N \in \mathbb{Z}^+$ such that $L - \epsilon < x_n < L + \epsilon = 0$ for all $n \geq N$. Therefore $x_N < 0$. Contradiction.

**Recall 1.**—$b_1, b_2, b_3, \cdots \to M$, where $b_j \neq 0$ for all $j \geq 1$ and $M \neq 0$. Want to prove: $\frac{1}{b_1}, \frac{1}{b_2}, \frac{1}{b_3}, \cdots \to \frac{1}{M}$.

**Proof.** Showed there exists $N_1 \in \mathbb{Z}^+$ such that $\frac{1}{|b_n|} < \frac{2}{|M|}$ for all $n \geq N_1$. Note that

$$\left|\frac{1}{b_n} - \frac{1}{M}\right| = \left|\frac{M - b_n}{b_n \cdot M}\right| = \frac{|M - b_n|}{|b_n||M|} = \frac{|b_n - M|}{|b_n||M|}. \tag{1.1}$$

Since $\frac{1}{|b_n|} < \frac{2}{|M|}$ for all $n \geq N_1$, we have $\frac{1}{|b_n||M|} < \frac{2}{|M|^2}$ for all $n \geq N_1$. Let an arbitrary $\epsilon > 0$ be given. Since $\{b_n\}$ converges to $M$ by assumption, there exists $N_2 \in \mathbb{Z}^+$ such that $|b_n - M| < \frac{|M|^2\epsilon}{2}$ for all $n \geq N_2$. Set $N = \max\{N_1, N_2\}$. Then we have $\frac{|b_n - M|}{|b_n||M|} < \frac{2}{|M|^2} \cdot \frac{|M|^2}{2} \cdot \epsilon = \epsilon$ for all $n \geq N$. Going back to equation (1.1), we have $\left|\frac{1}{b_n} - \frac{1}{M}\right| < \epsilon$ for all $n \geq N$.

**Example 11** (Exam 1, 7.)**.**—$a_n \leq x_n \leq b_n$ for all $n \geq 1$.

**Proof.** We are assuming that $\lim_{n\infty} a_n = L$ and $\lim_{n\infty} b_n = L$. Given $\epsilon > 0$, there exists $N_1$ such that $|a_n - L| < \epsilon$ for all $n \geq N_1$ and there exists $N_2 \in \mathbb{Z}^+$ such that $|b_n - L| < \epsilon$ for all $n \geq N_2$. Want to prove that $\lim_{n\infty} x_n = L$. We have $L - \epsilon < a_n < L + \epsilon$ for all $n \geq N_1$ and $L - \epsilon < b_n < L + \epsilon$ for all $n \geq N_2$. Let $N = \max\{N_1, N_2\}$ Then $L - \epsilon < a_n \leq x_n \leq b_n < L + \epsilon$ for all $n \geq N$, which proves that $\lim_{n\infty} x_n = L$.

**Theorem.** *Let $a, b \in \mathbb{R}$ with $a < b$. Assume $\{x_n\}$ is a sequence with $a \leq x_n \leq b$ for all $n \in \mathbb{Z}^+$ and that $\{x_n\}$ converges to $L$. Then $a \leq L \leq b$.*

**Proof.** (sketch) Look at the sequence $x_1 - a, x_2 - a, x_3 - a, \cdots \to L - a$, where $0 \leq x_n - a$ for all $n \geq 1$. Then (number 8 on Exam 1) $0 \leq L - a$. To get the other inequality look at $b - x_1, b - x_2, \cdots \to b - L$.

**Recall 2** (Archimedean Property of $\mathbb{R}$)**.**—Given any $x \in \mathbb{R}$, there exists $N \in \mathbb{Z}^+$ such that $x < N$.

*Corollary.*—Given any $x \in \mathbb{R}$, there exists $N \in \mathbb{Z}^+$ such that $-N < x$.

**Proof.** If $x \geq 0$, just pick $N = 1$. If $x < 0$ then we use the Archimedean property to find $N \in \mathbb{Z}^+$ such that $-x < N \implies x > -N$.

*Corollary.*—If $t > 0$, there exists $N \in \mathbb{Z}^+$ such that $0 < \frac{1}{N} < t$.

**Proof.** By the Archimedean property, there exists $N \in \mathbb{Z}^+$ such that $\frac{1}{t} < N \implies 1 < t \cdot N$ by (i3) $\implies 0 < \frac{1}{N} < t$ by (i6).

**Definition.**—$\mathbb{Z}^+ = \{1, 2, 3, \dots\}$. A nonempty subset $A \subseteq \mathbb{Z}$ contains a least element if there exists $q \in A$ such that $q \leq a$ for all $a \in A$.

**Proposition 4** (Well-ordering property)**.**—Every nonempty subset $S$ of the positive integers $\mathbb{Z}^+$ contains a least element, i.e., there exists $q \in S$ such that $q \leq a$ for all $a \in S$. (equivalent to induction). Read Appendix C.

*Corollary.*—If $y > 0$, there exists $n_y \in \mathbb{Z}^+$ such that $n_y - 1 \leq y < n_y$.

**Proof.** Consider the set $E_y = \{m \in \mathbb{Z}^+ \mid y < m\}$. $E_y$ is nonempty by the Archimedean property. By Well-ordering, there exists an element $n_y \in E_y$ such that $y < n_y$ and $n_y \leq a$ for all $a \in E_y$. Note in particular $n_y - 1 \notin E_y$ since $n_y - 1 < n_y$, i.e., $n_y - 1 \leq y$.

**Theorem Density Theorem.** *If $x$ and $y$ are real numbers with $x < y$, then there exists a rational number $r \in \mathbb{Q}$ such that $x < r < y$.*

**Proof.** First assume that $x > 0$. Since $0 < y - x$, it follows from Corollary 2 that there exists $n \in \mathbb{Z}^+$ such that $0 < \frac{1}{n} < y - x$. Then $1 < n(y - x) = ny - nx$. Therefore, we have: $nx + 1 < ny$. If we apply Corollary 3 to $nx > 0$ there exists $m \in \mathbb{Z}^+$ such that $m - 1 \leq nx < m$. Then $m \leq nx + 1 < ny$. So $nx < m < ny$ and $x < \frac{m}{n} < y$. Assume $x$ and $y$ are arbitrary elements in $\mathbb{R}$ with $x < y$. By Corollary 1, there exists $N \in \mathbb{Z}^+$ such that $-N < x < y \implies 0 < x + N < y + N$. By what was already proven, there exists $r_1 \in \mathbb{Q}$ such that $x + N < r_1 < y + N$. Then $x < r_1 - N < y$. We are done since $r_1 - N$ is a rational number.

**Recall 3** (Density Theorem)**.**—$x < y$, there exists $r \in \mathbb{Q}$ such that $x < r < y$.

**Proposition 5.**—Let $L$ be a fixed real number. Then there exists an infinite sequence of rational numbers $r_1, r_2, r_3, \dots$ that converges to $L$.

**Proof.** Use Density Theorem and Squeeze Theorem. Let $a_n = L$ for $n = 1, 2, 3, \dots$ and $b_n = L + \frac{1}{n}$ for $n \geq 1$. So $\lim_{n\infty} b_n = L$ and $a_n < b_n$ for all $n \geq 1$. For each $n \in \mathbb{Z}^+$, use the Density Theorem to pick $r_n \in \mathbb{Q}$ such that $a_n < r_n < b_n$. By the Squeeze Theorem $\lim_{n\infty} r_n = L$.

**Example 12** (Square roots)**.**—Let $a \in \mathbb{R}^+$. We will show that there exists a unique positive real number $s$ such that $s^2 = a$. This unique real number $s$ is denoted by $\sqrt{a}$.

**Proof.** Strategy: Construct an infinite sequence $s_1, s_2, s_3, \dots$ in $\mathbb{R}$ that converges to a real number $s$ with the desired properties. Let $s_1$ be any fixed positive real number and define

$s_{n+1} := \frac{1}{2}\left(s_n + \frac{a}{s_n}\right)$ for $n = 1, 2, 3, \ldots$. Claim: $0 < s_n$ for $n = 1, 2, 3, \ldots$. Claim: $s_n^2 \geq a$ for $n \geq 2$.

$$2s_{n+1} = s_n + \frac{a}{s_n} = \frac{s_n^2 + a}{s_n} \implies 2s_n s_{n+1} = s_n^2 + a$$

or $s_n^2 - 2s_n \cdot s_{n+1} + a = 0$, where $s_n$ represents a real number solution. The discriminant, $(-2s_{n+1})^2 - 4 \cdot 1 \cdot a \geq 0$, leads to $4s_{n+1}^2 \geq 4a$ or $s_{n+1}^2 \geq a$ for $n \geq 1$. This implies $s_n^2 \geq a$ for $n \geq 2$.

For $n \geq 2$, the expression $s_n - s_{n+1} = s_n - \frac{1}{2}s_n - \frac{a}{2s_n}$ simplifies to $\frac{1}{2}s_n - \frac{a}{2s_n} = \frac{s_n^2 - a}{2s_n} \geq 0$, given that $s_n > 0$ for all $n \geq 1$ and $s_n^2 - a \geq 0$ for all $n \geq 2$.

This establishes the inequality $0 < \cdots \leq s_5 \leq s_4 \leq s_3 \leq s_2$, as $s_n - s_{n+1} \geq 0$ implies $s_{n+1} \leq s_n$. Therefore, according to the Completeness Axiom, the sequence $s_1, s_2, s_3, \ldots$ converges. Let's denote the limit as $s$, i.e., $\lim_{n\to\infty} s_n = s$.

**Recall 4.**—Consider $a \in \mathbb{R}^+$ as a fixed positive real number. Choose $s_1 > 0$ arbitrarily. Define $s_{n+1} = \frac{1}{2}\left(s_n + \frac{a}{s_n}\right)$ for $n \in \mathbb{Z}^+$.

**Proof.** At a certain point in the proof, we had $s_n$ for $n \in \mathbb{Z}^+$, satisfying the quadratic equation $s_n^2 - 2s_{n+1}s_n + a = 0$. If $x \in \mathbb{R}$ satisfies $x^2 + bx + c = 0$, then $b^2 \geq 4c$. Taking $x = s_n \in \mathbb{R}$ for $1 \cdot x^2 - 2s_{n+1}x + a = 0$ leads to $4s_{n+1}^2 \geq 4a$ or $s_{n+1}^2 \geq a$ for all $n \geq 1$. We have demonstrated that $s_2 \geq s_3 \geq s_4 \geq \cdots \geq 0$. The sequence $s_1, s_2, s_3, \ldots$ defined this way converges to some real number $s \geq 0$ by the Completeness Axiom.

Let's complete the square of the quadratic equation: $x^2 + bx = -c$, so $x^2 + bx + \left(\frac{b}{2}\right)^2 = \frac{b^2}{4} - c$. Then $0 \leq \left(x + \frac{b}{2}\right)^2 = \frac{b^2}{4} - c \implies \frac{b^2}{4} \geq c \implies b^2 \geq 4c$.

**Claim**: $s^2 = a$. We have $s_n^2 \to s^2$ since $s_n \to s$, and $s_{n+1} \to s$, $a \to a$. Then $s_n^2 - 2s_{n+1}s_n + a = 0 \implies s^2 - 2s^2 + a = 0$ or $s^2 = a$.

**Example 13.**—Let $a = 8$. Choose $s_1 = 3$. Then $s_2 = \frac{1}{2}\left(3 + \frac{8}{3}\right) = \frac{1}{2}\left(\frac{9}{3} + \frac{8}{3}\right) = \frac{17}{6}$. Calculate $s_3$ as rational numbers, and continue to find $s_4$, and so on. Then approximate $\sqrt{8}$ as a rational decimal. Verify that $\sqrt{8} < \cdots < s_3 < s_2 < s_1$.

**Definition.**—Given $a \in \mathbb{R}^+$, a positive real number $s$ is said to be *a* square root of $a$ if $s^2 = a$.

**Proposition 6.**—Assume $s, t$ are both positive real numbers with $s^2 = a$ and $t^2 = a$. **Claim**: $s = t$, i.e., positive square roots are uniquely defined. We set $\sqrt{a} := s$, where $s$ is the unique positive real number such that $s^2 = a$.

**Proof.** Assume, without loss of generality, that $0 < s < t$. ($0 < s^2 < t^2$ would lead to a contradiction $0 < a < a$.) Therefore, $s^2 < t^2$, contradicting the assumption that $t$ is a positive real number satisfying $t^2 = a$.

s1. $|t| = \sqrt{t^2}$, $\forall t \in \mathbb{R}\backslash\{0\}$.

i) If $t > 0$: $|t| = t \overset{?}{=} \sqrt{t^2}$ by definition is a positive real number such that when squared, it equals $t^2$. This is true since $(t)^2 = t^2$.

ii) If $t < 0$: $|t| = -t \stackrel{?}{=} \sqrt{t^2}$. This is true since $(-t)^2 = t^2$ and $-t > 0$.

s2. If $0 < s \leq t$, then $0 < \sqrt{s} \leq \sqrt{t}$.

**Proof.** By contradiction: Assume $0 < \sqrt{t} < \sqrt{s}$. This implies $0 < (\sqrt{t})^2 < (\sqrt{s})^2$, or $0 < t < s$. Contradiction.

s3. $\sqrt{s} \cdot \sqrt{t} = \sqrt{st}$, $\forall s, t \in \mathbb{R}^+$.

**Proof.** By definition, we are looking for the positive real number such that when squared, it equals $st$. Note that $\left(\sqrt{s} \cdot \sqrt{t}\right)^2 = (\sqrt{s})^2 \left(\sqrt{t}\right)^2 = s \cdot t$, and since $\sqrt{s} \cdot \sqrt{t} > 0$, we are done.

Set $0 := \sqrt{0}$. Thus, $x \mapsto \sqrt{x}$ is a function defined on the interval $[0, \infty)$.

**Example 14** (Cube root again)**.**—Assume $b > 0$ is fixed. Let $t_1 > 0$ be chosen arbitrarily. Define recursively: $t_{n+1} = \frac{1}{3}\left[2t_n + \frac{b}{t_n^2}\right]$ for $n = 1, 2, 3, \ldots$.

**Example 15.**—$b = 2$; $t_1 = 2$. $t_2 = \frac{1}{3}\left[2 \cdot 2 + \frac{2}{2^2}\right] = \frac{1}{3}$
$[\ 4 + 1_{2 = \frac{1}{3}\left[\frac{9}{2}\right] = \frac{3}{2}}$. Find $t_3$ and $t_4$ as rational numbers.

**Homework**: Prove the cubic formula converges similarly to the square root formula.

**Definition.**—A positive integer $n$ is a perfect square if $n = d^2$, where $d \in \mathbb{Z}^+$.

**Example 16.**—$1, 4, 9, 16, 25, 36, 49$ are the first 7 perfect squares listed in increasing order.

**Recall 5** (Corollary to 3)**.**—If $y \in \mathbb{R}^+$, then there exists $n_y \in \mathbb{Z}^+$ such that $n_y - 1 \leq y < n_y$.

**Theorem.** *If $m \in \mathbb{Z}^+$ is not a perfect square, then $\sqrt{m}$ is irrational.*

**Proof.** Let $n$ be an integer with $n < \sqrt{m} < n + 1$. Since $m$ is not a perfect square, $\sqrt{m}$ is not an integer, allowing us to use Corollary 3 as a strict inequality. Goal: Prove that $\alpha = \sqrt{m} - n$ is irrational. Assume instead that $\alpha = \sqrt{m} - n$ is rational. By assumption, we have $0 < \sqrt{m} - n < 1$. Therefore, $\alpha = \frac{p}{q}$, where $p, q \in \mathbb{Z}^+$ and $0 < p < q$. Assume this is carried out so that $q$ is as small as possible, using the well-ordering principle. Then $\frac{q}{p} = \frac{1}{\sqrt{m}-n} \cdot \frac{\sqrt{m}+n}{\sqrt{m}+n} = \frac{\sqrt{m}+n}{m-n^2} = \frac{\alpha+2n}{m-n^2}$, since $\alpha + 2n = \sqrt{m} + n$. Solve for $\alpha$:

$$\frac{q(m-n^2)}{p} = \alpha + 2n \implies \alpha = \frac{(m-n^2)q}{p} - 2n = \frac{(m-n^2)q - 2np}{p} = \frac{r}{p},$$

where $r \in \mathbb{Z}^+$ and $p \in \mathbb{Z}^+$. We have contradicted the well-ordering principle because $p < q$ and $q$ was assumed to be the smallest possible, but we have another rational number for $\alpha$ with a smaller denominator. Thus, $\alpha$ is irrational. We know $\sqrt{m} = \alpha + n$, which is the sum of an irrational and integer (rational) number. In Homework #1, Problem 3, it was proven that this sum is irrational, i.e., $\sqrt{m}$ is irrational.

*Corollary.*—$\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{6}, \sqrt{7}, \sqrt{8}, \sqrt{10}, \ldots$ are *all* irrational.

**Recall 6** (Density Theorem)**.**—Between any two real numbers, there is a rational number.

**Theorem One corollary to the Density Theorem.** *If $x$ and $y$ are real numbers with $x < y$, then there exists an irrational number $z$ such that $x < z < y$.*

**Proof.** If we apply the Density Theorem to the real numbers $\frac{x}{\sqrt{2}} < \frac{y}{\sqrt{2}}$, we obtain a rational number $r \neq 0$ such that $\frac{x}{\sqrt{2}} < r < \frac{y}{\sqrt{2}}$. Multiplying through by $\sqrt{2}$, we get $x < r \cdot \sqrt{2} < y$. **Claim**: $z = r\sqrt{2}$ is *irrational.* By Homework #1, Problem 4, the product of a nonzero rational and irrational number is irrational. Therefore, $z$ is irrational.

# § 1 Subsequences

**Example 17.**—$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots, \frac{1}{10}, \ldots$. Pick: $\frac{1}{2}, \frac{1}{5}, \frac{1}{10}, \ldots$.

**Definition.**—Let $\{x_n\}$ be a sequence, and let $\{p_n\}$ be a strictly increasing sequence of positive integers. Then $\{x_{p_n}\}$ is called a subsequence of $\{x_n\}$.

**Example 18.**—From the previous example, if $p_1 = 3, p_2 = 7, p_3 = 8, p_4 = 17, \ldots$, then $\frac{1}{3}, \frac{1}{7}, \frac{1}{8}, \frac{1}{17}, \ldots$.

**Definition.**—A sequence is *increasing* if $x_n \leq x_{n+1}$ for all $n \in \mathbb{Z}^+$. A sequence is decreasing if $x_n \geq x_{n+1}$ for all $n \in \mathbb{Z}^+$. A sequence is *monotone* if it is either increasing or decreasing.

**Proposition 7.**—Given Definition § 1, we claim that $n \leq p_n$ for $n = 1, 2, 3, 4, \ldots$.

**Proof.** (By induction). For the base case $Q(1)$, we have $p_1 \geq 1$ since $p_1 \in Z^+$. Now, let's assume the inductive step $Q(k) : p_k \geq k$ holds for some arbitrary $k \in Z^+$. We aim to show that $Q(k) \implies Q(k+1)$.

Consider $Q(k+1) : p_{k+1} \geq k+1$. Assuming $p_k \geq k$, we have $p_k + 1 \geq k+1$. Since $p_{k+1} > p_k$ and $p_{k+1} \geq p_k + 1$ (due to working with integers), it follows that $p_{k+1} \geq p_k + 1 \geq k+1$.

**Theorem Monotone Subsequence Theorem.** *If $\{x_n\}$ is any given sequence of real numbers, then there exists a subsequence of $\{x_n\}$ that is monotone.*

**Proof.** We will refer to the $p$-th term $x_p$ as a "peak" if $x_p \geq x_n$ for all $n > p$. If $x_p$ is a peak, then $x_p \geq x_{p+n}$ for all $n \geq 1$. If $x_p$ is not a peak, then there exists $m > p$ such that $x_p < x_m$. We consider two cases.

**Case 1**: $\{x_n\}$ has infinitely many peaks. In this scenario, we list the peaks as $x_{p_1}, x_{p_2}, x_{p_3}, \ldots$. Since each term is a peak, we have $x_{p_1} \geq x_{p_2} \geq x_{p_3} \geq x_{p_4} \geq \ldots$, forming a decreasing (monotone) subsequence of $\{x_n\}$.

**Case 2**: $\{x_n\}$ has only a finite number of peaks. In this case, we list the peaks as $x_{m_1}, x_{m_2}, \ldots, x_{m_k}$. Let $s_1 := m_k + 1$. Since $x_{s_1}$ is not a peak, there exists $s_2 > s_1$ such that $x_{s_1} < x_{s_2}$. Continuing in this way, we obtain a strictly increasing (monotone) subsequence $x_{s_1} < x_{s_2} < x_{s_3} < x_{s_4} < \ldots$.

**Example 19.**—(a) A sequence that has subsequences converging to 1, 2, and 3: $1, 2, 3, 1, 2, 3, 1, 2, 3, \ldots$. (b) A sequence that has subsequences converging to $\pm\infty$: $1, -1, 2, -2, 3, -3, \ldots$. (c) A sequence with subsequences that are strictly increasing, strictly decreasing, and constant: $1, -1, 0, 2, -2, 0, \ldots$. (d) An unbounded sequence that has a convergent subsequence: $1, 1, 1, 2, 1, 3, 1, 4, \ldots$. (e) A sequence that has no convergent subsequence: $1, 2, 3, 4, 5, \ldots$.

**Theorem Bolzano-Weierstrass Theorem.** *Every bounded sequence has a convergent subsequence.*

**Proof.** If $\{x_n\}$ is bounded, then it is bounded below and above, i.e., there exist $a$ and $b$ in $R$ such that $a \leq x_n \leq b$ for all $n \geq 1$. By the theorem proved previously, $\{x_n\}$ has a monotone (increasing or decreasing) subsequence, say $x_{p_1}, x_{p_2}, \ldots$. By the Completeness Axiom, this subsequence converges. Thus, the Bolzano-Weierstrass Theorem is equivalent to the Completeness Axiom.

**Recall 7** (Theorem 5)**.**—Let $a$ and $b$ be real numbers with $a < b$. Assume $\{x_n\}$ is a sequence with $a \leq x_n \leq b$ for all $n \in Z^+$ and $\{x_n\}$ converges to $L$. Then $a \leq L \leq b$.

**Theorem.** *If $\{x_n\}$ is a sequence of numbers that converges to $L$, then every subsequence of this sequence also converges to $L$.*

**Proof.** Given $\epsilon > 0$, there exists $N \in Z^+$ such that $|x_n - L| < \epsilon$ for all $n \geq N$. Let $x_{p_1}, x_{p_2}, x_{p_3}, \ldots$ be any given subsequence of $\{x_n\}$. We want to show that this subsequence also converges to $L$. Since $p_1 < p_2 < p_3 < \ldots$ is an infinite sequence of strictly increasing positive integers, we know that $n \leq p_n$ for all $n \in Z^+$. Note that $|x_{p_n} - L| < \epsilon$ for all $n \geq N$.

**Theorem.** *Let $\{x_n\}$ be a sequence with distinct subsequences $x_{p_1}, x_{p_2}, x_{p_3}, \ldots$ converging to $L$ and $x_{q_1}, x_{q_2}, x_{q_3}, \ldots$ converging to $M$ with $L \neq M$. Then $\{x_n\}$ does not converge.*

**Proof.** (by Contradiction): Assume $\{x_n\}$ converges to some real number $K \in R$. Then, by Theorem 11, $x_{p_1}, x_{p_2}, \ldots$ converges to $K$ and $x_{q_1}, x_{q_2}, \ldots$ converges to $K$. This leads to a contradiction, as $L$ and $M$ were assumed to be distinct limits.

**Definition.**—A sequence $\{x_n\}$ is a Cauchy sequence if, for each $\epsilon > 0$, there exists $N \in Z^+$ such that $|x_m - x_n| < \epsilon$ for all $m, n \geq N$.

**Theorem.** *If $\{x_n\}$ is a convergent sequence of real numbers, then it is a Cauchy sequence.*

**Proof.** Since $\{x_n\}$ is convergent to $L$, for any $\epsilon > 0$, there exists $N \in Z^+$ such that $|x_n - L| < \frac{\epsilon}{2}$ for all $n \geq N$. Also, $|x_m - L| < \frac{\epsilon}{2}$ for all $m \geq N$. Assume $m, n \geq N$. Then,

$$\begin{aligned} |x_m - x_n| &= |x_m - L + L - x_n| \\ &\leq |x_m - L| + |x_n - L| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

**Lemma.**—*If $\{x_n\}$ is a Cauchy sequence, then it is bounded.*

**Proof.** Let $\epsilon = 1$. Since $\{x_n\}$ is Cauchy, there exists $N$ such that $|x_m - x_n| < 1$ for all $m, n \geq N$. Note that $|x_m - x_N| < 1$ for all $m \geq N$. Then $|x_m| = |x_m - x_N + x_N| \leq |x_m - x_N| + |x_N| < 1 + |x_N|$. Let $M = \max\{|x_1|, |x_2|, \ldots, |x_{N-1}|, |x_N| + 1\}$. Claim: $|x_m| \leq M$ for all $m \geq 1$.

**Theorem.** *A sequence $\{x_n\}$ of real numbers is convergent if and only if it is a Cauchy sequence. ( $\Longrightarrow$ ) Done. ($\Longleftarrow$) Assume $\{x_n\}$ is a Cauchy sequence. We wish to show this sequence converges to some real number $L \in R$.*

**Proof.** By the lemma just proved, the sequence $\{x_n\}$ is bounded since it is Cauchy. By the Bolzano-Weierstrass Theorem, there is a subsequence $x_{p_1}, x_{p_2}, x_{p_3}, \ldots$ of $\{x_n\}$ that converges, say to $L \in R$. Our goal is to show that the original sequence $\{x_n\}$ also converges to $L$.

Given $\epsilon > 0$, since $\{x_n\}$ is Cauchy, there exists $N \in Z^+$ such that $|x_m - x_n| < \frac{\epsilon}{2}$ for all $m, n \geq N$. Since $x_{p_1}, x_{p_2}, \ldots$ converges to $L$, there exists $K \in Z^+$ such that $K \geq N$ and $K$ is equal to one of the $p_j$'s so that $|x_K - L| < \frac{\epsilon}{2}$.

Since $K \geq N$, we know that $|x_m - x_K| < \frac{\epsilon}{2}$ for all $m \geq N$. We also have $|x_K - L| < \frac{\epsilon}{2}$. Now, we want $|x_m - L| = |x_m - x_K + x_K - L| \leq |x_m - x_K| + |x_K - L| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ for all $m \geq N$.

**Remark 3.**—Cauchy sequences are valid for complex numbers and metric spaces using a definition of distance. Monotone sequences don't make sense for complex numbers because they are not ordered, but Cauchy sequences can apply because of the definition of distance. A metric space is complete if it contains all the limits of its Cauchy sequences.

**Example 20.**—Consider the sequence $c_n = \sum_{k=1}^{\infty} \frac{1}{k^2}$. The terms are calculated as follows: $c_1 = 1$, $c_2 = 1 + \frac{1}{4} = \frac{5}{4}$, $c_3 = 1 + \frac{1}{4} + \frac{1}{9} = \frac{49}{36}$, and so on. This sequence is strictly increasing, denoted as $c_1 < c_2 < c_3 < \ldots$. It can be demonstrated that this is a Cauchy sequence, implying convergence. The convergence of this sequence is proven to be $\frac{\pi^2}{6}$, a result initially established by Euler.

Sources for the exam include homework, theorems discussed in class, applications covered in class, and the theorems from the course textbook.

*Theorems/Results whose proofs you should know*:

**Recall 8** (Corollary 2)**.**—If $t > 0$, there exists $N \in \mathbb{Z}^+$ such that $0 < \frac{1}{N} < t$.

**Recall 9** (Corollary 3)**.**—If $y > 0$, there exists $n_y \in \mathbb{Z}^+$ such that $n_y - 1 \leq y < n_y$.

**Example 21** (Quick Application of Corollary 2)**.**—The sequence $\{\frac{1}{n}\}_{n=1}^{\infty}$ converges to 0.

**Proof.** To prove this, it is required to show that, given $\epsilon > 0$, there exists $N \in \mathbb{Z}^+$ (find Waldo) such that $\frac{1}{n} = |\frac{1}{n}| < \epsilon$ for all $n \geq N$. By Corollary 2, there exists $N \in \mathbb{Z}^+$ such that $0 < \frac{1}{N} < \epsilon$. The claim is $\frac{1}{n} \leq \frac{1}{N}$ for all $n \geq N$ since $0 < N \leq n \implies n \geq N$ by (i10). Thus $\frac{1}{n} \leq \frac{1}{N} < \epsilon \implies \frac{1}{n} < \epsilon$ for all $n \geq N$.

**Example 22** (Slight strengthening of Bernoulli's inequality)**.**—Assume $x > -1$ and $x \neq 0$. Prove $(1+x)^n > 1+nx$ for each positive integer $n \geq 2$.

**Proof.** *Base case (n=2):* $(1+x)^2 > 1+2\cdot x \implies 1+2x+x^2 > 1+2x$. Note that $x^2 > 0$ if $x \neq 0$ by (i5), so $1+2x+x^2 > 1+2x+0$ by VIII.

*Induction Step:* Assume that $(1+x)^k > 1+kx$ for some positive integer $k \in \mathbb{Z}^{\geq 2}$. The goal is to prove $(1+x)^{k+1}$. Since $x > -1$, we have that $1+x > 0$, so $(1+x)^{k+1} > (1+kx)(1+x)$.

**Recall 10** (Proposition 5)**.**—Let $L \in \mathbb{R}$ be fixed. Then there exists an infinite sequence of rational numbers $r_1, r_2, \ldots$ that converges to $L$ (requires the squeeze theorem and density theorem).

**Recall 11** (Theorem 8)**.**—For real numbers $x$ and $y$ where $x < y$, there exists an irrational number $z$ such that $x < z < y$.

**Recall 12** (3 Proofs)**.**—If $\{x_n\}$ is convergent, then it is Cauchy. If $\{x_n\}$ is Cauchy, then it is bounded. If $\{x_n\}$ is Cauchy, then it converges.

**Definition.**—A set $S \subseteq \mathbb{R}$ is considered an "interval" if $S$ contains at least 2 points, and for any $x, y \in S$, every real number between $x$ and $y$ also belongs to $S$.

There are 9 different forms for how an interval looks:

1. $(a,b) = \{x \in \mathbb{R} \mid a < x < b\}$ - Bounded open interval, $a < b$, $a, b \in \mathbb{R}$
2. $[a,b) = \{x \in \mathbb{R} \mid a \leq x < b\}$ - Bounded, half-open
3. $(a,b] = \ldots$
4. $[a,b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$ - Bounded, closed
5. $(-\infty, a) = \{x \in \mathbb{R} \mid x < a\}$ - Unbounded open interval
6. $(-\infty, a]$ - Unbounded closed interval
7. $(b, \infty)$
8. $[b, \infty)$
9. $(-\infty, \infty) = \mathbb{R}$ - Unbounded, both open and closed

**Recall 13** (Bolzano-Weierstrass)**.**—If $x_n \in [a,b]$ for all $n \in \mathbb{Z}^+$, then $\{x_n\}$ has a subsequence $\{x_{p_n}\}$ that converges to a limit $L$ with $a \leq L \leq b$, i.e. $L \in [a,b]$.

**Example 23.**—$f(x) = \frac{1}{x}$ on $(0,1)$ is unbounded. A function that is continuous on $[a,b]$ is always bounded.

(codomain = $\mathbb{R}$). We'll focus on functions whose domains are intervals. Say $f : I \to \mathbb{R}$, where $I$ is an interval.

**Example 24.**—Polynomials (addition, multiplication, subtraction), rational functions (addition, multiplication, subtraction, division), algebraic functions (addition, multiplication, subtraction, division, roots), transcendental functions $\sin x, e^x$.

**Example 25.**—The domain of $\tan(x)$ is all reals except: $\{\dots, -\frac{3\pi}{2}, -\frac{-\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \dots\}$. Consider $\tan(x)$ on $(-\frac{\pi}{2}, \frac{\pi}{2})$. The domain of $\arcsin(x)$ is $[-1, 1]$.

**Definition.**—$I$ = interval; $f : I\mathbb{R}$. Let $J$ be a subinterval of $I$. The function $f$ is *increasing* on $J$ if $f(x) \leq f(y)$ for any $x, y \in J$ with $x < y$.

**Definition.**—$I$ is an interval; $f : I\mathbb{R}$, $J$ is a subinterval.

a) The function $f$ is bounded above on $J$ if $\exists M \in \mathbb{R}$ such that $f(x) \leq M$ for all $x \in J$.

b) The function $f$ is *bounded* on $J$ if $\exists M \in \mathbb{R}^+$ such that $|f(x)| \leq M$ for all $x \in J$.

**Recall 14.**—A sequence $\{x_n\}$ converges to $L \in \mathbb{R}$ if $\forall \epsilon > 0$ there $\exists N \in \mathbb{Z}^+$ such that $|x_n - L| < \epsilon$ *for all* $n \geq N$.

**Remark 4.**—Among the 9 types of intervals, there are 4 of these that are *"open"*:

i) $(a, b)$, $a < b$

ii) $(a, \infty)$, $a \in \mathbb{R}$

iii) $(-\infty, b)$, $b \in \mathbb{R}$

iv) $(-\infty, \infty) = \mathbb{R}$

**Definition.**—Let $I$ be an open interval containing $c$, and $f$ be a function defined on $I$ except possibly at $c$. The function $f$ has a limit $L$ at $c$ if, for every $\epsilon > 0$, there exists $\delta > 0$ ($\delta \in \mathbb{R}^+$) such that $|f(x) - L| < \epsilon$ for all $x \in I$ satisfying $0 < |x - c| < \delta$.

**Remark 5.**—We typically choose $\delta > 0$ small enough such that all $x$ satisfying $0 < |x - c| < \delta$ are in $I$, i.e., $(c - \delta, c + \delta) \subseteq I$. Choose $\delta \leq \min\{|b - c|, |c - a|\}$.

**Note 6** (Short-hand notation)**.**—The limit of a sequence: $\lim_{n\to\infty} x_n = L$. The limit of a function at $c$: $\lim_{x\to c} f(x) = L$.

**Note 7.**—$f'(c) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c}$ is actually the limit of $g(x) = \frac{f(x)-f(c)}{x-c}$. Assume $f(x)$ is defined on the open interval $I$. $g(x) \to L$ is defined on all of $I$ except at $x = c$.

**Example 26** (Exam #7)**.**—Fact 3: $1 + na \leq (1 + a)^n$ for all $n \in \mathbb{Z}^+$. Follows from Bernoulli's inequality since $-1 < 0 < a$.

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**Definition.**—Let $I$ be an open interval containing $c$, and $f$ is defined on $I$ except possibly at $c$. The function $f$ has a limit $L$ at $c$, written $\lim_{x\to c} f(x) = L$, if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x \in I$ with $0 < |x - c| < \delta$, we have $|f(x) - L| < \epsilon$.

**Example 27.**—Let $f(x) = x$ on $I = \mathbb{R}$. Let $c \in \mathbb{R}$. We wish to prove that $\lim_{x\to c} x = c$.

**Proof.** Given $\epsilon > 0$, we need to find some $\delta > 0$ such that $|x - c| < \epsilon$ for all $x \in I$ with $0 < |x - c| < \delta$. Just choose $\delta = \epsilon$, and you're done.

**Remark 6.**—Given $\epsilon$, you have to go out and find $\delta$; typically, $\delta(\epsilon)$ depends on $\epsilon$.

Assuming the set-up above, with some given $L \in \mathbb{R}$, what does it mean if $\lim_{x\to c} f(x) \neq L$?

It means: $\exists\epsilon > 0$ such that for all $\delta > 0$, we have $|f(x) - L| \leq \epsilon$ for some $x \in I$ with $0 < |x - c| < \delta$.

**Theorem.** *Let $I$ be an open interval containing $c$, and suppose $f$ is defined on $I$ except possibly at $c$.*

*a) $\lim_{x\to c} f(x) = L$ iff for every sequence $\{x_n\}$ in $I\backslash\{c\}$ that converges to $c$, the sequence $\{f(x_n)\}$ converges to $L$. Visualize: $a < x_n < b$ for all $n \in \mathbb{Z}^+$; $x_n \neq c$ for all $n \in \mathbb{Z}^+$.*

**Example 28.**—$x_n = c + \frac{(b-c)}{2n}$ for $n = 1, 2, 3, \ldots$ on $a < x_n < b$.

**Proof.** ( $\Longrightarrow$ ) Assume $\lim_{x\to c} f(x) = L$. Let $\{x_n\}$ be a sequence of the type described above. Want to show: Given $\epsilon > 0$, there exists $N \in \mathbb{Z}^+$ such that $|f(x_n) - L| < \epsilon$ for all $n \geq N$. Given: For every $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - L| < \epsilon$ for all $x$ in $I$ with $0 < |x - c| < \delta$. Also given: We have $x_1, x_2, x_3, \cdots \in I\backslash\{c\}$, and given any $\epsilon_1 > 0$, there exists $N_1 \in \mathbb{Z}^+$ such that $|x_m - c| < \epsilon_1$ for all $m \geq N_1$. Set $\epsilon_1 = \delta$. Then $\exists N_1 \in \mathbb{Z}^+$ such that $0 < |x_m - c| < \delta$ for all $m \geq N_1$. Claim: $N = N_1$ works; i.e., we claim that $|f(x_m) - L| < \epsilon$ for all $m \geq N_1 = N$.

**Remark 7** (Chain of events)**.**—Start with $\epsilon > 0$. Then you have $\delta$. Set $\epsilon_1 = \delta$. Then you get $N_1$. Use $N = N_1$ to finish the proof.

**Theorem p.85.** $P$ *: The function $f$ has a limit $L$ at $x = c$, i.e., $\lim_{x\to c} f = L$ for $f : I\backslash\{c\} \to \mathbb{R}$. $Q$ : for each sequence $\{x_n\}$ in $I\backslash\{c\}$ that converges to $c$, the sequence $\{f(x_n)\}$ converges to $L$. Last time: $P \Longrightarrow Q$ proved. Now $Q \Longrightarrow P$. We will prove the contrapositive, i.e., $\neg P \Longrightarrow \neg Q$. Not $P$: There exists an $\epsilon > 0$ such that for all $\delta > 0$, there is a point $x \in I$ such that $0 < |x - c| < \delta$ and $|f(x) - L| \geq \epsilon$. Not $Q$: There exists a sequence $\{x_n\}$ in $I\backslash\{c\}$ that converges to $c$ where the sequence $\{f(x_n)\}$ does not converge to $L$. [**??**]*

**Proof.** Not $P$: There exists an $\epsilon > 0$ such that for all $\delta > 0$ there is a point $x \in I$ such that $0 < |x - c| < \delta$ and $|f(x) - L| \geq \epsilon$. For this $\epsilon > 0$, for each $\delta = \frac{1}{n}$, $n \in \mathbb{Z}^+$, there is a point $x_n \in I$ such that $0 < |x_n - c| < \frac{1}{n}$ and $|f(x_n) - L| \geq \epsilon$. Note that $x_n \in I\backslash\{c\}$ for all $n \in \mathbb{Z}^+$, and $\lim_{n\to\infty} x_n = c$.

**Theorem.** *$I$ is an open interval containing some point $c$. Two functions $f$ and $g$ defined on all of $I$ except possibly at $c$. Suppose that $\lim_{x\to c} f(x) = S$ and $\lim_{x\to c} g(x) = T$.*

*a) The function $f+g$ has a limit at $c$, and $\lim_{x\to c}[f(x)+g(x)] = S+T$.*

*Proof (Using Q).* Let $\{x_n\}$ be a sequence in $I\backslash\{c\}$ that converges to $c$. Want to prove that $\lim_{n\to\infty}[f(x_n)+g(x_n)] = S+T$. We know that $\lim_{n\to\infty} f(x_n) = S$ by Theorem **??** ($P \implies Q$), and $\lim_{n\to\infty} g(x_n) = T$ by Theorem **??** ($P \implies Q$). By Theorem 4(b), we have $\lim_{n\to\infty}[f(x_n)+g(x_n)] = S+T$. $\lim_{x\to c}[f(x)+g(x)] = S+T \leftarrow$ This now follows from Theorem **??** ($Q \implies P$).

*Proof of a) using only the definition of limits of functions.* Assumption: For any given $\epsilon > 0$, there exists $\delta_1 > 0$ such that for all $x \in I$ with $0 < |x-c| < \delta_1$, $|f(x)-S| < \frac{\epsilon}{2}$. Similarly, there exists $\delta_2 > 0$ such that for all $x \in I$ with $0 < |x-c| < \delta_2$, $|f(x)-T| < \frac{\epsilon}{2}$. Set $\delta = \min\{\delta_1, \delta_2\}$ and observe that $\delta > 0$. Then, for all $x \in I$ with $0 < |x-c| < \delta$, we have:

$$|f(x)+g(x)-(S+T)| = |(f(x)-S)+(g(x)-T)| \leq |f(x)-S|+|g(x)-T| < \frac{\epsilon}{2}+\frac{\epsilon}{2} = \epsilon$$

.

*e) It should be assumed that $g(x) \neq 0$ for all $x \in I\backslash\{c\}$. Also, $\lim_{xc} g(x) = T \neq 0$ is given, where $g(x) = x$ on $R$ and $c = 0$. Consider the expression $\frac{f(x_n)}{g(x_n)}$.*

**Definition.**—An open interval is characterized by $\lim_{x\to c^+} f(x)$. $\epsilon > 0$, there exists $\delta > 0$ such that $\ldots$ $|f(x)-L| < \epsilon$ for all $x$ with $x \in (c, c+\delta)$ or $0 < x-c < \delta$ because $c < x < c+\delta$.

**Definition.**—Consider an interval $I$ (not necessarily open). Let $f : I \to R$ be a function, and let $c \in I$. We say that $f$ is continuous at $c$ if $\lim_{x\to c} f(x) = f(c)$ if $c$ is not an endpoint of $I$. $\lim_{x\to c^+} f(x) = f(c)$ if $c$ is the left endpoint [Assuming that the left endpoint of $I$ is in $I$]. Similarly, $\lim_{x\to c^-} f(x) = f(c)$ if $c$ is the right endpoint [Assuming that the right endpoint of $I$ is in $I$]. These three limits are assumed to exist for the given $c$!

**Remark 8.**—Many favorable properties arise when a function $f : [a,b]R$ is continuous on the interval $I = [a,b]$.

By Theorem **??**, Definition § 1 can be expressed as follows:

For every sequence $\{x_n\}$ in $I\backslash\{c\}$ converging to $c$, the sequence $\{f(x_n)\}$ converges to $f(c)$.

If $f$ is continuous at every point $c \in I$, we say that $f$ is continuous on $I$.

**Theorem.** *Let $f$ and $g$ be functions defined on the interval $I$, where $c \in I$ and $k$ is a fixed constant. If $f$ and $g$ are continuous at $c \in I$, then $f+g$, $f-g$, $kf$, and $fg$ (i.e., $fg(x) = f(x) \cdot g(x)$) are all continuous at $c$.*

$$\lim_{xc} f(x) = f(c), \qquad \lim_{xc} g(x) = g(c), \qquad \lim_{xc}[f(x)+g(x)] = f(c)+g(c)$$

*If $g(x) \neq 0$ for all $x \in I$, then $\frac{f}{g} : IR$ is continuous at $c$. Thus $\lim_{xc} \frac{f}{g} = \frac{f(c)}{g(c)}$.*

**Example 29** (Polynomials)**.**—Consider $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x^1 + a_0 x^0$, where $n \in Z^{\geq 0}$. The function is continuous on $I$, and $f_0(x) = a_0 \in R$. To prove that $f_i(x)$ is separately continuous on $I$, we can use induction. Is $f(x) = k$ continuous on $I = R$? Yes, because $\lim_{xc} k = k \implies |f(x) - L| < \epsilon \implies |k-k| < \epsilon$. $f(x)$ is continuous, as $\lim_{xc} x = c$. Apply Theorem 18 and use induction to prove continuity for all polynomials.

**Example 30** (Rational functions)**.**—These are functions of the form $\frac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are both polynomials, and $g(x) \neq 0$.

According to Theorem **??**, Definition § 1 is synonymous with the statement:

For every sequence $\{x_n\}$ in $I \backslash \{c\}$ that converges to $c$, the sequence $\{f(x_n)\}$ converges to $f(c)$.

If $f$ is continuous at each point $c \in I$, we say that $f$ is continuous on $I$.

**Theorem Spivak p.122.** *If $f$ is continuous on $[a,b]$ with $a < b$ and $f(a) < 0 < f(b)$, then there is some $x \in (a,b)$ such that $f(x) = 0$. In our book: If $f(a) < f(b)$ and $v \in R$ such that $f(a) < v < f(b)$, then some $c \in (a,b)$ such that $f(c) = v$.*

**Lemma p. 99.**—*If $S$ is a nonempty subset of $R$ that is bounded above and $B = \sup S$ [$B$ exists from Theorem (2) in the Supremums and Infimums handout], then there is a sequence $\{x_n\}$ in $S$, i.e. $x_j \in S$ for all $j \in Z^+$, such that $\lim_{n\infty} x_n = B$.*

**Proof.** If $\epsilon = 1$, then $\exists x_1 \in S$ such that $B - 1 < x_1 \leq B$ by definition of $B$ and according to Theorem (1) from supremums and infimums handout. If $\epsilon = \frac{1}{2}$, then $\exists x_2 \in S$ such that $B - \frac{1}{2} < x_2 \leq B$. If $\epsilon = \frac{1}{n}$ for $n \in Z^+$, then $\exists x_n \in S$ such that $B - \frac{1}{n} < x_n \leq B$. Use Squeeze Theorem!

**Example 31.**—Consider $f(x) = 0$ for $x \in \{c_1, c_2, c_2\}$. $S = \{x \in [a,b] : f(x) < 0\} = [a, c_1) \cup (c_2, c_2)$ and $\sup S = c_3$. Also, $S \subset [a,b]$.

p.97 #4 HW. p.97 #7: Let $f : RR$ be continuous on $R$, … $f(x) = 0$ for all rational $x$ in $R$. Prove $f(x) = 0$ for all $x \in R$. p.107 #3: $f(x) = x^3 + 2x - 1$ on $[0,1]$, $f(0) = -1$, $f(1) = 2$. For the Bisection method, compute each $f(*)$ as rational numbers. p.107 #16(a): Assume that $\exists a, b \in I$ with $a < b$ such that $f(a) \neq f(b)$ and $f$ is continuous on $I$.

**Theorem Spivak's Theorem 2.** *If $f$ is continuous on $[a,b]$ and $a < b$, then $f$ is bounded above on $[a,b]$, i.e. $\exists M \in R$ such that $f(x) \leq M$ for all $x \in [a,b]$.*

**Proof.** To prove the contrapositive, which says: If $f$ is not bounded above on $I$, then $\exists$ a point $z \in [a,b]$ where $f(x)$ is not continuous. If $f(x)$ is bounded above on $I = [a,b]$, then $\exists M \in R$ such that $f(x) \leq M$ for all $x \in I$. If $f(x)$ is not bounded above on $I$, then $\forall M \in R$ there exists $x \in I$ such that $M < f(x)$. For each positive integer $n \in Z^+$, there exists $x_n \in I$ such that $f(x_n) > n$. So now we have constructed an infinite sequence $\{x_n\}$ in $I$. By the Bolzano-Weierstrass Theorem, $\{x_n\}$ has a subsequence $\{x_{p_n}\}$ that converges to a limit $z$ with $a \leq z \leq b$. Since $f(x_{p_n}) > p_n \geq n$ for all $n \in Z^+$ and so the sequence $f(x_{p_1}), f(x_{p_2}), \ldots$ is unbounded. If $\{y_n\}$ converges, then $\{y_n\}$ is bounded. Contrapositive: If $\{y_n\}$ is not bounded, then $\{y_n\}$ does not converge. Therefore, $f(x_{p_1}), f(x_{p_2}), \ldots$ does not converge and so $f$ is not continuous at $z$.

**Definition.**—If $S$ is a non-empty subset of $R$ and $f : SR$ is a real-valued function whose domain is $S$, we let $f(S)$ denote the set of all values in $R$ taken on by $f$ for $x \in S$, i.e. $f(S) := \{f(x) : x \in S\}$.

**Note 8.**—If the function $f$ is bounded above on $S$, i.e. $\exists M \in R$ such that $f(x) \leq M$ for all $x \in S$, then the set $f(S)$ is bounded above. Similarly, $f$ is bounded below on $S$ iff $f(S)$ is bounded below.

**Definition.**—Let $I$ be an interval, let $f : IR$, and let $c \in I$. The function $f$ is said to obtain its absolute maximum value on $I$ at $c$ if $f(x) \leq f(c)$ for all $x \in I$.

**Example 32.**—$f(x) = x^2$ on $I = [0,1)$ does not have an absolute maximum value on $I$.

**Example 33.**—$f(x) = \begin{cases} \frac{1}{x} & \text{for } x \in (0,1] \\ 0 & \text{for } x = 0 \end{cases}$ does not have an absolute maximum on $[0,1]$.

**Theorem Spivak Theorem 3.** *If $f$ is continuous on $[a,b]$, then $\exists d \in [a,b] = I$ such that $f(x) \leq f(d)$ for all $x \in I$.*

**Recall 15** (Spivak Theorem 3)**.**—If $f$ is continuous on the closed and bounded interval $[a,b]$, then $.\exists.$ some number $d \in [a,b]$ such that $f(x) \leq f(d)$ for all $x \in [a,b]$, i.e. $f$ obtains its absolute maximum value on $I = [a,b]$.

**Proof.** By Spivak's Theorem 2, we know that the set $S = f([a,b])$ is bounded above and therefore we have a uniquely defined number $\beta \in R$ such that $\beta = \sup\{f([a,b])\}$. Note that $f(x) \leq \beta$ for all $x \in I$. We will prove $.\exists.$ a point $d \in [a,b]$ such that $f(d) = \beta$ and then we're done. For each positive integer $n \in Z^+$, $\exists d_n \in [a,b]$ such that $\beta - \frac{1}{n} < f(d_n) \leq \beta$. See Theorem 1 on p.3 of the Supremums & Infimums handout. Note that $d_1, d_2, d_3, \ldots$ is an infinite sequence of points in $[a,b]$. By Bolzano-Weierstrass, there is a subsequence $\{d_{q_n}\}$ that you know converges to a point $d \in [a,b]$. Since $f$ is continuous at every point in $I$, it is continuous at $d$. This implies that: $\lim_{n\infty} f(d_{q_n}) = f(d)$. Note that $\lim_{n\infty} f(d_{q_n}) = \beta$ from this and Squeeze Theorem. Therefore $f(d) = \beta$.

# § 2 Uniform Continuity

**Definition** (Handout Definition 2).—The function $f$ defined on the interval $I$: $f : IR$ is continuous at $c \in I$ if for every $\epsilon > 0$ there exists $s > 0$ such that $|f(x) - f(c)| < \epsilon$ for all $x \in I$ that satisfy $|x - c| < \delta$.

**Remark 9.**—Applies equally well to an endpoint $c \in I$.

Suppose a function $f$ is continuous on an interval $I$, i.e., $f$ is continuous at every point $\in I$. According to the definition, for each $c \in I$ and each $\epsilon > 0$, there exists $\delta(\epsilon, c) > 0$ such that $|f(x) - f(c)| < \epsilon$ for all $x \in I$ satisfying $|x - c| < \delta(\epsilon, c)$.

**Example 34.**—Consider $f(x) = x^2$. For $\epsilon = 0.4$, a narrower $\delta$ is required at $f(2) = 4$ than at $f(1) = 1$.

**Definition.**—Let $I$ be an interval. A function $f : IR$ is uniformly continuous on $I$ if for every $\epsilon > 0$, there exists $\delta > 0$ such that $|f(y) - f(x)| < \epsilon$ for all $x, y \in I$ satisfying $|y - x| < \delta$.

**Remark 10.**—If a function is uniformly continuous on an interval, then the function is continuous at every point in the interval, but the converse may not be true.

**Definition.**—Let $I$ be an interval. A function $f : IR$ is uniformly continuous on $I$ if $\forall \epsilon > 0$ there exists $\delta > 0$ such that $|f(y) - f(x)| < \epsilon$ for all $x, y \in I$ that satisfy $|y - x| < \delta$.

**Theorem.** *If $f : IR$ is uniformly continuous on $I$, then $f$ is continuous on $I$, i.e., $f$ is continuous at each point $c \in I$.*

**Proof.** Take $x = c \in I$ and show $f$ is continuous at $c$. Note that $f$ is continuous at $c$ if $\forall \epsilon > 0$ there exists $\delta > 0$ such that $|f(y) - f(c)| < \epsilon$ for all $y \in I$ satisfying $|y - c| < \delta$.

**Remark 11.**—The converse statement is not true! However, we do have the following:

**Theorem.** *If $f : [a, b]R$ is continuous on the closed & bounded interval $[a, b]$, then $f$ is uniformly continuous on $[a, b]$.*

**Example 35** (Recall + Negation).—$f : IR$. $f$ is uniformly continuous on $I$ if $\forall \epsilon > 0$ there exists $\delta > 0$ such that $|f(y) - f(x)| < \epsilon$ for all $x, y \in I$ satisfying $|y - x| < \delta$. Negation: $f$ is not uniformly continuous on $I$ if $\exists \epsilon > 0$ such that $\forall \delta > 0$ $|f(y) - f(x)| \geq \epsilon$ for some $x, y \in I$ satisfying $|y - x| < \delta$.

**Proof.** Assume $f$ is continuous on $I = [a, b]$ but not uniformly continuous on $I$. Seek to obtain a contradiction. For the $\epsilon > 0$ above, let $\delta = \frac{1}{n}$ $(n \in Z^+)$ and we know $\exists x_n, y_n \in I$ with $0 \leq |y_n - x_n| < \frac{1}{n}$ such that $|f(y_n) - f(x_n)| \geq \epsilon$. Thus, we generate 2 infinite sequences

x$_n\}$ and $\{y_n\}$ both in $I$. By the Bolzano-Weierstrass Theorem, .$\exists$. a subsequence $\{x_{p_n}\}$ that converges to some point $z \in I$. Note that $y_{p_n} = x_{p_n} + (y_{p_n} - x_{p_n})$ for each $n \in Z^+$. By construction, $\lim_{n\infty}(y_{p_n} - x_{p_n}) - x_{p_n}) = 0$. Therefore $\lim_{n\infty} y_{p_n} = z + 0 = z$. Since $f$ is continuous at $z \in I$, the sequence $\{f(y_{p_n}) - f(x_{p_n})\}$ converges to $f(z) - f(z)$= 0 and $f|(y_{p_n}) - f(x_{p_n})| \geq \epsilon > 0$. Contradiction. $\lim\limits_{n\infty} |y_n - x_n| = 0 \implies \lim\limits_{n\infty}(y_n - x_n) = 0$.

**Remark 12.**—The theorem could also have been proved using $-\frac{1}{n} \leq y_n - x_n \leq \frac{1}{n}$ and the Squeeze theorem.

**Definition.**—Let $I$ be an interval, $f : I \to \mathbb{R}$, and $c \in I$. The function $f$ is said to be differentiable at $c$ if the limit $\lim_{x\to c} \frac{f(x)-f(c)}{x-c}$ exists. The domain of $g(x) = \frac{f(x)-f(c)}{x-c}$ is $I\backslash\{c\}$. If $f$ is differentiable at $c$, we denote its derivative at $c$ as $f'(c)$.

**Theorem.** *Let $I$ be an interval, $f : I \to \mathbb{R}$, and $c \in I$. The function $f$ is differentiable at $c$ with derivative $f'(c) = L$ if and only if for each sequence $\{x_n\}$ in $I\backslash\{c\}$ converging to $c$, the sequence $\{\frac{f(x_n)-f(c)}{x_n-c}\}$ converges to $L$.*

**Theorem.** *Under the same hypotheses as above, if $f$ is differentiable at $c$, then $f$ is continuous at $c$. Therefore, if $f$ is differentiable on an interval $J$, then $f$ is continuous on $J$ as well.*

**Proof.** Observe that for all $x \in I\backslash\{c\}$, $f(x) - f(c) = (x - c) \cdot \left(\frac{f(x)-f(c)}{x-c}\right)$. Consider an arbitrary sequence $\{x_n\}$ in $I\backslash\{c\}$ that converges to $c$. Then $\lim_{n\to\infty}(x_n - c) = 0$ and $\lim_{n\to\infty} \left(\frac{f(x_n)-f(c)}{x_n-c}\right) = L \in \mathbb{R}$. So $\lim_{n\to\infty}[f(x_n) - f(c)] = 0 \implies \lim_{n\to\infty} f(x_n) = f(c)$.

**Theorem Product Rule, Quotient Rule.** *Refer to the proof on your own. Also, Chain Rule.*

**Theorem.** *If $f(x) = x^n$, where $n$ is a fixed positive integer, then $f'(c) = n \cdot c^{n-1}$ for any given $c \in \mathbb{R}$.*

**Proof.** Examine $\lim_{x\to c} \frac{x^n-c^n}{x-c}$. Utilize the identity:

$$\begin{aligned} x^n - c^n &= (x - c)(x^{n-1} + x^{n-2} \cdot c + x^{n-3} \cdot c^2 + \cdots + x^2 \cdot c^{n-3} + x \cdot c^{n-2} + c^{n-1}) \\ &= x^n + x^{n-1}c + \cdots + x^2c^{n-2} + xc^{n-1} - x^{n-1}c - \cdots - x^2c^{n-2} - xc^{n-1} - c^n \end{aligned}$$

For $n = 1$: $x - c = x - c$, then $x^2 - c^2 = (x - c)(x + c)$, then $x^3 - c^3 = (x - c)(x^2 + xc + c^2)$. Therefore, $\lim_{x\to c} \frac{x^n-c^n}{x-c} = \lim_{x\to c}[x^{n-1} + x^{n-2}c + \cdots + xc^{n-2} + c^{n-1}] = n \cdot c^{n-1}$.

**Definition.**—Let $I$ be an interval, $f : I \to \mathbb{R}$, and $c \in I$.

d) The function $f$ has a relative maximum value at $c$ if there exists $\delta > 0$ such that $f(x) \leq f(c)$ for all $x \in I$ satisfying $|x - c| < \delta$, i.e., $c - \delta < x < c + \delta$.

Let $f : [a, b] \to \mathbb{R}$ with $a < b$.

**Theorem Interior-Extremum Theorem.** *Let $c \in I$ be such that $a < c < b$, i.e., $c$ is an interior point of $I$. Assume that $f$ has a relative maximum/minimum value at $c$. If the derivative of $f$ at $c$ exists, then $f'(c) = 0$.*

**Lemma.**—*Let $g$ be defined on $I\backslash\{c\}$ and assume that $\lim_{x\to c} g(x) = L > 0$, i.e., given any $\epsilon > 0$, there exists $\delta > 0$ such that $|g(x) - L| < \epsilon$ for all $x \in I$ satisfying $0 < |x - c| < \delta$. Then, there exists $\delta_1 > 0$ such that $0 < g(x)$ for all $x$ satisfying $0 < |x - c| < \delta$.*

*Proof (sketch).* Set $\epsilon = L$. Then, $\exists \delta_1 > 0$ such that $|g(x) - L| < L$ for all $x \in I$ satisfying $0 < |x - c| < \delta$, $-L < g(x) - L < L$ or $0 < g(x) < 2L$.

*Proof of Theorem 28.* Let $g(x) = \frac{f(x)-f(c)}{x-c}$. Note that $g(x)$ is defined on $I \backslash \{c\}$. Since the derivative of $f$ at $c$ exists and is equal to $f'(c)$, we have $\lim_{x \to c} g(x) = f'(c)$. Now, assume that $f'(c) > 0$. We wish to obtain a contradiction. By the lemma, there exists $\delta_1 > 0$ such that $0 < g(x) = \frac{f(x)-f(c)}{x-c}$ for all $x$ satisfying $0 < |x - c| < \delta_1$ or $c - \delta_1 < x < c$. Assume $c < x < c + \delta_1$. Then $0 < x - c$, and so $0 < \left(\frac{f(x)-f(c)}{x-c}\right) \cdot (x - c) = f(x) - f(c)$, i.e., $f(c) < f(x)$ for all $x$ with $c < x < c + \delta_1$, a contradiction.

**Theorem Rolle's Theorem.** *Suppose that $f$ is continuous on $I = [a, b]$, $f'$ exists for every $x \in (a, b)$, and $f(a) = f(b) = 0$. Then there exists at least one point $c \in (a, b)$ such that $f'(c) = 0$.*

**Recall 16** (Rolle's theorem)**.**—i) $f$ is continuous on $I = [a, b]$, ii) $f'$ exists $\forall x \in (a, b)$. $f(a) = f(b) = 0$, iii) $f(a) = f(b) = 0$. Then $\exists c \in (a, b)$ such that $f'(c) = 0$.

**Proof.** Case 1: If $f(x) = 0$ for all $x \in I$, then $f'(x) = 0$ for all $x \in (a, b)$. Case 2: Assume $f(t) > 0$ for some $t \in (a, b)$. By the Extreme Value Theorem, $\exists c \in (a, b)$, where $f(x) < f(c)$ for all $x \in I$. By the Interior Extremum Theorem, $f'(c) = 0$. Case 3: Assume $f(t) < 0$ for some $t \in (a, b)$.

**Theorem Mean Value Theorem.** *Suppose that $f$ is continuous on $I = [a, b]$ and that $f'$ exists for every $x \in (a, b)$. Then there exists at least one point $c \in (a, b)$ such that $f(b) - f(a) = f'(c)(b - a)$.*

**Note 9.**—$f'(c) = \frac{f(b)-f(a)}{b-a}$ represents the slope of the secant line from $a$ to $b$.

**Proof.** Let's define the function $\phi(x)$ on the interval $I = [a, b]$ as

$$\phi(x) = f(x) - \left[f(a) + \frac{f(b) - f(a)}{b - a} \cdot (x - a)\right]$$

. Noteworthy is that:

i) $\phi(x)$ is continuous on $I$.

ii) $\phi(x)$ is differentiable on $(a, b)$, and its derivative is $\phi'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$.

iii) $\phi(a) = \phi(b) = 0$.

According to Rolle's Theorem, there exists $c \in (a, b)$ such that $\phi'(c) = 0$, or equivalently, there exists $c \in (a, b)$ where $f'(c) = \frac{f(b)-f(a)}{b-a}$.

*Corollary.*—Assume $f$ is continuous on $I = [a, b]$, $a < b$, and $f$ is differentiable on $(a, b)$ with $f'(x) = 0$ for all $x \in (a, b)$. Then, $f$ is constant on $I$.

**Proof.** We aim to demonstrate that $f(x) = f(a)$ for all $x \in I$. Suppose $a < x$ and apply the Mean Value Theorem (MVT) to $f$ on $[a, x]$. According to MVT, there exists $c$ with $a < c < x$ such that $0 = f'(c)(x - a) = f(x) - f(a) \implies f(x) = f(a)$.

*Corollary.*—Given that $f$ and $g$ are continuous on $[a, b]$ and both are differentiable on $(a, b)$ with $f'(x) = g'(x)$ for all $x \in (a, b)$, then there exists a constant $c$ such that $f = g + C$ for all $x \in I$.

**Proof.** Let's consider the function $h(x) = f(x) - g(x)$. Note that $h'(x) = f'(x) - g'(x) = 0$ for all $x \in (a, b)$. Applying the 1st Corollary to the Mean Value Theorem, we conclude that $h(x) = C$ for all $x \in I$.

# § 3 Riemann Integral

Suppose $[a, b]$ represents a closed bounded interval, $a < b$.

**Definition.**—A partition of $I$ is a finite, ordered set $P := (x_0, x_1, \ldots, x_n)$ of points in $I$ such that $a = x_0 < x_1 < x_2 < \cdots < x_n = b$.

**Definition.**—The norm of a partition $P$, denoted $||P||$, is defined as $P = \max\{x_i - x_{i-1} : 1 \leq i \leq n\}$, i.e., $||P||$ = the largest width among $x_i - x_{i-1}$.

**Definition.**—A tagged partition ${}^tP$ of an interval $[a, b]$ is composed of a partition $(x_0, x_1, \ldots, x_n)$ of $[a, b]$, along with a set of points known as tags that satisfy $x_{i-1} \leq t_i \leq x_i$ for $1 \leq i \leq n$. Additionally, $||{}^tP|| = ||P||$.

**Definition.**—Let $f : [a, b]R$ and ${}^tP = \{(t_i, [x_{i-1}, x_i]) : 1 \leq i \leq n\}$ be a tagged partition of $[a, b]$. The Riemann Sum $S(f, {}^tP)$ of $f$ associated with ${}^tP$ is given by: $S(f, {}^tP) = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1})$.

**Definition.**—A function $f : [a, b]R$ is termed Riemann Integrable on $[a, b]$ if there exists a number $L \in R$ such that for any $\epsilon > 0$, there exists $\delta > 0$ such that if ${}^tP$ is any tagged partition of $[a, b]$ with $||{}^tP|| < \delta$, then $|S(f, {}^tP) - L| < \epsilon$. The set of all Riemann-Integrable functions on $[a, b]$ will be denoted by $R[a, b]$.

**Theorem.** *If $f \in R[a, b]$, then the value $L$ above is uniquely determined.*

**Proof.** Assuming $L'$ and $L''$ both satisfy the definition, and for any given $\epsilon > 0$, there exists $\delta' > 0$ such that $|S(f, {}^tP_1) - L'| < \frac{\epsilon}{2}$ for all tagged partitions ${}^tP_1$ of $[a, b]$ with $||P|| < \delta'$, and similarly, there exists $\delta'' > 0$ such that $|S(f^tP_2) - L''| < \frac{\epsilon}{2}$ for all tagged partitions ${}^tP_2$ of $[a, b]$ with $||{}^tP_2|| < \delta''$. Let $\delta = \min\{\delta', \delta''\}$, and choose a specific tagged partition ${}^tP$ with $||{}^tP|| < \delta$. As $||{}^tP|| < \delta'$ and $||{}^tP|| < \delta''$, we have $|S(f, {}^tP) - L'| < \frac{\epsilon}{2}$ and $|S(f, {}^tP) - L''| < \frac{\epsilon}{2}$. By the triangle inequality, $|L' - L''| = |L'S(f, {}^tP)| + |S(f, {}^tP) - L''| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. The uniquely determined value $L$ is typically denoted by $\int_a^b f(x)\,dx$ or $\int_a^b f$.

Let $I = [a,b]$, $a < b$, $P = (x_0, x_1, \dots, x_n)$, $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$. Subintervals: (non-overlapping) $I_1 = [x_0, x_1], I_2 = [x_1, x_2], I_n = [x_{n-1}, x_n]$. Tags are chosen with $t_i \in I_i$ for $i = 1, 2, \dots, n$.

**Example 36.**—The function $f(x) = k =$ constant $\forall x \in [a,b]$ is in $R[a,b]$. Our guess is $\int_a^b f = k(b-a)$. If ${}^tP = \{(t_i, [x_{i-1}, x_i])\}_{i=1}^n$ is any partition of $[a,b]$, then

$$S(f,{}^t P) = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) = k(x_1 - x_0) + k(x_2 - x_1) + \cdots + k(x_n - x_{n-1}) = k[x_n - x_0] = k(b-a).$$

Hence, for any $\epsilon > 0$, we'll choose $\delta = 1$. Then, if $||{}^tP|| < 1$, we have $|S(f,{}^t P) - k(b-a)| = 0 < \epsilon$. Since $\epsilon > 0$ is arbitrary, we conclude that $f(x) = k$ is in $R[a,b]$.

**Example 37.**—Consider $c$ and $d$ as points in $[a,b]$. If $\phi : [a,b]R$ satisfies $\phi(x) = 1$ for all $x \in [c,d]$ (if $c = d$, then $[c,d] = \{c\}$) and $\phi(x) = 0$ elsewhere in $[a,b]$, then we claim that $\phi \in R[a,b]$ and $\int_a^b \phi = (d-c)$.

**Goal:** Given any $\epsilon > 0$, choose $\delta = \frac{\epsilon}{4}$ to show that if $||{}^tP|| < \delta$, then $|S(\phi,{}^t P) - (d-c)| \leq 2\delta = \frac{\epsilon}{2} < \epsilon$.

For a given tagged partition ${}^tP$ with $||{}^tP|| < \delta$, let $t_i$ be the first tag (reading from left to right) in $[c,d]$, and assume $t_{i+N}$ is the last tag in $[c,d]$, where $N \geq 0$. Note, for any $x_0 < x_1 < x_2$, we could have $t_1 = t_2$.

**Case 1:** We could actually have no tags in $[c,d]$. In this case, note that $S(\phi,{}^t P) = \sum_{i=1}^n \phi(t_i)(x_i - x_{i-1}) = 0$ since $\phi(t_i) = 0$ for all tags. We claim in this case that $d - c < 2\delta$.

**Proof.** Let $t_j < c$ and $d < t_{j+1}$, then $-c < -t_j$, so $d - c < t_{j+1} - t_j$. Now, $x_{j-1} \leq t_j \leq x_j \leq t_{j+1} \leq x_{j+1}$. Since $||{}^tP|| < \delta$, we have:

$$\begin{cases} x_j - x_{j-1} < \delta \\ x_{j+1} - x_j < \delta \end{cases} \implies d - c < t_{j+1} - t_j \leq x_{j+1} - x_{j-1} < 2\delta.$$

**Theorem.** *Suppose that $f, g \in R[a,b]$ and $k \in R$ is a constant. Then*

*a) $k \cdot f \in R[a,b)$ and $\int_a^b kf = k \int_a^b f$.*

*b) $f + g \in R[a,b]$ and $\int_a^b (f+g) = \int_a^b f + \int_a^b g$.*

*c) If $f(x) \leq g(x)$ for all $x \in [a,b]$, then $\int_a^b f \leq \int_a^b g$.*

*d) If $|f(x)| \leq M \in R^{\geq 0}$ for all $x \in [a,b]$, then $\left| \int_a^b f \right| \leq M(b-a)$.*

**Theorem p.199 of Bartle & Sherbert.** *Suppose $f$ and $g$ are in $R[a,b]$.*

*(a) $k \cdot f \in R[a,b]$*

*(b)* $f + g \in R[a,b]$

*(c)* *If* $f(x) \leq g(x)$ *for all* $x \in [a,b]$*, then* $\int_a^b f \leq \int_a^b g$.

*(4)* $\left|S(f,^t P) - \int_a^b f\right| < \frac{\epsilon}{2}$, *top of p.200.* $\left|\int_a^b f - S(f^t P)\right| < \frac{\epsilon}{2} \implies -\frac{\epsilon}{2} < \int_a^b f - S(f,^t P) < \frac{\epsilon}{2} \implies \int_a^b f - \frac{\epsilon}{2} < S(f,^t P)$.

**Recall 17.**—Squeeze Theorem for infinite sequences: $a_n \leq x_n \leq b_n$ for all $n \geq 1$. Assume that $\lim_{n\infty} a_n = \lim_{n\infty} b_n = L$, then $\lim_{n\infty} x_n = L$

Bottom of p.1 on Squeeze Theorem of Integrals handout: (4) $\left|S(\omega_{\epsilon/3},^t P_2) - \int_a^b \omega_{\epsilon/3}\right| < \frac{\epsilon}{3}$.

**Theorem Big Theorem.** *If* $f : [a,b]R$ *is continuous on* $[a,b]$ *with* $a < b$*, then* $f \in R[a,b]$.

**Proof.** First of all, we know $f$ is uniformly continuous on $[a,b]$, i.e., given any $\epsilon > 0$, there exists $\delta > 0$ such that if $u, v \in [a,b]$ and $|u - v| < \delta$, then $|f(u) - f(v)| < \frac{\epsilon}{(b-a)}$. Let $P = (x_0, x_1, \ldots, x_n)$ be a partition of $[a,b]$ such that $||P|| < \delta$. Subintervals: $I_1 = [x_0, x_1], I_2 = [x_1, x_2], \ldots, I_n = [x_{n-1}, x_n]$. Since $||P|| < \delta$, we have: $x_1 - x_0 < \delta, x_2 - x_1 < \delta, \ldots, x_n - x_{n-1} < \delta$. Let $u_i \in I_i$ be a point where $f$ attains its absolute minimum value on $I_i$. Let $v_i \in I_i$ be a point where $f$ attains its absolute maximum value on $I_i$. We're using the Extreme Value Theorem. Let $\alpha_\epsilon(x)$ be the step function defined by $\alpha_\epsilon(x) = f(u_i)$ for $x \in [x_{i-1}, x_i)$ for $i = 1, \ldots, n-1$, and we set $\alpha_\epsilon(x) = f(u_n)$ for $x \in [x_{n-1}, x_n]$. We define $\omega_\epsilon(x)$ the same way with $\omega_\epsilon(x) = f(v_i)$. Is $\alpha_\epsilon(x) \leq f(x)$ for all $x \in [a,b]$? Yes, also $f(x) \leq \omega_\epsilon(x)$ for all $x \in [a,b]$. We know $\alpha_\epsilon(x), \omega_\epsilon(x) \in R[a,b]$. $0 = \int_a^b 0 \leq \int_a^b (\omega_\epsilon - \alpha_\epsilon) = (f(v_1) - f(u_1)) \cdot (x_1 - x_0) + \cdots = \sum_{i=1}^n (f(v_i) - f(u_i)) \cdot (x_i - x_{i-1})$. Need to check $|u_i - v_i| \leq x_i - x_{i-1} < \delta$. Then $\sum_{i=1}^n (f(v_i) - f(u_i)) \cdot (x_i - x_{i-1}) < \sum_{i=1}^n \frac{\epsilon}{(b-a)} \cdot (x_i - x_{i-1}) = \frac{\epsilon}{(b-a)} \sum_{i=1}^n (x_i - x_{i-1}) = \frac{\epsilon}{(b-a)} \cdot (b-a) = \epsilon$.

1.) If $f : IR$ is differentiable at $c \in I$, then $f$ is continuous at $c$. ($I$ = interval)

2.) Product rule for derivatives: $(fg)'(c) = f'(c) \cdot g(c) + f(c) \cdot g'(c)$.

3.) Suppose that $f$ is continuous on $I = [a,b]$ and that $f$ is differentiable on $(a,b)$, and that $f'(x) = 0$ for all $x \in (a,b)$. Then $f(x)$ is constant on $I = [a,b]$. (Corollary to MVT)

4.) If $f \in R[a,b]$, then the limit value $L$ is uniquely determined.

5.) Theorem 7.1.4 parts (a), (b), (c) on p.199 of Bartle & Sherbert. $f, g \in R[a,b]$, $k$ is a constant.

(a) $\int_a^b kf = k \int_a^b f$

(b) $\int_a^b (f + g) = \int_a^b f + \int_a^b g$

(c) If $f(x) \le g(x)$ for all $x \in [a,b]$, then $\int_a^b f \le \int_a^b g$.

$S(kf,^t P) = \sum_{i=1}^n k \cdot f(t_i)(x_i - x_{i-1}) = k \sum_{i=1}^n f(t_i)(x_i - x_{i-1}) = kS(f,^t P)$.

*Proof of (a):* If $k = 0$, then $\int_a^b 0 = 0 = 0 \cdot \int_a^b f$. Since $f \in R[a,b]$, given any $\epsilon > 0$, there exists $\delta > 0$ such that if ${}^tP$ is any tagged partition with $||{}^tP|| < \delta$, then $|S(f,^t P) - \int_a^b f| < \frac{\epsilon}{|k|}$.

Want $|S(kf,^t P) - k \cdot \int_a^b f| = |k \cdot S(f,^t P) - k \cdot \int_a^b f| = |k| \cdot |S(f,^t P) - \int_a^b f| < |k| \cdot \frac{\epsilon}{|k|} = \epsilon$.

**Recall 18.**—Let $f : [a,b]R$ be continuous at $c \in [a,b]$ and suppose that $f(c) > 0$. Then there exists a positive number $m \in R^+$ and an interval $[u,v] \subset [a,b]$ such that $c \in [u,v]$ and $f(x) \ge m$ for all $x \in [u,v]$.

**Example 38.**—Suppose $f$ is continuous on $[a,b]$, that $f(x) \ge 0$ for all $x \in [a,b]$, and $\int_a^b f = 0$. Prove that $f(x) = 0$ for all $x \in [a,b]$.

*Proof (by contradiction).* Assume that there exists a point $c \in [a,b]$ such that $f(c) > 0$. To get a contradiction, I want to show that $\int_a^b f > 0$.

$$h(x) : \begin{cases} 0 & \text{on } [a,u) \\ m & \text{on } [u,v] \\ 0 & \text{on } (v,b] \end{cases}. \; h(x) \in R[a,b] \text{ and } h(x) \le f(x) \text{ for all } x \in [a,b]. \; f(x) \in R[a,b].$$

Then $0 < m(v-u) = \int_a^b h(x) \le \int_a^b f(x)$.

**Definition.**—A function $F$ satisfying $F'(x) = f(x)$ for all $x \in [a,b]$ is referred to as an antiderivative of $f$ on $[a,b]$. The definite integral is denoted as $\int_a^b f(x)\,dx = F(b) - F(a)$.

**Theorem Fundamental Theorem of Calculus (First Form).** *Consider two functions $f$ and $F$ on $[a,b]$ $(a < b)$ such that*

*(a) $F$ is continuous on $[a,b]$.*

*(b) $F'(x) - f(x)$ for all $x \in (a,b)$.*

*(c) $f \in \mathcal{R}[a,b]$.*

*Then $\int_a^b f = F(b) - F(a)$.*

**Example 39.**—Evaluate $\int_0^\pi \sin x\,dx$.

**Proof.** Given $\epsilon > 0$, since $f \in \mathcal{R}[a,b]$, there exists $\delta > 0$ such that for any tagged partition ${}^tP$ with $||{}^tP|| < \delta$, $|S(f,^t P) - \int_a^b f| < \epsilon$. Consider any partition $P$ with $||P|| < \delta$.

Assume, as usual, $I_i = [x_{i-1}, x_i]$ for $i = 1, \ldots, n$. Applying the Mean Value Theorem to $F$ on $I_i$ ($i$ fixed in the range $1 \le i \le n$), we can conclude that $\exists u_i \in (x_{i-1}, x_i)$ such that $F(x_i) - F(x_{i-1}) = F'(u_i) \cdot (x_i - x_{i-1})$. The tagged partition used here is ${}^uP = \{(u_i, [x_{i-1}, x_i])\}$.

Then $F(b)-F(x) = \sum_{i=1}^{n}(F(x_i)-F(x_{i-1})$ because $(F(x_1)-F(x_0))+(F(x_2)-F(x_1))+\cdots+(F(x_n)-F(x_{n-1})) = F(x_i)-F(x_{i-1})$. By MVT and condition (b), $F(b)-F(a) = \sum_{i=1}^{n} F'(u_i)(x_i - x_{i-1}) = \sum_{i=1}^{n} f(u_i)(x_i - x_{i-1}) = S(f, {}^u P)$. Thus $|S(f, {}^u P) - \int_a^b| < \epsilon$ since $||{}^u P|| < \delta$. $\implies$ $|F(b)-F(a)-\int_a^b f| < \epsilon$. Since $\epsilon > 0$ was arbitrary, we conclude that $F(b)-F(a) = \int_a^b f$.

**Theorem.** *If $f \in \mathcal{R}[a,b]$ and $[c,d] \subseteq [a,b]$, then the restriction of $f$ to $[c,d]$ is also in $\mathcal{R}[c,d]$.*

**Definition.**—If $f \in \mathcal{R}[a,b]$, then the function defined by $F(x) := \int_a^x f$ for any $x \in [a,b]$ is termed the indefinite integral of $f$, where $[c,d] = [a,x]$.

**Theorem.** *The indefinite integral $F(x)$ defined above is continuous on $[a,b]$.*

**Theorem Fundamental Theorem of Calculus (Second Form).** *Let $f \in \mathcal{R}[a,b]$ and assume $f$ is continuous at a point $c \in [a,b]$. Then the indefinite integral $F(x)$ is differentiable at $c$ with $F'(c) = f(c)$.*

**Theorem.** *If $f$ is continuous on $[a,b]$, then the indefinite integral $F(x)$ is differentiable on $[a,b]$ with $F'(x) = f(x)$ for all $x \in [a,b]$, i.e., the indefinite integral $F(x)$ is an antiderivative of $f(x)$.*

# Chapter 2

# REAL ANALYSIS

## § 1 The Real and Complex Number System

**Notation**:

$\mathbb{N} = \{0, 1, 2, 3, \dots\}$ - the set of natural numbers.
$\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$ - the set of integers.
$\mathbb{Q} = \{\frac{m}{n} : m, n \in \mathbb{Z}, n \neq 0\}$ - the set of rationals (denoted as $\mathbb{Q}$).
$\mathbb{R}$ - the set of all real numbers.
$\mathbb{C} = \{a + ib : a, b \in \mathbb{R}$ and $i^2 = -1\}$.

**Sets** - a collection of objects. The empty set is denoted as $\emptyset$ - a set that contains no objects. If $A \subset B$, it means that $a \in A \implies a \in B$ for all $a \in A$.

**Example 40.**—$A \subsetneq B$ - a proper subset if $A \subset B$ and there exists $x_0 \in B$ such that $x_0 \notin A$.

**Note 10.**—$A = B$ iff $A \subset B$ and $B \subset A$.

**Order on a set**: Let $S$ be a set. An order, denoted by '$<$' is a **relation** on $S$ satisfying:

(i) If $x, y \in S$ then one of the following must hold: $x < y$, $x = y$, $y < x$.

(ii) If $x, y, z \in S$ with $x < y$ and $y < z$ then $x < z$ (transitive property).

**Example 41.**—$S = \mathbb{N}$ with '$<$' ($x < y$ if $y - x > 0$) is an ordered set.

**Definition** (Upper bound)**.**—Let $(S, <)$ be an ordered set, $E \subset S$. We say that $\beta$ is an **upper bound of** $E$ if $s \leq \beta$ for all $s \in E$.

**Definition** (Lower bound)**.**—$\alpha$ is a **lower bound of** $E$ if $\alpha \leq s$ for all $s \in E$.

**Example 42.**—$(S = \mathbb{Q}, <)$ and

1. $E = \{x \in \mathbb{Q} : 0 \leq x < 1\}$. $\beta = 1$ is an upper bound of $E$ and $1 \notin E$. $\beta \geq 1$ are upper bounds of $E$. $\alpha = 0$ is a lower bound of $E$ and $0 \in E$. $\alpha \leq 0$ are lower bounds of $E$.

**Definition** (Least upper bound)**.**—Let $E \subset (S, <)$. We say that $\alpha \in S$ is the least upper bound or supremum of $E$ if:

(i) $\alpha$ is an upper bound of $E$, and

(ii) if $\gamma$ is an upper bound of $E$, then $\alpha \leq \gamma$ (or $\gamma$ is not an upper bound for any $\gamma < \alpha$).

**We write**: $\alpha = \text{lub}E$ or $\alpha = \sup E$.

**Example 43** (continued)**.**—$\sup E = 1 \notin E$, $\inf = 0 \in E$. The maximum of $E$ is the $\sup E$ and it belongs to $E$.

Rational numbers $\mathbb{Q}$ have gaps.

**Lemma.**—$\sqrt{2} \notin \mathbb{Q}$. *That is, $x^2 = 2$ has no solution in $\mathbb{Q}$.*

**Proof.** We will prove by contradiction (BWOC). Suppose $\sqrt{2} \in \mathbb{Q}$. Then $\sqrt{2} = \frac{m}{n}$, with $m, n \in \mathbb{Z}$, $n \neq 0$, and $\frac{m}{n}$ in least terms or $\gcd(m, n) = 1$. Then $2 = \frac{m^2}{n^2} \implies m^2 = 2n^2$. Then $m^2$ is even. **So $m$ is even**.

**Proof.** Suppose $m$ is odd. Then $m = 2k + 1$ for some $k \in \mathbb{Z}$. $\implies m^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1 \implies m^2$ is odd, a contradiction. So $m$ must be even.

Then $m = 2l$ for some $l \in \mathbb{Z}$. Then $(2l)^2 = 2n^2 \implies 4l^2 = 2n^2 \implies 2l^2 = n^2$. So $n^2$ is even and hence $n$ is even. This is a contradiction to $\frac{m}{n}$ being in least terms. So $\sqrt{2} \notin \mathbb{Q}$.

**Example 44.**—Prove that $\sup E = 1$ if $E = \{x \in \mathbb{Q} : 0 \leq x < 1\}$.

(i) Clearly $x < 1$ for all $x \in E$. So 1 is an upper bound of $E$.

(ii) If $\gamma < 1$ then $\gamma$ is not an upper bound of $E$. If $\gamma \leq 0$, then clearly $\gamma$ is not an upper bound of $E$. If $0 < \gamma < 1$, then $x = \frac{\gamma+1}{2}$. But $\gamma < x \in E$ by construction, so $\gamma$ is not an upper bound. Therefore $\sup E = 1$.

**Theorem Uniqueness of supremum.** *Let $(S, <)$ be an ordered set. If $E \subset S$ has a supremum (or infimum) in $S$, then it is unique.*

**Proof.** Let $\alpha_1 = \sup E$ and $\alpha_2 = \sup E$. Then by definition $\alpha_1$ and $\alpha_2$ are upper bounds of $E$. But $\alpha_1 = \sup E$ and $\alpha_2$ is an upper bound of $E$, so $\alpha_1 \leq \alpha_2$. Switching roles of $\alpha_1$ and $\alpha_2$, we get $\alpha_2 \leq \alpha_1$. Combining these two, $\alpha_1 = \alpha_2$. So $\sup E$ is unique.

**Example 45.**—$A := \{x \in \mathbb{Q} : x > 0 \text{ and } x^2 \leq 2\}$. $B := \{x \in \mathbb{Q} : x > 0 \text{ and } x^2 \geq 2\}$. $A \cap B = \{x \in \mathbb{Q} : x > 0 \text{ and } x^2 = 2\} = \emptyset$ - shown.

1. $A$ is bounded above in $\mathbb{Q}$ ($B$ is bounded below in $\mathbb{Q}$). If $a \in A$ and $b \in B$ then $a \leq b$.

2. There is no upper bound of $A$ in $A$. Let $p > 0 \in \mathbb{Q}$ be fixed and arbitrary. Set $q := p - \frac{p^2-2}{p+2} = \frac{2(p+2)}{p+2}$. $q^2 - 2 = \frac{4p^2+8p+4}{(p+2)} - 2 = \frac{4p^2+8p+4-2p^2-8p-8}{(p+2)^2} = \frac{2(p^2-2)}{(p+2)^2}$. So, $q^2 < 2$ if and only if $p^2 < 2$. Therefore $q \in A$ if and only if $p \in A$.

   Suppose $p \in A$ is an upper bound of $A$. Then $q = \frac{2p+2}{p+2} > p$ and $q \in A$. Therefore $p$ cannot be an upper bound of $A$.

3. $A$ has no least upper bound or supremum in $\mathbb{Q}$. $B$ is the set of upper bounds of $A$. Repeating the argument of 2. for $B$, there is no lower bound of $B$ in $B$. That means $A$ has no least upper bound in $\mathbb{Q}$.

**Definition** (Least upper bound property)**.**—Let $(S, <)$ be an ordered set. Then $S$ is said to have the least upper bound property if every nonempty subset of $S$ that is bounded above has a supremum in $S$, i.e., $\forall E \subset S$ with $E \neq \emptyset$ and bounded above, $\sup E \in S$.

**Example 46.**—The ordered set $(\mathbb{Q}, <)$ (usual ordering) does not possess the least upper bound property.

*Reason.* Consider $A = \{x \in \mathbb{Q} : x > 0 \text{ and } x^2 \leq 2\} \subset \mathbb{Q}$.

- $A \neq \emptyset$ since $1 \in \mathbb{Q}$.
- $A$ is bounded above (by $B$).

However, $\sup A$ does not exist in $\mathbb{Q}$.

**Theorem.** *Let $(S, <)$ have the least upper bound property. Suppose $B \subset S$ with $B \neq \emptyset$ and $B$ is bounded below. Set $L$ = set of all lower bounds of $B$. Then $\alpha = \sup L$ exists in $S$, and $\alpha = \inf B$.*

**Proof.** First, we want to show:

(i) $L$ is nonempty, and

(ii) $L$ is bounded above.

$L$ is nonempty since $B$ is bounded below, and hence $\exists l \in S$ such that $l \leq b$ for all $b \in B$. By definition of $L$, $l \leq b$ for all $l \in L$ and $\forall b \in B$ and $B \neq \emptyset$. So $L$ is bounded above. By the least upper bound property of $S$, $\sup L \in S$. Define $\alpha = \sup L$. **We claim**:

(a) $\alpha$ is a lower bound of $B$, and

   **Proof.** $\alpha = \sup L$ and if $b \in B$ then $\alpha \geq l$ for all $l \in L$. But $l \leq b$ for $b \in B$, so $b$ is an upper bound of $L$. Therefore, $\alpha \leq b$ since $\alpha = \sup L$.

(b) $\gamma > \alpha$ is not a lower bound of $B$.

   **Proof.** If $\gamma > \alpha$ then $\gamma \notin L$, and so $\alpha = \inf B$.

(a), (b) $\implies \alpha = \inf B$.

**Definition** (Field).—A field $F$ is a set with two operations:

- addition '+'
- multiplication '$\cdot$'

Satisfying the following axioms:

(A1) $x + y \in F$, ... (A5).

(M1) $x \cdot y \in F$, ... (M5).

(D) $x \cdot (y + z) = x \cdot y + x \cdot z$ for all $x, y, z \in F$.

**Proposition 8.**—Let $(F, +\cdot)$ be a field. Then for $x, y, z \in F$,

(a) If $x + y = x + z$, then $y = z$. (Cancellation law)

(b) If $x + y = x$, then $y = 0$.

(c) If $x + y = 0$, then $y = -x$.

(d) $-(-x) = x$.

**Proof.** Cases:

(a) $y A4{=} 0 + y A5{=} (-x + x) + y A3{=} -x + (x + y) A3{=} -x + (x + z) \equiv (-x + x) + z A5{=} 0 + z A4{=} z$

(b) Taking $z = 0$ in (a), we get $x + y = x + 0 \implies y = 0$.

(d) Let $x \in F$. Then $-x \in F$ such that $x + (-x) = 0$. Since $-x \in F$, $\exists -(-x) \in F$ such that $-(-x) + (-x) = 0$. So $x$ and $-(-x)$ are additive inverses of $-x$, by (c) $x = -(-x)$.

Done.

**Proposition 9.**—Let $(F, +, \cdot)$ be a field with $x, y, z \in F$. Then

(a) If $x \neq 0$ and $xy = xz$ then $y = z$.

**Proof.** Let $x \neq 0$ and $xy = xz$. $y M4{=} 1 \cdot y M5{=} (\frac{1}{x} \cdot x) \cdot y M3{=} \frac{1}{x}(xy) \equiv \frac{1}{x}(xz) M3{=} (\frac{1}{x} \cdot x) \cdot z M5{=} 1 \cdot z M4{=} z$

(b) If $x \neq 0$ and $xy = x$ then $y = 1$.

(c) If $x \neq 0$ and $xy = 1$ then $y = \frac{1}{x}$.

(d) If $x \neq 0$, then $\frac{1}{1/x} = x$.

**Proposition 10.**—Let $x, y, z \in (F, +, \cdot)$ field. Then

(a) $0 \cdot x = 0$

(b) If $x \neq 0$ and $y \neq 0$ then $xy \neq 0$.

(c) $(-x)y = -(xy) = x(-y)$.

(d) $(-x)(-y) = xy$.

**Proof.** Cases:

(a) $0 \cdot x = (0+0) \cdot x \overset{D}{=} 0 \cdot x + 0 \cdot x$. Then $0 \cdot x = 0$ (Proposition 8(b)).

(b) We'll prove by contradiction. Let $x \neq 0$, $y \neq 0$ but $xy = 0$. Then $1 \overset{M4}{=} 1 \cdot 1 = (x \cdot \frac{1}{x})(y \cdot \frac{1}{y})$ because $\exists \frac{1}{x}, \frac{1}{y} \in F$ such that $x \cdot \frac{1}{x} = 1$ and $y \cdot \frac{1}{y} = 1$. Then $(x \cdot \frac{1}{x})(y \cdot \frac{1}{y}) \overset{M2-M3}{=} (xy)(\frac{1}{x} \cdot \frac{1}{y}) \equiv 0(\frac{1}{x} \cdot \frac{1}{y}) \overset{(a)}{=} 0 \implies 1 = 0$, a contradiction to the assumption that $1 \neq 0$ (M4). So $xy \neq 0$.

(c) Need to show $(-x)y$ is an additive inverse of $xy$. $(-x)y + xy \overset{D}{=} (-x+x)y \overset{A5}{=} 0 \cdot y \overset{(a)}{=} 0 \implies (-x)y = -(xy)$ by Proposition 8(c). Similarly, $-(xy) = x(-y)$.

(d) $(-x)(-y) \overset{(c)}{=} -(x(-y)) \overset{(c)}{=} -[-(xy)] \overset{\text{Prop 8(d)}}{=} xy$

Done.

Field, ordered set $\} \implies$ Ordered field.

**Definition.**—An ordered field $(F, +, \cdot, <)$ is a field that is also an ordered set and satisfies the following conditions:

(i) If $y < z$, then $x + y < x + z$ for all $x \in F$.

(ii) If $x > 0$ and $y > 0$, then $xy > 0$.

**Example 47.**—$(\mathbb{Q}, +, \cdot, <)$ is an ordered field with the following operations:

$+$: $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$ for $\frac{a}{b}, \frac{c}{d} \in \mathbb{Q}$.

$\cdot$: $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$ for $\frac{a}{b}, \frac{c}{d} \in \mathbb{Q}$.

$<$: $\frac{a}{b} < \frac{c}{d}$ if $ad < bc$.

**Proposition 11.**—Let $(F, +, \cdot, <)$ be an ordered field. Then

(a) If $x > 0$, then $-x < 0$ (and if $x < 0$, then $-x > 0$).

(b) If $x > 0$ and $y < z$, then $xy < xz$.

(c) If $x < 0$ and $y < z$, then $xy > xz$.

(d) If $x \neq 0$, then $x^2 = x \cdot x > 0$. In particular, $1 > 0$.

(e) If $x > 0$, then $\frac{1}{x} > 0$.

(f) If $0 < x < y$, then $\frac{1}{x} > \frac{1}{y}$.

**Proof.** Cases:

(a) Let $x > 0$. Then $0 \overset{A5}{=} -x + x \overset{\text{(i)OF}}{>} -x + 0 \overset{A4}{=} -x \implies 0 > -x$. Similarly, ...

(b) Let $x > 0$ and $y < z$. Since $y < z \implies z - y > y - y = 0$, then $x(z-y) \overset{(a)}{>} 0$. Therefore, $xz = xz + 0 = xz - xy + xy = x(z-y) + xy \overset{(i)}{>} 0 + xy = xy \implies xy < xz$.

(c) Similar to (b).

(d) Let $x \neq 0$. Then $x > 0$ or $x < 0$ (ordered set). If $x > 0$, then $x^2 = x \cdot x \overset{\text{OF}(ii)}{>} 0$. Let $x < 0$. Then $-x > 0 \implies 0 < (-x)(-x) \overset{\text{Prop 10(d)}}{=} x \cdot x = x^2$. In particular, $1 \neq 0$, so $1^2 = 1 > 0$.

(e) Let $x > 0$. Suppose that $\frac{1}{x} \geq 0$. Then $-\frac{1}{x} \geq 0$. Then $0 = 0 \cdot x \leq (-\frac{1}{x}) \cdot x = -1 \implies 0 \leq -1$ $(1 \leq 0)$, a contradiction to $0 < 1$.

Done.

**Theorem Real field.** *There exists an ordered field, denoted $(\mathbb{R}, +, \cdot, <)$, which has the least upper bound property. Moreover, $\mathbb{R}$ contains $\mathbb{Q}$ as a subfield.*

**Note 11.**—$\mathbb{R}$ is called the real field, and its elements are called real numbers.

**Theorem.** *Cases:*

*(a) Let $x, y \in \mathbb{R}$ with $x > 0$. Then there exists $n \in \mathbb{N}\backslash\{0\}$ such that $nx > y$ (Archimedean property of $\mathbb{R}$).*

*(b) Let $x, y \in \mathbb{R}$ with $x < y$. Then $\exists q \in \mathbb{Q}$ such that $x < q < y$ (Denseness of $\mathbb{Q}$ in $\mathbb{R}$).*

*Special case: For each $y \in \mathbb{R}$, $\exists n \in \mathbb{N}\backslash\{0\}$ such that $n > y$ (taking $x = 1 > 0$).*

**Proof.** (a) Let $x, y \in \mathbb{R}$ with $x > 0$. Suppose, by contradiction, that $n \cdot x \geq y$ for all $n \in \mathbb{N}\backslash\{0\}$. Set $A := \{nx : n \in \mathbb{N}\backslash\{0\}\}$. $A$ is nonempty since $x = 1 \cdot x \in A$. $A$ is bounded above by $y$ since $nx \leq y$ for all $n \in \mathbb{N}$. Since $\mathbb{R}$ has the least upper bound property, $\sup A \in \mathbb{R}$. $\alpha := \sup A \in \mathbb{R}$. Now, $x > 0$, so $-x < 0$. $\alpha - x < x + 0 = \alpha = \sup A \implies \alpha - x$ is not an upper bound of $A \implies \exists m \in \mathbb{N}$ such that $m \cdot x \in A$ and $\alpha - x < mx < \alpha$. $\implies \alpha = \alpha - x + x < mx + x = (m+1)x \in A$, a contradiction to $\alpha = \sup A$. Therefore, $\exists n \in \mathbb{N}\backslash\{0\}$ such that $nx > y$.

(b) Want to construct $q = \frac{m}{n}$, $m, n \in \mathbb{Z}$, and $n \neq 0$ such that $x < \frac{m}{n} < y$. Since $x < y$, $y - x > 0$. By part (a), $\exists n \in \mathbb{N}\backslash\{0\}$ such that $n(y-x) > 1$. Next, we find $m$. Since $1 > 0$ by (a), $\exists k_1, k_2 \in \mathbb{N}\backslash\{0\}$ such that $k_1 > ny$ and $k_2 > -nx$ ($\implies -k_2 < nx$). Combining $-k_2 < nx < ny < k_1$. We're done if $\exists m \in \mathbb{Z}$ such that $nx < m < ny$. Define $S := \{j \in \mathbb{Z} : -k_2 \leq j \leq k_1 \text{ and } j > nx\}$. Then $S$ is finite, $S \neq \emptyset$ since $k_1 \in S$, $S$ is bounded below by $-k_2$. So $\inf S = \min S$ exists, say $m = \min S$. $\implies m > nx$ by construction. Since $m = \min S$, $m - 1 \leq nx$. Then $m = (m-1) + 1 \leq nx + 1 < nx + n(y-x) = ny$. So $nx < m < ny$ and hence $x < \frac{m}{n} < y$, where $q = \frac{m}{n} \in \mathbb{Q}$.

Done.

**Theorem.** *Cases:*

1. *For every positive real number $x$ and every $n \in \mathbb{N}\backslash\{0\}$, there exists a unique $y \in \mathbb{R}$, $y > 0$, such that $y^n = x$ ($y$ is the $n^{th}$ root of $x$ or $y = x^{1/n}$).*

2. *For every $n \in \mathbb{N}\backslash\{0\}$, there exists a unique $y \in \mathbb{R}$ (specifically, $y = 0$) such that $0^n = 0$ (or $0^{1/n} = 0$).*

3. *If $0 < a < b$, then $0 < a^{1/n} < b^{1/n}$ (monotonicity of $n^{th}$ root).*

**Proof.** Uniqueness of $n^{\text{th}}$ root. Case: When $x = 0$. Suppose $y^n = 0$ given fixed $n$ but $y \neq 0$. Then $y^2 \neq 0$, and by induction $y^k \neq 0$ for all $k \in \mathbb{N}\backslash\{0\}$, a contradiction to the fact that $y^n = 0$ for some fixed $n \in \mathbb{N}$. So $y = 0$ is the only $n^{\text{th}}$ root for $x = 0$.

Case: When $x > 0$. Suppose $\exists y_1, y_2 \in \mathbb{R}$ with $y_1, y_2 > 0$ and $y_1 \neq y_2$ such that $y_1^n = x = y_2^n$. WLOG assume $y_1 < y_2$. If $0 < a < b$ then $0 < a^k < b^k$ for any $k \in \mathbb{N}\backslash\{0\}$ and $a, b \in \mathbb{R}$. By induction: Base case: $a < b$. Suppose $a^{k-1} < b^{k-1}$. Then $a^k = a \cdot a^{k-1} < a \cdot b^{k-1} \underset{a<b}{<} b \cdot b^{k-1} = b^k$. By induction $0 < a^k < b^k$ for all $k \in \mathbb{N}\backslash\{0\}$. Then $y_1^n < y_2^n$, a contradiction to $y_1^n = y_2^n$.

Existence of $n^{\text{th}}$ root: If $x = 0$, then $0^n = 0$ and therefore it follows from the uniqueness that $y = 0$. Let $x > 0$. If $n = 1$, then $y^n = y^1 = x$. Assume $n \geq 2$. Define $S = \{t \in \mathbb{R} : t^n < x\}$. $S \neq \emptyset$ because $0 \in S$. $S$ is bounded above: find an upper bound of $S$. Let $\alpha := 1 + x$. Want to show that $\alpha$ is an upper bound of $S$, i.e., $\alpha \geq t$ for all $t \in S$. Equivalently, we show that if $\alpha < t$ then $t \notin S \implies t^n > (1+x)^n = (1+x)(1+x)^{n-1} > (1+x) \cdot 1 \underset{1>0}{>} x \implies t \notin S \implies \alpha$ is an upper bound of $S$ or $S$ is bounded above $\implies \sup S \in \mathbb{R}$.

*Corollary.*—For $a, b \in \mathbb{R}$ with $a \geq 0$ and $b \geq 0$, we have $(ab)^{1/n} = a^{1/n} \cdot b^{1/n}$.

**Proof.** By the uniqueness of the $n^{\text{th}}$ root, it is sufficient to show $(a^{1/n} \cdot b^{1/n})^n = ab$. Now, $ab = (a^{1/n})^n \cdot (b^{1/n})^n = a^{1/n} \cdot \ldots \cdot a^{1/n} \cdot b^{1/n} \cdot \ldots \cdot b^{1/n} = (a^{1/n} b^{1/n})$ repeated $n$ times, using commutativity and associativity.

## § 1.1 Decimal Expansion of Real Numbers

Let $x \in \mathbb{R}$ with $x > 0$. The decimal expansion of $x$ is a number of the form $n_0.n_1n_2n_3\ldots$, where $n_i$ is defined inductively as follows: Let $n_0$ be the largest integer such that $n_0 \leq x$. Let $n_1$ be the largest integer such that $n_0 + \frac{n_1}{10} \leq x$. Once $n_0, \ldots, n_{k-1}$ are chosen, $n_k$ is chosen as the largest integer such that

$$n_0 + \frac{n_1}{10} + \frac{n_2}{10^2} + \ldots + \frac{n_{k-1}}{10^{k-1}} + \frac{n_k}{10^k} \leq x.$$

**Note 12.**—By construction, $n_k \in \{0, 1, 2, \ldots, 8, 9\}$.

**Remark 13.**—Let $x \in \mathbb{R}$ with $x > 0$. Let $n_0.n_1n_2\ldots n_k\ldots$ be a decimal expansion of some real number. Define $E := \{n_0 + \frac{n_1}{10} + \frac{n_2}{10^2} + \ldots + \frac{n_k}{10^k} : k \in \mathbb{N}\}$. Then $x = \sup E$ if and only if $x = n_0.n_1n_2n_3\ldots$.

## § 1.2 Extended Real Number System

The extended real number system is denoted as $\mathbb{R}^{\#} := \mathbb{R} \cup \{-\infty, \infty\}$. The order on the set $\mathbb{R}^{\#}$ is such that $-\infty < x < \infty$ for all $x \in \mathbb{R}$. $\mathbb{R}^{\#}$ is not a field as it is not closed under usual addition. We use the conventions $x + \infty = \infty$, $x - \infty = -\infty$, $\frac{1}{\infty} = 0$, $\frac{1}{-\infty} = 0$. For $x \in \mathbb{R}^{\#}$, $x > 0$ implies $x \cdot (+\infty) = \infty$ and $x \cdot (-\infty) = -\infty$. For $x \in \mathbb{R}^{\#}$, $x < 0$ implies $x \cdot (+\infty) = -\infty$ and $x \cdot (-\infty) = \infty$. For any $A \subset \mathbb{R}^{\#}$, $\sup A \in \mathbb{R}^{\#}$, $\inf A \in \mathbb{R}^{\#}$. In particular, if $A$ is not bounded above, then $\sup A = \infty$. If $A$ is not bounded below, then $\inf A = -\infty$. Also, $\sup \emptyset = -\infty$ and $\inf \emptyset = \infty$ because every real number is an upper bound for the empty set, and every real number is a lower bound for the empty set.

## § 1.3 Complex Field

The dictionary order $(a, b) < (c, d)$ if $\begin{cases} a < c, \\ a = c, \quad b < d \end{cases}$.

**Definition.**—The set of all complex numbers is denoted as $\mathbb{C} := \{(a, b) : a, b \in \mathbb{R}\}$, where $(a, b)$ is an ordered pair, i.e., $(a, b) \neq (b, a)$ if $a \neq b$. If $z = (a, b) \in \mathbb{C}$, we write $a = \Re(z)$ and $b = \Im(z)$.

The addition operation $+_{\mathbb{C}}$ is defined as $z_1 = (a, b) \in \mathbb{C}$ and $z_2 = (c, d) \in \mathbb{C}$, then $z_1 +_{\mathbb{C}} z_2 = (a +_{\mathbb{R}} c, b +_{\mathbb{R}} d)$. The multiplication operation $\cdot_{\mathbb{C}}$ is defined as $z_1 \cdot_{\mathbb{C}} z_2 = (ac - bd, bc + ad)$. The additive identity is $0_{\mathbb{C}} = (0, 0)$. The multiplicative identity is $1_{\mathbb{C}} = (1, 0)$. Thus, $(\mathbb{C}, +_{\mathbb{C}}, \cdot_{\mathbb{C}})$ is a field.

**Recall 19.**—The set $(\mathbb{C}, +_{\mathbb{C}}, \cdot_{\mathbb{C}})$ is a field.

**Note 13.**—$\mathbb{R}$ is a subfield of $\mathbb{C}$, and we can define $f : \mathbb{R} \to \mathbb{C}$ by $f(a) = (a, 0)$, allowing us to identify $a \in \mathbb{R}$ as $a \in \mathbb{C}$.

**Definition.**—We define the imaginary unit $i$ as $i = (0, 1) \in \mathbb{C}$.

**Theorem.** $i^2 = -1$.

**Proof.** $i^2 = i \cdot i = (0, 1) \cdot_{\mathbb{C}} (0, 1) = (0 - 1, 0) = (-1, 0) \in \mathbb{C} = -1 \in \mathbb{R}$.

Given $z = (a, b) \in \mathbb{C}$, we can write $z = a + bi$. Indeed, $a + bi = (a, 0) +_{\mathbb{C}} (b, 0) \cdot_{\mathbb{C}} (0, 1) = (a, 0) +_{\mathbb{C}} (0, b) = (a, b)$.

**Definition** (Conjugate)**.**—For $z = a + bi = (a, b) \in \mathbb{C}$, the conjugate of $z$ is denoted as $z \in \mathbb{C}$, where $z = a - bi = (a, -b)$.

**Theorem Conjugate.** *Let $z, w \in \mathbb{C}$, $z = a + ib$, $w = c + id$, where $a, b, c, d \in \mathbb{R}$. Then*

*(a)* $z + w = z + w$

*(b)* $z \cdot w = z \cdot w$

*(c)* $z + z = 2\Re(z)$ *and* $z - z = 2i\Im(z)$

*(d)* $z \cdot z \in \mathbb{R}$, $z \cdot z \geq 0$, *and* $z \cdot z = 0 \Leftrightarrow z = 0$

*(e)* $z = z$

*(f)* $z \in \mathbb{R} \implies z = z$.

**Proof.** Cases:

(a) $z + w = (a + ib) + (c + id) = (a + c) + i(b + d) = (a + c) - i(b + d) = a + c - ib - id = (a - ib) + (c - id) = a + ib + c + id = z + w$.

(b) (skipped)

(c) (skipped)

(d) $z \cdot z = (a + ib)(a - ib) = a^2 + b^2 \in \mathbb{R}$. If $a \in \mathbb{R}$ and $b \in \mathbb{R}$, then $a^2 \geq 0$ and $b^2 \geq 0$, implying $z \cdot z \geq 0$. Now, suppose $z = 0$. Then $z \cdot z = 0 \cdot z = 0$. Conversely, suppose $z \cdot z = 0$. Then $a^2 + b^2 = 0$, implying $a = 0$ and $b = 0$. So $z = 0 + 0i = 0$.

(e) (skipped)

(f) (skipped)

Done.

**Remark 14.**—The map $f : \mathbb{C} \to \mathbb{C}$ given by $f(z) = z$ preserves the field properties of $\mathbb{C}$ but is not an identity mapping since $f(i) = -i$.

**Definition** (Absolute Value, Modulus)**.**—Given $z \in \mathbb{C}$, the modulus or absolute value of $z$ is denoted as $|z| = (z \cdot z)^{1/2}$, which is well-defined by the $n^{\text{th}}$ root theorem since $0 \leq z \cdot z \in \mathbb{R}$.

**Note 14.**—If $x \in \mathbb{R}$, then $|x| = x \cdot x^{1/2} = (x^2)^{1/2} \overset{x\in\mathbb{R}}{=} ((-x)^2)^{1/2} \implies |x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$

$|x|^2 = x^2 = (-x)^2$ for all $x \in \mathbb{R}$. However, this identity is not true in $\mathbb{C}$. Take $z = i$, so $|z|^2 = |i|^2 = i \cdot i = i(-i) = 1$. But $z^2 = i \cdot i = i^2 = -1$.

**Theorem.** *Let $z, w \in \mathbb{C}$. Then*

*(a)* $|z| \geq 0$ *and* $|z| = 0 \Leftrightarrow z = 0$

*(b)* $|z| = |z|$

*(c)* $|z \cdot w| = |z| \cdot |w|$

*(d)* $|\Re(z)| \leq |z|$ *and* $|\Re(z)| = |z| \Leftrightarrow z \in \mathbb{R}$

*(e)* $|z + w| \leq |z| + |w|$ *and equality holds* $\Leftrightarrow z = \alpha w$ *or* $w = \alpha z$ *for some* $\alpha \in \mathbb{R}$.

**Proof.** Cases:

(a) For $z + w$, we have:

$$\begin{aligned} z + w &= (a + ib) + (c + id) \\ &= (a + c) + i(b + d) \\ &= (a + c) - i(b + d) \\ &= a + c - ib - id \\ &= (a - ib) + (c - id) \\ &= a + ib + c + id \\ &= z + w. \end{aligned}$$

(b) (skipped)

(c) (skipped)

(d) For $z \cdot z = (a + ib)(a - ib) = a^2 + b^2 \in \mathbb{R}$. If $a \in \mathbb{R}$ and $b \in \mathbb{R}$, then $a^2 \geq 0$ and $b^2 \geq 0$, implying $z \cdot z \geq 0$. Now suppose $z = 0$. Then $z \cdot z = 0 \cdot z = 0$. Suppose $z \cdot z = 0$. Then $a^2 + b^2 = 0$, which implies $a = 0$ and $b = 0$. So $z = 0 + 0i = 0$.

(e) (skipped)

(f) (skipped)

Done.

**Remark 15.**—Let $f : CC$ be defined by $f(z) = z$. By Theorem, $f(z + w) = f(z) + f(w)$ for all $z, w \in C$ and $f(z \cdot w) = f(z) \cdot f(w)$ for all $z, w \in C$ and $f(0) = 0$ and $f(1) = 1$. This implies the conjugate preserves the field properties of $C$. However, $f$ is not an identity mapping since $f(i) = -i$.

**Definition** (Absolute value, modulus)**.**—Given $z \in C$, the modulus or absolute value of $z$ is defined as $|z| := (z \cdot z)^{\frac{1}{2}}$, which is well-defined by the $n^{\text{th}}$ root theorem since $0 \leq z \cdot z \in R$.

**Note 15.**—If $x \in R$, then $|x| = x \cdot x^{1/2} = (x^2)^{\frac{1}{2}} x \in R = ((-x)^2)^{\frac{1}{2}} \implies |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}.$

$|x|^2 = x^2 = (-x)^2$ for all $x \in R$. However, this identity does not hold in $C$. For example, take $z = i$, so $|z|^2 = |i|^2 = i \cdot i = i(-i) = 1$. But $z^2 = i \cdot i = i^2 = -1$.

**Theorem.** *Let $z, w \in C$. Then:*

*(a)* $|z| \geq 0$ *and* $|z| = 0z = 0$.

*(b)* $|z| = |z|$

*(c)* $|z \cdot w| = |z| \cdot |w|$

*(d)* $|\Re(z)| \leq |z|$ *and* $|\Re(z)| = |z| z \in R$.

*(e)* $|z + w| \leq |z| + |w|$ *and equality holds* $z = \alpha w$ *or* $w = \alpha z$ *for some* $\alpha \in R$.

**Proof.** Cases:

(a) Since $z \cdot z \geq 0$ and $|z| = (z \cdot z)^{1/2}$, $|z| \geq 0$ follows by the $n^{\text{th}}$ root theorem. By the previous theorem, $z \cdot z = 0z = 0$. So $|z| = 0z = 0$.

(b) $|z| = (z \cdot z)^{\frac{1}{2}} = (z \cdot z)^{\frac{1}{2}} = |z|$.

(c) $|z \cdot w|^2 = (z \cdot w)(z \cdot w) = (z \cdot w)(z \cdot w) = (z \cdot z) \cdot (w \cdot w) = |z|^2 \cdot |w|^2 = (|z| \cdot |w|)^2$. By the uniqueness of the square root, $|z \cdot w| = |z| \cdot |w|$.

(d) $|\Re(z)|^2 = \Re(z) \cdot (\Re(z)) = (\Re(z))^2 \leq (\Re(z))^2 + (\Im(z))^2 = |z|^2$. By the monotonicity of the square root, $|\Re(z)| \leq |z|$.

(e) $|z + w|^2 = (z + w)(z + w) = (z + w)(z + w) = z \cdot z + w \cdot z + z \cdot w + w \cdot w = |z|^2 + w \cdot z + z \cdot w + |w|^2 = |z|^2 + 2\Re(w \cdot z) + |w|^2 \leq |z|^2 + 2|\Re(w \cdot z)| + |w|^2$ since $x \leq |x| (d) \leq |z|^2 + 2|z \cdot w| + |w|^2 (b) + (c) = |z|^2 + 2|z| \cdot |w| + |w|^2 = (|z| + |w|)^2$ for all $x \in R$. By the monotonicity of roots, $|z + w| \leq |z| + |w|$.

Done.

**Theorem Schwartz Inequality.** *Let $z_1, z_2, \ldots, z_n, w_1, \ldots, w_n \in \mathbb{C}$. Then*

$$\left| \sum_{i=1}^{n} z_i \cdot \overline{w_i} \right|^2 \leq \left( \sum_{i=1}^{n} |z_i|^2 \right) \left( \sum_{i=1}^{n} |w_i|^2 \right).$$

**Proof.** Let $Z := \sum_{i=1}^n |z_i|^2 \in \mathbb{R}$, $W := \sum_{i=1}^n |w_i|^2 \in \mathbb{R}$, and $P := \sum_{i=1}^n z_i \cdot \overline{w_i} \in \mathbb{C}$. Claim: $ZW \geq |P|^2$. Note that $W \geq 0$. Then $W = 0 \Leftrightarrow |w_i| = 0 \Leftrightarrow w_i = 0$. In this case, $P = \sum_{i=1}^n z_i \cdot 0 = 0$, and the claim is satisfied.

Take $W > 0$: Then

$$
\begin{aligned}
0 &\leq \sum_{i=1}^n |Wz_i - Pw_i|^2 \\
&\overset{?}{\geq} W(WZ - |P|^2) \\
&= \sum_{i=1}^n (Wz_i - Pw_i)\overline{(Wz_i - Pw_i)} \\
&= \sum_{i=1}^n \left[W^2 z_i \cdot \overline{z_i} - W\overline{P} z_i \overline{w_i} - PW w_i \overline{z_i} + P\overline{P} w_i \cdot \overline{w_i}\right] \\
&= \sum_{i=1}^n \left[W^2 |z_i|^2 - W\overline{P} z_i \overline{w_i} - PW w_i \overline{z_i} + |P|^2 \cdot |w_i|^2\right] \\
&= W^2 \sum_{i=1}^n |z_i|^2 - W\overline{P} \sum_{i=1}^n z_i \cdot \overline{w_i} - PW \sum_{i=1}^n w_i \overline{z_i} + |P|^2 \sum_{i=1}^n |w_i|^2 \\
&= W^2 Z - W\overline{P}P - PW\overline{P} + |P|^2 W \\
&= W^2 Z - W|P|^2 \\
&= W(WZ - |P|^2) \geq 0 \implies WZ - |P|^2 \geq 0
\end{aligned}
$$

## § 1.4 Euclidean spaces

**Definition** (Vector space over a field)**.**—Let $(F, +_F, \cdot_F)$ be a field. A vector space $V$ over a field $F$ is a nonempty set $V$ with two operations: vector addition $+$ and scalar multiplication $\cdot$, satisfying the following:

1. $V$ satisfies (A1)-(A5) of field axioms with '+'.

2. For all $a \in F$, $\in V$, $a \in V$.

3. For all $a \in F$, $, \in V$, $a(+) = a + a$.

4. For all $a, b \in F$, $\in V$, $(a +_F b) = a + b$.

5. For all $a, b \in F$, $\in V$, $(a \cdot_F b) = a(b)$.

6. $1_F =$ for all $\in V$.

Define $\mathbb{R}^k = \underbrace{\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}}_{k \text{ times}}$ for $k \geq 1$.

For $= (x_1, \ldots, x_k), = (y_1, \ldots, y_k) \in \mathbb{R}$, define addition $+$ and scalar multiplication $\cdot$ as follows:

$$+ = (x_1 + y_1, x_2 + y_2, \ldots, x_k + y_k)$$
$$a = (ax_1, ax_2, \ldots, ax_k) \quad \text{for } a \in \mathbb{R}.$$
$$= (0, 0, \ldots, 0).$$

**Theorem.** $(\mathbb{R}^k, +, \cdot)$ *is a vector space over the field of* $\mathbb{R}$.

**Proof.** Skip.

**Definition.**—For $, \in \mathbb{R}^k$, the inner product or scalar product is defined as:

$$\cdot = x_1y_1 + x_2y_2 + \cdots + x_ky_k = \sum_{i=1}^{k} x_iy_i \in \mathbb{R}$$

**Definition** (Norm or modulus or absolute value)**.**—

$$|| = \left( \sum_{i=1}^{k} x_i^2 \right)^{\frac{1}{2}}$$

Other notation $|||| = \left( \sum_{i=1}^{k} x_i^2 \right)^{1/2}$.

**Theorem.** *Let* $, , \in \mathbb{R}^k$ *and* $a \in \mathbb{R}$. *Then*

*(a)* $|| \geq 0$ *and* $|| = 0 \Leftrightarrow =$.

*(b)* $|a| = |a| \cdot ||$

**(c)* $| \cdot | \leq || \cdot ||$

*(d)* $| + | \leq || + ||$

*(e)* $| - | \leq | - | + | - |$

**Proof.** Cases:

(a) $||^2 = \sum_{i=1}^{k} x_i^2 \geq 0$, since $x_i^2 \geq 0$ for $x_i \in \mathbb{R}$. $\implies$ $|| \geq 0$ using the monotonicity of roots. Suppose that $|| = 0$ and suppose by contradiction that $x_i \neq 0$ for some $i \in \{1, \ldots, k\}$. $\implies$ $x_i^2 > 0$ and $\sum_{j=1}^{k} x_j^2 > 0$, a contradiction. If $=$, then $x_i = 0$ for all $i = 1, \ldots, k$. So $|| = \left( \sum_{i=1}^{k} 0^2 \right)^{1/2} = 0$.

(b) $|a|^2 = \sum_{i=1}^{k} (ax_i)^2 = a^2 \sum_{i=1}^{k} x_i^2 = a^2(||)^2 \implies |a| = \sqrt{a^2}\sqrt{||^2} = |a|||$.

*(c) WTS: $||^2||^2$. Recall Schwartz inequality

$$\left|\sum_{i=1}^{n} z_i \cdot w_i\right|^2 \leq \sum_{i=1}^{n} |z_i|^2 \sum_{i=1}^{n} |w_i|^2.$$

$$|\cdot|^2 = \left|\sum_{i=1}^{k} x_i y_i\right|^2 \text{Schwartz}\leq \left(\sum_{i=1}^{n} |x_i|^2\right)\left(\sum_{i=1}^{n} |y_i|^2\right) x_i, y_i \in \mathbb{R}$$

$$= \left(\sum_{i=1}^{k} x_i^2\right)\left(\sum_{i=1}^{k} y_i^2\right) = ||^2||^2.$$

By the monotonicity of the root, $|\cdot| \leq ||||$.

(d) Want to show: $|+|^2 \leq (||+||)^2 = ||^2 + 2||\cdot|| + ||^2$.

$$\begin{aligned}|+|^2 &= \sum_{i=1}^{k}(x_i+y_i)^2 = \sum_{i=1}^{k}(x_i^2 + 2x_iy_i + y_i)^2 = \sum_{i=1}^{k} x_i^2 + 2\sum_{i=1}^{k} x_iy_i + \sum_{i=1}^{k} y_i^2 \\ &\leq ||^2 + 2|\cdot| + ||^2 (c)\leq ||^2 + 2||\cdot|| + ||^2 = (||+||)^2\end{aligned}$$

since $\cdot \in \mathbb{R}$ and $a \leq |a|$ for all $a \in \mathbb{R}$. This implies $|+| \leq ||+||$ by the monotonicity of roots.

(e) $|-| = |+-| = |-+-| (d)\leq |-| + |-|$.

Done.

Observe that $|\cdot| = ||\cdot||$ if $x, y \in \mathbb{R}$ or if $x, y \in \mathbb{C}$. Not true in $\mathbb{R}^2$:

Compare the result of (c) in $\mathbb{C}$ versus in $\mathbb{R}^2$. Take $z = a+ib, w = c+id \in \mathbb{C}$. Then

$$\begin{aligned}|z\cdot w|^2 &= |(a+ib)\cdot(c+id)|^2 = |(ac-bd)+i(ad+bc)|^2 = (ac-bd)^2 + (ad+bc)^2 \\ &= (ac)^2 + (bd)^2 + (ad)^2 + (bc)^2 = (a^2+b^2)(c^2+d^2) = |z|^2\cdot|w|^2 \implies |z\cdot w| = |z|\cdot|w|.\end{aligned}$$

Take $= (a,b)$, $= (c,d)$ in $\mathbb{R}^2$.

$$\begin{aligned}|\cdot|^2 &= (ac+bd)^2 = (ac)^2 + 2abcd + (bd)^2 \\ &\cdots ? \cdots \\ &\leq |(a^2+b^2)\cdot(c^2+d^2) = ||\cdot||\end{aligned}$$

If $(a,b),(c,d) \in \mathbb{C}$, then $|(a,b)\cdot(c,d)| = |(a,b)|\cdot|(c,d)|$.

If $(a,b),(c,d) \in \mathbb{R}^2$, then $|(a,b)\cdot(c,d)| \leq |(a,b)||(c,d)|$.

**Proof.** $0 \leq p^2 + q^2 - 2pq = (p-q)^2 \implies 2pq \leq p^2 + q^2$.

$$\begin{aligned}|(a,b)\cdot(c,d)|^2 &= |ac+bd|^2 = (ac+bd)^2 \\ &= (ac)^2 + 2abcd + (bd)^2 \\ &\leq a^2c^2 + b^2c^2 + a^2d^2 + b^2d^2 \\ &= (a^2+b^2)(c^2+d^2) = |(a,b)|^2|(c,d)|^2 \implies |(a,b)\cdot(c,d)| \leq |(a,b)||(c,d)|\end{aligned}$$

monotonicity of roots.

# § 2 Topology

**Definition** (Function)**.**—$A, B$ are sets. $f : A \to B$ is a function or a mapping if for each $x \in A$, $\exists! f(x) \in B$.

- $A =$ domain of $f$.
- $B =$ target set of $f$.
- $f(A) =$ Image of $A$ under $f = \{y \in B : \exists x \in A$ such that $f(x) = y\}$.

**Definition** (one-to-one and onto)**.**—$f : A \to B$ is 1-1 iff $f(a_1) = f(a_2) \implies a_1 = a_2$ (or iff $\forall a_1, a_2 \in A$, $a_1 \neq a_2 \implies f(a_1) \neq f(a_2)$. $f : A \to B$ is onto iff $\forall b \in B$, $\exists a \in A$ such that $f(a) = b$ OR $f(A) = B$.

**Definition** (Preimage or inverse image)**.**—Suppose $E_1 \subset B$. Then the preimage of $E_1$:

$$f^{-1}(E_1) = \{a \in A : f(a) \in E_1\}.$$

If $b \in B$, then $f^{-1}(b) = \{a \in A : f(a) = b\} \subset A$.

**Remark 16.**—One-to-one correspondence means both one-to-one and onto.

**Definition** (Cardinality)**.**—We say that a set $A$ has no more elements than a set $B$ if $\exists$ a 1-1 map $f : A \to B$. In this case, we write $|A| \leq |B|$, where $|\cdot|$ denotes the cardinality of $A$ (number of elements).

**Theorem Cantor-Schröder-Bernstein.** *If $\exists$ two 1-1 functions $f : A \to B$ and $g : B \to A$, then $\exists$ a 1-1 and onto function $h : A \to B$. i.e., $|A| \leq |B|$ and $|B| \leq |A| \implies |A| = |B|$.*

**Proof.** Skip.

**Definition** (Equivalence relation via cardinality)**.**—$A$ and $B$ have the same cardinality if $\exists$ a 1-1 and onto function $f : A \to B$. We write $|A| = |B|$. This is an equivalence relation.

(i) $|A| = |A|$ since $id : A \to A$ is 1-1 and onto.

(ii) $|A| = |B|$ then $|B| = |A|$. If $f : A \to B$ is 1-1 and onto, then $f^{-1} : B \to A$ is 1-1 and onto.

(iii) If $|A| = |B|$ and $|B| = |C|$, then $|A| = |C|$. There exist $f : A \to B$ and $g : B \to C$ both 1-1 and onto. Then $g \circ f : A \to C$ is 1-1 and onto.

Notation $J := \mathbb{N} \backslash \{0\}$ and $J_n := \{1, 2, \ldots, n\}$.

**Definition.**—Let $A$ be any set. We say

(a) $A$ is finite if $A \sim J_n$ or $|A| = |J_n|$ for some $n \in \mathbb{N}$.

(b) $A$ is infinite if $A$ is not finite.

(c) $A$ is countable if $|A| = |J|$ i.e., there exists a 1-1 and onto mapping $f : A \to J$.

(d) $A$ is uncountable if $A$ is neither finite nor countable.

(e) $A$ is at most countable if $A$ is finite or countable.

**Example 48.**—$|\mathbb{Z}| = |\mathbb{N}|$. Let $f : \mathbb{Z} \to \mathbb{N}$ be defined by

$$f(z) = \begin{cases} 2z - 1 & \text{if } z \in \mathbb{Z}_+ \\ -2z & \text{if } z \in \mathbb{Z}_- \end{cases}$$

The function $f$ maps elements from $\mathbb{Z}$ to $\mathbb{N}$ as shown below:

$$\begin{array}{cccccccc} \mathbb{Z} & 0 & 1 & -1 & 2 & -2 & 3 & -3 \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ \mathbb{N} & 1 & 3 & 2 & 5 & 4 & 7 & 6 \end{array}$$

Let $f : J \xrightarrow[\text{onto}]{1-1} \mathbb{Z}$ be defined by

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ -\frac{n-1}{2} & \text{if } n \text{ is odd} \end{cases}$$

.

**Note 16.**—Some notes:

- A finite set has <u>more</u> elements than its proper subsets.
- The empty set is a finite set.
- A set and its proper subsets can have the same cardinality.

**Example 49.**—$|\mathbb{N}| f? = |2n : n \in \mathbb{N}| = |2n - 1 : n \in \mathbb{N}|$

**Example 50.**—$|\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)| = |\mathbb{R}|$. Let $f : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$ by $f(x) = \tan x$ is 1-1 and onto.

**Proposition 12.**—$S$ is infinite $\Leftrightarrow \exists f : \mathbb{N} \to S$ that is 1-1.

**Proof.** $\Longrightarrow$ Suppose $S$ is infinite. We need to construct $f : \mathbb{N} \to S$ that is 1-1. Since $S$ is not finite, $\exists s_0 \in S$. Define $f_0 : \{0\} \to S$ by $f_0(0) = s_0$. By construction, $f_0$ is 1-1. $S$ is infinite, so $\exists s_1 \in S (s_1 \neq s_0)$. Define $f_1 : \{0, 1\} \to S$ by $f_1(0) = s_0$ and $f_1(1) = s_1$. Then $f_1$ is 1-1. Since $S$ is infinite, $\exists s_2 \in S$, $s_2 \neq s_0$, $s_2 \neq s_1$. Define $f : \{0, 1, 2\} \to S$ by $f_2(0) = s_0$,

$f_2(1) = s_1$, $f_2(2) = s_3$. Inductively, $\exists s_n \in S$ such that $s_n \neq s_j$ for $j = 0, \dots, n-1$ and $f_n : \{0, 1, \dots, n\} \to S$ by

$$\begin{aligned} f_n(0) &= s_0 \\ f_n(1) &= s_1 \\ &\vdots \\ f_n(n) &= s_n. \end{aligned}$$

Then $f : \mathbb{N} \to S$ defined by $f(n) = f_n(n) = s_n$ is 1-1 (by construction).

**Proof.** We aim to show $\inf B \leq \inf A$. By the definition of inf, $\gamma \leq \inf B$ for all $\gamma$ that are lower bounds of $B$. Since $x \in A \implies x \in B$, $\gamma$ is a lower bound of $A$. Choosing $\gamma = \inf B$, we find that $\inf B$ is a lower bound of $A$. Therefore, $\inf B \leq \inf A$.

**Example 51.**—We know by the definition of infimum, $b \leq \inf B$ for all $b \in B$. However, since $A \subset B$, $a \leq \inf B$ for all $a \in A$. This implies $\inf B$ is a lower bound of $A$. By the definition of $\inf A$, $\inf B \leq \inf A$.

**Example 52** (HW2.2).—Let $S := \{a + b : a \in A, b \in B\}$. We want to prove $\sup S = \sup A + \sup B$.

**Proof.** Since $A \neq \emptyset$, $B \neq \emptyset$, and $A, B \subset \mathbb{R}$ are bounded, and $\mathbb{R}$ has the least upper bound property, we have $\sup A, \sup B \in \mathbb{R}$.

Firstly, note that $S \neq \emptyset$ since $A \neq \emptyset$ and $B \neq \emptyset$. Now, for all $a \in A$ and $b \in B$, we have $a \leq \sup A$ and $b \leq \sup B$. Then $a + b \leq \sup A + b$ and $a + b \leq \sup A + \sup B$ for $b \in B$ fixed and $\forall a \in A$. This implies $\sup A + \sup B$ is an upper bound of $S$, so $\sup S \in \mathbb{R}$.

We need to show:

(i) $\sup A + \sup B$ is an upper bound of $S$ (checked).

(ii) If $\gamma$ is an upper bound of $S$, then $\gamma \geq \sup A + \sup B$.

Let $\gamma$ be an upper bound of $S$. Then $\gamma \geq s$ for all $s \in S$. Since $s = a + b$ for some $a \in A$ and $b \in B$, we have $\gamma \geq a + b \implies b \leq \gamma - a \implies \gamma - a$ is an upper bound for $B$. Then $\gamma - a \geq \sup B$ (by definition of sup), which implies $\gamma - \sup B \geq a \implies \gamma - \sup B$ is an upper bound for $A$. This, in turn, implies $\gamma - \sup B \geq \sup A$ (by definition of sup), leading to $\gamma \geq \sup A + \sup B$. Therefore, $\sup S = \sup A + \sup B$.

**Example 53.**—Let $A \subset \mathbb{R}$ be nonempty and bounded below. Define $-A := \{-a \in \mathbb{R} : a \in A\}$. We want to prove $-\inf A = \sup(-A)$.

**Proof.** $A \subset \mathbb{R}$ being nonempty and bounded below implies $\inf A \in \mathbb{R}$. By the definition of infimum, $a \geq \inf A$ for all $a \in A$. This implies $-a \leq -\inf A$ for all $-a \in -A$ (denoted by $(*)$). Since $-A$ is bounded above, $\sup(-A) \in \mathbb{R}$ since $-A \neq \emptyset$.

We want to show:

(i) $-\inf A$ is an upper bound of $-A$ (by $(*)$).

(ii) If $\gamma$ is an upper bound of $-A$, then $\gamma \geq -\inf A$.

Let $\gamma$ be an upper bound of $-A$. Then $\gamma \geq -a$ for all $-a \in -A \implies -\gamma \leq a$ for all $a \in A \implies -\gamma$ is a lower bound of $A$. Since $\inf A$ is the greatest lower bound of $A$, we have $\inf A \geq -\gamma \implies -\inf A \leq \gamma$. Therefore, $-\inf A = \sup(-A)$.

**Example 54.**—The inequality $||z| - |w|| \leq |z - w|$ holds for all $z, w \in \mathbb{C}$.

**Proof.** The inequality $|x| \leq c$ is equivalent to $-c \leq x \leq c$.
To show: (1) $|z| - |w| \leq |z - w|$ and (2) $-|z - w| \leq |z| - |w|$.
(1):

$$|z| \mathbb{C} \text{ field} = |z - w + w|$$
$$\Delta \leq |z - w| + |w|$$

This implies $|z| - |w| \leq |z - w|$.
(2):

$$|w| = |w - z + z|$$
$$\Delta = |w - z| + |z|$$

This implies $|w| - |z| \leq |z - w| \implies -(|w| - |z|) \geq -|z - w| \implies |z| - |w| \leq -|z - w|$.
Combining (1) and (2), $||z| - |w|| \leq |z - w|$.

**Recall 20.**—From last class: $S$ is infinite $\Leftrightarrow \exists f : \mathbb{N} \to S$ that is 1-1.

**Proof.** $\Leftrightarrow$ Suppose there exists $f : \mathbb{N} \to S$ that is 1-1. Assume, to the contrary, that $S$ is finite, i.e., $|S| = |J_n|$ for some $n \in \mathbb{N}$. This implies $\exists j : S 1 - 1 \text{onto} J_n$. Now, $J_{n+1} i 1 - 1 \mathbb{N} f 1 - 1 S 1 - 1 \text{onto} J_n$. If $A \subset B$, then the inclusion map $i : AB$ defined by $i(x) = x$ is a 1-1 map. Then $\exists h = j(f(i)) : J_{n+1} J_n$ is 1-1, leading to a contradiction since $|J_{n+1}| = n + 1$ and $|J_n| = n$.

**Proposition 13.**—A set $S$ is infinite $\Leftrightarrow \exists S' \subsetneq S$ such that $|S'| = |S|$.

**Proof.** Skip.

**Theorem.** *Every infinite subset of a countable set is countable.*

**Proof.** Let $A$ be a countable set, and let $E \subset A$ be infinite. To show: $E$ is countable, i.e., $\exists j : \mathbb{N} E$ that is 1-1 and onto. A set is countable if $\exists s : \mathbb{N} A$ that is 1-1 and onto.

Let $n_1 := \min\{n \in \mathbb{N} : s(n) = s_n \in\}$, and this set is well-defined since $E$ is infinite. Define $n_2 := \min\{n \in \mathbb{N} : n > n_1 \text{ and } s(n) = s_n\}$. Clearly, $n_1 < n_2$. Suppose $n_1 < n_2 < \cdots < n_{k-1}$. Then $n_k := \min\{n \in \mathbb{N} : n > n_{k-1} \text{ and } s(n) = s_n \in E\}$. We have $n_1 < n_2 < \cdots < n_{k-1} < n_k$, and $\{s_{n_1}, s_{n_2}, s_{n_3}, \ldots, s_{n_{k-1}}, s_{n_k}\} \neq E$. Now, define $f : \mathbb{N} E$ by $f(k) = s_{n_k}$. Then $f$ is 1-1. So, $|\mathbb{N}| f \text{ 1-1} \leq |E| E \subset A \leq |A| \text{countable} = |\mathbb{N}|$. Therefore, $|\mathbb{N}| = |E|$, so $E$ is countable.

## § 3 Unions and Intersections

Let $A$ and $B$ be arbitrary subsets of $\Omega$.

$$A \cup B = \{x \in \Omega : x \in A \text{ or } x \in B\}.$$

$$A \cap B = \{x \in \Omega : x \in A \text{ and } x \in B\}.$$

Let $A$ be any set (index set). Consider a collection of subsets of $\Omega$, denoted as $\{E_\alpha\}_{\alpha \in A}$.

$$\bigcup_{\alpha \in A} E_\alpha = \{x \in \Omega : x \in E_{\alpha_0} \text{ for some } \alpha_0 \in A\}.$$

$$\bigcap_{\alpha \in A} E_\alpha = \{x \in \Omega : x \in E_\alpha \text{ for all } \alpha \in A\}.$$

## § 4 Properties

**Remark 17.**—Let $A$ and $B$ be subsets of $\Omega$. The following properties hold:

- $A \cup B = B \cup A$, $A \cap B = B \cap A$.
- $(A \cup B) \cup C = A \cup (B \cup C)$.
- $(*)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
- $A, B \subset A \cup B$, $A \cap B \subset A, B$.
- $A \subset B$ implies $A \cup B = B$ and $A \cap B = A$.
- $\emptyset$ acts as a neutral element, i.e., $A \cup \emptyset = A$ and $A \cap \emptyset = \emptyset$.
- $*$ $\emptyset$ is a subset of every set.

**Example 55.**—

$$\bigcup_{x \in \mathbb{R}} (-\infty, x] = \mathbb{R}.$$

**Proof.** $\subset$ Let $a \in \bigcup_{x \in \mathbb{R}} (-\infty, x]$. Then $a \in (-\infty, x_0] \subset \mathbb{R}$ for some $x_0 \in \mathbb{R}$. This implies $a \in \mathbb{R} \implies \bigcup_{x \in \mathbb{R}} (-\infty, x] \subset \mathbb{R}$.

$\supset$ Let $a \in \mathbb{R}$. Then $a \in (-\infty, a]$. So $a \in \bigcup_{x \in \mathbb{R}} (-\infty, a]$. Therefore $\mathbb{R} \subset_{x \in \mathbb{R}} (-\infty, x]$. Hence $\bigcup_{x \in \mathbb{R}} (-\infty, x] = \mathbb{R}$.

**Example 56.**—

$$\bigcap_{x\in(0,1)}(0,x)=\emptyset.$$

There exists no $a\in\mathbb{R}$ such that $\bigcap_{x\in(0,1)}(0,x)=a$. If $a\leq 0$, then $a\notin(0,x)$ for any $x\in(0,1)$, so $a\notin\bigcap_{x\in(0,1)}(0,x)$. Let $a>0$. Then $a\notin(0,x)$ for any $x\in(0,1)$. So $a\notin\bigcap_{x\in(0,1)}(0,x)$. Let $0<a<1$. Then $a\notin(0,\frac{a}{2})$. So $a\notin\bigcap_{x\in(0,1)}(0,x)$. Therefore, $\bigcap_{x\in(0,1)}(0,x)=\emptyset$.

**Example 57** (*)**.**—Let $A:=\{x\in\mathbb{R}:0<x\leq 1\}=(0,1]$. For $x\in A$, define $E_x=\{y\in\mathbb{R}:0<y<x\}$. Then $E_x\subset E_z\Leftrightarrow 0<x\leq z\leq 1$.

# § 5 Countable Unions of Countable Sets

**Theorem.** *Let $\{E_n\}_{n\in\mathbb{N}}$ be a sequence of countable sets. Then $S:=\bigcup_{n\in\mathbb{N}}E_n$ is countable (countable union of countable sets is countable).*

**Example 58.**—$\mathbb{N}^2=\mathbb{N}\times\mathbb{N}=\bigcup_{n\in\mathbb{N}}\mathbb{N}\times\{n\}$.

**Proof.** We need to show that there exists $f:\bigcup_{n\in\mathbb{N}}E_n\to\mathbb{N}$ that is injective. For each $n\in\mathbb{N}$, write $E_n:=\{x_{n,1},x_{n,2},x_{n,3},\dots,x_{n,k},\dots\}$. Define $f$ as follows:

$$\begin{aligned}
f(g_1)&:x_{1,1},x_{1,2},x_{1,3},\dots,x_{1,k},\dots=E_1.\\
f(g_2)&:x_{2,1},x_{2,2},x_{2,3},\dots,x_{2,k},\dots=E_2.\\
f(g_3)&:x_{3,1},x_{3,2},x_{3,3},\dots,x_{3,k},\dots=E_3.\\
&\vdots\\
f(g_m)&:x_{m,1},x_{m,2},x_{m,3},\dots,x_{m,k},\dots=E_m.
\end{aligned}$$

This array contains all elements of $S$. We can rewrite $S=\bigcup_{n\in\mathbb{N}}E_n$ as $S=\bigcup_{n\in\mathbb{N}}\bigcup_{m\in\mathbb{N}}g_n(m)$. Define $f:\bigcup_{n\in\mathbb{N}}E_n\to\mathbb{N}$ by

$$f(g_n(m))=10^{m+n}+m\implies f\text{ is injective}\implies\bigcup E_n\text{ is countable.}$$

**Theorem.** *Let $A$ be a countable set. Let $B_n:=\{(x_1,\dots,x_n):x_i\in A\}$. Then $B_n$ is countable.*

**Proof.** By induction. Since $A$ is countable, $B_1:=\{x:x\in A\}=A$. Suppose that $B_{k-1}$ is countable for $k\geq 2$. Then $B_k=B_{k-1}\times A=\{(x_1,\dots,x_{k-1},a):a\in A\text{ and }(x_1,\dots,x_{k-1})\in B_{k-1}\}=\bigcup_{a\in A}B_{k-1}\times\{a\}$ implies $B_k$ is countable as a countable union of countable sets. Therefore, $B_n$ is countable for any $n\in\mathbb{N}$.

*Corollary.*—$\mathbb{Q}$ is countable.

**Proof.** We will show that $\mathbb{Q}$ is a countable union of countable sets. For a fixed $n \in \mathbb{N}$, define $S_n := \{\frac{m}{n} : m \in \mathbb{Z}\}$. Then $S_n$ is countable since $\mathbb{Z}$ is countable.

**Claim**: $\mathbb{Q} = \bigcup_{n\in\mathbb{N}} S_n$.

**Proof of Claim**:

$\subset$ Let $x \in \mathbb{Q}$. Then $x = \frac{m}{n}$ with $m, n \in \mathbb{Z}$ and $n \neq 0$. If $n \in \mathbb{N}$, then $x = \frac{m}{n} \in S_n$, so we're done. If $-n \in \mathbb{N}$, then $x = \frac{m}{n} = \frac{-m}{-n} \implies x \in S_{-n\in\mathbb{N}} \implies x \in \bigcup_{n\in\mathbb{N}} S_n \implies \mathbb{Q} \subset \bigcup_{n\in\mathbb{N}} S_n$.

$\supset$ Let $x \in \bigcup_{n\in\mathbb{N}} S_n$. Then $\exists n_0 \in \mathbb{N}$ such that $x \in S_{n_0}$, i.e., $x = \frac{m}{n_0}$ for some $m \in \mathbb{Z}$. $\implies x \in \mathbb{Q} \implies \bigcup_{n\in\mathbb{N}} S_n \subset \mathbb{Q}$.

So $\mathbb{Q} = \bigcup_{n\in\mathbb{N}} S_n$ - a countable union of countable sets. Therefore, $\mathbb{Q}$ is countable.

**Theorem.** *Let $A := \{0,1\}^{\mathbb{N}} = \{f : \mathbb{N} \to \{0,1\}\}$, the set of all sequences of 0's and 1's. Then $A$ is uncountable.*

**Proof.** We will use the following idea: If every countable subset of $A$ is a proper subset of $A$, then $A$ is uncountable (otherwise, $A \subsetneq A$ - a contradiction).

Let $E \subset A$ be a countable set. This means we can enumerate elements of $E$ as follows:

$$\begin{aligned} &s_1 : 0,1,0,1,0,0,0,0,1,0,\ldots \\ &s_2 : 0,0,0,0,0,0,0,0,0,1,\ldots \\ &s_3 : 0,1,1,1,1,0,0,1,0,0,\ldots \\ &\quad\vdots \\ &s_k : 0,0,1,0,1,0,0,0,1,\ldots \end{aligned}$$

- these are all elements of $E$.

We will construct an element $s^* \in A$ such that $s^* \notin E$. Define

$$s^*(k) = 1 - s_k(k) = \begin{cases} 1 & \text{if } s_k(k) = 0, \\ 0 & \text{if } s_k(k) = 1. \end{cases}$$

Then $s^*$ is a sequence of 0's and 1's, and hence, $s^* \in A$. By construction, $s^* \notin s_k$ for all $k \in \mathbb{N}$ in at least one place. This implies $s^* \in A$ but $s^* \notin E \implies E$ is a proper subset of $A$. Since $E$ is arbitrary, $A$ must be uncountable. This is Cantor's diagonalization method.

*Corollary.*—The set of real numbers, $\mathbb{R}$, is uncountable.

**Proof.** This will be demonstrated later in the chapter on topology.

**Theorem.** *The interval $(0,1) \subset \mathbb{R}$ is uncountable.*

**Proof.** Let $E \subset (0,1)$ be a countable set. We aim to show that $E \subsetneq (0,1)$. As $E$ is countable, its elements can be enumerated. Also, the elements of $E$ can be expressed as decimal expansions. Consider the enumeration:

$$\begin{aligned}
&s_1 : 0.a_{11}a_{12}a_{13}a_{14}\dots \\
&s_2 : 0.a_{21}a_{22}a_{23}a_{24}\dots \\
&s_3 : 0.a_{31}a_{32}a_{33}a_{34}\dots \\
&\quad\vdots \\
&s_k : 0.a_{k1}a_{k2}a_{k3}a_{k4}\dots \\
&\quad\vdots
\end{aligned}$$

Construct $s^* \in (0,1)$ as follows: $s^* := 0.s_1^* s_2^* s_3^* \dots$ with $s_1^* \neq a_{11}, s_2^* \neq a_{22}, s_3^* \neq a_{33}, \dots, s_k^* \neq a_{kk}$, and $s_i^* \neq 9$ for all $i \in \mathbb{N}$. Then $s^* \in (0,1)$ and $s^* \notin E$, implying $E \subsetneq (0,1)$. Therefore, $(0,1)$ is uncountable.

## § 5.1 Metric Spaces

**Definition** (Metric).—Let $X$ be any set. Then $\rho : X \times X \to \mathbb{R}$ is a metric if it satisfies the following:

(i) $\rho(a,b) > 0$ for all $a, b \in X$ and $\rho(a,b) = 0 \Leftrightarrow a = b$.

(ii) $\rho(a,b) = \rho(b,a)$ - symmetric.

(iii) $\rho(a,b) \leq \rho(a,c) + \rho(c,b)$ - triangle inequality.

**Example 59.**—$X = \mathbb{R}$ and $\rho(a,b) = |a-b|_{\mathbb{R}}$.

**Example 60.**—$X = \mathbb{C}$ and $\rho(z,w) = |z-w|_{\mathbb{C}} = \left[(z-w)\overline{(z-w)}\right]^{\frac{1}{2}}$.

**Example 61.**—$X = \mathbb{R}^k$ with $\mathbf{x} = (x_1, \dots, x_k), \mathbf{y} = (y_1, \dots, y_k)$ and $\rho(\mathbf{x}, \mathbf{y}) = |\mathbf{x} - \mathbf{y}|_{\mathbb{R}^k} = \left(\sum_{i=1}^k (x_i - y_i)^2\right)^{\frac{1}{2}}$.

**Example 62.**—$X$ be any set and define $\rho(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$ - discrete metric. Verify $\rho$ is a metric.

**Definition** (Convex Set).—Let $E \subset \mathbb{R}^k$. Then $E$ is convex if $\lambda\mathbf{x} + (1-\lambda)\mathbf{y} \in E$ for all $\mathbf{x}, \mathbf{y} \in E$ and for all $\lambda \in [0,1]$.

**Example 63.**—Open ball in $\mathbb{R}^k$: $B_r(\mathbf{x}) := \{\mathbf{y} \in \mathbb{R}^k : |\mathbf{x} - \mathbf{y}| < r\} \subset \mathbb{R}^k$ - ball centered at $\mathbf{x}$ with radius $r > 0$.

**Claim**: $B_r(\mathbf{x})$ is convex, so that $\lambda\mathbf{y} + (1-\lambda)\mathbf{z} \in B_r(\mathbf{x})$ for all $\mathbf{y}, \mathbf{z} \in B_r(\mathbf{x})$ and $0 \leq \lambda \leq 1$.

**Proof.** Let $\mathbf{y}, \mathbf{z} \in B_r(\mathbf{x})$ and $\lambda \in [0,1]$. Then

$$\begin{aligned}
|\mathbf{x} - (\lambda \mathbf{y} + (1-\lambda)\mathbf{z})|_{\mathbb{R}^k} &\overset{?}{<} r \\
&= |\mathbf{x} - \lambda \mathbf{y} - \mathbf{z} + \lambda \mathbf{z} + \lambda \mathbf{x} - \lambda \mathbf{x}| \\
&= |(\mathbf{x} - \mathbf{z}) + \lambda(\mathbf{x} - \lambda \mathbf{y}) + \lambda(\mathbf{x} + \mathbf{z})| \\
&= |(1-\lambda)(\mathbf{x} - \mathbf{z}) + \lambda(\mathbf{x} - \mathbf{y})| \\
&\overset{\Delta}{\leq} (1-\lambda)|\mathbf{x} - \mathbf{z}|_{\mathbb{R}^k} + \lambda|\mathbf{x} - \mathbf{y}|_{\mathbb{R}^k} \\
&\leq (1-\lambda)r + \lambda r = r \implies \lambda \mathbf{y} + (1-\lambda)\mathbf{z} \in B_r(\mathbf{x})
\end{aligned}$$

Thus $B_r(\mathbf{x})$ is convex.

**Definition** (Neighborhood)**.**—$N_r(x) = \{y \in X : \rho(x,y) < r\}$.

**Definition** (Limit Point)**.**—$p$ is a limit point of $E$ if for every $r > 0$, there exists $q \in E \subset X$ such that $q \in N_r(p) \backslash \{p\}$.

**Example 64.**—$X = (\mathbb{R}, |\cdot|)$ and $E := \{\frac{1}{n} : n \in \mathbb{N}\} \subset \mathbb{R}$. If $e \in E$, then $e = \frac{1}{n}$ for some $n \in \mathbb{N}$. Take $r := \min\{\frac{1}{n} - \frac{1}{n+1}, \frac{1}{n-1} - \frac{1}{n}\}$. Then $N_r(e)$ does not contain any point of $E$. $\implies$ $E$ does not contain any limit point of $E$. But $x = 0$ is a limit point of $E$. Indeed, let $r \in (0,1)$. Then $N_r(0)$ contains points of $E$.

**Remark 18.**—Limit points of a set do not necessarily belong to $E$.

**Notation**: $E' :=$ set of all limit points of $E$.

**Definition** (Closed Set)**.**—$E$ is closed if $E' \subset E$.

**Definition** (Interior Point)**.**—A point $p \in E$ is interior if there exists $r$ such that $N_r(p) \subset E$.

**Notation**: $E^\circ :=$ set of all interior points of $E$.

**Definition** (Open Set)**.**—If every point is an interior point, i.e., $E^\circ = E$.

$(X, \rho)$ is a metric space.

**Theorem.** *Every neighborhood in $X$ is an open set.*

**Proof.** Let $p \in X$ and let $N_r(p)$ be a neighborhood of $p$ in $X$. **Want to show**: Every point of $N_r(p)$ is an interior point of $N_r(p)$. Let $q \in N_r(p) \backslash \{p\}$. Then $\rho(p,q) < r$. So let $h = r - \rho(p,q)$. Then $N_h(q) \subset N_r(p)$. Indeed, if $s \in N_h(q)$, then $\rho(q,s) < h$, so

$$\rho(s,p) \overset{\Delta}{\leq} \rho(s,q) + \rho(q,p) < h + \rho(q,p) = r - \rho(p,q) + \rho(p,q) = r$$

implies $\rho(s,p) < r \implies s \in N_r(p) \implies N_h(q) \subset N_r(p)$. So $N_r(p)$ is open.

**Theorem.** *If $p$ is a limit point of $E \subset (X, \rho)$, then every neighborhood of $p$ contains infinitely many points of $E$.*

**Proof.** Constructed through induction. As $p \in E'$, for any fixed $r > 0$, there exists $e_1 \in E$ with $e_1 \in N_r(p)\backslash\{p\}$. Take $r_1 := \rho(p, e_1)$. Then there exists $e_2 \in E$ and $\rho_2 \in N_{r_1}(p)\backslash\{p\}$ such that $e_1 \neq e_2$. If $e_1, \ldots, e_n$ are chosen, take $r_n := \rho(p, e_n)$. Then there exists $e_{n+1} \in E$ and $e_{n+1} \in N_{r_n}(p)\backslash\{p\}$ such that $e_{n+1} \neq e_i$ for all $i = 1, \ldots, n$. By induction, there exists $\{e_n\}_{n=1}^{\infty} \in E$ such that $\{e_n\}_{n=1}^{\infty} \subset N_r(p)\backslash\{p\}$.

**Proof.** Book: By contradiction. Suppose $\exists r > 0$ such that $N_r(p)$ contains finitely many points, say $q_1, \ldots, q_n \in E$ such that $q_i \neq p$. Take $r_0 = 1 \leq i \leq n \text{min} \rho(p, q_i) > 0$, since $p \neq q_i$. Then $N_{r_0}(p) = \{p\}$ - a contradiction that $p$ is a limit point of $E$. This proves the theorem.

**Theorem.** *A set $E$ is open if and only if its complement $E^c$ is closed.*

**Proof.** ($\Leftarrow$) Suppose $E^c$ is closed. Let $x \in E$. Want to show: $\exists r > 0$ such that $N_r(x) \subset E$. Since $x \notin E^c$, $x$ is not a limit point of $E^c$. Hence, there exists $r > 0$ such that $N_r(x) \cap E^c = \emptyset$. $\implies N_r(x) \subset E \implies E$ is open.

($\Rightarrow$) Suppose $E$ is open. We want to show $F = E^c$ is closed, i.e., $F' \subset F$. Let $x \in F'$. Then for every $r > 0$, there exists $y \in F$ such that $y \in N_r(x) \setminus \{x\}$. $\implies y \in N_r(x)$ but $y \notin E$ because $y \in E^c$. $\implies N_r(x) \not\subset E \implies x$ is not an interior point of $E$ (open). $\implies x \notin E \implies x \in E^c$.

**Theorem de Morgan's law.** *Let $\{E_\alpha\}_{\alpha \in A}$ be a collection of subsets of $X$ and $A$ be any set. Then $\left(\bigcup_{\alpha \in A} E_\alpha\right)^c = \bigcap_{\alpha \in A} E_\alpha^c$. E.g., $(A \cup B)^c = A^c \cap B^c$.*

**Proof.** ($\subset$) Let $x \in \left(\bigcup_{\alpha \in A} E_\alpha\right)^c$. Then $x \notin \bigcup_{\alpha \in A} E_\alpha \implies x \notin E_\alpha$ for all $\alpha \in A$. This implies $x \in E_\alpha^c$ for all $\alpha \in A$. $\implies x \in \bigcap_{\alpha \in A} E_\alpha^c \implies \left(\bigcup_{\alpha \in A} E_\alpha\right)^c \subset \bigcap_{\alpha \in A} E_\alpha^c$.

($\supset$) Let $x \in \bigcap_{\alpha \in A} E_\alpha^c$. This implies $x \in E_\alpha^c$ for all $\alpha \in A \implies x \notin E_\alpha$ for all $\alpha \in A \implies x \notin \bigcup_{\alpha \in A} E_\alpha \implies x \in (\bigcup E_\alpha)^c \implies \bigcap_{\alpha \in A} E_\alpha^c \subset (\bigcup E_\alpha)^c$.

Therefore $\left(\bigcup_{\alpha \in A} E_\alpha\right)^c = \bigcap_{\alpha \in A} E_\alpha^c$.

**Theorem.** *Let $(X, \rho)$ be a metric space, and $A$ be a set of indices. Then*

*(i) $\{G_\alpha\}_{\alpha \in A}$ open $\implies \bigcup_{\alpha \in A} G_\alpha$ is open.*

*(ii) $\{F_\alpha\}_{\alpha \in A}$ closed $\implies \bigcap_{\alpha \in A} F_\alpha$ is closed.*

*(iii) If $G_1, \dots, G_n$ are open, then $\bigcap_{i=1}^n G_i$ is open.*

*(iv) If $F_1, \dots, F_n$ are closed, then $\bigcup_{i=1}^n F_i$ is closed.*

**Proof.** Cases:

(i) Let $x \in \bigcup_{\alpha \in A} G_\alpha$ be arbitrary. WTS: $x$ is an interior point of $\bigcup_{\alpha \in A} G_\alpha$ i.e., $\exists r > 0$ such that $N_r(x) \subset \bigcup_{\alpha \in A} G_\alpha$. $x \in \bigcup_{\alpha \in A} G_\alpha \implies \exists \alpha_0 \in A$ such that $x \in G_{\alpha_0}$. $G_{\alpha_0}$ open $\implies \exists r_{\alpha_0} > 0$ such that $N_{r_{\alpha_0}}(x) \subset G_{\alpha_0} \subset \bigcup_{\alpha \in A} G_\alpha$. Thus $x$ is an interior point of $\bigcup_{\alpha \in A} G_\alpha$, and hence $\bigcup_{\alpha \in A} G_\alpha$ is open.

(ii) Note that $\bigcap_{\alpha \in A} F_\alpha = (\bigcup F_\alpha^c))^c$ is closed, since $F_\alpha^c$ is open and $\bigcup F_\alpha^c$ is open by (i).

(iii) Let $G_1, \dots, G_n$ be open. WTS: $\bigcap_{i=1}^n G_i$ is open, i.e., every point of $\bigcap_{i=1}^n G_i$ is an interior point. Let $x \in \bigcap_{i=1}^n G_i$ be arbitrary. $G_i$ open $\implies \exists r_i > 0$ such that $N_{r_i}(x) \subset G_i$ for all $i = 1, \dots, n$. Take $r = 1 \leq i \leq n \min r_i > 0$ since $r_i > 0$ for all $i = 1, \dots, n$. Then $N_r(x) = \bigcap_{i=1}^n N_{r_i}(x) \subset G_i$ for every $i = 1, \dots n \implies N_r(x) \subset \bigcap_{i=1}^n G_i \implies x$ is an interior point of $\bigcap_{i=1}^n G_i \implies \bigcap_{i=1}^n G_i$ is open.

Done.

**Remark 19.**—Counterexamples:

1. Intersection of infinitely many open sets need not be open.

   **Example 65.**—Take $X = \mathbb{R}$ with $\rho(x,y) = |x-y|$. Take $G_n := (-\frac{1}{n}, \frac{1}{n})$ - open for all $n \in \mathbb{N}$. But $\bigcap_{i=1}^{\infty} G_i \overset{?}{=} \{0\}$ - closed (a set with finitely many elements is closed). Clearly, $\{0\} \subset \bigcap_{i=1}^{\infty} G_i \checkmark$. $\subset$ Let $x \in \bigcap i = 1^{\infty} G_i$. $x \in \mathbb{R}$ (an ordered field), so we need to show $x \neq 0$. Let $x > 0$. Then by the Archimedean principle, $\exists n_0 \in \mathbb{N}$ such that $n_0 x > 1$ i.e., $x > \frac{1}{n_0}$, a contradiction to $x < \frac{1}{n}$ for every $n \in \mathbb{N}$. So $x \not< 0$. Let $x < 0$. Then $-x > 0$. By the Archimedean principle, $\exists n_1 \in \mathbb{N}$ such that $n_1(-x) > 1$ i.e., $x < -\frac{1}{n_1}$, a contradiction to $x > -\frac{1}{n}$ for every $n \in \mathbb{N}$. So $x = 0 \implies \bigcap_{i=1}^{\infty} G_i = \{0\}$.

2. The union of infinitely many closed sets may not be closed.

   **Example 66.**—$\bigcup_{n \in \mathbb{N}} [-1 + \frac{1}{n}, 1 - \frac{1}{n}] = (-1, 1) \subset \mathbb{R}$ open, and $[-1 + \frac{1}{n}, 1 - \frac{1}{n}]$ is closed $\forall n \in \mathbb{N}$.

**Recall 21.**—$E'$ = the set of limit points of $E$. (E.g., $E = \{\frac{1}{n}\}_{n \in \mathbb{N}}$, $E' = \{0\}$.). $E = E \cup E'$ = the closure of $E$.

**Theorem.** *Let $E \subset (X, \rho)$. Then*

*(a) $E$ is closed.*

*(b) $E = E \Leftrightarrow E$ is closed.*

*(c) $E \subset F$ for every closed set $F \subset X$ containing $E$, or $E \subset F$ ($E$ is the smallest closed set containing $E$).*

**Proof.** Cases:

(a) We aim to demonstrate that $(E)^c$ is open. Let $x \in (E)^c$ be arbitrary. This implies $x \notin E = E \cup E' \implies x \notin E$ and $x \notin E'$, indicating that $x$ is not a limit point of $E$. Consequently, there exists $r > 0$ such that $N_r(x) \cap (E \cup E') = \{x\}$, but as $x \notin E$ and $x \notin E'$, we have $N_r(x) \cap E = \emptyset$. This leads to $N_r(x) \subset (E)^c \implies x$ is an interior point of $E$. Thus, $(E)^c$ is open, implying $E$ is closed.

(b) $\implies$ by (a). $\Leftarrow$ Suppose $E$ is closed. Want to show: $E \subset E \checkmark$ and $E \subset E$. Since $E$ is closed, $E' \subset E$. Then $E = E \cup E' \subset E \cup E = E$. Therefore, $E = E$.

(c) Let $F \subset X$ be closed and $E \subset F$. WTS: $E \subset F$. $F \subset X$ closed $\implies F' \subset F = F$. Since $E \subset F$, we only need to show that $E' \subset F$. Let $x \in E'$. Then $\exists r > 0$ such that $N_r(x) \cap (E \backslash \{x\}) \neq \emptyset$. This implies $N_r(x) \cap (F \backslash \{x\}) \neq \emptyset$ (since $E \subset F$). That means $x$ is a limit point of $F$, i.e., $x \in F' \implies x \in F' \cup F = F \implies E \subset F$.

Done.

Relative open sets

Let $X$ be a metric space, and $E \subset X$ is open if, for each $e \in E$, there exists $r_e > 0$ such that

$$N_r(e) = \{x \in X : \rho(x, e) < r_e\} \subset E.$$

If $E \subset Y \subset X$, then we say $E$ is open in $Y$ if, for each $e \in E$, there exists $r_e > 0$ such that

$$N_{r_e}^Y(e) := \{y \in Y : \rho(y, e) < r_e\} \subset E.$$

**Theorem.** *Let $Y \subset X$, where $X$ is a metric space. Then $E \subset Y$ is open relative to $YE = Y \cap G$ for some open set $G$ in $X$.*

**Example 67.**—$X = R$ and $Y = [0, 1]$. Then $(\frac{1}{2}, 1] = [0, 1] \cap (\frac{1}{2}, 2)$ is open in $[0, 1]$ but not open in $R$.

**Proof.** $\Longrightarrow$ Suppose $E \subset Y$ is open relative to $Y$. NTS: $\exists$ an open set $G$ of $X$ such that $E = Y \cap G$. Since $E$ is open relative to $Y$, for each $e \in E$, $\exists r_e > 0$ such that $N_{r_e}^Y(e) \subset E$. Set $G := \bigcup_{e\in E} V_e$, where $V_e = N_r(e)$ is an open neighborhood of $e$ in $X$. This implies $G$ is open in $X$. We need to show: $E = Y \cap G$. $\subset$: $E \subset Y$ (given). We only need to show $E \subset G$. Let $e \in E$. Then $e \in V_e$ and so $e \in \bigcup_{e\in E} V_e = G$. So $E \subset Y \cap G$. $\supset$ $Y \cap G = Y \cap \left(\bigcup_{e\in E} V_e\right) = \bigcup_{e\in E}(V_e \cap Y) \subset E$ by definition of relative open sets, since $V_e$ is a neighborhood of $X$ and $V_d \cap Y$ is a neighborhood of $e$ in $Y$. $\Longrightarrow$ $Y \cap G = E$.

Suppose $\exists G$ open in $X$ such that $E = Y \cap G$. We will show $E$ is open in $Y$. Let $e \in E$, NTS: $e$ is an interior point of $E$ of $E$). Then $e \in Y \cap G$. But $G$ is open in $X$, so $\exists r > 0$ such that $N_r(e) \subset G$. This implies $N_r(e) \cap Y \subset G \cap Y = E$. Therefore $e$ is an interior point of $E$, and $E$ is open in $Y$.

## § 5.2 Compact sets

$(X, \rho)$ metric space is $X^{m.s.}$.

**Definition** (Open cover)**.**—Let $\{G_\alpha\}$ be a collection of open sets of $X$ and $E \subset X$. We say that $\{G_\alpha\}$ is an open covering of $E$ if $E \subset \bigcup_\alpha G_\alpha$.

**Remark 20.**—Fitting people under umbrellas in the rain (umbrella neighborhood).

**Definition** (Compact set)**.**—$E \subset X$ is a compact set if every open cover of $E$ has a finite subcover of $E$; that is, if $\{G_\alpha\}$ is an open cover of $X$, $\exists \alpha_1, \ldots, \alpha_n$ such that $E \subset \bigcup_{i=1}^n G_{\alpha_i}$.

**Theorem.** *Suppose $K \subset Y \subset X$. Then $K$ is compact relative to $XK$ is compact relative to $Y$.*

**Proof.** $\Longrightarrow$ Suppose $K$ is compact relative to $X$. NTS: $K$ is compact relative to $Y$. Suppose $\{V_\alpha\}_\alpha$ be an open cover of $K$ with $V_\alpha$ open in $Y$. This means $V_\alpha = Y \cap G_\alpha$ for some open set $G_\alpha$ in $X$, for each $\alpha$. We have $K \subset \bigcup_\alpha V_\alpha = \bigcup_\alpha(Y \cap G_\alpha) = Y \cap (\bigcup_\alpha G_\alpha) \subset \bigcup_\alpha G_\alpha \Longrightarrow \{G_\alpha\}$ is an open cover of $K$ in $X$. But $K$ is compact in $X$, $\exists \alpha_1, \ldots, \alpha_n$ such that $K \subset \bigcup_{i=1}^n G_{\alpha_i}$. Since $K \subset Y$, $K \subset (\bigcup_{i=1}^n G_{\alpha_i}) \cap Y = \bigcup_{i=1}^n (G_{\alpha_i} \cap Y) = \bigcup_{i=1}^n V_{\alpha_i} \implies K$ has a finite subcover in $Y$. So $K$ is compact in $Y$.

Suppose $K$ is compact in $Y$. NTS: $K$ is compact in $X$. Let $\{G_\alpha\}$ be an open cover of $K$ in $X$, so $G_\alpha$ is open in $X$ (i.e., $K \subset \bigcup G_\alpha$). Take $V_\alpha = Y \cap G_\alpha$ - open in $Y$. Then $\bigcup V_\alpha = \bigcup (Y \cap G_\alpha) = Y \cap (\bigcup_\alpha G_\alpha) = Y \cap (\bigcup_\alpha G_\alpha) \supset K$ and $K \subset Y \implies \{V_\alpha\}$ is an open cover of $K$ in $Y$. But $K$ is compact in $Y$, so $\exists \alpha_1, \dots, \alpha_n$ such that $K \subset \bigcup_{i=1}^n V_{\alpha_i} = \bigcup_{i=1}^n (Y \cap G_{\alpha_i}) = Y \cap (\bigcup_{i=1}^n G_{\alpha_i}) \subset \bigcup_{i=1}^n G_{\alpha_i} \implies K$ is compact in $X$.

**Theorem.** *If $K \subset X^{m.s.}$ is compact, then $K$ is closed.*

**Proof.** Let $K \subset X$ be compact. We will show $K^c$ is open. Let $x \in K^c$. For $y \in K$, let $V_y = N_{\frac{1}{2}\rho(x,y)}(y)$ - neighborhood of $y$ with radius $\frac{1}{2}\rho(x,y)$. Then $K \subset \bigcup_{y \in K} V_y$ i.e., $\{V_y\}_{y \in K}$ is an open cover of $K$. $K$ compact $\implies \exists y_1, y_2, \dots, y_n \in K$ such that $K \subset V_{y_1} \cup \dots \cup V_{y_n}$. Set $W_{y_i} := N_{\frac{1}{2}\rho(x,y_i)}(x)$ - neighborhood of $x$ with radius $\frac{1}{2}\rho(x,y_i)$ for $i = 1, \dots n$ and set $W := \bigcap_{i=1}^n W_{y_i} \Rightarrow$ open. Also, $V_{y_i} \cap W = \emptyset$ for all $i$, and $K \subset V_{y_1} \cup \dots \cup V_{y_n} \subset \bigcup_{i=1}^n (V_{y_i} \cap W) \subset K^c \implies K^c$ is open. Therefore, $K$ is closed.

**Theorem.** *Let $F \subset K \subset X^{m.s.}$. If $F$ is closed and $K$ is compact, then $F$ is compact.*

**Proof.** Let $\{V_\alpha\}$ be an open cover of $F$ i.e., $F \subset \bigcup V_\alpha$. Since $F$ is closed, $F^c$ is open. So $K \subset (\bigcup V_\alpha) \cup F^c$ - an open cover of $K$. $K$ compact $\implies \exists \alpha_1, \dots, \alpha_n$ such that $K \subset V_{\alpha_1} \cup \dots \cup V_{\alpha_n} \cup F^c$. This implies $F \subset V_{\alpha_1} \cup \dots \cup V_{\alpha_n} \cup F^c$ and $F \cap F^c = \emptyset \implies F \subset V_{\alpha_1} \cup \dots \cup V_{\alpha_n} \implies F$ is compact.

*Corollary.*—$F \subset X^{m.s.}$ closed and $K \subset X^{m.s.}$ compact, then $F \cap K$ is compact.

**Proof.** $K$ compact $\implies K$ closed. $F$ and $K$ closed $\implies F \cap K$ is closed. $F \cap K \subset K$ and $K$ compact $\implies F \cap K$ is compact.

**Theorem.** *Let $\{K_\alpha\}$ be a collection of nonempty compact sets of $X$. If every finite intersection of this collection is nonempty, then $\bigcap K_\alpha \neq \emptyset$.*

**Proof.** By contradiction, suppose $\bigcap K_\alpha = \emptyset$. Let $K_1 \subset \{K_\alpha\}$ be fixed. Suppose if $x \in K_1$ then $x \notin K_\alpha$ for some $\alpha$. That is, if $x \in K_1$ then $x \in K_\alpha^c$ for some $\alpha$. So, $K_1 \subset \bigcup K_\alpha^c$ - an open cover of $K_1$ but $K_1$ is compact, so $\exists \alpha_1, \dots, \alpha_n$ such that $K_1 \subset \bigcup_{i=1}^n K_{\alpha_i}^c = (\bigcap_{i=1}^n K_{\alpha_i})^c \implies K_1 \cap (\bigcap_{i=1}^n K_{\alpha_i}) = \emptyset \implies K_1 \cap K_{\alpha_1} \cap \dots \cap K_{\alpha_n} = \emptyset$, a contradiction to finite interaction being nonempty. So $\bigcap K_\alpha \neq \emptyset$.

**Theorem.** *Assume $K \subset \mathbb{R}^k$ is a compact set, and $E \subset K$ is an infinite subset. Then, $E$ necessarily contains a limit point within $K$.*

**Proof.** Let us assume, for the sake of contradiction, that $K$ does not possess a limit point of $E$. This implies that for each $q \in K$, there exists a neighborhood $V_q$ of $q$ containing at most one point $q$ from $E$ (if $q \in E$). Consequently, we have $E \subset \bigcup_{q \in E} V_q$ and $K \subset \bigcup_{q \in K} V_q$. Since $E$ is infinite and $V_q$ contains at most one point of $E$, no finite subcover of $E$ exists, contradicting the compactness of $K$. Thus, we conclude that $E$ must have a limit point in $K$.

**Theorem.** *For a sequence of closed intervals $\{I_n\}$ in $\mathbb{R}$ such that $I_n \supset I_{n+1}$ $(n = 1, 2, 3, \dots)$, it follows that $\bigcap_n I_n \neq \emptyset$.*

**Proof.** Consider $I_n := [a_n, b_n]$ for $n = 1, \dots,$ and define $E := \{a_n : n = 1, 2, \dots\}$. As $E$ is nonempty and bounded above, let $\sup E \in \mathbb{R}$ be denoted as $x := \sup E$. Then, $x \geq a_n$ for all $n \in \mathbb{N}$. To establish $x \leq b_m$ for all $m \in \mathbb{N}$, observe that $a_n \leq b_m$ for all $n, m$. This implies $\sup\{a_n\} \leq b_m$ for all $m$, i.e., $x \leq b_m$ for all $m$. Hence, $x \in [a_n, b_n]$ for all $n \in \mathbb{N} \implies x \in \bigcap [a_n, b_n]$.

**Definition** ($k$-cell)**.**—Denoted as $\{\mathbf{x} = (x_1, \dots, x_k) : a_i \leq x_i \leq b_i \, (i = 1, \dots, k)\}$, where a 1-cell is an interval $[a_1, b_1]$, a 2-cell is a rectangle with corners $(a_1, a_2)$ and $(b_1, b_2)$, and a 3-cell is a box, and so on.

**Theorem.** *For a sequence of $k$-cells $\{I_n\}$ (in $\mathbb{R}^k$) satisfying $I_n \supset I_{n+1}$ for $n = 1, 2, 3, \dots,$ the intersection $\bigcap_n I_n$ is nonempty.*

**Proof.** Let $I_n := \{\mathbf{x} = (x_1, \dots, x_k) : a_{n,j} \leq x_j \leq b_{n,j} \, (j = 1, \dots, k)\}$ for $n = 1, 2, \dots$. Define $I_{n,j} = [a_{n,j}, b_{n,j}] \subset \mathbb{R}$ for a fixed $j$. For each fixed $j$, $\{I_{n,j}\}_n$ satisfies $I_{n,j} \supset I_{n+1,j}$ for $n = 1, 2, \dots$. According to Theorem 70, $\bigcap_n I_{n,j} \neq \emptyset$ implies $\exists x_j^* \in I_{n,j}$ for all $n = 1, 2, \dots$. Consider $\mathbf{x}^* = (x_1^*, \dots, x_k^*)$. Then, $\mathbf{x}^* \in I_n$ for all $n = 1, \dots,$ implying $\bigcap_n I_n \neq \emptyset$.

**Theorem.** *Every $k$-cell is compact.*

**Proof.** Skip.

**Example 68.**—A 1-cell or $[a, b] \subset \mathbb{R}$ is compact.

**Theorem.** *Let $E \subset \mathbb{R}^k$. The following statements are equivalent.*

*(i) $E$ is closed and bounded.*

*(ii) $E$ is compact.*

*(iii) Every infinite subset of $E$ has a limit point in $E$.*

**Remark 21.**—Some remarks:

(I) (i) $\Leftrightarrow$ (ii) in $\mathbb{R}^k$ - Heine-Borel Theorem.

(II) (ii) $\Leftrightarrow$ (iii) holds in a general metric space.

(III) (i) $\not\Rightarrow$ (ii) or (iii) in a metric space.

**Example 69.**—Consider $X = \mathbb{Q}$ and set $E := \{q \in \mathbb{Q} : 2 < q^2 < 3\}$. Then $E$ is closed and bounded but not compact in $\mathbb{Q}$.

**Example 70.**—Take $X = \mathbb{R}$ with the discrete metric. $E := \{x \in \mathbb{R} : 0 \leq x \leq 1\}$. Then $E$ is closed and bounded but not compact.

**Proof.** We will show: (i) $\implies$ (ii), (ii) $\implies$ (iii), and (iii) $\implies$ (i).

(i) $\implies$ (ii): Suppose $E \subset \mathbb{R}^k$ is closed and bounded. $E$ being bounded implies the existence of a $k$-cell $K$ such that $E \subset K$. Since $E$ is closed and $K$ is compact, $E$ is compact.

(ii) $\implies$ (iii) follows from Theorem 69.

(iii) $\implies$ (i): We will prove by contradiction. Suppose $E$ is not bounded. Then, for each $n = 1, 2, \ldots$, there exists $\mathbf{x}_n \in E$ such that $|\mathbf{x}_n|_{\mathbb{R}^k} > n$. Define $S := \{\mathbf{x}_n \in E : |\mathbf{x}_n|_{\mathbb{R}^k} > n\} \subset E$. This set $S$ is infinite and does not have a limit point in $E$, leading to a contradiction. Now, suppose $E$ is not closed. This implies $\exists \mathbf{x}_0 \in \mathbb{R}^k \backslash E$ such that $\mathbf{x}_0$ is a limit point of $E$. For each $n = 1, 2, \ldots$, there exists $\mathbf{x}_n \in E$ such that $|\mathbf{x}_n - \mathbf{x}_0|_{\mathbb{R}^k} < \frac{1}{n}$ (neighborhood of $\mathbf{x}_0$). Define $S := \{\mathbf{x}_n \in E : |\mathbf{x}_n - \mathbf{x}_0| < \frac{1}{n}\}$. This set $S$ is infinite, and $\mathbf{x}_0$ is a limit point of $S$, but $\mathbf{x}_0 \notin E$. We're done if there are no other limit points of $S$. Suppose $\mathbf{y} \in \mathbb{R}^k$ and $\mathbf{x}_0 \neq \mathbf{y}$. Then, $|\mathbf{x}_0 - \mathbf{y}_0|_{\mathbb{R}_k} = |\mathbf{x}_0 - \mathbf{x}_n + \mathbf{x}_n - \mathbf{y}_0|_{\mathbb{R}^k} \Delta\leq |\mathbf{x}_0 - \mathbf{x}_n|_{\mathbb{R}^k} + |\mathbf{x}_n - \mathbf{y}|_{\mathbb{R}^k} < \frac{1}{n} + |\mathbf{x}_n - \mathbf{y}|_{\mathbb{R}^k} \implies |\mathbf{x}_n - \mathbf{y}|_{\mathbb{R}^k} > |\mathbf{x}_0 - \mathbf{y}|_{\mathbb{R}^k} - \frac{1}{n} > \frac{1}{2}|\mathbf{x}_0 - \mathbf{y}|_{\mathbb{R}^k}$ for some $n$ large, implying $\mathbf{y}$ is not a limit point of $E$, a contradiction.

**Theorem Weierstrass.** *Every bounded infinite subset of $\mathbb{R}^k$ has a limit point in $\mathbb{R}^k$.*

**Proof.** Consider $E \subset \mathbb{R}^k$ as a bounded infinite subset. Since $E$ is bounded, $E \subset K$ for a compact $k$-cell $K$. According to Theorem 69, $E$ has a limit point in $K \subset \mathbb{R}^k$.

**Definition** (Perfect set)**.**—A subset $E \subset X$ is perfect if

(a) $E$ is closed (i.e., $E' \subset E$), and

(b) every point of $E$ is a limit point of $E$ (i.e., $E \subset E'$).

i.e., $E = E'$.

**Example 71.**—The interval $[a, b] \subset \mathbb{R}$ is a perfect set.

**Theorem.** *For a nonempty and perfect subset $P \subset \mathbb{R}^k$, $P$ is uncountable.*

**Proof.** Suppose, for the sake of contradiction, that $P$ is countable. Enumerate the elements of $P$ as $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$. As $P$ is perfect, it has an infinite number of elements. We will construct neighborhoods as follows: Let $V_1 := N_r(\mathbf{x}_1)$, where $\mathbf{x}_1 \in P$ and $P$ is perfect, implying $\mathbf{x}_1$ is a limit point of $P$. Thus, $\exists \mathbf{x} \in V_1$ such that $\mathbf{x} \in V_1 \cap P$. Since $\mathbf{x} \in V_1$ and $V_1$ is open, $\exists r' > 0$ such that $N_{r'}(\mathbf{x}) \subset V_1$. Define $r_1 := \min\{\frac{r'}{2}, \frac{|\mathbf{x} - \mathbf{x}_1|}{2}\}$. Now, let $V_2 := N_{r_1}(\mathbf{x})$. Then

- $V_2 \subset V_1$,
- $\mathbf{x}_1 \notin V_2$,
- $V_2 \cap P \neq \emptyset$.

Inductively, we construct $\{V_n\}$:

(i) $V_{n+1} \subset V_n$,

(ii) $\mathbf{x}_n \notin V_{n+1}$,

(iii) $V_{n+1} \cap P \neq \emptyset$.

Now, let $K_n := V_n \cap P$, where $V_n$ is closed and $P$ is closed. Then $K_n$ is closed and hence compact (since $V_n$ is bounded). Also, by construction $K_n \supset K_{n+1}$ for $n = 1, 2, \ldots$. This implies $\bigcap K_n \neq \emptyset$ (Theorem 68 Corollary). We know $\mathbf{x}_n \notin V_{n+1}$. So $\mathbf{x}_n \notin K_{n+1}$ and so $\mathbf{x}_n \notin \bigcap_n K_n \subset P$ and $\mathbf{x}_n \in P$. Therefore $\bigcap K_n = \emptyset$, a contradiction. So $P$ is uncountable.

**Definition** (Cantor set)**.**—An example of a perfect set in $\mathbb{R}$ that contains no segment.

$$\begin{aligned} &E_0 : [0, 1] \\ &E_1 : [0, \frac{1}{3}] \cup [\frac{2}{3}, 1] \\ &E_2 : [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{6}{9}, \frac{7}{9}] \cup [\frac{8}{9}, 1] \end{aligned}$$

**Note 17.**—Some notes:

- No endpoints are removed,
- $E_{n+1}$ contains only points of $E_n$ i.e., $E_n \supset E_{n+1}$,
- $P = \bigcap_n E_n \neq \emptyset$ - $P$ is the Cantor set.

Properties of $E_n$ and $P$:

1. $E_n$ is the union of $2^n$ intervals of length $\frac{1}{3^n}$.
2. $P$ is compact.
3. $P$ contains no segment.
4. $P$ is a perfect set - hence uncountable.

## § 5.3 Connected set

**Definition** (Separated sets)**.**—Sets $A, B \subset X^{m.s.}$ are separated if $A \cap \bar{B} = \emptyset$ and $\bar{B} \cap A = \emptyset$.

**Example 72.**—For $X = \mathbb{R}$ with the usual metric, $A = (0, 1)$ and $B = (1, 2)$ are separated sets. However, $A = (0, 1]$ and $B = (1, 2)$ are not separated since $A \cap \bar{B} = (0, 1] \cap [1, 2] = \{1\} \neq \emptyset$.

**Definition** (Connected set)**.**—A set $E \subset X$ is connected if $E$ is not a union of two nonempty separated sets.

**Theorem.** *$E \subset \mathbb{R}$ is connected $\Leftrightarrow$ for any $x, y \in E$ with $x < z < y$, we have $z \in E$.*

**Proof.** $\implies$ Suppose $x, y \in E$ and $\exists z \in (x, y)$ but $z \notin E$. We will show that $E$ is not connected by constructing two nonempty separated sets. Let $A := E \cap (-\infty, z)$ and $B := E \cap (z, \infty)$. Then $A \neq \emptyset$ since $x \in E \cap (-\infty, z)$, and $B \neq \emptyset$ since $y \in E \cap (z, \infty)$. Now, $A \subset (-\infty, z]$ and $B \subset [z, \infty)$. So, $A \cap B = \emptyset$ and $B \cap A = \emptyset$. Also, $E = A \cup B$. Therefore, $E$ is a union of two nonempty separated sets and hence not connected.

$\Leftarrow$ Suppose $E$ is not connected. We will show that if $x, y \in E$, $\exists z \in (x, y)$ such that $z \notin E$. Since $E$ is not connected, there exist nonempty sets $A$ and $B$ such that $E = A \cup B$, $A \cap B = \emptyset$, and $A \cap B = \emptyset$. Let $x \in A$, $y \in B$, and WLOG $x < y$. Define $z := \sup(A \cap [x, y])$. Then $z$ is well-defined since $x \in A \cap [x, y]$ and $A \cap [x, y]$ is bounded by $y$. By Theorem 2.28 (book), $z \in A \cap [x, y]?{=}A \cap [x, y] = A \cap [x, y] \implies z \in A \implies z \notin B$ since $A \cap B = \emptyset$. This implies $z \neq y$ (since $y \in B$), but the least upper bound $z \leq y$, so $z < y$. If $z \notin A$, then $z \neq x$ ($x \in A$). This means $z > x$. Combining $x < z < y$. Since $z \notin A$ and $z \notin B$, $z \notin E = A \cup B$. We're done. If $z \in A$, then we repeat the argument with $z$ instead of $x$ and $y \in B$ to find $z_1 \in (z, y)$ such that $z_1 \notin E$.

## § 6 Numerical sequences and series

**Definition** (Convergent sequences)**.**—Let $(X, \rho)$ be a metric space. A sequence $\{p_n\} \subset X$ is said to be convergent to $p \in X$ if for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\rho(p_n, p) < \epsilon$ for all $n \geq N$.

**Note 18.**—The index $N$ depends on both $\epsilon$ and $p$.

Notation: If $\{p_n\}$ converges to $p \in X$, we write $p_n \underset{n\to\infty}{\longrightarrow} p$ or $\lim_{n\to\infty} p_n = p$.

**Example 73.**—For $\{\frac{1}{n} : n = 1, 2, \dots\} \subset \mathbb{R}$, $\frac{1}{n} \underset{n\to\infty}{\longrightarrow} 0$.

**Proof.** Let $\epsilon > 0$ be given. We want to show: $\exists N \in \mathbb{N}$ such that $|\frac{1}{n} - 0| < \epsilon$ for all $n \geq N$. Take $N > \frac{1}{\epsilon}$. Then for $n \geq N$, $|\frac{1}{n} - 0| = \frac{1}{n} < \epsilon$. So $\frac{1}{n} \underset{n\to\infty}{\longrightarrow} 0$.

**Example 74.**—$\{\frac{1}{n}\}_n$ does not converge in $(0, 1)$. - space matters.

**Example 75.**—$\{\frac{1}{n}\}_n$ does not converge in $\mathbb{R}$ with discrete metric.

**Theorem.** *Let $\{p_n\} \subset (X, \rho)^{m.s.}$. Then*

*(a) $p_n \underset{n\to\infty}{\longrightarrow} p$ in $X$ $\Leftrightarrow$ every neighborhood of $p$ contains all but finitely many points of $\{p_n\}$.*

*(b) If $p_n \underset{n\to\infty}{\longrightarrow} p$ and $p_n \underset{n\to\infty}{\longrightarrow} p'$, then $p' = p$ (uniqueness of limit).*

*(c) If $\{p_n\}$ converges, then $\{p_n\}$ is bounded.*

*(d) If $E \subset X$ and $p \in E'$, then there exists a sequence $\{p_n\} \subset E$ such that $p_n \underset{n\to\infty}{\longrightarrow} p$.*

**Proof.** Cases:

(a) $p_n \underset{n\to\infty}{\longrightarrow} p \iff$ every neighborhood of $p$ contains all but finitely many points of $\{p_n\}$.

$\implies$ Suppose $p_n \underset{n\to\infty}{\longrightarrow} p$. Let $N_r(p) = \{x \in X : \rho(x,p) < r\}$ be any neighborhood of $p$. Since $p_n \underset{n\to\infty}{\longrightarrow} p$, for $\epsilon = r$, $\exists N \in \mathbb{N}$ such that $\rho(p_n, p) < \epsilon = r$ for all $n \geq N$. This implies $N_r(p)$ contains all except possibly $N-1$ elements of $\{p_n\}$.

$\Leftarrow$ Let $\epsilon > 0$. Need to show: $\exists N \in \mathbb{N}$ such that $\rho(p_n, p) < \epsilon$ for all $n \geq N$. By assumption, $\epsilon$-neighborhood $N_\epsilon(p)$ contains all but finitely many points of $\{p_n\}$, i.e., $\exists N \in \mathbb{N}$ such that $\rho(p_n, p) < \epsilon$ for all $n \geq N$. This implies $p_n \underset{n\to\infty}{\longrightarrow} p$.

(b) $p_n \underset{n\to\infty}{\longrightarrow} p$ and $p_n \underset{n\to\infty}{\longrightarrow} p' \implies p' = p$.

Claim: $\rho(p,p') = 0 \implies p' = p$. We will show: $\rho(p,p) < \epsilon$ for any $\epsilon > 0$. Let $\epsilon > 0$. Since $p_n \underset{n\to\infty}{\longrightarrow} p \implies \exists N_1 \in \mathbb{N}$ such that $\rho(p_n, p) < \frac{\epsilon}{2}$ for all $n \geq N_1$. Also $p_n \underset{n\to\infty}{\longrightarrow} p' \implies \exists N_2 \in \mathbb{N}$ such that $\rho(p_n, p') < \frac{\epsilon}{2}$ for all $n \geq N_2$. Then $\rho(p,p') \leq \rho(p,p_n) + \rho(p_n,p') < \frac{\epsilon}{2} + \frac{\epsilon}{2}$ for all $n \geq N = \max\{N_1, N_2\}$. This implies $\rho(p,p') < \epsilon$. Since $\epsilon > 0$ is arbitrary, $\rho(p,p') = 0$ and hence $p = p'$.

(c) If $\{p_n\}$ converges, then $\{p_n\}$ is bounded ($\exists N_r(p) \supset \{p_n\}$).

$\{p_n\}$ converges $\implies \exists p \in X$ such that $p_n \underset{n\to\infty}{\longrightarrow} p$. $p_n \underset{n\to\infty}{\longrightarrow} p \implies$ for $\epsilon = 1$, $\exists N \in \mathbb{N}$ such that $\rho(p_n, p) < 1$ for all $n \geq N$. Set $r := \max\{1, \rho(p_1,p), \rho(p_2,p), \ldots, \rho(p_{n-1},p)\}$. Then $p_n \in \mathbb{N}$ for all $n \in \mathbb{N}$ implies that $\{p_n\}$ is bounded.

(d) $E \subset X$ and $p \in E' \implies \exists \{p_n\} \subset E$ such that $p_n \underset{n\to\infty}{\longrightarrow} p$.

$p$ is a limit point of $E \implies$ every neighborhood of $p$ contains at least one point of $p$. This implies for each $n = 1, 2, \ldots$, $\exists p_n \in E$ such that $\rho(p_n, p) < \frac{1}{n}$. Now we will show: $p_n \underset{n\to\infty}{\longrightarrow} p$. Let $\epsilon > 0$. Then for $N > \frac{1}{\epsilon}$, $\rho(p_n, p) < \frac{1}{n} \leq \frac{1}{N} < \epsilon$ for all $n \geq N$, which implies $p_n \underset{n\to\infty}{\longrightarrow} p$.

Done.

**Theorem.** *Let $\{s_n\}, \{t_n\} \subset \mathbb{C}$. Then if $s_n \underset{n\to\infty}{\longrightarrow} s$ and $t_n \underset{n\to\infty}{\longrightarrow} t$:*

*(a)* $s_n + t_n \underset{n\to\infty}{\longrightarrow} s + t$

*(b)* $c \cdot s_n \underset{n\to\infty}{\longrightarrow} c \cdot s$ *for any* $c \in \mathbb{C}$

*(c)* $s_n \cdot t_n \underset{n\to\infty}{\longrightarrow} s \cdot t$

*(d)* $\frac{1}{s_n} \underset{n\to\infty}{\longrightarrow} \frac{1}{s}$ *whenever* $s_n \neq 0$ *for all* $n$ *and* $s \neq 0$ *(*$\implies \frac{t_n}{s_n} \underset{n\to\infty}{\longrightarrow} \frac{t}{s}$ *by (c)).*

**Proof.** Cases:

(a) Let $\epsilon > 0$ be given. Need to show: $\exists N \in \mathbb{N}$ such that $|(s_n + t_n) - (s+t)|_{\mathbb{C}} < \epsilon$ for all $n \geq N$.

Then $s_n \underset{n\to\infty}{\longrightarrow} s \implies \exists N_1 \in \mathbb{N}$ such that $|s_n - s|_{\mathbb{C}} < \frac{\epsilon}{2}$ for all $n \geq N_1$. Also $t_n \underset{n\to\infty}{\longrightarrow} t \implies \exists N_2 \in \mathbb{N}$ such that $|t_n - t|_{\mathbb{C}} < \frac{\epsilon}{2}$ for all $n \geq N_2$. For $n \geq \max\{N_1, N_2\} = N$

$$|s_n + t_n - s - t|_{\mathbb{C}} \overset{\Delta}{\leq} |s_n - s|_{\mathbb{C}} + |t_n - t|_{\mathbb{C}} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

(b) For you.

(c) Let $\epsilon > 0$. Need to show: $\exists N \in \mathbb{N}$ such that $|s_n t_n - st| < \epsilon$ for all $n \geq N$.

Then $s_n \underset{n\to\infty}{\longrightarrow} s \implies \exists N_1 \in \mathbb{N}$ such that $|s_n - s|_{\mathbb{C}} < \frac{\epsilon}{2M}$ for all $n \geq N_1$ and $t_n \underset{n\to\infty}{\longrightarrow} t \implies \exists N_2 \in \mathbb{N}$ such that $|t_n - t|_{\mathbb{C}} < \frac{\epsilon}{2(|s|+1)}$ for all $n \geq N_2$. For $n \geq \max\{N_1, N_2\} = N$

$$\begin{aligned} |s_n t_n - st|_{\mathbb{C}} &\leq |s_n - s|_{\mathbb{C}} \cdot |tn|_{\mathbb{C}} + |s|_{\mathbb{C}} \cdot |t_n - t|_{\mathbb{C}} \\ &\leq M|s_n - s|_{\mathbb{C}} + |s| \cdot |t_n - t|_{\mathbb{C}} \\ &< M \cdot \frac{\epsilon}{2M} + |s| \cdot \frac{\epsilon}{2(|s|+1)} < 1 \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \end{aligned}$$

which implies that $s_n t_n \underset{n\to\infty}{\longrightarrow} st$.

(d) Let $\epsilon > 0$. Need to show: $\exists N \in \mathbb{N}$ such that $|\frac{1}{s_n} - \frac{1}{s}|_{\mathbb{C}} < \epsilon$ for all $n \geq N$.

$$\left|\frac{1}{s_n} - \frac{1}{s}\right|_{\mathbb{C}} = \frac{|s_n - s|_{\mathbb{C}}}{|s|_{\mathbb{C}} \cdot |s_n|_{\mathbb{C}}}$$

Need to control $|s_n|$. Then $s_n \underset{n\to\infty}{\longrightarrow} s \implies \exists N_1 \in \mathbb{N}$ such that $|s_n - s|_{\mathbb{C}} < \frac{|s|_{\mathbb{C}}}{2}$ for all $n \geq N_1$. Claim: For $n \geq N_1$, $|s_n|_{\mathbb{C}} > \frac{|s|_{\mathbb{C}}}{2}$. Suppose not: ($|s_n| \leq |s|/2$)

$$|s|_{\mathbb{C}} = |s - s_n + s_n|_{\mathbb{C}} \leq |s - s_n|_{\mathbb{C}} + |s_n|_{\mathbb{C}} < \frac{|s|_{\mathbb{C}}}{2} + \frac{|s|_{\mathbb{C}}}{2} = |s|_{\mathbb{C}}$$

- a contradiction. Now, $\exists N_2 \in \mathbb{N}$ such that $|s_n - s| \leq \epsilon \cdot \frac{|s|^2_{\mathbb{C}}}{2}$ for all $n \geq N_2$. Take $N = \max\{N_1, N_2\}$. Then for $n \geq N$,

$$\left|\frac{1}{s_n} - \frac{1}{s}\right|_{\mathbb{C}} = \frac{|s_n - s|_{\mathbb{C}}}{|s|_{\mathbb{C}} \cdot |s_n|_{\mathbb{C}}} < \frac{\epsilon \cdot |s|^2_{\mathbb{C}} \cdot 2}{2 \cdot |s|_{\mathbb{C}} \cdot |s|_{\mathbb{C}}} = \epsilon,$$

which implies $\frac{1}{s_n} \underset{n\to\infty}{\longrightarrow} \frac{1}{s}$.

**Theorem.** *Cases:*

*(a) Suppose $(X, \rho)$ is a compact metric space. If $\{p_n\} \subset X$, then $\exists$ a subsequence $\{p_{n_k}\}$ of $\{p_n\}$ such that $\{p_{n_k}\}$ converges.*

*(b) Every bounded sequence in $\mathbb{R}^k$ has a convergent subsequence. ($\{p_n\}$ bounded in $\mathbb{R}^k \implies \exists$ a compact $k$-cell $K$ such that $\{p_n\} \subset K(a) \Longrightarrow \exists$ a convergent subsequence.)*

**Definition** (Range of a sequence)**.**—$\{p_n\} := \{p_n : n \in \mathbb{N}\}$. Eg: (1) $p_n = (-1)^n \implies$ Range $= \{-1, 1\}$ - finite $\implies$ closed. (2) $p_n = \frac{1}{n} \implies$ Range $= \{\frac{1}{n}\}$.

**Proof.** Case 1: Suppose the range of $\{p_n\}$, say $E$, is finite. Then $\exists p \in \{p_n\}$ such that $p_{n_1} = p_{n_2} = \cdots = p_{n_k} = p \implies \exists \{p_{n_k}\}$ of $\{p_n\}$ such that $p_{n_k} p$.

Case 2: $E$ is infinite. Since $E \subset X^{\text{compact}}$ is infinite, $E$ has a limit $p$ in $X$.

**Note 19.**—While every neighborhood of $p$ contains infinitely many points of $E$, this may not be a subsequence.

$p \in E'$ and $E$ is infinite, so $\exists p_{n_1} \in E$ such that $\rho(p, p_{n_1}) < 1$. Next, $\exists p_{n_2} \in E$ with $n_1 < n_2$ such that $\rho(p, p_{n_2}) < \frac{1}{2}$. Inductively, $\exists p_{n_k}$ with $n_1 < n_2 < \cdots < n_k$ such that $\rho(p, p_{n_k}) < \frac{1}{2^{k-1}}? \Longrightarrow p_{n_k} p$ as $k \infty$.

Notation: $E^*$ = set of all subsequential limits $= \{p \in X : \exists \{p_{n_k}\}$ such that $p_{n_k} p\}$.

Eg: $\{(-1)^n\} \implies E^* = \{-1, 1\}$.

**Theorem.** *Let $\{p_n\} \subset X$ and $E^* = \{p \in X : \exists \{p_{n_k}\}$ such that $p_{n_k} p\}$.*

**Proof.** We will show $(E^*)' \subset E^*$. Let $q \in (E^*)'$ i.e, $q$ is a limit point of $E^*$. We will construct a subsequence $\{p_{n_k}\}$ such that $p_{n_k} q$. Let $n_1 \in \mathbb{N}$ be such that $p_{n_1} \neq q$ (if no such index exists then $E^* = \{q\}$ - closed). Set $\delta := \rho(q, p_{n_1}) > 0$ because $p_{n_1} \neq q$. Let $n_2 > n_1$ such that $\rho(p_{n_2}, q) < \frac{\delta}{2}$. Inductively, let $n_1 < n_2 < \cdots < n_k$ such that $\rho(p_{n_k}, q) < \frac{\delta}{2^{k-1}}$. This implies $p_{n_k} q$ as $k \infty \implies q \in E^* \implies E^*$ is closed.

## § 6.1 Cauchy sequence

**Definition.**—For a metric space $(X, \rho)$, a sequence $\{p_n\}_n \subset X$ is Cauchy if $\forall \epsilon > 0$, $\exists N = N(\epsilon) > 0$ such that $\rho(p_n, p_m) < \epsilon$ for all $n, m \geq N$.

**Example 76.**—Consider $X = \mathbb{R}$ with the usual metric. The sequence $\{\frac{1}{n}\}_n$ is a Cauchy sequence.

**Proof.** Let $\epsilon > 0$ be fixed. We want to show that there exists $N \in \mathbb{N}$ such that $\left|\frac{1}{n} - \frac{1}{m}\right| < \epsilon$ for all $n, m \geq N$. Using the triangle inequality, we have

$$\left|\frac{1}{n} - \frac{1}{m}\right| \leq \left|\frac{1}{n}\right| + \left|\frac{-1}{m}\right| = \frac{1}{n} + \frac{1}{m} < \frac{\epsilon}{2} + \frac{\epsilon}{2}$$

for $n, m > \frac{2}{\epsilon} = N$. This implies $\{\frac{1}{n}\}_n$ is Cauchy.

Examples of Cauchy sequences that do not converge:

1. $s_n = (1+\frac{1}{n})^n \in \mathbb{Q}$ and $s_n \to e \notin \mathbb{Q}$, so $s_n$ does not converge in $\mathbb{Q}$.
2. $\{\frac{1}{n}\}_{n\geq 2} \subset (0,1)$ is Cauchy but does not converge in $(0,1)$.

Done.

**Definition.**—For $E \subset X^{m.s.}$, the diameter of $E$ is defined as $\operatorname{diam} E = \sup \rho(p,q)$ for $p, q \in E$.

**Example 77.**—For $X = \mathbb{R}$, if $E = (0,1)$, then $\operatorname{diam} E = 1$. For $X = \mathbb{R}^2$ and $E = [0,1]\times[0,1]$, we have $\operatorname{diam} E = \sqrt{2}$.

Given $\{p_n\} \subset X^{m.s.}$, let $E_n := \{p_n, p_{n+1}, \dots\}$. Then $\{p_n\}$ is Cauchy if and only if $\lim_{n\to\infty} \operatorname{diam} E_n = 0$.

**Theorem.** *Cases:*

*(a) If $E \subset (X,\rho)$, then $diam\, E = diam\, \overline{E}$.*

*(b) If $\{K_n\}_n \subset X$ is a sequence of compact sets such that*

- *$K_n \supset K_{n+1}$, and*
- *$\lim_{n\to\infty} diam\, K_n = 0$,*

*then $\bigcap K_n$ contains exactly one point.*

**Proof.** Cases:

(a) $\geq$ Note that if $A \subset B$, then $\operatorname{diam} A \leq \operatorname{diam} B$. Then since $E \subset \overline{E}$, we have $\operatorname{diam} E \leq \operatorname{diam} \overline{E}$.

$\leq$ We will show: $\operatorname{diam} \overline{E} \leq \operatorname{diam} E + \epsilon$ for all $\epsilon > 0$. Let $\overline{p}, \overline{q} \in \overline{E}$.

(i) If $\overline{p}, \overline{q} \in E$, then
$$\sup_{\overline{p},\overline{q}\in\overline{E}} \rho(\overline{p},\overline{q}) = \sup_{p,q\in E} \rho(p,q).$$

(ii) Suppose $\overline{p}, \overline{q} \in E'$. Then $\exists p, q \in E$ such that $\rho(\overline{p}, p) \leq \frac{\epsilon}{2}$ and $\rho(\overline{q}, q) \leq \frac{\epsilon}{2}$. Then
$$\rho(\overline{p},\overline{q}) \leq \rho(\overline{p},p) + \rho(p,q) + \rho(q,\overline{q}) \leq \epsilon + \rho(p,q).$$

(iii) Suppose $\overline{p} \in E'$ and $\overline{q} \in E$. Then $\exists p \in E$ such that $\rho(\overline{p}, p) < \frac{\epsilon}{2}$. Then
$$\rho(\overline{p},\overline{q}) \leq \rho(\overline{p},p) + \rho(p,\overline{q}) \leq \frac{\epsilon}{2} + \operatorname{diam} E \leq \epsilon + \operatorname{diam} E.$$

Then $\sup_{\overline{p},\overline{q}\in\overline{E}} \rho(\overline{p},\overline{q}) \leq \epsilon + \sup_{p,q\in E} \rho(p,q) \implies \operatorname{diam} \overline{E} \leq \epsilon + \operatorname{diam} E$. Since $\epsilon > 0$ is arbitrary, therefore $\operatorname{diam} \overline{E} \leq \operatorname{diam} E$. Hence $\operatorname{diam} \overline{E} = \operatorname{diam} E$.

(b) We have $\bigcap_n K_n \neq \emptyset$. Suppose by contradiction that there exist at least two points in $K = \bigcap_n K_n$. Then $\operatorname{diam} K > 0$ and $K \subset K_n$ for all $n \implies 0 < \operatorname{diam} K \leq \operatorname{diam} K_n \to 0$ for all $n$ - a contradiction since $\lim_{n\to\infty} \operatorname{diam} K_n = 0$.

Done.

**Theorem.** *Cases:*

*(a) If $(X, \rho)$ is a metric space, then every convergent sequence is Cauchy.*

*(b) If $(X, \rho)$ is a compact metric space, then a Cauchy sequence converges.*

*(c) In $\mathbb{R}^k$, every Cauchy sequence converges.*

**Definition.**—A metric space $(X, \rho)$ is called a complete metric space if every Cauchy sequence in $X$ converges.

**Proof.** Cases:

(a) Let $\{p_n\} \subset X$ be such that $p_n \to p$ in $X$. **WTS**: For any $\epsilon > 0$, we can find $N \in \mathbb{N}$ such that $\rho(p_n, p_n) < \epsilon$ for all $n, m \geq N$. Since $p_n \to p$, there exists $N \in \mathbb{N}$ such that $\rho(p_n, p) < \frac{\epsilon}{2}$ for all $n \geq N$. Then $\rho(p_n, p_m) \leq \rho(p_n, p) + \rho(p, p_m) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ for all $n, m \geq N$. This implies $\{p_n\}$ is Cauchy.

(b) Let $\{p_n\} \subset X^{\text{compact } m.s.}$ be Cauchy. **NTS**: There exists $p \in X$ such that $p_n \to p$. Let $E_n := \{p_n, p_{n+1}, \dots\}$. Then $\lim_{n\to\infty} \operatorname{diam} E_n = \lim_{n\to\infty} \overline{E}_n = 0$. Also, $E_n \supset E_{n+1}$ and hence $\overline{E}_n \supset \overline{E}_{n+1}$ for all $n$. Since $\overline{E}_n \subset X^{\text{compact}}$, $\overline{E}_n$ is compact. Then $\bigcap \overline{E}_n = \{p\}$ (with $p$ unique) by Theorem 81(b). **Claim**: $p_n \to p$. Let $\epsilon > 0$. Since $\lim_{n\to\infty} \operatorname{diam} E_n = 0$, there exists $N \in \mathbb{N}$ such that $\operatorname{diam} E_n < \epsilon$ for all $n \geq N$. Now $p \in \overline{E}_n$ for $n \geq N \implies \rho(p, q) < \epsilon$ for all $q \in \overline{E}_n$ $(\subset E_n) \implies \rho(p, p_n) < \epsilon$ for all $n \geq N \implies p_n \to p$.

(c) Let $\{\mathbf{x}_n\} \subset \mathbb{R}^k$ be Cauchy, $E_n := \{\mathbf{x}_n, \mathbf{x}_{n+1}, \dots\}$. Then $\{\mathbf{x}_n\} \subset \mathbb{R}^k$ Cauchy $\implies \lim_{n\to\infty} \operatorname{diam} E_n = 0 \implies \exists N \in \mathbb{N}$ such that $\operatorname{diam} E_n < 1$ for all $n \geq N$. Define $R := \max\{|\mathbf{x}_1|, \dots, |\mathbf{x}_{N-1}|, 1\} + 1$. Then $\{\mathbf{x}_n\}_n \subset B_R(\mathbf{0})$. This implies $\{\mathbf{x}_n\}$ is bounded $\implies \exists$ a compact $k$-cell $K$ such that $\{\mathbf{x}_n\} \subset K$. By part (b), $\{\mathbf{x}_n\}$ converges in $K$, hence in $X$.

Done.

**Definition.**—A sequence $\{s_n\}$ is defined as follows:

1. It is monotonically increasing if $s_n \leq s_{n+1}$ for all $n$.
2. It is monotonically decreasing if $s_n \geq s_{n+1}$ for all $n$.

A sequence is said to be monotonic if it is either monotonically increasing or monotonically decreasing.

**Theorem.** *Suppose $\{s_n\} \subset \mathbb{R}$ is monotonic. Then $\{s_n\}$ converges if and only if $\{s_n\}$ is bounded.*

**Proof.** $\Longrightarrow$ If $\{s_n\}$ converges, then by Theorem 77, $\{s_n\}$ is bounded. Suppose $\{s_n\}$ is bounded and assume it is monotonically increasing. Let $E = \text{range}\{s_n : n \in \mathbb{N}\}$. The nonempty and bounded set $E$ implies that $s := \sup E \in \mathbb{R}$. We will show that $s_n \to s$. Let $\epsilon > 0$ be fixed. There exists $N \in \mathbb{N}$ such that $s - \epsilon < s_N < s$. The monotonically increasing property implies $s - \epsilon < s_N \leq s_n < s$ for all $n \geq N$. This implies $s_n \to s$.

**Definition** (Upper and Lower limits ($\liminf$ and $\limsup$))**.**—Let $E$ be the set of all subsequential limits of $\{s_n\} \subset \mathbb{R}$, i.e., $E := \{s \in \mathbb{R} : \exists \{s_{s_k}\} \subset \{s_n\}$ such that $s_{n_k} \to s\}$. Define $s^* := \sup E$ and $s_* := \inf E$, where $s^*, s_* \in \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$.

**Example 78.**—Consider the sequence $s_n = 3-\frac{1}{2}, 5+\frac{1}{3}, 3-\frac{1}{4}, 5+\frac{1}{5}, \ldots$. The set $E = \{3, 5\}$, and $s^* = 5, s_* = 3$. Define $\inf\{s_n : n \geq 1\} = 3 - \frac{1}{2} = y_1$, $\inf\{s_n : n \geq 2, 3\} = 3 - \frac{1}{4} = y_2 = y_3$, and so on. Note that $\{y_k\}$ is a monotonically increasing sequence bounded above by 5, implying convergence. Therefore, $s_* = \sup\{\inf\{s_n : n \geq k\} : k \in \mathbb{N}\}$.

**Recall 22.**—If $A \subset B$, then $\inf A \geq \inf B$ and $\sup A \leq \sup B$.

In general, $s_* = \sup\{\inf\{s_n : n \geq k\} : k \in \{1, \ldots\}\} = \lim_{k\to\infty}(\inf\{s_n : n \geq k\})$ and $s^* = \sup\{\inf\{s_n : n \geq k\} : k \in \{1, 2, \ldots\}\} = \lim_{k\to\infty}(\sup\{s_n : n \geq k\})$. In the book, $s^*, s_* \in E$. Notation: $s^* = \sup E = \limsup_{n\to\infty} s_n$ and $s_* = \inf E = \liminf_{n\to\infty} s_n$.

**Note 20.**—$s_* = \inf E \leq \sup E = s^* \implies \liminf\limits_{n\to\infty} s_n \leq \limsup\limits_{n\to\infty} s_n$.

**Theorem.** *Let $\{s_n\} \subset \mathbb{R}$. Then $\lim_{n\to\infty} s_n = s \iff \liminf_{n\to\infty} s_n = \limsup_{n\to\infty} s_n = s$ (also holds when $s = \pm\infty$).*

**Proof.** $\Longrightarrow$ Suppose $s_n \to s$. We will show:

$$\liminf_{n\to\infty} \checkmark \leq \limsup_{n\to\infty} s_n \leq \lim_{n\to\infty} s_n \leq \liminf_{n\to\infty} s_n \leq \limsup_{n\to\infty} s_n \checkmark$$

1. We will show (1): Idea: show $\limsup_{n\to\infty} s_n \leq \lim_{n\to\infty} s_n + \epsilon$ for $\epsilon > 0$ arbitrary. Let $\epsilon > 0$ be fixed. $s_n \to s \implies \exists N \in \mathbb{N}$ such that $|s_n - s| < \epsilon$ for all $n \geq N \implies -\epsilon < s_n - s < \epsilon$ for all $n \geq N \implies s_n < s + \epsilon$ for all $n \geq N \implies \sup\{s_n : n \geq N\} \leq s + \epsilon$. If $m > N$, then $\sup\{s_n : n \geq m\} \leq \sup\{s_n : n \geq N\}$. This means $\sup\{s_n : n \geq m\} \leq s + \epsilon \implies \lim_{n\to\infty}(\sup\{s_n : n \geq m\}) \leq s + \epsilon \implies \limsup_{m\to\infty} s_m \leq s + \epsilon \implies \limsup_{m\to\infty} s_m \leq s = \lim_{m\to\infty} s_m$.

2. Repeat the argument for (2). Then $\liminf_{n\to\infty} s_n = \limsup_{n\to\infty} s_n = \lim_{n\to\infty} s_n$ – $(*)$.

Suppose $(*)$ holds. We need to show $\lim_{n\to\infty} s_n = \underline{\underline{s}}$. Let $\epsilon > 0$ be given. $\limsup_{n\to\infty} s_n = s \implies \lim_{k\to\infty}(\sup\{s_n : n \geq k\}) = s \implies \exists \underline{\underline{K}}_1 \in \mathbb{N}$ such that $|y_k - s| < \epsilon$ for all $k \geq K_1$. This implies $-\epsilon < y_k - s < \epsilon$ for all $k \geq K_1$, which implies $y_k < s + \epsilon$ for all

$k \geq K_1 \implies \sup\{s_n : n \geq K\} < s + \epsilon$ for all $k \geq K_1 \implies s_n < s + \epsilon$ for all $n \geq K_1$ – (3). Again, $\liminf_{n\to\infty} s_n = s \implies \lim_{k\to\infty}(\inf\{s_n : n \geq k\}) = s \implies \exists K_2 \in \mathbb{N}$ such that $|z_k - s| < \epsilon$ for all $k \geq K_2$. This implies $-\epsilon < z_k - s < \epsilon$ for all $k \geq K_2 \implies z_k - s > -\epsilon$ or $s_k > s - \epsilon$ for all $k \geq K_2 \implies \inf\{s_n : n \geq k\} > s - \epsilon$ for all $k \geq K_2 \implies s_n > s - \epsilon$ for all $n \geq K_2$ – (4). For $K \geq \max\{K_1, K_2\}$, $s - \epsilon < s_n < s + \epsilon$ or $-\epsilon < s_n - s < \epsilon \implies s_n \to s$.

**Theorem Theorem 3.20 book.** *Important examples - check!*

## § 6.2 Series - of real numbers

Let $\{a_n\} \subset \mathbb{R}$ be a sequence. Define

$$\begin{aligned} s_1 &:= a_1 \\ s_2 &:= a_1 + a_2 \\ s_3 &:= a_1 + a_2 + a_3 \\ &\vdots \\ s_n &:= a_1 + a_2 + \cdots + a_n \end{aligned}$$

- $n^{\text{th}}$ partial sum of? Then $\{s_n\} \subset \mathbb{R}$ is well defined. If $s_n \to s$ in $\mathbb{R}$, we say $\sum_{n=1}^{\infty} a_n = s$ and that $\sum_{n=1}^{\infty} a_n$ converges. Otherwise, $\sum_{n=1}^{\infty} a_n$ diverges.

**Recall 23.**—$\{s_n\} \subset \mathbb{R}$ Cauchy $\iff$ $\{s_n\} \subset \mathbb{R}$ is convergent.

**Theorem.** *$\sum_{n=1}^{\infty} a_n$ converges $\iff$ $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that $|s_n - s_{m-1}| < \epsilon$ for every $m \geq n > N$, where $s_n := a_1 + \cdots + a_n$.*

**Note 21.**—Taking $m = n$, we get $|a_m| < \epsilon$ for all $m > N \implies a_m \to 0$ as $m \to \infty$, i.e., if $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = 0$. ($\implies$ if $\lim_{n\to\infty} a_n \neq 0$, then $\sum a_n$ diverges.)

**Example 79.**—The series $\sum \frac{1}{n}$ diverges, and $\lim_{n\to\infty} \frac{1}{n} = 0$.

**Theorem.** *Let $a_n \geq 0$ for all $n \in \mathbb{N}$. Then $\sum_{n=1}^{\infty} a_n$ converges if and only if $s_n$ is bounded.*

**Proof.** The proof follows from the monotonic bounded sequence theorem since $s_n = a_1 + \cdots + a_n \leq a_1 + \cdots + a_{n+1}$, i.e., $s_n \leq s_{n+1}$ (monotonically increasing).

**Theorem Comparison test.** *Cases:*

*(a) If $|a_n| \leq c_n$ for some $n \geq N$ for some $N \in \mathbb{N}$, and if $\sum_{n=1}^{\infty} c_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.*

*(b) If $0 \leq d_n \leq a_n$ for $n \geq N$ for some $N \in \mathbb{N}$, and if $\sum_{n=1}^{\infty} d_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.*

**Proof.** Cases:

(a) Let $\epsilon > 0$ be fixed. NTS: $\exists N_0 \in \mathbb{N}$ such that $|\sum_{k=m}^{n} a_k| < \epsilon$ for all $n \geq m > N_0$. Then $\sum_{n=1}^{\infty} c_n$ converges $\implies$ $\exists N \in \mathbb{N}0$ such that $\sum k = m^n c_k = |\sum_{k=m}^{n} c_k| < \epsilon$ for all $n \geq m > N_0$. Given $a_n \leq |a_n| \leq c_n$ for $n \geq N$. Then $|\sum_{k=m}^{n} a_k| \Delta \leq \sum_{k=m}^{n} |a_k| \leq \sum_{k=m}^{n} c_k < \epsilon$ for $n \geq m > N_2 = \max N, N_0$. This implies $\sum_{k=1}^{\infty} a_k$ converges.

(b) If $\sum a_n$ converges, then by (a), since $0 \leq d_n \leq a_n$, $\sum_{n=1}^{\infty} d_n$ converges, contradiction. Hence the proof.

Done.

**Theorem Geometric series.** *Let $0 \leq x < 1$. Then $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$. If $x \geq 1$ then $\sum_{n=0}^{\infty} x^n$ diverges.*

**Proof.** Assume $x \neq 1$. Then $\sum_{k=0}^{n} x^k = 1 + x + x^2 + \cdots + x^n \implies (1-x)\sum_{k=0}^{n} x^k = \sum_{k=0}^{n} x^k - \sum_{k=1}^{n+1} x^k = 1 - x^{n+1}$. Therefore, $\lim_{n\infty} \frac{1-x^{n+1}}{1-x} = \frac{1}{1-x} \lim_{n\infty}(1 - x^{n+1}) = \frac{1}{1-x}$ if $|x| < 1$ i.e., $0 \leq x < 1$.

**Theorem.** *Suppose $a_n \geq 0$ and $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$. Then $\sum_n a_n$ converges $\iff$ $\sum_k 2^k a_{2^k}$ converges.*

**Proof.** Let $s_n = a_1 + a_2 + \cdots + a_n$ and $t_n = a_1 + 2a_2 + 2^2 a_{2^2} + \cdots + 2^k a_{2^k}$. **Claim:** $\{s_n\}$ is bounded $\iff$ $\{t_k\}$ is bounded. **Case**: $n < 2^k$.

$$\begin{aligned} s_n &\leq a_1 + a_2 + \underbrace{\leq a_2}_{a_3} + a_4 + \underbrace{\leq a_4}_{a_5} + \underbrace{\leq a_4}_{a_6} + \underbrace{\leq a_4}_{a_7} + a_8 + \underbrace{= a_n}_{\cdots} + a_{2^k} + \underbrace{\leq a_{2^k}}_{\cdots} + a_{2^{k+1}-1} + \cdots + a_n \\ &\leq a_1 + 2a_2 + 2^2 a_{2^2} + \cdots + 2^k a_{2^k} = t_k \implies \underline{s_n \leq t_k}. \end{aligned}$$

**Case**: $n > 2^k$.

$$\begin{aligned} s_n &= a_1 + a_2 + \underbrace{\geq a_4}_{a_3} + a_4 + \underbrace{\geq a_8}_{a_5} + \underbrace{\geq a_8}_{a_6} + \underbrace{\geq a_8}_{a_7} + \underbrace{\geq a_8}_{a_8} + \cdots + a_{2^{k-1}} + \cdots + a_{2^k} + \cdots + a_n \\ &\geq \frac{1}{2} a_1 + a_2 + 2a_4 + 4a_8 + \cdots + 2^{k-1} a_{2^k} \implies 2s_n \geq a_1 + 2a_2 + 2^2 a_{2^2} + 2^3 a_{2^3} + \cdots + 2^k a_{2^k} = t_k. \end{aligned}$$

This implies $\underline{s_n \geq \frac{t_k}{2} \text{ for } n > 2^k}$ $\implies$ $\{s_n\}$ is bounded $\iff$ $\{t_k\}$ is bounded.

**Recall 24.**—Suppose $a_n \geq 0$ and $a_n \geq a_{n+1}$ for all $n$. Then $\sum_n a_n$ converges $\iff$ $\sum 2^k a_{2^k}$ converges.

**Theorem.** *p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$, and diverges if $p \leq 1$.*

**Proof.** If $p \leq 0$, then $\lim_{n\infty} \frac{1}{n^p} \neq 0$, $\sum \frac{1}{n^p}$ diverges. If $p > 0$, then $\frac{1}{n} \geq \frac{1}{n+1} \implies \frac{1}{n^p} \geq \frac{1}{(n+1)^p}$ by monotonicity of power $p$ $\implies$ $\{\frac{1}{n^p}\}$ is monotonically decreasing, and $\frac{1}{n^p} \geq 0$ for all $n$ (Theorem 90 can be applied). Note that $\sum_k 2^k \frac{1}{(2^k)^p} = \sum_k 2^{k(1-p)} = \sum_k [2^{(1-p)}]^k$. This series converges (geometric) if $1 - p < 0$ i.e., $p > 1$, and diverges if $1 - p \geq 0$.

**Theorem** log $p$**-series.** $\sum_{n=1}^{\infty} \frac{1}{n(\log n)^p}$ *converges if* $p > 1$ *and diverges if* $p \leq 1$.

**Proof.** We will use Theorem 90. Clearly $n(\log n)^p \geq 0$ for $n \geq 2$. Assume $\log n$ is monotonically increasing. Then using the monotonicity of power $p$, $\frac{1}{\log n} \geq \frac{1}{(\log(n+1))^p \cdot (n+1)}$ for all $n \geq 2$. We can apply Theorem 90. Then $\sum_k 2^k \cdot \frac{1}{2^k (\log 2^k)^p} = \sum_k \frac{1}{(k \log 2)^p} = \frac{1}{(\log 2)^p} \sum_k \frac{1}{k^p}$ is $p$-series that converges if $p > 1$ and diverges if $p \leq 1$ by Theorem 91.

**Definition** $(e)$**.**—$e := \sum_{n=0}^{\infty} \frac{1}{n!}$ $(0! = 1)$

We will justify that the series converges. We will show: $\{s_n\}$ is monotonically increasing and bounded above. $s_n := 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} \implies s_n \leq s_{n+1}$ for all $n$.

$$\begin{aligned}
s_n &= 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots + \frac{1}{n!} \\
&= 1 + 1 + \frac{1}{2} + \underbrace{< \frac{1}{2^2}}_{\frac{1}{2\cdot 3}} + \underbrace{< \frac{1}{2^3}}_{\frac{1}{2\cdot 3\cdot 4}} + \cdots + \underbrace{< \frac{1}{2^{n-1}}}_{\frac{1}{2\cdot 3\cdot \cdots \cdot (n-1)\cdot n}} \\
&< 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{n-1}} \\
&= 1 + \sum_{k=0}^{n-1} \frac{1}{2^k} < 1 + \sum_{k=1}^{\infty} \frac{1}{2^k} = 1 + \frac{1}{1 - \frac{1}{2}} = 3
\end{aligned}$$

This implies $\{s_n\}$ is bounded above by 3 for all $n$ or $s_n < 3$ for all $n$. Therefore $\sum_{n=0}^{\infty} \frac{1}{n!}$ converges.

**Theorem.** $\lim_{n\infty} (1 + \frac{1}{n})^n = e$.

**Proof.** Skip.

**Theorem.** $e \in \mathbb{R} \setminus \mathbb{Q}$, *where* $e = \sum_{n=0}^{\infty} \frac{1}{n!}$.

**Proof.** Suppose by contradiction that $e \in \mathbb{Q}$. Then $e = \frac{p}{q}$, where $p, q \in \mathbb{N}$ (since $e > 0$). Consider

$$\begin{aligned}
e - s_n &= \sum_{k=0}^{\infty} \frac{1}{k!} - \sum_{k=0}^{n} \frac{1}{k!} = \sum_{k=n+1}^{\infty} \frac{1}{k!} \\
&= \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \frac{1}{(n+3)!} + \dots \\
&= \frac{1}{(n+1)!} \left[ 1 + \frac{1}{n+2} + \frac{1}{(n+2)(n+3)} + \dots \right] \\
&< \frac{1}{(n+1)!} \left[ 1 + \frac{1}{n+1} + \frac{1}{(n+1)^2} + \dots \right] \\
&= \frac{1}{(n+1)!} \sum_{k=0}^{\infty} \frac{1}{(n+1)^k} = \frac{1}{(n+1)!} \sum_{k=0}^{\infty} \left( \frac{1}{n+1} \right)^k \\
&= \frac{1}{(n+1)!} \cdot \frac{1}{1 - \frac{1}{n+1}} = \frac{1}{(n+1)!} \cdot \frac{1}{1 - \frac{1}{n+1}} = \frac{1}{(n+1)n!}
\end{aligned}$$

i.e., $e-s_n < \frac{1}{n!n}$ for all $n \geq 1$. Therefore $0 < e-s_q < \frac{1}{q!q}$. This implies $0 < q!(e-s_q) < \frac{1}{q} \leq 1$ (since $q \geq 1$). Since $e = \frac{p}{q}$, $p = eq \in \mathbb{N}$, so $q!e \in \mathbb{N}$. Now $q!s_q = q!(1+1+\frac{1}{2\cdot 1}+\frac{1}{3\cdot 2\cdot 1}+\cdots+\frac{1}{q!}) \in \mathbb{N}$. Therefore, $q!(e-s_q) \in \mathbb{N}$, which is a contradiction since $0 < q!(e-s_q) < \frac{1}{q} \leq 1$. Therefore $e \in \mathbb{R} \setminus \mathbb{Q}$.

## § 6.3 The Root and Ratio test for Series

**Theorem Root test.** *Consider $\sum_n a_n$ and define $\alpha = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$. Then*

*(a) $\sum_n a_n$ converges if $\alpha < 1$.*

*(b) $\sum_n a_n$ diverges if $\alpha > 1$.*

*(c) Test is inconclusive if $\alpha = 1$.*

**Proof.** Cases:

(a) Suppose

$$\begin{aligned}\alpha &= \limsup_{n\to\infty} \sqrt[n]{|a_n|} < 1\\ &= \lim_{m\to\infty} \Big(\sup\{\sqrt[n]{|a_n|} : n \geq m\}\Big) < 1\end{aligned}$$

Let $\epsilon > 0$ such that $\alpha < \alpha + \epsilon < 1$. Then $\exists N \in \mathbb{N}$ such that $\sup\{\sqrt[n]{|a_n|} : m \geq N\} < \alpha + \epsilon \implies \sqrt[n]{|a_n|} < \alpha + \epsilon$ for all $n \geq N$. This implies $|a_n| < (\alpha+\epsilon)^n$ for all $n \geq N$. Since $0 < \alpha + \epsilon < 1$, $\sum_{n\geq N}(\alpha+\epsilon)^n$ converges as a geometric series. By comparison test, $\sum_n a_n$ converges.

(b) Suppose $\alpha = \limsup_{n\to\infty} \sqrt[n]{|a_n|} > 1$. Then $\alpha = \sup\{s \in \mathbb{R} : \exists \sqrt[n_k]{|a_{n_k}|} \to s\}$. This implies $\exists$ a subsequence $\{\sqrt[n_k]{|a_{n_k}|}\}$ such that $\sqrt[n_k]{|a_{n_k}|} \to \alpha > 1$. Infinitely many terms of the subsequence, and hence of sequence $\{\sqrt[n]{|a_n|}\}$, are greater than 1. So $\lim_{n\to\infty} \sqrt[n]{|a_n|} \neq 0$ and hence $\lim_{n\to\infty} a_n \neq 0$. So $\sum_n a_n$ diverges.

(c) Consider (i) $\sum \frac{1}{n}$ (diverges) and (ii) $\sum \frac{1}{n^2}$ (converges).

(i) $\lim_{n\to\infty} n^{\frac{1}{n}} = 1 \implies \lim_{n\to\infty} \frac{1}{n^{\frac{1}{n}}} = 1$.

(ii) $\lim_{n\to\infty} \frac{1}{n^{\frac{2}{n}}} = \lim_{n\to\infty} \frac{1}{(n^{\frac{1}{n}})^2} = 1$.

Done.

**Example 80.**—Consider $0, 0, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{3}{8}, \frac{7}{8}, \frac{7}{16}, \frac{15}{16}, \frac{15}{32}, \frac{31}{32}, \ldots$. Then $\sup\{a_n : n \geq 1\} = 1$, $\inf\{a_n : n \geq 3\} = \frac{1}{4}$.

**Example 81.**—Look at Example 3.34 in the book: $\liminf |a_n|^{\frac{1}{n}}$ and $\limsup |a_n|^{\frac{1}{n}}$.

**Theorem Ratio test.** *Consider $\sum_n a_n$ with $a_n \neq 0$ for all $n$ (enough for sufficiently large $n$). Then*

*(a) $\sum a_n$ converges if $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$.*

*(b) $\sum a_n$ diverges if $\left|\frac{a_{n+1}}{a_n}\right| \geq 1$ for $n \geq N$ (fixed $N \in \mathbb{N}$).*

*(*) If $\liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \leq 1 \leq \limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$, the test gives no information.*

**Proof.** Cases:

(a) Let $a^* := \limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$. Let $\epsilon > 0$ be such that $a^* < a^* + \epsilon < 1$. *NTS*: $\exists$ a convergent series $\sum_n b_n$ such that $|a_n| \leq b_n$ for $n$ large. Since $a^* < a^* + \epsilon < 1$, $\exists N \in \mathbb{N}$ such that $\left|\frac{a_{n+1}}{a_n}\right| < a^* + \epsilon$ for $n \geq N$. In particular, $|a_{N+1}| < |a_N|(a^* + \epsilon) \implies |a_{N+2}| < |a_{N+1}|(a^* + \epsilon) < |a_N|(a^* + \epsilon)^2$ and further

$$|a_{N+k}| < |a_N|(a^* + \epsilon)^k \implies |a_n| < |a_N|(a^* + \epsilon)^{n-N} = \underbrace{\text{fixed}} |a_N|(a^* + \epsilon)^{-N}(a^* + \epsilon)^n.$$

But $\sum (a^* + \epsilon)^n$ is a convergent geometric series since $a^* + \epsilon < 1$, so by the comparison test, $\sum a_n$ converges.

(b) $\left|\frac{a_{n+1}}{a_n}\right| \geq 1$ for $n \geq N$ implies that $|a_{N+k}| \geq |a_N| \neq 0$ for all $k \implies \lim_{k\to\infty} |a_{N+k}| \neq 0 \implies \lim_{n\to\infty} a_n \neq 0 \implies \sum a_n$ diverges.

**Theorem.** *$\sum c_n$ with $c_n \geq 0$ for all $n$ and $c_n \neq 0$ for large $n$. Then*

$$\liminf_{n\to\infty} \frac{c_{n+1}}{c_n} \leq \liminf_{n\to\infty} \sqrt[n]{c_n} \leq \limsup_{n\to\infty} \sqrt[n]{c_n},$$

$$\text{and} \quad \limsup_{n\to\infty} \sqrt[n]{c_n} \leq \limsup_{n\to\infty} \frac{c_{n+1}}{c_n}.$$

**Remark 22.**—The Root test is more powerful than the Ratio test:

$$1 + 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \dots$$

So $\limsup_n \sqrt[n]{a_n} = \frac{1}{2} < 1 \implies$ converges. Then $\limsup_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$ (Test is inconclusive).

## § 6.4 Power series

**Definition.**—$\sum_{n=0}^{\infty} c_n z^n$; $c_n \in \mathbb{R}, z \in \mathbb{C}$.

Q: Under what assumptions on $c_n$ and $z$ itself does the series converge/diverge?

**Theorem.** *Consider $\sum c_n z^n$. Define $\alpha := \limsup_{n\to\infty} \sqrt[n]{|c_n|}$ and $R := \frac{1}{\alpha}$. Then $\sum c_n z^n$ converges if $|z| < R$ and diverges if $|z| > R$.*

**Proof.** Let $a_n := c_n z^n$. Then $\limsup_{n\to\infty} \sqrt[n]{|a_n|} = \limsup_{n\to\infty} \sqrt[n]{|c_n| \cdot |z|^n} = |z| \underbrace{\limsup_{n\to\infty} \sqrt[n]{|c_n|}}_{\alpha} = |z| \cdot \alpha$. By the Root test, $\sum c_n z^n$ converges if $|z| \cdot \alpha < 1$ or $|z| < \frac{1}{\alpha} = R$ and diverges if $|z| > \frac{1}{\alpha} = R$.

**Example 82.**—Consider $\sum_n 1 \cdot x^n$ with $1 \to c_n$. Then $\alpha = \lim_{n\to\infty}(1) = 1$. Converges if $|x| < \frac{1}{\alpha} = 1 = R$, diverges if $|x| > \frac{1}{\alpha} = 1 = R$.

**Recall 25.**—$\lim_{n\to\infty} \sqrt[n]{n} = 1$.

**Example 83.**—Consider $\sum \frac{2^n}{n^2} \cdot x^n$. Then $\alpha = \lim_{n\to\infty} \sqrt[n]{\frac{2^n}{n^2}} = \lim_{n\to\infty} \frac{2}{\sqrt[n]{n^2}} = 2 \implies$

$$\underbrace{R = \frac{1}{2}}_{\text{radius of convergence}}.$$

# Chapter 3

# COMPLEX ANALYSIS

## § 1 Complex Numbers: An Introduction

### § 1.1 Exploring the Complex Set $\mathbb{C}$

Consider the set of complex numbers denoted by $\mathbb{C}$. This set forms a vector space over the real numbers, denoted as $\mathbb{R}$, with a dimensionality of 2. Every complex number, represented as $z$, can be expressed in the form $a + bi$, where $a$ and $b$ are real numbers, and $i$ is the imaginary unit defined as $\sqrt{-1}$.

**Definition** (Addition).—For two complex numbers $z_1 = (a, b)$ and $z_2 = (c, d)$, their sum is given by $z_1 + z_2 = (a + c, b + d)$.

**Definition** (Multiplication).—The product of two complex numbers $z_1 = (a, b)$ and $z_2 = (c, d)$ is calculated as $z_1 \cdot z_2 = (ac - bd, ad + bc)$. In the standard form $a + bi$, this multiplication translates to $(a + bi)(c + di) = (ac - bd) + (ad + bc)i$.

**Definition** (Polar Coordinates).—Further exploration of polar coordinates will simplify multiplication operations.

**Example 84.**—The number $5i$ is classified as purely imaginary, while $\pi$ is purely real.

**Example 85.**—If we perform an addition operation by a complex number $z \in \mathbb{C}$, can this process be reversed? The reversibility property holds for addition.

**Definition** (Additive Identity).—The sum of a complex number $z$ and the additive identity 0 is equal to $z$, where $0 = (0, 0) = 0 + 0i$.

**Definition** (Additive Inverse).—The additive inverse of a complex number $z = x + yi$ is $-z = -x - yi$.

**Definition** (Multiplicative Identity).—The product of a complex number $z$ and the multiplicative identity 1 is equal to $z$, where $1 = 1 + 0i$.

**Definition** (Multiplicative Inverse).—For $z = x + yi$, the inverse of $z$ is $\frac{1}{z} = \frac{1}{x+yi} = \frac{x-yi}{x^2+y^2} = \frac{x}{x^2+y^2} - \frac{y}{x^2+y^2}i$.

All these operations collectively turn $\mathbb{C}$ into a field.

**Note 22.**—$\mathbb{C}$ stands as the smallest field encompassing both $\mathbb{R}$ and $i$, where $i^2 = -1$.

**Theorem Fundamental Theorem of Algebra.** *Every univariate polynomial with coefficients in $\mathbb{C}$ possesses a root in $\mathbb{C}$, establishing $\mathbb{C}$ as algebraically closed.*

## § 1.2 Navigating the Complex Plane

Let $z = x + yi$ where $i^2 = -1$. The complex plane involves the synergy of Analysis, Geometry, and Algebra. The parallelogram rule provides a geometric interpretation for addition, and multiplying by $i$ corresponds to a $\pi/2$ rotation on the complex plane.

**Definition** (Complex Conjugate).—For $z = x+yi$, the complex conjugate of $z$ is $\overline{z} = x-yi$. Complex conjugation acts as a reflection across the $\mathbb{R}$-axis.

**Definition** (Absolute Value).—The absolute value or modulus of $z$ is $|z| = \sqrt{z \cdot \overline{z}} = \sqrt{x^2 + y^2}$.

**Definition** (Triangle Inequality).—For complex numbers $z_1$ and $z_2$, $|z_1 + z_2| \leq |z_1| + |z_2|$.

**Definition** (Polar Coordinates).—If $z = x + yi$, polar coordinates are given by $z = r(\cos\theta + i\sin\theta)$, where $r = |z|$ and $\theta = \arg z$.

**Definition** (Euler's Formula).—$e^{i\theta} = \cos\theta + i\sin\theta$.

## § 1.3 Cubic Polynomials and Their Roots

Cubic polynomials, like $ax^3 + bx^2 + cx + d = 0$, lack a general formula akin to the quadratic formula. Complex numbers play a crucial role in finding roots, and the Fundamental Theorem of Algebra ensures that every polynomial in $\mathbb{C}$ has at least one root.

### Sequences and Series

A sequence $\{z_n\}$ converges to $z$ if and only if both $\Re z_n \to \Re z$ and $\Im z_n \to \Im z$. The concept of convergence holds significance in comprehending the topological and analytic aspects of $\mathbb{C}$.

**Definition** (Cauchy Sequence).—A sequence $\{z_n\}$ is a Cauchy sequence if, for every $\epsilon > 0$, there exists $N > 0$ such that $n, m > N$ implies $|z_n - z_m| < \epsilon$.

Cauchy sequences serve as a foundational concept in the exploration of convergence in complex analysis.

**Note 23.**—The inequalities $|\Re z|, |\Im z| \leq |z| \leq |\Re z| + |\Im z|$ highlight relationships between the real and imaginary parts of a complex number. The representation $z = \Re z + (\Im z)i$ emphasizes its structure. Additionally, $|\Re z| = \sqrt{(\Re z)^2} \leq \sqrt{(\Re z)^2 + (\Im z)^2} = |z|$ holds, and the triangle inequality further implies $|z| \leq |\Re z| + |\Im z i| = |\Re z| + |\Im z|$.

**Proposition 14.**—The sequence $z_n z$ (convergence of $z_n$ to $z$) if and only if both $\Re z_n \Re z$ and $\Im z_n \Im z$.

Thus, $z_n z$ is equivalent to $|z_n - z| 0$. The inequality $0 \leq |\Re(z_n - z)| \leq |z_n - z| 0$ demonstrates that if $z_n z$, then $\Re z$ and $\Im z$ also converge respectively. For $\Re z_n \Re z$, $|\Re z_n - \Re z| 0$ implies their equality. This equality is used to deduce that if $\Re z_n \Re z$ and $\Im z_n \Im z$, then $z_n z$.

**Example 86.**—Consider a sequence $\{z_n\}$ where $z^n 0$ for $|z| < 1$, $z^n \infty$ for $|z| > 1$, and is not convergent for $|z| = 1$ (except $z = 1$). The expression $|z^n - 0| = |z^n| = |z|^n 0$ illustrates this behavior.

**Example 87.**—Does $\frac{n}{n+i} 1$? Rewriting, $\left|\frac{n}{n+i} - 1\right| = \left|\frac{n-n-i}{n+i}\right| = \left|\frac{-i}{n+i}\right| 0$. It can be stated that $\left|\frac{-i}{n+i}\right| = 1 \cdot \left|\frac{1}{n+i}\right| = \sqrt{\frac{1}{n+i}\frac{1}{n-i}} = \sqrt{\frac{1}{n^2+1}} 0$. Thus, the sequence converges to 1.

When evaluating if a sequence $\{z_n\}$ converges, consider:

1. Propose a limit $L$.
2. Verify $|z_n - L| 0$.
3. If step 2 fails, revisit step 1.

Possibly, one may only be concerned with the convergence of the sequence without determining the limit.

**Theorem Proposition 1.4.** *The sequence $\{z_n\}$ converges if and only if, for every $\epsilon > 0$, there exists $N > 0$ such that $n, m > N$ implies $|z_n - z_m| < \epsilon$. This is known as a Cauchy sequence.*

Skipping the proof, the discussion on series will continue next time.

In logic, the example "For every person $A$, there is a person $B$ such that $A$ loves $B$; everybody loves somebody," is presented. Additionally, the challenge to the assertion that $\{z_n\}$ is Cauchy involves showing, for $\epsilon = 1$, the need to find $N = 10,000,000$, $n, m > 10,000,000$, such that $|z_{10,000,100} - z_{10,000,001}| < \epsilon = 1$.

The term "region" is defined as an open and connected set.

In Chapter 1, it is established that $z_n z$ is equivalent to both $\Re z_n \Re z$ and $\Im z_n \Im z$. Sequences are defined as lists of numbers in $\mathbb{C}$, and series are discussed in relation to sequences.

**Definition.**—A series $\sum_{k=1}^{\infty} z_k$ converges if the sequence of partial sums $\{s_n\}$ converges, where $S_n = \sum_{k=1}^{n} z_k$ is a $\mathbb{C}$-number.

Properties such as the sum and difference of convergent sequences being convergent are noted. The Divergence Test is introduced to check if $\lim_{k\infty} z_k \neq 0$, indicating divergence. Examples illustrate the application of these concepts.

**Set Classification in $\mathbb{C}$**

(*) $D(z_0, r) = \{z : |z - z_0| < r\}$ represents the open disk with radius $r$, centered at $z_0$. In Cartesian coordinates ($z = (x, y)$, $z_0 = (x_0, y_0)$), it satisfies $(x - x_0)^2 + (y - y_0)^2 = r^2$.

- Complement: $\tilde{S} = \mathbb{C} \backslash S$, where $S \subset \mathbb{C}$.

(*) $S$ is open if every point in $S$ is contained in a disk within $S$.

(*) $S$ is closed if $\tilde{S} = \mathbb{C} \backslash S$ is open. The boundary $\partial S$ is defined as $\{z \in \mathbb{C} : (\forall r > 0)((S \cap D(z, r) \neq \emptyset) \wedge (\tilde{S} \cap D(z, r) \neq \emptyset))\}$.

- $\bar{S}$ denotes the closure of $S$, defined as $S \cup \partial S$. $S$ is bounded if there is $N$ such that $S \subset D(0, N)$.

(*) $S$ is compact if and only if it is both closed and bounded.

(*) $S$ is connected if it is not disconnected.

(*) $S$ is disconnected if it is the union of two disjoint open sets.

(*) The line segment between $z_1$ and $z_2$ includes all points in between.

- Avoid using letters other than $z, w, \alpha, \beta$ for $\mathbb{C}$-numbers (not $x, y, r, a, b, n$).

**Definition.**—A set $S \subset \mathbb{C}$ is polygonally connected if any two points can be connected by a polygonal path.

(*) A region is a connected open set.

**Proposition 15.**—A region is polygonally connected.

**Note 24.**—Regions may have points connected by a polygonal path composed of horizontal and vertical lines.

**Remark 23.**—It is recommended to use only $z, w, \alpha, \beta$ for $\mathbb{C}$-numbers.

**Continuous Functions**

**Definition.**—A function $f(z)$ (taking values in $\mathbb{C}$) is continuous near $z_0$ if $f$ is defined on some disk $D(z_0, r)$, and the convergence $z_n z_0$ implies $f(z_n) f(z_0)$ if and only if, for every $\epsilon > 0$, there exists a $\delta > 0$ such that $|z - z_0| < \delta |f(z) - f(z_0)| < \epsilon$.

For $z = x + yi$, $f(z) = f(x, y) = u(x, y) + v(x, y)i$ is continuous if and only if $u(x, y)$ and $v(x, y)$ are continuous, analogous to $z_n z \iff \Re \wedge \Im$.

**Example 88.**—Consider polynomials $f(x, y) = x^2 + y^2 - 2xyi$ and $f(z) = z^{-1}$ on $\mathbb{C} \backslash \{0\}$.

Sums and products of continuous functions are continuous. Quotients of continuous functions are continuous everywhere except where the denominator is 0.

For future reference, $f \in C^n$ means that $u$ and $v$ have continuous partial derivatives of the $n$-th order.

For future: Important: Uniform convergence. $f_n(x) = x^n$ on $[0, 1]$.

$$f(x) = \lim_{n\infty} x^n$$

Convergence of $f(x)$ is not uniform as it is not continuous. (*) Uniform limit of continuous functions is continuous.

Series:

**Theorem Weierstrass M-Test.** *$f_k$ is continuous on $D$ = region for $k \in \mathbb{N}$. If $|f_k(z)| \leq M_k$ for every $z \in D$ and $\sum M_k$ converges, then $\sum_{k=1}^{\infty} f_k(z)$ converges to a continuous function on $D$.*

The importance of continuous functions lies in the fact that if $f$ is continuous and $C$ is compact and connected, then $f(C)$ is also compact and connected.

$$(*)\ f(x, y) = u(x, y) + v(x, y)i$$

**Theorem.** *If $u, (x, y)$ is real-valued and $u_x, u_y \equiv 0$ on a region, then $u$ = constant.*

Know open, closed, starred, and the definitions. Know thm 1.10.

**Review: Weierstrass M-Test and Series Tests**

Comments on HW1: Revisit problem 26 for Tuesday and post HW for Tuesday. For problem 26, utilize the triangle inequality and the squeezing theorem (valid for real numbers). Refer to Exercise 8: $|z_1 - z_1| \leq |z_1| + |z_2|$.

Question: Is $z_1 < z_2$ or $z_1 > z_2$? No, there is no order. Lexicographical order, such as $(1, 2) < (1, 3) < (2, 3)$, is not relevant. Instead, consider $|z| \in \mathbb{R}$.

**Example 89.**—For $|z| < 2$, it implies $\sqrt{z\bar{z}} < 2 - 4 < z\bar{z} < 4$.

The squeezing theorem naturally incorporates order.

Analytic Pause: What is analytic?

**Example 90.**—$e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k$ for every $x$. $e^x$ is analytic, meaning it is equal to its Taylor expansion on a domain.

**Example 91.**—Consider $e^{-x^2}$; its derivative is $-2xe^{-x^2}$. At $x = 0$, the value is 0 (potentially not this function).

There exist functions that are not analytic but possess a Taylor expansion, yet they are not equal. What defines an analytic polynomial? $P(x, y) = U(x, y) + v(x, y)i$. Initially, for a polynomial, $u(x, y), v(x, y)$ are considered as polynomials.

**Example 92.**—$P(x,y) = xy$, and $P(x,y) = x^2 - y^2 + 2xyi$.

Alternatively, we can discuss $z$ as a variable: $P(z) = z^2 = x^2 - y^2 + 2xyi$ with $z = x + yi$. The ability to express the function as a polynomial in the complex variable $z$ is what qualifies it as an analytic polynomial.

**Example 93.**—Consider $P(x,y) = x^2 + y^2 - 2xyi$; it is not analytic.

**Definition.**—A polynomial $P(x,y)$ in $\mathbb{R}$ variables $x,y$ is analytic if there exist complex numbers $\alpha_k \in \mathbb{C}$ such that $P(x,y) = \alpha_0 + \alpha_1 z + \alpha_2 z^2 + \cdots + \alpha_N z^N$ with $z = x + yi$.

**Example 94.**—For instance, $P(x,y) = x = \frac{z+\bar{z}}{2} \neq \sum_{k=0}^{\infty} \alpha_k z^k$, making it not analytic. Refer to the example on page 22. The expression $x^2 + y^2 - 2xyi$ is not analytic.

Why can't we just set $y = 0$? This is because equality must hold for all $x, y$ where the function is defined. The allowed set for this equality is a region.

Let $f(x,y) = u(x,y) + v(x,y)i$, where $(x,y) = z = x + yi$. Observe that $u(x,y), v(x,y)$ are real-valued functions of two variables. We have $u_x = \frac{\partial u}{\partial x}$, $u_y = \frac{\partial u}{\partial y}$, $v_x, v_y$. Thus, $f_x = u_x + v_x i$ and $f_y = u_y + v_y i$.

**Proposition 16.**—A polynomial $P(x,y)$ is analytic if and only if $P_y = iP_x$.

**Example 95.**—For example, $P(x,y) = u(x,y) + v(x,y)i$, so $P_y(x,y) = u_y + v_y i = iP_x = i(u_x + v_x i) = -v_x + u_x i$.

**Definition** (Cauchy-Riemann Equations)**.**—$f = u + vi$ is analytic if and only if $u_x = v_y$ and $u_y = -v_x$.

**Example 96.**—Let's apply these equations to test functions. Consider $f_1(x,y) = x^2 - y^2 + 2xyi$. We know that $f_1$ is analytic because we showed it is equal to $z^2$. Now, let $f_2(x,y) = x^2 + y^2 - 2xyi$. Applying the Cauchy-Riemann equations, $u_x = 2x$, $u_y = 2y$, $v_x = -2y$, and $v_y = -2x$. While $u_y$ and $v_x$ satisfy the equations, $u_x$ and $v_y$ do not, indicating that $f_2$ is not analytic.

**Definition.**—$f : \mathbb{C}\mathbb{C}$ is differentiable at $z_0 \in \mathbb{C}$ if $f(z_0)$ exists (is defined), $f$ is defined on an open disk $D(z_0, r)$, and $\lim\limits_{h0} \dfrac{f(z_0+h) - f(z_0)}{h} = f'(z_0)$ exists with $h \in \mathbb{C}$.

**Example 97.**—Consider $f(z) = \bar{z}$, $f(x,y) = x - yi$. So $u(x,y) = x$ and $v(x,y) = -y$.

$$\lim_{h0} \frac{f(z+h) - f(z)}{h} = \lim_{h0} \frac{\overline{z+h} - z}{h} = \lim_{h0} \frac{\bar{h}}{h},$$

$$\frac{\bar{h}}{h} = \begin{cases} 1 & h \in \mathbb{R} \\ -1 & h \in i\mathbb{R} \\ \cdots & \text{other cases} \end{cases},$$

$$\frac{\bar{i}}{i} = -1 \cdot \frac{i}{i}$$

The limit does not exist, hence $f$ is not complex differentiable anywhere.

**Proposition 17.**—The sum and product of differentiable functions are also differentiable. The quotient is also differentiable provided the denominator is not equal to zero.

**Proposition 18.**—Polynomials are differentiable.

## § 1.4 Power Series

**Definition.**—$\{a_n\}$ is a sequence of $\mathbb{R}$-numbers. $\sup a_n =$ the smallest number larger than all $a_n$.

**Example 98.**—For instance, $a_n = -1/n$ for $n \geq 1$. The $\sup a_n = 0$. Note that for no $n$ do we have $a_n = 0$.

**Example 99.**—Another example is $b_n = (-1)^n = \begin{cases} -1 & \text{odd} \\ 1 & \text{even} \end{cases}$, so $\sup b_n = 1$.

**Definition.**—$\overline{\lim_{n\infty}} a_n =$ limit superior or $\limsup_{n\infty} a_n = \lim_{n\infty} \left( \sup_{k\geq n} a_k \right)$. For instance, at $n = 1$, it is $\sup_{k\geq 1} a_k$ and at $n = 2$, it is $\sup_{k\geq 2} a_k$.

**Example 100.**—For example, if $a_1$ is strictly greater (or equal) to all $a_n$, then $\sup_{k\geq 1} a_k = a_1$, while $\sup_{k\geq 2} a_k < a_1$.

$\overline{\lim_{n\infty}} a_n$ always exists.

**Example 101.**—For instance, $\overline{\lim_{n\infty}} b_n = 1$.

**Example 102.**—Another example is $\overline{\lim_{n\infty}} a_n = 0$.

<u>In fact</u>: If $\{a_n\}$ is a convergent sequence, then $\overline{\lim}\ a_n = \lim a_n$.

**Example 103.**—Consider $a_n = 1/n$, where $\sup_{k\geq 1} a_n = 1$, $\sup_{k\geq 2} a_n = 1/2$, and $\sup_{k\geq n} 1/n = 1/n$.

<u>Take away</u>: (*) $\overline{\lim_n}\ a_n = \lim_n a_n$ (if $\{a_n\}$ is convergent). (*) $\overline{\lim_n}\ a_n$ is a number when $\{a_n\}$ is bounded <u>even</u> if the sequence doesn't converge.

**Properties of $\overline{\lim_n}\ a_n = L$**

1. For every $N$, every $\epsilon > 0$ there is a $k \geq N$ such that $a_k \geq L - \epsilon$.
2. For every $\epsilon > 0$ there is an $N$ such that $a_k \leq L + \epsilon$ for every $k > N$.

3. For $c \geq 0$ we have $\overline{\lim}\, ca_n = c \cdot L$.

**Definition.**—A power series $\displaystyle\sum_{k=0}^{\infty} c_k z^k$, where $c_k \in \mathbb{C}$ and $z$ is a complex variable.

**Theorem.** *If $f(z) = \sum_{n=0}^{\infty} C_n z^n$ has convergence $R > 0$, then $C_n = \frac{f^{(n)}(0)}{n!}$.*

**Proof.** We have $f(z) = \sum_{k=0}^{\infty} C_k z^k$ and $f(0) = C_0$. For $f'(z)$, we differentiate and obtain $f'(z) = \sum_{k=1}^{\infty} kC_k z^{k-1}$. Setting $z = 0$, we get $f'(0) = C_1$. Similarly, differentiating $k$ times gives $f^{(k)}(0) = k! \cdot C_k$.

*Corollary.*—Power series expansions are unique, and they are the Maclaurin series at $z = 0$. You don't need too much information about the function; it's enough to know the values on a sequence $z_k \to 0$.

## § 2 Analyticity and Cauchy-Riemann equations

**Proposition 19.**—If $f = u + iv$ is differentiable at $z$, then $f_x$ and $f_y$ exist and satisfy the Cauchy-Riemann equations: $u_x = v_y$ and $u_y = -v_x$.

**Proof.** Assume $h \to 0$ on $\mathbb{R}$.

$$\frac{f(x+h,y) - f(x,y)}{h} \xrightarrow{h \to 0} f_x.$$

Assume $h \to 0$ on $i\mathbb{R}$, $h = i\eta$, $\eta \to 0$ on $\mathbb{R}$.

$$\frac{f(x,y+h) - f(x,y)}{i\eta} = \frac{f_y}{i}.$$

*Redo proof. Uniqueness Theorem.*

$$f(z) = \sum_{k=0}^{\infty} C_k z^k,$$

where $z_n \to 0$, $f(z_n) = 0$. Then $f(z) \equiv 0$ (i.e., all coefficients $C_k = 0$).

By assumption, $f(z_n) = 0$ is defined on some disk of positive radius, so $f$ is continuous at $z = 0$.

$$f(z_k) \to f(0).$$

$$f(0) = \sum_{k=0}^{\infty} C_k(0^k) = C_0 = 0.$$

What do we have left?

$$f(z) = \sum_{k=1}^{\infty} C_k z^k = z \sum_{k=0}^{\infty} C_{k+1} z^k.$$

Consider:

$$\frac{f(z)}{z} = \sum_{k=0}^{\infty} C_{k+1} z^k.$$

$$\lim_{n\to\infty} \frac{f(z_n)}{z_n} = \lim_{n\to\infty} 0 = 0.$$

Repeat:

$$\frac{f(z)}{z^2} = \sum_{k=0} C_{k+1} z^k.$$

**Definition.**—$f$ is analytic at $z$ if it's differentiable on a neighborhood of $z$ and is analytic on set $S$ if it's analytic for each $z \in S$. Differentiable at every point in that 'hood. At a point $z$, $\lim_{h\to 0} \frac{f(z+h)-f(z)}{h}$ exists (*) then $f$ satisfies the Cauchy-Riemann equations. Analyticity at a point $z$ is pretty strong because we insist something happens on a whole 'hood of $z$.

## § 2.1 Exponential, Sine, and Cosine Functions

Starting with $e^z$: We begin by considering $e^x$, where $x \in \mathbb{R}$, and extend it to the complex plane $z \in \mathbb{C}$. Key Properties:

$$e^{z_1+z_2} = e^{z_1} \cdot e^{z_2}$$

**Definition.**—To generalize $e^z$, define a function $f(z)$ satisfying:

(1) $f(x) = e^x$,

(2) $f(z_1 + z_2) = f(z_1) \cdot f(z_2)$.

Derivation for $f(z)$: Assume $z = x + iy$. Applying (2) to $z$ yields $f(x+iy) = e^x \cdot f(iy)$. Setting $f(iy) = A(y) + iB(y)$ results in $f(z) = e^x A(y) + e^x B(y)i$. Additional Condition:

(3) $f$ is analytic.

This leads to the differential equation $A'' = -A$, whose solutions form a 2-dimensional vector space: $A(y) = \alpha \cos y + \beta \sin y$. Solution for $f(z)$:

$$f(z) = e^x \cos y + ie^x \sin y$$

where $z = x+iy$. The condition $f(x+iy) = f(x) \cdot f(iy)$ ensures $f(iy) = e^{iy} = \cos y + i \sin y$. Double Angle Formula: Deriving $\cos 2y$ and $\sin 2y$ from the product of $f(iy)$ with itself. Next, explore hyperbolic sine and cosine, expressing them in terms of the exponential function. Discuss the mapping $z \mapsto e^z = e^x e^{iy}$, where $e^{iy} = \cos y + i \sin y$. Investigate when $e^z$ is 0 or 1.

## § 2.2 Line Integrals and Entire Functions

Introduce properties of line integrals and define a class of "nice curves."

**Definition.**—For a smooth curve $C$ represented by $z(t)$ on $a \leq t \leq b$:

$$\int_C f(z)\,dz = \int_a^b f(z(t))\dot{z}(t)\,dt$$

Discuss differentiability, piecewise differentiability, and smoothness of curves. Define the concept of curves being smoothly equivalent.

**Proposition 20.**—If $C_1$ and $C_2$ are smoothly equivalent, then $\int_{C_1} f = \int_{C_2} f$.

Introduce the concept of $-C$ and prove that $\int_{-C} f = -\int_C f$. Explore examples involving line integrals with functions like $z^k$ and demonstrate linearity with the sum of functions.

**Example 104.**—Consider $f(x) = x^2 + iy^2$ with $C : z(t) = t + it$ on $0 \leq t \leq 1$. Then, $\dot{z}(t) = 1 + i$, and $\int_C f(z)\,dz = \int_0^1 (t^2 + it)(1 + i)\,dt = \int_0^1 (1 + i)^2 t^2\,dt = (1 + i)^2 \int_0^1 t^2\,dt = t^3 = \frac{t^3}{3}\Big|_0^1 = \frac{2i}{3}$.

The linearity of line integrals is expressed through the following proposition:

**Proposition 21.**—For smooth curves $C$, and continuous functions $f$ and $g$ on $C$, and $\alpha \in \mathbb{C}$:

$$\int_C (\alpha f + g)\,dz = \alpha \int_C f\,dz + \int_C g\,dz$$

Note: The notation $\alpha << \beta$ is used here to mean $|\alpha| \leq |\beta|$, with the understanding that this does not imply an order on $\mathbb{C}$.

These concepts and properties provide a foundation for understanding line integrals and entire functions. Further exploration and applications can be built upon this groundwork.

**Lemma.**—*Let $G(t)$ be a $\mathbb{C}$-valued function. If $\left|\int_a^b G(t)\,dt\right| \leq \int_a^b |G(t)|\,dt$, then for a given inequality in a book:*

$$\int_a^b G(t)\,dt \ll \int_a^b |G(t)| \implies \left|\int_a^b G(t)\,dt\right| \leq \left|\int_a^b |G(t)|\,dt\right| = \int_a^b |G(t)|\,dt.$$

**Proof.** Let $e^{-i\theta} \int_a^b G(t)\,dt = Re^{i\theta} \cdot e^{-i\theta}$ for $R > 0$, and $\int_a^b e^{-i\theta} g(t)\,dt = R \in \mathbb{R}$. Since $e^{-i\theta} G(t) = A(t) + iB(t)$, we obtain

$$\int_a^b A(t)\,dt + i\int_a^b B(t)\,dt = R \implies i\int_a^b B(t)\,dt = 0.$$

Recap: $R = \int_a^b e^{-i\theta} G(t)\,dt = \int_a^b A(t)\,dt$. Define $A(t) = \Re(e^{-i\theta} G(t))$. Recall that $\Re z \leq |\Re z| \leq |z|$. Then, $A(t) \leq |A(t)| \leq |e^{-i\theta} G(t)|$. Therefore, $R = \int_a^b A(t)\,dt \leq \int_a^b |A(t)|\,dt \leq \int_a^b |e^{-i\theta} G(t)|\,dt = \int_a^b |G(t)|\,dt$.

**Example 105.**—For real-valued functions, $\left|\int_a^b f(x)\,dx\right| \neq \int_a^b |f(x)|\,dx$. As an illustration: $\int_{-1}^1 x\,dx = 0 \neq \int_{-1}^1 |x|\,dx = 1$.

**Theorem** $M$-$L$ **Formula.** *Assume $|f(z)| \leq M$ for each $z \in C$, where $l(C)$ is the length of $C$ denoted by L. The conclusion is that $\left|\int_C f(z)\,dz\right| \leq M \cdot L$.*

**Proposition 22.**—Assume $f$ and $f_n$ are continuous and smooth on $C$. If $f_n \to f$ uniformly, then $\int_C f = \lim_{n\to\infty} \int_C f_n \in \mathbb{C}$.

**Example 106.**—Consider the sequence $f_n(x) = x^n$. It converges pointwise to $\begin{cases} 0 & x \in [0,1) \\ 1 & x = 1 \end{cases}$ but not uniformly.

**Proposition 23** (New Fundamental Theorem of Calculus)**.**—If $f = F'$ and $F$ is analytic, and $C : z(t)$ on $a \leq t \leq b$, then

$$\int_C f\,dz = F(z(b)) - F(z(a)) = \int_a^b f(z(t))\dot{z}(t)\,dt = \int_a^b F'(z(t))\dot{z}(t)\,dt.$$

**Proof.** Consider a curve $\lambda(t)$, where $\dot{\lambda}(t) = \lim_{h\to 0} \frac{\lambda(t+h)-\lambda(t)}{h}$ for $h \in \mathbb{R}$. Now, consider $\gamma(t) = F(z(t))$ on $a \leq t \leq b$. Then,

$$\begin{aligned} \dot{\gamma}(t) &= \lim_{h\to 0} \frac{F(z(t+h)) - F(z(t))}{h} \\ &= \frac{F(z(t+h)) - F(z(t))}{z(t+h) - z(t)} \cdot \frac{z(t+h) - z(t)}{h}. \end{aligned}$$

If $|h| < \delta$, then $\dot{z}(t) \neq 0$ (because $C$ is smooth), and $z(t+h) - z(t) \neq 0$. Thus, $\dot{\gamma}(t) = F'(z(t)) \cdot \dot{z}(t)$. This is crucial as $\dot{\gamma}(t) = \int_C f\,dz$.

Consider:

$$\int_C f(z)\,dz = \int_a^b f(z(t))\dot{z}(t)\,dt = F'(z(t))\dot{z}(t) = \int_a^b \dot{\gamma}(t)\,dt = \gamma(b) - \gamma(a) = F(z(a)) - F(z(b)).$$

**Example 107.**—

1. Consider $\frac{z^{k+1}}{k+1}$. Since $\frac{z^{k+1}}{k+1}$ is analytic throughout $C$, $z^k$ is the derivative of an analytic function throughout $C$. By the FTC, $\int_C z^k\,dz = 0$.
2. For $\int_C z^k\,dz = 0$, use the parametrization of $C$ on $0 \leq \theta \leq 2\pi$.

## § 2.3 Closed Curve Theorem for Entire Functions

**Definition.**—A curve $C$ is closed if $C : z(t)$ for $a \leq t \leq b$ and $z(a) = z(b)$. A curve $C$ is simple closed if $z(s) = s(t) \implies s = t$ or $s, t = a, b$.

**Example 108.**—An infinity-shaped curve is closed but not simple closed.

**Theorem.** *For an entire function $f$, if $R$ is a rectangle and $\Gamma$ is the boundary of $R$, then $\int_C f(z)\,dz = 0$.*

First:

**Lemma.**—*If $f(z) = \alpha + \beta z$ and $R, \Gamma$ are as defined above, then $\int_\Gamma f(z)\,dz = 0$.*

**Proof.** Let $\Gamma$ be a counterclockwise curve. By the FTC, $F(z) = \alpha z + \frac{\beta}{2} z^2$. Therefore, $\int_\Gamma f(z)\,dz = F(z(b)) - F(z(a)) = 0$.

*Proof of Theorem.* Let $\Gamma$ be a counterclockwise curve along the boundary of rectangle $R$. Consider $I = \int_\Gamma f(z)\,dz$. Partition the rectangle into 4 pieces with counterclockwise boundaries $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$. Then,

$$\int_\Gamma f(z)\,dz = \sum_{i=1}^{4} \int_{\Gamma_i} f(z)\,dz.$$

For some $\Gamma^{(i)}$, there is a property $\frac{|I|}{4} \le \left|\int_{\Gamma^{(i)}} f(z)\,dz\right|$ due to the above lemma and the triangle inequality. Choose $R^{(1)} \supset R^{(2)} \supset R^{(3)} \supset \cdots$ (*). We know $\mathrm{Diam} R^{(k+1)} = \frac{1}{2} \mathrm{diam} R^{(k)}$ and $\frac{|I|}{4^k} \le \left|\int_{\Gamma^{(k)}} f(z)\,dz\right|$ with (*) nested compact sets $\cap_k R^{(k)} \ne \emptyset$. Pick $z_0 \in \cap_k R^{(k)}$.

Know: $f$ is analytic. $\frac{f(z)-f(z_0)}{z-z_0} \xrightarrow{z \to z_0} f'(z_0)$. Take a linear approximation of $f$: $f(z) = f(z_0) + f'(z_0)(z - z_0) + \epsilon_z(z - z_0)$, where $\epsilon_z(z - z_0)$ is an error term and has the property that $\epsilon_z \to 0$ as $z \to z_0$.

By previous lemma: $\int_{\Gamma^{(k)}} f(z)\,dz = \int_{\Gamma^{(k)}} [f(z_0) + f'(z_0)(z - z_0)]\,dz + \int_{\Gamma^{(k)}} \epsilon_z(z - z_0)\,dz = \int_{\Gamma^{(k)}} \epsilon_z(z - z_0)\,dz$.

Assume that $s$ is the longest side of $\Gamma$. Then,

$$\int_{\Gamma^{(k)}} |dz| = l(\Gamma^{(k)}) = \text{length of } \Gamma^{(k)} \le \frac{4s}{2^k}.$$

Now, $\int_{\Gamma^{(k)}} \epsilon_z(z - z_0)\,dz$, assuming $z \in \Gamma^{(k)}$, gives $|z - z_0| \le \frac{\sqrt{2}s}{2^k}$. Take $\epsilon > 0$, find $N$, $|z - z_0| \le \frac{\sqrt{2}\cdot s}{2^N}$, then $|\epsilon_z| \le \epsilon$. Take $k > N$. Then $|z - z_0| < \frac{\sqrt{2}s}{2^k} < \frac{\sqrt{2}s}{2^N} \implies |\epsilon_z| \le \epsilon$.

$$\left|\int_{\Gamma^{(k)}} f(z)\,dz\right| = \left|\int_{\Gamma^{(k)}} \epsilon_z(z - z_0)\,dz\right| \le \epsilon \cdot \frac{\sqrt{2}s}{2^k} \cdot \frac{4s}{2^k}$$

by the $M$-$L$ theorem. The conclusion is $\frac{|I|}{4^k} \le \epsilon \cdot \frac{\sqrt{2}s}{2^k} \cdot \frac{4s}{2^k} \implies |I| \le 4\sqrt{2}s^2 \cdot \epsilon \implies I = 0$.

**Theorem.** *For an entire function $f$ and a special simple closed curve $\Gamma$, $\int_\Gamma f(z)\,dz = 0$.*

Difference with FTC: We don't assume $f = F'$, where $F'$ is analytic.
Q: What is $\ln 5$? A: It's a number $x$ such that $e^x = 5$.

**Theorem.** *If $f$ is entire, then $f$ is everywhere the derivative of an analytic function, i.e., there exists an $F$ such that $F' = f$. So, $F(z) = \int_0^z f(\zeta)\, d\zeta$.*

**Example 109.**—A curve $C$ is closed if $C : z(t)$ for $a \le t \le b$, $z(a) = z(b)$. A curve $C$ is simple closed if $z(s) = z(t) \implies s = t$ or $s, t = a, b$. $f$ is an entire function, $R$ is a rectangle, $\Gamma$ is the boundary of $R$, $\implies \int_\Gamma f(z)\, dz = 0$. If $\Gamma$ is a special simple closed curve. $M$-$L$ formula: Let $f$ be continuous, $|f(z)| \le M$ on $C$ for each $z \in \mathbb{C}$ and $l(c) = L$. Then $\left|\int_C f(z)\, dz\right| \le M \cdot L$.

**Theorem Integral Theorem.** *If $f$ is an entire function, then $f$ is everywhere the derivative of an analytic function, i.e., there exists $F$ such that $F'(z) = f(z)$ for each $z$.*

**Recall 26** (FTC)**.**—If $f = F'$ and $C : z(t)$ on $a \le t \le b$, then $\int_C f\, dz = F(z(b)) - F(z(a))$.

**Proof.** Define $F(z) = \int_0^z f(\zeta)\, d\zeta = \int_{C_1} f(\zeta)\, d\zeta + \int_{C_2} f(\zeta)\, d\zeta$, where $C_1$ is the real component of $z$ and $C_2$ is the imaginary component of $z$. Examine $F(z+h)$. Claim: $F(z+h) = F(z) + \int_z^{z+h} f(\zeta)\, d\zeta$. LHS and RHS agree on sticks. By the Rectangle theorem, they are equal. $\implies \frac{F(z+h)-F(z)}{h} = \frac{1}{h}\int_z^{z+h} f(\zeta)\, d\zeta$. Observe: $\frac{1}{h}\int_z^{z+h} f(z)\, d\zeta = f(z)$. $\implies$ $\frac{F(z+h)-F(z)}{h} - f(z) = \frac{1}{h}\int_z^{z+h}[f(\zeta) - f(z)]\, d\zeta$. We want to show $\lim_{h\to 0}\left(\frac{F(z+h)-F(z)}{h} - f(z)\right) = 0$. Let $|h|$ be very small. Fix $\epsilon > 0$. There's a $\delta > |h| > 0$ such that $|f(\zeta) - f(z)| < \epsilon$ for $\zeta$ in the path from $z$ to $z+h$ because $f$ is continuous. Then $\left|\frac{F(z+h)-F(z)}{h} - f(z)\right| \le \frac{1}{|h|}\left|\int_z^{z+h}[f(\zeta) - f(z)]\, d\zeta\right| \le \frac{1}{|h|} \cdot \epsilon \cdot |h| = \epsilon$ by the $M$-$L$ formula.

*Corollary.*—If $f$ is an entire function and $C$ is a smooth closed curve, then $\int_C f(z)\, dz = 0$.

## § 3 Cauchy Integral Formula and Taylor Expansion

**Theorem Rectangle Theorem II.** *If $f$ is an entire function and $a \in \mathbb{C}$, let $g(z) = \begin{cases} \frac{f(z)-f(z)}{z-a} & z \ne a \\ f'(a) & z = a \end{cases}$. Then $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = f'(a)$. Set $z = a + h$ so $z - a = h$, then $\lim_{z\to a} \frac{f(z)-f(a)}{z-a} = f'(a)$. Let $\int_\Gamma g(z)\, dz = 0$, where $\Gamma$ is the boundary of rectangle $R$.*

*Case 1: $a \notin R$, then $g(z)$ is analytic on $R$, so by the previous rectangle theorem (check).*

*Case 2: $a \in R$. Divide $R$ so that $a$ is not on the corner of two rectangles. $\int_\Gamma g(z)\, dz = \sum_{k=1}^{9} \int_{\Gamma_k} g(z)\, dz$. Also, want the special rectangle containing $a$ (call it $\Gamma_1$) so that $l(\Gamma_1) < \epsilon$. Observe: $g$ is continuous at $z = a$. For every $z \in \Gamma_1$, $|g(z)| \le M$. So $\left|\int_\Gamma g(z)\, dz\right| = \left|\sum_{k=1}^{9} \int_{\Gamma_k} g(z)\, dz\right| \le M \cdot \epsilon + \left|\sum_{k=2}^{9} \int_{\Gamma_k} g(z)\, dz\right| = M \cdot \epsilon + 0 \implies \int_\Gamma g(z)\, dz = 0$. Because $\Gamma_2, \ldots \Gamma_9$ doesn't contain $a$.*

*Corollary.*—Let $g(z) = \begin{cases} \frac{f(z)-f(z)}{z-a} & z \ne a \\ f'(a) & z = a \end{cases}$. The Integral theorem and closed curve theorem apply to $g$ as well. $g$ is the derivative of an analytic function, and $\int_C g = 0$, where $C$ is a smooth closed curve.

**Proposition 24** (Cauchy's Integral Formula)**.**—If $f$ is an entire function, $a \in \mathbb{C}$, and $|a| < R$, where $C : Re^{i\theta}$ on $0 \leq \theta \leq 2\pi$, then $f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a}\, dz$.

**Proof.** By Corollary 13, $\int_C g(z)\, dz = 0$, where $g$ is defined as before. Then $\int_C \frac{f(z)}{z-a}\, dz \implies \int_C \frac{f(z)}{z-a}\, dz = \int_C \frac{f(a)}{z-a}\, dz = f(a) \int_C \frac{1}{z-a}\, dz = (?) 2\pi i f(a)$. Prove $\int_C \frac{1}{z-a}\, dz = 2\pi i$ next.

**Lemma.**—*For $a \in \mathbb{C}$, if $C_\rho$ is a circle of radius $\rho$ centered at $\alpha$, with $a$ within $C$, then $\int_{C_\rho} \frac{1}{z-a}\, dz = 2\pi i$. Observe: $\alpha = a = 0$.*

**Proof.** Let's first consider $z$ on $C_\rho$, $\int_{C_\rho} \frac{1}{z-\alpha}\, dz$. $z - \alpha : \rho e^{i\theta}$ and $\dot{z}(\theta) - i\rho^{i\theta}$. Then $\int_0^{2\pi} \frac{i\rho e^{i\theta}}{\rho e^{i\theta}}\, d\theta = i2\pi$.

$\int_{\text{unit circle}} \frac{1}{z}\, dz \neq 0$ shows $\ln z \neq$ analytic.

**Recall 27.**—For $n > 1$, $\int_{C_\rho} \frac{1}{(z-\alpha)^n}\, dz = 0$. $\frac{1}{z-a} = \frac{1}{(2-\alpha)+(\alpha-a)} = \frac{1}{z \cdot \alpha}\left(\frac{1}{1-\omega}\right)$. $\omega = \frac{a-\alpha}{z-\alpha}$. $\frac{1}{1-\omega} = \sum_{k=0}^{\infty} \omega^k$ for $|\omega| < 1$. So $|\omega| = \left|\frac{a-\alpha}{z-\alpha}\right| = \frac{|a-\alpha|}{\rho} < 1$. Then $\frac{1}{z-\alpha}\left[1 + \frac{a-\alpha}{z-\alpha} + \left(\frac{a-\alpha}{z-\alpha}\right)^2 + \ldots\right] = \frac{1}{z-\alpha} + \frac{a-\alpha}{(z-\alpha)^2} + \ldots$. So $\int_{C_\rho} \frac{1}{z-a}\, dz = \sum \int$ above $= \int_{C_\rho} \frac{1}{z-\alpha}\, dz = 2\pi i$.

**Example 110.**—a) $f$ entire and odd $\implies$ $f(z) = -f(-z)$, so $f(z) = \frac{f(z)+f(z)}{2} = \frac{f(z)-f(-z)}{2}$. Since $f$ is entire, $f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} z^k = \frac{1}{2}\left[\sum_{k=0}^{\infty} \frac{f^{(k)}(0) z^k}{k!} - \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} z^k (-1)^k\right]$. Thus, $f(z) = \frac{1}{2}\left[\frac{f(0)}{0!} + \frac{f'(0)z}{1!} + \frac{f''(0)z^2}{2!} + \frac{f'''(0)z^3}{3!} + \frac{f^{(4)}(0)z^4}{4!} + \frac{f^{(5)}z^5}{5!} + \cdots - (\ldots)\right]$ and so on.

**Example 111.**—$P(z)\infty$ if $z\infty$. $z\infty$ means $|z|\infty$. Want to show $|z|\infty$. Then $|P(z)|\infty$. $P(z) = \sum_{n=0}^{N} a_n z^n$. $\lim_{|z|\infty} |P(z)| \leq \sum_{n=0}^{N} |a_n| \cdot |z^n|$ by the triangle inequality. Pull out $z^N$ so $P(z) = z^N \sum_{n=0}^{N} a_n z^{N-n}$. Then $|P(z)| = |z^N| \cdot |\sum_{n=0}^{N} \frac{a_n}{z^{n-N}}|$. Know $a_n \neq 0$ and $N \neq 0$. By the triangle inequality, $|a_N| + \sum_{n=0}^{N-1} |\frac{a_n}{z^{n-N}}| \geq |\sum_{n=0} \frac{a_n}{z^{n-N}} \geq |a_N| - \sum_{n=0}^{N-1} |\frac{a_n}{z^{n-N}}|$.

**Theorem Taylor expansion for Entire Functions.** *If $f$ is an entire function, then $f$ has a power series expansion, and $f^{(k)}(0)$ exists, so $f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} z^k$.*

*Corollary.*—Entire functions are infinitely differentiable and equal to their power series everywhere.

**Example 112.**—$e^z \equiv \sum_{k=0}^{\infty} \frac{1}{k!} z^k$, is identically equal to.

*Corollary.*—Same but for power series centered at $a$: If $f$ is an entire function, then $f(z) \equiv \sum_{k=0}^{\infty} \frac{f^{(k)}}{k!}(z-a)^k$. Look over the proof carefully.

**Proposition 25.**—If $f$ is entire, then the function $g(z) = \begin{cases} \frac{f(z)-f(a)}{z-a} & z \neq a \\ f'(a) & z = a \end{cases}$ is entire.

**Proof.** To show that $g(z)$ is entire, we need to demonstrate that $g(z)$ is holomorphic in $\mathbb{C}$, i.e., it has complex derivatives of all orders.First, consider the case where $z \neq a$. Then, $g(z) = \frac{f(z)-f(a)}{z-a}$ is the quotient of two holomorphic functions ($f(z)$ and $z-a$) in the domain where $z \neq a$. Hence, $g(z)$ is holomorphic for $z \neq a$.

Now, let's examine the case when $z = a$. We need to show that the limit

$$\lim_{h\to 0} \frac{g(a+h)-g(a)}{h}$$

exists. Recall the definition of $g(z)$ when $z \neq a$:

$$g(z) = \frac{f(z)-f(a)}{z-a}$$

Now, consider $g(a+h)$:

$$g(a+h) = \frac{f(a+h)-f(a)}{h}$$

Hence,

$$\frac{g(a+h)-g(a)}{h} = \frac{\frac{f(a+h)-f(a)}{h} - \frac{f(a)}{h}}{h} = \frac{f(a+h)-f(a)}{h^2}$$

The limit of this expression as $h \to 0$ exists because $f(z)$ is entire, and thus, its second derivative exists and is continuous. Therefore, $g(z)$ is holomorphic at $z = a$.

By the Identity Theorem, since $g(z)$ is holomorphic in a neighborhood of every point in $\mathbb{C}$, it is holomorphic everywhere in $\mathbb{C}$.

Now, by Corollary 13, the Integral Theorem and the Closed Curve Theorem apply to $g$ as well. Therefore, $g$ is the derivative of an analytic function, and $\int_C g = 0$, where $C$ is a smooth closed curve.

Now, let's move on to Cauchy's Integral Formula and its proof.

**Proposition 26** (Cauchy's Integral Formula)**.**—If $f$ is an entire function, $a \in \mathbb{C}$, and $|a| < R$, where $C : Re^{i\theta}$ on $0 \leq \theta \leq 2\pi$, then

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a}\, dz$$

**Proof.** We'll prove this using the Rectangle Theorem II, which states that if $f$ is an entire function and $a \in \mathbb{C}$, then

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = f'(a)$$

Let $g(z) = \frac{f(z)-f(a)}{z-a}$ for $z \neq a$ and $g(a) = f'(a)$. By the Rectangle Theorem II, we have

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = f'(a)$$

which implies

$$\lim_{z\to a} g(z) = f'(a)$$

Now, let's consider the integral

$$\int_C g(z)\,dz = \int_C \frac{f(z)}{z-a}\,dz - \int_C \frac{f(a)}{z-a}\,dz$$

Using Corollary 13, we know that $\int_C g(z)\,dz = 0$ for a closed curve $C$. Therefore,

$$\int_C \frac{f(z)}{z-a}\,dz = \int_C \frac{f(a)}{z-a}\,dz$$

Now, consider a circle of radius $\rho$ centered at $a$ (denoted as $C_\rho$). By the Lemma mentioned earlier, we know that

$$\int_{C_\rho} \frac{1}{z-a}\,dz = 2\pi i$$

Therefore,

$$\int_C \frac{f(z)}{z-a}\,dz = f(a)\int_C \frac{1}{z-a}\,dz = f(a)\cdot 2\pi i$$

which implies

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,dz$$

Hence, Cauchy's Integral Formula is established.

*Corollary.*—If $f$ is an entire function and $f(z_1),\ldots,f(z_N)=0$, then $g_N(z) = \frac{f(z)}{(z-z_1)\cdots(z-z_N)}$ is an entire function. Define $g_N(z_1) = \lim_{z\to z_i}$.

**Proof.** By the preceding proposition, $g_1(z) = \frac{f(z)-f(z_1)}{z-z_1} = \frac{f(z)}{z-z_1}$ is entire. Inductively, $g_k(z) = \frac{g_{k-1}(z)-g_{k-1}(z_k)}{z-z_k}$ is entire $\implies g_N$ is entire.

**Example 113.**—Let $f(z) = z^2 - a^2$. Then, $g_2(z) = \frac{z^2-a^2}{(z-a)(z+a)} = 1$.

**Example 114.**—Consider $f(z) = e^z - 1$. The roots of $f(z) = 0$ are $z = 0$ and $z = 2\pi k i$. Define $g_1(z) = \frac{e^z-1}{z}$.

## § 3.1 Liouville's Theorem and Fundamental Theorem of Algebra

**Theorem Liouville's Theorem.** *A bounded entire function is constant.*

**Proof.** Fix $a, b \in \mathbb{C}$ and let $C$ be a circle of radius $R$ (chosen large enough to contain $a$ and $b$) centered at 0. If $f$ is constant, then $|f(a) - f(b)| = 0$. This equals $\left|\frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,dz - \frac{1}{2\pi i}\int_C \frac{f(z)}{z-b}\,dz\right| = \left|\frac{1}{2\pi i}\int_C \frac{f(z)(z-b)-f(z)(z-a)}{(z-a)(z-b)}\right|$. By Cauchy's integral formula, this is $\left|\frac{1}{2\pi i}\int_C \frac{f(z)(a-b)}{(z-a)(z-b)}\right|$. Since $\frac{1}{|z-a|} \leq B$ and $|z| - |a| \leq |z-a|$, we have $z \in \mathbb{C}$, $0 < R\cdot|a| = z$. Assume $|f(z)| \leq M$. Using the $M$-$L$ formula, $\left|\frac{1}{2\pi i}\int_C \frac{f(z)(a-b)}{(z-a)(z-b)}\right| \leq \frac{M|a-b|2\pi R}{2\pi(R-|a|)}$, where the length of $C$ is $2\pi R$.

**Theorem Liouville's Theorem Part 2.** Note: *(Not on exam). If $f$ is bounded by a polynomial of degree $N$, then $f$ is a polynomial of degree $\leq N$.*

**Theorem Fundamental Theorem of Algebra!.** *If $P(z) =$ nonconstant polynomial, then it has a root $z_0 \in \mathbb{C}$ such that $P(z_0) = 0$. This implies that $\mathbb{C}$ is algebraically closed.*

**Example 115.**—$p(x) = x^2 + 1$ has no root in $\mathbb{R}$ since $p(z) = (z-i)(z+i)$, where $i^2 = -1$.

**Theorem.** *An odd polynomial has a root in $\mathbb{R}$.*

*Proof. (FTA).* By contrapositive, assume $P(z) \neq 0$ for any $z$ and conclude $P(z) \equiv$ constant. Let $f(z) = \frac{1}{P(z)}$, which is analytic. If $P(z) = 0 \implies$ Not bounded. Bounded $\implies$ not zero. From homework #26: when $P \neq$ constant, then $|P(z)| \to \infty$ as $|z| \to \infty$. By Liouville's Theorem, $f$ is constant!.

*Proof. (FTA).* Contradiction. Assume $P(z) \neq 0$ and $P(z) \neq$ constant for any $z$, and conclude $P(z) \equiv$ constant. Let $f(z) = \frac{1}{P(z)}$, which is analytic. If $P(z) = 0 \implies$ Not bounded. Bounded $\implies$ not zero. When $P \neq$ constant, then $|P(z)| \to \infty$ as $|z| \to \infty$. By Liouville's Theorem, $f$ is constant! Contradiction.

Read Remarks (1)-(3) following Fundamental Theorem of Algebra.

*Corollary.*—Every polynomial $P(z) = az^N + a_{N-1}z^{N-1} + \cdots + a_0 = a_N(z-z_1)(z-z_2)\cdots(z-z_N)$ can be factored into $N$ linear roots.

Consider: A new polynomial $P_0 = (z-z_1)\cdots(c-z_N) = \frac{P(z)}{a_N} = 1 \cdot z^N + C_{N-1}z^{N-1} + \cdots + C_0$. Q: Do the coefficients $C_k$ tell me something about the roots?

**Example 116.**—$(z-2)(z-3) = z^2 - (2+3)z + (-2)(-3)$. Here $C_1 = 2+3 =$ sum of roots and $C_0 = (-2)(-3) =$ product of roots.

**Example 117.**—$(z-2)(z-3)(z-5) = z^3 - (2+3)z^2 + (-2)(-3)z - 5z^2 + (-1)(-1)(5)(2+3)z + (-2)(-3)(-5) = z^3 - (2+3+5)z^2 + (2\cdot 3 + 2\cdot 5 + 3\cdot 5)z + (-2)(-3)(-5)$.

In general: $P_0(z) = (z-z_1)\cdots(z-z_N) = \sum_{k=0}^N C_k z^k$, where $C_N = 1$. The coefficients $C_k$ come from

$$C_k = \pm \sum_{j_1}^{j_{N-k}} z_{j_1}\cdots z_{j_{N-k}} = \pm \sum_{j=1}^{N-k}\prod_{k=1}^{N-k} z_k$$

and $C_0 = \pm z_1 \cdot z_N$ and $C_{N-1} \pm \sum_{j=1}^N z_j$. $(z-z_1)(z-z_2)$ Review

**Theorem.** *$f$ is analytic on the disk $D$, $a \notin \Gamma$, $\int_\Gamma f\,dz = 0 = \int_\Gamma \frac{f(z)-f(a)}{z-a}\,dz$.*

**Recall 28** (Rectangle Theorem 1)**.**—$f$ is entire, $\Gamma = 2R$, $\implies \int_\Gamma = 0$.

**Recall 29** (Rectangle Theorem 2)**.**—$f$ is entire, $\int_\Gamma f = 0$.

**Theorem.** *$f$ is analytic on $D$, then there are $F, G$ also analytic on $D$. $F' = f$, $G'(z) = \begin{cases} \frac{f(z)-f(a)}{z-a} & z \neq a \\ f'(a) & z = a \end{cases}$.*

Analogue: §5.8: If $f$ is analytic on $D$ then $g$ is analytic on $D$.

**Recall 30** (Liouville Theorem)**.**—Bounded entire function = constant. $f(z) = \cos x$.

FTC I: If $f = F'$ and $F$ is analytic on $C$, then $\int_C f(z)\,dz = F(z(a)) - F(z(b))$. FTC II: $C : z(t)$, $a \leq t \leq b$, can find $F(z) = \int_0^z f(\zeta)\,d\zeta$. $\int_C f(z)\,dz = \int_a^b f(z(t))\dot{z}(t)\,dt$, $\dot{z}(t)$ exists $\neq 0$. $C : z(t)$, $a \leq t \leq b$. $c_1 \sim c_2$ then $\int_{c_1} f = \int_{c_2} f$, $f$ = continuous. If we had: $f$ entire FTC. LHS $= -(F(b) - F(a))$, RHS $= F(a) - F(b)$. $-\int_C f(z)\,dz = \int_{-C} f(z)\,dz$. Calculus II: $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$. $C : z(t)$ on $a \leq t \leq b$, then $-C : z(b+a-t)$ on $a \leq t \leq b$, $\omega(t) = z(b+a-t) = z(l(t))$, where $l(t) = b+a-t$. $\dot{\omega}(t) = \dot{z}(l(t)) \cdot \frac{dl}{dt} = -\dot{z}(l(t))$. (is a negative involution)

**Recall 31** (ML Formula)**.**—$|f(z)| \leq M$, $z \in C$, $l(c)$ = length $= L$, $\implies \left|\int_C f(z)\,dz\right| \leq M \cdot L$. $\Leftarrow \left|\int_a^b G(t)\,dt\right| \leq \int_a^b |G(t)|\,dt$, where $G$ is continuous and $\mathbb{C}$-valued.

## § 4 Analyticity on $D(\alpha; r)$

**Recall 32** (Rectangle Theorem)**.**—$f$ is analytic on $D(\alpha; r)$ and $R$ is a rectangle $\subset D(\alpha; r)$, $\Gamma$ = boundary of the disk $\implies \int_\Gamma f\,dz = \int_\Gamma g_a\,dz = 0$, where $g_a(z) = \begin{cases} \frac{f(z)-f(a)}{z-a} & z \neq a \\ f'(a) & z = a \end{cases}$.

**Theorem Closed Curve Theorem.** *$C \subset D$ is a closed smooth curve, $f$ is analytic on $D$, $a \in D(\alpha, r) \implies \int_C f = \int_c g_a = 0$.*

**Theorem Cauchy $\int$-theorem.** *Suppose $f$ is analytic on $D(\alpha; r)$ on the disk. $0 < \rho < r$, $|a - \alpha| < \rho$. $f(a) = \frac{1}{2\pi i}\int_{C_\rho} \frac{f(z)}{z-a}\,dz$. (*)*

**Definition** (Power Series)**.**—$f$ is analytic on $D(\alpha, r) \implies \exists C_k$ such that $f(z) = \sum_{k=0}^\infty C_k(z-\alpha)^k$ for every $z \in D(\alpha, r)$.

# § 5 Analyticity on Open Region

**Definition** (Power Series).—If $f$ is analytic on an open domain $D$, then there exist coefficients $C_k$ such that $f(z) = \sum_{k=0}^{\infty} C_k(z-\alpha)^k$ holds for every $z \in D(\alpha, r)$.

**Recall 33.**—The series $\frac{1}{1-z} = \sum_{k=0}^{\infty} z_k$ converges for $|z| < 1$ and diverges for $|z| > 1$. Similarly, $\frac{1}{1-2z} = \sum_{k=0}^{\infty} 2^k z^k$ converges for $|2z| < 1$ or $|z| < \frac{1}{2}$ and diverges for $|z| > \frac{1}{2}$. The best power series convergence tests rely on a comparison to geometric series.

**Example 118.**—Consider $f(z) = \frac{1}{z-1}$, which is analytic on $\mathbb{C}\backslash\{1\}$. The power series expansion about $z = 2$ is $\frac{1}{z-1} = \sum_{k=0}^{\infty}(-1)^k(z-2)^k$, converging for $|z-2| < 1$ and diverging for $|z-2| > 1$.

# § 6 Uniqueness, MVT, Maximum Modulus Theorem, Critical Points, and Saddle Points

**Definition.**—$f$ is analytic at $z$ if $f$ is differentiable on $D(z, r)$ for some $r > 0$. This implies the existence of the limit $\lim_{h\to 0} \frac{f(\omega+h)-f(\omega)}{h}$, making $f$ differentiable at $\omega$ and satisfying the Cauchy-Riemann equations.

**Proposition 27.**—If $f$ is analytic at $z = \alpha$, then $g_\alpha$ is also analytic at $z = \alpha$.

**Theorem.** *If $f$ is analytic at $z = \alpha$, then $f$ is infinitely differentiable.*

**Theorem.** *Suppose $f$ is analytic on an open connected region $D$. Let $z_n \in D$ with $z_n \neq z_m$ for $n \neq m$. If $z_n \to z_0 \in D$ and $f(z_n) \equiv 0$, then $f \equiv 0$ on $D$. Here, $z_0$ is an accumulation point.*

**Proof.** The function $f$ has a power series expansion about $z_0$ that converges on the disk $D(z_0, r)$. The proof involves showing that if $f(z_n) = 0$ for a sequence $z_n \to z_0$, then $f$ is identically zero on $D(z_0, r)$.

The expansion $\sum_{k=0}^{\infty} C_k(z-a)^k$, denoted as a power series about $z = a$, is characterized by the coefficients $C_k = \frac{f^k(a)}{k!}$.

**Example 119.**—Consider the series $\sum_{k=0}^{\infty} \frac{e}{k!}(z-1)^k = e\sum_{k=0}^{\infty} \frac{1}{k!}(z-1)^k = e^1 e^{z-1} = e^z$.

**Recall 34.**—If $z_k \to 0$, $z_k \neq z_l$, $k \neq l$, and $f$ is analytic on the disk $D(0, r)$, then $f(z_k) = 0$ implies $f \equiv 0$ on the disk, a result used in the proof of Theorem **??**.

**Proof.** Demonstrate that $f(z) = \sum_{k=0}^{\infty} C_k z^k$ implies $C_k = 0$ for any $k$. Establish $f(0) = C_0 \stackrel{?}{=} 0$. Show $\lim_{k\to\infty} f(z_k) = 0$ and $z_k \to 0 = C_0 = f(0)$ using the continuity of $f$. Introduce $g_1(z) = \frac{f(z)}{z} = \sum_{k=1}^{\infty} C_k z^{k-1}$ and prove $g_1(z_k) = \frac{f(z_k)}{z_k} \equiv 0$, leading to $\lim_{k\to\infty} g_1(z_k) = 0 = C_1$.

**Proof.** For a power series about $z_0$ with an accumulation point $z_n \to z_0$, given by $P(z) = \sum_{k=0}^{\infty} C_k(z-z_0)^k$, the positive radius of convergence ensures $P(z_n) = 0$ implies $C_k = 0$ for all $k$, concluding $P(z) \equiv 0$. The uniqueness stems from the equation (**??**), and it is enough for this equation to hold on infinitely many points accumulating to $z_0$. Illustrate with the development of $e^z = e^{x+iy}$ and other examples involving entire functions and discrete sets.

*Corollary.*—For analytic functions $f$ and $g$ on domain $D$, if $f(z) = g(z)$ on a set with an accumulation point, then $f \equiv g$ on $D$.

However, exceptions exist, such as $f(z) = \sin z$ having zeros on a discrete set.

**Example 120.**—Consider $f(z) = \sin\left(\frac{1}{z}\right)$ where $f(\frac{1}{2\pi k}) = \sin(2\pi k)$.

Recall that if $P(z)$ is a non-constant polynomial, then $|P(z)| \underset{|z|\to\infty}{\longrightarrow} \infty \iff P(z) \to \infty$ as $z \to \infty$. This is particularly relevant for entire polynomials.

**Theorem.** *If $f$ is an entire function and $f(z) \to \infty$ as $z \to \infty$, then $f$ is a polynomial.*

The proof involves showing that $f(z) \to \infty$ implies $f(z)$ has finitely many zeros, and consequently, $f$ is a polynomial. The notion of closed bounded sets being compact is utilized.

**Theorem Maximum Modulus Theorem.** *Let $f$ be an analytic and nonconstant function on an open set $D$. Then, for every $z_0 \in D$ and $\delta > 0$, there exists $\omega \in D(z_0, \delta)$ such that $|f(\omega)| > |f(z_0)|$.*

**Recall 35.**—A point $z$ is considered a <u>relative maximum</u> if $|f(z)| \geq |f(\omega)|$ for every $\omega \in D(z, \delta)$, with $\delta$ small enough.

**Recall 36** (<u>Mean Value Theorem</u>)**.**—For an analytic function $f$, $\alpha \in D$, and a curve $C : D(\alpha, r) \subset D$, we have $f(\alpha) = \frac{1}{2\pi}\int_0^{2\pi} f(\alpha + re^{i\theta})\, d\theta$.

**Proof.** Applying the Mean Value Theorem with $\alpha = z_0$, we get $|f(z_0)| \leq \frac{1}{2\pi}\int_0^{2\pi} |f(z_0 + re^{i\theta})|\, d\theta \leq \frac{1}{2\pi} \cdot 2\pi \cdot \max |f(z_0 + re^{i\theta})|$, where $[0, 2\pi]$ is compact, and $|f(z + re^{i\theta})|$ is continuous. This implies there exists $\omega_r = z + re^{i\theta}$ for some $\theta$ where the maximum is achieved. Thus, $|f(z_0)| \leq |f(\omega_r)|$. If equality holds for all $r$, then $|f(z_0 + re^{i\theta})| \equiv$ constant on some disk, making $f$ constant on that disk (Theorem 3.7).

(*) If $f$ is analytic on $D$, a bounded and open set, and $f$ is continuous on $D$, then $|f|$ is also continuous and attains a maximum. By the Maximum Modulus Theorem, it must attain this maximum on the boundary $\partial D$.

**Theorem Minimum Modulus Theorem.** *For a nonconstant and analytic function $f$, if $z \in D$ is a relative minimum, then $f(z_0) = 0$.*

**Proof.** Assume $f$ is analytic, $f(z_0) \neq 0$, and show $f$ is nonconstant. Define $g(z) = \frac{1}{f(z)}$, which is also analytic in some neighborhood $D(z_0, \delta)$. Since $z_0$ is both a relative minimum and maximum, it leads to a contradiction. If $|f(z_0)| \leq |f(z)|$ for $z$ close to $z_0$, then $\frac{1}{|f(z)|} \leq \frac{1}{|f(z_0)|}$.

**Theorem.** *For a nonconstant, analytic function $f$, if $D$ is a closed disk containing a relative maximum at $z_0 \in \partial D$, then $f'(z_0) \neq 0$.*

**Theorem.** *$z_0$ is a saddle point of $f$ if and only if $f'(z_0) = 0$ and $f(z_0) \neq 0$. The idea of a saddle point for $|f|$ is also introduced.*

In §6.3, the Maximum Modulus Theorem, Critical Points, Saddle Points, and other related concepts are discussed.

# § 7 Open Mapping Theorem & Schwartz Lemma

**Definition.**—A function $f$ is continuous at $z_0$ if, for every sequence $z_n \to z_0$, $f(z_n) \to f(z_0)$.

**Theorem.** *A function $f$ is continuous at $z_0$ if, for every open disk $D(f(z_0), \epsilon)$, there exists another disk $D(z_0, \delta)$ such that $f^{-1}(D(f(z_0), \epsilon)) \supseteq D(z_0, \delta)$.*

**Example 121.**—Consider the function $f(z) \equiv 0$. The inverse image of $D(0, 1)$ is $\mathbb{C}$, which is an open set. However, for a continuous function, the image of an open set is not necessarily open, as seen with $f(z) \equiv 0 : f(\mathbb{C}) = \{0\}$, which is not open.

**Theorem Open Mapping.** *If $f$ is a non-constant entire function, then the image of an open set under $f$ is also open.*

**Example 122.**—Let $U$ be an open set, and $f(x) = x^2$ on $[0, \infty)$ or $(-\infty, \infty)$. Similarly, $f(z) = z^2$ can be considered.

**Example 123.**—For Homework 6, Question 9, it is required to directly prove that $e^z$ attains max/min on $\partial D$. The solution, however, does not involve $e^z$ explicitly but focuses on the specific function.

**Definition.**—The Weierstrass Function is defined as $f(x) = \sum_{n=0}^{\infty} a_n \cos(b^n \pi x)$, and it is $\infty$-differentiable for $x \in [0, 1]$. Although $f(x)$ is continuous, it is Differentiable Nowhere.

Post-Homework: Due Tuesday, April 11 (Chapter 7). Midterm 3: Covers Chapters 2, 3, 6, and includes some review exercises. Understanding the derivation of the Cauchy-Riemann equations from analyticity is crucial.

**Theorem Open Mapping Theorem.** *For a function $f : \mathbb{C} \to \mathbb{C}$, nonconstant and entire, the image of an open set under $f$ is also open.*

**Proof.** To show that $f$ is analytic, it must be proven that $f(\alpha) = 0$ implies there is $r, \epsilon > 0$ such that $D(\alpha, r) \subset f(D(\alpha, \epsilon))$. Assuming $f(\alpha) = 0$, the function $f$ is nonconstant, and a suitable $r$ is chosen such that $|f(z)| \neq 0$ for $z \in C_r = \partial D(\alpha, r)$. Then, $\epsilon = \frac{\min_{\zeta \in C_r} |f(\zeta)|}{2}$. For any $\omega \in D(0, \epsilon)$, define $g(z) = f(z) - \omega$, and it is shown that $g(z)$ must have a minimum in $D(\alpha, r)$.

(*) A deep dive into Schwartz's Lemma in §7.2 and an example following the concept of Bilinear Transformation is recommended. Readers are advised to read Examples 1 and 2, skipping Proposition 7.3 at the end of §7.1.

**Theorem <u>Morera's Theorem</u>.** *If $f$ is continuous on $D$ and $\int_\Gamma f(z)\,dz = 0$ for every $\Gamma = \partial R$, then $f$ is analytic on $D$.*

**Remark 24.**—The actual proof shows that if $\int_\Gamma f = 0$ for $\Gamma = \partial R$, where $R$ is a rectangle with sides parallel to the real and imaginary axes, then $f$ is analytic.

**Theorem.** *If $f_n$ is analytic on an open domain $D$ and $f_n \rightrightarrows f$, then $f$ is analytic on $D$.*

**Proof.** First, establish that $f$ is continuous. Applying Morera's Theorem to show that $\int_\Gamma f = 0$, where $\Gamma = \partial R$ and $R \subset D$, involves proving that $\int_\Gamma \lim_{n\to\infty} f_n = \lim_{n\to\infty} \int_\Gamma f_n = 0$. The key to this equality is uniform convergence.

**Theorem.** *For a continuous function $f$ on an open set $D$, analytic on $D$ except possibly on some line segment $L$, the conclusion is that $f$ is analytic on $D$.*

**Proof.** We employ Morera's Theorem once again. Assume, without loss of generality, that $L \subset \mathbb{R}$ by pre-composing with the transformation $z \mapsto Az + B$. Our goal is to show that $\int_\Gamma f = 0$ for $\Gamma = \partial R$ within $D$.

<u>Case 1</u>: If $\Gamma \subset D \setminus L$, then since $f$ is analytic, it follows that $\int_\Gamma f = 0$.

<u>Case 2</u>: If $\Gamma \cap L \neq \emptyset$, consider a small $\epsilon$ and truncate $\Gamma$ by $i\epsilon$ to obtain $\Gamma_\epsilon$. Then, $\int_\Gamma f = \lim_{\epsilon\to 0} \int_{\Gamma_\epsilon} f$. As $\epsilon \to 0$, $\int_a^b f(x+i\epsilon)\,dx$ converges to $\int_a^b f(x)\,dx$ due to the continuity of $f$.

<u>Case 3</u>: If $\Gamma$ is split into $\Gamma_1$ and $\Gamma_2$ so that $\int_\Gamma f = \int_{\Gamma_1} f + \int_{\Gamma_2} f = 0$.

Thus, in all cases, we have $\int_\Gamma f = 0$ for $\Gamma = \partial R$ in $D$.

**Recall 37** (Open Mapping Theorem)**.**—If $f$ is a nonconstant entire function, then the image of an open set under $f$ is also open.

**Recall 38** (Morera's Theorem)**.**—If $f$ is continuous on an open set $D$, and $\int_\Gamma f(z)\,dz = 0$ for every closed curve $\Gamma = \partial R$ in $D$, then $f$ is analytic on $D$.

**Recall 39** (Definition)**.**—For a sequence of functions $\{f_n\}$ defined on $D$, $f$ converges uniformly on compact sets ($f_n \rightrightarrows f$) if, for every compact set in $D$, $f_n$ converges uniformly to $f$ on that compact set.

**Recall 40** (Theorem)**.**—If $f$ is analytic on an open domain $D$, and $f_n \rightrightarrows f$, then $f$ is analytic on $D$.

**Recall 41** (Theorem)**.**—If $f$ is continuous on an open set $D$ and analytic on $D$ except possibly on some line segment $L \subset D$, then $f$ is analytic on $D$.

**Theorem Schwartz' Reflection Principle.** *Let $f$ be analytic on $D$ and continuous on $\overline{D}$, where $f(x) \in \mathbb{R}$ for $x \in \mathbb{R}$. If $\partial D \cap \mathbb{R} = L$, a line segment, then the conclusion is:*

$$g(z) = \begin{cases} f(z) & z \in D \cup L \\ \overline{f(\overline{z})} & z \in D^* \end{cases}$$

*is analytic on $D^*$.*

**Proof.** To prove that $g$ is analytic on $D^*$, consider the difference quotient:

$$\frac{g(z+h) - g(z)}{h} \triangleq \overline{\frac{f(\overline{z+h}) - f(\overline{z})}{h}} = \overline{f'(\overline{z})}.$$

*Corollary.*—If $f$ is analytic on a region symmetric with respect to the $\mathbb{R}$-axis and $f(x) \in \mathbb{R}$ for $x \in \mathbb{R}$, then $\overline{f(\overline{z})} = f(z)$.

**Example 124.**—Visualize the maps involving rotation, reflection, and composition on different axes. Utilize rotations by $R_{-\theta}$, multiplication by $e^{-i\theta}$, reflections, and the inverse mapping to solve the problem.

Analytic means: Observe the reflections about an analytic arc, where $\omega \mapsto \omega^\wedge$ represents a reflection about $\lambda$. Consider extensions of functions, e.g., $f : z \mapsto (f(z^*))^\wedge$.

**Remark 25** (Riddle)**.**—Find a curve such that if either nail is removed, it will fall, but it is hanging with both.

**Example 125.**—The set $\mathbb{C}\backslash\mathbb{R}$ is not connected.

**Example 126.**—Consider the annulus $A = \{z : 1 < |z| < 3\}$.

**Example 127.**—The set $\{z : \Re(z) > 0\}\backslash\{y = \sin(1/x), x \in (0, 1]\}$ is simply connected.

**Recall 42.**—A polygonal path is a finite connected chain of horizontal and vertical line segments.

Objective: Define $\log z$. $\log z = \int_{z_0}^{z} \frac{1}{\zeta}\, d\zeta + \log z_0$, aiming to be analytic on some large set. $\int_{C_1 - C_2} \frac{1}{z}\, dz = 2\pi i$.

Claim 1: A nontrivial closed polygonal path has an equal # of horizontal and vertical line segments (and positive).

Claim 2: A closed polygonal path is the sum of finitely many rectangles. sum: touching edges cancel.

*Proof of Claim 1.* Assume $\gamma(a) = \gamma(b)$ is a corner, it's the North-West corner. Conclusion: $\#h = \#v > \ 0$.

*Proof of Claim 2.* By induction on # of sides.

**Theorem.** *If $f$ is analytic on a simply connected domain $D$, and $\Gamma$ is a simple closed polygonal path $\subset D$, then $\int_\Gamma f = 0$.*

**Theorem.** *If $f$ is analytic on a simply connected region $D$, then there is an anti-derivative $F$, i.e., $F' = f$.*

**Example 128** (Well define).—Consider the statement: "My office hours are 1 hour after class. Let's say class is at 11 AM. 49 hours ago started class."

**Theorem.** *If $f$ is analytic on a simply connected $D$, and $C$ is a simple closed smooth curve, then $\int_C f = 0$.*

Consider branches of $\log z$. Fix $z_0 = 1$, $\log(1) = 0$. The domain is open and simply connected $\mathbb{C}\backslash\{z : \Re z \leq 0\}$. $f(z) = \int_1^z \frac{d\zeta}{\zeta} + 0$ is an analytic branch of $\log z$ with $-\pi < \arg z < \pi$. Alternatively, consider $0 < \arg z < 2\pi$.

Another inverse function: $e^{f(z)} = z$, where $f(z)$ is an analytic branch of $\log z$.

**Theorem.** *Suppose $g$ is the inverse of $f$ at $z_0 \in \mathbb{C}$, and $g$ is continuous at $z_0$. If $f$ is differentiable at $g(z_0)$ and $f'(g(z_0)) \neq 0$, then $g$ is also differentiable at $z_0$ and $g'(z_0) = \frac{1}{f'(g(z_0))}$.*

**Definition** (Inverse function).—$f(g(z)) = z$. Apply the chain rule: $f'(g(z))g'(z) = 1$, so $g'(z) = \frac{1}{f'(g(z))} \implies f(z) = e^z, f'(z) = e^z$. If there is an inverse function $g(z)$ to $e^z$, $g'(z) = \frac{1}{e^{g(z)}} = \frac{1}{z}$.

**Theorem.** *$D$ is simply connected (open, connected, no holes), $0 \notin D$, $z_0 \in D$. Fix $\log z_0 = \omega_0$. Define $f(z) = \int_{z_0}^z \frac{1}{\zeta}\,d\zeta + \log(z_0)$, and $f$ is an analytic branch of $\log z$ on $D$.*

**Note 25.**—There are infinitely many $\omega$ for which $e^\omega = z_0$.

**Proof.** The function $\frac{1}{\zeta}$ is analytic on $D$ (with $0 \notin D$), so $f$ is well-defined because any two paths yield the same value. The derivative $f'(z) = \frac{1}{z}$ is analytic. To show $e^{f(z)} = z$, consider the function $g(z) = ze^{-f(z)}$. To prove that $g \equiv 1$, we show $g'(z) = 0$. We find $g'(z) = e^{-f(z)} - e^{-f(z)}(z)\frac{1}{z} = 0$, implying $g$ is a constant function. Since $g(z_0) = 1$, it follows that $g \equiv 1$.

**Remark 26.**—$\int_{z_0}^{z_0} \frac{1}{\zeta}\,d\zeta = 0$ because $0 \notin D$.

Next, let's examine branches of $\log z$. Fix $z_0 = 1$, $\log(1) = 0$. The domain is now open and simply connected as $\mathbb{C}\backslash\{z : \Re z \leq 0\}$. The function $f(z) = \int_1^z \frac{d\zeta}{\zeta} + 0$ serves as an analytic branch of $\log z$ with $-\pi < \arg z < \pi$. Alternatively, consider $0 < \arg z < 2\pi$.

Now, consider another inverse function: $e^{f(z)} = z$, where $f(z)$ is an analytic branch of $\log z$.

**Theorem.** *Suppose $g$ is the inverse of $f$ at $z_0 \in \mathbb{C}$, and $g$ is continuous at $z_0$. If $f$ is differentiable at $g(z_0)$ and $f'(g(z_0)) \neq 0$, then $g$ is also differentiable at $z_0$, and $g'(z_0) = \frac{1}{f'(g(z_0))}$.*

**Definition** (Inverse function).—$f(g(z)) = z$. Applying the chain rule, $f'(g(z))g'(z) = 1$. This leads to $g'(z) = \frac{1}{f'(g(z))}$. Thus, for $f(z) = e^z$, $f'(z) = e^z$. If an inverse function $g(z)$ to $e^z$ exists, $g'(z) = \frac{1}{e^{g(z)}} = \frac{1}{z}$.

**Theorem.** *$D$ is simply connected (open, connected, no holes), $0 \notin D$, $z_0 \in D$. Fix $\log z_0 = \omega_0$. Define $f(z) = \int_{z_0}^{z} \frac{1}{\zeta} d\zeta + \log(z_0)$, and $f$ is an analytic branch of $\log z$ on $D$.*

**Note 26.**—There are infinitely many $\omega$ for which $e^{\omega} = z_0$.

# § 8 Isolated Singularities

Classification of singularities, Riemann's Principle, and Casorati-Weierstrass Theorem
Want to focus on behavior of an analytic function near an <u>isolated</u> <u>singularity</u>.

**Definition.**—The deleted neighborhood of $\{z : 0 < |z - z_0| < \delta\}$, means $z \neq z_0$.

**Definition.**—$f$ has an isolated singularity at $z_0$ if for some $\delta$, $f$ is analytic on the punctured-$\delta$-'hood of $z_0$ (but not at $z_0$).

$f$ is not continuous at $z_0$.

**Example 129.**—

- Removable singularity: $f(z) = \begin{cases} \sin z & z \neq 3 \\ 0 & z = 2 \end{cases}$.
- Pole of order 1: $g(z) = \frac{1}{z-3}$
- Essential singularity: $e^{1/z}$, wild

**Definition.**—$z \neq z_0$, $f(z) = \frac{A(z)}{B(z)}$ in some deleted 'hood of $z_0$ and $A(z) =$ analytic, $A(z_0) \neq 0$, and $B(z_0) = 0$ then we say $z_0$ is a pole. If $B(z)$ has a zero of order $k$. $\begin{cases} \frac{B(z)}{(z-z_0)^k} & z \neq z_0 \\ \lim_{zz_0} \uparrow & \dots \end{cases}$, $z \neq z_0$.

**Definition.**—If there is an analytic function $g$ and $z_0$ and $g \equiv f$ on some deleted 'hood of $z_0$ then we say $z_0$ is a <u>removable singularity</u> of $f$.

**Definition.**—Assume $z_0$ is an isolated singularity of $f$ and it's not removable and not a pole. Then we call it essential.

**Theorem Riemann's Principle of Removable Singularities.** *Assume $z_0$ is an isolated singularity and $\lim_{zz_0}(z - z_0)f(z) = 0$. <u>Conclusion</u>: $z_0$ is removable.*

**Proof.** Consider $h(z) = \begin{cases} (z-z_0)f(z), & z \neq 0 \\ 0, & z = 0 \end{cases}$. Observe $h(z)$ is analytic on the deleted 'hood of $z_0$ and continuous at $z_0$. Conclusion: $h$ is analytic also at $z_0$. Now: Since $h(z) = 0$, $g(z) = \frac{h(z_0)}{z-z_0}$ also analytic by modified version of Corollary 5.9. $g(z) = f(z)$ except at $z_0$. Conclusion: $z_0$ is removable.

*Corollary.*—If $f$ is bounded on a deleted 'hood of an isolated singularity $z_0$, then it's removable.

**Proof.** $|(z-z_0)f(z)| \leq M \cdot |z-z_0| \underset{zz_0}{\longrightarrow} 0$ by Riemann's principle.

**Definition.**—$z \neq z_0$, $f(z) = \frac{A(z)}{B(z)}$, $A$ and $B$ are analytic on 'hood of $z_0$. $A(z_0) \neq 0$ and $B(z_0) = 0$. Then we say $f$ has a pole at $z_0$.

**Theorem.** *$f$ analytic on a deleted 'hood of $z_0$ and $\exists k \in Z_{>0}$ such that $\lim_{zz_0}(z-z_0)^k f(z) \neq 0$ has a removable singularity and $\lim_{zz_0}(z-z_0)^{k+1} f(z) = 0$. Then $f$ has a pole of order $k$ at $z_0$.*

**Proof.** $g(z) = \begin{cases} (z-z_0)^k f(z) & z \neq z_0 \\ 0 & z = z_0 \end{cases} \implies g$ is analytic. Since $g(z_0) = 0 \implies A(z) = \frac{g(z)}{z-z_0}$ is analytic at $z_0$ (by Corollary 5.9). $f(z) = \frac{A(z)}{(z-z_0)^k} = A(z) \cdot \frac{1}{(z-z_0)^k}$.

**Example 130.**—$f(z) = \frac{1}{z^2+25}$, $C \backslash \{\pm 5i\}$, two isolated singularties. $z_0 = 5i$, $A(z) = ?$ $z - z_0 = z - 5i$ so $\frac{1}{z^2+25} = \frac{A(z)}{z-5i}$, where $A(z) = \frac{1}{z+5i}$. Conclusion: $z_0$ is essential $\lim_{zz_0}(z-z_0)^{k+1} f(z) \not\emptyset$ for all $k$.

**Theorem Casorati-Weierstrass.** *If $f$ has an essential singularity at $z_0$ and $D$ is any deleted 'hood of $z_0$. Then $R = \{f(z) : z \in D\}$. Conclusion $\bar{R} = C$ (i.e. $R$ is dense).*

Actual Theorem: Picard's Theorem: With the assumptions as above, $R$ is all of $C$ except maybe 1-point.

**Example 131.**—$e^{1/z}$, wild!

**Proof.** Suppose $\bar{R} \neq C$. Means: $\exists$ small ball centered at $\omega_0$ of radius $\delta$, $R \cap B(\omega_0, \delta) = \emptyset$. Pick $z \in D$, then $|f(z) - \omega_0| \leq \delta \left| \frac{1}{f(z)-\omega_0} \right| < \frac{1}{\delta}$ Bounded for any $z \in D$. Conclusion: 1) $\frac{1}{f(z)-\omega_0}$ is analytic on $D$. 2) $\frac{1}{f(z)-\omega_0}$ has a removable singularity. $\implies g(z) = \begin{cases} \frac{1}{f(z)-\omega_0} & z \neq z_0 \\ \lim & z = z_0 \end{cases}$ is analytic and $f(z) = \omega_0 + \frac{1}{g(z)}$ for $z \neq z_0$. Either $g(z_0) \neq 0$ and so $f$ has a removable singularity or $g(z_0) = 0$ and $f$ has a pole at $z_0$.

§9.1: Laurent Series:

Let $M_k \in C$. $\sum_{k=-\infty}^{\infty} M_k = \sum_{k=0}^{\infty} M_k + \sum_{k=1}^{\infty} M_{-k}$. $L$ means both sums converge and their sum is $L$.

$\sum_{k=-\infty}^{\infty} C_k z^k$ in general will converge on an annulus.

3-types of annuli:

- $\{z : 0 < |z| < R\}$
- $\{z : r < |z| < R\}$
- $C\backslash\{0\}$. $|z| \neq 0$.

$\sum_{k=0}^{\infty} C_k z^k$ converges for $|z| < R$. For $\sum_{k=1}^{\infty} C_{-k} z^{-k} = \sum_{k=1}^{\infty} C_{-k}(\frac{1}{z})^k$. $|frac1z| < 1000$. $\frac{1}{1000} < |z|$. $\frac{1}{1000} < |z|$.

**Recall 43.**—If $\lim_{k>0}|C_k|^{1/k} = L$ then $\sum_{k=0}^{\infty} C_k z^k$ converges on disk of radius $R = \frac{1}{L}$ centered at $z = 0$.

**Recall 44.**—$\sum_{k=-\infty}^{\infty} C_k z^k = \sum_{k=1}^{\infty} C_{-k}\left(\frac{1}{z}\right)^k + \sum_{k=0}^{\infty} C_k z^k$. For this to converge, we demand each piece converge, so we will get an annulus.

**Theorem.** *$f(z) = \sum C_k z^k$ converges on the annulus $D = \{z : R_1 < |z| < R_2\}$, where $(*)$ $R_2 = \frac{1}{\lim_{k\infty}|C_k|^{1/k}}$ and $R_1 = \lim_{k\infty}|C_{-k}|^{1/k}$. In particular: If $R_1 < R_2$ then we get a domain of convergence. 3 types of annuli: $C\backslash\{0\} : \{z : 0 < |z| < \infty\}$, punctured disk $\{z : 0 < |z| < R\}$, standard: $\{z : R_1 < |z| < R_2\}$, $R_1 < R_2$. $\{z : R_1 < |z| < \infty\}$.*

**Theorem.** *If $f$ is analytic on $A : R_1 < |z| < R_2$, then $f$ possesses a unique Laurent expansion on $A$, given by $f(z) = \sum_{k=-\infty}^{\infty} a_k z^k$, where $a_k = \frac{1}{2\pi i}\int_{C_R} \frac{f(z)}{z^{k+1}}, dz$ for $R_1 < R < R_2$ and the circle $C_R$ has a radius $R$ centered at $0$ (the same holds for $z = z_0$).*

3-types of isolated discontinuities: $\exists g(z)$ analytic. Removable: $f(z) = g(z)$ on some punctured neighborhood of $z = z_0$; pole: order $k$, $g(z) = (z - z_0)^k f(z)$ but $(z - z_0)^{k-1} f(z)$ Not analytic; essential $\uparrow$ never happens.

What happens in each case: Look at $k > 0$: $a_k = \frac{1}{2\pi i}\int_{C_R} f(z) z^k, dz$. If Removable: $a_k = 0$ for all $k > 0$. If pole of order $N$ then $a_k = 0$ for $k > N$. If essential then none of the above.

**Definition.**—$f(z) = \sum_{k=-\infty}^{\infty} C_k z^k = \sum_{k=1}^{\infty} C_{-k}\left(\frac{1}{z}\right)^k + \sum_{k=0}^{\infty} C_k z^k$, where the negative part is called principle part and positive part is Analytic part.

Assume $z = 0$ is an isolated singularity: Removable $\iff$ Laurent expansion has no principle part (i.e. $C_k = 0$ for $k > 0$). Pole of order $N$ $\iff$ Laurent expansion has $C_k = 0$ and $C_{-N} \neq 0$ for $k < -N$. Essential Singularity $\iff$ there are $\infty$-many negative terms $C_k \neq 0$, $k > 0$.

**Example 132.**—$f(z) = \frac{1}{z^5} + \frac{1}{z^4} + \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z}$. Check this is the Laurent expansion at $z = 0$. Then $z^5 f(z) = 1 + z + z^2 + z^3 + z^4$ analytic. $f(z) = \sum_{k=-N}^{-1} C_k z^k + \sum_{k=0}^{\infty} C_k z^k = z^N f(z)$.

**Definition.**—$\hat{C} = C \cup \infty$ is one point compactification of $C$. Warning: When we do $R$-calculus, we deal with $R \cup -\infty, \infty$, $-\infty \neq \infty$. Similarly, could consider $\hat{R} = R \cup \infty \subset \hat{C}$. Hausdorff.

Stereographic Projection: bijection between point of $C$ and sphere without its north pole $N = (0, 0, 1)$. Type 1: If $l$ is a line through $N$ and not parallel to $x$-$y$ plane, then $l$ intersects the $x$-$y$ plane in a unique point and intersects the sphere in one additional point (besides $N_1$). Type II: parallel to $x$-$y$ plane, i.e. tangent at $N$. Of type 1: 3-subtypes. Type E: equator  unit circle; Type L: lower  unit disk; Type U: Upper. Riemann Sphere. Type II corresponds to $\infty$. Parallel lines meet at the point at infinity. More generally, lines and circles  circles through $\infty$ and circles and map is conformal, i.e. angle preserving.

# Chapter 4

# NUMERICAL ANALYSIS

## § 1 Sources of Error

1. Transitioning from a continuous/infinite problem to a discrete/finite one.

   **Example 133.**—For a function $f'(a)$, the approximation involves fixing $h = 0.01$: $f'(a) \approx \frac{f(a+0.01)-f(a)}{0.01}$, referred to as truncation error.

2. Rounding errors due to the finite representation of numbers in computers.

   **Example 134.**—Approximating the area of a circle as $\pi r^2 \approx 3.14159 r^2$.

Emphasis in this course

1. Root finding: solving $f(x) = 0$ for $x$.
2. Interpolation: Estimating function values given several known values.
3. Integration: $\int_a^b f(x)\,dx = ?$
4. Applications: Differential Equations.

### § 1.1 Fundamental Calculus Tools

**Theorem Mean Value Theorem (MVT).** *If $f$ is continuous on $[a, b]$ and differentiable on $(a, b)$, then there exists $\xi$ in $(a, b)$ such that $f'(\xi) = \frac{f(b)-f(a)}{b-a}$.*

**Example 135.**—Estimating $\cos(0.01)$ using the Mean Value Theorem. Knowing $\cos(0) = 1$ and $\frac{d}{dx}[\cos x] = -\sin(x)$ with $-1 \leq \sin(x) \leq 1$. Applying MVT, $\cos(0.01) \approx 0.995$ with $|\text{Error}| \leq 0.005$.

**Theorem Taylor's Remainder Theorem.** *For a function $f(x)$ with $n + 1$ continuous derivatives on $[a, b]$ for some $n \geq 0$, and $x, x_0 \in [a, b]$, the function can be expressed as $f(x) = p_n(x) + R_n(x)$, where $p_n(x)$ is an $n$th degree polynomial and $R_n(x)$ is the remainder/error term.*

**Example 136.**—Using Taylor's Remainder Theorem to express $e^x$ with error estimation. For $x_0 = 0$, $e^x = 1 + x + \frac{1}{2!}x^2 + \cdots + \frac{1}{n!}x^n + R_n(x)$ with $R_n(x) = \frac{(x-x_0)^{n+1}}{(n+1)!} f^{(n+1)}(\xi_x)$.

**Example 137.**—Approximating $\cos(0.01)$ with an error less than $10^{-6}$ using Taylor's Remainder Theorem. Let $\cos(x) = p_n(x) + R_n(x)$ with $x_0 = 0$ and $p_n(x) = \sum_{k=0}^{n} \frac{(-1)^k}{(2k)!} x^{2k}$. Ensuring $|R_n(x)| \leq 10^{-6}$ for accuracy.

**Recall 45** (Taylor's Theorem)**.**—$f(x) = \sum_{k=0}^{n} p_n(x) \frac{(x-x_0)^k}{k!} f^{(k)}(x_0) + R_n(x) \frac{(x-x_0)^{n+1}}{(n+1)!} f^{(n+1)}(\xi_x)$, $\xi_x$ is between $x_0$ and $x$.

**Example 138.**—Bounding the error when using $p_3(x)$ to approximate $f(x) = \frac{1}{1-x}$ over the interval $[-\frac{1}{2}, 0]$.

**Theorem Intermediate Value Theorem.** *If $f$ is continuous on $[a, b]$ and $w$ lies between $f(a)$ and $f(b)$ or between $f(b)$ and $f(a)$, then there exists $c$ in $[a, b]$ such that $f(c) = w$.*

**Example 139.**—Proving that $f(x) = 2x + \cos(x)$ has only one zero using the Intermediate Value Theorem.

## § 2 Error and Big O Notation

**Precise Value:** $A$. **Approximation:** $A_n$ (with parameters $h$ and $n$ approaching zero and infinity, respectively).

**Example 140.**—$A = f'(x)$. $A_n = \frac{f(x+h)-f(x)}{h}$, where $h$ is infinitesimally small.

**Example 141.**—$A = \sum_{k=0}^{\infty} a_k$. $A_n = \sum_{k=0}^{n} a_k$.

- **Error:** $A - A_n$
- **Absolute Error:** $|A - A_n|$
- **Relative Error:** $\frac{|A-A_n|}{|A|}$.

**Example 142.**—$A = 10^6$. $A_n = 10^6 + 1$. Error $= -1$. Absolute Error $= 1$. Relative Error $= \frac{1}{10^6} = 10^{-6}$.

**Example 143.**—$A = 10^{-6}$. $A_n = 10^{-7}$. Absolute Error $= \frac{9}{10^7}$. Relative Error $= \frac{9}{10}$.

**Last time:** Given $A$ and approximations $A_h$ or $A_n$, we defined error, absolute error, and relative error. Convergence implies $\lim_{h\to 0} A_n = A$ (leading to $\lim_{h\to 0}\text{error} = 0$) and $\lim_{n\to\infty} A_n = A$ (leading to $\lim_{n\to\infty}\text{error} = 0$). When comparing two convergent methods, both approaching $A$, we often want to assess the rate or order of convergence.

A useful example to consider is the approximation of $A = \int_1^3 e^{x^2}\,dx$. **Method 1:** Using rectangles of width $h = \frac{3-1}{n} = \frac{2}{n}$ to obtain Riemann sums $A_h^1$ or $A_n^1$. **Method 2:** Using trapezoids of width $h$ to get sums $A_h^2$ or $A_n^2$.

**Definition.**—$A = A_h + \mathcal{O}(\beta(h))$ if there exists $C > 0$, independent of $h$, such that $|A - A_h| \le C\beta(h)$.

**Note 27.**—If $\lim_{h\to 0}\beta(h) = 0$, then $A_h \to A$.

**Definition.**—$A = A_n + \mathcal{O}(\beta(n))$ if there exists $C > 0$, independent of $n$, such that $|A - A_n| \le C\beta(n)$.

Sometimes, we also consider $a_n$, which represents the number of floating-point operations or the amount of memory used.

## § 2.1 Comparison of Methods

Suppose Method 1 has $A^1 \sim \mathcal{O}(\beta_1(h))$, and Method 2 has $A^2 \sim \mathcal{O}(\beta_2(h))$.

- If $\lim_{h\to 0}\frac{\beta_1(h)}{\beta_2(h)} = 0$, then Method 1 converges faster.
- If $\lim_{h\to 0}\frac{\beta_1(h)}{\beta_2(h)} = M \in (0,\infty)$, then both methods have the same rate of convergence.
- If $\lim_{h\to 0}\frac{\beta_1(h)}{\beta_2(h)} = \infty$, then Method 2 converges faster.

**Example 144.**—Consider $f$ smooth on $[a,b]$, and fix $x \in (a,b)$. Set $A = f'(x)$ and $A_h = \frac{f(x+h)-f(x)}{h}$. Show $A = A_h + \mathcal{O}(h)$. Using Taylor's Theorem, $f(x+h) = f(x) + hf'(x) + \frac{1}{2}h^2 f''(\xi)$, where $\xi$ is between $x$ and $x+h$. Thus, $f'(x) = \frac{f(x+h)-f(x)}{h} - \frac{1}{2}hf''(\xi)$. This implies $A = A_h - \frac{1}{2}hf''(\xi)$. Set $C = \frac{1}{2}\max_{a\le\xi\le b}|f''(\xi)|$. Then $|A - A_h| \le Ch$.

**Example 145.**—Suppose $f$ is smooth on $[a,b]$, and fix $x \in (a,b)$. $A = f'(x)$, $A_h = \frac{f(x+h)-f(x-h)}{2h}$. Show $A = A_h + \mathcal{O}(h^2)$. Using Taylor's Theorem: $f(x+h) = f(x) + hf'(x) + \frac{1}{2}h^2 f''(x) + \frac{1}{6}h^2 f'''(\xi_1)$, where $\xi_1$ is between $x$ and $x+h$. Also, $f(x-h) = f(x) - hf'(x) + \frac{1}{2}h^2 f''(x) - \frac{1}{6}h^3 f'''(\xi_2)$, where $\xi_2$ is between $x-h$ and $x$. Combining these,

$$A_h = f'(x) + \frac{1}{12}h^2[f'''(\xi_1) + f'''(\xi_2)].$$

Let $C = \frac{1}{12}\max_{a\le\xi\le b}|f'''(\xi)|$. Therefore, $|A - A_h| \le \frac{1}{6}Ch^2$.

# § 3 Computer Arithmetic

Computers are limited in their ability to represent numbers, using a fixed, finite number of digits.

**Definition** (Floating-point numbers)**.**—$x = \sigma \times f \times \beta^{t-p}$, where $\sigma = \text{sgn }(+,-)$, $f =$ fraction $= .d_1 d_2 \ldots d_m$, $\beta =$ base (typically $d \implies d_i \in 0, 1$ for all $i$), $t - p =$ exponent where $t$ variable, $p$ fixed (called the shift).

Storage: $x = 1 \text{ bit}\sigma \mid N \text{ bits}t \mid M \text{ bits}f$, so $M + N + 1$ bits.

**Example 146.**—$\beta = 2$, $p = 15$, $N = 5$, $M = 8$. $x = \pm 0.d_1 d_2 \ldots d_8 \times 2^{t-15}$. So $0 \leq t \leq 11111_2 = 2^0 + 2^1 + 2^2 + 2^3 + 2^4 = 31$. So $-15 \leq t - p \leq 16$ nearly equal range of positive and negative exponents.

1. $x = -0.25 = -[2^{-2}] = 0.01_2$. So $t - p = -1$ and $t = -1 + 5 = 14 = 2^3 + 2^2 + 2^1 = 1110_2$. Thus $x = \text{neg} \mid 01110 \mid 10000000$
2. $x = 50 = 2^5 + 2^4 + 2^1 = 110010._2$. So $t - p = 6$ implies $t = 6 + 15 = 21 = 2^4 + 2^2 + 2^0 = 10101_2$. Thus $t = \text{pos} \mid 10101 \mid 11001000$.

**Note 28** (Biggest Number)**.**—all 1's: $+.M11\ldots1 \times 2^{N11\ldots1-p}$

**Note 29** (Normalized)**.**—$d_1 = \beta - 1$.

**Note 30** (Overflow)**.**—exponent too large.

**Note 31** (Underflow)**.**—exponent too small.

**Definition** (Rounding Error)**.**—rounding a real number to a corresponding floating-point number.

Two methods: rounding and chopping (round down).

**Example 147.**—$\frac{1}{6} + \frac{1}{10}$ base 10 with 3-digit arithmetic.

1. Exact: $\frac{1}{6} + \frac{1}{10} = \frac{10+6}{60} = \frac{4}{15} = 0.26$
2. Chopping: $\frac{1}{6} = 0.16 \mapsto 0.166$, $\frac{1}{10} = 0.1$, add $= 0.266$,

   relative error $= \frac{|0.26-0.266|}{|0.26|} = \frac{|\frac{4}{15} - \frac{266}{1000}|}{4/15} = 0.0025$.
3. Rounding: $0.6666$, $\frac{1}{6} \mapsto 0.167$, add $= 0.267$,

   relative error $= \frac{|0.26-0.267|}{0.26} = 0.00125$.

**Note 32.**—:

1. If changing the precision (single to double) drastically changes the output, then rounding errors are having a large effect.

2. The order in which operations are performed makes a difference $(a+b)+c \neq a+(b+c)$ for some $a, b, c$.

3. Subtraction can lead to a major loss of accuracy if the two numbers are nearly equal.

$e^x = p(x) + R_x$ or $e^{-x}$ = alternating sign in polynomial or $= \frac{1}{e^x}$.

Goal: Bound the relative error when using floating-point arithmetic. $a \circledast b \mapsto fl[rd(e) * rd(b)]$

**Definition.**—Machine epsilon is defined by

$$\epsilon_{mach} = \max x \in \text{Computer \#s} 1 + x = 1 \text{ in computer arithmetic.}$$

**Note 33.**—$\epsilon_{mach}$ is not the smallest number in magnitude.

$\left|\frac{rd(x)-x}{x}\right| = O(\epsilon_{mach}) \leq c\epsilon_{mach}$, $x$ is a real number, $rd(x)$ is $fl$.

**Theorem Computer Arithmetic Error.** *Let* $* = +, -, \times, \div$. *Then,*

$$\frac{|x * y - fl(x * y)|}{|x * y|} = O(\epsilon_{mach}).$$

$*$ *Single arithmetic calculations have bounded relative error.*

## § 3.1 Approximation

Goal: Approximate $e^x$ and integrals involving $e^x$ using simple functions.

Book: $f(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt \equiv \operatorname{erf}(x)$.

**Example 148.**—$f(x) = \int_0^x t^2 e^{-t^2}\, dt$. Approximate over $(0, 2)$. Bound Error.

Strategy: $t^2 e^{-t^2} \approx p(t)$. Integrate $p(t)$ exactly.

$$e^t = p_k(t) + R_k(t) = 1 + t + \frac{1}{2}t^2 + \cdots + \frac{1}{k!}t^k c_t \text{ between 0 and } t + \frac{t^{k+1}}{(k+1)!}e^{c_t}$$

$$\implies t^2 e^{-t^2} = t^2\left[1 + (-t^2) + \frac{1}{2}(-t^2)^2 + \cdots + \frac{1}{k!}(-t^2)^k + c_{-t^2} \text{ between } -t^2 \text{ and } 0\frac{(-t^2)^{k+1}}{(k+1)!}e^{c_{(-t^2)}}\right]$$

$$= q_k(t)\sum_{i=0}^{k}(-1)^k \cdot \frac{t^{2k+2}}{k!} + r_k(t)\frac{(-1)^{k+1}t^{2k+4}}{(k+1)!}e^c$$

$$f(x) = \int_0^x t^2 e^{-t^2}\, dt = \int_0^x q_k(t) + r_k(t)\, dt = \sum_{i=0}^{k}(-1)^k \cdot \left.\frac{t^{2k+3}}{(2k+3)k!}\right|_0^x + \int_0^x r_k(t)\, dt$$

$$= \sum_{i=0}^{k}(-1)^k \frac{x^{2k+3}}{(2k+3)k!} + E_k(x)\int_0^x r_k(t)\, dt.$$

Bound $|E_k(x)|$: Let $x$ between 0 and 2, $t$ between 0 and $x$, $c$ between 0 and $-t^2 \implies c \in [-x^2, 0]$ with

$$E_k(x) = \frac{(-1)^{k+1}}{(k+1)!} \int_0^x > 0 \text{ for all } t > 0 t^{2k+4} e^c \, dt.$$

$*$ Naive bound: $\int_0^x t^{2k+4} e^c \, dt \le \max |t^{2k+4}| \cdot \max |e^c| \cdot |x - 0| = (2^{2k+4})(e^0)(2)$. So

$$|E_k(t)| \le \frac{2 \cdot 2^{2k+4}}{(k+1)!} 0$$

as $k \infty$.

**Recall 46.**—$\int_a^b f(x) \, dx \le \max |f(x)| \cdot (b - a)$.

MVT, $\xi_x \in (-x^2, 0)$,

$$E_k(x) = \frac{(-1)^{k+1}}{(k+1)!} e^{\xi_x} \int_0^x t^{2k+4} \, dt = \frac{(-1)^{k+1}}{(k+1)!} e^{\xi_x} \cdot \left. \frac{t^{2k+5}}{2k+5} \right|_0^x = \delta_k(x) \frac{(-1)^{k+1}}{(2k+5)(k+1)!} x^{2k+5} \le M |e^{\xi_x},$$

$M$ based on the original problem. For $x$ fixed; $\lim_{k\infty} \delta_k(x) = 0$. So

$$|E_k(x)| = \frac{1}{(k+1)!(2k+5)} \cdot |x|^{2k+5} \cdot e^{\xi_x} \le \frac{1}{(k+1)!(2k+5)} |x|^{2k+5} M$$

where $M = e^0 = 1$ because $\xi_x \in (-x^2, 0) \implies \xi_x \in (-4, 0)$ since $x < 2$. For $x$ in $(0, 2)$: $|x|^{2k+5} \le 2^{2k+5}$. Thus

$$|E_k(x)| \le \frac{2 \cdot 2^{2k+5}}{\text{fairly precise}(k+1)!(2k+5)} \le \frac{2^5 4^k}{7(k+1)!} = \frac{32}{7} \cdot \frac{4^k}{(k+1)!} \; (\forall k \ge 1).$$

$\therefore \forall x \in (0, 2)$:

$$f(x) = \int_0^x t^2 e^{-t^2} \, dt = \sum_{i=0}^k \frac{(-1)^k}{k!(2k+3)} x^{2k+3} + O \frac{4^k}{(k+1)!}$$

$$k = 10 : |E_k(x)| \le \frac{32}{7} \cdot \frac{4^{10}}{11!} \le 0.12009$$

$$k = 20 : |E_k(x)| \le 0.84 \times 10^{-8}$$

**Example 149.**—Find the order of accuracy when approximating $e^x$ with the rational function

$$r(x) = \frac{1 + \frac{1}{2}x}{1 - \frac{1}{2}x}$$

for $-1 \leq x \leq 0$. So $|e^x - r(x)| \leq$ ? Then

$$\begin{aligned}
e^x - r(x) &= 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots + \frac{1}{k!}x^k + \cdots - \frac{1+\frac{1}{2}x}{1-\frac{1}{2}x} \\
&= \frac{1}{1-\frac{1}{2}x}1 - \frac{1}{2}x1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + O(x^4) - 1 + \frac{1}{2}x \\
&= \frac{1}{1-\frac{1}{2}x}1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + O(x^4) - \frac{1}{2}x^2 - \frac{1}{4}x^3 - 1 - \frac{1}{2}x \\
&= \text{Exact Error}\frac{1}{1-\frac{1}{2}x}\left[-\frac{1}{12}\frac{1}{6} - \frac{1}{4}x^3 + O(x^4)\right]
\end{aligned}$$

For $x \leq 0$: $\frac{1}{1-\frac{1}{2}x} \leq 1 \implies |e^x - r(x)| \leq \frac{1}{12}|x|^3 + O(x^4) \implies e^x = r(x) + O(|x|^3)$, ($x^4 \leq |x|^3$ for $-1 \leq x \leq 0$).

For $x < -1$: Then $e^x = 1 + x + \frac{1}{2}x^2 + \downarrow$ won't cancel.$\frac{1}{6}x^3e^\xi$ for $\xi$ between 0 and $x$.

## § 3.2 Horner's Rule

Goal: Compute polynomials efficiently and more accurately.

$$\begin{aligned}
p(x) &= a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \qquad \text{(Standard/expanded form)} \\
&= a_0 + x(a_1 + a_2x + \cdots + a_nx^{n-1}) \\
&= a_0 + xa_1 + x(a_2 + \cdots + a_nx^{n-2}) \\
&\quad \vdots \\
\text{nested form} - \ &= a_0 + x(a_1 + x(a_2 + \cdots + x(a_{n-1} + a_nx)\ldots)).
\end{aligned}$$

**Note 34.**—:

1. Nested form $n+1$ multiplications, $n$ additions.

   Standard - same + cost of computing $x^2, x^3, \ldots, x^n$.

2. Typically more accurate.

3. Most accurate and efficient is fully factored form: $p(x) = a_n(x-c_1)^{m_1}(x-c_2)^{m_2}\ldots(x-c_k)^{m_k}$ where $m_1 + m_2 + \cdots + m_k = n$. Not known in general.

Horner's Method $a_0, a-1, \ldots, a_n$ and $x$ $px \leftarrow a_n$ $k = n-1$ to $0$ $px \leftarrow a_k + px \cdot x$ $px = p(x)$ A slight modification outputs $p'(x)$.

**Example 150.**—$p(x) = 1 - x + \frac{1}{2}x^2 - \frac{1}{4}x^5 = 1 + x(-1 + \frac{1}{2}x - \frac{1}{4}x^4) = 1 + x(-1 + x(\frac{1}{2} - \frac{1}{4}x^3))$.

**Example 151.**—$p(x) = 1 + x^2 + \frac{1}{2}x^4 = 1 + x^2(1 + \frac{1}{2}x^2)$.

## § 3.3 Approximating natural log

$$x = f \times \beta^{t-p}$$
$$\ln(x) = \ln(f \times \beta^{t-p}) = \ln(f) + (t-p)\ln(\beta)$$

For $\beta = 2$: $\frac{1}{2} \le f \le 1$. Assume $\ln(2)$ is known to high precision. We only need to approximate $\ln(f)$ for $\frac{1}{2} \le f \le 1$ to high precision for any $x \in (0, \infty)$. This implies only need a Taylor's series on a small interval, which implies we can bound error uniformly.

Method in book: $f_0 = \frac{3}{4}$, $|R_n(z)| \le \frac{1}{2}\frac{1}{3}^n$ for all $\frac{1}{2} \le z \le 1$. Set $n = 33$ in $p_n(z) \implies$ Error $\le 10^{-16}$.

Goal: use log properties to improve the convergence rate (based on problem 1.6.6).

1. Find $w$ such that $\frac{1-w}{1+w} = z$ for $\frac{1}{2} \le z \le 1$. This implies $w = \frac{1-z}{1+z}$ with $0 \le w \le \frac{1}{3}$ (shrunk domain size).
2. $\ln(z) = \ln\frac{1-w}{1+w} = \ln(1-w) - \ln(1+w)$.
3. Taylor's expansion for $\ln(1-w), \ln(1+w)$ centered at $w_0 = \frac{1}{6}$ (midpoint for $w$).

Integral form: $f(x) = p_n(x) + R_n(\xi_x)$, $p_n(x) = \sum_{k=0}^n \frac{(x-x_0)^k}{k!} f^{(k)}(x_0)$, $R_n(\xi_x) = \int_{x_0}^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)\,dt$. Then $y = \ln(x)$, $y' = \frac{1}{x} = x$, $y'' = -x^{-2}$, $y''' = 2x^{-3}$, $y^{(4)} = -6x^{-4}$, $\implies y^{(k)} = (-1)^{k-1}(k-1)!x^{-k}$, with $k \ge 1$.

$$\begin{aligned}
\ln(1+w) = \ln(1+w_0) + \frac{w-w_0}{1+w_0} &+ \sum_{k=2}^n \frac{(w-w_0)^k}{k!}(-1)^k(k+1)!(1+w_0)^{-k} \\
&+ \int_{w_0}^w (w-t)^n \cdot (-1)^n(n+1-1)!(1+t)^{-n-1}\,dt \\
&= \ln(1+w_0) + \frac{w-w_0}{1+w_0} + \sum_{k=2}^n \frac{(-1)^{k-1}(w-w_0)^k}{k(1+w_0)^k} + \int_{w_0}^w (-1)^n(w-t)^n(1-t)^{-n-1}\,dt
\end{aligned}$$
$$\ln(1-w) = \ln(1-w_0) - \frac{w-w_0}{1-w_0} + \sum_{k=2}^n \frac{(-1)^k}{k}\frac{(w-w_0)^k}{(1-w_0)^k} + \int_{w_0}^w (-1)^{n+1}(w-t)^n(1-t)^{-n-1}\,dt$$

4. Bound $R_n(w)$ over $[0, \frac{1}{3}]$. Note that $\int_a^b f(x)\,dx \le |b-a| \cdot \max_{a \le x \le b}|f(x)|$. So

$0 \le t \le \frac{1}{3}$ (lazy)

$$\begin{aligned}
\int_{w_0}^w (-1)^{n+1}(w-t)^n(1-t)^{-n-1}\,dt &\le \frac{1}{6} \cdot \max_t \frac{1}{1-t} \cdot \max_t \frac{w-t}{1-t}^n \\
&\le \frac{1}{6} \cdot \frac{3}{2} \cdot \frac{1/6}{5/6}^n \\
&= \frac{1}{4}\frac{1}{5}^n.
\end{aligned}$$

Combining: Total Error $\leq \frac{1}{6}\frac{1}{7}^n + \text{bigger}\frac{1}{4}\frac{1}{5}^n \leq \frac{1}{2}\frac{1}{5}^n$.
If error $\leq 10^{-16} \implies \frac{1}{2}\frac{1}{5}^n \leq 10^{-16} \implies n \geq \log_2(\frac{1}{2}\cdot 10^{-16}) \approx 22.46$.
Given $z$ in $[\frac{1}{2}, 1]$, $n \geq 23$ will give $10^{-16}$ accuracy for

$$\begin{aligned}
\ln(z) &= \ln(1-w) - \ln(1+w) \approx p_n(1-w) - q_n(1+w) \qquad \text{for} \\
p_n(x) &= \ln\frac{5}{6} + \sum_{k=1}^{n} \frac{(-1)^k}{k} \frac{x-\frac{1}{6}}{5/6}^k \\
q_n(x) &= \ln\frac{7}{6} + \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \frac{x-\frac{1}{6}}{\frac{7}{6}}^k
\end{aligned}$$

with $w = \frac{1-z}{1+z}$, need $\ln(2), \ln(\frac{5}{6}), \ln(\frac{7}{6})$ to high precision.

## § 3.4 Difference Approximation of Derivatives

$f'(x) = \lim_{h0} \frac{f(x+h)-f(x)}{h}$. Forward difference approximation $(h0^+)$ $D_h^+ f(x) = \frac{f(x+h)-f(x)}{h} = f'(x) + O(h)$ Backward difference approximation $(h0^-)$ $D_h^- f(x) = \frac{f(x)-f(x-h)}{h} = f'(x) + O(h)$ Central difference approximation $D_h f(x) = \frac{f(x+h)-f(x-h)}{2h} = f'(x) + O(h^2) = \frac{1}{2}D_h^+ f(x) + D_h^- f(x)$

**Note 35.**—:

(a) Prone to cancellation error due to subtraction in the numerator, after threshold roundoff dominates.

(b) Using more nodes can increase the accuracy.

**Example 152.**—$f'(x) = \frac{8f(x+h)-8f(x-h)-f(x+2h)+f(x-2h)}{12h} + O(h^4)$.

### Calculating Rates of Convergence

Suppose $E = (h^p)$ for some $p$. This implies $E \approx Ch^p$ for some constant $C$ when $h0$. Pick $h_1, h_2 > 0$ with $h_1 \neq h_2$. So $E_1 \approx Ch_1^p$ and $E_2 \approx Ch_2^p$ imply

$$\begin{aligned}
\frac{E_1}{E_2} &\approx \frac{h_1}{h_2}^p, \\
\log\frac{E_1}{E_2} &\approx p\log\frac{h_1}{h_2}, \\
p &\approx \frac{\log(E_1/E_2)}{\log(h_1/h_2)}.
\end{aligned}$$

Approximating for several $h_i$ values lets us estimate $p$. Book: $h_1 = 2h_2 \implies \frac{E_1}{E_2} \approx 2^p$. If $p = 1$: $\frac{E_1}{E_2} \approx 2^1 = 2$ (error halves when $hh/2$) $p = 2$: $\frac{E_1}{E_2} \approx 2^2 = 4$ (error error/4 as $hh/2$)

**Second derivatives**

Centered difference approximation

$$f''(x) = \frac{f(x+2h) - 2f(x) + f(x-h)}{h^2} + O(h^2) \equiv D_h^2 f(x) = D_h^- D_h^+ f(x) = D_h^+ D_h^- f(x).$$

**Proof.** Observe,

$$\begin{aligned} D_h^+ D_h^- f(x) &= D_h^+ \frac{f(x) - f(x+h)}{h} = \frac{1}{h} D_h^+ f(x) - D_h^+ f(x-h) \\ &= \frac{1}{h} \frac{f(x+h) - f(x)}{h} - \frac{f(x-h+h) - f(x-h)}{h} \\ &= \frac{1}{h^2} f(x+h) - f(x) - f(x) + f(x+h) = D_h^2 f(x) \end{aligned}$$

*Accuracy.* Goal: $f''(x)$ everything at $x$.

$$\begin{aligned} D_h^2 f(x) &= \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} \\ &= \frac{1}{h^2} \Big[ f(x) + hf'(x) + \frac{1}{2} h^2 f''(x) + \frac{1}{6} h^3 f''(x) + \frac{1}{24} h^4 f^{(4)}(x) + O(h^5) \\ &\quad - 2f(x) + f(x) - hf'(x) + \frac{1}{2} h^2 f''(x) - \frac{1}{6} h^3 f'''(x) + \frac{1}{24} h^4 f^{(4)}(x) \Big] \\ &= \frac{1}{h^2} h^2 f''(x) + \frac{1}{12} h^4 f^{(4)}(x) + O(h^5) = f''(x) + O(h^2) \frac{1}{12} h^2 f^{(4)}(x) + O(h^5) \end{aligned}$$

**Note 36.**—:

$$\begin{aligned} D_h D_h f(x) &= f''(x) + O(2h)^2 = \frac{f(x+2h) - 2f(x) + f(x-2h)}{4h^2}, \\ D_h^2 f(x) &= f''(x) + \uparrow f^{(4)}(x) O(h^2). \end{aligned}$$

Error $= 0$ if $f(x)$ is cubic.

## § 3.5 Application: Euler's Methods for Initial Value Problems

**Objective**: Approximate the unknown function $y(t)$ over the interval $[t_0, T]$ using the differential equation

$$\begin{cases} y'(t) = f(t, y(t)) \\ y(t_0) = y_0 \end{cases}$$

where $t_0 < t \leq T$ and $f$ is a given function.

**Example 153.**—Consider the differential equation $\begin{cases} y'(t) = 4t \\ y(0) = 1 \end{cases}$. Solving this, we find $y(t) = 2t^2 + c$. Since $y(0) = 1$, we get $c = 1$, resulting in $\boxed{y(t) = 2t^2 + 1}$.

**Example 154.**—For the equation $\begin{cases} y'(t) = y(t) \\ y(0) = 2 \end{cases}$ with $f(t, y) = y$, an initial guess is $y(t) = e^t$. However, $y(0) = e^0 = 1 \neq 2$. The general solution $y(t) = ce^t$ satisfies $y'(t) = ce^t$, and choosing $c = 2$ gives $\boxed{y(t) = 2e^t}$.

**Example 155.**—For the equation $y'(t) = 3y(t)$, the general solution is $y(t) = ce^{3t}$ and $y'(t) = 3ce^{3t}$, satisfying the differential equation.

## Approximation

The approximation is given by $y'(t) = \frac{y(t+h)-y(t)}{h} + \frac{1}{2}hy''(t_n)$ where $t_n$ is between $t$ and $t + h$. Substituting into the differential equation $y' = f$:

$$\frac{y(t+h) - y(t)}{h} + \frac{1}{2}hy''(t_n) = f(t, y(t))$$

Solving for $y(t + h)$:

$$y(t + h) = y(t) + hf(t, y(t)) - \underbrace{\mathcal{O}(h^2)}_{\frac{1}{2}h^2y''(t_n)}$$

**Idea**: Given $y(t)$, we can estimate $y(t + h)$ with $\mathcal{O}(h^2)$ accuracy. Similarly, given $y(t + h)$, we can estimate $y(t + 2h)$, and so on.

$$t + Nh = T$$

**Euler's Method**: Start with $y_0 = y(t_0)$ (given), choose $h > 0$, set $t_j = t_0 + jh$, and for $n = 0, 1, 2, \ldots$ (stop at $t = T$):

$$y_{n+1} = y_n + hf(t_n, y_n)$$

For $T = t_0 + Nh$, $|y(T) - y_N| = \mathcal{O}(h)$. The error decreases from $h^2$ to $h$ due to the accumulation of error (without rounding).

**Example 156.**—As an illustration, let's consider $y' + 4y = t$, $y(0) = 1$. We estimate $y(1)$ using $N = 4$ steps, setting $h = \frac{1}{4}$. The calculations yield $y(1) \approx \frac{3}{16}$.

## § 3.6 Linear Interpolation

**Objective**: Given data points $(x_k, y_k)$ with $x_i \neq x_j$ for $i \neq j$ and $y_k = f(x_k)$ for some unknown function $f$, determine a function $p$ such that

- $p(x_k) = y_k = f(x_k)$ for all $k$
- $p(x) \approx f(x)$ otherwise

* Typically, $f(x)$ is unknown.
* Consider a function $f(x)$ with known values at $x_0 < x_1 < x_2 < x_3 < x_4$.
* In a linear interpolation, $p(x)$ is piecewise linear, being linear on $[x_{k-1}, x_k]$ for all $k$. The expression for $p(x)$ is given by $p(x) = p_k(x)$ if $x_{k-1} \leq x \leq x_k$.

Let $x \in [x_{k-1}, x_k]$. $p_k(x)$ represents the secant line connecting $f(x_{k-1})$ and $f(x_k)$. The slope is $m = \frac{f(x_k)-f(x_{k-1})}{x_k-x_{k-1}}$, and the equation is $y - f(x_{k-1}) = m(x - x_{k-1})$. Thus,

$$y = f(x_{k-1}) + \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}(x - x_{k-1})$$

(in point-slope form) or

$$p_k(x) = \frac{x_k - x}{x_k - x_{k-1}} f(x_{k-1}) + \frac{x - x_{k-1}}{x_k - x_{k-1}} f(x_k)$$

(in parameterized form) for all $x \in [x_{k-1}, x_k]$.

**Theorem.** *If $f \in C^2[a, b]$ and $p(x)$ is the piecewise linear interpolant for nodes $a = x_0 < x_1 < \ldots < x_N = b$ ($N > 0$), then for all $x \in [a, b]$,*

$$|f(x) - p(x)| \leq \frac{1}{8}(\max_k |x_k - x_{k-1}|)^2 \max_{a \leq \zeta \leq b} |f''(\zeta)|.$$

**Note 37.—** (a) If $f$ is linear, then $f'' = 0 \implies p(x) = f(x)$ for all $x$.

(b) The error estimate only holds over $[a, b]$ where $x_0 = a, x_N = b$.

**Theorem Linear Interpolation with $N = 1$.** *Let $f \in C^2[x_0, x_1]$ and $p_1(x)$ be the linear polynomial that interpolates $f$ at $x_0$ and $x_1$. Then, for all $x \in [x_0, x_1]$,*

$$|f(x) - p(x)| \leq \frac{1}{2}|(x - x_0)(x - x_1)| \max_{x_0 \leq \zeta \leq x_1} |f''(\zeta)| \leq \frac{1}{8}(x_1 - x_0)^2 \max_{x_0 \leq \zeta \leq x_1} |f''(\zeta)|.$$

**Proof.** Let $E(x) = f(x) - p_1(x)$ (error at $x$). Define

$$\omega(x) = (x - x_0)(x - x_1), \quad G(x) = E(x) - \frac{\omega(x)}{\omega(t)} \text{auxiliary function}\, E(t),$$

where $x_0 < t < x_1$. Then

- $G(x_0) = E(x_0) = 0$
- $G(x_1) = E(x_1) = 0$
- $G(t) = 0$

(Rolle's Theorem (MVT) $\implies$ there exists $\eta_0$ in $(x_0, t)$ and $\eta_1$ in $(t, x_1)$ such that $G'(\eta_0) = G'(\eta_1) = 0$). Rolle's Theorem $\implies$ there exists $\xi_t$ in $(\eta_0, \eta_1)$ such that $G''(\xi_t) = 0$.

**Note 38.**—$G''(x) = f''(x) - \frac{2}{\omega(t)} E(t)$ and $p_1''(x) = 0$ and $w''(x) = 2$, $\omega$ is quadratic.

$G''(\xi_t) = 0 \implies f''(\xi_t) - \frac{2}{\omega(t)} E(t) = 0 \implies E(t) = \frac{1}{2}\omega(t) \cdot f''(\xi_t) \implies f(t) - p(t) = \frac{1}{2}(t - x_0)(t - x_1) f''(\xi_t)$. * exact error for any $t$ in $(x_0, x_1)$. Thus $|E(x)| \le \frac{1}{2} |\underbrace{\omega(x)}_{\text{[quadratic]}}| \cdot \max_{x_0 \le \xi \le x_1} |f''(\xi)|$. ✓ Then $x_c = \frac{x_0 + x_1}{2}$ and $|\omega(x)|$ is maximized over $[x_0, x_1]$ at the vertex $x_c$. So,

$$\begin{aligned}
|\omega(x_c)| &= \left| \omega\left(\frac{x_0 + x_1}{2}\right) \right| \\
&= \left| \left(\frac{x_0 + x_1}{2} - x_0\right)\left(\frac{x_0 + x_1}{2} - x_1\right) \right| \\
&= \left| \frac{x_1 - x_0}{2} \cdot \frac{x_0 - x_1}{2} \right| \\
&= \frac{1}{4} |x_1 - x_0|^2.
\end{aligned}$$

Implies $|E(x)| \le \frac{1}{2} \cdot \frac{1}{4} |x_1 - x_0|^2 \cdot \max_{x_0 \le \xi \le x_1} |f''(\xi)|^2$.

## § 3.7 Application - The Trapezoidal Rule

**Objective**: Approximate $\int_a^b f(x)\, dx$.

Consider a mesh $a = x_0 < x_1 < x_2 < \cdots < x_N = b$ dividing $[a, b]$ into $N$ subintervals $[x_{k-1}, x_k]$. Let $q_N(x)$ represent the piecewise linear interpolant of $f(x)$ ($f(x_k) = q_N(x_k)$).

**Proposition 28** (Trapezoidal Rule ($N$-subintervals))**.—**

$$\int_a^b f(x)\,dx = \sum_{k=1}^N \int_{x_{k-1}}^{x_k} f(x)\,dx \approx \sum_{k=1}^N \int_{x_{k-1}}^{x_k} p_k(x)\,dx = \sum_{k=1}^N \frac{1}{2}(x_k - x_{k-1})[f(x_k) + f(x_{k-1})].$$

Assuming $x_k - x_{k-1} = h$ for all $k$ (uniform mesh):

$$\begin{aligned}\int_a^b q_N(x)\,dx &= \frac{h}{2}\sum_{k=1}^N [f(x_{k-1}) - f(x_k)] \\ &= \frac{h}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{N-1}) + f(x_N)].\end{aligned}$$

**Notation**: $T_N(f) = \int_a^b q_N(x)\,dx$, $I(f) = \int_a^b f(x)\,dx$.

**Example 157.—**$I = \int_0^1 x^3\,dx = \frac{1}{4}$. $f(x) = x^3$. Let $h = \frac{1}{4} \implies \frac{b-a}{N} = h \implies \frac{1}{N} = \frac{1}{4} \implies N = 4$:

$$T_4 = \frac{h}{2} f(0) + 2f\frac{1}{4} + 2f\frac{1}{2} + 2f\frac{3}{4} + f(1) = \frac{17}{64} \approx 0.265625$$

**Proposition 29** (Accuracy)**.—**(uniform grid) $h = \frac{b-a}{N}$, $I(f) - I_N(f) = -\frac{b-a}{12} \cdot h^2 \cdot f''(\xi_h)$ for some $\xi_h$ in $[a,b]$. Thus, $|I(f) - I_N(f)| \leq \frac{b-a}{12} h^2 \max_{a \leq x \leq b} |f''(x)|$.

**Remark 27.—**Globally, we have $O(h^2)$ error, and locally we have $O(h^3)$ error.

**Example 158.—**$I = \int_0^1 x^3\,dx$. $h = \frac{b-a}{N}$. How small does $h$ need to be for $I_N$ to approximate $I$ with an error $\leq 10^{-6}$? **Goal**: $\frac{b-a}{12} h^2 \max_{a \leq x \leq b} |f''(x)| \leq 10^{-6}$.

$$\begin{aligned} f(x) &= x^3 \\ f'(x) &= 3x^2 \\ f''(x) &= 6x \\ \max_{0 \leq x \leq 1} |f''(x)| &= 6 \cdot 1 = 6. \\ \implies \frac{1-0}{12} \cdot h^2 \cdot 6 &\leq 10^{-6} \\ &\boxed{h \leq \sqrt{2} \cdot 10^{-3}.}\end{aligned}$$

More: $N = \frac{b-a}{h} = \frac{1}{\sqrt{2} \cdot 10^{-3}} = \frac{10^3}{\sqrt{2}} \approx 707.1$, so need $N \geq 708$.

## § 3.8 Solving Tridiagonal Linear Systems

**Example 159.—**Consider the system of equations

$$\begin{aligned} 6x_1 + x_2 + 0 + 0 &= 8 \\ 2x_1 + 4x_2 + x_3 + 0 &= 13 \\ 0 + x_2 + 4x_3 + 2x_4 &= 22 \\ 0 + 0 + x_3 + 6x_4 &= 27 \end{aligned}$$

for $x_1, x_2, x_3, x_4$. In matrix form:

$$A6100241001420016x_1x_2x_3x_4 = 8132227$$

Tridiagonal:

- Each equation for $x_i$ involves at most $x_{i-1}$ and $x_{i+1}$.
- The matrix form $A =$ has $a_{ij} = 0$ if $j > i+1$ or $j < i-1$.
- Only (length $n-1$), (length $n$), and (length $n-1$) need to be stored for lower diagonal, diagonal, and upper diagonal entries for $n$ unknowns $x_1, \dots, x_n$.

To solve, eliminate unknowns using Gaussian Elimination in matrix form.

- Eliminate $x_1$ from the second equation by subtracting $\frac{1}{3}$ of the first equation.

$$\begin{aligned} 2x_1 + 4x_3 + x_3 &= 13 \\ -\frac{1}{3}(6x_1 + x_2 &= 8) \end{aligned}$$

This results in $\frac{11}{3}x_2 + x_3 = \frac{31}{3}$. Replace the second equation with this new equation:

$$\begin{aligned} 6x_1 + x_2 &= 8 \\ \frac{11}{3}x_2 + x_3 &= \frac{31}{3} \\ x_2 + 4x_3 + 2x_4 &= 22 \\ x_3 + 6x_4 &= 27 \end{aligned}, \qquad \tilde{A}61000\frac{11}{3}1001420016, \qquad \tilde{b}8\frac{31}{3}2227$$

This system has the same solution as $A =$.

- Eliminate $x_2$ from the third equation by subtracting $\frac{3}{11}$ of the second equation and replace the third equation with the result:

$$\begin{aligned} x_2 + 4x_3 + 2x_4 &= 22 \\ -\frac{3}{11}\left(\frac{11}{3}x_2 + x_3 \right. &= \left. \frac{31}{3}\right) \\ \implies \frac{41}{1}1x_3 + 2x_4 &= \frac{211}{11}. \end{aligned}$$

$$\begin{aligned} 6x_1 + x_2 &= 8 \\ \frac{11}{3}x_2 + x_3 &= \frac{31}{3} \\ \frac{41}{11}x_3 + 2x_4 &= \frac{211}{11} \\ x_3 + 6x_4 &= 27 \end{aligned}, \qquad 61000\frac{11}{3}1000\frac{41}{11}20016, \qquad 8\frac{31}{3}\frac{211}{11}27$$

- Eliminate $x_3$ from the fourth equation by subtracting $\frac{11}{41}$ of the third equation:

$$\begin{aligned} x_3 + 6x_4 &= 27 \\ -\frac{11}{41}\left(\frac{41}{11}x_3 + 2x_4 \right.&= \left.\frac{211}{11}\right) \\ \implies \frac{224}{41}x_4 &= \frac{896}{41} \end{aligned}$$

$$\begin{aligned} 6x_1 + x_2 &= 8 \\ \frac{11}{3}x_2 + x_3 &= \frac{31}{3} \\ \frac{41}{3}x_3 + 2x_4 &= \frac{211}{11} \\ \frac{224}{41}x_4 &= \frac{896}{41} \end{aligned}, \qquad \tilde{A}61000\frac{11}{3}1000\frac{41}{11}2000\frac{224}{41}, \qquad \tilde{b}8\frac{31}{3}\frac{211}{11}\frac{896}{41}$$

Backward substitution: Given $x_{i+1}, x_{i+2}, \ldots, x_n$, find $x_i$ [$\tilde{A}$ upper triangular].

$$\frac{224}{41}x_4 = \frac{896}{41}, \qquad \boxed{x_4 = 4}$$

$$\begin{aligned} \frac{41}{11}x_3 + 2x_4 &= \frac{211}{11} \\ \frac{41}{11}x_3 + 2(4) &= \frac{211}{11} \\ \frac{41}{11}x_3 &= \frac{123}{11}, \qquad \boxed{x_3 = 3} \end{aligned}$$

$$\begin{aligned} \frac{11}{3}x_2 + x_3 &= \frac{31}{3} \\ \frac{11}{3}x_2 + 3 &= \frac{31}{3} \\ \frac{11}{3}x_2 &= \frac{22}{3}, \qquad \boxed{x_2 = 2} \end{aligned}$$

$$\begin{aligned} 6x_1 + x_2 &= 8 \\ 6x_1 + 2 &= 8 \\ 6x_1 &= 6, \qquad \boxed{x_1 = 1} \end{aligned}$$

The solution is $= 1234^T$.

**Theorem Algorithm ($n$ equations, $n$ unknowns, tridiagonal).**

$$A =$$

$\vec{\ell}$,,, # *Eliminate* ($A \mapsto \tilde{A}$ *for* $A =$ *tridiag*)

**Theorem.** *When a tridiagonal matrix exhibits diagonal dominance ($d_i > |u_i| + |\ell_i|$ for all $i = 1, \ldots, n$), the algorithm is ensured to complete successfully.*

**Note 39.—** (a) Diagonal dominance guarantees $\tilde{d_i}$ (preventing division by zero).

(b) Diagonal dominance is adequate but not necessary.

(c) The only error arises from roundoff.

## § 3.9 Application: Solving Simple Two-Point Boundary Value Problems

Objective: Approximate the unknown function $u(x)$ over the interval $[a, b]$ such that

$$\begin{cases} -u''(x) + u(x) = f(x), & a < x < b \\ u(a) = g_a \\ u(b) = g_b \end{cases}$$

where $f$ is a given function, and $g_a$, $g_b$ are prescribed values.

**Example 160.—** $\begin{cases} -u''(x) + u(x) = 2\sin(x), & 0 < x < \pi \\ u(0) = u(\pi) = 0 \end{cases}$

Let $u(x) = \sin(x)$, $u'(x) = \cos(x)$, and $u''(x) = -\sin(x)$.

Thus, $-u''(x) + u(x) = -(-\sin(x)) + \sin(x) = 2\sin(x) = f(x)\checkmark$.

Also, $u(0) = \sin(0) = 0\checkmark$ and $u(\pi) = \sin(\pi) = 0\checkmark$.

Approach: Divide $[a, b]$ into $n$ equal subintervals $[x_{k+1}, x_k]$ and approximate $u(x_k)$. Let $h = x_k - x_{k-1}$.

$$\begin{aligned} f(x_k) &= -D_h^2 u(x_k) u''(x_k) + u(x_k) \\ &= -\frac{u(x_k - h) - 2u(x_k) + u(x_k + h)}{h^2} + u(x_k) + \frac{1}{12} h^2 u^{(4)}(\xi_k) \\ &\approx -\frac{1}{h^2} u(x_k - h) + \frac{2}{h^2} + 1 u(x_k) - \frac{1}{h^2} u(x_k + h) \\ &= -\frac{1}{h^2} u(x_{k-1}) + \frac{2}{h^2} + 1 u(x_k) - \frac{1}{h^2} u(x_{k+1}). \end{aligned}$$

Let $u_k \approx u(x_k)$ be defined by (multiply by $h^2$) $-1 \cdot U_{k-1} + (2 + h^1) U_k - 1 \cdot U_{k+1} = h^2 f(x_k) \implies U_k$ for $k = 1, 2, \ldots, N-1$ defined by a tridiagonal system of linear equations with $U_0 = u(x_0) = u(a) = g_a$, $U_N = u(x_N) = u(b) = g_b$.

$*$ we only consider $U_k$ an unknown for $k = 1, \ldots, N-1$.

$k=1$: $-U_0+(2+h^2)U_1-U_2=h^2f(x_1)$, then $-U_0=-g_a$ (move to the right because it's known)

$$\boxed{(2+h^2)U_1-U_2=h^2f(x_1)+g_a}$$

$k=2$: $\boxed{-U_1+(2+h^2)U_2-U_3=h^2f(x_2)}$

$k=N-2$: $\boxed{-U_{N-3}+(2+h^2)U_{N-2}-U_{N-1}=h^2f(x_{N-2})}$

$k=N-1$: $-U_{N-2}+(2+h^2)U_{N-1}-U_N=h^2f(x_{N-1})$ with $U_N=g_b$.

$$\boxed{-U_{N-2}+(2+h^2)U_{N-1}=h^2f(x_{N-1})+g_b}$$

$|u(x_k)-U_k|\le O(h^2)$.

**Recall 47.**—$\begin{cases}-u''+u=f(x)\\ u(a)=g_a\\ u(b)=g_b\end{cases}$, $-D_h^2u+u=f$ with $D_h^2u(x)=\frac{u(x-h)-2u(x)+2u(x+h)}{h^2}\approx u''(x)+O(h^2)$.

**Example 161.**—$-u''(x)+u(x)=2\sin(x)+1$ on $[0,\pi]$ with $u(0)=u(\pi)=1$. [solution: $u(x)=\sin(x)+1$]. Using 6 intervals ($N=6$). $h=\frac{b-a}{N}=\frac{\pi}{6}$. So $x_0=0$, $x_1=\frac{\pi}{6}$, $x_2=\frac{\pi}{3}$, $x_3=\frac{\pi}{2}$, $x_4=\frac{2\pi}{3}$, $x_5=\frac{5\pi}{6}$, $x_6=\pi$. $u_0=u(0)=1$, $u_6=u(\pi)=1$, unknowns $u_k$ for $k=1,\dots,5$.

$$\begin{aligned}
&\begin{bmatrix}2+\frac{\pi^2}{6} & -1 & 0 & 0 & 0\\ -1 & 2+\frac{\pi^2}{6} & -1 & 0 & 0\\ 0 & -1 & 2+\frac{\pi^2}{6} & -1 & 0\\ 0 & 0 & -1 & 2+\frac{\pi^2}{6} & -1\\ 0 & 0 & 0 & -1 & 2+\frac{\pi^2}{6}\end{bmatrix}\begin{bmatrix}u_1\\u_2\\u_3\\u_4\\u_5\end{bmatrix}\\
&=\begin{bmatrix}h^2f(x_1)+g_a\\ h^2f(x_2)\\ h^2f(x_3)\\ h^2f(x_4)\\ h^2f(x_5)+g_b\end{bmatrix}\\
&=\begin{bmatrix}\frac{\pi^2}{36}\left(2\sin\left(\frac{\pi}{3}\right)+1\right)+1\\ \frac{\pi^2}{36}\left(2\sin\left(\frac{\pi}{3}\right)+1\right)+1\\ \frac{\pi^2}{36}\left(2\sin\left(\frac{\pi}{2}\right)+1\right)\\ \frac{\pi^2}{36}\left(2\sin\left(\frac{2\pi}{3}\right)+1\right)\\ \frac{\pi^2}{36}\left(2\sin\left(\frac{5\pi}{6}\right)+1\right)+1\end{bmatrix}\\
&=\begin{bmatrix}\frac{\pi^2}{18}+1\\ \frac{(\sqrt{3}+1)\pi^2}{36}\\ \frac{(\sqrt{2}+1)\pi^2}{36}\\ \frac{(\sqrt{3}+1)\pi^2}{36}\\ \frac{\pi^2}{18}+1\end{bmatrix}
\end{aligned}\tag{4.1}$$

Run algorithm:

$$= 1.47321.80201.87581.80201.4732$$

Graph of points with the exact curve.

# § 4 Root Finding

Goal: Given a function $f(x)$, find $\alpha$ such that $f(\alpha) = 0$.

Idea: Form a sequence $x_k{}_{k=0}^{\infty}$ such that $\lim_{x\infty} x_k = \alpha$.

Global Method $x_k\alpha$ for any initial guess $x_0$.

Local Method $x_k\alpha$ only if $x_0$ is close to $\alpha$.

- Global methods are typically slow (but reliable),
- Local methods are fast only if they work.
- To improve reliability, use a global method first for a few steps and use the output as an initial guess for a faster method.
- Double roots are hard to find.

$f(x) = 0$

## § 4.1 Bisection Method

Global approach for finding zeros where $f(x)$ changes signs. Idea: $f$ is continuous. Suppose $f(a) \cdot f(b) < 0$ (indicating different signs). By the Intermediate Value Theorem (IVT), there exists $\alpha$ in $(a, b)$ such that $f(\alpha) = 0$. Let $c = \frac{a+b}{2}$.

- If $f(c) \cdot f(a) < 0$, then there exists $\tilde{\alpha} \in (a, b)$ such that $f(\tilde{\alpha}) = 0$.
- If $f(c) = 0$, then $\tilde{\alpha} = c$, and $f(\tilde{\alpha}) = 0$ (Done).
- If $f(c) \cdot f(b) < 0$, then there exists $\tilde{\alpha} \in (c, b)$ such that $f(\tilde{\alpha}) = 0$.

We know $a < \alpha < b$ (with width $b - a$). Also, $a < \tilde{\alpha} \leq c$ or $c < \tilde{\alpha} < b$ (with width $\frac{b-a}{2}$). If $f(c) \neq 0$, let $\tilde{a} = a$, $\tilde{b} = c$, or $\tilde{a} = c$, $\tilde{b} = b$, and repeat. Eventually, the process converges to a root $\alpha^* \in (a^*, b^*]$ with $b^* - a^*$ arbitrarily small.

Bisection Method: Let $a_0 = a, b_0 = b$, where $f(a) \cdot f(b) < 0$. For $k = 0, 1, 2, \ldots$

$$c_k = \frac{a_k + b_k}{2} = a_k + \frac{1}{2}(b_k - a_k) \qquad \text{[stable form]}$$

If $f(c_k) = 0$: $\alpha = c_k$. Break. Verbatim if $|c - 0| < tol$.

If $f(a_k) \cdot f(c_k) < 0$, then $a_{k+1} = a_k$ and $b_{k+1} = c_k$, else $a_{k+1} = c_k, b_{k+1} = b_k$. Finally, $k = k + 1$, end.

**Theorem.** *Suppose $f$ is continuous on $[a, b]$ and $f(a) \cdot f(b) < 0$. Let $x_n = c_n = \frac{a_{n-1}+b_{n-1}}{2}$. Then there exists a root $\alpha \in [a, b]$ such that $|x_n - \alpha| \leq \left(\frac{1}{2}\right)^n (b - a)$. Let $\epsilon > 0$. Then $|x_n - \alpha| \leq \epsilon$ if $n \geq \frac{\log(b-a)-\log\epsilon}{\log 2}$.*

**Note 40.**—:

(a) The zero $\alpha$ is not known beforehand. The theorem guarantees the existence of $\alpha$.

(b) This method is generally considered slow.

(c) The error may not necessarily decrease on each step. For instance, $x_k$ might be closer to $\alpha$ than $x_{k+1}$.

We aim to solve $f(x) = 0$.

**Example 162.**—Consider $f(x) = x^3 - 3$ with interval $[a, b] = [0, 2]$. Let $\alpha = \sqrt[3]{3} \approx 1.44225$.

$$x_0 = \frac{a+b}{2} = 1.$$
$$f(0) = -3, \qquad \text{sign change} f(1) = -2, \qquad f(2) = 5$$
$$a_1 = 1, \qquad b_1 = 2, \qquad x_1 = \frac{1+2}{2} = \frac{3}{2} = 1.5.$$
$$f(1) = -2, \qquad f\left(\frac{3}{8}\right), \qquad f(2) = 5$$
$$a_2 = 1, \qquad b_2 = \frac{3}{2}, \qquad x_2 = \frac{1+\frac{3}{2}}{2} = \frac{5}{4} = \text{estimate than}$$

$x_1$ this is a worse 1.25

$$f(1) = -2, \qquad f\left(\frac{5}{4}\right) = -\frac{67}{64}, \qquad f\left(\frac{3}{2}\right) = \frac{3}{8}$$
$$a_3 = 1.25, \qquad b_3 = \frac{3}{2}, \qquad x_3 = \frac{11}{8} = 1.375$$
$$f\left(\frac{5}{4}\right) = -\frac{67}{64}, f\left(\frac{11}{3}\right) = -\frac{205}{512}, \qquad f\left(\frac{3}{2}\right)$$
$$a_4 = \frac{11}{8}, \qquad b_4 = \frac{3}{2}, \qquad x_4 = \frac{23}{16} = 1.4375$$
$$f\left(\frac{11}{8}\right) = -\frac{205}{512}, \qquad f\left(\frac{23}{16}\right) = -\frac{121}{4096}, \qquad f(32) = \frac{3}{8}$$
$$a_5 = \frac{23}{16}, \qquad b_5 = \frac{3}{2}, \qquad x_5 = \frac{47}{32} = 1.46875.$$

After 5 iterations:

- $|x_5 - \alpha| = 0.0265.$
- $f(x_5) = \frac{5519}{32768} \approx 0.16843$ (getting closer to zero).

$$\begin{aligned}
|x_N - \alpha| &\le \epsilon = 10^{-3}. \\
\implies N &\ge \frac{\log(b-a) - \log(10^{-3})}{\log 2} \\
&= \frac{\log(2) + 3\log(10)}{\log 2} \approx 10.966 \qquad \text{(is sufficient)}
\end{aligned}$$

**Note 41.**—Error does not necessarily decrease with each iteration (upper bound does).

## § 4.2 Newton's methods

Derivation and Examples.

Newton's method: Given $x_0$,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

for all $n = 0, 1, 2, \ldots$. Goal: $x_n \alpha$ as $n \infty$, where $f(\alpha) = 0$.

**Note 42.**—:

(a) local method. Behavior/success depends on $x_0$ and $f$.

(b) converges fast when it works (mostly). Repeated roots slow down convergence.

**Derivation**

Approach 1: Taylor's Expansion. Let $x_n \approx \alpha$. Center at $x_n$.

$$0 = f(\alpha) = f(x_n) + (\alpha - x_n)f'(x_n) + O(|\alpha - x_n|^2)$$

Solve for $x_n$:

$$\alpha - x_n \approx -\frac{f(x_n)}{f'(x_n)} \implies \alpha \approx x_n - \frac{f(x_n)}{f'(x_n)}.$$

Approach 2: Graphical. Tangent line at $x_0$, tangent line at $x_1$.

- Find the tangent line at $x_k$.
- Let $x_{k+1}$ be the $x$-intercept of the tangent line (easy root-finding problem).
- Repeat.

Tangent line:

$$y - f(x_k) = f'(x_k)(x - x_k)$$

$$y = f(x_k) + f'(x_k)(x - x_k)$$

$$0 = f(x_k) + f'(x_k)(x - x_k)$$

$$\implies x = x_k - \frac{f(x_k)}{f'(x_k)} \equiv x_{k+1}$$

**Example 163.**—$f(x) = x^3 - 1$, $\alpha = 1$.

$$\begin{aligned}
x_0 &= 2. \\
f'(x) &= 3x^2. \\
x_{k+1} &= x_k - \frac{f(x_k)}{f'(x_k)} = x_k - \frac{x_k^3 - 1}{3x_k^2}. \\
x_0 &= 2. \\
x_1 &= 2 - \frac{2^3 - 1}{3 \cdot 2^2} = \frac{17}{12} = 1.416 \\
x_2 &= \frac{17}{12} - \frac{\frac{17}{12}^3 - 1}{3 \cdot \frac{17}{12}^2} = \frac{5777}{5202} \approx 1.11053441. \\
x_3 &\approx 1.0106367 \\
x_4 &\approx 1.00011156.
\end{aligned}$$

**Example 164.**—$f(x) = \ln(x)$. $x_0 = e$.

$$\begin{aligned}
f'(x) &= \frac{1}{x} \\
x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)} = x_0 = \frac{\ln(x_0)}{1/x_0} = x_0 - x_0 \ln(x_0) = e - e(1) = 0. \\
x_2 &= x_1 - x_1 \ln(x_1), \qquad \ln(0) \text{ undefined}
\end{aligned}$$

**Example 165.**—$f(x) = \ln(x)$. $x_0 = e$.

$$\begin{aligned}
f'(x) &= \frac{1}{x} \\
x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)} = x_0 = \frac{\ln(x_0)}{1/x_0} = x_0 - x_0 \ln(x_0) = e - e(1) = 0. \\
x_2 &= x_1 - x_1 \ln(x_1), \qquad \ln(0) \text{ undefined}
\end{aligned}$$

**Example 166.**—$f(x) = x^2$, $x_0 = 1$, $\alpha = 0$.

$$\begin{aligned}
f'(x) &= 2x \\
x - \frac{f(x)}{f'(x)} &= x - \frac{x^2}{2x} = x - \frac{1}{2}x = \frac{1}{2}x \qquad \text{for } x \neq 0 \\
x_0 &= 1. \\
x_1 &= \frac{1}{2}. \\
x_2 &= \frac{1}{4}. \\
x_3 &= \frac{1}{8}.
\end{aligned}$$

$x_k = \left(\frac{1}{2}\right)^k 0$ as $k \infty$.

Find $x$ such that $f(x) = 0$, use $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ for $n = 0, 1, 2, \ldots$. Goal: $x_n \alpha$ for $f(\alpha) = 0$.

**Example 167.**—Consider a function $f(x)$ with $f'(p) = 0$ and $\alpha < p$ and an asymptote such that $f(x)0^+$ as $x\infty$. Then $x_0 < p \implies x_n\alpha$ and $x_0 > p \implies x_n\infty$ and for $x_0 = p$ the method is undefined since $f'(p) = 0$.

**Example 168.**—Consider a function with graph 2. Then $|x_0| < \beta \implies x_n\alpha$ and $x_0 = \beta \implies x_{2n+1} = -\beta, x_{2n} = \beta$ and $|x_0| > \beta \implies$ diverges.

**Example 169.**—Concave up function, $x_0 > 0 \implies x_n\alpha$ and $x_0 < 0 \implies x_n\alpha_{-1}$ and $x_0 = 0 \implies x_1$ undefined.

## § 4.3 How to Terminate Newton's Method

Objective: Establish criteria for determining $N$ such that $x_N \approx \alpha$ for the sequence $x_{n_{n=0}}^{N}$ obtained through Newton's Method.

Assumption: $x_n\alpha$ as $n\infty$.

IF the method converges, THEN we can identify the point when $|\text{error} x_N - \alpha| <$ TOL and terminate the search.

Assumption: $f'(\alpha) \neq 0$, and $f$ and $f'$ are continuous on $[a, b]$ with $a < \alpha < b$.

* Choose an interval $[\tilde{a}, \tilde{b}]$ such that
  - $\alpha \leq \tilde{a} < \alpha$, $\alpha < \tilde{b} \leq b$
  - $C|f'(\alpha)| = M \geq |f'(x)| \geq m = \frac{|f'(\alpha)|}{C}$, $C > 1$, for al $\tilde{a} < x < \tilde{b}$.
  - $x_n \in [\tilde{a}, \tilde{b}]$ for all $n$ large enough. Convergence: $|x_n - \alpha| < \epsilon$ for all $n$ large.

Idea: By the Mean Value Theorem (MVT), $\exists \xi_n$ between $x_n$ and $\alpha$ such that $f(x_n) - f(\alpha) = f'(\xi_n)(x_n - \alpha) \implies \boxed{(x_n - \alpha) = \frac{f(x_n)}{f'(x_n)}}$

Case 1: $|f(x_n)|$ small

$$\begin{aligned} |x_n - \alpha| &= \frac{|f(x_n)|}{|f'(x_n)|} \leq \frac{|f(x_n)|}{\min_{\tilde{a} \leq \xi \leq \tilde{b}} |f'(\xi)|} \\ &\leq \frac{|f(x_n)|}{m} = \frac{1}{m}|f(x_n)| \ < \text{TOL} \end{aligned}$$

Typically $<$ 5TOL. This implies termination at $N$ if $|f(x_N)| \leq m$TOL.

Criterion 2: $|x_m - x_n|$ small. $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \implies f(x_n) = f'(x_n)(x_n - x_{n+1})$. Last time:

$$x_n - \alpha = \frac{f(x_n)}{f'(x_n)} = \frac{f'(x_n)}{f'(\xi_n)}(x_n - x_{n+1}),$$
$$|x_n - \alpha| = \frac{f'(x_n)}{f'(\xi_n)} \cdot |x_n - x_{n+1}| \leq \frac{M}{m}|x_{n+1} - x_n|$$

implies termination at $N$ if $|x_N - x_{N-1}| < \frac{m}{M}\text{TOL} \leq \text{TOL}$.

**Note 43.—**:

- Both $|f(x_n)|$ and $|x_n - x_{n-1}|$ are computable and can be checked on each iteration.
- $N$ is not known beforehand.
- Conservatively, we terminate if $|f(x_n)| + |x_n - x_{n-1}| \leq \text{TOL}/5$.
- The test can lead to false conclusions if $x_n \not\alpha$.

* Need TOL small enough to rule out false answers but large enough to account for roundoff error.

## § 4.4 The Newton Error Formula

Objective: Classify the convergence speed of Newton's method when $x_n \alpha$.

**Theorem.** *Let $f \in C^2(I = [a,b])$ with $f(\alpha) = 0$ for some $\alpha \in I$ (not an endpoint). If $x_n \in I$ and $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ [Newton], then $\exists \xi_n$ between $\alpha$ and $x_n$ such that $(\alpha - x_{n+1}) = -\frac{1}{2}(\alpha - x_n)^2 \cdot \frac{f''(\xi_n)}{f'(x_n)}$.*

**Note 44.—**:

(a) Does not guarantee $x_{n+1} \in I$.

(b) error$|x_{n+1} - \alpha| = \frac{f''(\xi_n)}{2f'(x_n)} \cdot$ old error.$|x_n - \alpha|^2$

Let $M_n = \frac{f''(\xi_n)}{2f'(x_n)}$. If $M_n$ is bounded for all $n$, then the new error $= O(|\text{old error}|^2)$. If $|\text{old error}| < 1$, then $|\text{new error}|$ is much smaller $\implies$ fast convergence once $|x_n - \alpha|$ is small enough.

**Proof.** Taylor series, center at $x_n$: $0 = f(\alpha) = f(x_n) + (\alpha - x_n)f'(x_n) + \frac{1}{2}(\alpha - x_n)^2 f''(\xi_n)$ for some $\xi_n$ between $\alpha$ and $x_n$.

$$(x_n - \alpha)f'(x_n) - f(x_n) = \frac{1}{2}(\alpha - x_n)^2 f''(\xi_n).$$

$$(x_1 - \alpha) - \frac{f(x_n)}{f'(x_n)} = \frac{1}{2}(\alpha - x_n)^2 \frac{f''(\xi_n)}{f'(x_n)} \qquad \text{RHS}\checkmark$$

$$\text{LHS} = x_n - \alpha - \frac{f(x_n)}{f'(x_n)} = x_{n+1} - \alpha \qquad \checkmark$$

**Example 170.**—Suppose for some $f$

- $|f''(x)| \leq 10$ for all $x$,
- $|f'(x)| \geq 3$ for all $x$,
- $|x - \alpha| < \frac{1}{2}$.

a) Bound the error in the 1st three steps of Newton's methods.

$$\begin{aligned}
|x_1 - \alpha| &\leq \frac{|f''(\xi_1)|}{2|f'(x_0)|} \cdot |x_0 - \alpha|^2, \qquad |f'(x)| \geq 3 \implies \frac{1}{|f'(x_0)|} \leq \frac{1}{3} \\
&\leq \frac{10}{2 \cdot 3} \cdot \frac{1}{2}^2 = \frac{5}{12} = 0.416. \\
|x_2 - \alpha| &\leq \frac{10}{2 \cdot 3}|x_1 - \alpha|^2 \\
&\leq \frac{10}{6} \cdot \frac{5}{12}^2 = \frac{125}{432} = 0.2893518 \\
|x_3 - \alpha| &\leq \frac{10}{6} \cdot \frac{125}{432}^2 \approx 0.13954 \\
&\vdots \\
|x_6 - \alpha| &\leq 5.135 \times 10^{-6} \\
&\vdots \\
|x_8 - \alpha| &\leq 3.219 \times 10^{-21}
\end{aligned}$$

b) How small does the initial error need to be to guarantee convergence? $|x_n - \alpha| 0$. Need $|x_0 - \alpha| < \frac{6}{10}$.

**Note 45.**—The method may converge for $x_0$ further from $\alpha$. It will converge for $|x_0 - \alpha| < \frac{6}{10}$.

Let $\frac{f''(\xi)}{2f'(x)} \leq M$, $e_n = |x_n - \alpha|$. Relate $|x_0 - \alpha| = e_0$ to $M$.

$$\begin{aligned}
e_1 &\leq Me_0^2 = M(Me_0)^2 \\
e_2 &\leq Me_1^2 = M \cdot (M)^2(Me_0)^{2^2} = M(Me_0)^4 \\
e_3 &\leq M(Me_0)^8.
\end{aligned}$$

In general, $\boxed{e_n \leq M(Me_0)^{2^n}}$ for all l$n \geq 0$. Then $\lim_{n\infty} e_n = 0$ if $\boxed{|Me_0| < 1}$. In the example, $M = \frac{10}{6}$, so

$$\frac{10}{6} e_0 < 1 \implies e_0 < \frac{6}{10}, \qquad \boxed{|x_0 - \alpha| < \tfrac{6}{10}}$$

If $x_0 \in (\alpha - \frac{6}{10}, \alpha\frac{6}{10})$, then $x_n\alpha$ as $n\infty$.

**Note 46.**—The method may converge for $x_0$ further from $\alpha$. It will converge for $M|x_0 - \alpha| < 1$.

**Definition.**—Let $x_n\alpha$ such that

$$\lim_{n\infty} \frac{|\alpha - x_{n+1}|}{|\alpha - x_n|^p} = C$$

for some nonzero (finite) constant $C$ and some $p$. The number $p$ is the order of the convergence for the sequence.

**Example 171** (Newton's)**.**—$\frac{|\alpha - x_{n+1}|}{|\alpha - x_n|^2} = \frac{1}{2}\frac{f''(\xi_n)}{f'(x_n)}$. Suppose $f'(\alpha) \neq 0$ and $f''$ is bounded. $f'$ constant $\implies f'(x_n f'(\alpha)$. $|f''|$ bounded. $f''$ constant $\implies f''(\xi_n)f''(\alpha)$. $\xi_n$ between $x_n$ and $\alpha$. Limit holds by squeeze theorem. $f''(x_n)f''(\alpha) \implies f''(\xi_n)f''(\alpha)$. $(\xi_n\alpha)$. Thus,

$$\begin{aligned}
\lim_{n\infty} \frac{|\alpha - x_{n+1}|}{|\alpha - x_n|^2} &= \frac{1}{2}\frac{f''(\alpha)}{f'(\alpha)} = C \in (0, \infty) \\
&\leq \frac{1}{2}\frac{\max}{|f'(\alpha)|}.
\end{aligned}$$

If $f''$ constant and $f'(\alpha) \neq 0, f''(\alpha) \neq 0$, then $p = 2$.

**Example 172.**—For $f(x) = x^2$, Newton's method converges for all $x_0$ with $p = 1$.

## § 4.5 Newton's Method: Theory and Convergence

Objective: Establish conditions ensuring the convergence of Newton's method to $\alpha$.

**Theorem Optimistic.** *Assume $f$ is twice continuously differentiable for all $x \in (-\infty, \infty)$, with $f(\alpha) = 0$ for some $\alpha$. Define the ratio*

$$M = \frac{\sup_{x\in\mathbb{R}} |f''(x)|}{2\inf_{x\in\mathbb{R}} |f'(x)|}. \qquad \textit{book}: \frac{\max}{\min}, \textit{correcting for asymptotes}$$

*Assume $M < \infty$. Then, for any $x_0$ such that $M|\alpha - x_0| < 1$, Newton's method converges. Moreover, $|\alpha - x_n| \leq M(M|\alpha - x_0|)^{2^n}$.*

* *Need $x_0$ close enough to have $M|\alpha - x_0| < 1$ (only need $M|\alpha - x_k| < 1$ for some $k \geq 0$).*

*Optimistic $\implies$ only need local estimates for $M$, not global.*

With such strong assumptions, we can find how big $n$ needs to be. Goal: $|x_N - \alpha| \leq \epsilon$. Observe

$$\begin{aligned} M(M|\alpha - x_0|)^{2^n} &\leq \epsilon \\ (M|\alpha - x_0|)^{2^n} &\leq M\epsilon \\ 2^n {<}\, 0 \log_2({<}\, 1 M|\alpha - x_0|) &\leq \log_2(M\epsilon) \\ 2^n &\geq \frac{\log_2(M\epsilon)}{\log_2(M|\alpha - x_0|)} \end{aligned}$$

$$\boxed{n \geq \log_2\left(\frac{\log_2(M\epsilon)}{\log_2(M|\alpha - x_0|)}\right).}$$

**Example 173.**—Consider $f$ such that

- $f(a) = 0$ for $\alpha \in [2, 3]$,
- $f'(x) \geq 3$ and $0 \leq f''(x) \leq 5$ for all $x \in \mathbb{R}$.

Show the method converges for $y_0 = \frac{5}{2}$. How many iterations to get $10^{-4}$ accuracy.

$$M = \frac{\sup |f''|}{2\inf |f'|}$$

$M \leq \frac{5}{2\cdot 3} = \frac{5}{6}$. Then $|x_0 - \alpha| \leq \frac{1}{2}$ ($x_0$ midpoint of $[2, 3]$). $M|x_0 - \alpha| \leq \frac{5}{6} \cdot \frac{1}{2} = \frac{5}{12} < 1$ implies convergence. Need

$$n \geq \log_2\left(\frac{\log_2(\frac{5}{6} \cdot 10^{-4})}{\log_2(5/12)}\right) \approx 3.4234$$

Need $n \geq 4$ iterations.

$$\begin{aligned} M|x_0 - \alpha| &> 1 \\ \frac{5}{6} \cdot |x_0 - \alpha| &> 1 \\ |x_0 - \alpha| &> \frac{6}{5}. \end{aligned}$$

$\alpha$ in $[0, 4]$, $x_0 = 2$. $|x_0 - \alpha| \leq 2$. Run bisection with tol $= \frac{6}{5}$. $x_N$ with $|x_N - \alpha| < \frac{5}{6}$. Use $x_N$ for Newton, this is for guaranteed convergence.

**Example 174.**—$f$ smooth and monotone with root $x = \alpha$. Newton's method may not converge for all $x_0$. Let

$$\begin{aligned} f(x) &= \tan^{-1}(x) \\ f'(x) &= \frac{1}{1+x^2} > 0 \qquad \forall t \\ f(0) &= 0 \\ f''(x) &= -\frac{2x}{(1+x^2)^2} \qquad \text{(bounded)} \end{aligned}$$

($M = \infty$ since $\inf_{x\in\mathbb{R}} |f'(x)| = 0$ and $0 < \sup_{x\in\mathbb{R}} |f''(x)| < \infty$). If $|x_0|$ is large enough, then $x_n \not\to 0$.

**Proof.**

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\ &= x_n - \frac{\tan^{-1}(x_n)}{1/(x^2+1)} \\ &= x_n - (x_n^2 + 1)\tan^{-1}(x). \end{aligned}$$

Suppose $\tan^{-1}(|x_n|) \geq 1$. ($|a - b| \geq |a| - |b|$ inverse triangle inequality). Then,

$$\begin{aligned} |x_{n+1}| &= |x_n - (x_n^2 + 1)\tan^{-1}(x_n)| \\ &\geq |(x_n^2 + 1)\tan^{-1}(x_n)| - |x_n| \\ &\geq |x_n|^2 + 1 - |x_n| \\ &\geq |x_n|^2 - |x_n| \\ &= |x_n|(|x_n| - 1). \end{aligned}$$

If $|x_{n+1}| \geq |x_n|$, the method will not converge. Solving:

$$\begin{aligned} |x_n| \cdot (|x_n| - 1) &\geq > 0|x_n| \\ |x_n| - 1 &\geq 1 \\ |x_n| &\geq 2. \end{aligned}$$

For $|x_n| \geq 2$ and $|\tan^{-1}(x_n)| \geq 1$,

we have $|x_{n+1}| \geq |x_n|$ $(\geq \cdots \geq |x_0|)$. Pick $x_0$ such that $|x_0| \geq 2$ and $|\tan^{-1}(x_0)| \geq 1$.

**Theorem Local Result.** *Let $f \in C^2(I)$, where $\alpha \in I = [a,b] \subset \mathbb{R}$ is a root. Assume $f'(\alpha) \neq 0$, and let $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$. For $x_0$ sufficiently close to $\alpha$, we have $\lim_{n\infty} x_n = \alpha$ and $\lim_{n\infty} \frac{|\alpha - x_{n+1}|}{|\alpha - x_n|^2} = -\frac{f''(\alpha)}{2f'(\alpha)}$.*

*Proof idea.* There exists $\tilde{I} = [\tilde{a}, \tilde{b}] \subseteq [a,b]$ such that

- $\alpha \in \tilde{I}$,
- $M = \frac{\max_{x\in\tilde{I}} |f''(x)|}{\min_{x\in\tilde{I}} |f'(x)|} < \infty$,
- $x_{n+1}$ is in $\tilde{I}$ for $x_n$ in $\tilde{I}$,
- $M|\alpha - x_0| < 1$ for $x_0 \in \tilde{I}$.

$M < \infty$: $f''$ continuous on $\tilde{I}$ (bounded), $f'(\alpha) \neq 0 \implies |f'| \geq \frac{|f'(\alpha)|}{2}$ on $\tilde{I}$. Error

$$|\alpha - x_{n+1}| \sim M(M|\alpha - x_n|)^2 \leq |x_0 - \alpha|$$

- small enough for $M|\alpha - x_n| < 1$
- small enough for $|x_{n+1} - \alpha| \leq |x_n - \alpha| \implies x_{n+1}$ in $\tilde{I}$.

Key: only need $M$ for $x$-values near $\alpha$, implies need to start closer to $\alpha$. (May converge otherwise. This is for a guarantee).

Objective: Approximate $f'(x_n)$ in Newton's method to avoid calculating $f'$.

Idea: For $x_n \alpha$, $x_{n-1}$ is close to $x_n$. Let $h_n = x_{n-1} - x_n$. Then,

$$f'(x_n) = \lim_{h0} \frac{f(x+h) - f(x)}{h} \approx \frac{f(x_n + h_n) - f(x_n)}{h_n} = \frac{f(x_{n-1}) - f(x_n)}{x_{n-1} - x_n}$$

Secant Method: $x_{n+1} = x_n - f(x_n) \cdot \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}$ for $x_0, x_1$ given.

**Note 47.**—Reuses the values $f(x_n), f(x_{n-1})$ to avoid additional overhead in approximating $f'$.

Geometrically, the secant method uses the secant line for $x_{n-1}, x_n$ to find $x_{n+1}$ instead of the tangent line at $x_n$.

Stopping Criteria:

- $|f(x_n)|$ small
- $|x_n - x_{n-1}|$ small
  $\alpha - x_n = c_n(x_n - x_{n-1})$ and $c_n 1$ as $n \infty$ if $x_n \alpha$.

Error Formula: $\alpha - x_{n+1} = -\frac{1}{2}(\alpha - x_n)(\alpha - x_{n-1})\frac{f''(\xi_n)}{f'(\eta_n)}$ for some $\xi_n, \eta_n$ such that

$$\min\{\alpha, x_n, x_{n-1}\} \leq \xi_n, \eta_n \leq \max\{\alpha, x_n, x_{n-1}\}.$$

Rate of convergence: $p = \frac{1+\sqrt{5}}{2} \approx 1.618$.

(Optimistic) convergence: $M = \frac{\sup_{x\in\mathbb{R}}|f''(x)|}{2\inf_{x\in\mathbb{R}}|f'(x)|} \in [0, \infty)$. Then $x_n \alpha$ if $M(\max\{|x_0 - \alpha|, |x_1 - \alpha|\}) < 1$. See §3.11.3 for proofs + local analysis. $*$ convergence for $f$ nice and $x_0, x_1$ are close enough to $\alpha$. Slower rate than Newton's, but less work per step.

**Example 175.**—$f(x) = 2^x - 4$, $\alpha = 2$. $x_0 = 0$, $x_1 = 1$. Find $x_3$.

$$x_2 = x_1 - f(x_1) \cdot \frac{x_1 - x_0}{f(x_1) - f(x_0)} = 1 - (-2) \cdot \frac{1}{-2 - (-3)} = 1 + 2(1) = 3.$$
$$x_3 = x_2 - f(x_2) \cdot \frac{x_2 - x_1}{f(x_2) - f(x_1)} = 3 - (4) \cdot \frac{3 - 1}{4 - (-2)} = \frac{5}{3}.$$

$|\alpha - x_7| \approx 2.091 \times 10^{-6}$. $|\alpha - x_8| \approx 3.422 \times 10^{-10}$.

**Example 176.**—$f(x)$ such that

- $f(a) = 0$ for $0 < \alpha < 8$, $f'(\alpha) \neq 0$, $f(0)f(8) < 0$.
- $|f''(x)| \leq 6$
- $|f'(x)| \geq 2$.

How many steps of the bisection method are needed to generate $x_0, x_1$ such that secant method converges? $M \leq \frac{6}{2\cdot 2} = \frac{3}{2}$. Need $\frac{3}{2}|x_i - \alpha| < 1$ so $|x_i - \alpha| < \frac{2}{3}$ for $i = 0, 1$. Not possible on $[0, 8]$ without more information. Bisection: $|x_N - \alpha| < \frac{2}{3}$ for $N \geq \frac{\log(b-a) - \log(\frac{2}{3})}{\log 2} \approx 3.58$ for $b - a = 8$. $\tilde{x}_0$ requires/based on 4 iterations of bisection. $\tilde{x}_1$ requires 5 iterations. Need 5 iterations of Bisection to generate $\tilde{x}_0, \tilde{x}_1$ for secant method.

## § 4.6 Fixed-Point Iterations

Goal: Develop an abstract convergence tool for root-finding methods.

**Definition.**—$x = \beta$ is a fixed point of $g(x)$ if $g(\beta) = \beta$.

Idea: Solve $g(x) = x$ for $x = \beta$ i.e., solve root-finding$g(x) - x = 0$.

**Example 177.**—$f(x) = x^2$, $\alpha - 0$ root.

$$\begin{aligned} f(x) &= x \\ x^2 &= x \\ x(x-1) &= 0 \\ x &= 0, 1 \end{aligned}$$

$\beta = 0, 1$ fixed points.

Relationship to root-finding:

$$\begin{aligned} f(x) &= 0 \\ f(x) + x - x &= 0 \\ g(x)f(x) + x) &= x \end{aligned}$$

$x = \alpha$ is a zero of $f(x)$ if and only if $x = \alpha$ is a fixed
point of $g(x) = f(x) + x$. We can go from $f(x) = 0$ to $g(x) = \alpha$. Non-unique:

**Example 178.**—$f(x) = 3x^2 - 2x = 0$. $g(x)3x^3 - x = x$.

$$3x^3 = 2x, \qquad x^3 = \frac{2}{3}x, \qquad x = g(x)3\frac{2}{3}x$$

Fixed-Point Iteration/Solver:
Want $x = g(x)$.
Choose $x_0$.
Let $x_{n+1} = g(x_n)$ for $n = 0, 1, 2, \ldots$.
(Hopefully $x_n \beta$).

**Example 179.**—Let $f(x) = 0$. Apply Newton's Method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

This is a fixed-point iteration for $g(x) = x - \frac{f(x)}{f'(x)}$. Observe: $x = x - \frac{f(x)}{f'(x)}$, $0 = -\frac{f(x)}{f'(x)} \implies f(x) = 0$ if $f'(x) \neq 0$.

* We can analyze root-finding methods with the form $x_{n+1} = g(x_n)$.

**Theorem.** *Let $g \in ab$ with $a \leq g(x) \leq b$ for all $a \leq x \leq b$. Then* *(a) $g$ has at least one fixed-point in $[a,b]$.*

*(b) If there exists $\gamma < 1$ such that $|g(x) - g(y)| \leq \gamma|x - y|$ for all $x, y$ in $[a,b]$, then*

*a) the fixed-point $\alpha$ is unique*

*b) The iteration $x_{n+1} = g(x_n)$ converges to $\alpha$ for any $x_0$ in $[a,b]$.*

*c) $|\alpha - x_n| \leq \frac{\gamma^n}{1-\gamma}|x_1 - x_0|$.*

*(c) If $g \in 1ab$ with $x \in \max_{[a,b]} |g'(x)| = \gamma < 1$. Then,*

*a) $\alpha$ is unique*

*b) $x_{n+1} = g(x_n)\alpha$ for any $x_0 \in [a,b]$.*

*c) $|\alpha - x_n| \leq \frac{\gamma^n}{1-\gamma}|x_1 - x_0|$.*

*d) $\lim_{n\infty} \frac{\alpha - x_{n+1}}{\alpha - x_n} = g'(\alpha)$.*

$*$ *Yields a local convergence theorem.*

- *verify $a \leq g(x) \leq b$ for all $x \in [a,b]$ where $\alpha \in [a,b]$ for some $a, b$.*
- *Verify $|g'(x)| \leq \gamma < 1$ for all $a \leq x \leq b$.*

*(Find $a, b$)*

**Example 180.**—Let $g(x) = \frac{1}{2}\cos(x)$, $[a,b] = [0, \frac{1}{2}]$. $\alpha = \frac{1}{2}\cos(\alpha)$.

- $g(x) = \frac{1}{2}\cos(x)$, $0 \leq \frac{1}{2}\cos(x) \leq \frac{1}{2}$ for $0 \leq x \leq \frac{1}{2}$. ✓
- $g'(x) = -\frac{1}{2}\sin(x)$. $|g'(x)| \leq \frac{1}{2} = \gamma < 1$. ✓

$\implies$ $\alpha$ exists such that $\alpha = g(\alpha) = \frac{1}{2}\cos(\alpha)$. Letting $x_{n+1} = g(x_n)$ and $x_0 \in [0, \frac{1}{2}]$, we have $x_n\alpha \approx 0.450184$. How many iterations are needed to guarantee $|x_n - \alpha| <$ TOL for $x_0 = 0$?

$$|\alpha - x_n| \leq \frac{\gamma^n}{1-\gamma}|x_1 - x_0|.$$

$\gamma = \frac{1}{2}$. $x_1 = g(x_0) = \frac{1}{2}\cos(0) = \frac{1}{2}$.

$$\frac{\gamma^n}{1-\gamma}|x_1 - x_0| = \frac{\frac{1}{2}^n}{1/2}\frac{1}{2} = \frac{1}{2^n}.$$

$\frac{1}{2^n} <$ TOL, $2^n > \frac{1}{\text{TOL}}$, $n > \log_2 \frac{1}{\text{TOL}}$. $x - \frac{1}{2}\cos x = 0$.

**Theorem rate.** $x_{n+1} = g(x_n)$. *Suppose $g \in pab$ with $\alpha \in [a,b]$, $g(\alpha) = \alpha$. If $g'(\alpha) = g''(\alpha) = \cdots = g^{(p-1)}(\alpha) = 0$ but $g^{(p)}(\alpha) \neq 0$, then $x_n\alpha$ with order $p$ for $x_0$ sufficiently close to $\alpha$.*

**Example 181.**—Show Newton's method has rate at least 2 for $f'(\alpha) \neq 0$, $f(\alpha) = 0$.

$$g(x) = x - \frac{f(x)}{f'(x)}$$

Then

$$g'(x) = \frac{f(x)f''(x)}{[f'(x)]^2}.$$

$$g'(\alpha) = \frac{f(\alpha)f''(\alpha)}{[f'(\alpha)]^2} = 0, \qquad \text{(assuming } f'' \text{ okay)}$$

Review: Solve $f(x) = 0$ – root finding. or solve $g(x) = x$ – fixed point problem. To relate the two ideas, $g(x) = x + f(x)$. Start with $x_0$: update $x_{n+1} = g(x_n)$. $g(x) = x - \frac{f(x)}{f'(x)}$.

**Theorem.** *Let $g \in ab$, $a \leq g(x) \leq b$ for all $x \in [a, b]$. There is a $\gamma < 1$ such that*

$$|g(x) - g(y)| \leq \gamma |x - y|.$$

*(a) unique $\alpha$ such that $g(\alpha) = \alpha$.*

*(b) The iteration of $x_{n+1} = g(x_n)$ converges to $\alpha$ for any $x_0[a, b]$.*

*(c) $|\alpha - x_n| \leq \frac{\gamma^n}{1-\gamma}|x_1 - x_0|$.*

*If $g \in 1ab$ with $\max_x |g'(x)| = \gamma < 1$ then*

$$\alpha \textit{ is uniqueconvergencedistance} + \lim_{n\infty} \frac{\alpha - x_{n+1}}{\alpha - x_n} = g'(\alpha)$$

## § 4.7 Special Topics

### Multiple Roots

Suppose $f'(\alpha) = 0$

**Lemma.**—*If $f \in kab$ (derivatives up to $k-1$) are all 0 at $x$, $f(\alpha) = f'(\alpha) = \cdots = f^{(k)}(\alpha) = 0$ and $f^{(n)}(\alpha) \neq 0$, then $f(x) = (x - \alpha)^k F(x)$, $F(\alpha) \neq 0$.*

**Example 182.**—Suppose $f(x) = (x-\alpha)^2 F(x)$, $F(\alpha) \neq 0$, with

$$\begin{aligned}
g(x) &= x - \frac{f(x)}{f'(x)} \\
&= x - \frac{(x-\alpha)^2 f(x)}{2(x-\alpha)F(x) + (x-\alpha)^2 F'(x)} \\
&= x - \frac{(x-\alpha)F(x)}{2F(x) + (x-\alpha)F'(x)} \\
g'(x) &= 1 - \frac{[f(x) - (x-\alpha)F'(x)][\text{Denom}] - (x-\alpha)F(x)[\text{Denom}]'}{[\text{Denom}]^2} \\
&= 1 - \frac{F(\alpha)\text{Denom}[\alpha]}{\text{Denom}[\alpha]^2} \\
&= 1 - \frac{F(\alpha)}{\text{Denom}[\alpha]} = 1 - \frac{F(\alpha)}{2F(\alpha)} = \boxed{\tfrac{1}{2}} \neq 0
\end{aligned}$$

Modified Newton's Method

Let $u(x) = f(x)/f'(x)$. $u(x)$ has root $\alpha$ with multiplicity 1. Apply Newton's method to $u(x)$ to find $\alpha$.

$$\begin{aligned}
f(x) &= (x-\alpha)^k F(x) \\
u(x) &= \frac{(x-\alpha)^k F(x)}{k(x-\alpha)^{k-1}F(x) + (x-\alpha)^k F'(x)} \\
&= \frac{x-\alpha)F(x)}{kF(x) + (x-\alpha)F'(x)} \\
u(\alpha) &= \frac{0}{kF(\alpha)} = 0 \\
u'(x) &= (\text{same thing as above}) \\
u'(\alpha) &= \frac{F(\alpha)}{\text{Denom}(\alpha)} = \frac{F(\alpha)}{kF(\alpha)} = \frac{1}{k} \neq 0 \\
x_{n+1} &= x - \frac{u(x_n)}{u'(x_n)} \qquad \text{converges to } \alpha
\end{aligned}$$

## § 4.8 Hybrid Algorithm

**Objective**: Combine the Bisection Method with the Secant Method.

**Approach**: The $k$th interval $[a_k, b_k]$ is such that either $x_k = a_k$ or $b_k$, and $x_{k-1} = b_k$ or $a_k$.

Assuming $f(a_1)f(b_1) < 0$, set $x_0 = a_1$ and $x_1 = b_1$. For $k = 1, 2, 3, \ldots$, apply the Secant Method:

$$c = x_k - f(x_k) \cdot \frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})}$$

If $a_k \leq c \leq b_k$, update the brackets. If $f(a_k)f(c) < 0$, update as follows:

$$\begin{cases} a_{k+1} &= a_k \\ b_{k+1} &= c \end{cases}$$

Otherwise, if $f(b_k)f(c) < 0$, update as:

$$\begin{cases} a_{k+1} &= c \\ b_{k+1} &= b_k \end{cases}$$

and set $x_{k+1} = c$. If the Secant Method fails, resort to bisection with $c = a_k + \frac{1}{2}(b_k - a_k)$.

Again, if $f(a_k)f(c) < 0$, update the brackets:

$$\begin{cases} a_{k+1} &= a_k \\ b_{k+1} &= c \end{cases}$$

Otherwise, update as:

$$\begin{cases} a_{k+1} &= c \\ b_{k+1} &= b_k \end{cases}$$

and set $x_{k+1} = c$.

# § 5 Interpolation + Approximation

**Objective**: Find a simpler function $p(x)$ to approximate $f(x)$ using a set of data $x_i, y_i = f(x_i)$, along with information about derivatives.

## § 5.1 Lagrange Interpolation

**Objective**: Given $n+1$ data values $y_i = f(x_i)$ for distinct nodes $\{x_i\}_{i=1}^{n}$, find a polynomial of degree $\leq n$ such that $p_n(x_i) = y_i$ for all $i = 0, 1, \ldots, n$.

**Example 183.**—Consider $f(x) = |x|$. For $x_0 = -1$, $x_1 = 0$, and $x_2 = 1$, find $p_2(x)$. Assume $p_2(x) = x^2$:

$$\begin{aligned} p_2(x) &= a_0 + a_1 x + a_2 x^2 \\ 1 = f(-1) &= p_2(-1) = a_0 - a_1 + a_2 \\ 0 = f(0) &= p_2(0) = a_0 \\ 1 = f(1) &= p_2(1) = a_0 + a_1 + a_2 \end{aligned}$$

This results in $a_0 = 0$, $a_1 = 0$, and $a_2 = 1$, leading to $p_2(x) = x^2$.

**Alternative Construction**: Utilize Lagrange Basis Functions $L_i^{(n)}$ for $p_n$ such that:

$$p_n(x) = \sum_{i=0}^{n} L_i^{(n)}(x) \cdot f(x_i)$$

The Lagrange Basis Functions are defined as:

$$L_i^{(n)}(x) = \prod_{\substack{k \neq i \\ k=0}}^{n} \frac{x - x_k}{x_i - x_k}$$

where the product excludes $k = i$. These functions ensure $p_n(x_i) = f(x_i)$.

**Theorem Lagrange Interpolation Existence + Uniqueness.** *Let $x_i \in I$, $0 \leq i \leq n$, with $x_i \neq x_j$ for all $i \neq j$. There exists a unique polynomial of degree $\leq n$ such that $p_n(x_i) = f(x_i)$ for a given function $f \in C(I)$.*

**Proof. Existence:** $p_n(x) = \sum_{i=0}^{n} L_i^{(n)}(x) f(x_i)$.

**Uniqueness:** Assume there exists another polynomial $q(x)$ such that $q(x_i) = f(x_i)$ for all $i = 0, \ldots, n$. Let $r(x) = p_n(x) - q(x) \implies r$ has degree $\leq n$. Then $r(x)$ has degree $\leq n$ and:

$$r(x_i) = p_n(x_i) - q(x_i) = f(x_i) - f(x_i) = 0$$

for all $i = 0, 1, \ldots, n \implies r(x)$ has $n + 1$ zeros. Therefore, $r(x) = 0$ since degree $n$ polynomials have at most $n$ distinct zeros unless they are the zero polynomial (Fundamental Theorem of Algebra). $r(x) = 0 \implies p_n(x) - q(x) = 0$ implies $p_n(x) = q(x) \implies p_n$ is unique.

**Note**:

(a) For large $n$, computing $p_n(x)$ numerically is challenging.

(b) $n \to \infty$ does not guarantee $p_n(x) \approx f(x)$, even for some "nice" functions $f(x)$.

**Newton Interpolation and Divided Difference**

Objective: Efficiently formulate Lagrange Interpolation polynomials using Newton's interpolation method. The approach allows partial factoring akin to Horner's method.

**Theorem.** *Let $p_n(x)$ interpolate $f$ at the nodes $x_i$, $i = 0, 1, 2, \ldots, n$. Here, $p_{n+1}(x)$ interpolates $f$ at $x_i$, $i = 0, 1, 2, \ldots, n+1$, where $x_i \neq x_j$. Then $p_{n+1}(x) = p_n(x) + a_{n+1}\omega_n(x)$, with*

$$\begin{aligned}\omega_n(x) &= \prod_{i=0}^{n}(x - x_i) \\ a_{n+1}(x) &= \frac{f(x_{n+1}) - p_n(x)}{\omega_{n+1}(x)} \\ p_0(x) &= a_0 = f(x_0)\end{aligned}$$

* *Constructs $p_{n+1}$ given $p_n$ when one extra node is added.*
* *Can be implemented recursively.*
* *Avoids repetitive computations: $w_{n+1} = (x - x_{n+1})\omega_n(x)$.*

*Corollary.*—Given $a_k$ and $\omega_n$,

$$\begin{aligned}p_n(x) &= a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \cdots + a_n(x - x_0)(x - x_1)\ldots(x - x_{n-1}) \\ &= \sum_{k=0}^{n} a_k\omega_{k-1}(x).\end{aligned}$$

for $\omega_{-1}(x) \equiv 1$.

$$p_n(x) = a_0 + (x - x_0)a_1 + (x - x_1)a_2 + \cdots + (x - x_{n-1})a_n \sim \text{Horner's}$$

* $a_{k_{k=0}}^{n}$ called divided differences and are used to evaluate $p_n(x)$.

Algorithm 4.1 constructs $a_k$.

Algorithm 4.2 evaluates $p_n(xx)$ given $x_k$. $\omega_n$ is built/included in the algorithm using the partial factoring/Horner's approach. $X = x_{i_{i=0}}^{n}$ fixed nodes for defining $p_n$.

**Example 184** (Constructing $p_n(x)$ by hand)**.**—via divided difference tables. Find $p_2(x)$ for $f(x) = \log_2 x$, $x_0 = 1, x_2 = 2, x_3 = 4$.

Define for $j \geq 1$:

$$\boxed{(j,k)} f_j(x_k) \equiv \frac{f_{j-1}(x_{k+1}) - f_{j-1}(x_k)}{x_{k+j} - x_k}$$

Insert Table from class!!!

$$\begin{aligned}
f_1(x_0) &= \frac{f_0(x_1) - f_0(x_0)}{x_1 - x_0} = \frac{f(2) - f(1)}{2-1} = \frac{1}{1} = 1. \\
f_1(x_1) &= \frac{f_0(x_2) - f_0(x_1)}{x_2 - x_1} = \frac{f(4) - f(2)}{4-2} = \frac{2-1}{2} = \frac{1}{2}. \\
f_2(x_0) &= \frac{f_1(x_1) - f_1(x_0)}{x_2 - x_0} = \frac{\frac{1}{2} - 1}{4-1} = -\frac{1}{6}.
\end{aligned}$$

$f_2(x_1)$ not defined. So $x_{2+1} = x_3$.

$$\begin{aligned}
p_2(x) &= f_0(x_0) + f_1(x_0)(x - x_0) + f_2(x_0)(x - x_0)(x - x_1) \\
&= 0 + 1 \cdot (x-1) + -\frac{1}{6}(x-1)(x-2) \\
&= (x-1)1 + (x-2)-\frac{1}{6} \qquad \text{evaluate} \\
&= -\frac{1}{6}(x-1)(x-2) \\
p_2(x) &= f_0(x_2) + f_1(x_1)(x - x_2) + f_2(x_0)(x - x_2)(x - x_1) \\
&= 2 + \frac{1}{2}(x-4) - \frac{1}{6}(x-4)(x-2) \\
&= -\frac{1}{6}(x-1)(x-8).
\end{aligned}$$

**Example 185** (continued)**.**—Now interpolate at $x_0 = 1, x_1 = 2, x_2 = 4, x_3 = 8$.

-x$_2$)
$= p_2(x) + f_3(x_0)(x - x_0)(x - x_1)(x - x_2)$
$= p_2(x) + \frac{1}{56}(x-1)(x-2)(x-4)$
$= \frac{1}{168}(x-1)(3x^2 - 46x + 248)$
$= \frac{1}{168}(x-1)(248 + x(3x - 46))$
$= \frac{1}{56}x^3 - \frac{7}{24}x^2 + \frac{7}{4}x - \frac{31}{21}$.

## § 5.2 Interpolation Error

Objective: Bound the error over the entire interval.

**Theorem.** *Let $f \in C^{n+1}([a,b])$, and let the nodes $x_k \in [a,b]$ for $0 \le k \le n$ with $x_i \ne x_j$. Then for each $x \in [a,b]$, there exists $\xi_x$ in $[a,b]$ such that $f(x) - p_n(x) = \frac{\omega_n(x)}{(n+1)!} f^{(n+1)}(\xi_x)$, where $\omega_n(x) = \prod_{k=0}^{n}(x - x_k)$.*

*Corollary.*—$\max_{x\in[a,b]} |f(x) - p_n(x)| \le \frac{\max_{x\in[a,b]} |\omega_n(x)|}{(n+1)!} \max_{x\in[a,b]} |f^{(n+1)}(x)|$.

**Note 48.**—:

(a) The error can blow up since $f^{(n+1)}(x)$ may become unbounded (especially for $n$ large).

(b) Chebyshev Interpolation: choose $x_k$ to minimize $|\omega_n(x)|$. The nodes are non-uniformly spaced.

(c) $|\omega_n(x)| \le (b-a)^{n+1}$ since $x, x_k \in [a,b]$. Fact: $\lim_{n\infty} \frac{c^n}{n!} = 0$ for any fixed real number $c$.

**Example 186.**—What is the error for cubic interpolation to $f(x) = \sqrt{x}$ on the interval $[1,4]$ using equally spaced nodes?

$$\begin{aligned}
\omega_n(x) &= (x-x_0)(x-x_1)(x-x_2)(x-x_3) \\
&= (x-1)(x-2)(x-3)(x-4) \\
\max_{1\le x\le 4} \omega_3(x) &= \frac{9}{16} \qquad \text{at } x = \frac{5}{2} \\
\min_{1\le x\le 4} \omega_3(x) &= -1 \qquad \text{at } x = \frac{5}{2} \pm \frac{\sqrt{5}}{2}. \\
\implies \max_{1\le x\le 4} |\omega_3(x)| &= +1 \\
f^{(4)}(x) &= -\frac{15}{16} \cdot \frac{1}{\sqrt{x^7}} \\
\max_{1\le x\le 4} |f^{(4)}(x)| &= +\frac{15}{16} \cdot \frac{1}{\sqrt{1^7}} = \frac{15}{16}. \\
\implies f(x) - p_3(x) &\le |\omega_3| \frac{1}{4!} \cdot f^{(4)} \frac{15}{16} = \frac{5}{128} \approx 0.039.
\end{aligned}$$

**Example 187.**—If we want to use a table of values to interpolate the sine function on the interval $[0,\pi]$, how many points are needed for $10^{-6}$ accuracy with linear interpolation (piecewise)? $f^{(2)}(x) = -\sin(x) \implies |f^{(2)}(x)| \le 1$. On the $k^{\text{th}}$ subinterval:

$$\begin{aligned}
f(x) - p_1(x) &= \frac{\omega_1(x)}{2!} f^{(2)}(\xi_x) \\
\omega_1(x) &= (x - x_{k-1})(x - x_1)
\end{aligned}$$

On the $k^{\text{th}}$ subinterval $[x_{k-1}, x_k]$:

$$\begin{aligned}
\omega_1(x) &= (x - x_{k-1})(x - x_k) \\
&\le \frac{x_{k-1} + x_k}{2} - x_{k-1}\frac{x_{k-1} + x_k}{2} - x_k \\
&= \frac{x_k - x_{k-1}}{2}\frac{x_{k-1} - x_k}{2} = \frac{h^2}{5} \\
\implies |f(x) - p_1(x)| &\le \frac{h^2}{4} \cdot \frac{1}{2!} = \frac{h^2}{8}.
\end{aligned}$$

Want $\frac{h^2}{8} \le \text{TOL} = 10^{-6}$

$$\begin{aligned}
h &\le \sqrt{8 \cdot 10^{-6}} = \frac{1}{250\sqrt{2}} \approx 0.00283. \\
\pi = nh \implies n &= \frac{\pi}{h} \ge \frac{\pi}{(1/250\sqrt{2})} = 250\pi\sqrt{2} \approx 1110.72.
\end{aligned}$$

**Example 188** (continued)**.**—Cubic interpolation?

$$\begin{aligned}
\omega_3(x) &= (x - x_k)(x - x_{k+1})(x - x_{k+2})(x - x_{k+3}) \\
\max \omega_3 \left(\frac{x_k + x_{k+3}}{2}\right) &= \left(\frac{3h}{2}\right)\left(\frac{h}{2}\right)\left(-\frac{h}{2}\right)\left(-\frac{3h}{2}\right) = \frac{9}{16}h^4 \\
\max \omega_3 \left(\frac{x_k + x_{k+1}}{2}\right) &= \left(\frac{h}{2}\right)\left(-\frac{h}{2}\right)\left(-\frac{3h}{2}\right)\left(-\frac{5h}{2}\right) = -\frac{15}{16}h^4 \\
\frac{5h^4}{128} \le 10^{-6} \implies h^4 &\le \frac{128}{5 \times 10^{-6}} \implies h \le \sqrt[4]{\frac{128}{5 \times 10^{-6}}} = \frac{\sqrt[4]{10}}{25} \approx 0.07113 \\
\pi = nh \implies n &\ge \frac{\pi}{h} \ge \frac{25\pi}{4\sqrt{10}} \approx 44.17
\end{aligned}$$

## § 5.3 Hermite Interpolation

Objective: Interpolate derivative data in addition to function values.

**Theorem.** *Given the nodes $x_i$, $1 \le i \le n$, and a differentiable function $f(x)$, if the nodes are distinct, then there exists a unique polynomial $H_n$ of degree $\le 2n - 1$ such that $H_n(x_i) = f(x_i)$, $H_n'(x_i) = f'(x_i)$, $1 \le i \le n$.*

**Proof.** (Idea) Define basis functions $h_k, \tilde{h}_k$ polynomials with degree $2n-1$ such that

$$\begin{cases} h_k(x_j) = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \ne k \end{cases} \\ h_k'(x_j) = 0 \end{cases}, \qquad \begin{cases} \tilde{h}_k(x_j) = 0 \\ \tilde{h}_k'(x_j) = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \ne k \end{cases} \end{cases}.$$

Then, $H_n(x) = \sum_{k=1}^n f(x_k)h_k(x) + f'(x_k)\tilde{h}_k(x)$.

Homework: $h_k(x) = 1 - 2L_k^{(n)}(x_k)(x - x_k)L_k^{(n)}(x)^2$, $\tilde{h}_k(x) = (x - x_k)L_k^{(n)}(x)^2$.

**Theorem Error.** *Let $f \in 2nab$ and let the nodes $x_k \in [a, b]$ for all $k$, $1 \leq k \leq n$, with $x_i \neq x_j$ for $i \neq j$. Then, for each $x \in [a, b]$, $\exists \xi_x \in [a, b]$ such that*

$$f(x) - H_n(x) = \frac{\psi_n(x)}{(2n)!} f^{(2n)} \xi_x$$

*where*

$$\psi_n(x) = \prod_{k=1}^{n} (x - x_k)^2.$$

**Note 49.**—:

(a) If $|f^{(m)}|$ is bounded for all $m$, error tends to zero as $n$ tends to infinity. Otherwise, the error may diverge similarly to Lagrange interpolation.

(b) Chebyshev Interpolation: choose $x_k$ to minimize $|\omega_n(x)|$. The nodes are non-uniformly spaced.

(c) $|\omega_n(x)| \leq (b - a)^{n+1}$ since $x, x_k \in [a, b]$. Fact: $\lim_{n\infty} \frac{c^n}{n!} = 0$ for any fixed real number $c$.

**Example 189.**—What is the error for cubic interpolation to $f(x) = \sqrt{x}$ on the interval $[1, 4]$ using equally spaced nodes?

$$\begin{aligned}
\omega_n(x) &= (x - x_0)(x - x_1)(x - x_2)(x - x_3) \\
\max_{1 \leq x \leq 4} \omega_3(x) &= \frac{9}{16} \qquad \text{at } x = \frac{5}{2} \\
\min_{1 \leq x \leq 4} \omega_3(x) &= -1 \qquad \text{at } x = \frac{5}{2} \pm \frac{\sqrt{5}}{2}. \\
\implies \max_{1 \leq x \leq 4} |\omega_3(x)| &= +1 \\
f^{(4)}(x) &= -\frac{15}{16} \cdot \frac{1}{\sqrt{x^7}} \\
\max_{1 \leq x \leq 4} |f^{(4)}(x)| &= +\frac{15}{16} \cdot \frac{1}{\sqrt{1^7}} = \frac{15}{16}. \\
\implies f(x) - p_3(x) &\leq |\omega_3| \frac{1}{4!} \cdot f^{(4)} \frac{15}{16} = \frac{5}{128} \approx 0.039.
\end{aligned}$$

**Example 190.**—If we want to use a table of 1000 values of $f(x)$ to interpolate, then $n = 999$.

$$f^{(4)}(x) = -\frac{15}{16} \cdot \frac{1}{\sqrt{x^7}} \implies \max_{1 \leq x \leq 4} |f^{(4)}(x)| = \frac{15}{16} \cdot \frac{1}{\sqrt{1^7}} = \frac{15}{16}.$$

$$\max_{1\le x\le 4}|f(x)-p_3(x)| \le \frac{5}{128}\cdot\frac{15}{16}\approx 0.039.$$
$$\frac{4^4}{3840}\max_{1\le \xi\le 4}|f^{(4)}(\xi)| \le 10^{-6}$$
$$\implies \max_{1\le \xi\le 4}|f^{(4)}(\xi)| \le 2560 \implies \max_{1\le x\le 4}|f(x)-p_3(x)| \le 0.039.$$

Therefore, if $|f^{(4)}(\xi)|$ is bounded, the error for cubic interpolation tends to zero as the number of nodes tends to infinity.

## § 5.4 Piecewise Polynomial Interpolation

Objective: Use piecewise polynomials of low degree to avoid needing to bound $|f^{(n)}(\xi)|$ for large $n$ in the error result. This avoids large oscillations caused by high-degree polynomials.

Idea:

* For piecewise degree $d$ polynomials, we need $n = md$ for some integer $M$. Each subinterval is width $dh$ and defines a polynomial with degree $\le d$.
* Piecewise Lagrange: the interpolating function is continuous.
* Piecewise Hermite: the interpolant function is $C^1$ (differentiable since it avoids corners when gluing pieces together).

**Theorem Error for piecewise degree** $d$**.** *Let $f \in d+1ab$ and let $q_d$ be the piecewise polynomial interpolant of degree $d$ to $f$ on $[a,b]$ using $n+1$ equally spaced nodes, where $x_k$, $0\le k\le n$, $x_i \ne x_j$ for $i\ne j$, $n = Md$ ($M$ is an integer). Then*

$$|f(x)-q_x(x)| \le C_d h^{d+1}\max_{a\le\xi\le b}|f^{(d+1)}(\xi)|$$

*for all $a\le x\le b$, where $h = x_k - x_{k-1}$.*

$C_1 = \frac{1}{8}$, $C_2 = \frac{1}{9\sqrt{3}}$, $C_3 = \frac{1}{24}$.

**Example 191.**—Construct the piecewise cubic interpolating polynomial for $f(x) = \log_2(x)$ on the interval $[\frac{1}{8}, 8]$ using the nodes $x_0 = \frac{1}{8}, x_1 = \frac{1}{4}, x_2 = \frac{1}{2}, x_3 = 1, x_4 = 2, x_5 = 4, x_6 = 8$.

**Note 50.**—$6 = n = 3\cdot 2$, $d = 3$, $M = 2$ (two pieces).

$$p_1(x) = \frac{64}{7}x^3 - \frac{56}{3}x^2 + 14x - \frac{94}{21}$$
$$p_2(x) = \frac{1}{56}x^3 - \frac{7}{24}x^2 + \frac{7}{4}x - \frac{31}{21}.$$

**Example 192.**—Using piecewise cubic interpolation, how many nodes are needed to approximate $\ln(x)$ over $[1, 10]$ within $10^{-8}$ error using equally spaced nodes.

$$f^{(4)}(x) = -\frac{6}{x^4} \implies \max_{1\le\xi\le10} |f^{(4)}(\xi)| = 6.$$

$$\begin{aligned}
C_d h^4 \cdot 6 &\le 10^{-8} \\
\frac{1}{24} h^4 \cdot 6 &\le 10^{-8} \\
h^4 &\le 4 \times 10^{-8} \\
h &\le \frac{1}{30\sqrt{2}} \approx 0.01414. \\
n &= md. \\
h &= \frac{b-a}{n} = \frac{10-1}{m3} = \frac{3}{m}. \\
M &\ge \frac{3}{h} = 150\sqrt{2} \approx 212.13. \\
M &= 213.
\end{aligned}$$

$\boxed{n = 3M = 639}$.

## § 5.5 Introduction to Splines

Objective: Develop smooth piecewise polynomial interpolation functions using only given function values (no derivative information).

**Definition.**—$q_d$ is a spline of degree $d$ if:

- each piece (defined on $[x_{k-1}, x_k]$) is a polynomial of degree $\le d$
- $q_d(x_k) - f(x_k)$ for all $k = 0, 1, \ldots, n$, $x_j \ne x_i$ for $i \ne j$ (interpolation)
- $\lim_{xx_k^-} q_d^{(i)}(x) = \lim_{xx_k^+} q_d^{(i)}(x)$ for all $i = 0, 1, \ldots, M$ (smoothness)
  $M$ is the degree of smoothness.

In practice:

(a) $M = d - 1$ (cubic $\implies$ 2 continuous derivatives) $\sim$ existence

(b) $d$ is odd $\implies$ conditions can be added at both $x_0, x_n$ to ensure existence and uniqueness of the spline.

**Example 193.**—For what value of $k$ is the following a spline function?

$$q(x) = \begin{cases} x^2 - x^2 + kx + 1, & 0 \le x \le 1 \\ -x^3 + (k+2)x^2 - kx + 3, & 1 \le x \le 2 \end{cases}$$

$q$ is piecewise cubic with 2 pieces.

$$\begin{aligned} &\lim_{x1^-} q(x) = 1 - 1 + k + 1 = k + 1. \\ &\lim_{x1^+} q(x) = -1(k+2) - k + 3 = 4. \\ &\qquad\qquad \implies k + 1 = 4 \qquad \text{for continuity} \\ &\qquad\qquad \implies \boxed{k = 3} \end{aligned}$$

Check $q'$ and $q''$ (since $d = 3$, $M = 2$)

$$\begin{aligned} \lim_{x1^-} q'(x) &= \lim_{x1^-} 3x^2 - 2x + k = 3 - 2 + k = k + 1. \\ \lim_{x1^+} q'(x) &= \lim_{x1^+} -3x^2 + 2(k+2)x - k = -3 + 2(k+2) - k = k + 1 \\ \lim_{x1^-} q''(x) &= \lim_{x1^-} 6x - 2 = 6 - 2 = 4. \\ \lim_{x1^+} q''(x) &= \lim_{x1^+} -6x + 2(k+2) = -6 + 2k + 4 = -2 + 2k. \\ k = 3 &\implies -2 + 2(3) = -2 + 6 = 4. \\ &\boxed{k = 3} \end{aligned}$$

Then $q \in 202$ and $q$ is cubic on $[0,1]$ and $[1,2]$.

**Example 194.**—Construct a quadratic spline interpolating $(-1,0)$, $(0,1)$, and $(1,3)$. $x_0 = -1, x_1 = 0, x_2 = 1.$ $\implies$ 2 pieces, $p_1$ on $[-1,0]$ and $p_2$ on $[0,1]$.

$$q_2(x) = \begin{cases} p_1(x) = a_1 + b_1 x + c_1 x^2, & -1 \le x < 0, \\ p_2(x) = a_2 + b_2 x + c_2 x^2, & 0 \le x \le 1. \end{cases}$$

$$\begin{aligned} p_1(-1) = 0 &\implies 1_1 - b_1 + c_1 = 0 \\ p_1(0) = 1 &\implies a_1 = 1 \\ p_2(0) = 1 &\implies a_2 = 1 \\ p_2(1) = 3 &\implies a_2 + b_2 + c_2 = 3 \end{aligned}$$

4 equations, 6 unknowns. $a_1 = 1, a_2 = 1$.

$$\begin{aligned} p_1'(0) &= p_2'(0) \\ b_1 + 2c_1 x\big|_{x=0} &= b_2 + 2c_2 x\big|_{x=0} \\ b_1 &= b_2 \end{aligned}$$

$$\begin{cases} -b_1 + c_1 &= -1 \\ b_2 + c_2 &= 2 \quad \text{3 equations, 4 unknowns} \implies \text{infinitely many choices.} \\ b_1 - b_2 &= 0 \end{cases}$$

Suppose $q_2'(-1) = 0$ (Arbitrary) $\sim$ best choice: $f'(-1)$ complete spline.

$$\begin{aligned} p_1'(-1) = 0 \implies b_1 + 2c_1 x\big|_{x=-1} &= 0 \\ b_1 - 2c_1 &= 0 \end{aligned}$$

$$\begin{cases} -b_1 + c_1 &= -1 \\ b_1 - 2c_1 &= 0 \\ b_1 - b_2 &= 0 \\ b_2 + c_2 &= 2 \end{cases}$$

$$\begin{aligned} -c_1 &= -1 \\ &\boxed{c_1 = 1} \\ b_1 - 2 &= 0 \\ &\boxed{b_1 = 2} \\ 2 - b_2 &= 0 \\ &\boxed{b_2 = 2} \\ 2 + c_2 &= 2 \\ &\boxed{c_2 = 0} \end{aligned}$$

$$q_2(x) = \begin{cases} 1 + 2x + x^2, & -1 \le x < 0 \\ 1 + 2x, & 0 \le x \le 1 \end{cases}$$

## § 5.6 Cubic B-splines

Objective: Construct piecewise cubic splines efficiently.

Idea: Define basis functions $B_i(x)$ such that

$$q_3(x) = \sum_i c_i B_i(x)$$

for some constants $c_i$ [determined by given function values $f(x_k)$].

* Smoothness built into $B_i(x)$.

Define a single reference function $B(x)$ by

$$B(x) = \begin{cases} 0 & x \leq -2 \\ (x+2)^3 & -2 < x \leq -1 \\ 1 + 3(x+1) + 3(x+1)^2 - 3(x+1)^3 & -1 < x \leq 0 \\ 1 + 3(1-x) + 3(1-x)^2 - 3(1-x)^3 & 0 < x \leq 1 \\ (2-x)^3 & 1 < x \leq 2 \\ 0 & x > 2 \end{cases}$$

Then, $B(x)$ is $C^2$, symmetric, and only locally defined.

$$\begin{aligned} B(2) &= B(-2) = 0 \\ B(-1) &= B(1) = 1 \\ B(0) &= 4 \\ B'(-2) &= B'(0) = B'(2) = 0 \\ B'(-1) &= 3 \\ B'(1) &= -3 \end{aligned}$$

Using translations and scaling: $B_i(x) \equiv B\left(\frac{x-x_i}{h}\right)$ for $h = x_j - x_{j-1}$ (uniform spacing)

- $B_i$ is centered at $x_i$
- $B_i$ is zero for $x \notin [x_{i-2}, x_{i+2}]$.

**Construction $q_3(x)$ (cubic spline)**

Let $x_i = a + ih$, $h = \frac{b-a}{n}$. Given $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$ and $f(x_0), f(x_1), \ldots, f(x_n)$. Introduce extra nodes

$$\begin{aligned} x_{-3} &= a - 3h, & x_{-2} &= a - 2h, & x_{-1} &= a - h \\ x_{n+1} &= b + h, & x_{n+2} &= b + 2h, & x_{n+3} &= b + 3h \end{aligned}$$

Let $\boxed{q_3(x) = \sum_{i=-1}^{n+1} c_i B_i(x)}$ $\sim$ extended because $B_{-1}(x)$ is nonzero for $x$ in $[a,b]$ ($B_{-2}(x) = 0$ for all $x \in [a,b]$).

Goal: $\boxed{q_3(x_k) = f(x_k) \text{ for all } 0 \leq k \leq n}$.

$*$ $q_3$ is $C^2$ by construction using $B$-splines.

$$
\begin{aligned}
f(x_k) &= q_3(x_k) \\
&= \sum_{i=-1}^{n+1} c_i B_i(x_k) \\
&= c_{k-1}B_{k-1}(x_k) + c_k B_k(x_k) + c_{k+1}B_{k+1}(x_k) \\
(*) &= c_{k-1}B(-1) + c_k B(0) + c_{k+1}B(1) \\
&= c_{k-1} + 4c_k + c_{k+1}.
\end{aligned}
$$

Matrix Form

$$41141141 \ddots \ddots \ddots 14114 c_0 c_1 c_2 \vdots c_{n-1} c_n = f(x_0) \boxed{-c_{-1}} f(x_1) f(x_2) \vdots f(x_{n-1}) f(x_n) \boxed{-c_{n+1}}$$

$$
\begin{cases}
n+1 \text{ equations} & \text{(interpolation)} \\
n+3 \text{ unknowns} & c_{-1} = ?,\ c_{n+1} = ?
\end{cases}
$$

This is a cubic spline for any choice of $c_{-1}, c_{n+1}$ (satisfies the definition).
What is the best choice?

### Complete Splines

Assume $f'(x_0)$ and $f'(x_n)$ are given.

$$
\begin{aligned}
f'(x_0) = q_3'(x_0) &= c_{-1}B_{-1}'(x_0) + c_0 B_0'(x_0) + c_1 B_1'(x_0) \\
&= c_{-1}\frac{1}{h}B'(1) + c_1 \frac{1}{h}B'(-1) \\
&= -\frac{3}{h}c_{-1} + \frac{3}{h}c_1. \\
\implies c_{-1} &= -\frac{h}{3}q_3'(x_0) + c_1
\end{aligned}
$$

Choose $q_3'(x_0) = f'(x_0)$.
So $c_{-1} = -\frac{h}{3}f'(x_0) + c_1$.

$$
\begin{aligned}
\implies 1^{\text{st}} \text{ eqn}: \qquad 4c_0 + c_1 &= f(x_0) - c_{-1} \\
&= f(x_0) + \frac{h}{3}f'(x_0) - c_1 \\
&\implies \boxed{4c_0 + 2c_1 = f(x_0) + \tfrac{h}{3}f'(x_0)} \\
\text{Last eqn}: \qquad & \boxed{2c_{n-1} + 4c_n = f(x_n) - \tfrac{h}{3}f'(x_n)}
\end{aligned}
$$

$\implies$ $n+1$ equations, $n+1$ unknowns $(c_0, \ldots, c_n)$ $\implies$ has 1 unique solution.

- $*$ Finding a complete cubic spline is equivalent to solving an $(n+1)\times(n+1)$ system of linear equations that is diagonally dominant and tri-diagonal.

**Evaluation**

Given $c_0, c_1, \ldots, c_n$ $(c_{-1}, c_{n+1})$, then $q_3(x) = \sum_{i=1}^{n+1} c_i B_i(x)$. Suppose $x \in [x_{k-1}, x_k]$ for some $1 \le k \le n$.

$$\implies q_3(x) = c_{k-2}B_{k-2}(x) + c_{k-1}B_{k-1}(x) + c_k B_k(x) + c_{k+1}B_{k+1}(x).$$

For a uniform grid: $k = \lfloor \frac{x-x_0}{h} \rfloor + 1$, round down to nearest integers (floor).

**Theorem Convergence.** *If $f \in 4ab$ with $\max(x_k - x_{k-1}) \le h$ for all $k$, then for all $a \le x \le b$,*

$$|f(x) - q_3(x)| \le \frac{5}{384} h^4 \max_{a\le\xi\le b} |f^{(4)}(\xi)|.$$

*Moreover,*

$$|f^{(k)}(x) - q_3^{(k)}(x)| \le C_k h^{4-k} \max_{a\le\xi\le b} |f^{(4)}(\xi)|$$

*for some constants $C_1, C_2, C_3$.*

$$\implies |f'(x) - q'(x)| \le C_1 h^3 \max |f^{(4)}(\xi)|$$

**Note 51.**—:

(a) Accuracy consistent with piecewise cubic but better approximates derivatives due to smoothness.

(b) see Example 4.8 for building $q_3(x)$ using $B$-splines.

(c) The construction with $B$-splines can be generalized/adapted to any degree spline. lines are $C^0$, quadratic are $C^1$

(d) If $f'(x_0), f'(x_N)$ are unknown, they can be approximated using finite differences for complete splines. Need to be $O(h^4)$.

(e) Common alternative construction finds $q_3''(x)$ by solving a system of equations (tri-diagonal & diagonally dominant) (called moments) and then integrate twice locally and choose constants based on interpolation constraints. $f(x_{k-1}), f(x_k)$

# § 6 Numerical Integration

Goal: Approximate $I(f) = \int_a^b f(x)\,dx$. Idea: $I(f) \approx I_n(f) \equiv \sum_{i=0,1}^n \omega_i f(x_i)$ for some

- nodes/abscissas $x_i$
- weights $\omega_i$

Then, $\lim_{n\infty} I_n(f) = I(f)$ for $f$ sufficiently nice.

**Definition.**—$I_n(f)$ called a quadrature rule.

**Example 195.**—Trapezoidal Rule (§2.5)

$$T_n(f) = \frac{h}{2} f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)$$

$x_{i\,i=0}^{\,n}$ uniformly spaced, $\omega_0 = \frac{h}{2}, \omega_n = \frac{h}{2}, \omega_i = h$ for $i = 1, \ldots, n-1$.

$$|I(f) - I_n(f)| \leq \frac{b-a}{12} h^2 \max_{a \leq \xi \leq b} |f''(\xi)| 0$$

as $h0$ $(n\infty)$ for $h = \frac{b-a}{n}$. Better method is corrected trapezoidal rule.

## § 6.1 A review of the Definite Integral

- Choose $n > 0$.
- Pick $n-1$ points such that $x_i^{(n)} \in (a,b)$ with $x_i^{(n)} \neq x_j^{(n)}$, $x_0^{(n)} = a$, $x_n^{(n)} = b$.
- Let $h_i^{(n)} = x_i^{(n)} - x_{i-1}^{(n)}$ and $h^{(n)} = \max_i h_i^{(n)}$.
- Define $\boxed{I_n(f, \vec{\eta}^{\,(n)} = \sum_{i=1}^n f(\eta_i{}^{(n)}) h_i^{(n)}}$ for some points $\vec{\eta}^{\,(n)}$ such that $x_{i-1}^{(n)} \leq \eta_i{}^{(n)} \leq x_i^{(n)}$ (arbitrary).

**Definition** (Riemann Integral).—Suppose $\lim_{n\infty} h^{(n)} = 0$. Then $C = \int_a^b f(x)\,dx$ exists for a given $a, b, f$ if $\lim_{n\infty} I_n(f, \vec{\eta}^{\,(n)}) = C$ independent of the choices for $\vec{\eta}^{\,(n)}$ and $x_i^{(n)}$.

**Example 196.**—:

- $L_n(f)$ - left-endpoint rule $\eta_i = x_{i-1}$
- $R_n(f)$ - right-endpoint rule $\eta_i = x_i$

- $M_n(f)$ - midpoint rule $\eta_i = \frac{x_{i-1}+x_i}{2}$

* Riemann sums integrate a piecewise constant-valued approximation/interpolant of $f$ exactly, where $p_0(x) = f(\eta_i)$ for all $x_{i-1} < x < x_i$.

* They are all quadrature rules $I_n(f) = \sum_{i=i_0}^{n} \omega_i f(x_i)$ for $x_i = \eta_i^{(n)}$, $\omega_i = h_i^{(n)}$, $i_0 = 1$.

* converges for $f$ continuous.

* Typically say it converges slowly due to the inaccuracy of a piecewise constant approximation.

**Theorem.** *Suppose a quadrature rule $I_n$ integrates all polynomials of degree $\leq N$ exactly. Let $f(x) = p_N(x) + R_N(x)$ for $p_N$ a polynomial of degree $\leq N$ (Ex. Taylor's Remainder Theorem). Then*

$$I(f) - I_n(f) = I_n(R_N) - I_n(R_N).$$

**Proof.**

$$\begin{aligned} I(f) - I_n(f) &= Ip_N + R_N - I_n p_N + R_N \\ &= I(p_N) + I(R_N) - = I(p_N)I_n(p_N) - I_n(R_N) \\ &= I(R_N) - I_n(R_N). \end{aligned}$$

* To gauge how accurate a given rule $I_n(f)$ is, we see what the largest degree $N$ is such that $I(p) = I_n(p)$ for all polynomials $p$ of degree $\leq N$. $L_n, R_n : N = 0$. $T_n, M_n : N = 1$.

## § 6.2 The Midpoint Rule

Goal: show it is more accurate than $L_n, R_n$, and $T_n$.

Riemann sums: $\int_{x_{i-1}}^{x_i} f(x)\, dx \approx \int_{x_{i-1}}^{x_i} f(\eta_i)\, dx = f(\eta_i)(x_i - x_{i-1})$, Midpoint rule: $\eta_i = \frac{x_{i-1}+x_i}{2}$.

$f(x) = f(\eta_i) + (\eta_i - x)f'(\eta_i) + \frac{1}{2}(\eta_i - x)^2 f''(\xi_i)$ by Taylor's Theorem.

$$\int_{x_{i-1}}^{x_i} f(x)\,dx - \int_{x_{i-1}}^{x_i} f(\eta_i)\,dx = \int_{x_{i-1}}^{x_i} (1)(\eta_i - x)f'(\eta_i) + (2)\frac{1}{2}(\eta_i - x)^2 f''(\xi_i(x)\xi_i)$$

$$\begin{aligned}
(1) &= f'(\eta_i)\int_{x_{i-1}}^{x_i} (\eta_i - x)\,dx = f'(\eta_i)\eta_i x - \frac{1}{2}x^2\Big|_{x_{i-1}}^{x_i} \\
&= f'(\eta_i)\eta_i(x_i - x_{i-1}) - \frac{1}{2}x_i^2 - x_{i-1}^2 \\
&= f'(\eta_i)\eta_i(x_i - x_{i-1}) - \frac{1}{2}(x_i - x_{i-1})(x_i + x_{i+1}) \\
&= f'(\eta_i)(x_i - x_{i-1})\eta_i - \frac{x_i + x_{i+1}}{2} \\
&= \begin{cases} 0 & \text{if } \eta_i = \frac{x_{i-1}+x_i}{2}, \\ O(h^2) & \text{otherwise.} \end{cases} \\
(2) &\leq \int_{x_{i-1}}^{x_i} \frac{1}{2}h^2 \max_{a\leq\xi\leq b} |f''(\xi)|\,dx = \frac{1}{2}h^3 \max_{a\leq\xi\leq b} |f''(\xi)|
\end{aligned}$$

$\int_{x_{i-1}}^{x_i} 1\,dx = h$ Integral MVT

$$(2) = \frac{1}{24}(x_i - x_{i-1})^3 f''(\xi_i')$$

for some $\xi_i'$.

$$\begin{aligned}
\int_{x_{i-1}}^{x_i} f(x)\,dx - \int_{x_{i-1}}^{x_i} f(\eta_i)\,dx &= (1) + (2) \\
(1) &= \begin{cases} 0 & \text{if } \eta_i = \frac{x_{i-1}+x_i}{2}, \\ O(h^2) & \text{otherwise.} \end{cases} \\
(2) &= v\frac{1}{24}(x_i - x_{i-1})^3 f''(\xi_i') \qquad \text{for some } \xi_i' \\
I(f) - I_n(f) &= \int_a^b f(x)\,dx - \sum_{i=1}^n f(\eta_i)h \\
&= \sum_{i=1}^n \int_{x_{i-1}}^{x_i} [f(x) - f(\eta_i)]\,dx = c\sum_{i=1}^n h^3 = ch^3 n = ch^2(b-a) \\
&\leq \begin{cases} O(h^2) & \text{if } \eta_i = \frac{x_{i-1}+x_i}{2} \\ O(h) & \text{otherwise} \end{cases}
\end{aligned}$$

Exact result:

$M_n:\ I(f) - I_n(f) = \frac{b-a}{24}h^2 f''(\xi)$ for some $a \leq \xi \leq b$.

$T_n : T(f) - T_n(f) = \frac{b-a}{12} h^2 f''(\xi)$

* $*$ $M_n$ is typically considered more accurate than $L_n, R_n, T_n$.
* $L_n, R_n$ exact for constant functions.
* $I_n, M_n$ are exact for linear functions.
* $M_n$ uses a piecewise constant interpolant, but is exact for linear all functions.
* $*$ $\exists$ integral fules more accurate than $M_n$ despite having the same form $I_n = \sum_{i=1}^{n} \omega_i f(x_i)$.

Motivation: Incorporate piecewise polynomial interpolants instead of relying on Riemann sums (assuming piecewise constant). Let $M_n$ denote the integration of a piecewise constant (discontinuous) interpolant, where $p_0(\eta_i) = f(\eta_i)$.

**Example 197.**—Utilize the midpoint rule with $h = \frac{1}{4}$ to estimate $\int_0^1 \ln(1+x)\,dx = 2\ln 2 - 1$. Determine the required smallness of $h$ to ensure $|I - M_n(f)| \leq 10^{-6}$.

$$\begin{aligned} M_4(f) &= \sum_{i=1}^{4} \ln(1+\eta_i) \cdot h \\ &= \frac{1}{4} \ln 1 + \frac{1}{3} + \ln 1 + \frac{3}{8} + \ln 1 + \frac{5}{8} + \ln 1 + \frac{7}{8} \\ &= \frac{1}{4} \ln \frac{19305}{4096} \approx 0.387588 \\ I &= 2\ln 2 - 1 \approx 0.38629 \end{aligned}$$

$$\begin{aligned} \frac{b-a}{24} h^2 \max_{a \leq \xi \leq b} |f''(\xi)| &\leq 10^{-6} \\ \frac{1}{24} h^2 \max_{0 \leq \xi \leq 1} = 1 - \frac{1}{(1+x)^2} &\leq 10^{-6} \\ h^2 &\leq 24 \cdot 10^{-6} \\ \boxed{h \leq \tfrac{\sqrt{6}}{500} \approx 0.0004899} \end{aligned}$$

$n = \frac{1}{h}$, $n \geq 204.1$, $\boxed{n \geq 205 \text{ for } M_n}$.

## § 6.3 Enhancing the Trapezoidal Rule

Objective: Derive an $O(h^4)$ version of $T_n$. Define $T_n^c(f) = T_n(f) = -\frac{1}{12} h^2 f'(b) - f'(a)$, which can be approximated with the right choice of difference quotients $\implies$ don't need $f'$.

$$\begin{aligned}
\int_a^b f(x)\,dx - T_1^c(f) &= §2.5 \boxed{\textstyle\int_a^b f(x)\,dx - T_1(f)} + \frac{1}{12}(b-a)^2 f'(b) - f'(a) \\
&= -\frac{1}{12}(b-a)^3 f''(\xi) + \frac{1}{12}(b-a)^3 f''(\eta)^{\text{MVT}} \\
&= \frac{1}{12}(b-a)^2 f''(\eta) - f''(\xi) \qquad |\eta - \xi| \le b-a \\
&= \frac{1}{12}(b-a)^4 f'''(c)
\end{aligned}$$

In composite form: $|I(f) - T_n^c(f)| \le ch^3 \max_{a\le c\le b} |f'''(c)|$ lost a power when summing. Lazy analysis $\implies$ gained one order of accuracy. Error $= O(h^2)$ using a more refined analysis (Grad HW). $T_n^c$ is equivalent to exactly integrating the piecewise cubic Hermite interpolate, where interior derivatives cancel in the summation (requires uniform spacing).

**Example 198.**—Apply the trapezoidal rule and the corrected trapezoidal rule to approximate

$$I = \int_0^1 \ln(1+x)\,dx = 2\ln 2 - 1$$

with $h = \frac{1}{4}$.

$$\begin{aligned}
T_4(f) &= \frac{h}{2}\ln(1+0) + 2\ln 1 + \frac{1}{4} + 2\ln 1 + \frac{1}{2} + 2\ln 1 + \frac{3}{4} + \ln(1+1) \\
&\approx 0.3836995. \\
T_4^c(f) &= T_4(f) - \frac{1}{12}h^2 \frac{1}{1+x}\bigg|_{x=1} - \frac{1}{1+x}\bigg| x_{x=0} \\
&= T_4(f) + \frac{1}{384} \approx 0.386303676. \\
I &= 2\ln 2 - 1 \approx 0.38629436. \\
|I - T_4| &\approx 0.00259. \\
|I - T_4^c| &\approx 0.00000932.
\end{aligned}$$

### Error Estimation

The correction term is easily computable $\frac{1}{12}h^2[f'(b) - f'(a)]$ when compared to the exact error

$$\frac{1}{12}h^3 \sum_{i=1}^n f''(\xi_i)$$

and approximate the error to high order (for $f$ nice). $\implies$ Bound the approximate error when choosing $h$

$$\frac{1}{12}h^2|f'(b) - f'(a)| \le \text{TOL}.$$

## § 6.4 Simpson's Rule

Objective: Develop a more accurate method by exactly integrating a piecewise quadratic interpolant.

$$\int_a^b f(x)\,dx \approx \int_a^b p_2(x)\,dx \qquad \text{where}$$
$$p_2(a) = f(a)$$
$$*\ p_2\frac{a+b}{2} = f\frac{a+b}{2}$$
$$p_2(b) \approx f(b)$$

$*$ increased accuracy by using midpoint.

$$p_2(x) = f(a)L_0(x) + f\frac{a+b}{2}L_1(x) + f(b)L_2(x)$$

for the Lagrange basis functions $L_j$. For
$h = \frac{b-a}{2}$,

$$\int_a^b L_0(x)\,dx = \frac{h}{3}, \qquad \int_a^b L_1(x)\,dx = \frac{4h}{3}, \qquad \int_a^b L_2(x)\,dx = \frac{h}{3}.$$

Thus

$$\begin{aligned}
\int_a^b f(x)\,dx &\approx \int_a^b p_2(x)\,dx \\
&= f(a)\int_a^b L_0(x)\,dx + f\frac{a+b}{2}\int_a^b L_1(x)\,dx + f(b)\int_a^b L_2(x)\,dx \\
&= \frac{h}{3}f(a) + 4f\frac{a+b}{2} + f(b) \\
&\equiv S_2(f).
\end{aligned}$$

Letting $p_2$ be the piecewise quadratic interpolant for a uniform mesh $x_i = a + ih$, $i = 0, 1, \ldots, n = 2m$, we have

$$S_n(f) = \frac{h}{3}f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)$$

.

$*$ $n$ must be an even number, $n = 2m$.

**Example 199.**—Apply Simpson's rule with $h = \frac{1}{4}$ to approximate $\int_0^1 \frac{1}{1+x^3}\,dx$.

$$\begin{aligned}
S_4 \frac{1}{1+x^3} &= \frac{1/4}{3} f(0) + 4f\frac{1}{4} + 2f\frac{1}{2} + 4f\frac{3}{4} + f(1) \\
&= \frac{1}{12} 1 + 4\frac{64}{65} + 2\frac{8}{9} + 4\frac{64}{91} + \frac{1}{2} \\
&= \frac{82141}{98280} \approx 0.8357855. \\
\int_0^1 \frac{1}{1+x^3}\,dx &= \frac{1}{3}\ln 2 + \frac{1}{9}\sqrt{3}\pi \\
&\approx 0.8356488
\end{aligned}$$

### Convergence

In the context of Gaussian Quadrature, the error is expressed as follows:

Interpolation: $f(x) - p(x) = \frac{\omega_2(x)}{3!} f^{(3)}(\xi_x)$ for some $\xi_x$ in $(a, b)$, where $\omega_2(x) = (x - a)\left(x - \frac{a+b}{2}\right)(x - b)$.

$$\begin{aligned}
|I(f) - S_2(f)| &= \left| \int_a^b (f(x) - p_2(x))\,dx \right| = \left| \int_a^b \frac{\omega_2(x)}{3!} f^{(3)}(\xi_x)\,dx \right| \\
&\le \frac{(b-a)^3}{6} \int_a^b \max_{a\le\xi\le b} \left|f^{(3)}(\xi)\right| dx = \frac{(b-a)^4}{6} \max_{a\le\xi\le b} \left|f^{(3)}(\xi)\right| \\
&\implies |I(f) - S_{2m}(f)| \le \frac{(b-a)h^3}{5} \max_{a\le\xi\le b} \left|f^{(3)}(\xi)\right|.
\end{aligned}$$

This is better than $T_n$ with $\mathcal{O}(h^2)$ accuracy, and it is exact for quadratics since $f^{(3)} = 0$.

The goal is to show that the method has $\mathcal{O}(h^4)$ error, similar to $T_n^c$.

$$\begin{aligned}
&I(x^3) = \int_a^b x^3\,dx = S_2(x^3), \quad x_0 = a,\ x_1 = \frac{a+b}{2},\ h = \frac{b-a}{2},\ x_2 = b \\
&I(x^3) = \frac{1}{4} x^4\Big|_a^b = \frac{1}{4}[b^4 - a^4].
\end{aligned}$$

$$
\begin{aligned}
S_2(x^3) &= \frac{h}{3}[f(a) + 4f(x_1) + f(b)] \\
&= \frac{(b-a)}{6}[a^3 + 4\left(\frac{a+b}{2}\right)^3 + b^3] \\
&= \frac{b-a}{12}[2a^3 + (a+b)^3 + 2b^3] \\
&= \frac{b-a}{12}[2a^3 + a^3 + 3a^2b + 3ab^2 + b^3 + 2b^3] \\
&= \frac{b-a}{12}[2a^3 + a^3 + 3a^2b + 3ab^2 + b^3 + 2b^3] \\
&= \frac{b-a}{4}[a^3 + a^2b + ab^2 + b^3] \\
&= \frac{1}{4}[a^3b + a^2b^2 + ab^3 + b^4 - a^4 - a^3b - a^2b^2 - ab^3] \\
&= \frac{1}{4}[b^4 - a^4] = \frac{1}{4}x^4\Big|_a^b = \int_a^b x^3\,dx = I(x^3).
\end{aligned}
$$

This required choosing $x_1 = \frac{a+b}{2}$.

**Theorem.** *Simpson's Rule is exact for all cubic functions.*

**Theorem.** $|I(f) - S_2(f)| = \mathcal{O}((b-a)^5)$.

**Proof.** Define the Hermite cubic interpolant $q_3(x)$ by

- $q_3(a) = f(a)$
- $q_3(b) = f(b)$
- $q_3\left(\frac{a+b}{2}\right) = f\left(\frac{a+b}{2}\right)$
- $q_3'\left(\frac{a+b}{2}\right) = f'\left(\frac{a+b}{2}\right)$.

Then, $\exists c$ such that, for $a \le x \le b$, $f(x) - q_3(x) = c\,\psi(x)f^{(4)}(\xi_x)$ with $\psi(x) = (x-a)\left(x - \frac{a+b}{2}\right)^2(x-b)$ (interpolation error). Note $|\psi(x)| \le (b-a)^4$.
Thus,

$$
\begin{aligned}
|I(f) - S_2(f)| &= |I(f) - S_2(q_3)| \quad \text{— since function values agree} \\
&= \left|\int_a^b [f(x) - q_3(x)]\,dx\right| \quad \text{— } S_2(q_3) = I(q_3) \text{ – cubic} \\
&\le c(b-a)^4 \int_a^b \left|f^{(4)}(\xi_x)\right|\,dx \\
&\le c(b-a)^5 \max_{a\le\xi\le b}\left|f^{(4)}(\xi)\right|.
\end{aligned}
$$

*Corollary.*—$|I(f)-S_{2m}(f)| \le ch^4(b-a)\max_{a\le\xi\le b}\left|f^{(4)}(\xi)\right|$.

$I(f)-S_2(f) = -\frac{1}{90}\left(\frac{b-a}{2}\right)^5 f^{(4)}(\xi)$ and $|I(f)-S_{2m}(f)| \le \frac{b-a}{180}h^4\max_{a\le\xi\le b}\left|f^{(4)}(\xi)\right|$.

**Example 200.**—How small of an $h$ is needed to have $|I(f)-S_{2m}(f)| \le$ q $10^{-6}$ for $I(f)=\int_0^1 \ln(1+x)\,dx = 2\ln 2 - 1$?

$$\begin{aligned}
\frac{b-a}{180}h^4 \max_{a\le\xi\le b}\left|f^{(4)}(\xi)\right| &\le 10^{-6}\\
\frac{1}{180}h^4 \max_{0\le\xi\le 1}\left|-\frac{6}{(x+1)^4}\right| &\le 10^{-6}\\
\frac{1}{180}h^4\cdot 6 &\le 10^{-6}\\
h^4 &\le \frac{180}{6}\cdot 10^{-6} = \frac{3}{100,000}\\
h &\le \frac{1}{10}\sqrt[4]{\frac{3}{10}} \approx 0.074008.
\end{aligned}$$

$2m = \frac{1}{h} = 13.15$, $2m \ge 14$ — next biggest even number $\boxed{n\ge 14}$, $M_n(f) \implies n \ge 205$.

## § 6.5 Gaussian Quadrature

Goal: $I_n = \sum_{i=1}^n \omega_i^{(n)} f(x_i^{(n)})$ on $(a,b)$. Choose the weights and nodes such that $I_n(p) = I(p)$ for $p$ a polynomial of degree $\le d$ with $d$ as big as possible. Idea: $f(x) = p_d(x) + R(x)$, Error $= I(R) - I_n(R)$ should be smaller when $d$ is maximized (Taylor's series: $R0$ as $d\infty$).

$$\begin{aligned}
M_1(f) &= \int_a^b f\left(\frac{a+b}{2}\right)dx = (b-a)f\left(\frac{a+b}{2}\right)\\
M_2(f) &= \frac{b-a}{2}f\left(a+\frac{b-a}{4}\right) + \frac{b-a}{2}f\left(a+\frac{3}{4}(b-a)\right)
\end{aligned}$$

writing $M_n(f) = I_n(f) = \sum_{i=0}^n \omega_i^{(n)} f(x_i^{(n)})$, we have the weights and nodes depend on $a,b$.

* Make one rule on the interval $[-1,1]$ and describe a way to adapt the rule to $[a,b] \implies$ only need one list of weights and nodes.

Substitution $a\le x\le b \longleftrightarrow -a\le u\le 1$, let $\boxed{x = a+\frac{b-a}{2}(u+1)} = \frac{b-a}{2}u + \frac{a+b}{2}$, so $x(-1)=a$ and $x(1)=b$. Then $u = -1+2\frac{x-a}{b-a} = \frac{2x-a-b}{b-a}$, so $u(a)=-1$ and $u(b)=1$. Thus $du = \frac{2}{b-a}\,dx$ and $\boxed{dx = \frac{b-a}{2}\,du}$.

$$\int_a^b f(x)\,dx = \int_{-1}^{1} f\left(a + \frac{b-a}{2}(u+1)\right) \cdot \frac{b-a}{2}\,du$$

Let $F(u) = f\left(a + \frac{b-a}{2}(u+1)\right) \cdot \frac{b-a}{2}$. Then

$$I(u) = \int_a^b f(x)\,dx = \int_{-1}^{1} F(u)\,du.$$

Replacing $f(x)$ with $F(u)$, we integrate $F(u)$ over the interval $[-1, 1]$.

$*$ The rescaling preserves the definite integral.

We only need rules on $-1 \le u \le 1$.

Gaussian Quadrature: $G_n(f) = \sum_{i=1}^{n} \omega_i^{(n)} F(u_i^{(n)})$ for $-1 \le u_i^{(n)} \le 1$ for all $i$.

| $n$ | $u_i^{(n)}$ | $\omega_i^{(n)}$ |
|---|---|---|
| 1 | $0$ | $2$ |
| 2 | $\pm\sqrt{\frac{1}{3}}$ | $1$ |
| 3 | $0, \pm\sqrt{\frac{3}{5}}$ | $\frac{8}{9}, \frac{5}{9}$ |
| 4 | $\pm\sqrt{\frac{3}{7} - \frac{2}{7}\sqrt{\frac{6}{5}}}, \pm\sqrt{\frac{3}{7} + \frac{2}{7}\sqrt{\frac{6}{5}}}$ | $\frac{18+\sqrt{30}}{36}, \frac{18-\sqrt{30}}{36}$ |

**Example 201.—**

$$\begin{aligned}
G_3(f) &= \sum_{i=1}^{3} \omega_i^{(3)} F(u_i^{(3)}) \\
&= \omega_1^{(3)} F(u_1^{(3)}) + \omega_2^{(3)} F(u_2^{(3)}) + \omega_3^{(3)} F(u_3^{(3)}) \\
&= \frac{5}{9} F\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} F(0) + \frac{5}{9} F\left(\sqrt{\frac{3}{5}}\right)
\end{aligned}$$

$G_n(f)$ is precise if $f$ is a polynomial of degree $d \le 2n - 1$.

**Example 202.—**$G_3(x^p)$ is precise for $0 \le p \le 2(3) - 1 = 5$ — Same number of points as Simpson's rule $S_1$ ($u = 01, 0, 1,\ 0 \le p \le 3$).

**Example 203.—**

$$
\begin{aligned}
I &= \int_0^1 \ln(1+x)\,dx = 2\ln(2) - 1 \\
f(x) &= \ln(1+x), \quad a = 0, \quad b = 1 \\
F(u) &= \ln\left(1 + \frac{1}{2}(u+1)\right)\left(\frac{1-0}{2}\right) = \ln\left(\frac{u+3}{2}\right)\cdot\frac{1}{2} \\
G_1(f) &= 2\cdot\frac{1}{2}\ln\left(\frac{0+3}{2}\right) = \ln\left(\frac{3}{2}\right) \quad \text{(same as } M_1(f)\text{)}.
\end{aligned}
$$

$$
\begin{aligned}
G_2(f) &= 1\cdot\frac{1}{2}\ln\left(\frac{\sqrt{\frac{1}{3}}+3}{2}\right) + 1\cdot\frac{1}{2}\ln\left(\frac{-\sqrt{\frac{1}{3}}+3}{2}\right) \\
&= \frac{1}{2}\ln\left(\frac{13}{6}\right).
\end{aligned}
$$

$$
\begin{aligned}
G_3(f) &= \frac{5}{9}\cdot\frac{1}{2}\ln\left(\frac{-\sqrt{\frac{3}{5}}+3}{2}\right) + \frac{8}{9}\cdot\frac{1}{2}\ln\left(\frac{3}{2}\right) + \frac{5}{9}\cdot\frac{1}{2}\ln\left(\frac{\sqrt{\frac{3}{5}}+3}{2}\right) \\
&= \ln\left[\left(\frac{7}{5}\right)^{5/18}\left(\frac{3}{2}\right)^{13/18}\right].
\end{aligned}
$$

| Rule | Error |
|---|---|
| $G_1$ | 0.01917 |
| $G_2$ | 0.000306 |
| $G_3$ | 0.0000060605 |

Simpson's rule with error $\leq 6.1\times 10^{-6} \implies n \geq 10$ (8.5978), 11 points using $x_0 = a$, but $G_3$ used 3 points.

## § 6.6 Goal

Determine how to choose the nodes $x_i^{(n)}$ and weights $\omega_i^{(n)}$ to ensure $\int_{-1}^1 p(x)\,dx = G_n(p) = \sum_{i=1}^n \omega_i^{(n)} f(x_i^{(n)})$ for all polynomials $p$ with degree $\leq N = 2n-1$.

**Lemma.**—*There are no sets of nodes and weights such that $G_n$ is exact for all polynomials with* $\deg \leq N = 2n$.

**Proof.** Suppose there exist nodes $\{x_i^{(n)}\}$ and weights $\{w_i^{(n)}\}$ such that $G_n$ is exact for all $x^k$, $k = 0, 1, \ldots, N = 2n$.

Define $P(x) = \prod_{j=1}^{n}(x - x_j^{(n)})^2$. So $P(x) \geq 0$ on $[-1,1]$, and $\int_{-1}^{1} P(x)\,dx > 0$. However,

$$G_n(P) = \sum_{j=1}^{n} \omega_i^{(n)} P(x_i^{(n)}) = 0,$$

which is a contradiction since $P$ has degree $2n$.

Finding weights to ensure $N = 2n-1$ (given nodes).

**Lemma.**—*Suppose $G_n$ is exact for all $x^k$, $k = 0, 1, \ldots, N = 2n-1$. Then the weights must satisfy*

$$\omega_i^{(n)} = \int_{-1}^{1} L_i^{(n)}(x)\,dx$$

*for*

$$L_i^{(n)}(x) = \prod_{\substack{k=1 \\ k\neq i}}^{n} \frac{x - x_k^{(n)}}{x_i^{(n)} - x_k^{(n)}}$$

*(polynomials with degree $= n-1$).*

* Given nodes, we can find the weights. Nodes $\Longrightarrow$ $L_i^{(n)}(x)$.

Idea for finding nodes: use orthogonal polynomials.

**Definition.**—The functions $f, g$ are orthogonal over the interval $[-1,1]$ if $\int_{-1}^{1} f(x)g(x)\,dx = 0$.

**Definition.**—Legendre family of orthogonal polynomials $\{\phi_j\}_{j=0}^{n}$ are polynomials that

- have degree $\phi_j = j$
- $\int_{-1}^{1} \phi_j(x)\phi_i(x)\,dx = \begin{cases} 0 & \text{if } j \neq i \\ \frac{2}{2j+1} & \text{if } j = i \end{cases}$
- All roots of $\phi_j$ are distinct and are all in the interval $[-1,1]$.
- $\{\phi_j\}$ forms a basis: there exist constants $c_j$ such that

  $$p(x) = \sum_{i=0}^{n} c_j \phi_j(x)$$

  for all polynomials with degree $\leq n$.

$\phi_0(x) = 1$
$\phi_1(x) = x$
$(j+1)\phi_{j+1}(x) = (2j+1)x\phi_j(x) - j\phi_{j-1}(x)$

$$\implies \phi_2(x) = \frac{1}{2}(3x^2-1), \quad \phi_3(x) = \frac{1}{2}(5x^3-3x), \quad \phi_4(x) = \frac{1}{8}(35x^4-30x^2+3), \ldots$$

**Lemma.**—*If* $\deg(p) \leq n$, *then* $\int_{-1}^{1} p(x)\phi_i(x)\,dx = 0$ *for all* $i > n$.

**Proof.**

$$p(x) = \sum_{i=0}^{n} c_j\phi_j(x)$$

$$\implies \int_{-1}^{1} p(x)\phi_i(x)\,dx = \int_{-1}^{1}\sum_{j=0}^{n} c_j\phi_j(x)\,dx = \sum_{j=0}^{n} c_j\int_{-1}^{1}\phi_j(x)\phi_i(x)\,dx$$

$$\implies \int_{-1}^{1} p(x)\phi_i(x)\,dx = 0$$

**Theorem.** *For* $N = 2n-1$, *there exist points* $\{x_i^{(n)}\}$ *(and corresponding weights* $\omega_i^{(n)}$*) such that*

$$G_n(x^k) = I(x^k) = \int_{-1}^{1} x^k\,dx$$

*for all* $k = 0, 1, \ldots, N = 2n$.

**Proof.** Choose $\{x_i^{(n)}\}$ to be the $n$ distinct zeros of the function $\phi_n$ (Legendre basis functions). $\phi_n$ has degree $n$, so it has $n$ zeros. Let $p(x)$ have degree $\leq 2n-1$. Then, $p(x) = \psi_n(x)Q(x)+R(x)$ for $Q, R$ with degrees $\leq n-1$ and $\psi_n$ with degree $n$ defined by $\psi_n(x) = \prod_{i=1}^{n}(x - x_i^{(n)})$.

$$\left.\begin{aligned}\psi_n(x_i^{(n)}) &= 0\\ p(x_i^{(n)}) &= R(x_i^{(n)})\end{aligned}\right\} \implies p(x_i^{(n)}) = R(x_i^{(n)}) = \sum_{j=1}^{n} R(x_i^{(n)})L_i^{(n)}(x)$$

By Lagrange interpolation, $R(x) = \sum_{i=1}^{n} R(x_i^{(n)})L_i^{(n)}(x)$ (exact for polynomials with degree $\leq n-1$).

Also, $\psi_n(x_i^{(n)}) = 0 \implies p(x_i^{(n)}) = R(x_i^{(n)})$.

$$\implies \int_{-1}^{1} p(x)\,dx = \int_{-1}^{1}\left(\psi_n(x)Q(x)+R(x)\right)\,dx = \int_{-1}^{1}\psi_n(x)Q(x)\,dx\bigg|_{(1)} + \int_{-1}^{1} R(x)\,dx\bigg|_{(2)}$$

$$(1) = 0 \text{ (by lemma)}, \quad (2) = G_n(p).$$

**Error Expression**

Assume $H_n$ is the Hermite interpolant of $f$ at the Gaussian nodes $\{x_i^{(n)}\}$, with $H_n$ having a degree of at most $2n-1$. The error formula is given by:

$$f(x) - H_n(x) = \frac{1}{(2n)!}\psi_n(x) f^{(2n)}(\xi_x)$$

where $\psi_n(x) = \prod_{i=1}^n (x - x_i^{(n)})^2$. Integrating and applying the Mean Value Theorem (MVT), the error in the integral approximation is expressed as:

$$I(f) - G_n(f) = G_n(H_n) = I(H_n) = \boxed{\frac{2^{2n+1}(n!)^4}{(2n+1)[(2n)!]^3} f^{(2n)}(\xi)}$$

Here, $\square = K_n \underbrace{M_n}_{\frac{\sqrt{n}}{n+\frac{1}{2}}\left(\frac{e}{16n}\right)^{2n}}$ for $0.7 \le K_n \le 1.04$. Assuming $f^{(2n)}$ is bounded, the error experiences exponential decay.

For various values of $n$:

$$\begin{aligned}
n &= 2: & M_n &\approx 2.9 \times 10^{-5} \\
n &= 3: & M_n &\approx 1.6 \times 10^{-8} \\
n &= 4: & M_n &\approx 4.7 \times 10^{-12} \\
n &= 8: & M_n &\approx 5.7 \times 10^{-28} \\
n &= 16: & M_n &\approx 1.7 \times 10^{-64}
\end{aligned}$$

## § 6.7 Lagrange Interpolation Error

**Theorem.** *Let $f \in n+1ab$ and let the nodes $x_k \in [a,b]$ for $0 \le k \le n$ with $x_i \ne x_j$. Then, for each $x \in [a,b]$, there exists $\xi_x \in [a,b]$ such that:*

$$f(x) - p_n(x) = \frac{\omega_n(x)}{(n+1)!} f^{(n+1)}(\xi_x)$$

*where $\omega(x) = \prod_{k=0}^n (x - x_k)$.*

Previously, the proof was presented for $n = 1$. Below is the proof for $n = 3$.

*Proof for $n = 3$.* Choose $t \in [a, b]$, $t \neq x_k$. Define the function:

$$G(x) = E(x) - \frac{\omega(x)}{\omega(t)} E(t)$$

where $E(x) = f(x) - p_3(x)$ and $\omega(x) = \prod_{k=0}^{3} (x - x_k)$. Then,

- $G(x_k) = E(x_k) - \frac{\omega(x_k)}{\omega(t)} E(t) = 0 - 0 = 0$
- $G(t) = E(t) - \frac{\omega(t)}{\omega(t)} = 0$.

This implies that $G$ has 5 zeros: $x_0, x_1, x_2, x_3, x_4$. Applying Rolle's Theorem, $G'(x)$ has 4 zeros: $c_1, c_2, c_3, c_4$. Further, $G''$ has 3 zeros $\xi_1, \xi_2, \xi_3 \implies G'''$ has 2 zeros $\implies G^{(4)}$ has 1 zero $G^{(4)}(\xi) = 0$, where $\xi$ is in $[a, b]$.

$$\begin{aligned} G^{(4)}(x) &= f^{(4)}(x) - 4! \frac{E(t)}{\omega(t)} \\ E^{(4)}(x) &= f^{(4)}(x) - \underbrace{p_3^{(4)}(x)}_{=0} = f^{(4)}(x). \\ \omega^{(4)}(x) &= 4! \\ 0 = G^{(4)}(\xi) &= f^{(4)}(\xi) - 4! \cdot \frac{E(t)}{\omega(t)} \\ \implies E(t) &= \frac{\omega(t)}{4!} f^{(4)}(\xi) \checkmark \end{aligned}$$

Here, $t$ in $[a, b]$, and $\xi$ depends on $t$.

# Chapter 5

# ALTERNATE ANALYSIS

We make use of fundamental set definitions:

If $A \subset B$, it implies that for any $x \in A$, $x \in B$. The notation $\subseteq$ is not utilized.

$A = B$ signifies that $A \subset B$ and $B \subset A$.

Assume familiarity with rational numbers, denoted as $\mathbb{Q}$. However, rational numbers prove insufficient for calculus and analysis.

Consider the example:

**Example 204.**—The equation $p^2 - 2 = 0$ has no solution for $p \in \mathbb{Q}$.

**Proof.** Assume, for the sake of contradiction (BWOC), that $p^2 = 2$ for some $p \in \mathbb{Q}$. Then $p = \frac{m}{n}$, where $m, n$ are integers, $n \neq 0$, and $m$ and $n$ are not both even. Therefore, $\left(\frac{m}{n}\right)^2 = 2$, leading to $m^2$ being even. Consequently, $m$ is even (if $r$ is prime and $r \mid a \cdot b$, where $a$ and $b$ are integers, then $r$ divides either $a$ or $b$). Hence, $m = 2k$ for some integer $k$. Thus, $(2k)^2 = 2n^2$, implying $2k^2 = n^2$. This, in turn, results in $n^2$ being even, which implies $n$ is even. As a result, both $m$ and $n$ are even, leading to a contradiction. Therefore, there is no $p \in \mathbb{Q}$ such that $p^2 - 2 = 0$.

This example demonstrates that $\mathbb{Q}$ may not always provide solutions to such equations. Another crucial inadequacy of $\mathbb{Q}$ for analysis is characterized as follows:

Let $A = \{p \in \mathbb{Q} \mid p^2 < 2\}$ and $B = \{p \in \mathbb{Q} \mid p^2 > 2\}$.

$A$ has no largest number, and $B$ has no smallest number.

**Proof.** For any $p > 0$, define $q = p - \frac{p^2-2}{p+2} = \frac{2p+2}{p+2}$ $(*)$ so $q > 0$. Also,

$$\begin{aligned} q^2 - 2 &= \frac{(2p+2)^2}{(p+2)^2} - 2 = \frac{(2p+2)^2 - 2(p+2)^2}{(p+2)^2} \\ &= \frac{4p^2 + 8p + 4 - 2p^2 - 8p - 8}{(p+2)^2} \\ q^2 - 2 &= \frac{2p^2 - 4}{(p+2)^2} = \frac{2(p^2-2)}{(p+2)^2} \qquad (**) \end{aligned}$$

If $p \in A$, $p > 0$, then $p^2 - 2 < 0$, so $q > p$ by $(*)$. Also, $q^2 - 2 < 0$ by $(**)$, so $q \in A$. Thus, $A$ has no largest element. If $p \in B$, $p > 0$, then $p^2 - 2 > 0$, so $q < p$ by $(*)$. Also, if $q^2 - 2 > 0$ by $(**)$, so $q \in B$.

**Definition.**—Let $S$ be a set. An order on $S$ is a relation, denoted $<$, with the properties:

(i) If $x, y \in S$, then exactly one of the following is true: $x < y$ or $y < x$ or $x = y$.

(ii) If $x, y, z \in S$ and $x < y$ and $y < z$, then $x < z$.

Similar definitions hold for $>, \leq, \geq$.

**Definition.**—An ordered set is a set on which an order is defined.

**Definition.**—Let $S$ be an ordered set and $E \subset S$. If there exists $\beta \in S$ such that $x \leq \beta$ for every $x \in E$, then we say $\beta$ is an upper bound of $E, and E is bounded above.$ Similar definitions apply for lower bounds.

**Definition.**—Let $S$ be an ordered set, and suppose $E \subset S$ is bounded above. If there exists $\alpha \in S$ such that

(i) $\alpha$ is an upper bound of $E$

(ii) If $\gamma < \alpha$, then $\gamma$ is not an upper bound of $E$

(or)

(ii)′ If $\gamma$ is an upper bound of $E$, then $\alpha \leq \gamma$.

Then $\alpha$ is the least upper bound of $E$. Also called the supremum of $E$, denoted $\alpha = \sup E$. Similar definitions apply for infimum, greatest lower bound, $\inf E$.

**Example 205.**—Consider $A$ and $B$ from the previous example (with only positive elements). $A$ is bounded above. The set of upper bounds of $A$ is $B$. Since $B$ has no least element, then $A$ has no least upper bound, and $\sup A$ does not exist in $\mathbb{Q}$.

**Definition.**—An ordered set $S$ has the least upper bound property if every nonempty subset $E \subset S$ that is bounded above has a supremum in $S$.

A note: $\mathbb{Q}$ does not have the least upper bound property, as illustrated in the previous example.

**Theorem.** *Let $S$ be an ordered set with the least upper bound property. If $B \subset S$, $B$ is nonempty, and bounded below, then $\inf B$ exists in $S$. In particular, $\inf B = \alpha$, where $\alpha = \sup L$ and $L$ is the set of all lower bounds of $B$.*

**Proof.** Let $B$ and $L$ be as described. Since $B$ is bounded below, $L$ is nonempty. Also, $L$ is bounded above because $B$ is nonempty, and every element of $B$ is an upper bound for $L$. Since $S$ has the least upper bound property, $\alpha = \sup L$ exists in $S$. We claim $\alpha = \inf B$.

If $\gamma < \alpha$, then $\gamma$ is not an upper bound for $L$ (because $\alpha = \sup L$), so $\gamma \notin B$. Thus, $\alpha$ is a lower bound for $B$.

If $\alpha < \beta$, then $\beta \notin L$ since $\alpha = \sup L$. So $\beta$ is not a lower bound for $B$. Thus, $\alpha = \inf B$.

Fields and axioms: 1.12-1.16

**Example 206.**—$\mathbb{Q}$ is a field.

**Definition.**—An ordered field is a field $F$ that is also an ordered set, satisfying

(i) $x + y < x + z$ if $x, y, z \in F$ and $y < z$.

(ii) If $x, y \in F$ and $x > 0$ and $y > 0$, then $xy > 0$.

**Note 52.**—$\mathbb{Q}$ is an ordered field.

**Theorem.** *The following are true in any ordered field:*

a. *If $x > 0$, then $-x < 0$ and vice versa.*

b. *If $x > 0$ and $y < z$, then $xy < xz$.*

c. *If $x < 0$ and $y < z$, then $xy > xz$.*

d. *If $x \neq 0$, then $x^2 > 0$. Thus, $1 > 0$.*

e. *If $0 < x < y$, then $0 < \frac{1}{y} < \frac{1}{x}$.*

**Proof.** :

a. If $x > 0$, then $0 = -x + x > -x + 0 = -x$. So $-x < 0$.

b. If $z > y$, then $z-y > 0$, so $x(z-y) > 0$. Thus, $xz = x(z-y)+xy > 0+xy = xy$.

d. If $x > 0$, then $x^2 = x \cdot x > 0$. If $x < 0$, then $-x > 0$, so $(-x)^2 > 0$. But $(-x)^2 = x^2$, so $x^2 > 0$. Also, $1 > 0$ because $1 = 1^2$.

HW Chap1: 8, Grad: Let $S$ be a nonempty subset of $\mathbb{R}$, bounded above and below, and let $b < 0$. Show that $\inf(bS) = b \sup S$ and $\sup(bS) = b \inf S$.

**Note 53.**—$bS = \{bs \mid s \in S\}$.

Last time: least-upper-bound property, ordered fields

**Theorem.** *There exists an ordered field with the least upper bound property, called the real numbers, denoted $\mathbb{R}$, and $\mathbb{R}$ contains $\mathbb{Q}$ as a subfield.*

**Note 54.**—$\mathbb{R}$ contains $\mathbb{Z}$ and $\mathbb{N}$.

**Note 55.**—The least upper bound property is equivalent to completeness (in $\mathbb{R}^1$).

**Note 56.**—Proof using cuts.

**Theorem.** *:*

*a. (Archimedean Property) If $x, y \in \mathbb{R}$ and $x > 0$, then there exists a positive integer $n$ such that $nx > y$.*

*b. (Denseness) If $x, y \in \mathbb{R}$ and $x < y$, then $\exists p \in \mathbb{Q}$ such that $x < p < y$.*

*Proof of (a).* Let $x, y \in \mathbb{R}$ and $x > 0$. Define the set $A = \{nx \mid n \in \mathbb{N}\}$. By way of contradiction, suppose (a) is false. Then $y$ is an upper bound for $A$. Since $A$ is nonempty, $A$ has a least upper bound in $\mathbb{R}$, say $\alpha = \sup A$. Since $x > 0$, $\alpha - x < \alpha$, so $\alpha - x$ is not an upper bound for $A$. Thus, $\alpha - x \leq a$ for some $a \in A$. That is, $\alpha - x \leq mx$ for some $m \in \mathbb{N}$. This implies $\alpha \leq (m+1)x$, which contradicts that $\alpha$ is an upper bound for $A$. Thus, (a) is true.

**Note 57.**—(a) is equivalent to: If $x \in \mathbb{R}$ and $x > 0$, then $\exists n \in \mathbb{N}$ such that $\frac{1}{n} < x$.

**Note 58.**—(a) is equivalent to: $\mathbb{N}$ is unbounded above.

**Theorem.** *For every real $x > 0$ and positive integer $n > 0$, $\exists!$ positive real $y$ such that $y^n = x$.*

**Proof.** The uniqueness is evident as $0 < y_1 < y_2$ implies $y_1^n < y_2^n$.

To demonstrate existence, define $E = \{t \in \mathbb{R} \mid t^n < x\}$.

Let $t = \frac{x}{x+1}$; we have $0 < t < 1$. Thus, $t^n < t < x$, so $t \in E$, making $E$ nonempty.

To show that $E$ is bounded above, suppose $t \in \mathbb{R}$ and $t > x + 1$. Then $t^n > t > x$, so $t \notin E$. Thus, $x + 1$ is an upper bound for $E$.

Consequently, $E$ has a least upper bound, denoted $y = \sup E$. To show $y^n = x$, we need to demonstrate that $y^n \not< x$ and $y^n \not> x$. We employ the inequality $b^n - a^n < (b-a)nb^{n-1}$ $(*)$ when $0 < a < b$, which is derived from $b^n - a^n = (b-a)(b^{n-1} + b^{n-2}a + \cdots + ba^{n-2} + a^{n-1})$.

Suppose $y^n < x$. Pick $h$ such that $0 < h < 1$ and $h < \frac{x-y^n}{n(y+1)^{n-2}}$. Using $a = y$, $b = y + h$ in $(*)$, we get

$$(y+h)^n - y^n < hn(y+h)^{n-1} < hn(y+1)^{n-1} < x - y^n.$$

Thus, $(y+h)^n < x$, so $y + h \in E$. However, $y + h > y$, contradicting $y = \sup E$. Thus, $y^n \not< x$.

Suppose $y^n > x$. Set $k = \frac{y^n - x}{ny^{n-1}}$, clearly $0 < k$. Also, $ky^{n-1} \leq kny^{n-1} = y^n - x < y^n$. So $k < y$, $0 < k < y$.

To see that $y - k$ is an upper bound of $E$, suppose $t \geq y - k$. Then

$$\begin{aligned} y^n - t^n &\leq y^n - (y-k)^n \\ &< kny^{n-1} \\ &= y^n - x \end{aligned}$$

Thus $t^n > x$, so $t \notin E$. Consequently, $y - k$ is an upper bound of $E$. However, $y - k < y$, contradicting $y = \sup E$. Thus, $y^n \not> x$, so $y^n = x$.

Last time: properties of $\mathbb{R}$

**Note 59.**—:

- $\mathbb{R}$ can be represented via infinite decimal expansion
- Extended real numbers consist of $\mathbb{R}$ with $\pm\infty$ and some algebra, such as $x + \infty = +\infty$, $\frac{x}{+\infty} = 0$, $x \cdot (\infty) = \pm\infty$ depending on the sign of $x$.

One deficiency of $\mathbb{R}$ is not all polynomials have zeros in $\mathbb{R}$. This can be fixed by considering the complex field.

**Definition.**—A complex number is an ordered pair $(a, b)$, where $a, b \in \mathbb{R}$. The set of complex numbers is a field with $+, \cdot$ defined by $(a, b) + (c, d) = (a + c, b + d)$ and $(a, b) \cdot (c, d) = (ac - bd, ad + bc)$.

**Definition.**—$i = (0, 1)$

**Theorem.** $i^2 = -1 = (-1, 0)$

**Theorem.** $(a, b) = a + bi$

**Definition.**—If $a, b \in \mathbb{R}$ and $z = a+bi$, then the complex conjugate of $z$ is $z = a-bi$. The absolute value or modulus of $z$ is $|z| = \sqrt{zz} = \sqrt{a^2 + b^2}$.

**Note 60.**—$zz \geq 0$ and $zz = 0$ only when $z = 0$.

**Proposition 30.**—Properties:

(i) $|\Re z| \leq |z|$

(ii) $|z + w| \leq |z| + |w|$

Schwarz Inequality: If $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are complex, then

$${\sum_{j=1}^{n} a_j b_j}^{2} \leq \sum_{j=1}^{n} |a_j|^2 \cdot \sum_{j=1}^{n} |b_j|^2.$$

**Example 207.**—Suppose $S \subset R$ bounded above and $a > 0$. Show $\sup(aS) = a \sup S$.

**Recall 48.**—$aS = \{as \mid s \in S\}$

Let $\alpha = \sup S$. Want to show $\sup(aS) = a\alpha$. To do this, show $a\alpha$ is an upper bound of $aS$, and if $\gamma$ is an upper bound of $aS$, then $a\alpha \leq \gamma$. Let $x \in aS$. Then $x = a \cdot s$ for some $s \in S$. Thus $s \leq \alpha$, so $as \leq a\alpha$, $x \leq a\alpha$. So $a\alpha$ is an upper bound of $aS$. Suppose $\gamma$ is an upper bound of $aS$. Then $\gamma \geq as$ for all $s \in S$, so $\frac{\gamma}{a} \geq s$ for all $s \in S$. Thus $\frac{\gamma}{a}$ is an upper bound of $S$, so $\frac{\gamma}{a} \geq \alpha$ since $\alpha = \sup S$. So $\gamma \geq a\alpha$. Thus $a\alpha$ is the l.u.b. (or sup) of $aS$.

# Chapter 6

# FUNCTIONAL ANALYSIS

**Definition.**—A vector space over a field $K$ is a nonempty set $X$ of elements (called vectors) together with algebraic operations of vector addition and scalar multiplication, which satisfy axioms (p. 50-51).

**Note 61.**—In this course, $K$ is always either $\mathbb{R}$ or $\mathbb{C}$.

**Example 208** ($\mathbb{R}^2$ and $\mathbb{R}^3$)**.**—These vectors can be visualized as directed line segments, and we have some intuition.

**Example 209** ($X = \mathbb{R}^n$)**.**—$x = (\xi_1, \dots, \xi_n)$, $\xi_i \in \mathbb{R}$.

**Example 210** ($X = \mathbb{C}^n$)**.**—$x = (\xi_1, \dots, \xi_n)$, $\xi_i \in \mathbb{C}$.

**Example 211** ($X = \ell^\infty$)**.**—Vectors are sequences $x = (\xi_1, \xi_2, \dots)$ satisfying $\sup\{|\xi_i| \mid i = 1, 2, \dots\} < \infty$.

**Example 212** ($X = C[a, b]$)**.**—Vectors $x = x(t)$ are continuous functions on the interval $[a, b]$.

**Definition.**—A subspace of a vector space $X$ is a nonempty subset $Y$ such that $\forall x, y \in Y$ and $\forall \alpha, \beta \in K$, then $\alpha x + \beta y \in Y$.

**Note 62.**—A subspace $Y$ is itself a vector space.

**Definition.**—A linear combination of vectors $x_1, \dots, x_n$ in $X$ is a vector of the form $\alpha_1 x_1 + \cdots + \alpha_n x_n$, with $\alpha_n \in K$.

**Definition.**—If $M \subset X$ is a subset of $X$, the set of all linear combinations of vectors of vectors in $M$ is called the span of $M$, denoted $\operatorname{span}(M)$.

**Note 63.**—$\operatorname{span}(M)$ is a subspace.

**Definition.**—Consider a finite set $M = \{x_1, \dots, x_n\}$ and the equation $\alpha_1 x_1 + \cdots + \alpha_n x_n = 0$ $(*)$. If $(*)$ holds only for $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$, then $M$ is linearly independent; otherwise, it is dependent.

An infinite set $M$ is linearly independent if every finite subset is linearly independent.

**Definition.**—If $X$ is a vector space and $\mathcal{B}$ is a linearly independent subset such that $\operatorname{span}(\mathcal{B}) = X$, then $\mathcal{B}$ is a basis for $X$ (Hamel basis).

**Definition.**—A vector space $X$ is finite-dimensional if there is a natural number $n$ such that $X$ contains a set of $n$ linearly independent vectors, whereas any set of $n+1$ vectors is linearly dependent. In this case, $n$ is the dimension of $X$. If $X$ is not finite-dimensional, it is infinite-dimensional.

**Note 64.**—$X = \{0\}$ has dimension 0.

*Corollary.*—Every finite-dimensional vector space has a basis.

**Theorem.** *Every vector space has a basis (requires axiom of choice).*

Last time: vector spaces and their (algebraic) properties.

To motivate the definition of a norm on a vector space, consider $\mathbb{R}^2$.

**Definition.**—A normed vector space is a vector space $X$ with a norm $||\cdot||$ defined on it. A norm $||\cdot||$ on a vector space $X$ is a real-valued function on $X$, with values denoted by $||x||$, which satisfies:

N1) $||x|| \geq 0$

N2) $||x|| = 0$ if and only if $x = 0$

N3) $||\alpha x|| = |\alpha|\, ||x||$ for all $x \in X$, for all $\alpha \in K$

N4) $||x+y|| \leq ||x|| + ||y||$ for all $x, y \in X$ (Triangle inequality).

**Note 65.**—The norm defines a metric on $X$ by $d(x, y) = ||x - y||$.

**Example 213** ($\mathbb{R}^3$ with Euclidean norm)**.**—For $x = (\xi_1, \xi_2, \xi_3)$, $||x|| = \sqrt{\xi_1^2 + \xi_2^2 + \xi_3^2}$.

**Example 214** ($\mathbb{R}^n$)**.**—For $x = (\xi_1, \ldots, \xi_n)$, define $||x||_2 = \sqrt{\sum_{i=1}^n \xi_i^2}$.

**Example 215** ($X = C[a, b]$)**.**—For $x = x(t)$, define $||x|| = \max\limits_{a \leq t \leq b} |x(t)|$. Notation: this is also denoted $||\cdot||_\infty$.

**Example 216** ($X = \ell^\infty$)**.**—For $x = (\xi_i)_{i=1}^\infty$, define norm $||x|| = \sup\{|\xi_i| \mid i = 1, \ldots\}$.

**Definition.**—A sequence of vectors $(x_n)_{n=1}^\infty$ in a normed vector space $X$ is convergent if there exists $x \in X$ such that $\lim_{n\infty} ||x_n - x|| = 0$. Notation: we write $x_n x$.

**Recall 49.**—$\lim\limits_{n\infty} ||x_n - x|| = 0$ means: for every $\epsilon > 0$, there exists $N > 0$ such that if $n \geq N$, then $||x_n - x|| < \epsilon$.

**Definition.**—A sequence of vectors $(x_n)_{n=1}^{\infty}$ is Cauchy if for every $\epsilon > 0$, there exists $N > 0$ such that if $m, n > N$, then $||x_m - x_n|| < \epsilon$.

**Definition.**—A normed vector space is complete if every Cauchy sequence in $X$ is convergent in $X$.

**Definition.**—A complete normed vector space is called a Banach space.

**Example 217** ($X = \mathbb{R}^n$, with Euclidean norm $||\cdot||_2$)**.**—For $x = (\xi_1, \dots, \xi_n)$, $||x||_2 = \sqrt{\sum_{i=1}^n \xi_i^2}$ is a norm. N1-N3 easy. For $x, y = (\eta_1, \dots, \eta_n)$. To show $||x + y|| \leq ||x|| + ||y||$, show

$$
\begin{aligned}
||x+y||^2 &\leq (||x|| + ||y||)^2 = ||x||^2 + ||y||^2 + 2||x||\,||y||. \\
||x+y||^2 &= \sum_{i=1}^n (\xi_i + \eta_i)^2 = \sum_{i=1}^n (\xi_i^2 + \eta_i^2 + 2\xi_i\eta_i) \\
&= \sum_{i=1}^n \xi_i + \sum_{i=1}^n \eta_i + 2\sum_{i=1}^n \xi_i\eta_i = ||x||^2 + ||y||^2 + 2\sum_{i=1}^n \xi_i\eta_i \\
&\leq ||x||^2 + ||y||^2 + 2\sum_{i=1}^n |\xi_i\eta_i| \qquad \text{Cauchy-Schwarz ineqaulity} \\
&\leq ||x||^2 + ||y||2 + 2\sqrt{\sum_{i=1}^n |\xi_i|^2}\sqrt{\sum_{i=1}^n |\eta_i|^2} \\
&= ||x||^2 + ||y||^2 + 2||x|| \cdot ||y|| = (||x|| + ||y||)^2
\end{aligned}
$$

To show $\mathbb{R}^n$ is complete with this norm: Let $(x_m)_{m=1}^{\infty}$ be a Cauchy sequence in $\mathbb{R}^n$. Notation: $x_m = (\xi_1^m, \dots, \xi_n^m)$. Let $\epsilon > 0$. There $\exists N > 0$ such that if $m, r > N$, then $||x_m - r_r|| < \epsilon$. So $||x_m - x_r||^2 < \epsilon^2$. So $\sum_{i=1}^n (\xi_i^m - \xi_i^r)^2 < \epsilon^2$ $(*)$. For each $i$, $(\xi_i^m - \xi_i^r)^2 < \epsilon^2$, so $|\xi_i^m - \xi_i^r| < \epsilon$. So $(\xi_i^m)_{m=1}^{\infty}$ is a Cauchy sequence of real numbers, here convergent since $\mathbb{R}$ is complete. Thus $\lim_{m\infty} \xi_i^m = \xi_i$ for each $i$. Define $x = (\xi_1, \dots, \xi_n)$. Let $r\infty$ in $(*)$ to get $\sum_{i=1}^n (\xi_i^m - \xi_i)^2 \leq \epsilon^2$ and $||x^m - x||^2 \leq \epsilon^2$ and $||x^m - x|| \leq \epsilon$. Thus $x_m x$.

Last time: normed vector spaces, $\mathbb{R}^n$ is complete.

Define vector space $s$ = set of all sequences (bounded or unbounded). $x \in s$, $x = (\xi_1, \xi_2, \dots)$ or $x = (\xi_i)_{i=1}^{\infty}$. Can define a metric by

$$
d(x, y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|\xi_i - \eta_i|}{1 + |\xi_i - \eta_i|},
$$

where $x = (\xi_i)_{i=1}^{\infty}, y = (\eta_i)_{i=1}^{\infty}$. Is there a norm $||\cdot||$ on $S$ such that $d(x, y) = ||x - y||$ for all $x, y \in S$.

**Theorem Translation Invariance.** *On vector space $X$, a metric $d$ induced by a norm is translation invariant: $d(x+a, y+a) = d(x, y)$ and $d(\alpha x, \alpha y) = |\alpha| d(x, y)$ for all $x, y, a \in X$, and all scalar.*

**Proof.** Let $x, y, a \in X$ and $\alpha$ scalar. $d(x+a, y+a) = ||(x+a)-(y+a)|| = ||x-y|| = d(x, y)$ and $d(\alpha x - \alpha y) = ||\alpha x - \alpha y|| = ||\alpha(x-y)|| = |\alpha|\ ||x-y|| = |\alpha| d(x, y)$.

**Recall 50.**—For $x = (x_1, x_2) \in \mathbb{R}^2$, $||x||_2 = \sqrt{x_1^2 + x_2^2}$ and $||x||_\infty = \max\{|x_1|, |x_2|\}$. So

$$S_2 = x \in \mathbb{R}^2 ||x||_2 = 1 = (x_1, x_2) x_1^2 + x_2^2 = 1$$

and

$$S_\infty = x \in \mathbb{R}^2 ||x||_\infty = 1 = (x_1, x_2)|x_1| = 1 \text{ or } |x_2| = 1.$$

**Definition.**—Consider normed spaces $X$ and $Y$. A mapping $T : XY$ is termed an isometry if it preserves length, meaning $||Tx||_Y = ||x||_X$ for all $x \in X$. If $T$ is also an isomorphism, we refer to $X$ and $Y$ as isomorphically isometric, implying they are essentially the same.

**Theorem.** *Let $X$ be a normed space. Then there exists a completion of $X$.*

**Lemma.**—*Consider a set $x_1, x_2, \ldots, x_n$ that is linearly independent in a normed vector space $X$. Then, there exists $c > 0$ such that for every set of scalars $\alpha_1, \ldots, \alpha_n$, the inequality $(*)$ $||\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n|| \geq c(|\alpha_1| + \cdots + |\alpha_n|)$ holds.*

**Proof.** Set $s = |\alpha_1| + \cdots + |\alpha_n|$. If $s = 0$, then $(*)$ always holds for any $c > 0$. If $s > 0$, consider $(**)$ $||\beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_n x_n|| \geq c$ for all $\sum_{j=1}^n |\beta_j| = 1$, where $\beta_j = \frac{\alpha_j}{s}$.

Suppose, for the sake of contradiction, that $(**)$ is not true for all scalars such that $\sum_{j=1}^n |\beta_j| = 1$. Thus, there exists a sequence $(y_n)_{m=1}^\infty$, where $y_m = \beta_1^m x_1 + \beta_2^m x_2 + \cdots + \beta_n^m x_n$, with $\sum_{j=1}^n |\beta_j^m| = 1$, and $||y_m|| 0$ as $m 0$.

Proceeding with a construction of subsequences and applying the Bolzano-Weierstrass theorem, we obtain a subsequence $(y_{n,m})_{m=1}^\infty$ of $(y_m)$ such that $y_{n,m} = \gamma_1^m x_1 + \gamma_2^m x_2 + \cdots + \gamma_n^m x_n$, where $\sum_{j=1}^n |\gamma_j^m| = 1$ and $\lim_{m\infty} \gamma_j^m = \beta_j$. Define $y = \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_n x_n$. Thus, $y_{n,m} y$, where $\sum_{j=1}^n |\beta_j| = 1$. However, $||y_{n,m}|| 0$, and $||y|| \neq 0$, leading to a contradiction.

**Theorem.** *Every finite-dimensional subspace $Y$ of a normed space $X$ is complete and closed.*

*Corollary.*—Every finite-dimensional normed space is a Banach space.

In infinite-dimensional spaces, certain phenomena occur that do not hold in finite-dimensional spaces. For instance, every finite-dimensional vector space is complete, but this does not generalize to all infinite-dimensional spaces.

HW 2.3: 1,3,10, G2; HW 2.4: 1,2,8, G6.

Recall: Every finite-dimensional normed space is complete. Use Lemma: If $x_1, \dots, x_n$ is linearly independent in $X$, then $\exists c > 0$ such that $||\alpha_1 x_1 + \cdots + \alpha_n x_n|| \geq c(|\alpha_1| + \cdots + |\alpha_n|)$.

**Definition.**—Consider a vector space $X$, and let $|| \cdot ||$ and $|| \cdot ||_0$ be two norms on $X$. The norms are said to be equivalent if there are positive numbers $a$ and $b$ such that $\forall x \in X$, $a||x||_0 \leq ||x|| \leq b||x||_0$.

**Remark 28.**—Note that $a||x||_0 \leq ||x|| \leq b||x||_0 \implies \frac{1}{b}||x|| \leq ||x||_0 \leq \frac{1}{a}||x||$ and $||x_n - x||0$ implies $||x_n - x||_0 0$ + vice versa.

**Theorem.** *Let $X$ be a finite-dimensional vector space. If $|| \cdot ||, || \cdot ||_0$ are norms on $X$, they are equivalent.*

**Proof.** Let $X$ be finite-dimensional, and let $||\cdot||, ||\cdot||_0$ be norms on $X$. It is sufficient to show $\exists$ constants $c_1, c_2 > 0$ such that $||x||_0 \leq c_1||x||$ and $||x|| \leq c_2||x||_0$ for all $x \in X$.

Let $x_1, \dots, x_n$ be a basis for $X$. Set $k = \max ||x_j||_0 j = 1, \dots, n > 0$ and let $c > 0$ be given by the Lemma for $|| \cdot ||$. Let $x \in X$. Then $x = \alpha_1 x_1 + \cdots + \alpha_n x_n$. Then

$$\begin{aligned} ||x||_0 &\leq ||\alpha_1 x_1 + \cdots + \alpha_n x_n||_0 \leq |\alpha_1|\,||x_1||_0 + \cdots + |\alpha_n|\,||x_n||_0 \leq k(|\alpha_1| + \cdots + |\alpha_n|) \\ &\leq \frac{k}{c}||\alpha_1 x_1 + \cdots + \alpha_n x_n|| \leq \frac{k}{c}||x|| = c_1||x|| \end{aligned}$$

A similar argument shows that $||x|| \leq c_2||x||_0$.

**Example 218.**—An example of a normed space with two non-equivalent norms. The space must be infinite-dimensional by the previous result. Consider $X = C[a, b]$ with norms $|| \cdot ||_\infty$ and $|| \cdot ||_1$. These norms are not equivalent; $X$ with $|| \cdot ||_\infty$ is complete (a Banach space), but $X$ with $|| \cdot ||_1$ is not complete.

**Definition.**—A subset $M$ of a normed space (or metric space) $X$ is compact if every sequence in $M$ has a subsequence that converges in $M$.

**Lemma.**—*A compact set $M$ is closed and bounded. The converse is not generally true, but it holds for compact subsets of $R$ by the Bolzano-Weierstrass theorem.*

**Theorem.** *Let $X$ be a finite-dimensional normed space. A subset $M$ is compact if and only if $M$ is closed and bounded.*

**Example 219.**—A subset $M$ of a normed space that is closed and bounded but not compact. Let $X = \ell^2$ with $|| \cdot ||_2$. Define

$$\begin{aligned} e_1 &= (1, 0, 0, \dots) = (\delta_{1j})_{j=1}^\infty \\ e_2 &= (0, 1, 0, \dots) = (\delta_{2j})_{j=1}^\infty \\ e_n &= (\delta_{nj})_{j=1}^\infty \end{aligned}$$

The set $M = e_1, e_2, \ldots$ is bounded because $||e_n|| = 1$ for $n = 1, \ldots$. Also, $||e_m - e_n|| = \sqrt{2}$ for all $m \neq n$. Thus, $M$ is closed, but $M$ is not compact, as $(e_n)_{n=1}^{\infty}$ is a sequence in $M$ with no convergent subsequence.

- A set is compact if and only if it is closed and bounded.
- The converse is true in a finite-dimensional normed space.

**Theorem Riesz Lemma.** *Let $Y, Z$ be subspaces of a normed space $X$, and suppose $Y$ is a closed proper subspace of $Z$. Then, for every $\theta \in (0,1)$, there exists $z \in Z$ such that $||z|| = 1$ and $||z - y|| \geq \theta$ for all $y \in Y$.*

**Proof.** Let $\theta \in (0,1)$. Choose $v \in Z \backslash Y$ and set $a = \inf_{y \in Y} ||v - y||$. Since $a > 0$, there exists $y_0 \in Y$ such that $a < ||v - y_0|| \leq \frac{a}{\theta}$. Define $z = \frac{1}{||v-y_0||}(v - y_0)$. Then, $z \in Z$ and $||z|| = 1$. For any $y \in Y$, we have

$$\begin{aligned} ||z - y|| &= \left\| \frac{1}{||v - y_0||}(v - y_0) - y \right\| \\ &= \frac{1}{||v - y_0||} ||v - y_0 - y||. \end{aligned}$$

Now, for any $y \in Y$, $y_1 = y_0 + \frac{1}{||v-y_0||} y \in Y$, so $||v - y_1|| \geq a$. Therefore,

$$\begin{aligned} ||z - y|| &\geq \frac{1}{||v - y_0||} a \\ &\geq \theta, \end{aligned}$$

implying $||z - y|| \geq \theta$.

**Theorem.** *Let $X$ be a normed space. Then $X$ is finite dimensional if and only if the closed unit ball $M = x \in X ||x|| \leq 1$ is compact.*

**Proof.** $\Longrightarrow$ Suppose that $X$ is finite dimensional. Then $M$ is closed and bounded, hence compact by the previous result.

Suppose $M$ is compact. Assume, for the sake of contradiction, that $X$ is infinite dimensional. Choose $x_1 \in M$ such that $||x_1|| = 1$, and set $X_1 = \text{span}\{x_1\}$. By Riesz's lemma, there exists $x_2 \in X$ such that $||x_2|| = 1$ and $||x_2 - x|| \geq \frac{1}{2}$ for all $x \in X_1$. In particular, $||x_2 - x_1|| \geq \frac{1}{2}$. Set $X_2 = \text{span}\{x_1, x_2\}$. By Riesz's lemma again, there exists $x_3 \in X \backslash X_2$ such that $||x_3|| = 1$ and $||x_3 - x_2|| \geq \frac{1}{2}$ for all $x \in X_2$. In particular, $||x_3 - x_2|| \geq \frac{1}{2}$ and $||x_3 - x_1|| \geq \frac{1}{2}$. Continue this process by induction to get a sequence $(x_n)_{n=1}^{\infty}$ in $M$. This sequence cannot have a convergent subsequence because $||x_m - x_n|| \geq \frac{1}{2}$ for all $m \neq n$. This leads to a contradiction to $M$ being compact.

## § 0.1 Linear Operators

**Definition.**—A linear operator $T$ is a mapping such that

i) The domain $D(T)$ is a vector space, and the range $R(T)$ lies in a vector space with the same field of scalars.

ii) For all $x, y \in D(T)$, and scalars $\alpha$, $T(x+y) = Tx + Ty$ and $T(\alpha x) = \alpha Tx$.

**Definition.**—The null space of $T$, denoted $N(T)$, is $N(T) = x \in D(T)Tx = 0$.

**Theorem.** *Let $T : D(T) \subset XY$ be a linear operator from $X$ into $Y$. Then*

*a. $R(t)$ is a subspace of $Y$.*

*b. If* $\dim D(T) = n < \infty$, *then* $\dim R(T) \leq n$.

*c. $N(T)$ is a subspace of $X$.*

**Example 220.**—For any vector space $X$, the identity operator $I$ and the zero operator $T = 0$ (that is, $Tx = 0$ for all $x \in X$) are linear operators.

**Example 221** ($X = ab$)**.**—The integral operator $T : XX$,

$$(Tx)(t) = \int_a^t x(s)\,ds.$$

The differential operator $T : D(T) \subset XX$,

$$(Tx)(t) = x'(t).$$

But $D(T) = x(t) \in abx'(t) \in ab$.

**Definition.**—Let $T : D(T) \subset XY$ be one-to-one. Then the inverse operator $T : R(T) \subset YX$ is defined by $Ty = x$ if $Tx = y$.

**Theorem.** *Let $X, Y$ be vector spaces, and let $T : D(T) \subset XY$ be a linear operator.*

*a) $T$ exists if and only if $Tx = 0$ implies $x = 0$.*

*b) If $T$ exists, it is a linear operator.*

*c) If* $\dim D(T) = n < \infty$, *and $T$ exists, then* $\dim R(T) = n$.

**Note 66.**—If $T : XY$ and $S : YZ$ are invertible, and $ST : XZ$ is defined, then $(ST) = TS$. (like inverse of the product of two invertible square matrices)

**Definition.**—Let $X, Y$ be normed spaces, and let $T : D(T) \subset XY$ be a linear operator. $T$ is bounded if there exists $c \geq 0$ such that $||Tx|| \leq c||x||$ for all $x \in D(T)$.

**Definition.**—The smallest $c$ that works is called the norm of $T$, denoted $||T||$. For $x \neq 0$, we have $\frac{||Tx||}{||x||} \leq c$ for all $x \in D(T), x \neq 0$. So

$$||T|| = \sup_{\substack{x \neq 0 \\ x \in D(T)}} \frac{||Tx||}{||x||}.$$

Alternatively,

$$||T|| = \sup_{\substack{x \neq 0 \\ x \in D(T)}} \frac{||Tx||}{||x||} = \sup_{\substack{x \neq 0 \\ x \in D(T)}} \left\| T\left(\frac{x}{||x||}\right) \right\| = \sup_{\substack{||y||=1 \\ y \in D(T)}} ||Ty|| = \sup_{\substack{||x||=1 \\ x \in D(T)}} ||Tx||.$$

**Note 67.**—If $T$ is bounded, then $||Tx|| \leq ||T|| \, ||x||$.

**Note 68.**—$||T||$ satisfies the properties of a norm and is a norm on the vector space of all bounded linear operators from $X$ to $Y$.

**Example 222.**—The identity operator $||I|| = 1$ and the zero operator $||0|| = 0$.

**Example 223.**—$T : \mathbb{R}^n \mathbb{R}^m$ by $Tx = Ax$, where $A$ is an $m \times n$ matrix. For any norms on $\mathbb{R}^m$ and $\mathbb{R}^n$, $T$ is a bounded linear operator.

**Example 224** $(X = C[0,1). - - -with—— \cdot ||_\infty]$ Let $K(t,\tau)$ be continuous on $[0,1] \times [0,1]$. So $\exists K_0 \geq 0$ such that $|K(t,\tau)| \leq K_0$ for all $t, \tau \in [0,1]$. Define $T : XX$ by $(Tx)(t) = \int_0^t K(t,\tau)x(\tau)\,d\tau$. One can show $T$ is linear. Also, $T$ is bounded because

$$\begin{aligned}
||Tx||_\infty &= \sup_{0 \leq t \leq 1} \left| \int_0^t K(t,\tau)x(\tau)\,d\tau \right| \\
&\leq \sup_{0 \leq t \leq 1} \int_0^t |K(t,\tau)|\,|x(\tau)|\,d\tau \\
&\leq \sup_{0 \leq t \leq 1} \int_0^t K_0 ||x||\,d\tau \\
&= \sup_{0 \leq t \leq 1} K_0 ||x|| t \\
&= K_0 ||x||_\infty.
\end{aligned}$$

So $||T|| \leq K_0$.

**Example 225** $(X = C[0,1). - - -with—— \cdot ||_\infty]$ $D(T) = C^1[0,1]$. $(Tx)(t) = x'(t)$. Consider $x_n(t) = t^n$. Then $||x_n|| = 1$ for $n = 1, 2, \ldots$ but $||Tx_n||_\infty = ||nt^{n-1}|| = n$. So there is no $c$ such that $\frac{||Tx_n||}{||x_n||} \leq c$ for all $n$. Thus, $T$ is unbounded.

Let $x = (x_1, \dots, x_n) \in \mathbb{R}^n$.

$$\begin{aligned}
&\frac{1}{\sqrt{n}}\|x\|_1 \leq \|x\|_2 \\
&\iff \frac{1}{\sqrt{n}}(|x|_1 + \cdots + |x_n|) \leq \sqrt{|x_1|^2 + \cdots + |x_n|^2} \\
&\iff (|x_1| + \cdots + |x_n|)^2 \leq n(|x_1|^2 + \cdots + |x_n|^2)
\end{aligned}$$

Case $n = 2$: $(|x_1| + |x_2|)^2 \leq 2(|x_1|^2 + |x_2|^2)$.

Observe: $\pm 2ab$

leq $\text{a}^2 + b^2$ for all $a, b \in \mathbb{R}$ because $0 \leq (a+b)^2$ and $0 \leq (a-b)^2$.

For $n = 2$, $(|x_1| + |x_2|)^2 = |x_1|^2 + |x_2|^2 + 2|x_1|\,|x_2| \leq 2(|x_1|^2 + |x_2|^2)\checkmark$.

For general $n$, by induction.

Assume for $n = k$:

$$(||x||_1)^2 \leq k(|x_1|^2 + \cdots + |x_k|^2)$$

For $n = k + 1$:

$$\begin{aligned}
(|x_1| + \cdots + |x_k| + |x_{k+1}|)^2 &= (|x_1| + \cdots + |x_k|)^2 + |x_{k+1}^2| + 2(|x_1| + \cdots + |x_k|)|x_{k+1}| \\
&\leq k(|x_1| + \cdots + |x_k|) + |x_{k+1}|^2 + 2|x_1|\,|x_{k+1}| \\
&\quad + \cdots + 2|x_k|\,|x_{k+1}| \\
&\leq k(|x_1|^2 + \cdots + |x_k|^2) + |x_{k+1}|^2 \\
&\quad + (|x_1|^2 + |x_{k+1}|^2) + \cdots + (|x_k|^2 + |x_{k+1}|^2) \\
&= k(|x_1|^2 + \cdots + |x_k|^2) + (|x_1|^2 + \cdots + |x_k|^2) \\
&\quad + (k+1)|x_{k+1}|^2 \\
&= (k+1)(|x_1|^2 + \cdots + |x_k|^2) + (k+1)|x_{k+1}|^2 \checkmark
\end{aligned}$$

**Example 226** ($X = \ell^\infty$ with $|| \cdot ||_\infty$)**.**—Given $x = (\xi_1, \xi_2, \dots) \in \ell^\infty$, define $Tx = (\xi_2, \xi_3, \dots)$ as the left shift operator. Easy to show $T$ is linear, $D(T) = X$, so $T : XX$. Also $||Tx||_\infty = \sup\{|\xi_i|\}_{i=2}^\infty \leq ||x||_\infty$. This implies $T$ is bounded, and $||T|| \leq 1$. Actually $||T|| = 1$, since, for example, for $x = (0, 2, 0, \dots)$

$$||Tx||_\infty = ||(2, 0, 0, \dots)||_\infty = 2 = ||x||_\infty.$$

Thus, it can't be possible for $||T|| < 1$.

**Theorem.** *Let $T : D(T) \subset XY$ be a linear operator. If $X$ is finite dimensional, then $T$ is bounded.*

**Recall 51.**—To show $T$ is bounded, show there is a constant $C$ such that $||Tx|| \leq C||x||$ for all $x \in D(T)$.

**Recall 52.**—$\exists c > 0$ such that for any scalars $\xi_1, \ldots, \xi_n$, we have $||\xi_1 e_1 + \cdots + \xi_n e_n|| \geq c(|\xi_1| + \cdots + |\xi_n|)$.

**Proof.** Assume $T$ is linear, $X, Y$ are normed spaces, and $X$ is finite dimensional. Let $n$ be the dimension of $X$, and $\{e_1, \ldots, e_n\}$ be a basis for $X$. Let $B = \max\{||Te_1||, \ldots, ||Te_n||\}$. Let $x \in D(T)$. Then $x = \xi_1 e_1 + \cdots + \xi_n e_n$ for some scalars $\xi_1, \ldots, \xi_n$. Then

$$\begin{aligned} ||Tx|| &= ||T(\sum_{i=1}^{n} \xi_i e_i)|| \\ &\leq \left|\left|\sum_{i=1}^{n} \xi_i T e_i\right|\right| \leq |\xi_1|\,||Te_1|| + \cdots + |\xi_n|\,||Te_n|| \\ &\leq B(|\xi_1| + \cdots + |\xi_n|) \leq \frac{B}{c}\left|\left|\sum_{i=1}^{n} \xi_i e_i\right|\right| = \frac{B}{c}||x||. \end{aligned}$$

Thus $||Tx|| \leq \frac{B}{c}||x||$ for all $x \in D(T)$. So $T$ is bounded.

**Definition.**—Let $T : D(T) \subset XY$ be any operator (not necessarily linear), where $X$ and $Y$ are normed vector spaces. We say $T$ is continuous at $x_0 \in D(T)$ if for all $\epsilon > 0$, there exists $\delta > 0$ such that $x \in D(T)$ and $||x - x_0|| < \delta$, then $||Tx - Tx_0|| < \epsilon$. A continuous operator is continuous at every $x \in D(T)$.

**Theorem.** *Let $T : D(T) \subset XY$ be linear. Then*

*a. $T$ is continuous if and only if $T$ is bounded.*

*b. If $T$ is continuous at one point $x_0 \in D(T)$, then $T$ is continuous on $D(T)$.*

*Proof of a.* | Assume $T$ is bounded, and $||T|| \neq 0$. Let $x_0 \in D(T)$. Let $\epsilon > 0$. Set $\delta = \frac{\epsilon}{||T||}$. If $x \in D(T)$ and $||x - x_0|| < \delta$, then

$$||Tx - Tx_0|| = ||T(x - x_0)|| \leq ||T|| \cdot ||x - x_0|| < ||T|| \cdot \delta = ||T|| \cdot \frac{\epsilon}{||T||} = \epsilon.$$

Thus $T$ is continuous at $x_0$. Since $x_0 \in D(T)$ is arbitrary, $T$ is continuous for every $x \in D(T)$.

$\Longrightarrow$ | Assume $T$ is continuous. Fix $x_0 \in D(T)$, so $T$ is continuous at $x_0$. Let $\epsilon = 1$. Then there exists $\delta > 0$ such that if $||x - x_0|| \leq \delta$, then $||Tx - Tx_0|| \leq 1$. Let $y \in D(T)$, $y \neq 0$. Set $x = x_0 + \frac{\delta}{||y||}y$. Then $||x - x_0|| = \left|\left|x_0 + \frac{\delta}{||y||}y - x_0\right|\right| = \delta$. Thus $||Tx - Tx_0|| \leq 1$. But observe

$$||Tx - Tx_0|| = ||T(x - x_0)|| = \left|\left|T\left(\frac{\delta}{||y||}y\right)\right|\right| = \frac{\delta}{||y||}\,||Ty||.$$

So $\frac{\delta}{||y||}\,||Ty|| \leq 1$, so $||Ty|| \leq \frac{1}{\delta}||y||$.

**Recall 53.**—$T$ bounded $\iff$ $\exists C$ such that $||Tx$
——$\leq C||x||$ for all $x \in D(T)$.

*Proof of b.* Suppose $T$ is continuous at $x_0$. By proof of $\implies|$ in a., this implies $T$ is bounded. By $|$ in a., this implies $T$ is continuous on all of $D(T)$.

*Corollary.*—Let $T : D(T) \subset XY$ be a bounded linear operator.

a. If $x_n x$ and $x_n, x \in D(T)$, then $Tx_n Tx$.

b. $N(T)$ is closed.

*Proof of a.* Assume $x_n x$, $x_n, x \in (D)T$, and $T$ is bounded. Then

$$||Tx_n - Tx|| = ||T(x_n - x)|| \leq ||T||\, ||x_n - x||^{\to 0}.$$

So $||Tx_n - Tx|| \to 0$, so $Tx_n Tx$.

*Proof of b.* Suppose $T$ is a bounded operator. Let $(x_n)_{n=1}^{\infty}$ be a sequence in $N(T)$, with $x_n \to x$ for $x \in X$. Therefore, $Tx_n \to Tx$. However, since $Tx_n = 0$ for all $n = 1, 2, \ldots$, we have $Tx = 0$. Consequently, $x$ belongs to $N(T)$, proving that $N(T)$ is closed.

**Definition.**—A linear functional $f$ is a linear operator having its domain as $D(f) \subset X$, where $X$ represents a vector space, and its range space is either the field of scalars $R$ or $C$. Notation-wise, we use $f$, $g$, and $h$.

**Note 69.**—A linear functional is synonymous with a linear operator.

**Definition.**—A bounded linear functional is a linear functional that acts as a bounded linear operator.

Thus, for a bounded linear functional, the following hold:

- There exists $c \geq 0$ such that $|f(x)| \leq c||x||$, implying
$$||f|| = \sup_{\substack{x \neq 0 \\ x \in D(f)}} \frac{|f(x)|}{||x||} = \sup_{\substack{||x||=1 \\ x \in D(f)}} |f(x)|.$$
- $|f(x)| \leq ||f||\, ||x||$.

**Example 227** ($X = R^3$ with Euclidean norm)**.**—Fix $a = (\alpha_1, \alpha_2, \alpha_3) \in R^3$. Define $f(x)$ for $x = (\xi_1, \xi_2, \xi_3) \in R^3$ by $f(x) = x \cdot a = \xi_1\alpha_1 + \xi_2\alpha_2 + \xi_3\alpha_3$. Clearly, $f$ is a linear functional. By a theorem, we know $f$ is bounded. We have: $|f(x)| = |x \cdot a| \leq ||x|| \cdot ||a||$ by Cauchy-Schwarz. This implies $||f|| \leq ||a||$. To show $||f|| = ||a||$, find $x$ such that $|f(x)| = ||a||\, ||x||$. Take $x = a$:

$$|f(x)| = |f(a)| = a \cdot a = ||a||^2 = ||a||\, ||a|| = ||a||\, ||x||.$$

Thus, $||f|| = ||a||$.

**Example 228** ($X = C[a,b]$ with $||\cdot||_\infty$)**.**—Define $f(x)$ for $x(t) \in X$ by $f(x) = \int_a^b x(t)\,dt$. Clearly, $f$ is a linear functional. To show that $f$ is bounded,

$$|f(x)| = \int_a^b x(t)\,dt \leq \int_a^b |x(t)|\,dt \leq \int_a^b ||x||_\infty\,dt = \boxed{(b-a)}\,||x||_\infty.$$

Thus, $f$ is bounded, and $||f|| \leq b-a$. To see that $||f|| = b-a$, consider $x(t) \equiv 1$. Then $||x||_\infty = 1$, and $|f(x)| = \int_a^b 1\,dt = b-a = (b-a)||x||$. So, $||f|| = b-a$.

**Example 229** ($X = C[a,b]$ with $||\cdot||_\infty$)**.**—Fix $c \in [a,b]$. Define $f(x) = x(c)$ (point evaluation); $f$ is linear. It is easy to show that $f$ is bounded.

**Example 230** ($X = C[0,1]$ with $||\cdot||_1$)**.**—Then $||x||_1 = \int_0^1 |x(t)|\,dt$. Recall that this is a normed space, but not complete. Define the linear functional by $f(x) = x(0)$. We claim that $f$ is not bounded (there is no constant $C$ such that $|f(x)| \leq C||x||_1$ for all $x$). Consider

$$x_n(t) = \begin{cases} 0 & \frac{1}{n} \leq t \leq 1 \\ -nt+1 & 0 \leq t \leq \frac{1}{n} \end{cases}.$$

Then $|f(x_n)| = |x_n(0)| = 1$ for all $n = 1, 2, \ldots,$ and $||x_n||_1 = \int_0^1 |x_n(t)|\,dt = \frac{1}{2n}$ for $n = 1, 2 \ldots.$ There is no $C$ such that $|f(x_n)| = 1 \leq C\frac{1}{2n} = C||x_n||_1$ for all $n$.

HW 2.8: 3, G6.

Last time: examples of linear functionals.

**Example 231** ($X = \ell^2$)**.**—Fix $a = (\alpha_j) \in \ell^2$. So $\sum_{j=1}^{\infty} |\alpha_j|^2 < \infty$. Define $f : XR$ or $C$ by $f(x) = \sum_{j=1}^{\infty} \alpha_j \xi_j$, where $x = (\xi_j)$. Clearly, $f$ is linear. $f$ is bounded because

$=1^{\infty}|\xi_j|^2 = ||a||_2 \cdot ||x||_2$.

Thus, $f$ is bounded and $||f|| \leq ||a||_2$. Notice for $x = a$ we get $|f(a)| = ||a||_2 \cdot ||a||_2$. So, $||f|| = ||a||_2$. ✓

**Definition.**—Let $X$ be a vector space. The vector space $X^* = ff$ is a linear functional on $X$ is called the algebraic dual space of $X$. $X^*$ is a vector space.

What about $(X^*)^*$? This is called the second algebraic dual space. Fix $x \in X$. Define $g_x \in X^{**}$ by: For any $f \in X^*$, define $g_x(f) = f(x)$. The claim is that $g_x$ is linear: for any $f_1, f_2 \in X^*$ and scalars $\alpha, \beta$,

$$\begin{aligned} g_x(\alpha f_1 + \beta f_2) &= (\alpha f_1 + \beta f_2)(x) \\ &= \alpha f_1(x) + \beta f_2(x) \\ &= \alpha g_x(f_1) + \beta g_x(f_2). \end{aligned}$$

This defines a mapping $C : XX^{**}$ by $Cx = g_x$. $C$ is a linear operator, called the canonical mapping. Are there any elements of $X^{**}$ besides those in $R(C)$? If $X^{**} = R(C)$, we say $X$ is algebraically reflexive.

Linear operators on finite-dimensional vector spaces. Let $X, Y$ be finite-dimensional vector spaces, and let $T : XY$ be a linear operator. Let $E = e_1, \ldots, e_n$ be a basis for $X$ and $B = b_1, \ldots, b_r$ be a basis for $Y$. Let $x \in X$. Then $x = \sum_{i=1}^{n} \xi_i e_i$. We have

$$y = Tx = \sum_{j=1}^{n} \xi_k T e_k = \sum_{k=1}^{n} \xi_k y_k,$$

where $y_k = Te_k$. On the other hand, $y$ and $y_k$ have unique representations in terms of basis $B$:

$$y = \sum_{j=1}^{r} \eta_j b_j, \qquad y_k = Te_k = \sum_{j=1}^{r} \tau_{jk} bj.$$

Thus:

$$\sum_{j=1}^{r} \eta_j b_j = y = \sum_{k=1}^{n} \xi_k y_k = \sum_{k=1}^{n} \xi_k \sum_{j=1}^{r} \tau_{jk} b_j = \sum_{j=1}^{r} \sum_{k=1}^{n} \xi_k \tau_{jk} b_j$$

Thus $\eta_j = \sum_{k=1}^{n} \xi_k \tau_{jk}$. Define matrix $T_{EB} = (\tau_{jk})_{r \times n}$. Set $\tilde{x} = \xi_1 \vdots \xi_n$ and $\tilde{y} = \eta_1 \vdots \eta_r$; then $\tilde{y} = T_{EB} \tilde{x}$.

**Example 232** (2.3-2).—Consider a sequence $(x_k)$ in $c_0$, where $x_k = (\xi_j^k)_{j=1}^{\infty}$ with $\xi_j^k 0$ as $j\infty$. Suppose $x_k x$ in $\ell^{\infty}$, where $x = (\xi_j)_{j=1}^{\infty}$, meaning $||x_k - x||_{\infty} 0$. Show that $x \in c_0$, implying $\xi_j 0$. Let $\epsilon > 0$...

**Example 233** (2.3-3).—Let $x_n = (1, \frac{1}{2}, \dots, \frac{1}{n}, 0, 0, 0, \dots)$. Clearly, each $x_n \in Y$, and $x_n x \in \ell^{\infty}$, where $x = (1\frac{1}{2}, \frac{1}{3}, \dots) \notin Y$ because $||x_n - x||_{\infty} = \frac{1}{n+1}$.

**Example 234** (2.3-10).—Consider a Schauder basis $(e_n)_{n=1}^{\infty}$. Set $Y = \sum_{k=1}^{n} q_k e_k q_k \in Q$. Let $x \in X$ and $\epsilon > 0$. Then

$$x - \sum_{k=1}^{n} q_k e_k \leq x - n\alpha ek + \sum_{k=1}^{n} (\alpha_k - q_k) e_k$$

$\epsilon/2$.

**Example 235.**—Let $X = p(x)p(x)$ is a polynomial, $||p||_1 = \int_0^1 |p(x)|\, dx$, and $||p||_{\infty} = \max_{0 \leq x \leq 1} |p(x)|$. Consider $f_n(x) = x^n$. Then $||f_n||_{\infty} = 1$ for all $n$, but $||f_n||_1 = \frac{1}{n+1} 0$. So there is no constant $c > 0$ such that $||p||_{\infty} \leq c||p||_1$ for all $p \in X$.

Last time: Linear operators on finite-dimensional spaces, matrix representations.

**Definition** (Dual Basis).—Given a vector space $X$ with basis $e_1, \dots, e_n$, there is a unique set of linear functionals $f_1, \dots, f_n$ such that $f_i(e_j) = \delta_{ij}$. This is a basis for the dual space $X^*$.

Given vector spaces $X, Y$, in linear algebra we define $L(X, Y) =$ the vector space of all linear operators from $X$ to $Y$. If $X, Y$ are normed spaces, we define $B(X, Y) =$ the vector space of all bounded linear operators from $X$ to $Y$. $B(X, Y)$ is a normed space with norm

$$||T|| = \sup_{x \neq 0 x \in X} \frac{||Tx||}{||x||} = \sup_{||x||=1 x \in X} ||Tx||.$$

The dual space of $X$ is denoted as $X' = B(X, R)$ or $B(X, C)$.

**Theorem.** *$B(X, Y)$ is complete (hence a Banach space) if $Y$ is complete.*

*Corollary.*—The dual space $X'$ is a Banach space.

# § 1 Hilbert spaces

We motivate the definition of an inner product by considering the dot product in $R^3$.

- Geometrically, the dot product is defined as $x \cdot y = ||x||\ ||y|| \cos\theta$, where $\theta = \arccos \frac{x \cdot y}{||x||\ ||y||}$.

- Algebraically, the dot product of $x = (\xi_1, \xi_2, \xi_3)$ and $y = (\eta_1, \eta_2, \eta_3)$ is $x \cdot y = \xi_1\eta_1 + \xi_2\eta_2 + \xi_3\eta_3$, which can be easily extended to $R^n$.
- $||x||^2 = x \cdot x$.
- Parallelogram law: $||x + y||^2 + ||x - y||^2 = 2(||x||^2 + ||y||^2)$.

**Definition.**—Let $X$ be a vector space. An inner product on $X$ is a mapping $\cdot, \cdot : X \times XK$ ($R$ or $C$) such that for all $x, y, z \in X$ and scalars $\alpha \in K$,

(a) $x + y, z = x, z + y, z$

(b) $\alpha x, y = \alpha x, y$.

(c) $x, y = y, x$.

(d) $x, x \geq 0$ and $x, x = 0 \iff x = 0$.

The space $X$ is then called an inner product space. A Hilbert space is a complete inner product space.

**Remark 29.**—$\cdot, \cdot$ is sesquilinear.

- An inner product defines a norm by $||x|| = \sqrt{x, x}$.
- If $X$ is real, then $x, y = y, x$.
- Not every norm is defined by an inner product.
- If $|| \cdot ||$ is defined by an inner product, then $|| \cdot ||$ satisfies
  the parallelogram law.
- If a norm satisfies the parallelogram law, then it comes from an inner product (polarization identity).

Hilbert spaces have certain important properties not found in general normed spaces or Banach spaces.

- Hilbert space $H$ can be represented as $H = M \oplus M^{\perp}$.
- Hilbert spaces may have an orthonormal basis (countable).
- Riesz Theorem characterizes all bounded linear functionals.
- Can define the adjoint of a linear operator.

**Example 236.**—$R^n, C^n$ with the dot product and Euclidean norm.

**Example 237.**—$\ell^2 = x = (\xi_j)_{j=1}^{\infty} \sum_{j=1}^{\infty} |\xi_j|^2 < \infty$, $||x||_2 = \sqrt{\sum_{j=1}^{\infty} |\xi_j|^2}$, $x, y = \sum_{j=1}^{\infty} \xi_j \eta_j$, $x = (\xi_j), y = (\eta_j)$.

**Example 238.**—$C[a,b]$ with $f, g = \int_a^b f(x)g(x)\, dx$ is an inner product space with norm $||f||_2 = \sqrt{\int_a^b |f(x)|^2\, dx}$. This is not complete. The completion is the Hilbert space $L^2(a,b)$.

Last time: inner product, Hilbert space. We noted in $R^3$ the dot product is related to the Euclidean norm by

$$||x|| = \sqrt{x, x}. \qquad (*)$$

**Lemma.**—*Consider the inner product space $(X, \cdot, \cdot)$, where $X$ is equipped with the inner product $\cdot, \cdot$. Define the norm $||\cdot||$ according to the expression $(*)$. Then, the following hold:*

a. *For any vectors $x, y \in X$, the Schwarz inequality is satisfied: $|x, y| \leq ||x||\, ||y||$. Equality holds only if $x$ is a scalar multiple of $y$.*

b. *The triangle inequality is satisfied: $||x + y|| \leq ||x|| + ||y||$.*

*Proof of a.* The inequality is trivially true when $y = 0$. Assume $y \neq 0$. For any scalar $\alpha$,

.

Take $\alpha = \frac{y,x}{y,y}$. Then,

$$0 \leq x, x - \frac{y, xx, y}{y, y}$$
$$0 \leq x, x - \frac{|x, y|^2}{y, y}$$

Thus, $|x, y|^2 \leq x, xy, y \implies x, y \leq ||x||\, ||y||$.

Equality holds if $x, y = ||x||\, ||y||$, which implies $\frac{y,x}{y,y} = \frac{x,y}{x,x}$. Solving for $\alpha$ yields $\alpha = \alpha$, and substituting this back gives $y, x = x, y$. Since the inner product is conjugate symmetric, this implies $x$ is a scalar multiple of $y$.

[Proof of b.] Consider

$$\begin{aligned} ||x + y||^2 &= x + y, x + y \\ &= x, x + x, y + y, x + y, y \\ &\leq ||x||^2 + ||y||^2 + |x, y| + |y, x| \\ &\leq ||x||^2 + ||y||^2 + 2||x||\, ||y|| \qquad \text{(Schwarz inequality)} \\ &= (||x|| + ||y||)^2. \end{aligned}$$

So, $||x + y|| \leq ||x|| + ||y||$.

Let $X$ be a normed space, $M$ a subset, and $x \in X$. The distance $\delta$ from $x$ to $M$ is defined by

$$\delta = \inf_{\tilde{y} \in M} ||x - \tilde{y}||.$$

Is there a vector $y \in M$ such that $||x - y|| = \delta$? If so, is it unique?

**Example 239** (In $R^2$).—The distance $\delta$ from a point $x$ to an open line segment $M$ is always defined, but there may be no vector $y \in M$ such that $||x - y|| = \delta$.

**Example 240.**—If the boundary of $M$ has a concave circular boundary with a center $x$, then there are infinitely many $y \in M$ such that $||x - y|| = \delta$.

**Definition.**—Let $x, y \in X$. The line segment from $x$ to $y$ is $z = \alpha x + (1 - \alpha)y 0 \leq \alpha \leq 1$.

**Definition.**—$M \subset X$ is convex if whenever $x, y \in M$, then the line segment from $x$ to $y$ is in $M$.

**Remark 30.**—Any set can be made convex by taking its convex hull.

**Theorem.** *Let $X$ be an inner product space, and let $M \neq 0$ be a convex set which is complete. Then for all $x \in X$, there exists a unique $y \in M$ such that $||x - y|| = \delta = \inf_{\tilde{y} \in M} ||x - y||$.*

*a. Existence.* By the definition of inf, there is a sequence $(y_n) \in M$ such that $\delta_n \delta$, where $\delta_n = ||x - y_n||$. Claim: $(y_n)$ is Cauchy. Let $v_n = x - y_n$. So $||v_n|| = \delta_n$ and $y_n - y_m = v_m - v_n$. We have:

$$||v_n + v_m|| = ||y_n + y_m - 2x|| = 2 \left|\left| \left(\frac{1}{2}y_n + \frac{1}{2}y_m\right) - x \right|\right| \geq 2\delta.$$

By the parallelogram law,

$$\begin{aligned} ||y_n - y_m||^2 = ||v_n - v_m||^2 &= -||v_n + v_m||^2 + 2(||v_n||^2 + ||v_m||^2) \\ &\leq -(2\delta)^2 + 2(\delta_n^2 + \delta_m^2) 0 \end{aligned}$$

as $n, m \infty$. So $(y_n)$ is Cauchy. Since $M$ is complete, $y_n y$, and $y \in M$. Also, $||x - y|| \geq \delta$. Also, $||x - y|| \leq ||x - y_n|| + ||y_n - y|| = \delta_n + ||y_n - y|| \delta$. So $||x - y|| \leq \delta$, so $||x - y|| = \delta$.

[b. Uniqueness.] Suppose there exist $y_1, y_2 \in M$ such that $||x - y_1|| = \delta$ and $||x - y_2|| = \delta$. Let $z_1 = x - y_1$ and $z_2 = x - y_2$. By the previous lemma, since $M$ is complete, $z_1 \perp M$ and $z_2 \perp M$. Subtracting these orthogonality conditions, we get $z_1 - z_2 \perp M$. But $z_1 - z_2 = y_2 - y_1$, which implies $y_2 - y_1 \perp M$. Since $y_1$ and $y_2$ are in $M$, this implies $y_2 - y_1 \perp (y_2 - y_1)$, which means $y_2 - y_1 = 0$. Therefore, $y_1 = y_2$, and the solution is unique.

**Lemma.**—*Let $X$ be an inner product space, and let $Y$ be a complete subspace. (Thus $Y$ is a nonempty, complete, convex subset). Fix $x \in X$ and let $y \in Y$ be the unique vector in $Y$ closest to $x$. Then $z = x - y$ is orthogonal to $Y$. That is, $z, \tilde{y} = 0$ for all $\tilde{y} \in Y$.*

**Recall 54** (Last time).—Let $X$ be inner product space.

(a) Let $M \neq \emptyset$ be complete, convex subset of $X$, and $x \in X$. [Then there exists unique vector in $M$ closest to $x$.] Then $\exists! y \in M$ such that

$$\delta = \inf_{\tilde{y} \in M} ||x - \tilde{y}|| = ||x - y||.$$

(b) If $Y$ is a complete subspace of $X$ and $x \in X$, then $\exists! y \in Y$ such that $z = x - y \perp Y$ and $||z|| = \delta$.

**Definition.**—Let $H$ be a Hilbert space and $Y$ a subspace. The orthogonal complement of $Y$, denoted $Y^\perp$, is the set of all vectors orthogonal to $Y$.

$$Y^\perp = z \in Hz, y = 0 \ \forall y \in Y$$

This is a specific instance of the more general definition:

**Definition.**—If $M \neq \emptyset$ is a subset of $H$, the annihilator of $M$, denoted $M^\perp$, is the set

$$M^\perp = z \in Hz, y = 0 \ \forall y \in M.$$

An alternative definition is given by:

**Note 70.**—For $M \subset X$, $M^\circ = f \in X' f(x) = 0 \ \forall x \in M$.

**Definition.**—A vector space $X$ is said to be the direct sum of two subspaces $Y$ and $Z$ if each $x \in X$ has a unique representation $x = y + z$, $x \in X, y \in Z$. Notation: $X = Y \oplus Z$.

**Example 241.**—Let $Y$ and $Z$ be any two non-parallel, intersecting subspaces of $R^2$, then $Y \oplus Z = R^2$.

**Theorem.** *Let $H$ be a Hilbert space and $Y$ a closed subspace. Then $H = Y \oplus Z$, where $Z = Y^\perp$.*

**Proof.** Let $x \in H$. Since $H$ is complete and $Y$ is closed, then $Y$ is complete. By the previous theorem, $\exists! y \in Y$ such that $z = x - y \in Y^\perp$. Clearly $x = y + z$, and $y \in Y, z \in Y^\perp$ $(*)$. To see this is unique, suppose $x = y + z$ and $x = y_1 + z_1$, where $y, y_1 \in Y$, $z, z_1 \in Y^\perp$. Then $y - y_1 = z_1 - z$. But $y - y_1 \in Y$, so $z_1 - z \in Y \cap Y^\perp$. Thus $z_1 - z = 0$, so $z = z_1$ and $y = y_1$.

**Notation**: The vector $y \in Y$ is called the orthogonal projection of $x$ onto $Y$.

**Note 71.**—$(*)$ defines a linear operator on $H$ by $Px = y$. $P$ is a bounded linear operator on $H$. Also $P^2 x = Px$, so $P^2 = P$. $N(P) = Y^\perp$.

**Remark 31.**—In Hilbert space $H$, an idempotent operator $P$ will have $R(P) \oplus N(P) = H$ (requires $R(P)$ is closed, maybe $P$ is bounded), but not necessarily in Banach space.

Notice that if $x \in Y$, then $x \perp Y^\perp$, so $x \in (Y^\perp)^\perp$. Thus $Y \subset (Y \perp)^\perp$.

**Lemma.**—*Let $H$ be a Hilbert space and $Y$ a closed subspace. Then $Y = (Y^\perp)^\perp$.*

**Proof.** We know $Y \subset (Y^\perp)^\perp$, need to show $(Y^\perp)^\perp \subset Y$. Let $x \in (Y^\perp)^\perp$. By the previous result, since $Y$ is closed and hence complete, $x = y + z$, where $y \in Y, z \in Y^\perp$. Thus $z = x - y \in (Y^\perp)^\perp$, so $z \in Y^\perp \cap (Y^\perp)^\perp$, so $z = 0$. Thus $x = y$, so $x \in Y$.

Last time: orthogonal complements, direct sums, projections. **Next**: sets and sequences of orthogonal vectors. Motivation by considering $R^3$.

For $x \in R^3$, $x = (x_1, x_2, x_3)$ and basis $e_1, e_2, e_3$. Then $x = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3$, where $\alpha_1 = x_1 = x, e_1$, $\alpha_2 = x_2 = x, e_2$, $\alpha_3 = x_3 = x, e_3$.

**Definition.**—An orthogonal set $M$ in an inner product space is a subset $M \subset X$ where elements are pairwise orthogonal. If all elements have norm 1, we say $M$ is orthonormal. If $M$ is countable, we can write it as an orthogonal or orthonormal sequence $(x_n)_{n=1}^{?}$.

**Note 72.**—$x \perp y \iff ||x+y||^2 = ||x||^2 + ||y||^2$.

**Lemma.**—*An orthogonal set is linearly independent.*

**Proof.** Assume $M$ is an orthogonal set in inner product space $X$. Let $e_1, \dots, e_n \subset M$. Suppose $\alpha_1 e_1 + \dots + \alpha_n e_n = 0$. For any $k = 1, \dots, n$,

$$\begin{aligned} 0 = 0, e_k &= \alpha_1 e_1 + \cdots + \alpha_n e_n, e_k \\ &= \alpha_1 e_1, e_k + \cdots + \alpha_k e_k, e_k + \cdots + \alpha_n e_n, e_k \\ &= \alpha_k e_k, e_k = \alpha_n ||e_k||^2, \end{aligned}$$

so $\alpha_k = 0$. Thus $e_1, \dots, e_n$ is linearly independent.

**Example 242.**—$R^n$, $e_1 = (1, 0, \dots, 0), e_2 = (0, 1, 0, \dots, 0)$, $e_1, \dots, e_n$ is an orthonormal set.

**Example 243.**—$\ell^2$, $e_1 = (1, 0, 0, \dots), e_2 = (0, 1, 0, \dots)$. Then $(e_k)_{k=1}^\infty$ is orthonormal.

**Example 244.**—$L^2(0, 2\pi)$, $f, g = \int_0^{2\pi} f(x)g(x)\,dx$. Define $u_n(t) = \cos nt$, $n = 0, 1, \dots$. Can show

$$u_n, u_m = \int_0^{2\pi} \cos nt \cos mt\,dt = \begin{cases} 0 & n \neq m \\ \pi & n = m = 1, 2, \dots. \\ 2\pi & n = m = 0 \end{cases}$$

Thus $(u_n)_{n=0}^\infty$ is an orthogonal sequence. $e_0 = \frac{1}{\sqrt{2\pi}}, e_n = \frac{1}{\sqrt{\pi}} u_n$, then $(e_n)$ is orthonormal.

**Remark 32.**—Gram-Schmidt process can be used to construct an orthonormal set.

Let $e_1, \ldots, e_n$ be orthonormal, and let $x \in e_1, \ldots, e_n$. Then $x = \sum_{k=1}^n x, e_k e_k$.

**Proof.** We know $x = \sum_{j=1}^n \alpha_j e_j$. Fix $k$. Then $x, e_k = \sum_{j=1}^n \alpha_j e_j, e_k = \sum_{j=1}^n \alpha_j e_j, e_k = \alpha_k$ ✓

Next, let's assume that $Y_n = e_1, \ldots, e_n \subset X$, and $x \notin Y_n$. We can define $y = \sum_{k=1}^n x, e_k e_k$ and set $z = x - y$.

Claim: $z \perp Y_n$. In other words, $y$ is the unique vector in $Y_n$ closest to $x$.

Let $\tilde{y} \in Y_n$. Then $\tilde{y} = \sum_{j=1}^n \alpha_j e_j$. Now, we can proceed with the calculations:

$$\begin{aligned} z, \tilde{y} &= x - y, \tilde{y} = x, \tilde{y} - y, \tilde{y} \\ &= x, \sum_{j=1}^n \alpha_j e_j - \sum_{k=1}^n x, e_k e_k, \sum_{j=1}^n \alpha_j e_j \\ &= \sum_{j=1}^n x, \alpha_j e_j - \sum_{k=1}^n x_j, e_k e_k, \sum_{j=1}^n \alpha_j e_j \\ &= \sum_{j=1}^n \alpha_j x, e_j - \sum_{k=1}^n x, e_k \cdot e_k, \alpha_k e_k \\ &= \sum_{j=1}^n \alpha_j x, e_j - \sum_{k=1}^n \alpha_k x, e_k = 0 \end{aligned}$$

Also note, since $z \perp y$, $||x||^2 = ||z+y||^2 = ||z||^2 + ||y||^2$. So $||z||^2 = ||x||^2 - ||y||^2$. Also,

$$\begin{aligned} ||y||^2 = y, y &= \sum_{k=1}^n x, e_k e_k, \sum_{j=1}^n x, e_j e_j \\ &= \ldots \\ &= \sum_{k=1}^n |x, e_k|^2. \end{aligned}$$

So $||z||^2 = ||x||^2 - \sum_{k=1}^n |x, e_k|^2$. Therefore, $\sum_{k=1}^n |x, e_k|^2 \leq ||x||^2$ ($\boxed{\text{Bessel's inequality}}$).

**Theorem.** *Let $(e_k)_{k=1}^\infty$ be an orthonormal sequence in an inner product space $X$. Then for any $x \in X$, $\sum_{k=1}^\infty |x, e_k| \leq ||x||^2$.*

Let $H$ be a Hilbert space, and let $(e_k)_{k=1}^\infty$ be an orthonormal sequence. We consider series of the form $\sum_{k=1}^\infty \alpha_k e_k$ $(*)$.

**Definition.**—The series $(*)$ exists and has a limit $s$ if $||s_n - s||0$ as $n\infty$, where $s_n = \sum_{k=1}^{n} \alpha_k e_k$.

**Theorem.** *:*

a. *The series $(*)$ converges if and only if the series $\sum_{k=1}^{\infty} |\alpha_k|^2 < \infty$.*

b. *If the series converges, then $\alpha_k = x, e_k$ where $x = \sum_{k=1}^{\infty} \alpha_k e_k$.*

c. *For any $x \in H$, the series $\sum_{k=1}^{\infty} x, e_k e_k$ converges.*

Last time: Bessel's Inequality: If $(e_k)_{k=1}^{\infty}$ is an orthonormal sequence in an inner product space $X$, and $x \in X$, then $\sum_{k=1}^{\infty} |x, e_k|^2 \leq ||x||^2$.

**Recall 55.**—Let $(e_k)_{k=1}^{\infty}$ be an orthonormal sequence in Hilbert space $H$, and consider $\sum_{k=1}^{\infty} \alpha_k e_k$ $(*)$.

a. The series $(*)$ converges if and only if the series $\sum_{k=1}^{\infty} |\alpha_k|^2$ converges.

b. If the series converges, let $x = \sum_{k=1}^{\infty} \alpha_k e_k$. Then $\alpha_k = x, e_k$, $k = 1, 2, \ldots$.

c. For any $x \in H$, define $\alpha_k = x, e_k$ (called Fourier coefficients). Then the series $(*)$ converges.

**Proof.** :

a. Let $s_n = \sum_{k=1}^{n} \alpha_k e_k$ and $\sigma_n = \sum_{k=1}^{n} |\alpha_k|^2$. For any $m, n$ with $n > m$,

$$||s_n - s_m||^2 = \sum_{k=m+1}^{n} \alpha_k e_k{}^{2} = \sum_{k=m+1}^{n} ||\alpha_k e_k||^2 = \sum_{k=m+1}^{n} |\alpha_k|^2 = |\sigma_n - \sigma_m|.$$

**Recall 56.**—$x \perp y \implies ||x + y||^2 = ||x||^2 + ||y||^2$.

Thus $(s_n)$ is Cauchy if and only if

$(\sigma_n)$ is Cauchy. Thus $\sum_{k=1}^{\infty} \alpha_k e_k$ converges $\iff$ $(s_n)$ converges

$\iff$ $(s_n)$ is Cauchy (because $H$ is complete)

$\iff$ $(\sigma_n)$ is Cauchy

$\iff$ $(\sigma_n)$ is convergent (because $R$ is complete)

$\iff$ $\sum_{k=1}^{\infty} |\alpha_k|^2$ converges.

b. Suppose the series converges, and let $x = \sum_{k=1}^{\infty} \alpha_k e_k$. Thus $s_n x$. Fix $j$. For $n \geq j$, $s_n, e_j = \sum_{k=1}^{n} \alpha_k e_k, e_j = \alpha_j$. Thus

$$x, e_j = \lim_{n\infty} s_n, e_j = \lim_{n\infty} \alpha_j = \alpha_j.$$

c. Let $x \in H$ and define $\alpha_k = x, e_k$ for $k = 1, \ldots$. By Bessel's inequality, $\sum_{k=1}^{\infty} |x, e_k|^2 \leq ||x||^2$. By part a., $\sum_{k=1}^{\infty} x, e_k e_k$ converges.

**Note 73.**—Bessel's is for orthonormal sequences. If $E = (e_k)$ with $k \in I$ is an uncountable orthonormal set in $H$, we can still form Fourier coefficients $x, e_k$ for any $x \in H$. There are uncountably many Fourier coefficients. Can we sum them? By the argument for Bessel's inequality, we have $\sum_{k=1}^{n} |x, e_k|^2 \leq ||x||^2$ for any finite subset $e_1, \ldots, e_n$ of $E$. Thus for any $m = 1, 2, \ldots$ the set $E_m = e_k \in E |x, e_k| > \frac{1}{m}$. The set of all nonzero Fourier coefficients is $\bigcup_{m=1}^{\infty} E_m$, hence countable.

Given $x \in H$, and $E = (e_k)$ an orthonormal set, is $\sum_{k=1}^{\infty} x, e_k e_k = x$?

Given $x \in H$, and $E = (e_k)$ an orthonormal set, the question is equivalent to asking if $\sum_{k=1}^{\infty} x, e_k e_k = x$.

**Definition.**—Let $X$ be a normed space. A subset $M$ is total in $X$ if $M$ is dense in $X$. That is, $M$ is total in $X \iff M = X$ (closure of span). If $X$ is an inner product space, an orthonormal (set, sequence) is a total orthonormal (set, sequence) if it is a total set in $X$.

The above question is the same as asking: Is $E = (e_k)$ a total orthonormal set in $H$?

(a) Every nontrivial Hilbert space has a total orthonormal set.

- For finite dimensions, use a basis, ....
- For separable $H$, use Gram-Schmidt and induction on a countable basis.
- Otherwise, use Zorn's lemma.

(b) In a nontrivial Hilbert space, all total orthonormal sets have the same cardinality.

**Lemma.**—*Let $H$ be a Hilbert space, and $M \neq \emptyset$ a subset. Then $M^{\perp} = 0$ if and only if $M$ is total.*

**Proof.** $\Leftarrow$: Suppose $M$ is total and $M \neq \emptyset$. Set $V = M$, so $V = H$. Let $x \in M^\perp$. It follows that $x \in V^\perp$. Also $x \in V = H$. Thus, there exists a sequence $(x_n)_{n=1}^\infty$ in $V$ such that $x_n x$. Note $x_n, x = 0$ for all $n$ since $x \in V^\perp$ and $x_n \in V$. So

$$||x||^2 = x, x = \lim_{n\infty} x_n, x = \lim_{n\infty} x_n, x = 0.$$

So $x = 0$, $M^\perp = 0$.

$\Rightarrow$: Suppose $M^\perp = 0$. Let $V = M$. If $x \in V^\perp$, then $x \in M^\perp$, so $V^\perp = 0$. Thus $V^\perp = 0$. By the previous result, $H = V \oplus V^\perp = V \oplus 0 = V$. Thus $M$ is total.

**Remark 33.**—If we are just in an inner product space (not Hilbert space), then $M^\perp = 0$.

**Recall 57.**—Let $(e_k)_{k=1}^\infty$ be an orthonormal sequence in Hilbert space. If $x \in H$, then $\sum_{k=1}^\infty |x, e_k|^2 \leq ||x||^2$ (Bessel's inequality). When this is an equality: $\sum_{k=1}^\infty |x, e_k|^2 = ||x||^2$ (Parseval's identity).

**Theorem.** *An orthonormal set $M$ in a Hilbert space $H$ is total if and only if Parseval's identity holds for all $x \in H$.*

**Proof.** $\Leftarrow$: Suppose Parseval's identity holds for all $x \in H$. Assume $M$ is not total. Then, by the previous result, $M^\perp = 0$, so $\exists x \in M^\perp$, $x \neq 0$. Thus $||x||^2 \neq 0$. But $x, e_k = 0$ for $k = 1, \ldots$ because $e_k \in M$ and $x \in M^\perp$. So $\sum_{k=1}^\infty |x, e_k|^2 = 0$, contradicting Parseval's identity.

$\Rightarrow$ (For the case $M$ is countable): Assume $M = (e_k)_{k=1}^\infty$ is a total orthonormal sequence. Let $x \in H$. Define $y = \sum_{k=1}^\infty x, e_k e_k$. Note we know $||y||^2 = \sum_{k=1}^\infty |x, e_k|^2$. Claim: $x - y \in M^\perp$ (note $M^\perp = 0$). For any fixed $j$,

$$x - y, e_j = x, e_j - \sum_{k=1}^\infty x, e_k e_k, e_j = x, e_j - \sum_{k=1}^\infty x, e_k e_k, e_j = x, e_j - x, e_j \cdot 1 = 0.$$

Thus the claim is true, so $x = y$, and Parseval's identity holds.

**Theorem.** *Let $H$ be a Hilbert space.*

a. *If $H$ is separable, then every orthonormal set is countable.*

b. *If $H$ contains a total orthonormal sequence (countable), then $H$ is separable.*

**Proof.** a. Assume $H$ is separable, and let $M$ be an orthonormal subset. Assume, for the sake of contradiction, that $M$ is uncountable. Let $B$ be any dense subset of $H$. If $x, y \in M$, $x$

$\neq y$, then

$$||x - y||^2 = ||x||^2 + ||y||^2 = 2.$$

Thus, the distance from $x$ to $y$ is $\sqrt{2}$. Let $N_x = z \in H||z - x|| < \frac{\sqrt{2}}{3}$. Thus, all neighborhoods of vectors in $M$ are mutually disjoint. For each $x \in M$, there exists $b_x \in B$ such that $b_x \in B \cap N_x$. Thus, $b_x \neq b_y$ if $x, y \in M$, $x \neq y$. Thus, $B$ is uncountable, a contradiction to $H$ being separable, which means $H$ has at least one countable dense subset. Thus, $M$ is countable.

b. Let $(e_k)_{k=1}^{\infty}$ be a total orthonormal sequence. Consider the set of all finite linear combinations of $e_k$'s with rational coefficients.

**Recall 58.**—Let $X$ be a normed space. A bounded linear functional $f$ is a bounded linear operator from $X$ to $R$ or $C$. For Banach spaces, we can sometimes characterize the dual space. For Hilbert spaces, it is easy.

**Example 245.**—Let $H$ be a Hilbert space with an element $z \in H$. A bounded linear functional $f$ on $H$ can be defined as $f(x) = x, z$. It can be proven that $f$ is both linear and bounded, with $||f|| = ||z||$. Consider the null space of $f$, denoted as $N(f) = x \in Hx, z = 0$. Then, the orthogonal complement of $N(f)$, denoted as $N(f)^{\perp}$, is equal to the span of $z$.

**Theorem Riesz Representation Theorem.** *For every bounded linear functional $f$ on a Hilbert space $H$, there exists a unique $z \in H$ such that $f(x) = x, z$ (1), and $||f|| = ||z||$ (2).*

**Proof.** We demonstrate:

a. The existence of $z \in H$ satisfying (1),

b. The uniqueness of $z$ in (a),

c. The satisfaction of (2) by $z$.

Note that the result is trivially true when $f \equiv 0$. Assume $f \not\equiv 0$. This implies $N(f) \neq H$, and consequently, $N(f)^{\perp} \neq 0$. Let $z_0 \in N(f)^{\perp}$ be a non-zero element. Define $z = \frac{f(z_0)}{z_0, z_0} z_0$, ensuring $z_0 \in N(f)^{\perp}$. For $x \in H$, let $v = f(x)z_0 - f(z_0)x$. Observe that $f(v) = 0$, indicating $v \in N(f)$, and consequently, $z_0 \perp v$.

$$\begin{aligned} 0 = v, z_0 &= f(x)z_0 - f(z_0)x, z_0 \\ &= f(x)z_0, z_0 - f(z_0)x, z_0, \end{aligned}$$

leading to

$$f(x) = \frac{f(z_0)}{z_0, z_0} x, z_0 = x, \frac{f(z_0)}{z_0, z_0} z_0 = x, z.$$

b. To show the uniqueness of $z$, suppose there exist $z_1$ and $z_2$ such that $f(x) = x, z_1 = x, z_1$ for all $x \in H$. This implies $x, z_1 - z_2 = 0$ for all $x \in H$. Therefore, $z_1 - z_2 = 0$, establishing $z_1 = z_2$.

c. In the case where $f \equiv 0$, we have $z = 0$, and $||f|| = 0 = ||z||$. Suppose $f \not\equiv 0$, implying $z \neq 0$. We can then show that $||z|| = ||f||$. Note that

$$||z||^2 = z, z = f(z) = |f(z)| \leq ||f|| \cdot ||z||.$$

This implies $||z|| \leq ||f||$. On the other hand, for all $x \in H$,

$$|f(x)| = |x, z| \leq ||x|| \cdot ||z||.$$

From this, it follows that $||f|| \leq ||z||$, ultimately resulting in $||f|| = ||z||$.

**Example 246** ($H = L^2(a, b)$)**.**—Let $H$ be the space of square-integrable functions on the interval $(a, b)$. Define the inner product $f, g = \int_a^b f(x)g(x)\,dx$, and the norm $||f|| = \sqrt{\int_a^b |f(x)|^2\,dx}$. If $F$ is a bounded linear functional on $H$, then there exists a unique $f \in H$ such that $F(g) = \int_a^b g(x)f(x)\,dx$ for all $g \in H$.

**Definition.**—Consider vector spaces $X$ and $Y$ over a field ($R$ or $C$). A sesquilinear form $h$ on $X \times Y$ is a mapping $h : X \times YR$ or $C$ satisfying the following conditions:

a. $h(x_1 + x_1, y) = h(x_1, y) + h(x_2, y)$,

b. $h(x, y_1 + y_2) = h(x, y_1) + h(x, y_2)$,

c. $h(\alpha x, y) = \alpha h(x, y)$,

d. $h(x, \alpha y) = \alpha h(x, y)$.

**Definition.**—Consider normed spaces $X$ and $Y$. A sesquilinear form $h$ on $X \times Y$ is considered bounded if there exists $c > 0$ such that $|h(x, y)| \leq c||x||\,||y||$. In such cases, the norm of $h$ is defined as

$$||h|| = \sup_{y \in Y0 x \in X0} \frac{|h(x, y)|}{||x||\,||y||}.$$

**Example 247.**—Let $H_1$ and $H_2$ be Hilbert spaces. Assume $S \in L(H_1, H_2)$ is a bounded linear operator. Define $h : H_1 \times H_2 R$ or $C$ by $h(x, y) = Sx, y_{H_1}$. It can be easily verified that $h$ is a sesquilinear form on $H_1 \times H_2$. Additionally,

$$\begin{aligned} |h(x, y)| = |Sx, y_{H_2}| &\leq ||Sx||_{H_2} \cdot ||y||_{H_2} \\ &\leq ||S||\,||x||_{H_1}\,||y||_{H_2} \end{aligned}$$

Therefore, $h$ is a bounded sesquilinear form. It can also be demonstrated that $||h|| = ||S||$.

**Example 248** (Gram-Schmidt).—Given vectors $x_1$ and $x_2$, define $v_1 = x_1$ and $e_1 = \frac{1}{||v_1||}v_1$. Then, set $v_2 = x_2 - x_2, e_1 e_1$ and $e_2 = \frac{1}{||v_2||}v_2$.

In the previous, we discussed the Riesz Representation Theorem, which states that for a bounded linear functional $f$ on a Hilbert space $H$, there exists a unique element $z \in H$ such that $f(x) = x, z$. We also introduced the concept of a bounded sesquilinear form $h$ on $H_1 \times H_2$, satisfying $|h(x, y)| \leq ||h||\,||x||\,||y||$.

For a bounded linear operator $S$ from $H_1$ to $H_2$, the sesquilinear form $h(x, y) = Sx, y$ is also bounded, and $||h|| = ||S||$. Now, we present the Representation Theorem, which asserts that for a bounded sesquilinear form $h$ on $H_1 \times H_2$, there exists a unique bounded linear operator $S$ from $H_1$ to $H_2$ such that $h(x, y) = Sx, y$ for all $x \in H_1$ and $y \in H_2$, and $||h|| = ||S||$.

**Theorem Representation Theorem.** *Let $H_1$ and $H_2$ be Hilbert spaces, and consider a bounded sesquilinear form $h : H_1 \times H_2 R$ or $C$. Then, there exists a unique bounded linear operator $S$ from $H_1$ to $H_2$ such that $h(x, y) = Sx, y$ for all $x \in H_1$ and $y \in H_2$, and $||h|| = ||S||$.*

**Proof.** Assume $h$ is a bounded sesquilinear form on $H_1 \times H_2$. Fix $x \in H_1$, and define a bounded linear functional on $H_2$ by $f(y) = h(x, y)$. By the Riesz Representation Theorem, there exists a unique element $z \in H_2$ such that $f(y) = y, z_{H_2}$ for all $y \in H_2$. Note that $z$ depends uniquely on $x$. Define $S : H_1 H_2$ by $Sx = z$. It can be verified that $S$ is linear. Also, $h(x, y) = Sx, y$. Additionally,

$$||S|| = \sup_{x \neq 0} \frac{||Sx||}{||x||} = \sup_{Sx \neq 0 x \neq 0} \frac{Sx, Sx}{||Sx||\,||x||} \leq \sup_{y \neq 0 x \neq 0} \frac{Sx, y}{||y||\,||x||} = \sup_{y \neq 0 x \neq 0} \frac{|h(x, y)|}{||y||\,||x||} = ||h||$$

This establishes that $S$ is bounded, and $||S|| \leq ||h||$. Moreover,

$$||h|| = \sup_{y \neq 0 x \neq 0} \frac{|h(x, y)|}{||x||\,||y||} = \sup_{||y|| \neq 0 ||x|| \neq 0} \frac{|Sx, y|}{||x||\,||y||} \leq \sup_{y \neq 0 x \neq 0} \frac{||Sx||\,||y||}{||x||\,||y||} = ||S||.$$

Hence, $||h|| = ||S||$. Finally, the uniqueness of $S$ is confirmed by the uniqueness in the Riesz Representation Theorem.

**Example 249** ($H_1 = H_2 = R^3$).—Consider $x = \xi_1 \xi_2 \xi_3, y = \eta_1 \eta_2 \eta_3$. Define $h(x, y) = (\xi_2 + \xi_3)\eta_1 + 2\xi_1 \eta_2 + \xi_2 \eta_3$. It can be shown that $h$ is sesquilinear and bounded. According to the theorem, there exists an operator $T : R^3 R^3$ such that $h(x, y) = Tx, y$. Therefore, $Tx = \xi_2 + \xi_3 2\xi_1 \xi_2$.

**Example 250** ($H_1 = H_2 = \ell^2$).—Let $x = (\xi_k)_{k=1}^{\infty}, y = (\eta_k)_{k=1}^{\infty}$. Define $h(x, y) = \xi_1 \eta_2 + \xi_2 \eta_3 + \ldots$. It can be shown that $h$ is sesquilinear and bounded (Cauchy-Schwartz p.14). Determine the operator $T$ such that $h(x, y) = Tx, y$.

Consider $H_1 = R^n$, $H_2 = R^m$ with the usual inner product $x, y = y^T x = x^T y$. If $A$ is an $m \times n$ matrix, it defines a linear operator $A : H_1 H_2$. In the complex case, $x, y = y^T x$ and yields $Ax, y = \cdots = x, A^T y$.

**Definition.**—Let $H_1$ and $H_2$ be Hilbert spaces, and let $T : H_1 H_2$ be a bounded linear operator. The Hilbert-adjoint operator $T^*$ of $T$ is the unique bounded linear operator satisfying $T^* : H_2 H_1$ such that

$$Tx, y_{H_2} = x, T^* y_{H_1}$$

for all $x \in H_1$ and $y \in H_2$.

**Theorem.** *The Hilbert-adjoint operator $T^*$ is not only guaranteed to exist but is also unique. Moreover, it satisfies $||T^*|| = ||T||$.*

Here are some properties of the adjoint operator:

$$\begin{aligned}
(S+T)^* &= S^* + T^* \\
(\alpha T)^* &= \alpha T^* \\
(T^*)^* &= T \\
||T^*T|| &= ||TT^*|| = ||T^2|| \\
(ST)^* &= T^* S^*
\end{aligned}$$

Thus, $(T^*T)^* = T^* T^{**} = T^* T$.

Recall the representation theorem for bounded sesquilinear forms discussed last time:

**Recall 59.**—Let $H_1$ and $H_2$ be Hilbert spaces, and let $T$ be a bounded linear operator from $H_1$ to $H_2$. Then, there exists a unique bounded linear operator $T^*$ from $H_2$ to $H_1$, referred to as the Hilbert-adjoint of $T$. It satisfies

$$Tx, y = x, T^* y$$

for all $x \in H_1$ and $y \in H_2$, with $||T^*|| = ||T||$.

**Proof.** Consider a bounded linear operator $T$ from $H_1$ to $H_2$. Define a function on $H_2 \times H_1$ by $h(y, x) = y, Tx$. This function is sesquilinear because the inner product is sesquilinear. Furthermore,

$$|h(y, x)| = |y, Tx| \leq ||y|| \cdot ||Tx|| \leq ||y||\, ||T||\, ||x||.$$

Hence, $h$ is bounded, and $||h|| \leq ||T||$. Additionally,

$$\begin{aligned}
||h|| &= \sup_{y \neq 0 x \neq 0} \frac{|h(y, x)|}{||x||\, ||y||} = \sup_{y \neq 0 x \neq 0} \frac{|y, Tx|}{||x||\, ||y||} \\
&\geq \sup_{Tx \neq 0 x \neq 0} \frac{|Tx, Tx|}{||x||\, ||Tx||} = \sup_{Tx \neq 0 x \neq 0} \frac{||x||^2}{||x||\, ||Tx||} \\
&= \sup_{x \neq 0} \frac{||Tx||}{||x||} = ||T||.
\end{aligned}$$

This implies $||h|| \geq ||T||$, so $||h|| = ||T||$. By the representation theorem, there exists a unique bounded linear operator $T^*$ from $H_2$ to $H_1$ such that $h(y,x) = T^*y, x_{H_1}$ and $||T^*|| = ||h||$, resulting in $||T^*|| = ||T||$. Additionally,

$$T^*x, y = y, Tx = h(y,x) = T^*y, x = x, T^*y$$

**Example 251** ($H_1 = H_2 = H = \ell^2$)**.**—Consider $x = (\xi_1, \xi_2, \dots), y = (\eta_1, \eta_2, \dots)$, where $x, y = \sum_{k=1}^{\infty} \xi_k \eta_k$. Define $T : HH$ by $Tx = (0, \xi_1, \xi_2, \dots)$ (the right shift operator). It is evident that $||T|| = 1$, $N(T) = 0$, and $R(T) = z = (\lambda_1, \lambda_2, \dots)\lambda_1 = 0$. For $x, y \in H$,

$$Tx, y = \sum_{k=2}^{\infty} \xi_{k-1}\eta_k = \sum_{j=1}^{\infty} \xi_j \eta_{j+1} = x, T^*y,$$

where $T^*y = (\eta_2, \eta_3, \dots)$ (the left shift). We observe that $N(T^*) = y = (\eta_1, \eta_2, \dots)\eta_k = 0$. Furthermore,

$$N(T^*) \perp R(T).$$

**Note 74.**—If $H_1 = H_2 = H$, and $T : HH$, then $T^* : HH$.

**Definition.**—Let $H$ be a Hilbert space, and let $T : HH$ be a bounded linear operator.

- $T$ is self-adjoint (or Hermitian) if $T = T^*$
- $T$ is unitary if $T$ is bijective and $T^* = T$
- $T$ is normal if $T^*T = TT^*$

**Example 252** (Square Matrices)**.**—A real square matrix $A = (a_{ij})_{n\times n}$ is symmetric when $A$ equals its transpose $A^T$. A complex square matrix is Hermitian if $A$ equals the conjugate transpose $A^T$. A real square matrix is orthogonal if its transpose is its inverse, denoted as $A^T = A$. A complex square matrix is unitary if its conjugate transpose is its inverse, represented as $A^T = A$. A complex square matrix is normal if the conjugate transpose times the matrix equals the matrix times the conjugate transpose, i.e., $A^T A = AA^T$.

**Theorem.** *Let $H$ be a Hilbert space, and let $U$ and $V$ be unitary operators on $H$. Then,*

*a. $U$ preserves length (isometric): $||Ux|| = ||x||$ for all $x \in H$.*

*b. $||U|| = 1$*

*c. $U$ is unitary*

*d. $UV$ is unitary*

*a.* $||Ux||^2 = Ux, Ux = x, U^*(Ux) = x, UUx = x, x = ||x||^2.$

*b.* Immediate from *a*.

*c.* $U$ is bijective, so $U$ is bijective. $(U)^* = (U^*)^* = U = (U)$. So $U$ is unitary.

*d.* $(UV)^* = V^*U^* = VU = (UV)$.

## § 2 Spectral Theory

We extend the theory of eigenvalues and eigenvectors from square matrices to operators on general normed spaces. Let's first review the concepts for matrices. Consider the equation $(*)$: $Ax = \lambda x$, where $x$ is $n \times 1$ in $C^n$ and $\lambda \in C$.

**Definition.**—An eigenvalue of $A$ is a $\lambda \in C$ such that $(*)$ has a nontrivial solution. A nonzero solution $x$ associated with $\lambda$ is called an eigenvector. The spectrum $\sigma(A)$ is the set of all eigenvalues of $A$. The resolvent set of $A$ is the set $\rho(A) = C\sigma(A)$.

**Note 75.**—$(*)$ is equivalent to $(A - \lambda I)x = 0$, which has a nonzero solution $\iff$ $A - \lambda I$ is singular $\iff$ $\det(A - \lambda I) = 0$. This determinant is a polynomial of degree $n$, so $A$ has at most $n$ distinct eigenvalues.

Last time: spectral theory in finite dimensions (equivalently, eigenvalues and eigenvectors of $n \times n$ matrices). The characteristic equation is $\det(A - \lambda I) = 0 = \det(\lambda I - A)$. An eigenvalue of $A$ exists $\iff$ it is a solution (or zero) of the characteristic equation. The spectrum of $A$ is $\sigma(A) = \lambda\lambda$ is an eigenvalue. A note: $\sigma(A)$ is a closed subset of $C$. The resolvent set is $\rho(A) = C\sigma(A)$, so $\rho(A)$ is an open subset of $C$. Another note: $\lambda \in \rho(A) \iff (A - \lambda I)$ exists. In order to generalize to possibly infinite-dimensional spaces, let's first define the resolvent set. Let $X$ be a complex normed space, and let $T : D(T) \subset XX$ be a linear operator. Define

$$\begin{aligned} T_\lambda &= T - \lambda I, && D(T_\lambda) = D(T) \\ R_\lambda &= T_\lambda = (T - \lambda I) && \text{if it exists} \end{aligned}$$

This is called the resolvent operator of $T$. Consider the following properties:

(R1) $(T - \lambda I)$ exists

(R2) $(T - \lambda I)$ is bounded

(R3) $(T - \lambda I)$ is densely defined (equivalently, $D(T - \lambda I)$ is dense in $X$ or, $R(T - \lambda I)$ is dense in $X$).

In finite dimensions, either all are true or none are true.

**Remark 34.**—$Ax = \lambda x \iff Ax = \lambda Ix \iff Ax - \lambda Ix = 0 \iff (A - \lambda I)x = 0$.

Define the following subsets of $C$:

$$\begin{aligned}
\rho(T) &= \lambda \in C\text{R1, R2, R3 are satisfied} \\
\sigma_p(T) &= \lambda \in C\text{R1 not satisfied} = \text{point spectrum : set of eigenvalues} \\
\sigma_c(T) &= \lambda \in C\text{R1, R3 satisfied, but R2 not satisfied} \\
\sigma_r(T) &= \lambda \in C\text{R1 satisfied, R3 not satisfied,} \\
\sigma(T) &= \sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T)
\end{aligned}$$

which is the spectrum of $T$. Therefore, $\sigma(T) = C\rho(T)$.

**Example 253** ($X = \ell^2$)**.**—For $x = (\xi_1, \xi_2, \dots)$ define $Tx = (0, \xi_1, \xi_2, \dots)$. $T$ is a bounded linear operator, $||T|| = 1$. Consider $\lambda = 0$:

$$R_\lambda(T) = (T - 0I) = T \qquad \text{exists (the left shift)}$$

But $D(T) = R(T) = y = (\eta_j)\eta_1 = 0$. This is not dense in $\ell^2$. So $0 \in \sigma_r(T)$.

For spectral properties of bounded linear operators, the following is useful.

**Theorem.** *Let $X$ be a Banach space, and let $T \in B(X, X)$. If $||T|| < 1$, then $\exists(I - T)$, $D(I - T) = X$, and $(I - T) = \sum_{j=0}^{\infty} T^j = I + T + T^2 + \dots$ $(*)$.*

**Proof.** Note: $||T^j|| \leq ||T||^j$. Note also geometric series $\sum_{j=0}^{\infty} \alpha^j$ converges absolutely if $0 \leq \alpha < 1$. Thus $\sum_{j=0}^{\infty} ||T||^j$ converges. Thus $\sum_{j=0}^{\infty} T^j$ is Cauchy in $B(X, X)$. Since $B(X, X)$ is complete, then the series converges, say $S = \sum_{j=0}^{\infty} T^j$. Observe

$$(I - T)(I + T + T^2 + \cdots + T^n) = (I + T + T^2 + \cdots + T^n)(I - T) = (I - T^{n+1})$$

Let $n\infty$: $(I - T)S = S(I - T) = I$.

Last time: resolvent set and spectrum for linear operators on normed spaces, $T_\lambda = T - \lambda I$, $R_\lambda(T) = (T - \lambda I)$. If $T$ is a linear operator on a Banach space $X$, and $||T|| < 1$, then $(I - T)$ exists and is bounded, $D(I - T) = X$, and $(I - T) = I + T + T^2 + \cdots$.

**Theorem.** *Let $T$ be a bounded linear operator on a complex Banach space $X$. Then $\rho(T)$ is an open subset of $C$ (Hence $\sigma(T)$ is closed in $C$).*

**Proof.** Note it is true if $\rho(T) = \emptyset$, so assume $\rho(T) \neq 0$. [To show $\rho(T)$ is open, show for every $\lambda \in \rho(T)$ there is an open disk/neighborhood containing $\lambda$ which is also in $\rho(T)$]. Fix $\lambda_0 \in \rho(T)$. Observe if $\lambda \in C$,

$$\begin{aligned}
T - \lambda I &= (T - \lambda_0 I) - (\lambda - \lambda_0)I \\
&= (T - \lambda_0 I)[I - (\lambda - \lambda_0)(T - \lambda_0 I)]
\end{aligned}$$

Thus, $T_\lambda = T_{\lambda_0} V$, where $V = I - (\lambda - \lambda_0)(T - \lambda_0 I)$. By the previous theorem, $V$ is invertible, $V$ is bounded, and $D(V) = X$ if $||(\lambda - \lambda_0)R_{\lambda_0}|| < 1$ and $|\lambda - \lambda_0| \cdot ||R_{\lambda_0}|| < 1$. This is true if $|\lambda - \lambda_0| < \frac{1}{||R_{\lambda_0}||}$. For these $\lambda$, $\lambda \in \rho(T)$. Thus $\rho(T)$ is open in $C$.

This is also true in the more general case of unbounded (closed) operators on complex normed spaces.

**Note 76.**—When $T$ is bounded, then $\rho(T) \neq \emptyset$, in fact $\rho(T) \supset \lambda|\lambda| > ||T||$. To see this, suppose $|\lambda| > ||T||$. Then $T - \lambda I = -\lambda(I - \frac{1}{\lambda}T)$. But $||\frac{1}{\lambda}T|| = \frac{1}{|\lambda|}||T|| < 1$. Thus $(I - \frac{1}{\lambda}T)$ exists, is bounded, and domain $= X$. So the same is true for $T - \lambda I$, hence $\lambda \in \rho(T)$. Later we show when $T$ is bounded, $\sigma(T) \neq \emptyset$.

Next, we define a class of unbounded linear operators important for applications.

**Definition.**—Let $X, Y$ be normed spaces and $T : D(T) \subset XY$. $T$ is closed if whenever $(x_n) \subset D(T)$ and $x_n x$ in $X$ and $Tx_n y$ in $Y$, it implies $x \in D(T)$ and $Tx = y$.

**Note 77.**—This is not the same as a bounded operator. However, if $T$ is bounded and $D(T) = X$, then $T$ is closed.

**Theorem.** *$T$ is closed if and only if its graph $G(T)$ is closed in the normed space $X \times Y$, where $G(T) = (x, y)x \in D(T), y = Tx$ and $X \times Y = (x, y)x \in X, y \in Y$.*

$$||(x, y)|| = ||x||_X^2 + ||y||_Y^2{}^{1/2}$$

**Theorem Closed Graph Theorem.** *Let $T : XY$ be a closed operator, $X, Y$ Banach spaces. If $D(T)$ is closed, then $T$ is bounded.*

**Example 254** ($X = 01$)**.**—$||\,||_\infty$, $D(T) = 101$, $(Tx)(t) = x'(t)$. Previously we verified $T$ is unbounded. We claim $T$ is closed. Suppose $(x_n) \subset D(T)$ satisfies $x_n(t)x(t)$ in $X$ and $Tx_n y$. We need to show $x \in D(T)$ and $Tx = y$. Then $Tx_n y$ means $x_n'(t)y(t)$. Observe

$$\int_0^t y(\tau)\,d\tau = \int_0^t \lim_{n\infty}(Tx_n)(\tau)\,d\tau = \lim_{n\infty}\int_0^t x_n'(\tau)\,d\tau = \lim_{n\infty} x_n(t) - x_n(0) = x(t) - x(0).$$

So $x(t) = x(0) + \int_0^t y(\tau)\,d\tau$. Thus $x$ is differentiable and $x'(t) = y(t) \in 01$, so $x \in 101 = D(T)$ and $Tx = y$. ✓ $T$ is closed, and so are several related operators.

$$\begin{aligned} &T_1 x = x', && D(T_1) = x \in 101x(0) = 0 \\ &T_0 x = x', && D(T_0) = x \in 101x(0) = 0 = x(1) \\ &T_k x = x', && D(T_k) = x \in 101x(1) = kx(0) \end{aligned}$$

$T$ is onto but not invertible ($Tx = 0$ for $x(t) \equiv$ constant). $T_1$ is onto and invertible: $Tx = 0$ implies $x(t) \equiv 0$. Also, if $y \in X$, then $Tx = y$ for $x(t) = \int_0^t y(\tau)\,d\tau$. $T_0$ is invertible but not onto.

**Theorem.** *If $T : D(T) \subset XY$ is a closed operator, and $T^{-1}$ exists, then $T^{-1}$ is also closed.*

**Note 78.**—The operator $T$ is considered closed if, for any sequence $(x_n) \subset D(T)$ converging to $x$ in $X$ and $Tx_n y$ in $Y$, it implies $x \in D(T)$ and $Tx = y$.

**Remark 35.**—In this context, $x_nx$ denotes $||x_n - x||0$. Another form of convergence is $x_n{}^wx$, indicating $f(x_n)f(x)$ for all $f \in X^*$. It's essential to note that this does not necessarily imply $x_nx$ due to the inequality

$$|f(x_n) - f(x)| = |f(x_n - x)| \le ||f||\,||x_n - x||.$$

**Proof.** Assume $T$ is closed, and $T^{-1}$ exists. Note that $D(T^{-1}) = R(T)$. Suppose $(y_n) \subset D(T^{-1})$ satisfying $y_ny$ in $Y$ and $T^{-1}y_nx$ in $X$. We need to demonstrate $y \in D(T^{-1})$ and $T^{-1}y = x$. For each $n$, $y_n \in D(T^{-1}) = R(T)$, so there exists $x_n \in D(T)$ such that $Tx_n = y_n$. Thus, $T^{-1}y_n = x_n$. Consequently, $x_nx$ and $Tx_ny$. Since $T$ is closed, we conclude $x \in D(T)$ and $Tx = y$. Therefore, $y \in R(T) = D(T^{-1})$ and $T^{-1}y = x$. ✓

An alternative argument using the graph: Observe

$$(x, y) \in G(T) \subset X \times Y \iff y = Tx \iff x = T^{-1}y \iff (y, x) \in G(T^{-1}) \subset Y \times X.$$

Thus, $G(T)$ is a closed subset of $X \times Y$ if and only if $G(T^{-1})$ is a closed subset of $Y \times X$.

**Recall 60.**—Resolvent and spectrum are defined for closed operators $T : D(T) \subset XX$. It can be shown that $\rho(T)$ is an open subset of $C$.

Next, consider the spectrum for the differential operators $T, T_0, T_1, T_k$.

**Example 255** $(X = 01, D(T) = 101)$**.**—Let $T : D(T) \subset XX$ be given by $(Tx)(t) = x'(t)$. The claim is: $\sigma(T) = \sigma_p(T) = C$. If $\lambda \in C$, then $(T - \lambda I)x = 0$ implies $x'(t) - \lambda x(t) = 0$, and $x(t) = e^{\lambda t}$ is a nontrivial solution for any $\lambda \in C$.

**Example 256** ($T_1$ where $D(T_1) = x \in 101x(0) = 0$)**.**—If $\lambda \in C$, then $(T_1 - \lambda I)x = 0$ implies $x'(t) - \lambda x(t) = 0$ and $x(0) = 0$. So the only solution is $x(t) \equiv 0$. Thus, $(T_1 - \lambda I)$ exists. Considering $y(t) = (T_1 - \lambda I)x(t)$ implies $x'(t) - \lambda x(t) = y(t)$ and $x(0) = 0$. The solution is given by $(*)$:

$$\begin{aligned}
\frac{d}{dt}e^{-\lambda t}x(t) &= e^{-\lambda t}y(t)\\
e^{-\lambda t}x(t) &= \int_0^t e^{-\lambda s}y(s)\,ds\\
x(t) &= e^{\lambda t}\int_0^t e^{-\lambda t}y(s)\,ds \qquad (*)
\end{aligned}$$

Thus, for every $y \in X$, $(T_1 - \lambda I)y$ exists and is given by $(*)$, i.e., $D(T_1 - \lambda I) = X$. Also,

$$\begin{aligned}
(T_1 - \lambda I)y &= x\\
&= e^{\lambda t}\int_0^t e^{-\lambda s}y(s)\,ds\\
&= \max_{0\le t\le 1} e^{\lambda t}\max_{0\le t\le 1}\int_0^t e^{-\lambda s}y(s)\,ds \le ky.
\end{aligned}$$

Thus, $(T_1 - \lambda I)$ is bounded, and $\lambda \in \rho(T_1)$. Therefore, $\rho(T_1) = C$.

**Example 257** ($T_0$ where $D(T_0) = x \in 101x(0) = 0, x(1) = 0$)**.**—Similar to the argument for $T_1$, $T_0 - \lambda I$ is invertible, and $(T_0 - \lambda I)$ is bounded for all $\lambda \in C$. However, if $y \in D(T_0 - \lambda I) = R(T_0 - \lambda I)$, then $y = (T_0 - \lambda I)x$, so $x'(t) - \lambda x(t) = y(t)$, $x(0) = 0$, $x(1) = 0$. Therefore, $x(t) = e^{\lambda t} \int_0^t e^{-\lambda s} y(s)\, ds$. So, $0 = x(t) = e^{\lambda} \int_0^1 e^{-\lambda s} y(s)\, ds$. Thus, $\int_0^t e^{-\lambda s} y(s)\, ds = 0$. Hence, $D(T_0 - \lambda I) = y \in 01 \int_0^1 e^{-\lambda s} y(s)\, ds = 0$. This set is not dense in 01, so $\sigma(T_0) = \sigma_r(T_0) = C$.

**Example 258** ($T_k$ where $D(T_k) = x \in 101x(1) = k\, x(0)$)**.**—The spectrum is given by

$$\sigma(T_k) = \sigma_p(T_k) = \lambda = \ln k + 2n\pi i n = 0, \pm 1, \pm 2, \ldots.$$

For $\lambda \notin \sigma_p(T_k)$,

$$(T_k - \lambda I) = \frac{e^{\lambda t}}{k - e^{\lambda}} k \int_0^t e^{-\lambda s} y(s)\, ds + e^{\lambda} \int_t^1 e^{-\lambda s} y(s)\, ds.$$

Last time: closed operators, resolvent + spectrum for differential operators.

**Recall 61** (The definition of adjoint of a bounded operator)**.**—If $T : H_1 H_2$ is a bounded linear operator and $D(T) = H_1$, then $T^* : H_2 H_1$ is the unique bounded linear operator satisfying $D(T^*) = H_2$ and

$$Tx, y_{H_2} = x, T^* y_{H_1}.$$

For general linear operators, possibly unbounded, $T, S$ are adjoint to each other if $Tx, y = x, Sy$. If $T$ is densely defined (i.e., $D(T)$ is dense in $H_1$), then there exists a unique maximal operator $S$ adjoint to $T$, which is defined to be the adjoint $T^*$. Let $T : D(T) \subset HH$ be a densely defined (but possibly unbounded) linear operator. The adjoint operator $T^*$ is defined by $D(T^*) = y \in H$there exists a unique $y^* \in H$ satisfying $Tx, y = x, y^*\ \forall x \in D(T)$. For $y \in D(T^*)$, define $T^* y = y^*$. Thus, $T^*$ satisfies $Tx, y = x, T^* y\ \forall x \in D(T), \forall y \in D(T^*)$.

**Recall 62.**—The differential operators $T_0, T_1, T_k, T$ are not densely defined in $X = ab$. However, they are densely defined in $H = L^2(0,1)$ with $f, g = \int_0^1 f(t)g(t)\, dt$, where $||f|| = \sqrt{\int_0^1 |f(t)|^2\, dt}$.

To define the domain, we need to define the Sobolev spaces $H^1(0,1)$ analogous to 101. If $f \in H^1(0,1)$, it means

$$f(t) = c + \int_0^t g(s)\, ds, \qquad g \in L^2(0,1).$$

$$101 \subset H^1(0,1) \subset C(0,1) \subset L^2(0,1)$$

$$H_0^1(0,1) = f \in H^1(0,1) f(0) = 0 = f(1)$$

**Example 259** ($H = L^2(0,1), T_1 x = x'$)**.**—$D(T_1) = x \in H^1(0,1) x(0) = 0$. Then $T_1$ is densely defined, so $T_1^*$ exists. Aside: Observe, if $f(t) =$ constant, then $\int_0^1 f(t)g'(t)\,dx = 0$ for all $g \in H_0^1(0,1)$.

**Theorem Fundamental Lemma of Calculus of Variations.** *If $\int_0^1 f(t)g'(t)\,dt = 0$ for all $g \in H_0^1(0,1)$, then $f(t) =$ constant.*

By definition, $T_1 x, y = x, T_1^* y \ \forall x \in D(T_1), \forall y \in D(T_1^*)$. Set $z(t) = T_1^* y$. So

$$\int_0^1 x'(s)y(s)\,ds = \int_0^1 x(s)z(s)\,ds$$

for all $x \in D(T_1)$. Set $w(t) = \int_0^t z(s)\,ds$, so $w'(t) = z(t)$. So,

$$\int_0^1 x'(s)y(s)\,ds = \int_0^1 x(s)w'(s)\,ds = x(s)w(s)\big|_0^1 - \int_0^1 x'(s)w(s)\,ds$$

$$(*) \qquad \int_0^1 x'(s)[y(s) + w(s)]\,ds - x(1)w(1) = 0$$

is true $\forall \in D(T_1)$, so it's also true $\forall x \in H_0^1(0,1)$. By the Fundamental Lemma of the Calculus of Variations (FLCV), $y(t) + w(t) = c =$ constant. So $y(t) = c - \int_0^1 z(s)\,ds$ and $y \in H^1(0,1)$, and $y'(t) = -z(t)$, so $T_1^* y = -y'$. Also, from $(*)$, $\int_0^1 x'(s) \cdot c\,ds - x(1)w(1) = 0$ implies $x(1)c - x(1)w(1) = 0$ for all $x \in D(T_1)$. Since $x(1)$ may not be 0, then $c = w(1)$. Thus, $y(t) = c - w(t) = w(1) - w(t)$ so $y(1) = 0 \implies D(T_1^*) = y \in H^1(0,1), y(1) = 0$.

**Example 260** (Test 2 #4)**.**—Let $T : H_1 H_2$ be a bounded linear operator. If $M_1 \subset H_2$, $M_2 \subset H_2$, and $T(M_1) \subset M_2$, then $T^*(M_2^\perp) \subset M_1^\perp$.

**Proof.** Let $y \in M_2^\perp$. We need to show $0 = x, T^* y$ for all $x \in M_1$. Note that $Tx \in M_2$ if $x \in M_1$. Therefore, $\forall x \in M_1$, $x, T^* y = Tx, y = 0$ because $Tx \in M_2$ and $y \in M_2^\perp$. Hence, $T^* y \in M_1^\perp$ is true for all $y \in M_2^\perp$, so

$$T^*(M_2^\perp) \subset M_1^\perp.$$

# Chapter 7

# FOURIER ANALYSIS

Consider a solid cylinder with perfectly insulated sides, extending from $x = 0$ to $x = c$. Let $u(x, t)$ represent the temperature at position $x$ and time $t$. Subject to appropriate assumptions, we formulate a mathematical model known as the heat equation partial differential equation (PDE):

$$u_t(x, t) = ku_{xx}(x, t), \tag{7.1}$$

where $0 < x < c$ and $t > 0$. The boundary conditions are $u_x(0, t) = 0$ and $u_x(c, t) = 0$ for $t > 0$, and the initial conditions are given by $u(x, 0) = f(x)$ for $0 < x < c$. We assume the thermal diffusivity $k > 0$. This derivation is presented in section 22 of the textbook, and we will employ a solution method outlined in section 36.

Idea: We seek solutions to the PDE (1) and the associated boundary conditions. Since (1) and the boundary conditions are linear and homogeneous, the superposition principle allows us to consider linear combinations of solutions. Our goal is to determine a specific linear combination that also satisfies the initial conditions.

Idea: We employ separation of variables, assuming solutions in the form $u(x, t) = X(x)T(t)$, where $X, T \neq 0$. Substituting this into (1) and applying the boundary conditions, we arrive at

$$\frac{T'(t)}{kT(t)} = \frac{X''(x)}{X(x)}. \tag{7.2}$$

As the left-hand side is a function of only $t$ and the right-hand side is a function of only $x$, both must be constant. Setting $\frac{T'(t)}{kT(t)} = -\lambda = \frac{X''(x)}{X(x)}$, where $\lambda \in \mathbb{R}$, yields two separate ODEs.

**Note 79.**—In accordance with the boundary conditions, we have $X'(0)T(t) = 0$ and $X'(c)T(t) = 0$.

This leads to the Sturm-Liouville boundary value problem:

$$X''(x) + \lambda X(x) = 0, \qquad X'(0) = 0, \qquad X'(c) = 0, \tag{7.3}$$

and a separate ODE for $T(t)$:

$$T'(t) + k\lambda T(t) = 0. \tag{7.4}$$

Equation (5) has a solution for any $\lambda$, i.e., $T(t) = e^{-k\lambda t}$. Equation (4) has solutions only for certain values of $\lambda$. We explore (4) for different $\lambda$'s.

When $\lambda = 0$, $X''(x) = 0$, $X'(0) = 0$, $X'(c) = 0$. The solution is $X(x) = Ax + B$, and by satisfying the boundary conditions, $X(x)$ becomes any constant multiple of 1.

When $\lambda > 0$, assuming $\lambda = \alpha^2$ where $\alpha > 0$, the general solution is $X(x) = c_1 \cos \alpha x + c_2 \sin \alpha x$. Satisfying the boundary conditions yields $X(x) = c_1 \cos \alpha x$, where $\alpha = n\pi/c$ for $n \in \mathbb{N}$. Thus, $X(x) = \cos\left(\frac{n\pi}{c}x\right)$ and any constant multiple.

When $\lambda < 0$, setting $\lambda = -\alpha^2$ for $\alpha > 0$, the general solution is $X(x) = c_1 e^{\alpha x} + c_2 e^{-\alpha x}$. Satisfying the boundary conditions results in the trivial solution $X(x) = 0$.

Set $\lambda_0 = 0$, then $\lambda_n = \left(\frac{n\pi}{c}\right)^2$ for $n \in \mathbb{N}$. The solutions are $X_0(x) = 1$, $T_0(t) = 1$, $X_n(x) = \cos \frac{n\pi}{c}x$, and $T_n(t) = e^{-k\frac{n^2\pi^2}{c^2}t}$. Therefore, $u_0(x,t) = 1$ and $u_n(x,t) = \cos\left(\frac{n\pi}{c}x\right) e^{-k\frac{n^2\pi^2}{c^2}t}$ solve (1) and the boundary conditions. By superposition, the general solution is expressed as

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{c}x\right) e^{-k\frac{n^2\pi^2}{c^2}t},$$

where the coefficients $A_0, A_1, \ldots$ are to be determined. The question arises whether such a solution can also satisfy the initial conditions, and further exploration is needed to determine $A_0, A_1, \ldots$ for which $A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi}{c}x = f(x)$.

## § 1 Fourier Series

Consider the finite interval $(a,b)$.

**Definition.**—$f(x)$ is piecewise continuous (PWC) if $f$ is continuous for all points on $(a,b)$ except for a finite set, and if those points, and the one-sided limits at endpoints, exist. $C_p(a,b)$ is the set of all functions that are PWC on the interval $(a,b)$.

**Example 261.**—$y = \tan x$ is not PWC on $(-\pi/2, \pi/2)$ because $\lim_{x\to\pi/2^-} f(x)$ does not exist.

**Example 262.**—$f(x) = \begin{cases} 0 & \text{for } -1 < x < 0 \\ 1 & \text{for } 0 \le x < 1 \end{cases}$ is PWC.

**Example 263.**—$f(x) = \sin(1/x)$ on $(0,1)$ is not PWC.

**Note 80.**—If $f \in C_p(a,b)$, then $\int_a^b f(x)\,dx$ exists. If $f, g \in C_p(a,b)$, then $c_1 f(x) + c_2 g(x) \in C_p(a,b)$. Thus, $C_p(a,b)$ is a vector space and an example of function spaces. If $f, g \in C_p(a,b)$, then $f \cdot g \in C_p(a,b)$.

We can define an inner product $(f,g) = \int_a^b f(x)g(x)\,dx$, and we can define orthogonal functions as $f \perp g \iff (f,g) = 0$. The norm (length) is defined as $||f|| = \sqrt{(f,f)} = \left(\int_a^b f^2(x)\,dx\right)^{1/2}$. Thus, $C_p(a,b)$ is an infinite-dimensional normed vector space.

Let $f \in C_p(0,\pi)$. Assuming $f$ has a Fourier cosine series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \tag{7.5}$$

for $0 < x < \pi$, we can determine the coefficients $a_i$ by integrating term by term. $\int_0^\pi f(x)\,dx = \int_0^\pi \frac{a_0}{2}\,dx + \sum_{n=1}^\infty \int_0^\pi a_n \cos nx\,dx$. Since $\int_0^\pi a_n \cos nx\,dx = \frac{1}{n}\sin nx\Big|_0^\pi = 0$, we get $a_0 = \frac{2}{\pi}\int_0^\pi f(x)\,dx$.

In the study of, $\int_0^\pi \cos mx \cdot \cos nx\,dx = \begin{cases} 0 & \text{for } m \neq n \\ \pi/2 & \text{for } m = n \end{cases}$. Rewriting $f$ as $f(x) = \frac{a_0}{2} + \sum_{m=1}^\infty a_m \cos mx$ and multiplying by $\cos nx$, we integrate:

$$\int_0^\pi f(x)\cos nx\,dx = \frac{a_0}{2}\int_0^\pi \cos nx\,dx + \sum_{m=1}^\infty a_m \int_0^\pi \cos mx \cdot \cos nx\,dx.$$

$\int_0^\pi f(x)\cos nx\,dx = a_n\frac{\pi}{2}$, so $a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos nx\,dx$. We express $f(x)$ as $f(x) \sim \frac{a_0}{2} + \sum_{n=1}^\infty a_n \cos nx$, where $a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos nx\,dx$ for $n \in \mathbb{N}$.

**Example 264.**—Consider $f(x) = x$ on $(0,\pi)$, $f \in C_p(0,\pi)$. Then

$$a_0 = \frac{2}{\pi}\int_0^\pi x\,dx = \frac{2}{\pi}\frac{x^2}{2}\Big|_0^\pi = \pi.$$

Set $x = u$ and $\cos nx = dv$ so that

$$a_n = \frac{2}{\pi}\int_0^\pi x\cos nx\,dx = \frac{2}{\pi}x\frac{1}{n}\sin nx\Big|_0^\pi - \frac{2}{\pi}\frac{1}{n}\int_0^\pi \sin nx\,dx = \frac{2}{\pi}\frac{1}{n^2}\cos nx\Big|_0^\pi = \frac{2}{n^2\pi}\left((-1)^n - 1\right).$$

So $x \sim \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^\infty \frac{1}{n^2}((-1)^n - 1)\cos nx$, where even index terms are 0. Thus, we re-index $n = 2k-1$ with

$$x \sim \frac{\pi}{2} + \frac{2}{\pi}\sum_{k=1}^\infty \frac{-2}{(2k-1)^2}\cos(2k-1)x.$$

Suppose $f$ possesses a Fourier sine series

$$f(x) = \sum_{n=1}^\infty b_n \sin nx$$

for $0 < x < \pi$. A formula for $b_n$ can be derived similarly to the cosine series. In the homework, it is demonstrated that $\int_0^\pi \sin mx \sin nx\, dx = \begin{cases} 0 & \text{if } m \neq n \\ \pi/2 & \text{if } m = n \end{cases}$.

Express $f(x) = \sum_{m=1}^\infty b_m \sin mx$. Multiply by $\sin nx$ and integrate:

$$\int_0^\pi f(x) \sin nx\, dx = \sum_{m=1}^\infty b_m \int_0^\pi \sin mx \sin nx\, dx = b_n \frac{\pi}{2},$$

thus $b_n = \frac{2}{\pi} \int_0^\pi f(x) \sin nx dx$.

**Example 265.**—Calculate the Fourier sine series for $f(x) = x$ on $(0, \pi)$

$$b_n = \frac{2}{\pi} \int_0^\pi x \sin nx\, dx = \frac{2}{\pi} \left( x \frac{-1}{n} \cos nx \Big|_0^\pi + \frac{1}{n} \int_0^\pi \cos nx\, dx \right) = \frac{2}{\pi} \frac{-\pi}{n} (-1)^n = \frac{2}{n} (-1)^{n+1}$$

So $x \sim \sum_{n=1}^\infty \frac{(-1)^{n+1}}{n} \sin nx$.

**Note 81.**—Observe that even though $f(x)$ is only defined on $(0, \pi)$, the Fourier sine or cosine series is defined for all $x \in (-\infty, \infty)$. Thus, on $(0, \pi)$, the series is 'equal' to $f(x)$. What does the series look like outside the interval?

**Note 82.**—Note that every term in the cosine series is an even function, and every term in the sine series is odd.

**Example 266.**—$f(x) = x$ on $(0, \pi)$, $f(x) =$ cosine series. So on $(-\pi, \pi)$, the cosine series will be the even extension of $f$. It can also be extended periodically on the entire $x$-axis.

**Example 267.**—Assuming on $[0, \pi)$ that the Fourier sine series converges. Then on $(-\infty, \infty)$.

Fourier cosine + sine series on $0 < x < \pi$.

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^\infty a_n \cos nx, \qquad a_n = \frac{2}{\pi} \int_0^\pi f(s) \cos ns\, ds \tag{7.6}$$

$$f(x) \sim \sum_{n=1}^\infty b_n \sin nx, \qquad b_n \frac{2}{\pi} \int_0^\pi f(s) \sin ns\, ds \tag{7.7}$$

with even, odd, periodic extensions.

Thus, we can construct a Fourier series on $-\pi < x < \pi$ for any even or odd function.

What about other functions? Let $f \in C_p(-\pi, \pi)$. Observe $f(x) = g(x) + h(x)$, where $g(x) = \frac{f(x)+f(-x)}{2}$ and $h(x) = \frac{f(x)-f(-x)}{2}$. Then $g$ is even, and $h$ is odd. Thus, $g(x) \sim$

$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$ and $h(x) \sim \sum_{n=1}^{\infty} b_n \sin nx$. If the series converge on $0 < x < \pi$, then they also converge on $-\pi < x < \pi$. So

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{7.8}$$

on $-\pi < x < \pi$. Set $x = -s$. Observe that

$$\begin{aligned} a_n &= \frac{2}{\pi} \int_0^{\pi} g(x) \cos nx \, dx = \frac{1}{\pi} \left[ \int_0^{\pi} f(x) \cos nx \, dx + \int_0^{\pi} f(-s) \cos ns \, ds \right] \\ &= \frac{1}{\pi} \left[ \int_0^{\pi} f(x) \cos nx \, dx + \int_{-\pi}^{0} f(x) \cos nx dx \right] = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx. \end{aligned}$$

Similarly,

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx.$$

Equation (7.8) is called the Fourier series for $f$ on $-\pi < x < \pi$. If $f$ is even, it reduces to Fourier cosine series on $0 < x < \pi$. If $f$ is odd it reduces to Fourier sine series on $0 < x < \pi$.

**Recall 63.**—$\cos(A - B) = \cos A \cos B + \sin A \sin B$.

From equation (7.8),

pi f(s) $\sin ns \, ds$
$\sim \frac{1}{2\pi} \int_{-\pi}^{\pi} f(s) \, ds + \frac{1}{\pi} \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} f(s) \left[ \cos ns \cos nx + \sin ns \sin nx \right] \, ds$
$\sim \frac{1}{2\pi} \int_{-\pi}^{\pi} f(s) \, ds + \frac{1}{\pi} \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} f(s) \cos n(s - x) \, ds.$

This will be useful later in the convergence proof.

**Example 268.**—$f(x) = \begin{cases} 0 & -\pi < x < 0 \\ \pi & 0 < x < \pi \end{cases}$. Then

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, dx = \frac{1}{\pi} \int_{-\pi}^{\pi} \pi \, dx = \pi, \\ a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx = \frac{\pi}{\pi} \int_0^{\pi} \cos nx \, dx = 0, \\ b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx = -\frac{1}{n} \cos nx \Big|_0^{\pi} = \begin{cases} \frac{2}{n} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}, \\ f(x) &\sim \frac{\pi}{2} + \sum_{n=1}^{\infty} b_n \sin nx = \frac{\pi}{2} + 2 \sum_{k=1}^{\infty} \frac{1}{2k-1} \sin(2k-1)x. \end{aligned}$$

**Note 83.**—$f(x) - \frac{\pi}{2}$ is odd, only requiring terms from the Fourier sine series.

Consider more general intervals of the form $-c < x < c$. Let $f \in C_p(-c, c)$. Apply a change of variables to the interval $-\pi < s < \pi$. Define $g(s) = f(\frac{cs}{\pi})$, so $-c < \frac{cs}{\pi} < c$. Use $x = \frac{cs}{\pi}$, $s = \frac{\pi x}{s}$. $f(x) = f(\frac{cs}{\pi}) = g(s) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos ns + b_n \sin ns]$ on $-\pi < s < \pi$, where $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\frac{cs}{\pi}) \cos ns \, ds$ and $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\frac{cs}{\pi}) \sin ns \, ds$. By a change of variables,

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi x}{c} + b_n \sin \frac{n\pi x}{c}\right]$$

where $a_n = \frac{1}{\pi} \frac{\pi}{c} \int_{-c}^{c} f(x) \cos n\pi x c \, dx$ and $b_n = \frac{1}{c} \int_{-c}^{c} f(x) \sin \frac{n\pi x}{c} \, dx$.

## § 2 Convergence of Fourier Series

The objective is to establish a Fourier theorem, determining conditions on $f(x)$ that ensure the convergence of the Fourier series for $f$. Additionally, an analysis of the convergence behavior will be conducted. The theory will be developed initially for the interval $-\pi < x < \pi$ and subsequently extended to $-c < x < c$.

**Recall 64.**—$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$, provided the limit exists.

Notation for one-sided limits: $g(x_0+) = \lim_{x \to x_0^+} g(x)$ and $g(x_0-) = \lim_{x \to x_0^-} g(x)$.

**Definition.**—Suppose $f(x_0+)$ exists. Then the <u>right-hand derivative</u> of $f$ at $x_0$ is

$$f'_+(x_0) = \lim_{x \to x_0^+} \frac{f(x) - f(x_0+)}{x - x_0}, \tag{7.9}$$

provided the limit exists. Similarly, if $f(x_0-)$ exists, then the <u>left-hand derivative</u> is

$$f'_-(x_0) = \lim_{x \to x_0^-} \frac{f(x) - f(x_0-)}{x - x_0}, \tag{7.10}$$

if it exists.

**Note 84.**—$f(x_0)$ need not be defined. Some texts require $f(x_0)$ to be defined.

**Note 85.**—If the ordinary derivative exists at $x_0$, then $f$ is continuous at $x_0$, and $f'_+(x_0) = f'_-(x_0) = f'(x_0)$.

The converse is not necessarily true: $f(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$, so $f'_+(0) = 0$ and $f'_-(0) = 0$, but $f'(0)$ does not exist, and $f$ is not continuous at $x = 0$.

**Note 86.**—Usual derivative rules apply to right + left-hand derivatives.

**Definition.**—$C_p^1(a,b) = \{f \in C_p(a,b) : f' \in C_p(a,b)\} = \{f : f \text{ is piecewise smooth (PWS)}\}$.

**Theorem.** *If $f \in C_p^1(a,b)$, then at each point $x_0 \in (a,b)$, the one-sided derivatives exist, and*

$$f'_+(x_0) = f'(x_0+), \qquad\qquad f'_-(x_0) = f'(x_0-). \tag{7.11}$$

**Proof.** If $f$ is PWS, then $f$ and $f'$ are continuous on the interiors of subintervals. It is sufficient to prove this at the endpoints, assuming $f, f'$ are continuous on $(a,b)$. We will show $f'_+(a)$ exists and is equal to $f'(a+) = \lim_{x\to a^+} f'(x)$. Let $s \in (a,b)$. Since $f'$ is continuous on $(a,b)$, by the mean value theorem (MVT), $\frac{f(s)-f(a+)}{s-a} = f'(c)$ for some $c \in (a,s)$. As $s \to a^+$, then $c \to c^+$. This tells us $f'_+(a) = \lim_{s\to a+} \frac{f(s)-f(a+)}{s-a} = \lim_{s\to a+} f'(c) = \lim_{c\to a+} f'(c) = f'(a+)$. Similarly, $f'_-(b) = f'(b-)$.

**Example 269.**—$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$

**Note 87.**—$f(0+) = 0 = f(0-)$, so $f$ is continuous on $\mathbb{R}$ for all $x$.

For $x \neq 0$, $f$ is differentiable, and $f'(x) = 2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)$. One-sided limits of $f'$ do not exist at $x = 0$, so $f'(0+)$ does not exist but $f'_+(0) = \lim_{x\to 0+} \frac{f(x)-f(0+)}{x-0} = \lim_{x\to 0+} x\sin\frac{1}{x} = 0$ as with $f'_-(0) = \cdots = 0$. In other words, $f \notin C_p^1(a,b)$ if $0 \in [a,b]$.

**Recall 65.**—We have numerical evidence that Fourier coefficients satisfy $a_n \to 0$ and $b_n \to 0$.

Geometric argument: If $f(x) = c$ is constant, then $a_n = \frac{2}{\pi}c\int_0^\pi \cos nx\,dx = 0$. When $f$ is not constant, it is almost constant over small subintervals.

Let $f \in C_p(a,b)$, and $a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos nx\,dx$ for $n = 0, 1, \ldots$, with the partial sum $S_N$ defined as $S_N(x) = \frac{a_0}{2} + \sum_{n=1}^N a_n \cos nx$. Then $\int_0^\pi [f(x) - S_N(x)]^2\,dx = \int_0^\pi [f(x)]^2\,dx + \int_0^\pi [S_N(x)]^2\,dx - 2\int_0^\pi f(x)S_N(x)$

, dx.$For$J$_N$: $\int_0^\pi S_N(x)\cdot 1\,dx = \int_0^\pi \left[\frac{a_0}{2} + \sum_{n=1}^N a_n\cos nx\right]dx = \frac{a_0}{2}$ and

$$\int_0^\pi S_N(x)\cdot\cos nx\,dx = \int_0^\pi \left[\frac{a_0}{2} + \sum_{m=1}^N a_m\cos mx\right]\cos nx\,dx = \frac{\pi}{2}a_n.$$

So $J_N = \int_0^\pi [S_N(x)]^2\,dx = \int_0^\pi S_N(x)S_N(x)\,dx$ and

$$J_N = \frac{a_0}{2}\int_0^\pi S_N(x)\,dx + \sum_{n=1}^N a_n\int_0^\pi S_N(x)\cos nx\,dx = \frac{\pi}{2}\left[\frac{a_0^2}{2} + \sum_{n=1}^N a_n^2\right].$$

So $0 \leq \int_0^\pi [f(x) - S_N(x)]^2 \, dx = \int_0^\pi [f(x)]^2 \, dx - \frac{\pi}{2}\left[\frac{a_0^2}{2} + \sum_{n=1}^N a_n^2\right]$. So

$$0 \leq \frac{a_0^2}{2} + \sum_{n=1}^{N} a_n^2 \leq \frac{2}{\pi}\int_0^\pi [f(x)]^2 \, dx \tag{7.12}$$

for all $N = 1, 2, \ldots$. This is known as Bessel's inequality. The partial sums form a non-decreasing sequence bounded above; hence, it converges. Thus, the series (sequence of partial sums) converges.

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 < \infty.$$

Thus, $a_n^2 \to 0$, so $a_n \to 0$.

## § 3 Fourier Theorem

Previously discussed: one-sided derivatives, Bessel's inequality: $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n^2 \leq \frac{2}{\pi}\int_0^\pi [f(x)]^2 \, dx$, which implies $\lim_{n\to\infty} a_n = 0$. This also holds for Fourier sine coefficients $b_n$.

**Lemma Riemann-Lebesgue.**—*If $G(u)$ is piecewise continuous on $0 < u < \pi$, then* $\lim_{N\to\infty} \int_0^\pi G(u) \sin\left(\frac{u}{2} + Nu\right) du = 0.$

**Proof.** Observe $\sin\left(\frac{u}{2} + Nu\right) = \sin\frac{u}{2}\cos Nu + \cos\frac{u}{2}\sin Nu$. Hence

$$\begin{aligned}\int_0^\pi G(u)\sin\left(\frac{u}{2} + Nu\right) du &= \frac{\pi}{2}\cdot\frac{2}{\pi}\int_0^\pi \left[G(u)\sin\frac{u}{2}\right]\cos Nu \, du + \frac{\pi}{2}\cdot\frac{2}{\pi}\int_0^\pi \left[G(u)\cos\frac{u}{2}\right]\sin Nu \, du \\ &= \frac{\pi}{2}a_n + \frac{\pi}{2}b_n,\end{aligned}$$

where $a_n$ is the Fourier cosine coefficient for $G(u)\sin\frac{u}{2}$, and $b_n$ is the Fourier sine coefficient for $G(u)\cos\frac{u}{2}$. By the previous result, $a_n \to 0$ and $b_n \to 0$ as $N \to \infty$.

**Definition** (Dirichlet Kernel).—$D_N(u) = \frac{1}{2} + \sum_{n=1}^{N} \cos nu$. Observe:

(a) $D_N(u)$ is continuous, even, and has a period of $2\pi$.

(b) $\int_0^\pi D_N(u) \, du = \frac{\pi}{2}$.

(c) $D_N(u) = \dfrac{\sin\left(\frac{u}{2} + Nu\right)}{2\sin\frac{u}{2}}$, $u \neq 2n\pi$, $n = 0, \pm 1, \pm 2, \ldots$.

To see this: $\sin(A+B) = [\sin A\cos B + \cos A\sin B]$ and $\sin(A-B) = [\sin A\cos B - \cos A\sin B]$ and $2\sin A\cos B = [\sin(A+B)+\sin(A-B)]$. So,

$$\begin{aligned}
2\sin\frac{u}{2}D_N(u) &= \sin\frac{u}{2} + 2\sum_{n=1}^{N}\sin\frac{u}{2}\cos nu \\
&= \sin\frac{u}{2} + \sum_{n=1}^{N}\left[\sin\left(\frac{u}{2}+nu\right)+\sin\left(\frac{u}{2}-nu\right)\right] \\
&= \sin\frac{u}{2} + \left[\left(\sin\frac{3}{2}u - \sin\frac{u}{2}\right) + \left(\sin\frac{5}{2}u - \sin\frac{3}{2}u\right) + \cdots\right. \\
&\quad \left. + \left(\sin\frac{2N+1}{2}u - \sin\frac{2N-1}{2}u\right)\right] \\
&= \sin\left(\frac{u}{2}+Nu\right),
\end{aligned}$$

which is a telescoping sum.

**Lemma.**—*Suppose $g(u)$ is piecewise continuous on $(0,\pi)$ and $g'_+(0)$ exists. Then*

$$\lim_{N\to\infty}\int_0^\pi g(u)D_N(u)\,du = \frac{\pi}{2}g(0+)$$

.

**Proof.** Since $g(u) = g(u) - g(0+) + g(0+)$, define $I_N$ and $J_N$ by

$$\int_0^\pi g(u)D_N(u)\,du = \int_0^\pi [g(u)-g(0+)]D_N(u)\,du + \int_0^\pi g(0+)D_N(u)\,du = I_N + J_N.$$

Then $I_N = \displaystyle\int_0^\pi [g(u)-g(0+)]\frac{\sin\left(\frac{u}{2}+Nu\right)}{2\sin\frac{u}{2}}\,du = \int_0^\pi G(u)\sin\left(\frac{u}{2}+Nu\right)du$, where $G(u) = \dfrac{g(u)-g(0+)}{2\sin\frac{u}{2}}$. From the Riemann-Lebesgue lemma, we know that $\lim_{N\to\infty} = 0$ as long as $G(u)$ is piecewise continuous on $(0,\pi)$. One possible problem is at $u=0$, where the denominator is 0. $G(u)$ is piecewise continuous on $(0,\pi)$ as long as $G(0+) = \lim\limits_{u\to0+} G(u)$ exists. $G(0+) = \lim\limits_{u\to0+}\dfrac{g(u)-g(0+)}{u}\cdot\dfrac{u}{2\sin\frac{u}{2}}$. But $\lim\limits_{u\to0+}\dfrac{g(u)-g(0+)}{u}$ exists because it is $g'_+(0)$, which is assumed to exist. And $\lim\limits_{u\to0+}$

$\frac{}{u2\sin\frac{u}{2}=1}$. Also $J_N = g(0+)\int_0^\pi D_N(u)\,du = \frac{\pi}{2}g(0+)$.

Given $f\in C_p(-\pi,\pi)$, use notation $S(x) = \dfrac{a_0}{2} + \displaystyle\sum_{n=1}^{\infty}[a_n\cos nx + b_n\sin nx]$, where $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$ for $n=0,1,\ldots$ and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$. Also define the partial sum $S_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N}[a_n \cos nx + b_n \sin nx]$. Observe:

$$\begin{aligned} S_N(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx + \frac{1}{\pi}\sum_{n=1}^{N}\left[\cos nx \int_{-\pi}^{\pi} f(s)\cos ns\,ds + \sum_{n=1}^{N} f(s)\sin ns\,ds\right] \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx + \frac{1}{\pi}\sum_{n=1}^{N}\int_{-\pi}^{\pi} f(s)[\cos nx\cos ns + \sin nx \sin ns]\,ds \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx + \frac{1}{\pi}\sum_{n=1}^{N}\int_{-\pi}^{\pi} f(s)\cos n(s-x)\,ds. \end{aligned}$$

**Theorem.** *Let $f$ be piecewise continuous on $(-\pi,\pi)$ and periodic with a period of $2\pi$ on $\mathbb{R}$. The Fourier series for $f$ converges to the mean value*

$$\frac{f(x+) - f(x-)}{2} \tag{7.13}$$

*at each $x \in (-\infty,\infty)$ where both one-sided derivatives $f'_+(x)$ and $f'_-(x)$ exist.*

**Note 88.**—When $f$ is continuous at $x$, then equation (7.13) $= f(x)$, so the Fourier series converges to $f(x)$.

**Proof.** We want to show $S_N(x) \to \frac{f(x+)+f(x-)}{2}$. We will set $x \pm \pi$ because $f$, $D_N$ are periodic with $2\pi$:

$$\begin{aligned} S_N(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx + \frac{1}{\pi}\sum_{n=1}^{N}\int_{-\pi}^{\pi} f(s)\cos n(s-x)\,ds = \frac{1}{\pi}\int_{-\pi}^{\pi} f(s)\left[\frac{1}{2} + \sum_{n=1}^{N}\cos n(s-x)\right] dx \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(s)D_N(s-x)\,ds = \frac{1}{\pi}\int_{x-\pi}^{x+\pi} f(s)D_N(s-x)\,ds = \frac{1}{\pi}[I_N + J_N], \end{aligned}$$

where $I_N = \int_x^{x+\pi} f(s)D_N(s-x)\,ds$ and $J_N = \int_{x-\pi}^{x} f(s)D_N(s-x)\,ds$. Consider $I_N$: Let $u = s - x$, so $I_N = \displaystyle\int_0^{\pi} f(x+u)D_N(u)\,du = \int_0^{\pi} g(u)D_N(u)\,du$, where $g(u) = f(x+u)$. Observe $g$ is piecewise continuous on $(0,\pi)$. Also, $g'_+(0) = \displaystyle\lim_{u\to 0+}\frac{g(u)-g(0+)}{u} = \lim_{u\to 0+}\frac{f(x+u)-f(x+)}{u} = \lim_{v\to x+}\frac{f(v)-f(x+)}{v-x} = f'_+(x)$, which <u>exists</u>. We can apply the previous theorem, which says $\displaystyle\lim_{N\to\infty} I_N = \lim_{N\to\infty}\int_0^{\pi} g(u)D_N(u)\,du = \frac{\pi}{2}g(0+) = \frac{\pi}{2}f(x+)$. Similarly, for $J_N$ we get $\lim_{N\to\infty} J_N = \frac{\pi}{2}f(x-)$. $S_N(x) = \frac{1}{\pi}[I_N + J_N]$, so $S_N(x) \to \frac{1}{2}f(x+) + \frac{1}{2}f(x-)$.

<u>Last time</u>: Proved a Fourier theorem, conditions on $f(x)$ which guarantee Fourier series converges.

*Corollary.*—Let $f$ be piecewise smooth on $(-\pi, \pi)$ and let $F(x)$ be the $2\pi$-periodic extension to $(-\infty, \infty)$. At each $x \in (-\infty, \infty)$, the Fourier series (for $f(x)$ on $(-\pi, \pi)$) converges to $\frac{F(x+)+F(x-)}{2}$.

**Example 270.**—$f(x) = x$ on $(-\pi, \pi)$. $f$ is piecewise smooth on $(-\pi, \pi)$. Let $F(x)$ be the $2\pi$-periodic extension. The Fourier series for $f(x)$ is

$$S(x) = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(nx)$$

This is the Fourier series for the function $f(x) = x$ on the interval $(-\pi, \pi)$. We know that $f(x)$ is a piecewise smooth function, and we'll use the result we derived to find the Fourier series.

Since $f(x) = x$ is piecewise smooth on $(-\pi, \pi)$, the Fourier series converges to the mean value of the one-sided limits at each point where both $f'_+(x)$ and $f'_-(x)$ exist. In this case, we can find these one-sided limits:

$$f'_+(x) = 1 \quad \text{and} \quad f'_-(x) = -1$$

So, at each point where $f'_+(x)$ and $f'_-(x)$ exist, the Fourier series converges to the mean value:

$$\frac{f(x+) + f(x-)}{2} = \frac{1 + (-1)}{2} = 0$$

Therefore, the Fourier series for $f(x) = x$ converges to 0 for all $x$ where $f'_+(x)$ and $f'_-(x)$ exist.

This is a specific case where the Fourier series converges to the constant value 0, even though the function $f(x) = x$ itself is not constant. This behavior is a consequence of the piecewise smoothness conditions and the convergence properties of the Fourier series.

Let's explore further the convergence behavior of Fourier series. As a Fourier series is essentially a sequence of partial sums of functions, it is, in essence, a sequence of functions. To delve into its convergence, we initially focus on the convergence of sequences of functions.

Consider a sequence $\{g_n(x)\}_{n=1}^{\infty}$ of functions defined on the domain $E$, with $g(x)$ also being a function on $E$.

**Definition.**—We say that $\{g_n\}$ converges point-wise to $g(x)$ on $E$ if, for each fixed $x \in E$, $\lim_{n\infty} g_n(x) = g(x)$. In other words, for each $x \in E$ and any $\epsilon > 0$, there exists $N = N(\epsilon, x)$ such that if $n \geq N$, then $|g_n(x) - g(x)| < \epsilon$.

**Definition.**—We say that $\{g_n\}$ converges uniformly to $g$ on $E$ if, for any $\epsilon > 0$, there exists $N(\epsilon)$ such that if $n \geq N$, then $|g_n(x) - g(x)| < \epsilon$ for all $x \in E$.

**Note 89.**—Here, $N$ does not depend on $x$.

Consider the example: $g_n(x) = \begin{cases} 1 - nx & 0 \leq x \leq \frac{1}{n} \\ 0 & \frac{1}{n} \leq x \leq 1 \end{cases}$. $\{g_n\}$ converges point-wise to $g(x) = \begin{cases} 1 & x = 0 \\ 0 & 0 < x \leq 1 \end{cases}$ but not uniformly.

Additionally, if each $g_n(x)$ is continuous on $E$ and $g_n g(x)$ uniformly on $E$, then $g$ is continuous on $E$.

Now, let's apply these ideas to series of functions. Consider the sequence $\{f_n(x)\}_{n=1}^{\infty}$ on $a \leq x \leq b$, and define partial sums $S_N(x) = \sum_{n=1}^{N} f_n(x)$. The series $S(x) = \sum_{n=1}^{\infty} f_n(x)$ converges if the sequence $S_N(x)$ converges point-wise to $S(x)$.

**Note 90.**—Our Fourier theorem demonstrated point-wise convergence.

The series converges uniformly on $a \leq x \leq b$ if, for any $\epsilon > 0$, there exists $N$ such that if $n \geq N$, then $|\sum_{k=1}^{n} f_k(x) - S(x)| < \epsilon$.

To establish this type of convergence, it is often beneficial to utilize the Weierstrass M-test: If there is a convergent series (of positive real numbers) $\sum_{n=1}^{\infty} M_n < \infty$ such that $|f_n(x)| \leq M_n$ for each $n$ and all $x \in [a, b]$, then $\sum_{n=1}^{\infty} f_n(x)$ is uniformly convergent on $[a, b]$.

**Theorem.** *Let $f$ be a function satisfying:*

1. *$f$ is continuous on $-\pi \leq x \leq \pi$*
2. *$f(-\pi) = f(\pi)$*
3. *$f'$ is piecewise continuous on $-\pi < x < \pi$.*

*The Fourier series $\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$, where $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx \, dx$ and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, dx$, converges uniformly to $f(x)$ on $-\pi \leq x \leq \pi$.*

**Proof.** Observe that $|a_n \cos nx + b_n \sin nx| \leq |a_n \cos nx| + |b_n \sin nx| \leq |a_n| + |b_n| \leq 2\sqrt{a_n^2 + b_n^2}$. The result follows from the Weierstrass M-test once we verify that $\sum_{n=1}^{\infty} 2\sqrt{a_n^2 + b_n^2} < \infty$ (next).

**Definition** (Cauchy-Schwarz inequality).—Let $= p_1 \cdots p_N$ and $= q_1 \cdots q_N$ be vectors, then $|\langle p, q\rangle| \leq ||p|| \cdot ||q||$ on $R^N$ with Euclidean norm i.e.,

$$\left|\sum_{n=1}^{N} p_n q_n\right| \leq \sqrt{\sum_{n=1}^{N} p_n^2}\sqrt{\sum_{n=1}^{N} q_n^2}, \qquad \left|\sum_{n=1}^{N} p_n q_n\right|^2 \leq \sum_{n=1}^{N} p_n^2 \sum_{n=1}^{N} q_n^2.$$

**Lemma.**—*Assume the hypotheses on $f$ from the previous theorem. Then $\sum_{n=1}^{\infty} \sqrt{a_n^2 + b_n^2} < \infty$.*

**Proof.** Since $f'$ is piecewise continuous, its Fourier series exists with coefficients $\alpha_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)\cos nx$ and $\beta_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)\sin nx\,dx$. Observe: $\alpha_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)\,dx = \frac{1}{\pi}[f(\pi) - f(-\pi)] = 0$.

For $n = 1, 2, \ldots$

$$\int_{-\pi}^{\pi} f'(x)\cos nx\,dx = \frac{1}{\pi}\left[f(x)\cos nx\Big|_0^{\pi} + n\int_{-\pi}^{\pi} f(x)\sin nx\,dx\right] = nb_n,$$
$$\beta_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)\sin nx\,dx = \frac{1}{\pi}\left[f(x)\sin nx\Big|_{-\pi}^{\pi} - n\int_{-\pi}^{\pi} f'(x)\cos nx\,dx\right] = -na_n.$$

We have $T_N = \sum_{n=1}^{N}\sqrt{a_n^2 + b_n^2} = \sum_{n=1}^{N}\frac{1}{n}\sqrt{\alpha_n^2 + \beta_n^2}$. Then

$$T_N^2 = \left(\sum_{n=1}^{N}\frac{1}{n}\sqrt{\alpha_n^2 + \beta_n^2}\right)^2 \le \left(\sum_{n=1}^{N}\frac{1}{n^2}\right)\left(\sum_{n=1}^{N}(\alpha_n^2 + \beta_n^2)\right).$$

This converges to $\zeta(2)$. By Bessel's inequality, $\sum_{n=1}^{N}(\alpha_n^2 + \beta_n^2) \le \frac{1}{\pi}\int_{-\pi}^{\pi}[f'(x)]^2\,dx$.

Thus, $T_N^2$ are increasing and bounded above, so convergent. Thus, $T_N$ is convergent. So $\sum_{n=1}^{\infty}\sqrt{a_n^2 + b_n^2} < \infty$.

**Theorem.** *Consider a function $f$ that satisfies the following conditions on the interval $-\pi \le x \le \pi$:*

*a. $f$ is continuous.*

*b. $f(-\pi) = f(\pi)$.*

*c. $f'$ is piecewise continuous on $-\pi < x < \pi$.*

*Then, the Fourier series representation of $f(x)$ as $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n\cos nx + b_n\sin nx]$, where $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$ and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$, is differentiable at each $x$ in $-\pi < x < \pi$ where $f''(x)$ exists. In this case, $f'(x) = \sum_{n=1}^{\infty}[-na_n\sin nx + nb_n\cos nx]$.*

**Proof.** Fix $x$ such that $f''(x)$ exists, implying $f'$ is continuous at $x$. Applying the Fourier Theorem to $f'$ yields $f'(x) = \frac{\alpha_0}{2} + \sum_{n=1}^{\infty}[\alpha_n\cos ns + \beta_n\sin nx]$, where $\alpha_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)\cos nx\,dx$ and $\beta_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)\sin nx\,dx$. As shown earlier, $\alpha_0 = 0$, $\alpha_n = nb_n$, $\beta_n = -na_n$, for $n = 1, 2, \ldots$. The result follows.

**Example 271.**—Consider $f(x) = x$ on $0 < x < \pi$. The sine series $S(x) = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx$ does not converge for any $x$. On the other hand, the cosine series $C(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2}$ does converge. Note that the odd extension of $f(x) = x$ to $(-\pi, \pi)$ is different from the even extension $\tilde{f}$, leading to different Fourier series representations, $S(x)$ and $C(x)$, respectively.

**Example 272.**—For $f(x) = x^2$ on $-\pi < x < \pi$, the Fourier series coefficients are $A_n = \frac{4}{n^2}(-1)^n$. The Fourier series representation is $x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx$.

**Theorem.** *Let $f$ be piecewise continuous on $-\pi < x < \pi$ with Fourier series $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos nx + b_n \sin nx]$. The term-by-term integration is valid, meaning that for any $x \in [-\pi, \pi]$,*

$$\int_{-\pi}^{\pi} f(s)\,ds = \frac{a_0}{2}(x+\pi) + \sum_{n=1}^{\infty} \frac{1}{n}\left[a_n \sin nx - b_n\left(\cos nx + (-1)^{n+1}\right)\right].$$

*Note that the resulting series may not be a Fourier series.*

**Theorem.** *Assume $f$ is piecewise continuous on $-\pi < x < \pi$ with the Fourier series $f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$. Then term-by-term integration is valid. In other words, for any $x \in [-\pi, \pi]$,*

$$\int_{-\pi}^{\pi} f(s)\,ds = \frac{a_0}{2}(x+\pi) + \sum_{n=1}^{\infty} \frac{1}{n}\left[a_n \sin nx + b_n\left(\cos nx + (-1)^{n+1}\right)\right].$$

**Proof.** Define $F(x) = \int_{-\pi}^{x} f(s)\,ds - \frac{a_0}{2}x$. $F(x)$ is continuous on $-\pi \le x \le \pi$. If $f$ is continuous at $x$, then $F$ is differentiable at $x$ and $F'(x) = f(x) - \frac{a_0}{2}$. Thus, $F$ is piecewise smooth on $(-\pi, \pi)$ since $f$ is piecewise continuous there.

Also, $F(-\pi) = \frac{a_0\pi}{2}$ and $F(\pi) = \int_{-\pi}^{\pi} f(s)\,ds - \frac{a_0\pi}{2} = \pi a_0 - \frac{\pi a_0}{2} = \frac{\pi a_0}{2}$. So $F(-\pi) = F(\pi)$. We can apply our Fourier theorem to $F(x)$.

$$F(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty}(A_n \cos nx + B_n \sin nx) \tag{7.14}$$

on $-\pi \le x \le \pi$, where $A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\cos nx\,dx$ and $B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\sin nx\,dx$. For $n = 1, 2, \ldots$, set $u = F(x)$ and $dv = \cos nx\,dx$ so that

$$\begin{aligned}
A_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\cos nx\,dx \\
&= \frac{1}{\pi}F(x)\left(\frac{1}{n}\right)\sin nx\Big|_{-\pi}^{\pi} - \frac{1}{n\pi}\int_{-\pi}^{\pi}\left[f(x) - \frac{a_0}{2}\right]\sin nx\,dx \\
&= -\frac{1}{n\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = -\frac{b_n}{n} \\
B_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\sin nx\,dx \\
&= \frac{1}{\pi}\left[-F(x)\frac{1}{n}\cos nx\Big|_{-\pi}^{\pi} + \frac{1}{n}\int_{-\pi}^{\pi}\left(f(x) - \frac{a_0}{2}\right)\cos nx\,dx\right] \\
&= \frac{1}{n}a_n \\
A_0 &\colon \frac{\pi a_0}{2} = F(\pi) = \frac{A_0}{2} + \sum_{n=1}^{\infty}[A_n \cos \pi n + B_n \sin \pi n] \\
&\implies \frac{A_0}{2} = \frac{\pi a_0}{2} + \sum_{n=1}^{\infty} A_n(-1)^n
\end{aligned}$$

Plug in $A_n, B_n$ to $F(x)$, check.

## § 4 Derivation of Heat Equation

There are three types of models for heat transfer:

1. Conduction: Due to molecular activity.
   - Energy transfer from more active to less active particles.
   - Fourier's law.
2. Convection: Due to bulk transfer/motion of mass.
   - Newton's law of cooling.
3. Radiation: Electromagnetic waves, e.g., sun heating Earth.

We consider heat transfer in a solid body, where conduction is the appropriate model. Let $u(x, y, z, t)$ be the temperature at location $(x, y, z)$ and time $t$.

**Definition.**—Fourier's Law: The magnitude of flux is proportional to the magnitude of the directional derivative $\frac{du}{d}$. In other words, $\Phi(x, y, z, t) = -K\frac{du}{dn}$, for $K > 0$, where $K$ is the thermal conductivity.

**Definition.**—Let $\sigma$ be the specific heat, representing the energy required to raise the temperature of one unit of mass by one degree. Let $\delta$ be the mass density. In general, $K$, $\sigma$, $\delta$ may not be constant and may depend on $x, y, z$ or even $t$ or $u$. We usually assume they are constant for simplicity.

**Definition** (One-dimensional heat equation)**.**—Assume:

- The solid is a circular cylinder with a constant cross-sectional area $A$ in the $yz$-plane.
- Heat flows only parallel to the $x$-axis. Thus, $\Phi = \Phi(x, t)$ and $u = u(x, t)$.
- $K$, $\sigma$, $\delta$, $A$ are constant.
- Temperature is constant over a cross-section.
- Perfect insulation, so no heat escapes through the side of the cylinder.
- No heat is generated or

  lost inside the cylinder (no sources or sinks).

We derive a model by considering conservation of thermal energy in a small segment of width $\Delta x$. WLOG (without loss of generality), assume thermal energy flows from left to right. The conservation law states that the net rate of heat accumulation is equal to the rate of heat entering minus the rate of heat leaving. From the definition of specific heat, the rate of heat change per unit time is approximately $\sigma \cdot m \cdot u_t(x^*, t)$ on $x < x^* < x + \Delta x$, where $m = \delta A \Delta x$. So, $\sigma \delta A \Delta x u_t(x^*, t) \approx K \cdot A[u_x(x + \Delta x, t) - u_x(x, t)]$. Taking the limit as $\Delta x \to 0$, we get $u_t(x, t) = k u_{xx}(x, t)$, where $k = \frac{K}{\sigma\delta} > 0$ is the thermal diffusivity.

**Note 91.**—In this 1-$d$ model, thermal energy can only enter or leave through the boundaries at the left and right ends. The full mathematical model consists of:

1. The partial differential equation $u_t = k u_{xx}$ on $0 \leq x \leq c$.

2. Initial temperature distribution (IC): $u(x, 0) = f(x)$ on $0 < x < c$.

3. Two boundary conditions at $x = 0$ and $x = c$. For example, Dirichlet conditions: $u(0, t) = 0$ and $u(c, t) = 0$, or Neumann conditions: $u_x(0, t) = 0$ and $u_x(c, t) = 0$.

4. Third condition: $\Phi|_{x=0,c} = H(T - u|_{x=0,c})$.

**Example 273.**—Let $u(x)$ denote the steady-state temperature in a cylinder whose faces at $x = 0$ and $x = c$ are kept at constant temperatures $u = 0$ and $u = u_0 > 0$. Since it's steady state, $u_t = 0$, and we have $k u_{xx}(x, t) = 0$. This leads to $u(x, t) = u(x)$, and $u''(x) = 0$ with boundary conditions $u(0) = 0$ and $u(c) = u_0$. The solution is $u(x) = \frac{u_0}{c} x$.

## § 5 Model of a Vibrating Elastic String

**Definition.**—Consider a tightly stretched elastic string. Assumptions:

- Motion consists only of vertical displacements, which are small. $y(x, t) =$ vertical displacement at time $t$ of the point whose equilibrium position is $(x, 0)$ in the $xy$-plane.

- The string is perfectly flexible, so there is an elastic restoring force, but no resistance to bending.

- Horizontal component of tensile force is constant, $H > 0$.

$V(x, t) =$ vertical component of tensile force, exerted by the left part of the string on the right part of the string at $(x, y)$. At $(x, y)$, if sloped down, then $V$ is negative (downward) and $y_x(x, t)$ is positive. Then $y_x(x, t) = \frac{-V(x,t)}{H}$. On the other hand, if at $(x, y)$ the slope is upward, then $V$ is positive (upward) and $y_x(x, t)$ is negative. Again, $y_x(x, t) = \frac{-V(x,t)}{H}$. So $V(x, t) = -H y_x(x, t)$ and $V(x + \Delta x, t) = H y_x(x + \Delta x, t)$. Apply Newton's law $F = ma$. $\delta =$ density $=$ mass per unit length. So $\delta \Delta x =$ mass of a piece of the string. Then $\delta \Delta x y_{tt}(\bar{x}, t) = H y_x(x + \Delta x, t) - H y_x(x, t)$. Then the one-dimensional wave equation can be derived as $y_{tt}(x, t) = a^2 y_{xx}(x, t)$, where $a^2 = \frac{H}{\delta}$. Can apply separation of variables to the wave equation, just like we did previously for the heat equation.

**Example 274.**—At $x = 0$, flux into the cylinder is constant $\Phi_0$. At $x = c$, the temperature is held constant to value 0. Steady state means $u_t(x,t) = 0$. So $u(x,t) = u(x)$, and the PDE becomes ODE $u''(x) = 0$. The boundary conditions are: At $x = c$: $u(c) = 0$. At $x = 0$: $\Phi_0 = -Ku'(0)$. The general solution is $u(x) = Ax + B$.

**Example 275.**—$f(x) = \begin{cases} 0 & -\pi \le x \le 0 \\ \sin x & 0 \le x \le \pi \end{cases}$ continuous on $[-\pi,\pi]$, PWS, $f(-\pi) = 0 = f(\pi)$. By Theorem in §17, we get uniform convergence of the Fourier series $S(x) = \frac{1}{\pi} + \frac{1}{2}\sin x - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos 2nx}{4n^2-1}$. Observe $\left|\frac{\cos 2nx}{4n^2-1}\right| \le \frac{1}{4n^2-1} \le \frac{1}{3n^2} = M_n$ and $\sum M_n < \infty$. Theorem in §19 implies $S(x)$ is differentiable at all $x$ except $x = 0$ in $(-\pi,\pi)$ because $f''(x)$ exists except at $x = 0$. Graph $S'(x)$. Use the fact that $S'(x)$ is the Fourier series for $f'(x)$. $f'(x) = \begin{cases} 0 & -\pi < x < 0 \\ \cos x & 0 < x < \pi \end{cases}$.

**Example 276.**—$f(x) = x \sim S(x) =$ Fourier cosine series on $(0,\pi)$. $S(x)$ is the Fourier series for $g(x) = |x|$ on $(-\pi,\pi)$. $g$ is continuous on $[-\pi,\pi]$, satisfies $g(-\pi) = g(\pi)$, and $g$ is PWS. By Theorem, $S'(x)$ is Fourier series for $f'(x)$ and converges for all $x \ne 0$, so all $x \in (0,\pi)$ since $f''(x)$ exists for all $x$ except $x = 0$.

**Example 277.**—Theorem: Let $f$ be a function satisfying

i) $f$ is continuous on $0 \le x \le \pi$

ii) $f(0) = 0 = f(\pi)$

iii) $f'$ is PWC on $0 < x < \pi$.

Then the Fourier sine series for $f$ is differentiable at each point where $f''(x)$ exists.

**Example 278.**—$f(x) = x \sim S(x) =$ Fourier sine series on $(0,\pi)$. $S(x)$ is the Fourier series for the odd extension to $(-\pi,\pi)$, which is continuous on $(-\pi,\pi)$. The theorem says $S(x)$ converges to the mean value of the $2\pi$-periodic extension of $g(x)$.

**Example 279.**—$f(x) = \begin{cases} 0 \\ \sin x \end{cases}$

# § 6 The Fourier Method

(Previously did §36) Some ideas about linear combinations.

**Definition** (Function space)**.**—A vector space whose elements are functions.

**Example 280.**—$C_p(a,b)$, $C(a,b)$, $L^2(a,b)$.

**Definition** (Linear Operator)**.**—A linear operator $L$ on a function space $V$ is a map from $V$ to $V$ with the properties: For all $u_1, u_2 \in V$ and scalars $c_1, c_2$, we have $L(c_1u_1 + c_2u_2) = c_1L(u_1) + c_2L(u_2)$.

**Example 281.**—Differential operator: $L(u) = \frac{du}{dx}$.

**Example 282.**—Multiplication operator: $L(u) = f(x) \cdot u(x)$.

**Definition** (Superposition Principle).—If $L$ is a linear operator and $L(u_1) = 0$ and $L(u_2) = 0$, then $L(c_1, u_1 + c_2u_2) = 0$. This extends to any finite linear combination $L(c_1u_1 + c_2u_2 + \cdots + c_nu_n) = 0$.

**Example 283.**—$L(u) = u_t - ku_{xx} = 0$, with the function space defined by the BVP. Under suitable assumptions, we can extend to infinite linear combinations, which we need for the Fourier method (separation of variables). We did one example (§36) for a heat equation.

**Example 284** (A wave equation example).—$y_{tt}(x,t) = a^2y_{xx}(x,t)$ on $0 < x < c$ and $0 < t$ with BC $y(0,t) = 0 = y(c,t)$. IC is $y(x,0) = f(x)$ and $y_t(x,0) = 0$. Assume the solution has the form $y(x,t) = X(x)T(t)$. Plug in: $\frac{T''(t)}{a^2T(t)} = \frac{X''(x)}{X(x)} = -\lambda \implies X''(x) + \lambda X(x) = 0$, $X'(0) = 0$, $X'(c) = 0$. Also $T''(t) + \lambda a^2T(t) = 0$ and $T'(t) = 0$.

$\lambda = 0$: $X''(x) = 0 \implies X(x) = Ax + B$ but BC imply $X(x) = 0$.

$\lambda < 0$: $\lambda = -\alpha^2$, $X''(x) - \alpha^2X(x) = 0$. Then $X(x) = c_1e^{\alpha x} + c_2e^{-\alpha x}$. At $X(0) = 0$: $c_1 + c_2 = 0$, so $c_1 = -c_2$. At $X(c) = 0$: $c_1e^{\alpha c} - c_1e^{-\alpha c} = 0 = c_1\left[e^{\alpha c} - e^{-\alpha c}\right]$, which implies $c_1 = 0$, so $c_2 = 0$.

$\lambda > 0$: $\lambda = \alpha^2$, $X''(x) + \alpha^2X(x) = 0$ and $X(x) = c_1\cos\alpha x + c_2\sin\alpha x$. At $X(0) = 0$: $c_1 + 0 = 0$, so $c_1 = 0$. So $X(x) = c_2\sin\alpha x$. At $X(c) = 0$: $\sin(\alpha c) = 0$, so $\alpha = \pm\frac{n\pi}{c}$, $n = 1, 2, \ldots$. So $\lambda_n = \frac{n^2\pi^2}{c^2}$, $n = 1, 2, \ldots$ with corresponding solutions (eigenfunctions) $X_n(x) = \sin\frac{n\pi}{c}x$. Use $\lambda_n$ to get corresponding $T_n(t)$: $T''(t) + \frac{n^2\pi^2a^2}{c^2}T(t) = 0$, $T_n(t) = c_1\cos\frac{n\pi a}{c}t + c_2\sin\frac{n\pi a}{c}t$. So $T_n(t) = \cos\frac{n\pi a}{c}t$.

So $y_n(x,t) = \sin\frac{n\pi}{c}x\cos\frac{n\pi a}{c}t$ for $n = 1, 2, \ldots$ is a solution of PDE and BC and $y_t(x,0) = 0$. Thus the series (Infinite linear combination) also satisfies these. $y(x,t) = \sum_{n=1}^{\infty} B_n\sin\frac{n\pi}{c}x\cos\frac{n\pi a}{c}t$ satisfies PDE, BC, and $y_t(x,0) = 0$. Can we pick $B_n$ so $y(x,t)$ also satisfies $y(x,0) = f(x)$? That is, $f(x) \stackrel{?}{=} \sum_{n=1}^{\infty} B_n\sin\frac{n\pi}{c}x$? Yes, if $f(x)$ has a Fourier series.

Previously, we used the Fourier method (separation of variables) to solve two different problems:

1. Heat Equation with insulated BC. PDE: $u_t(x,t) = ku_{xx}(x,t)$, BC: $u_x(0,t) = 0 = u_x(c,t)$, IC: $u(x,0) = f(x)$. This leads to the BVP $X''(x) + \lambda X(x) = 0$, $X'(0) = 0$, $X'(c) = 0$. Eigenvalues: $\lambda_0 = 0$, $\lambda_n = \frac{n^2\pi^2}{c^2}$, $n = 1, 2, \ldots$. Eigenfunctions: $X_0(x) = 1$, $X_n(x) = \cos\frac{n\pi}{c}x$.

2. Wave equation. PDE: $y_{tt}(x,t) = a^2y_{xx}(x,t)$. BC: $y(0,t) = 0 = y(c,t)$, IC: $y(x,0) = f(x)$, $y_t(x,0) = 0$. This leads to the BVP: $X''(x) + \lambda X(x) = 0$, $X'(0) = 0$, $X'(c) = 0$. Eigenvalues: $\lambda_0 = 0$, $\lambda_n = \frac{n^2\pi^2}{c^2}$, $n = 1, 2, \ldots$. Eigenfunctions: $X_0(x) = 1$, $X_n(x) = \cos\frac{n\pi}{c}x$.

3. Wave equation. PDE: $y_{tt}(x,t) = a^2 y_{xx}(x,t)$. BC: $y(0,t) = 0 = y(c,t)$, IC: $y(x,0) = f(x)$, $y_t(x,0) = 0$. This leads to the BVP: $X''(x) + \lambda X(x) = 0$, $X(0) = 0$, $X(c) = 0$. Eigenvalues $\lambda_n = \frac{n^2\pi^2}{c^2}$ for $n = 1, 2, \ldots$. Eigenfunctions: $X_n(x) = \sin \frac{n\pi}{c} x$. We will study other BC's and BVP's later (Sturm-Liouville).

**Example 285.**—Heat Equation with zero temperature BC $u_t(x,t) = ku_{xx}(x,t)$, $u(0,t) = 0 = u(\pi,t)$, $u(x,0) = f(x)$. Apply Fourier Method, $X''(x)+\lambda X(x) = 0$, $X(0) = 0$, $X(\pi) = 0$, and $T'(t) + \lambda k T(t) = 0$. So $\lambda_n = n^2$, $X_n(x) = \sin(nx)$. $T_n(t) = e^{-n^2kt}$. So $u(x,t) = \sum_{n=1}^{\infty} B_n \sin(nx) e^{-n^2kt}$. From IC: $f(x) = \sum_{n=1}^{\infty} B_n \sin(nx)$. So $B_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx$.

**Example 286.**—Heat equation with nonzero temperature BC: $u_t(x,t) = ku_{xx}(x,t)$, $u(0,t) = 0$, $u(\pi,t) = u_0$, $u(x,0) = 0$. Note: BC is nonhomogeneous, so Fourier method doesn't apply directly (because Superposition Principle only applies to linear, homogeneous problems). Look for solution of form $u(x,t) = U(x,t) + \Phi(x)$ Plug in: $U_t(x,t) = u_t(x,t) = ku_{xx}(x,t) = kU_{xx}(x,t) + k\Phi''(x)$. At boundary conditions: $0 = u(0,t) = U(0,t) + \Phi(0)$ and $u_0 = u(\pi,t) = U(\pi,t) + \Phi(\pi)$. Suppose we select $\Phi(x)$ to satisfy $\Phi''(x) = 0$, $\Phi(0) = 0$, $\Phi(\pi) = u_0$. If such a $\Phi(x)$ is used, then $U(x,t)$ must satisfy $U_t(x,t) = kU_{xx}(x,t)$, $U(0,t) = 0$, $U(\pi,t) = 0$, $U(x,0) = -\Phi(x)$. We just solved this for $U(x,t)$. $\Phi''(x) = 0$ implies $\Phi(x) = Ax + B$ or $\Phi(x) = \frac{u_0}{\pi}x$. So $U(x,t) = \sum_{n=1}^{\infty} B_n \sin(nx) e^{-n^2kt}$, where $\sum_{n=1}^{\infty} B_n \sin(nx) = -\frac{u_0}{\pi}x$. We know $x = \sum_{n=1}^{\infty} 2\frac{(-1)^{n+1}}{n}\sin nx$, so $B_n = \frac{-u_0}{\pi}\cdot\frac{2(-1)^{n+1}}{n} = \frac{2u_0}{\pi n}(-1)^n$. So

$$u(x,t) = \frac{u_0}{\pi}x + \frac{2u_0}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin(nx)e^{-kn^2t}.$$

**Example 287.**—Two numerical examples:

1. $u_t = ku_{xx}$, $u_x(x,t) = 0 = u_x(1,t)$, $u(x,0) = x$.
   Solution $u(x,t) = \frac{1}{2} - \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}e^{-(2n-1)^2\pi^2kt}\cos(2n-1)\pi x$

2. $u_t = ku_{xx}$, $u(0,t) = 0 = u(\pi,t)$, $u(x,0) = x(\pi - x)$.
   Solution $u(x,t) = \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}e^{-(2n-1)^2kt}\sin(2n-1)x$.

# § 7 Generalized Vector Spaces

We aim to extend the concept of vectors in three-dimensional space to more general vector spaces, where the elements are functions. In three-dimensional space, vectors are defined as directed line segments. We can generalize various geometric aspects:

- Calculate the length of a vector.
- Determine the angle between vectors.
- Perform vector addition using the parallelogram method.

- Multiply a vector by a real number $a$: obtain a new vector in the same direction if $a > 0$, in the opposite direction if $a < 0$, and with a length equal to the original length multiplied by $|a|$.
- Given a vector and a plane in three-dimensional space, define the projection of the vector onto the plane. This involves moving to an equivalent vector with the tail in the plane and dropping a perpendicular line from the head to the plane. Connect these two points in the plane.

  **Note 92.**—The length of the projection is always less than or equal to the length of the original vector, being equal only if the original vector lies in the plane.

All these geometric properties and definitions can be expressed algebraically, which is crucial for extending them to higher dimensions and more general vector spaces.

**Definition.**—For a "geometric" vector, consider an equivalent vector with the tail at the origin $(0, 0, 0)$. Define an "algebraic" vector by assigning coordinates to the point at the head: $x = (x_1, x_2, x_3)$.

- Define addition of two vectors and multiplication by a real number $a$ component-wise: $x + y = (x_1 + y_1, x_2 + y_2, x_3 + y_3)$, $ax = (ax_1, ax_2, ax_3)$. This definition aligns with the geometric interpretation.
- Define algebraic length using the Euclidean norm: $||x|| = \sqrt{x_1^2 + x_2^2 + x_3^2}$. This is consistent with the geometric definition (Pythagorean Theorem).
- Define the dot product: $x \cdot y = x_1y_1 + x_2y_2 + x_3y_3$. This has important geometric interpretations. If $\theta$ is the "geometric" angle between vectors $x$ and $y$, then $\cos\theta = \frac{x \cdot y}{||x|| \cdot ||y||}$ (Law of cosines). Two vectors are orthogonal if and only if $x \cdot y = 0$.

  **Note 93.**—$||x|| = \sqrt{x \cdot x}$.

Recall the Euclidean basis vectors $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, and $e_3 = (0, 0, 1)$. Observe the following properties: $||e_i|| = 1$ for $i = 1, 2, 3$, and $e_i \cdot e_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$. Let $x = (x_1, x_2, x_3)$. Then $x = \displaystyle\sum_{i=1}^{3} x_i e_i = \sum_{i=1}^{3} (x \cdot e_i)e_i$. Also, $||x||^2 = \sum_{i=1}^{3} |x_i|^2 = \sum_{i=1}^{3} (x \cdot e_i)^2$. Additionally, if $y \cdot e_i = 0$ for $i = 1, 2, 3$, then $y = (0, 0, 0)$. Thus, $\{e_1, e_2, e_3\}$ forms an orthonormal basis of $\mathbb{R}^3$.

**Example 288.**—This concept extends to any orthonormal basis in any vector space. Consider in $\mathbb{R}^3$: $u_1 = \frac{1}{\sqrt{6}}(1, 2, 1)$, $u_2 = \frac{1}{\sqrt{21}}(2, 1, -4)$, $u_3 = \frac{1}{\sqrt{4}}(3, -2, 1)$. It can be verified that $\{u_1, u_2, u_3\}$ forms an orthonormal set. Define $W = \text{span}\{u_1, u_2\}$. For $x \in \mathbb{R}^3$, define $\hat{x} = \sum_{i=1}^{2} (x \cdot u_i)u_i = (x \cdot u_1)u_1 + (x \cdot u_2)u_2$. Then $\hat{x}$ is the orthogonal projection of $x$ onto $W$. In fact, $x = \sum_{i=1}^{3} (x \cdot e_i)e_i$.

**Example 289.**—Recall our earlier discussion on the partial differential equation $u_t(x,t) = ku_{xx}(x,t)$, $u_x(0,t) = 0$, $u_x(c,t) = 0$, and $u(x,0) = f(x)$. Applying the Fourier Method and seeking solutions of the form $u(x,t) = X(x)T(t)$, we plugged it into the PDE and derived solutions involving Fourier series.

**Example 290.**—Consider a partial differential equation in two variables: $u_{xx}(x,y) + u_{yy}(x,y) = 0$ on $0 < x < \pi$ and $0 < y < 2$, with specific boundary conditions. This problem involves finding solutions of the form $u(x,t) = X(x)Y(y)$. Further details on this example are provided separately.

In our previous discussion, we explored the geometry of the dot product in $\mathbb{R}^3$ and aimed to extend these concepts to general function spaces.

**Recall 66.**—We defined $C_p(a,b) = \{f : f \text{ is piecewise continuous on } [a,b]\}$, forming a vector space.

**Definition.**—We introduced an inner product on $C_p(a,b)$ as $(f,g) = \int_a^b f(x)\overline{g(x)}\,dx$.

**Recall 67.**—In $\mathbb{R}^3$, if $u = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$, then $u \cdot u = \sum a_i^2 = ||u||^2$.

Hence, $C_p(a,b)$ is an inner product space (infinite dimensional). The associated norm is $||f|| = \sqrt{(f,f)} = \left(\int_a^b [f(x)]^2\,dx\right)^{1/2}$.

**Definition.**—Two functions $f$ and $g$ are <u>orthogonal</u> if $(f,g) = 0$, i.e., $\int_a^b f(x)\overline{g(x)}\,dx = 0$.

A set of functions $\{\psi_n\}_{n=1}^\infty$ is <u>orthogonal</u> if $(\psi_m, \psi_n) = 0$ when $m \neq n$. If $\{\psi_n\}_{n=1}^\infty$ is an orthogonal set, we can define $\phi_n = \frac{1}{||\psi_n||}\psi_n$, resulting in an orthonormal set. This implies $(\phi_n, \phi_m) = \delta_{mn}$, where $\delta_{mn} = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$.

**Example 291.**—Consider $\psi_n(x) = \sin nx$ for $n = 1, 2, \ldots$ on $0 \leq x \leq \pi$. Then $(\psi_m, \psi_n) = \int_0^\pi \sin mx \sin nx\,dx = \begin{cases} 0 & \text{if } m \neq n \\ \frac{\pi}{2} & \text{if } m = n \end{cases}$. Therefore, $\{\psi_n\}_1^\infty$ is an orthogonal set. We can define $\phi_n(x) = \sqrt{\frac{2}{\pi}} \sin nx$, resulting in an orthonormal set.

**Example 292.**—Consider $\phi_n = \sqrt{\frac{2}{\pi}} \cos nx$ for $n = 1, 2, \ldots$ and $\phi_0(x) = \frac{1}{\sqrt{\pi}}$. Then $\{\phi_n\}_{n=0}^\infty$ forms an orthonormal set.

**Example 293.**—Define $\phi_0(x) = \frac{1}{\sqrt{2\pi}}$, $\phi_{2n-1}(x) = \frac{1}{\sqrt{\pi}} \cos nx$, and $\phi_{2n}(x) = \frac{1}{\sqrt{\pi}} \sin nx$ for $n = 1, 2, \ldots$. Thus, $\{\phi_n\}_{n=0}^\infty$ is an orthonormal set.

**Example 294.**—Consider the set $\{1, x, x^2, \ldots\}$ and use the Gram-Schmidt process to obtain an orthonormal set.

Given an orthonormal set $\{\phi_n\}_{n=1}^\infty$ in $C_p(a,b)$, can every $f \in C_p(a,b)$ be represented as an infinite series $\sum_{n=1}^\infty C_n\phi_n(x)$? We denote this as $f(x) \sim \sum_{n=1}^\infty C_n\phi_n(x)$ if they are equal for all but a finite number of $x$ values on $(a,b)$.

Suppose the answer is yes. Then $f(x) = \sum_{m=1}^\infty C_m\phi_m(x)$. Consequently,

$$\begin{aligned}(f(x), \phi_n(x)) &= \left(\sum_{m=1}^{\infty} C_m \phi_m(x), \phi_n(x)\right)\\ &= \sum_{m=1}^{\infty} C_m(\phi_m(x), \phi_n(x))\\ &= \sum_{m=1}^{\infty} C_m \delta_{mn}\\ &= C_n.\end{aligned}$$

Hence, $C_n = (f, \phi_n)$.

In the following, we explore the notion of convergence in mean and completeness for orthonormal sets.

Given an orthonormal set $\{\phi_n(x)\}_{n=1}^{\infty}$ in $C_p(a,b)$, is it possible to express any function $f$ (in $C_p(a,b)$ or in a subspace $W$ of $C_p(a,b)$) as $\sum_{n=1}^{\infty} C_n\phi_n(x)$? If yes, then $C_n = (f(x), \phi_n(x)) = \int_a^b f(x)\phi_n(x)\,dx$. The constants $C_n$ are referred to as Fourier constants or coefficients.

**Definition.**—A sequence of functions $\{S_N(x)\}_{N=1}^{\infty}$ in $C_p(a,b)$ is said to converge in the mean to the function $f(x)$ in $C_p(a,b)$ if $E_N = ||f - S_N||^2 = \int_a^b |f(x) - S_N(x)|^2\,dx$ satisfies $\lim\limits_{N\infty} E_n = 0$, or equivalently $\lim\limits_{N\infty} ||f - S_N|| = 0$.

**Note 94.**—This convergence is not the same as pointwise or uniform convergence.

**Definition.**—An orthonormal set $\{\phi_n\}_{n=1}^{\infty}$ is complete in a subspace $W$ of $C_p(a,b)$ if, for every $f \in W$, the partial sums of the generalized Fourier series $S_N(x) = \sum_{n=1}^{N} C_n\phi_n(x)$, $C_n = (f, \phi_n)$ converge in the mean to $f$. $||f - S_n||0$ as $N\infty$.

**Remark 36.**—Consider $C[a,b]$ as a vector space and a subspace of $L_2(a,b)$. Define $||f||_\infty = \max\limits_{a\le x\le b}|f(x)|$. Then $C[a,b]$ with $||\cdot||_\infty$ is a complete normed space. However, $C[a,b]$ with our norm is not complete. $||f||_2 = \sqrt{\int_a^b |f(x)|^2\,dx}$. But $L_2(a,b)$ is complete with $||\cdot||_2$.

Consider $\mathbb{R}^3$. Given a vector $\in \mathbb{R}^3$, which vector in $W = \text{span}\{,\} = xy$-plane is closest to ? If $= (x,y,z)$, then the closest vector in the $xy$-plane is $(x,y,0)$, that is $(\cdot) + (\cdot)$. So the closest vector to  is the Fourier sum.

Let's extend this to a general inner product space. Let $\{\phi_n\}_{n=1}^{\infty}$ be an orthonormal set in $C_p(a,b)$, and let $f \in C_p(a,b)$. Which vector in $\text{span}\{\phi_N\}_{n=1}^{\infty}$ is closest to $f$? Let $\Phi_n(x) = \sum_{n=1}^{N} \gamma_n\phi_n(x)$. Which choice of $\gamma_1, \ldots, \gamma_N$ minimizes $||f - \Phi_N||$? It is equivalent to minimizing $E = ||f - \Phi_N||^2$.

$$\begin{aligned} E = ||f - \Phi_n||^2 &= \left( f - \sum_{n=1}^N \gamma_n \phi_n, f - \sum_{n=1}^N \gamma_n \phi_n \right) \\ &= (f, f) + \left( \sum_{n=1}^N \gamma_n \phi_n, \sum_{n=1}^N \gamma_n \phi_n \right) - 2 \left( f, \sum_{n=1}^N \gamma_n \phi_n \right) \\ &= ||f||^2 + \sum_{n=1}^N \gamma_n^2 - 2 \sum_{n=1}^N \gamma_n C_n + \sum_{n=1}^N C_n^2 - \sum_{n=1}^N C_n^2 \\ &= ||f||^2 + \sum_{n=1}^N (\gamma_n - C_n)^2 - \sum_{n=1}^N C_n^2 \end{aligned}$$

The best approximation which minimizes $E$ is obtained by choosing $\gamma_n = C_n$: In this case, since $E \geq 0$, $\sum_{n=1}^N C_n^2 \leq ||f||^2$, called Bessel's Inequality.

**Theorem.** *If $\{C_n\}_{n=1}^\infty$ are the Fourier constants for $f \in C_p(a, b)$ (or any vector in any inner product space) with respect to some orthonormal set $\{\phi_n\}_{n=1}^\infty$, then* $\lim_{n\infty} C_n = 0$

**Proof.** $\sum_{n=1}^\infty C_n^2 \leq ||f||^2$. So $C_n^2 0$, so $C_n 0$.

Suppose $\{\phi_n\}_{n=1}^\infty$ is an orthonormal set in $W \subset C_p(a, b)$ and $f \in W$. Set $C_n = (f, \phi_n)$. Let $S_N(x) = \sum_{n=1}^N C_n \phi_n(x)$. So $||f - S_n(x)||^2 = ||f||^2 - \sum_{n=1}^N C_n^2$. If $\{\phi_n\}_{n=1}^\infty$ is complete, then $||f(x) - S_N(x)|| 0$, so $\sum_{n=1}^\infty C_n^2 = ||f||^2$, which is Parseval's equation.

**Remark 37.**—Whether an orthonormal set is complete is equivalent to Parseval's equation.

Last time: orthonormal sets, complete, Parseval's equation.

**Recall 68.**—Two fundamental BVP's:

1. $X''(x) + \lambda X(x) = 0$, $X'(0) = 0$, $X'(c) = 0$. Eigenvalues: $\lambda_0 = 0$, $\lambda_n = \left(\frac{n\pi}{2}\right)^2$ for $n = 1, 2, \ldots$. Eigenfunctions: $\psi_0(x) = 1$, $\psi_n(x) = \cos \frac{n\pi}{c} x$
2. $X''(x) + \lambda X(x) = 0$, $X(0) = 0$, $X(c) = 0$. Eigenvalues: $\lambda_n = \left(\frac{n\pi}{c}\right)^2$ for $n = 1, 2, \ldots$. Eigenfunctions $\psi_n(x) = \sin \frac{n\pi}{c} x$ for $n = 1, 2, \ldots$.

These arise by applying separation of variables to certain PDE's. For other more general PDE's or BC's, the method of separation of variables leads to more general BVP, which we now consider.

Consider the Sturm-Liouville problem (SL) on $a < x < b$:

$$(r(x) X'(x))' + [q(x) + \lambda p(x)] X(x) = 0.$$

with separated BC's $a_1 X(a) + a_2 X'(a) = 0$ and $b_1 X(b) + b_2 X'(b) = 0$, where $a_1, a_2$ not both 0 and $b_1, b_2$ not both 0. The functions $p, q, r$ and parameters $a_1, a_2, b_1, b_2$ are real and also independent of $\lambda$.

**Definition.**—The SL problem is regular if interval $(a, b)$ is bounded and

(i) $p, q, r, r'$ are continuous on $[a, b]$.

(ii) $p(x) > 0$ and $r(x) > 0$ on $[a, b]$.

Otherwise, the SL problem is singular.

**Definition.**—$\lambda$ is an eigenvalue (possibly complex) of (SL) if, for that value of $\lambda$, there is a nontrivial solution $X(x)$, which is called an eigenfunction. The spectrum of (SL) is the set of all eigenvalues.

[which will be assumed without proof] A regular SL problem has countably infinite many eigenvalues, $\lambda_1, \lambda_2, \ldots$. Next, we prove that eigenvalues and eigenfunctions of regular SL problems have many similar properties to those of problems (1) and (2).

**Recall 69.**—For $f, g \in C_p(a, b)$, $(f, g) = \int_a^b f(x)g(x)\,dx$. $f$ is orthogonal to $g \iff 0 = (f, g)$. Also, $||f||\sqrt{(f, f)} = \left(\int_a^b |f(x)|^2\,dx\right)^{1/2}$.

**Definition.**—Let $p \in C_p(a, b)$ satisfy $p(x) > 0$ and $f, g \in C_p(a, b)$. $f$ and $g$ are orthogonal with respect to the weight function $p$ if $(f, g) = \int_a^b p(x)f(x)g(x)\,dx = 0$. We define the weighted norm $||f|| = \left(\int_a^b p(x)|f(x)|^2\,dx\right)^{1/2}$.

Consider the SL problem: (1) $[r(x)X'(x)]' + [g(x) + \lambda p(x)]X(x) = 0$ and (2) $a_1X(a) + a_2X'(a) = 0$, $b_1X(b) + b_2X'(b) = 0$, where

(i) $p, r, r'$ continuous on $[a, b]$ and $q$ continuous on $(a, b)$.

(ii) $p(x) > 0$ and $r(x) > 0$ and $a < x < b$.

This includes all regular SL problems, plus some singular.

**Theorem.** *If $\lambda_m$ and $\lambda_n$ are distinct eigenvalues of (1)-(2), then the corresponding eigenfunction $X_m(x)$ and $X_n(x)$ are orthogonal with respect to $p(x)$.*

**Proof.** We have $[rX_m']' + [q + \lambda_m p]X_m = 0$ and $[rX_n']' + [q + \lambda_n p]X_n = 0$. Multiply by $X_n$ and $X_m$ respectively and subtract the equations to get

$$\begin{aligned}(\lambda_m - \lambda_n)pX_mX_n = X_m(rX_n')' - X_n(rX_m')' &= [X_m(rX_n')' + X_m'rX_n'] - [X_n(rX_m')' + X_m'rX_n'] \\ &= \frac{d}{dx}[X_m(rX_n') - X_n(rX_m')]\end{aligned}$$

Integrate from $a$ to $b$ $(\lambda_m - \lambda_n)\int_a^b pX_mX_n\,dx = r(x)\left[X_m(x)X_n'(x) - X_m'(x)X_n(x)\right]_a^b = r(b)\Delta(b) - r(a)\Delta(a)$, where

$$\Delta(x) = \left|X_m(x)X_n(x)X_m'(x)X_n'(x)\right|.$$

Left BC implies $a_x X_m(a) + a_2 X'_m(a) = 0$ and $a_1 X_n(a) + a_2 X'_n(x) = 0$ is

$$X_m(a) X'_m(a) X'_m(a) X'_n(a) a_1 a_2 = 00.$$

This implies $\Delta(a) = 0$. Similarly right BC implies $\Delta(b) = 0$. Thus

$$(\lambda_m - \lambda_n) \int_a^b p(x) X_m(x) X_n(x)\, dx = r(a)\Delta(a) - r(b)\Delta(b) = 0$$

. Since $\lambda_m - \lambda_n \neq 0$, so $\int_a^b p X_m X_n \, dx = \ 0$. If $r(a) = r(b)$ and BC become $X(a) = X(b)$ and $X'(a) = X'(b)$, then BC imply $\Delta(a) = \Delta(b)$, so again $r(a)\Delta(a) - r(b)\Delta(b) = 0$.

**Theorem.** *If $\lambda$ is an eigenvalue for SL problem in previous theorem, then $\lambda$ is real.*

**Proof.** Suppose $\lambda$ is an eigenvalue with corresponding eigenfunction $X(x)$. We write $\lambda = \alpha + i\beta, \alpha, \beta \in R$ and $X(x) = u(x) = iv(x)$, $u, v \in R$. Take conjugate of (1)-(2), use fact that $p, q, r, a_1, a_2, b_1, b_2$ are real. So $[r(x)X'(x)]' + [q(x) + \lambda p(x)]X(x) = 0$, $a_1 X(a) + a_2 X'(a) = 0$, and $b_1 X(b) + b_2 X'(b) = 0$. Thus $\lambda$ is also an eigenvalue, with corresponding eigenfunction $X(x)$. By way of contradiction, suppose $\lambda$ not real, so $\beta \neq 0$. Then $\lambda \neq \lambda$, so $X(x)$ and $X(x)$ are orthogonal with respect to weight function $p$. Thus $0 = \int_a^b p(x) X(x) X(x)\, dx = \int_a^b p(x)|X(x)|^2\, dx = \int_a^b p(x)(u(x)^2 + v(x)^2)\, dx$. This implies $u(x) \equiv 0$ and $v(x) \equiv 0$, so $X(x) \equiv 0$. Contradiction since $X(x)$ is an eigenfunction.

**Theorem.** *If $\lambda$ is an eigenvalue of regular SL problem, then it has a real eigenfunction. If $X, Y$ are eigenfunctions for same eigenvalue $\lambda$, then $Y(x) = cX(x)$, $c \neq 0$ $(*)$.*

**Proof.** Prove $2^{\text{nd}}$ statement first. Suppose $X(x), Y(x)$ are eigenfunctions for $\lambda \in R$. Define (linear combination) $Z(x) = Y'(a)X(x) - X'(a)Y(x)$. Thus $Z(x)$ satisfies: $[rZ']' + [q + \lambda p]Z = 0$ and $Z'(a) = 0$. Also: $a_1 X(a) + a_2 X'(a) = 0$ and $a_1 Y(a) + a_2 Y'(a) = 0$. This implies that $Z(a) = 0$, $X(a)X'(a)Y(a)Y'(a)a_1 a_2 = 00$. By Existence and Uniqueness Theorem for $2^{\text{nd}}$ order linear IVP, since 0 function is also a solution, $Z(x) \equiv 0$. Thus $Y'(a)X(x) - X'(a)Y(x) \equiv 0$. Must have either $Y'(a)$ and $X'(a)$ both zero or both nonzero. If $Y'(a) \neq 0$ and $X'(a) \neq 0$, then $(*)$ holds. If $Y'(a) = 0$ and $X'(a) = 0$, then $Y(a) \neq 0$ and $X(a) \neq 0$. In this case, define $W(x) = Y(a)X(x) - X(a)Y(x)$. Follow argument similar to $Z(x)$, conclude $W(x) \equiv 0$, so $Y(x) = \frac{Y(a)}{X(a)} X(x)$, so $(*)$ holds. Let $X(x) = U(x) + iV(x)$ be eigenfunction for $\lambda$, where $U(x), V(x)$ are real. Plug in to DE and see that $U(x)$ and $V(x)$ are also eigenfunctions for $\lambda$. Thus $V(x) = \beta U(x)$, so $\beta$ is real. Thus $X(x) = (1 + i\beta)U(x)$.

**Example 295.**—$[xX'(x)]' + \frac{\lambda}{x} X(x) = 0$. Let $x = e^s$ so $s = \ln x$. $X'(x) = \frac{dX}{dx} = \frac{dX}{ds} \cdot \frac{ds}{dx}$ or $\frac{d}{dx}[e^s X'(x)]$. $X'(x) = \frac{d}{dx} X(x) = \frac{dX}{dx} \cdot \frac{ds}{dx} = e^{-s} \frac{dX}{ds}$.

**Recall 70.**—Regular SL problems $[r(x)X'(x)]' + [q(x) + \lambda p(x)]X(x) = 0$ with $a_1 X(a) + a_2 X'(a) = 0$, $b_1 X(b) + b_2 X'(b) = 0$, where $p, q, r, r'$ continuous on $[a, b)$, $p > 0$, $r > 0$ on $[a, b]$. We know

- countably infinitely many real eigenvalues,
- each eigenvalue has a real eigenfunction (dimension of eigenspace is 1),
- eigenfunctions corresponding to distinct eigenvalues are orthogonal with respect to $p(x)$.

**Theorem.** *Let $\lambda$ be an eigenvalue of a regular SL problem. If $q(x) \leq 0$ on $[a,b]$ and $a_1a_2 \leq 0$, $b_1b_2 \geq 0$, then $\lambda \geq 0$.*

**Proof.** Let $X(x)$ be a real eigenfunction for eigenvalue $\lambda$. So $[rX']' + [q+\lambda p]X = 0$. Multiply by $X$ and integrate:

$$\begin{aligned}\lambda \int_a^b p(x)X^2(x)\,dx &= -\int_a^b [rX']'X\,dx + \int_a^b (-q)X^2\,dx \\ &= -r(x)X'(x)X(x)\big|_a^b + \int_a^b r(x)[X'(x)]^2\,dx + \int_a^b (-q)X^2\,dx \\ &= r(a)X(a)X'(a) - r(b)X(b)X'(b) + \int_a^b r(X')^2\,dx + \int_a^b (-q)X^2\,dx,\end{aligned}$$

using integration by parts. Consider $r(a)X(a)X'(a)$: If either $a_1 = 0$ or $a_2 = 0$, then $r(a)X(a)X'(a) = 0$. If $a_1 \neq 0$ and $a_2 \neq 0$, $r(a)X(a)X'(a) = -r(a)X(a)\left(\frac{a_1}{a_2}\right)X(a) = r(a)[X(a)]^2\left(-\frac{a_1}{a_2}\right) \geq 0$ because $-\frac{a_1}{a_2} > 0$. Similar argument shows $-r(b)X(b)X'(b) \geq 0$, so $\lambda \int_a^b p(x)[X(x)]^2\,dx \geq 0$, so $\lambda \geq 0$.

**Example 296.**—$X''(x) + \lambda X(x) = 0$ with $X'(0) = 0$ and $hX(c) + X'(c) = 0$, $h > 0$. This is a regular SL problem, $r(x) \equiv 1$, $p(x) \equiv 1$, $q(x) \equiv 0$, $a_1 = 0$, $a_2 = 1$, $b_1 = h$ and $b_2 = 1$.

Case $\lambda = 0$: Then $X''(x) = 0$. Genereal solution is $X(x) = Ax + B$. $X'(0)$ implies $A = 0$, so $X(x) = B$. Then $hX(c) + X'(c) = 0$ implies $hB = 0$, so $B = 0$. So $X(x) = 0$, $\lambda = 0$ not an eigenvalue.

Consider $\lambda > 0$, say $\lambda = \alpha^2$, where $\alpha > 0$. Thus $X''(x) + \alpha^2 X(x) = 0$. General solution is $X(x) = c_1\cos\alpha x + c_2 \sin\alpha x$. $X'(0) = 0$ implies $\alpha c_2 \cos(0) = 0$, so $c_2 = 0$. So $X(x) = c_1\cos\alpha x$. $hX(c) + X'(c) = 0$ implies $c_1 h\cos(\alpha c) - \alpha c_1 \sin(\alpha c) = 0 = h\cos(\alpha c) - \alpha\sin(\alpha c)$, so $\frac{h}{\alpha} = \tan(\alpha c)$ or $\frac{hc}{\alpha c} = \tan(\alpha c)$. so $\frac{hc}{x} = \tan x$ for $x = \alpha c$. If $x$ is a root, then $\alpha = \frac{x}{c}$ and $\lambda = \alpha^2$ is an eigenvalue. Plot $y = \tan x$ and $y = \frac{hc}{x}$: geometrically there are countably infinite number of solutions at the intersection points. As $n\infty$, $x_n \sim (n-1)\pi$. Then $\alpha_n = \frac{x_n}{c} \sim \frac{(n-1)\pi}{c}$ as $n\infty$. So $\lambda_n = \alpha_n^2$. Corresponding eigenfunction $X_n(x) = \cos\alpha_n x$, for $n = 1, 2, \ldots$. Also $\{\cos\alpha_n x\}_{n=1}^\infty$ is orthogonal. Note: $||X_n(x)||^2 = \int_0^c \cos^2\alpha_n x\,dx = \cdots = \frac{ch+\sin^2(\alpha_n c)}{2h}$. So $\phi_n(x) = \frac{1}{||X_n||}X_n(x) = \sqrt{\frac{2h}{ch+\sin^2(\alpha_n c)}}\cos(\alpha_n x)$, where $\{\phi_n\}_{n=1}^\infty$ is an orthonormal set.

**Example 297.**—Consider temperature in cylinder on $0 < x < 1$ with perfect insulation at $x = 0$ and at $x = 1$, surface heat transfer into a medium with temperature 0. Initial temperature distribution $f(x)$. The model: $u_t(x,t) = ku_{xx}(x,t)$, $u_x(0,t) = 0$, $u_x(1,t) = -hu(1,t)$, $u(x,0) = f(x)$. Newton's law of cooling $h[T - u(x,t)]$, $h > 0$. Do separation of variables: $u(x,t) = X(x)T(t)$. Get: $X''(x) + \lambda X(x) = 0$, $X'(0) = 0$, $hX(1) + X'(1) = 0$. We solved this last time: $(c = 1)$ $X_n(x) = \cos \alpha_n x$, $n = 1, 2, \ldots$, $\lambda_n = \alpha_n^2$, $\alpha_n$'s are roots of $\tan \alpha_n = \frac{h}{\alpha_n}$. Normalized eigenfunctions $\phi_n(x) = \sqrt{\frac{2h}{h+\sin^2 \alpha_n}} \cos \alpha_n x$. We also get $T_n(t) = e^{-\lambda_n kt} = e^{-\alpha_n^2 kt}$. So solution is

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-\alpha_n^2 kt} \sqrt{\frac{2h}{h + \sin^2 \alpha_n}} \cos \alpha_n x.$$

Using IC: $f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x)$, so $c_n = (f(x), \phi_n(x)) = \int_0^1 f(x)\phi_n(x)\,dx$.

Transverse vibrations of solid elastic beam. Consider elastic beam of length $l$ (much larger than cross sectional are). Let $w(x,t)$ = transverse displacement at time $t$ and position $x$. Under reasonable assumptions, can derive the Euler-Bernoulli beam equation:

$$m(x)w_{tt}(x,t) + \frac{\partial^2}{\partial x^2}[EI(x)w_{xx}(x,t)] = 0.$$

Assume all parameters are constant, we get $w_{tt}(x,t) + \beta w_{xxxx}(x,t) = 0$, $\beta = \frac{EI}{m} > 0$. Boundary Conditions:

Clamped: $w(\bar{x},t) = 0$, $w_x(\bar{x},t) = 0$ for $\bar{x} = 0$ or $l$.

Hinged: $w(\bar{x}) = 0$, $w_{xx}(\bar{x},t) = 0$ for $\bar{x} = 0$ or $l$.

Free: $w_{xx}(\bar{x},t) = 0$, $w_{xxx}(\bar{x},t) = 0$ for $\bar{x} = 0$ or $l$.

**Example 298.**—Consider Euler-Bernoulli beam hinge at both ends. $w_{tt}(x,t) + \beta w_{xxxx}(x,t) = 0$. BC $w(0,t) = 0 = w_x(0,t)$, $w(l,t) = 0 = w_{xx}(l,t)$. IC $w(x,0) = f(x)$. Apply Fourier method: $w(x,t) = X(x)T(t)$, so $T''(t) + \lambda T(t) = 0$ and $X'''' - \frac{\lambda}{\beta}X = 0$ with $X(0) = 0 = X''(0)$ and $X(l) = 0 = X''(l)$. Get $\lambda > 0$, assume $\frac{\lambda}{\beta} = \mu^4$ so that $X'''' - \mu^4 X = 0$. Characteristic euqation: $r^4 - \mu^4 = 0 = (r^2 - \mu^2)(r^2 + \mu^2) = (r - \mu)(r + \mu)(r^2 + \mu^2)$, with roots $r = \mu, -\mu, i\mu, -i\mu$. General solution: $X(x,t) = c_1 \sin \mu x + c_2 \cos \mu x + c_3 \sinh \mu x + c_4 \cosh \mu x$. BC: $X(0) = 0$: $c_2 + c_4 = 0$. $X''(0) = 0$: $-c_2 + c_4 = 0$. $\implies c_2 = 0 = c_4$. $X(l) = 0$: $c_1 \sin \mu l + c_2 \cos \mu l + c_3 \sinh \mu l + c_4 \cosh \mu l = 0$. $X''(l) = 0$: $-c_1 \sin \mu l - c_2 \cos \mu l + c_3 \sinh \mu l + c_4 \cosh \mu l = 0$.

$$01010 - 101 \sin \mu l \cos \mu l \sinh \mu l \cosh \mu l - sin \mu l - \cos \mu l - \sinh \mu l - \cosh \mu l c_1 c_2 c_3 c_4 = 0000.$$

$c_2 = 0 = c_4$, so $c_1 \sin \mu l + c_3 \sinh \mu l = 0$ and $-c_1 \sin \mu l + c_3 \sinh \mu l = 0$. Then $2c_3 \sinh \mu l = 0 \implies c_3 = 0$. Thus $c_1 \sin \mu l = 0$, $c_1 \neq 0$, so $\sin \mu l = 0$. So $\mu l = n\pi$, $n = 1, 2, \ldots$, or $\mu_n = \frac{n\pi}{l}$ and $\lambda_n = \beta \left(\frac{n\pi}{l}\right)^4$. Therefore $X_n(x) = \sin \frac{n\pi}{l} x$.

**Recall 71.**—Beam example $w_{tt}(x,t) + \beta w_{xxxx}(x,t) = 0$. Separate variables $x''''(x) + \frac{\lambda}{\beta}x(x) = 0$, where $\frac{\lambda}{\beta} = \mu^4$ with $X(0) = 0 = X'(0)$ and $X''(l) = 0 = X'''(l)$. Then $X'''' + \mu^4 X = 0$. $X(x) = c_1 \sin \mu x + c_2 \cos \mu x + c_3 \sinh \mu x + c_4 \cosh \mu x$.

Modifications for certain non-homogeneous problems (section 77, similar to section 39)

**Example 299.**—$u_t(x,t) = ku_{xx}(x,t)$ with $u(0,t) = 0$ and $Ku_x(1,t) = A$, $A > 0$; $u(x,0) = 0$. Non homogeneous BC. Similar to idea in section 39, suppose $u(x,t) = U(x,t) + \Phi(x)$. Try to select $\Phi(x)$ so that $U(x,t)$ is a solution to homogeneous PDE with homogeneous BC. If possible, then can apply Fourier Method to determine $U(x,t)$, and recover $u(x,t)$. Then $U(x,t) = u(x,t) - \Phi(x)$. So $U_t(x,t) = u_t(x,t) = ku_{xx}(x,t)$ and $U_t(x,t) = kU_{xx}(x,t) + k\Phi''(x)$. Then $U(0,t) = u(0,t) - \Phi(0) = -\Phi(0)$ and $U_x(1,t) = u_x(1,t) - \Phi'(1) = \frac{A}{K} - \Phi'(1)$. To make PDE and BC for $U(x,t)$ homogeneous, we want $\Phi(x)$ to satisfy:

i) $k\Phi''(x) = 0$

ii) $\Phi(0) = 0$

iii) $\frac{A}{K} - \Phi'(1) = 0$

So i) implies $\Phi(x) = Bx + C$. ii) implies $0 = B(0) + C$, so $C = 0$ so $\Phi(x) = Bx$. Now iii) implies $\Phi'(1) = \frac{A}{K}$ so $B = \frac{A}{K}$. So $\Phi(x) = \frac{A}{K}x$. Then $U(x,t)$ satisfies: $U_t(x,t) = kU_{xx}(x,t)$, $U(0,t) = 0$, $U_x(1,t) = 0$, and $U(x,0) = u(x,0) - \Phi(x) = -\frac{A}{K}x$. Next apply Fourier method to determine $U(x,t)$. $U(x,t) = X(x)T(t)$. $X''(x) + \lambda X(x) = 0$, $X(0) = 0$, and $X'(1) = 0$. By problem 72.1, $\lambda_n = \alpha_n^2$; $\alpha_n = \frac{(2n-1)\pi}{2}$ for $n = 1, 2, \ldots$; $X_n(x) = \phi_n(x) = \sqrt{2}\sin(\alpha_n x)$. Then $T_n(t) = e^{-\alpha_n^2 kt}$. So $U(x,t) = \sum_{n=1}^{\infty} c_n e^{-\alpha_n^2 kt}\phi_n(x)$. Then determine $c_n$ so that $-\frac{A}{K}x = \sum_{n=1}^{\infty} c_n\phi_n(x)$, $c_n = \sqrt{2}\frac{A}{K}\frac{(-1)^n}{\alpha_n^2}$. Thus $u(x,t) = \frac{A}{K}x + \frac{2A}{K}\sum_{n=1}^{\infty} e^{-\alpha_n^2 kt}\frac{(-1)^n}{\alpha_n^2}\sin(\alpha_n x)$.

**Example 300.**—$u_t(x,t) = u_{xx}(x,t)$, $u_x(0,t) = 0$, $u(1,t) = 4$, $u(x,0) = 0$. $u(x,t) = U(x,t) + \Phi(x)$. $U(x,t) = u(x,t) - \Phi(x)$. $U_t(x,t) = u_t(x,t) = u_{xx}(x,t) = U_{xx}(x,t) + \Phi''(x)$. $U_x(0,t) = u_x(0,t) - \Phi'(0) = -\Phi'(0)$. $U(1,t) = u(1,t) - \Phi(1) = 4 - \Phi(1)$. Select $\Phi(x)$ so that i) $\Phi''(x) = 0 \implies \Phi(x) = Ax + B$; ii) $\Phi'(0) = 0 \implies A = 0, \Phi(x) = B$; iii) $\Phi(1) = 4 \implies B = 4$, so $\Phi(x) = 4$. So $U(x,t)$ solves: $U_t(x,t) = U_{xx}(x,t)$, $U_x(0,t) = 0$, $U(1,t) = 0$, $U(x,0) = -4$.

**Example 301.**—$X''(x) + \lambda X(x) = 0$, $X(0) = 0$, $X'(1) + 2X(1) = 0$. Know $\lambda \geq 0$. Case $\lambda = 0$: only trivial solution, so not an eigenvalue. Case $\lambda > 0$: Assume $\lambda = \alpha^2$, $\alpha > 0$. $X(x) = c_1 \sin \alpha x + c_2 \cos \alpha x$. $X(0) = 0$ implies $0 = c_2$, so $X(x) = c_1 \sin \alpha x$, $X'(x) = c_1\alpha \cos \alpha x$. $X'(1) + 2X(1) = 0$ implies $c_1\alpha\cos\alpha + 2c_1 \sin\alpha = 0$. Want $c_1 \neq 0$, $\alpha\cos\alpha + 2\sin\alpha = 0$ means $\tan\alpha = -\frac{\alpha}{2}$. By considering graphs of $y = \tan x$ and $y = -\frac{x}{2}$, we see there are countable infinite number of solutions $\alpha_n$, $n = 1, 2 \ldots$. Then $\lambda_n = \alpha_n^2$, $X_n(x) = \sin(\alpha_n x)$. To get normalized eigenfunction $\phi_n(x) = \frac{1}{||X_n(x)||}X_n(x)$. $||X_n(x)||^2 = \int_0^1 \sin^2(\alpha_n x)\,dx = \int_0^1 \frac{1}{2}[1 - \cos 2\alpha_n x]\,dx = \frac{1}{2}x + \frac{1}{4\alpha_n}\sin 2\alpha_n x\Big|_0^1 = \frac{1}{2} - \frac{1}{4\alpha_n}\sin 2\alpha_n = \frac{1}{2} - \frac{1}{2\alpha_n}\sin\alpha_n\cos\alpha_n = \frac{1}{2} - \frac{1}{2}\left(-\frac{1}{2}\cos\alpha_n\right)\cos\alpha_n = \frac{1}{2} + \frac{1}{4}\cos^2\alpha_n = \frac{2+\cos^2\alpha_n}{4}$. So $\phi_n(x) = \frac{2}{\sqrt{2+\cos^2\alpha_n}}\sin(\alpha_n x)$.

**Example 302** (Beam Equation)**.**—Last time got to $X'''' - \frac{\lambda}{\beta}X = 0$, where $\frac{\lambda}{\beta} = \mu^4$. Then $X'''' - \mu^4 X = 0$ with $X(0) = 0 = X'(0)$, $X''(l) = X'''(l) = 0$, and $X(x) = c_1 \sin \mu x + c_2 \cos \mu x + c_3 \sinh \mu x + c_4 \cosh \mu x$. Then $X'(x) = \mu[c_1 \cos \mu x - c_2 \sin \mu x + c_3 \cosh \mu x + c_4 \sinh \mu x]$, $X''(x) = \mu^2[-c_1 \sin \mu x - c_2 \cos \mu x + c_3 \sinh \mu x + c_4 \cosh \mu x]$, and $X'''(x) = \mu^3[-c_1 \cos \mu x + c_2 \sin \mu x + c_3 \cosh \mu x + c_4 \sinh \mu x]$. $X(0) = 0$ implies $c_2 = -c_4$. $X'(0) = 0$ implies $c_1 = -c_3$. So $X''(l) = 0$ implies $c_3[\sin \mu l + \sinh \mu l] + c_2[\cos \mu l + \cosh \mu l] = 0$ and $X'''(l) = 0$ implies $-c_1[\cos \mu l + \cosh \mu l] + c_2[\sin \mu l - \sinh \mu l] = 0$.

$$\sin \mu l + \sinh \mu l \cos \mu l + \cosh \mu l - \cos \mu l - \cosh \mu l \sin \mu l - \sinh \mu l c_1 c_2 = 00.$$

This has a nontrivial solution only if determinant of matrix is zero. Then

$$(\sin \mu l + \sinh \mu l)(\sin \mu l - \sinh \mu l) + (\cos \mu l + \cosh \mu l)^2 = 0$$
$$\sin^2 \mu l - \sinh^2 \mu l + \cos^2 \mu l + \cosh^2 \mu l + 2 \cos \mu l \cosh \mu l = 0 = 2 + 2 \cos \mu l \cosh \mu l = 0$$

So $\cos \mu l \cosh \mu l = -1$.

# Chapter 8

# MATHEMATICAL ANALYSIS

**Recall 72** (Cauchy product)**.**—For two series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$, their Cauchy product series is given by $\sum_{n=0}^{\infty} c_n$, where

$$c_n = \sum_{k=0}^{n} a_k b_{n-k} = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0.$$

**Theorem.** *Assume:*

*(a)* $\sum_{n=0}^{\infty} a_n$ *converges absolutely,*

*(b)* $\sum_{n=0}^{\infty} a_n = A$*,*

*(c)* $\sum_{n=0}^{\infty} b_n = B$*,*

*(d)* $c_n = \sum_{k=0}^{n} a_k b_{n-k}$*.*

*Then* $\sum_{n=0}^{\infty} c_n$ *converges to* $AB$*.*

**Proof.** Define the sequences of partial sums $A_n := \sum_{k=0}^{n} a_k$, $B_n := \sum_{k=0}^{n} b_k$, $C_n := \sum_{k=0}^{n} c_k$. By assumption, $A_n \to A$ and $B_n \to B$. We aim to show $C_n \to AB$.

$$\begin{aligned}
C_n &= \sum_{k=0}^{n} c_k \\
&= a_0 b_0 + (a_0 b_1 + a_1 b_0) + \cdots + (a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0) \\
&= a_0(\underbrace{\quad B_n \quad}_{b_0+b_1+\cdots+b_n}) + a_1(\underbrace{\quad B_{n-1} \quad}_{b_0+\cdots+b_{n-1}}) + \cdots + a_n \underbrace{B_0}_{b_0} \\
&= a_0 B_n + a_1 b_{n-1} + a_2 B_{n-2} + \cdots + a_n B_0 \\
&= a_0(\beta_n + B) + a_1(\beta_{n-1} + B) + \cdots + a_n(\beta_0 + B) \\
&= (\underbrace{\quad A_n \quad}_{a_0+a_1+\cdots+a_n})B + a_0\beta_n + a_1\beta_{n-1} + \cdots + a_n\beta_0 \to AB
\end{aligned}$$

if $a_0\beta_n + a_1\beta_{n-1} + \cdots + a_n\beta_0 \to 0$ as $n \to \infty$, where $\beta_n = B_n - B$ and $B_n = \beta_n + B$.

Let $\gamma_n := \alpha_0\beta_n + \alpha_1\beta_{n-1} + \cdots + a_n\beta_0$. Fix $\epsilon > 0$. We want to show $\exists N \in \mathbb{N}$ such that $|\gamma_n| < \epsilon$ for all $n \geq N$. Since $\sum a_n$ converges absolutely, $\alpha := \sum_{n=0}^{\infty} |a_n| \in \mathbb{R}$. Then $|\gamma_n| = |a_0\beta_n + a_1\beta_{n-1} + \cdots + a_n\beta_0|$. Since $\beta_n \to 0$ as $n \to \infty$, $\exists N_1 \in \mathbb{N}$ such that $|\beta_n| < \epsilon$ for all $n \geq N_1$. For $n \geq N_1$,

$$
\begin{aligned}
|\gamma_n| &\leq |a_0\beta_n + \cdots + a_{n-N_1-1}\beta_{N_1+1}| + |a_{n-N_1}\beta_{N_1} + \cdots + a_n\beta_0| \\
&\leq |a_0| \underbrace{|\beta_n|}_{<\epsilon} + |a_1| \underbrace{|\beta_{n-1}|}_{<\epsilon} + \cdots + |a_{n-N_1-1}| \underbrace{|\beta_{N_1+1}|}_{<\epsilon} + |a_{n-N_1}\beta_{N_1} + \cdots + a_n\beta_0| \\
&< \epsilon \underbrace{\leq \alpha}_{\sum_{k=0}^{n-N_1-1}|a_k|} + |a_{n-N_1}\beta_{N_1} + \cdots + a_n\beta_0| < \epsilon\alpha + |a_{n-N_1}\beta_{N_1} + \cdots + a_n\beta_0|
\end{aligned}
$$

Keep $N_1$ fixed. Letting $n \to \infty$ and noting that $a_i \to 0$ as $i \to \infty$, we get $\lim_{n\to\infty} |\gamma_n| \leq \epsilon\alpha$. Since $\alpha > 0$ is fixed and $\epsilon > 0$ is arbitrary, $\lim_{n\to\infty} |\gamma_n| = 0$.

**Theorem.** *If $\sum a_n = A$, $\sum b_n = B$, $\sum c_n = C$, where $c_n = \sum_{k=0}^{n} a_k b_{n-k}$, then $C = AB$.*

## § 0.1 Rearrangement of an Infinite Series

Let $\sum a_n$ be rearranged as $\sum a'_n$, where $a'_n = a$ and $k_n$ is a bijection from $\mathbb{N}$ to $\mathbb{N}$.

**Example 303.**—Consider the series $a_1\ a_2\ a_3\ a_4\ a_5\ a_6\ a_7\ \cdots$,

Its rearrangement is $a_1\ a_3\ a_2\ a_5\ a_7\ a_4\ \cdots$,

Note that $a_1\ a_3\ a_5\ a_7\ \cdots\ a_{2n+1}\ \cdots\ a_2\ a_4\ a_6\ a_8\ \cdots$ is not a rearrangement.

**Example 304.**—Consider the alternating harmonic series: $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots = \ln 2$.

Rearranging, we get 2 odd terms followed by an even term: $1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \ldots$.

**Show:** The sum of the rearranged series is $\frac{3}{2}\ln 2$.

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \ldots.$$

Multiply by $\frac{1}{2}$:

$$\frac{\ln 2}{2} = 0 + \frac{1}{2} + 0 - \frac{1}{4} + 0 + \frac{1}{6} + 0 + \ldots.$$

Adding the two above:

$$\frac{3}{2}\ln 2 = 1 + 0 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \ldots,$$

so the sum of the rearranged series is $\frac{3}{2}\ln 2$.

**Theorem.** *Suppose $\sum a_n$ converges absolutely. Then every rearrangement $\sum a'_n$ of $\sum a_n$ converges, and they all converge to the same sum.*

**Proof.** Consider a rearrangement $\sum a'_n$. Let $s_n$ and $s'_n$ be sequences of partial sums of $\sum a_n$ and $\sum a'_n$ respectively. Assume $s_n \to s$.

**Goal:** Show $s'_n \to s$.

Let $\epsilon > 0$ be fixed. **Claim:** There exists $N \in \mathbb{N}$ such that $|s'_n - s| < \epsilon$ for all $n \geq N$.

Now,

$$\begin{aligned} |s'_n - s| &= |s'_n - s_n + s_n - s| \\ &\leq |s'_n - s_n| + |s_n - s| \end{aligned}$$

Since $\sum a_n$ converges absolutely, there exists $N_2 \in \mathbb{N}$ such that $m \geq n > N_2$

$$\sum_{i=n}^{m} |a_i| < \frac{\epsilon}{2}.$$

Choose $p \in \mathbb{N}$ such that $\{1, 2, \ldots, N_2\} \subset \{k_1, k_2, \ldots, k_p\}$. Then, for $n > p$,

$$\begin{aligned} |s'_n - s_n| &= |(a'_1 + a'_2 + \cdots + a'_n) - (a_1 + a_2 + \cdots + a_n)| \quad \text{– cancel terms} \\ &\leq \sum_{i=n}^{m} |a_i| < \frac{\epsilon}{2} \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, $s'_n \to s$.

# § 1 Continuity

From Calculus: Consider a function $f : \mathbb{R} \to \mathbb{R}$ where $f$ may not be defined at $p$. The limit of $f(x)$ as $x$ approaches $p$ is denoted as $\lim_{x \to p} f(x) = q$ if and only if, for every $\epsilon > 0$, there exists $\delta = \delta(\epsilon, p) > 0$ such that $0 < |x - p| < \delta$ implies $|f(x) - q| < \epsilon$. A function $f$ is continuous at $p \in \mathbb{R}$ if $\lim_{x \to p} f(x) = f(p)$ if and only if, for every $\epsilon > 0$, there exists $\delta = \delta(\epsilon, p) > 0$ such that for all $x \in \mathbb{R}$ with $|x - p| < \delta$, $|f(x) - f(p)| < \epsilon$.

In metric spaces: Consider a function $f : E \subset (X, d_X) \to (Y, d_Y)$.

**Definition** (Limit).—Let $p \in E'$. We say $\lim_{x \to p} f(x) = q$ if and only if, for every $\epsilon > 0$, there exists $\delta = \delta(\epsilon, p) > 0$ such that for all $x \in E$ with $0 < d_X(x, p) < \delta$, $d_Y(f(x), q) < \epsilon$. Thus, $X \supset E \supset B_\delta(p) \xrightarrow{f} f(B_\delta(p)) \subset Y$.

An alternative definition of the limit is given by:

**Theorem.** *Let $f : E \subset (X, d_X) \to (Y, d_Y)$ and $p \in E'$. Then $\lim_{x \to p} f(x) = q$ if and only if $\lim_{n \to \infty} f(x_n) = q$ for every sequence $\{x_n\} \subset E$ with $x_n \neq p$ and $x_n \to p$.*

Limit properties: Let $f, g : E \subset (X, d_X) \to (\mathbb{C}, |\cdot|_{\mathbb{C}})$. Then if $\lim_{x \to p} f(x) = A$ and $\lim_{x \to p} g(x) = B$ for $p \in E'$, the following hold:

1. $\lim_{x \to p} (f + g)(x) = A + B$
2. $\lim_{x \to p} (f \cdot g)(x) = A \cdot B$
3. $\lim_{x \to p} \left(\frac{f}{g}\right) = \frac{A}{B}$.

## § 1.1 Continuity

**Definition.**—Suppose $f : E \subset (X, d_X) \to (Y, d_Y)$ and $p \in E$. We say $f$ is continuous at $p$ if and only if, for every $\epsilon > 0$, there exists $\delta = \delta(\epsilon, p) > 0$ such that for all $x \in E$ with $d_X(x, p) < \delta$, $d_Y(f(x), f(p)) < \epsilon$.

- Continuity is a pointwise property.
- $f$ is continuous on $E$ if $f$ is continuous at every $p \in E$.
- $f$ is continuous at an isolated point $p \in E$.

**Example 305.**—For any fixed $\epsilon > 0$, take $\delta(\epsilon, p) = 0 < \delta$ such that $N_\delta(p) \setminus \{p\} = \emptyset \implies d_Y(f(p), f(p)) = 0 < \epsilon$.

**Theorem.** *Suppose* $f : (X, d_X) \to (Y, d_Y)$ *is continuous on* $X \iff f^{-1}(V) = \{x \in X : f(x) \in V\}$ *is open in* $X$ *for every open set* $V \subset Y$.

**Proof.** $\implies$ Suppose $f$ is continuous on $X$. Let $V \subset Y$ be open. NTS: $f^{-1}(V)$ is open in $X$. Let $p \in f^{-1}(V)$. NTS: $\exists \delta > 0$ such that $B_\delta(p) \subset f^{-1}(V)$. Since $p \in f^{-1}(V)$, $f(p) \in V$. So $V$ is open $\implies \exists \epsilon > 0$ such that $B_\epsilon(f(p)) \subset V$. Since $f$ is continuous, $\exists \delta = \delta(\epsilon, p) > 0$ such that $f(B_\delta(p)) \subset B_\epsilon(f(p)) \subset V \implies B_\delta(p) \subset f^{-1}(V) \implies p$ is an interior point of $f^{-1}(V) \implies f^{-1}(V)$ is open.

Let $p \in X$ and $\epsilon > 0$ be given. NTS: $\exists \delta = \delta(p, \epsilon) > 0$ such that for $x \in X$ with $d_X(x, p) < \delta$ implies $d_Y(f(p), f(x)) < \epsilon$. Let $v := B_\epsilon(f(p)) \subset Y$. Then $V$ is open. Therefore, $f^{-1}(V)$ is open in $X$ and $p \in f^{-1}(V)$. Since $f^{-1}(V)$ is open, $p$ is an interior point. So $\exists \delta > 0$ such that $B_\delta(p) \subset f^{-1}(V) \implies f(B_\delta(p)) \subset V = B_\epsilon(f(p)) \implies f$ is continuous on $X$ since $p \in X$ is arbitrary.

*Corollary.*—$f : (X, d_X) \to (Y, d_Y)$ is continuous if and only if $f^{-1}(W)$ is closed in $X$ for every closed set $W$ in $Y$.

**Proof.** Observe that

- $W$ is closed if and only if $V = W^c$ is open.
- $f$ is continuous on $X$ if and only if $f^{-1}(V)$ is open if and only if $[f^{-1}(V)]^c$ (which is $f^{-1}(V^c)$) is closed, with $V^c = W$.

To establish the equivalence, it is enough to show that $[f^{-1}(V)]^c = f^{-1}(V^c)$ for any $V$. Consider $x \in [f^{-1}(V)]^c$. This means $x \notin f^{-1}(V)$, which is equivalent to $f(x) \notin V$, implying $f(x) \in V^c$. Therefore, $x \in f^{-1}(V^c)$.

**Remark 38.**—Topological properties are preserved under continuous mappings in either direction:

(i) $f(\text{compact}) = \text{compact}$

(ii) $f(\text{connected}) = \text{connected}$

✓(iii) $f^{-1}(\text{open}) = \text{open}$

✓(iv) $f^{-1}(\text{closed}) = \text{closed}$

**Remark 39.**—Not necessarily true:

(i) $f(\text{open}) \neq \text{open}$

(ii) $f(\text{closed}) \neq \text{closed}$

(iii) $f^{-1}(\text{compact}) \neq \text{compact}$

(iv) $f^{-1}(\text{connected}) \neq \text{connected}$

(v) $f(\text{bounded}) \neq \text{bounded}$

(vi) $f^{-1}(\text{bounded}) \neq \text{bounded}$.

**Example 306.**— (i) $f(\text{open}) \neq \text{open}$

(a) constant function.

(b) $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \frac{1}{1+x^2}$. Take $U = (-1, 1) \subset \mathbb{R}$, which is open, but $f(U) = \left(\frac{1}{2}, 1\right]$ is not open.

(ii) $W = [0, \infty)$ is closed in $\mathbb{R}$, but $f(W) = (0, 1]$ is not closed.

**Theorem.** *Let $(X, d_X), (Y, d_Y), (Z, d_Z)$ be metric spaces. If $f : X \to Y$ is continuous at $p \in X$, and $g : f(X) \to Z$ is continuous at $f(p)$, then $(g \circ f) : X \to Z$ is continuous at $p$. In symbols, $X \ni p \xrightarrow{f} f(p) \in Y$ and $X \ni p \xrightarrow{g \circ f} g(f(p)) \in Z \xleftarrow{g} f(p) \in Y$.*

**Proof.** Let $p \in X$ and $\epsilon > 0$ be fixed. Since $g$ is continuous at $f(p)$, there exists $\eta > 0$ such that $g(B_\eta(f(p))) \subset B_\epsilon(g(f(p)))$. Since $f$ is continuous at $p$, there exists $\delta > 0$ such that $f(B_\delta(p)) \subset B_\eta(f(p))$. Therefore, $g(f(B_\delta(p))) \subset B_\epsilon(g(f(p)))$. Hence, $g \circ f$ is continuous at $p$.

**Theorem.** *If $f, g : (X, d_X) \to \mathbb{C}$ are continuous, then $f + g, f \cdot g, f/g$ (where defined) are continuous.*

**Proof.** Skipped.

**Theorem.** *Let $: (X, d_X) \to \mathbb{R}^k$ with $k \geq 1$ defined by $(x) = (f_1(x), f_2(x), \ldots, f_k(x))$ for $x \in X$, where $f_i : X \to \mathbb{R}$ for $i = 1, 2, \ldots, k$.*

*(a) is continuous on $X$ if and only if each $f_i$ is continuous on $X$.*

*(b) If and are continuous on $X$, then $+$ and $\cdot$ (interpreted as the inner product) are continuous on $X$.*

**Theorem.** *If $(X, d_X)$ is compact and $f : (X, d_X) \to (Y, d_Y)$ is continuous, then $f(X)$ is compact.*

**Proof.** Let $\{y_n\} \subset f(X)$ be a sequence. For each $n \in \mathbb{N}$, there exists $x_n \in X$ such that $f(x_n) = y_n$. Since $X$ is compact, there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $x_{n_k} \to x^* \in X$. By continuity of $f$, $f(x_{n_k}) \to f(x^*) \in f(X)$, which implies that $\{y_{n_k}\} = \{f(x_{n_k})\}$ has a convergent subsequence in $f(X)$. Therefore, $f(X)$ is compact.

**Recall 73** (Theorem 227)**.**—Let $(X, d_X), (Y, d_Y)$ be metric spaces. Then $f : X \overset{\text{compact}}{\longrightarrow} Y \implies f(X)$ is compact.

**Theorem.** $f : X^{compact} \overset{cts}{\longrightarrow} Y \implies f(X)$ *is closed and bounded.*

**Proof.** Every compact set is closed and bounded. By Theorem 227, $f(X)$ is compact, and hence it is closed and bounded.

**Theorem Extreme Value Theorem.** *Let $f : X^{compact} \longrightarrow \mathbb{R}$ be continuous. Then $f$ attains both maximum and minimum on $X$, i.e., there exist $p, q \in X$ such that $f(p) = \sup_{x\in X} f(x)$ and $f(q) = \inf_{x\in X} f(x)$.*

**Proof.** By Theorem 227 and Theorem 228, $f(X)$ is closed and bounded. Since $f(X) \subset \mathbb{R}$ is bounded and $\mathbb{R}$ has the least upper bound property, $\inf_{x\in X} f(x) \in \mathbb{R}$ and $\sup_{x\in X} f(x) \in \mathbb{R}$. Now, $f(X)$ is closed in $\mathbb{R}$, so $p, q \in X$ such that $f(p) = \sup_{x\in X} f(x)$ and $f(q) = \inf_{x\in X} f(x)$.

**Definition** (Homeomorphism)**.**—$f : X \longrightarrow Y$ is a homeomorphism if the following holds:

1. $f$ is bijective (one-to-one and onto).
2. $f$ is continuous on $X$.
3. $f$ is continuous on $Y$.

**Theorem.** *If $X$ is compact and $X$ is homeomorphic to $Y$, then $Y$ is compact.*

**Proof.** If $X$ is homeomorphic to $Y$, there exists a homeomorphism $f$ from $X$ to $Y$. Moreover, $f(X) = Y$. By Theorem 227, $f(X) = Y$ is compact.

**Definition** (Inverse Mapping)**.**—Suppose $f : X \longrightarrow Y$ is bijective. Then the inverse mapping $f : Y \longrightarrow X$ is defined by $f(f(x)) = x$ for all $x \in X$.

**Theorem.** *Let $f : X^{compact} \longrightarrow Y$ be bijective and continuous. Then $f : Y \longrightarrow X$ is continuous.*

**Proof.** We will show that $f(V)$ is open in $Y$ for any open set $V$ in $X$. Let $V \subset X$ be open. Then $V^c \subset X$ is closed and hence compact since $X$ is compact. Since $f$ is continuous, $f(V^c)$ is compact in $Y$. Then $f(V) \overset{?}{=} [f(V^c)]^c$ and hence is open, since $f(V^c)$ is closed. Hence, $f$ is continuous on $Y$.

Compactness and uniform continuity:

**Recall 74.**—$f : X \longrightarrow Y$. $f$ is continuous at $p \in X$ if and only if $\forall \epsilon > 0$, $\exists \delta = \delta(p, \epsilon) > 0$ such that for all $x \in X$ with $d_X(x, p) < \delta$ implies $d_Y(f(x), f(p)) < \epsilon$. – pointwise concept ($p$ dependent).

**Definition** (Uniform Continuity)**.**—$f : X \longrightarrow Y$ is uniformly continuous on $X$ if and only if $\forall \epsilon > 0$, $\exists \delta = \delta(\epsilon) > 0$ such that for all $x, p \in X$ with $d_X(x, p) < \delta$ implies $d_Y(f(x), f(p)) < \epsilon$.

**Example 307.**—$f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = x^2$ is continuous on $\mathbb{R}$.

**Proof.** Let $p \in \mathbb{R}$ and $\epsilon > 0$ be fixed. Need to Show: $\exists \delta = \delta(p, \epsilon) > 0$ such that for any $x \in \mathbb{R}$ with $|x-p| < \delta \implies |f(x)-f(p)| < \epsilon$. Now, $|f(x)-f(p)| = |x^2-p^2| = |x-p|\,|x+p|$. Take $|x - p| < 1$ (this is not $\delta$ yet). Then $|x + p|\Delta \leq |x| + |p| < 1 + |p|$. Therefore, $|f(x) - f(p)| = |x - p|\,|x + p| < \epsilon$ if $|x - p| < \frac{\epsilon}{2|p|+1}$. Then $\delta(\epsilon, p) = \min\{1, \frac{\epsilon}{2|p|+1}\}$.

**Example 308.**—$f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = x^2$. Then $f$ is uniformly continuous on $[-7, 7]$.

**Proof.** Let $\epsilon > 0$ be fixed. Need to Show: $\exists \delta = \delta(\epsilon) > 0$ such that $\forall x, p \in [-7, 7]$ such that $|x - p| < \delta \implies |f(x) - f(p)| < \epsilon$. Then $|f(x) - f(p)| = |x^2 - p^2| = |x + p|\,|x - p| \leq 14|x - p| < \epsilon$ if $|x - p| < \frac{\epsilon}{14} = \delta$.

**Example 309.**—$f(x) = \frac{1}{x^2}$ is continuous on $(0, \infty)$. Let $p > 0$ and $\epsilon > 0$ be fixed. Find $\delta$. Then $f(x) - f(p) = \frac{1}{x^2} - \frac{1}{p^2} = \frac{p^2-x^2}{p^2x^2} = \frac{(p-x)(p+x)}{p^2x^2}$. Take $|p - x| < \frac{p}{2}$. Also

$$|p| - |x| < |p - x| < \frac{p}{2} \implies -\frac{p}{2} < |p| - |x| < \frac{p}{2} \implies |x| < \frac{p}{2}$$

and $|x| < \frac{3p}{2}$ because $p > 0$. Finish on your own. (Uniformly continuous on $[a, \infty)$ for any $a > 0$).

**Example 310.**—Let $f(x) = \frac{1}{x^2}$. The function is uniformly continuous on $[a, \infty)$ for any $a > 0$.

**Proof.** Assume $a > 0$ and $\epsilon > 0$ are fixed. We aim to show that there exists $\delta = \delta(\epsilon) > 0$ such that for all $x, y \in [a, \infty)$ with $|x - y| < \delta$, it follows that $|f(x) - f(y)| < \epsilon$.

Starting with $|f(x)-f(y)| = \left|\frac{1}{x^2} - \frac{1}{y^2}\right| = \frac{|y^2-x^2|}{x^2y^2}$, we manipulate the expression to arrive at $|y - x| \cdot \left(\frac{1}{x^2y} + \frac{1}{xy^2}\right)$. This is further bounded by $|y - x|\frac{2}{a^3}$, which is less than $\epsilon$ when $|x - y| < \frac{\epsilon a^3}{2}$. Thus, $f$ is uniformly continuous on $[a, \infty)$ since $x, y \geq a$ implies $\frac{1}{x}, \frac{1}{y} \leq \frac{1}{a}$.

**Theorem.** *Let $f : X^{compact} \xrightarrow{cts} Y$. Then $f$ is uniformly continuous on $X$.*

**Proof.** Fix $\epsilon > 0$. We want to show that there exists $\delta = \delta(\epsilon) > 0$ such that for all $x, y \in X$ with $d_X(x, y) < \delta$, it implies $d_Y(f(x), f(y)) < \epsilon$.

Let $(*)$ : $f$ is continuous on $X$. For each $p \in X$, there exists $\eta(p) > 0$ such that for all $q \in X$ with $d_X(p, q) < \eta(p)$, $d_Y(f(p), f(q)) < \frac{\epsilon}{2}$.

Define $J(p) := \{q \in X \mid d_X(p, q) < \frac{\eta(p)}{2}\}$ for each $p \in X$. Then $\{J(p)\}_{p\in X}$ is an open cover of $X$. Since $X$ is compact, there exist $p_1, p_2, \ldots, p_n \in X$ such that $X \subset J(p_1) \cup J(p_2) \cup \cdots \cup J(p_n)$. Set $\delta := \frac{1}{2}\min\{\eta(p_1), \eta(p_2), \ldots, \eta(p_n)\} > 0$.

Let $p, q \in X$ with $d_X(p, q) < \delta$. Then $p \in J(p_m)$ for some $m \in \{1, \ldots, n\}$, implying $d_Y(f(p), f(p_m)) < \frac{\epsilon}{2}$. Also, $d_X(q, p_m) \leq d_X(p, q) + d_X(p, p_m) < \delta + \frac{\eta(p_m)}{2} \leq \eta(p_m)$, which leads to $d_Y(f(q), f(p_m)) < \frac{\epsilon}{2}$.

Combining these results, $d_Y(f(p), f(q)) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$, confirming the uniform continuity of $f$ on $X$.

**Theorem Compactness cannot be relaxed.** *Let $E \subseteq \mathbb{R}$ be a noncompact set. Then:*

1. *There exists a continuous function on $E$ that is not bounded.*

2. *There exists a continuous and bounded function on $E$ that does not achieve its maximum on $E$.*

3. *If $E$ is bounded, there exists a continuous and bounded function on $E$ that is not uniformly continuous.*

## § 1.2 Continuity and Connectedness

**Recall 75.**—$E \subset X^{m.s.}$ is a union of nonempty disjoint separated sets if there exist $A^{\neq\emptyset}, B^{\neq\emptyset} \subset X$ such that $E = A \cup B$, $\overline{A} \cap B = \emptyset$, $A \cap \overline{B} = \emptyset$. A set $E \subset X$ is connected if it is not a union of nonempty separated sets.

**Theorem.** *Let $f : X \xrightarrow{cts} Y$ and $E \subset X$ be connected. Then $f(E) \subset Y$ is connected.*

**Theorem Intermediate Value property.** *Let $f : [a, b] \xrightarrow{cts} \mathbb{R}$. If $f(a) < f(b)$ and $\exists y_0 \in (f(a), f(b))$, then $\exists x_0 \in (a, b)$ such that $f(x_0) = y_0$.*

**Proof.** Note that $[a, b]$ is connected. Then Theorem 234 implies that $f([a, b]) \subset R$ is connected. Since $f(a), f(b) \in$ connected$f([a, b])$ and $f(a) < y_0 < f(b)$, $y_0 \in f([a, b]) \implies \exists x_0 \in (a, b)$ such that $f(x_0) = y_0$.

**Example 311.**—Connectedness is more crucial than compactness. Consider $E = [-2, -1] \cup [1, 2]$, a compact subset of $\mathbb{R}$. Define $f : E \longrightarrow \mathbb{R}$ as $f(x) = \begin{cases} -1 & \text{for } -2 \leq x \leq -1 \\ 1 & \text{for } 1 \leq x \leq 2 \end{cases}$. Although $f(-2) < 0 < f(2)$, there exists $x_0 \in E$ such that $f(x_0) = 0$.

**Example 312.**—The Intermediate Value Property (IVP) is not guaranteed for a discontinuous function. Consider $f : [-1,1] \longrightarrow \mathbb{R}$ defined by $f(x) = \begin{cases} -1 & \text{for } -1 \leq x \leq 0 \\ 1 & \text{for } 0 < x \leq 1 \end{cases}$. Although $f(-1) < 0 < f(1)$, there exists $x \in (-1,1)$ such that $f(x) = 0$.

**Example 313.**—The converse of Theorem 235 does not hold; a function may not satisfy IVP, and $f$ does not necessarily have to be continuous. Consider $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$. The function $\sin x$ is continuous, and $|x \sin x| \leq |x|$ for all $x \in \mathbb{R}$. However, $f$ is not continuous at $x = 0$. For any $x_1, x_2 \in \mathbb{R}$ and any $f(x_1) < y_0 < f(x_2)$, there exists $x_0 \in (x_1, x_2)$ such that $f(x_0) = y_0$.

## § 1.3 Discontinuities (on $\mathbb{R}$)

The primary goal is to classify types of discontinuities into two categories:

- When $\lim_{x\to c} f(x) \neq f(c)$,
- When $\lim_{x\to c} f(x)$ does not exist.

Define the left-hand limit as $f : (a,b) \subset \mathbb{R} \longrightarrow \mathbb{R}$ with $x \in (a,b]$. Then $f(x_-) = q$ or $\lim_{t\to x_-} f(t) = q$ if and only if:

i) For all $\epsilon > 0$, there exists $\delta = \delta(x,\epsilon) > 0$ such that if $x-\delta < t < x$, then $|f(t)-q| < \epsilon$.

ii) For all sequences $(t_n) \subset (a,x)$ such that $t_n \to x$, it implies $f(t_n) \to q$.

Similarly, $f(x_+) = q$ or $\lim_{t\to x_+} f(t) = q$.

**Definition.**—A function has a discontinuity of type 1 at $x$ if $f(x_+)$ and $f(x_-)$ exist. This is also called a simple discontinuity. Otherwise, it has a type 2 discontinuity.

**Example 314.**—Consider $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$. The function $f$ is continuous for $x \neq 0$, but $f$ is not continuous at $x = 0$. Let $(x_n) \subset (0,\infty)$ such that $x_n \to 0$. Then $f(x_n) = 1$ for all $n$, so $f(x_n) \to 1$. Similarly, let $(q_n) \subset (0,\infty)$ such that $q_n \to 0$. Then $f(q_n) = -1$ for all $n$, so $f(q_n) \to -1$. This implies a simple discontinuity at $x = 0$ since $f(0_-) = -1$ exists and $f(0_+) = 1$. (as mentioned above)

**Example 315.**—Consider $f : \mathbb{R} \longrightarrow \mathbb{R}$ given by $f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$. The function $f$ is not continuous at $x = 0$. It has a type 2 discontinuity. To show this, find sequences $(t_n), (q_n) \subset (0,\infty)$ such that $t_n \to 0$ and $q_n \to 0$, but $\lim_{n\to\infty} f(t_n) \neq \lim_{n\to\infty} f(q_n)$. Take $t_n = \frac{1}{2\pi n + \frac{\pi}{2}} \implies f(t_n) = 1$ for all $n$, so $f(t_n) \to 1$. Take $q_n = \frac{1}{2\pi n + \frac{3\pi}{2}} \to 0$ and $f(q_n) = -1$ for all $n$, so $f(q_n) \to -1$. Thus, $f(0_+)$ does not exist.

**Example 316.**—Consider $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined as $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ -1 & \text{if } x \notin \mathbb{Q} \end{cases}$. The function $f$ is not continuous on $\mathbb{R}$. Fix $x^* \in \mathbb{R}$ and show that $f$ is not continuous at $x^*$. Since $\mathbb{Q}$ and $\mathbb{Q}^c$ are dense in $\mathbb{R}$, there exist $q_n \in \mathbb{Q}$ and $t_n \in \mathbb{Q}^c$ such that $q_n \to x^* \implies f(q_n) \to 1$ and $t_n \to x^* \implies f(t_n) \to -1$, indicating that $f$ is not continuous at $x^*$. This discontinuity is of type 2.

**Example 317** (Illustration of a continuous function)**.**—Let $f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right); & x \neq 0 \\ 0; & x = 0 \end{cases}$ ✓

It is continuous for $x \neq 0$ since both $\frac{1}{x}$ and $\sin x$ are continuous. Is it continuous at $x = 0$?

$\epsilon - \delta$: Consider any $\epsilon > 0$,

$$|f(x) - f(0)| = \left| x \sin\left(\frac{1}{x}\right) - 0 \right| = |x \sin\left(\frac{1}{x}\right)| = |x| \underbrace{\leq 1}_{\sin\left(\frac{1}{x}\right)} \leq |x| < \delta = \epsilon$$

This implies $f$ is continuous at $x = 0$.

**Definition** (Monotonic functions)**.**—$f : E \subset \mathbb{R} \to \mathbb{R}$

(i) $f$ is monotonically increasing ($\uparrow$) on $E$ if $x, y \in E$ and $x < y$ imply $f(x) \leq f(y)$.

(ii) $f$ is monotonically decreasing ($\downarrow$) on $E$ if $x, y \in E$ and $x < y$ imply $f(x) \geq f(y)$.

**Example 318.**—$f(x) = 1$; $E = (a, b)$

**Example 319.**—$f(x) = x$; $E = \mathbb{R}$

**Example 320.**—$f(x) = e^x$; $E = \mathbb{R}$.

**Theorem.** *Let $f : (a, b) \to \mathbb{R}$ be $\uparrow$. Then*

*(i) $f(x_-)$ and $f(x_+)$ exist for all $x \in (a, b)$. More precisely,* $\sup_{a<t<x} f(t) = f(x_-) \leq f(x) \leq f(x_+) = \inf_{x<t<b} f(t)$.

*(ii) If $a < x < y < b$ then $f(x_+) \leq f(y_-)$.*

**Example 321.**—$f(x) = \begin{cases} 1 & x \in (a, x_0) \\ 2 & x \in [x_0, b) \end{cases}$ with $f(x_-) = 1$ and $f(x_+) = 2$.

**Theorem.** *Let $f : (a, b) \subset \mathbb{R} \to \mathbb{R}$ be monotonic. Let $E := \{x \in (a, b) \mid f \text{ is discontinuous at } x\}$. Then $E$ is at most countable.*

**Proof.** Without loss of generality, assume $f$ is $\uparrow$. If $f$ is discontinuous at finitely many points, then $E$ is finite. Claim: There is a one-to-one correspondence between $E$ and $\mathbb{Q}$ (countable). Let $x \in E$. Since $f$ is $\uparrow$, the discontinuity is of the first kind. By Theorem 236(i), it follows that

$$f(x_-) < f(x_+).$$

Since $\mathbb{Q}$ is dense in $\mathbb{R}$, $\exists r(x) \in \mathbb{Q}$ such that $f(x_-) < r(x) < f(x_+)$. It remains to show that this correspondence is one-to-one, i.e., $x_1 \neq x_2 \implies r(x_1) \neq r(x_2)$ for any $x_1, x_2 \in E$. Assume $x_1 < x_2$ (WLOG). Theorem 236(ii) implies $f(x_{1+}) \leq f(x_{2-})$. Now, $f \uparrow$ with $x_1 < x_2$ implies

$$f(x_{1-}) < r(x_1) < f(x_{1+}) \leq f(x_{2-}) < r(x_2) < f(x_{2+}) \implies r(x_1) < r(x_2),$$

i.e., $r(x_1) \neq r(x_2)$. So $E$ is countable.

## § 2 Derivatives

**Definition.**—Let $f : [a, b] \to \mathbb{R}$ and $x \in [a, b]$. $f$ is said to be differentiable at $x$ if $f'(x) := \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ exists, or equivalently, if $f'(x) = \lim_{t\to x} \frac{f(t)-f(x)}{t-x}$. One-sided derivatives:

if exists

f'(x$_-$) $= \lim_{h\to 0^-} \frac{f(x+h)-f(x)}{h}$
$= \lim_{t\to x_-} \frac{f(t)-f(x)}{t-x}$ if exists

$$f'(x_+) = \lim_{h\to 0^+} \frac{f(x+h)-f(x)}{h} \quad = \lim_{t\to x_+} \frac{f(t)-f(x)}{t-x}$$

if exists
f'(x$_-$) $= \lim_{h\to 0^-} \frac{f(x+h)-f(x)}{h}$
$= \lim_{t\to x_-} \frac{f(t)-f(x)}{t-x}$ if exists

$f'(x)$ exists if and only if $f'(x_-) = f'(x_+)$.

**Example 322.**—$f(x) = |x|$ is continuous on $\mathbb{R}$ but not differentiable at $x = 0$. Indeed,

$$\lim_{h\to 0_+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0_+} \frac{|h|}{h} = 1$$
$$\lim_{h\to 0_-} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0_-} \frac{|-h|}{h} = -1$$

implies $f'(0)$ does not exist.

**Example 323.**—$f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right); & x \neq 0 \\ 0; & x = 0 \end{cases}$ is continuous on $\mathbb{R}$. Is $f$ differentiable at $x = 0$?

$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{h\sin\left(\frac{1}{h}\right)}{h} = \lim_{h\to 0} \sin\left(\frac{1}{h}\right)$$

does not exist, implying $f'(0)$ does not exist.

**Example 324.**—$f(x) = \begin{cases} x^2\sin\left(\frac{1}{x}\right); & x \neq 0 \\ 0; & x = 0 \end{cases}$. Is $f$ differentiable at $x = 0$?

$$f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{h^2\sin\left(\frac{1}{h}\right)}{h} = 0$$

because $x\sin\left(\frac{1}{x}\right)$ is continuous at $x = 0$ and $= 0$ at $x = 0$.

**Theorem.** *Let $f : [a, b] \to \mathbb{R}$ and $x \in [a, b]$. If $f$ is differentiable at $x$, then $f$ is continuous at $x$.*

**Proof.** Suppose $f$ is differentiable at $x$. NTS: $\lim_{t\to x} f(t) = f(x)$. Consider $\frac{f(t)-f(x)}{t-x}\cdot(t-x)$ for $t \neq x$. Using limit properties and $f'(x) = \lim_{t\to x} \frac{f(t)-f(x)}{t-x}$, we get

$$\lim_{t\to x}[f(t) - f(x)] = f'(x) \cdot 0 \implies \lim_{t\to x} f(t) = f(x)$$

implies $f$ is continuous at $x$.

**Remark 40.**—The converse is not true – see examples 323 and 324.

Properties of differentiable functions:

**Theorem.** *Let $f, g : [a, b] \to \mathbb{R}$. Suppose $f$ and $g$ are differentiable at $x \in [a, b]$. Then $f + g$, $f \cdot g$, and $\frac{f}{g}$ (where $g(x) \neq 0$) are differentiable at $x$. Moreover,*

*(i)* $(f + g)'(x) = f'(x) + g'(x)$

*(ii)* $(fg)'(x) = f(x) \cdot g'(x) + g(x) \cdot f'(x)$

*(iii)* $\left(\frac{f}{g}\right)'(x) = \frac{g(x)\cdot f'(x) - f(x)\cdot g'(x)}{[g(x)]^2}$

**Proposition 31** (Power rule)**.**—$f(x) = x^n \implies f'(x) = nx^{n-1}$ for $n \in \mathbb{R}$.

- If $n \in \mathbb{N}$, use product rule and induction.
- If $n \in \mathbb{Z}$, use quotient rule $\frac{1}{x^n}$.

- If $n \in \mathbb{Q}$, assume chain rule, then $n = \frac{p}{q}$ for $p, q$

$\in \mathbb{Z}$, $q \neq 0$. Let $y = x^{\frac{p}{q}} \implies y^q = x^p \implies qy^{q-1} \cdot y'(x) = px^{p-1}$ (chain rule) $\implies y'(x) = \frac{p}{q}\frac{x^{p-1}}{y^{q-1}} = \frac{p}{q} \cdot \frac{x^{p-1}}{x^{\frac{p}{q}(q-1)}} = \frac{p}{q} \cdot \frac{x^{p-1}}{x^{p-p/q}} x^{\frac{p}{q}-1} = \frac{p}{q} x^{\frac{p}{q}-1} = nx^{n-1} = y'(x)$.

- If $n \in \mathbb{R}$, $f(x) = x^n = e^{\ln x^n} = e^{n \ln x}$ (assuming differentiability of $e^x$ and $\ln x$) $\implies f'(x) = e^{n \ln x} \cdot \frac{n}{x} = x^n \cdot \frac{n}{x} = nx^{n-1} = f'(x)$ for $n \neq 0$.

**Theorem.** *Suppose*

- *$f$ is continuous on $[a, b]$,*
- *$f'(x)$ exists at some $x \in [a, b]$,*
- *$g$ is defined on some interval $I$ containing $f([a, b])$,*
- *$g$ is differentiable at $f(x)$.*

*Then $g \circ f$ is differentiable at $x$ and $(g \circ f)'(x) = g'(f(x)) \cdot f'(x)$.*

**Proof.** First, $f$ differentiable at $x$ means $f'(x) = \lim_{tx} \frac{f(t)-f(x)}{t-x}$ exists. This implies

$$f(t) - f(x) = [f'(x) + u(t)](t - x),$$

where $u(t)0$ as $tx$. Then $g$ differentiable at $f(x)$ implies $g'(f(x)) = \lim_{sf(x)} \frac{g(s)-g(f(x))}{s-f(x)}$ exists

$$\implies g(s) - g(f(x)) = [g'(f(x)) + v(s)](s - f(x)),$$

where $\boxed{v(s)0 \text{ as } sf(x)}$. NTS:

$$\lim_{tx} \frac{g(f(t)) - g(f(x))}{t - x} = g'(f(x)) \cdot f'(x).$$

Finish next time.

**Remark 41.**—The proof is a bit hand wavy, look for better proof.

**Recall 76.**—$g(f(x))' = g'(f(x)) \cdot f'(x)$.

**Proof.** Since $f$ is differentiable at $x$, $f'(x) = \lim_{tx} \frac{f(t)-f(x)}{t-x}$ exists.

So $\boxed{f(t) - f(x) = (t - x)[f'(x) + u(t)], \text{ where } u(t)0 \text{ as } tx}$ (1). $g$ differentiable at $f(x)$.

So $\boxed{g(s) - g(f(x)) = (s - f(x))[g'(f(x)) + v(s)],\ v(s)0 \text{ as } sf(x)}$ (2).

Want to show: $\lim_{tx} \frac{g(f(t))-g(f(x))}{t-x} = g'(f(x)) \cdot f'(x)$. Using (2), for $t \neq x$ with $s = f(t)$ $g(f(t)) - g(f(x)) = (f(t) - f(x))[g'(f(x)) + v(f(t))]$, where $v(f(t))0$ as $tx$ since $f(t)f(x)$ as $tx$ by continuity of $f$. missing CONTENT as $tx$. Therefore $\lim_{tx} \frac{g(f(t))-g(f(x))}{t-x} = g'(f(x)) \cdot f'(x)$.

**Remark 42.**—Rolle's theorem is a special case for the MVT for the existence of a zero slope secant line.

## § 2.1 Mean Value Theorem

Local max/min: Suppose $f : X^{m.s}R$. We say that $f$ has a local max (min) at $x \in X$ if $\exists \delta > 0$ such that $f(t) \leq (\geq) f(x)$ for all $t \in B_\delta(x)$.

**Theorem.** *Let $f : [a, b]R$. If*

- *$f$ has a local max (min) at $x \in (a, b)$,*
- *$f$ differentiable at $x$,*

*then $f'(x) = 0$.*

**Proof.** We will compute $f'(x_-)$ and $f'(x_+)$. Since $f$ has a local maximum at $x$, $\exists \delta > 0$ such that if $t \in (x - \delta, x + \delta)$ then $f(t) \leq f(x)$.

$f'(x_-)$: Let $t \in (x - \delta, x)$. Then

$$\frac{\boxed{f(t) - f(x)}^{\leq 0}}{< 0 t - x} \implies lim_{tx_-} \frac{f(t) - f(x)}{t - x} \geq 0 \qquad (f'(x_-))$$

$f'(x_+)$: Let $t \in (x, x + \delta)$. Then $\lim_{tx_+} \frac{(f(t)-f(x))^{\leq 0}}{(t-x)^{>0}}$ $(f'(x_+))$ But $f'(x)$ exists, so $f'(x_+) = f'(x_-) = 0 = f'(x)$.

**Theorem Generalized MVT.** *Let $f, g : [a, b]R$ be*

- *continuous on $[a, b]$,*
- *differentiable on $(a, b)$.*

*Then there exists $x \in (a, b)$ such that $[f(b) - f(a)]g'(x) = [g(b) - g(a)]f'(x)$.*

**Remark 43.**—MVT: $\frac{f(b)-f(a)}{b-a} = f'(x)$; $x \in (a, b)$, $f(b) - f(a) = (b - a)f'(x)$. Standard MVT: $g(x) = x$. Rolle's theorem: $g(x) = x$ and $f(b) = f(a)$.

**Proof.** Define $h(t) = [f(b) - f(a)]g(t) - [g(b) - g(a)]f(t)$ for $t \in [a, b]$. Then $h$ is continuous on $[a, b]$ and differentiable on $(a, b)$. Also,

$$\begin{aligned} h(a) &= [f(b) - f(a)]g(a) - [g(b) - g(a)]f(a) \\ &= f(b)g(a) - g(b)f(a) \\ h(b) &= [f(b) - f(a)]g(b) - [g(b) - g(a)]f(b) \\ &= f(b)g(a) - g(b) - f(a) = h(a) \end{aligned}$$

Now, if $h(b) = h(a) = h(t)$ for every $t \in [a, b]$, then $h'(t) = 0$ for every $t \in (a, b)$✓. Otherwise, $h(t)$ attains either a local maximum or a local minimum at some point $x \in (a, b)$. Then $h'(x) = 0$ by Theorem 241 i.e., $[f(b) - f(a)]g'(x) = [g(b) - g(a)]f'(x)$.

**Theorem Derivative and monotonicity.** *Suppose $f : [a, b]R$ continuous and differentiable on $(a, b)$. Then* *(i) if $f'(x) \geq 0$ for all $x \in (a,b)$ then $f$ is monotonically increasing on $(a,b)$,*

*(ii) if $f'(x) \leq 0$ for all $x \in (a,b)$ then $f$ is monotonically decreasing on $(a,b)$,*

*(iii) if $f'(x) = 0$ for all $x \in (a,b)$ then $f$ is constant.*

**Proof.** (i) Suppose $x_1, x_2 \in (a,b)$ with $x_1 < x_2$. NTS: $f(x_1) \leq f(x_2)$ if $f'(x) \geq 0$. By applying MVT on $[x_1, x_2]$, $\exists x \in (x_1, x_2)$ such that $\geq 0 f'(x) = \frac{f(x_2)-f(x_1)}{(x_2-x_1)>0} \implies f(x_2) - f(x_1) \geq 0$ i.e., $f(x_1) \leq f(x_2)$.

**Example 325.**—Examples of functions $f$: Derivative exists but Derivative not continuous.

**Theorem Intermediate value property for derivative.** *Suppose $f : [a,b]R$ is differentiable and $\exists \lambda$ such that $f'(a) < \lambda < f'(b)$. Then $\exists x \in (a,b)$ such that $f'(x) = \lambda$.*

**Note 95.**—$f'$ exists $\not\Longrightarrow$ $f'$ continuous, therefore IVT for continuous function cannot be applied.

**Recall 77** (IVP for $f'$)**.**—Suppose $f : [a,b]R$ differentiable, and $f'(a) < \lambda < f''(b)$ $(f'(a) \neq f'(b))$. Then $\exists x_0 \in (a,b)$ such that $f'(x_0) = \lambda$.

**Proof.** Define $h(x) = f(x) - \lambda x$ for $x \in [a,b]$. Then $h$ is differentiable on $[a,b]$ since $f$ is. Also $h'(x) = f'(x) - \lambda$. Then $h'(a) = f'(a) - \lambda < 0$ and $h'(b) = f'(b) - \lambda > 0$. (We do not know if $f'(x)$ is continuous, so IVT for continuous functions cannot be applied.) Therefore there exists a point $x \in (a,b)$ where $h$ achieves a local minimum. Since $h$ is differentiable, $f'(x) = 0$ or $f'(x) = \lambda$.

*Corollary.*—Suppose $f : [a,b]R$ is differentiable. Then $f'$ cannot have any simple discontinuity.

*Homework (proof).* By contradiction. Suppose $f'$ is discontinous at $x \in (a,b)$.
Case I: Let $f'(x_-) = f'(x_+) < f'(x)$ WLOG. Case II: Let $f'(x_-) < f'(x_+)$ WLOG.

*Corollary.*—Let $f : I \subset RR$ have bounded derivative. Then $f$ is uniformly continuous on $R$.

**Proof.** $f$ has bounded derivative on $I$, so $\exists M > 0$ such that $|f'(x)| \leq M$ for all $x \in I$. Let $\epsilon > 0$. NTS $\exists \delta > 0$ such that $\forall x, y \in I$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$. By MVT, $\exists c \in (x,y)$ (WLOG) such that $f(x) - f(y) = f'(c)(x-y)$ implies

$$|f(x) - f(y)| = |f'(c)|\,|x - y| \leq M|x - y| < \epsilon$$

if $|x - y| < \delta = \frac{\epsilon}{M}$. Therefore $f$ is uniformly continuous on $I$.

**Example 326.**—$f(x) = x$ is uniformly continuous on $R$ since $|f'(x)| = 1$ on $R$.

**Example 327.**—$f(x) = \sin x$ or $\cos x$ uniformly continuous on $R$ since $|f'(x)| = |\sin x|$ or $|\cos x| \leq 1$ on $R$.

**Example 328.**—$f(x) = \sqrt{x}$ on $[1, \infty)$ – calculate $|f'(x)|? \leq ?$

**Example 329.**—$f : I \subset RR$ is uniformly continuous on $I$ if and only if $\forall \epsilon > 0$, $\exists \delta > 0$ such that $\forall x, y \in I$ with $0 < |x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$.

Converse: $f$ is not uniformly continuous on $I$ if and only if $\exists \epsilon_0$ such that $\forall \delta > 0$, $\exists x_0, y_0 \in I$ with $|x_0 - y_0| < \delta$ but $|f(x_0) - f(y_0)| \geq \epsilon$.

**Example 330.**—$f(x) = \frac{1}{x}$ is not uniformly continuous on $(0, 1)$. Pick $\epsilon_0 = 1$. Fix $\delta > 0$ (arbitrary) such that $0 < \delta < 1$. Find $x_0, y_0$. Take $x_0 = \delta, y_0 = \frac{\delta}{2}$. Then $|x_0 - y_0| = \frac{\delta}{2}$ and $|f(x_0) - f(y_0)| = \frac{1}{\delta} - \frac{2}{\delta} = \frac{1}{\delta} > 1 = \epsilon_0$.

**Example 331.**—$f(x) = x^2$ is not uniformly continuous on $R$. Fix $\epsilon_0 = 1$ and $\delta > 0$. Let $x_0 = \frac{1}{\delta}, y_0 = \frac{1}{\delta} + \frac{\delta}{2}$, which implies $|x_0 - y_0| = \frac{\delta}{2} < \delta$. Then

$$|f(x_0) - f(y_0)| = \frac{1}{\delta^2} - \frac{1}{\delta} + \frac{\delta}{2}^2 = \frac{1}{\delta^2} - \frac{1}{\delta^2} - 2 \cdot \frac{1}{\delta} \cdot \frac{\delta}{2} - \frac{\delta^2}{4} = 1 + \frac{\delta^2}{4} > 1 = \epsilon_0.$$

## § 2.2 Higher order derivatives

**Example 332.**—$f(x) = \begin{cases} \sin \frac{1}{x}; & x \neq 0 \\ 0; & x = 0 \end{cases}$.

✓ continuous at $x \neq 0$.

? $x = 0$ $\lim_{x0} f(x) =$ does not exist $\implies$ not continuous at $x = 0$.

Note $\lim_{xc} f(x) = L$ if and only if $\forall$ sequences $(x_n) \subset R$ such that $x_n c$ implies $f(x_n) L$.

Then $x_n = \frac{1}{2\pi n} 0$ and $f(x_n) = \sin(2\pi n) = 0$ for all $n$, so 0. Also $y_n = \frac{1}{2\pi n + \frac{\pi}{2}} 0$ and $f(x_n) = \sin(2\pi n + \frac{\pi}{2}) = 1$ for all $n$, so 1. This implies $\lim_{x0} f(x)$ does not exist.

**Theorem Taylor's theorem (Calculus version).** *Suppose $f$ has $(n + 1)$-continuous derivatives on an open interval containing $a \in I \subset R$. Then for each $x \in I$,*

$$f(x) = n^{th} \textit{ degre polynomial} \sum_{k=0}^{\infty} \frac{f^{(k)}}{k!} (x - a)^k + R_{n+1}(x),$$

*where $R_{n+1}(x) := \frac{f^{(n+1)}(z}{(n+1)!} (x - a)^{n+1}$ for some $z$ between $a$ and $x$.*

Notation: (function space)

$$\begin{aligned} f : [a, b] R \text{ continuous} &= C([a, b]), \\ f : [a, b] R \text{ such that } f' \text{ is continuous} &= C^1([a, b]), \\ &\vdots \\ f : [a, b] R \text{ such that} f^{(k)} \text{ is continuous} &= C^k([a, b]), \end{aligned}$$

where $C([a, b]) \subset C^1([a, b]) \subset \cdots \subset C^k([a, b])$.

**Example 333.**—$e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots = \sum_{k=0}^{\infty}\frac{x^k}{k!}$ – Mclaurin seriesTaylor series at $a=0$

$R_{n+1}(x) := \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}$ for some $c$ between and 0 and $x$. $?R_{n+1}(x)0$ as $n\infty$ (ratio test).

$\sum_{k=0}^{\infty}\frac{x^k}{k!}$ converges by Ratio test: $\lim_{k\infty}\frac{|x|^{k+1}}{(k+1)!}\cdot\frac{k!}{|k|^k} = \lim_{k\infty}\frac{|x|}{k+1} = 0 < 1$ for all $x \in R$. $\lim_{n\infty}\frac{x^{n+1}}{(n+1)!} = 0$ and $|f^{(n+1)}(c)| = e^c$ is bounded for $c$ between 0 and $x$: $\implies R_{n+1}0$ as $n\infty$.

**Theorem Taylor's theorem.** *Let $f : [a,b]R$. Suppose*

- $f^{(n-1)}$ *continuous on* $[a,b]$,
- $f^{(n)}$ *exists* $\forall t \in (a,b)$.

*Let $\alpha, \beta \in [a,b]$ and define*

$$P(t) := \sum_{k=0}^{n-1}\frac{f^{(k)}(\alpha)}{k!}(t-\alpha)^k.$$

*Then $\exists x$ between $\alpha$ and $\beta$ such that*

$$f(\beta) = P(\beta) + \textit{Lagrange Remainder (1797)}\frac{f^{(n)}(x)}{n!}(\beta-\alpha)^n.$$

**Proof.** WLOG assume $\alpha \neq \beta$. Define $M := \frac{f(\beta)-P(\beta)}{(\beta-\alpha)^n}$. Define $g(t) = f(t) - P(t) - M(t-\alpha)^n$; $a \leq t \leq b$.

$$P(t) = f(\alpha) + f'(\alpha)(t-\alpha) + \frac{f''(\alpha)}{2!}(t-\alpha)^2 + \cdots + \frac{f^{(n-1)}(\alpha)}{(n-1)!}\cdot(t-\alpha)^{n-1}.$$

✓ $g^{(n-1)}$ continuous on $[a,b]$ and $g^{(n)}$ exists $\forall t \in [a,b]$.

$g(\alpha) = f(\alpha) - P(\alpha) - M\cdot 0 = 0$ since $(f(\alpha) = P(\alpha))$.

$g'(\alpha) = f'(\alpha) - P'(\alpha)) - n\cdot M\cdot 0$.

$\vdots$

$g^{(n-1)}(\alpha) = f^{(n-1)}(\alpha) - P^{(n-1)}(\alpha) - n!M\cdot 0$.

$\implies \boxed{g^{(k)}(\alpha) = 0 \text{ for } k = 0,\ldots,n-1.}$ Now,

$$g(\beta) = f(\beta) - P(\beta) - M(\beta-\alpha)^n = f(\beta) - P(\beta) - \frac{f(\beta)-P(\beta)}{(\beta-\alpha)^n}(\beta-\alpha)^n$$

We have: $g(x) = 0$ and $g(\beta) = 0$. Rolle's $\implies \exists x_1$ between $\alpha$ and $\beta$ such that $g'(x_1) = 0$. Again, $g'(\alpha) = 0$ and $g'(x_1) = 0$Rolle's $\implies \exists x_2$ between $x_1$ and $\alpha$ such that $g''(x_2) = 0$. Inductively, $\exists x_n$ between $x_{n-1}$ and $\alpha$ such that $g^{(n)}(x_n) = 0 \implies f^{(n)}(x_n) - = 0P^{(n)}(x_n) - M\cdot n! = 0 \implies M = \frac{f^{(n)}(x_n)}{n!} \implies \frac{f(\beta)-P(\beta)}{(\beta-\alpha)^n} = \frac{f^{(n)}(x_n)}{n!}$.

**Theorem Inverse function theorem.** *Suppose*

- $f : (a, b)(c, d)$ *is differentiable*
- $f$ *is surjective*
- $f'(x) \neq 0$ *for all* $x \in (a, b)$

*Then*

* $f$ *is a homeomorphism*
* $f$ *is differentiable on* $(c, d)$, *and*
* $(f)'(y) = \frac{1}{f'(f(y))}$.

*i.e.,* $f$ *is a diffeomorphism.*

**Example 334.**—$f(x) = \sin x$; $(-\frac{\pi}{2}, \frac{\pi}{2})$. $f'(x) = \cos x$, $f(x) = \sin x$.

$$(f)'(x) = \frac{1}{f'(f(x))} = \frac{1}{\cos(\sin x)} = \frac{1}{\sqrt{1 - [\sin(\sin x)]^2}} = \frac{1}{\sqrt{1 - x^2}}.$$

**Example 335.**—$f(x) = e^x$; $(-\infty, \infty)$. $f'(x) = e^x$, $f(x) = \ln x$; $(0, \infty)$.

$$(f)'(x) = \frac{1}{f'(f(x))} = \frac{1}{e(\ln x)} = \frac{1}{x}.$$

$C_0([a, b]) = f : [a, b]\text{cts}Rf(a) = 0,\ f(b) = 0.$

**Recall 78** (Inverse function theorem in $R$)**.**—Suppose

- $f : (a, b)(c, d)$ is differentiable
- $f$ is surjective
- $f'(x) \neq 0$ for all $x \in (a, b)$

Then

# $f$ is a homeomorphism

# $f$ is differentiable on $(c, d)$
$(f)'(y) = \frac{1}{f'(f(y))}$.

**Proof.** WLOG assume $f' > 0$ on $(a,b)$. First we will show that $f$ is injective. Let $a < t < s < b$ (i.e., $t \neq s$ WLOG). By MVT, $\exists x \in (t,s)$ such that $f(s) - f(t) = > 0 f'(x) > 0 (s-t) \implies f(s) > f(t) \implies f$ is injective. Then $f$ differentiable on $(a,b) \implies f$ continuous on $(a,b) \implies f$ is a homeomorphism ($f$ bijective, $f$ continuous $\implies f$ continuous). Now show $f$ is differentiable and $(*)$ holds. Note that $f$ differentiable on $(a,b)$, so for $t \neq x$

$$f'(x) = \lim_{tx} \frac{\neq 0 f(t) - f(x)}{\neq 0 t - x}$$

holds. This implies

$$\frac{1}{f'(x)} = \lim_{tx} \frac{t-x}{f(t)-f(x)}; \qquad x \in (a,b). \qquad -(1)$$

Fix $\epsilon > 0$. NTS: $\exists \eta > 0$ such that $\forall s \in (c,d)$ with

$$|s-y| < \eta \implies \frac{f(s)-f(y)}{s-y} - \frac{1}{f'(x)} < \epsilon,$$

where $\boxed{y = f(x)}$. For fixed $\epsilon > 0$, (1) implies that $\exists \delta > 0$ such that $\forall t \in (a,b)$ with $|t-x| < \delta$,

$$\frac{t-x}{f(t)-f(x)} - \frac{1}{f'(x)} < \epsilon \qquad -(2)$$

By using the continuity of $f$, $\exists \eta > 0$ such that $\forall s \in (c,d)$ with $|s-y| < \eta$, $|f(s)-f(y)| < \delta$. For $s \in (c,d)$ with $|s-y| < \eta$, using (2)

$$\begin{aligned} & \frac{f(s)-x}{f(f(s))-f(x)} - \frac{1}{f'(x)} < \epsilon, \qquad \boxed{y = f(x)} \\ \implies & \frac{f(s)-f(y)}{s-y} - \frac{1}{f'(x)} < \epsilon. \\ \implies & \lim_{sy} \frac{f(s)-f(y)}{s-y} = \frac{1}{f'(x)} = \boxed{(f)'(y)}. \end{aligned}$$

## § 2.3 Vector-valued function

: $[a,b] \subset RR^k$, $(t) = (\in Rf_1(t), \in Rf_2(t), \ldots, \in Rf_k(t)) \in R^k$

**Example 336.**—:

1. $(t) = (\cos t, \sin t)$ – unit circle
2. $(t) = (\cos t, \sin t, t)$ – helix

3. $(t) = \cos t + i \sin t$

Vector-valued functions enjoy similar (continuity).
Differentiability:

✓ is differentiable if and only if $f_i$ is differentiable for each $i = 1, \ldots, k$.

✓ + differentiable if + are.

✓✓ $\cdot$ (inner product) is differentiable if and are.

Caution: MVT and book! (example)L'Hospital's rule may not necessarily hold.

**Example 337.**—$f(x) = e^{ix} = \cos x + i \sin x$; $[0, 2\pi]$

✓ $f(0) = 1 = f(2\pi)$

✓ $f$ differentiable

✓ $|f'(x)| = |ie^{ix}| = 1 \neq 0 \implies f'(x) \neq 0$ for all $x \in (0, 2\pi)$.

## § 2.4 Mean Value Theorem estimate

**Theorem.** *Suppose*

- $: [a, b] R^k$ *continuous,*
- *differentiable on* $(a, b)$.

*Then* $\exists x \in (a, b)$ *such that* $|(b) - (a)|_{R^k} \leq (b - a)('(x))_{R^k}$.

**Proof.** Define $g(t) = \in R(\in R^k(t) - (b)) \cdot \in R^k(t)$; $t \in [a, b]$. Then

✓ $g$ continuous on $[a, b]$

✓ $g$ differentiable on $[a, b]$

by hypothesies on . By MVT, $\exists x \in (a, b)$ such that $\boxed{(b - a)g'(x) = g(b) - g(a)}$ – (1). Now,

$$\begin{aligned} g(b) - g(a) &= ((b) - (a)) \cdot (b) - ((b) - (a)) \cdot (a) \\ &= ((b) - (a)) \downarrow \text{IP} \cdot ((b) - (a)) \\ &= |(b) - (a)|^2_{R^k}, \qquad (||^2 = \cdot) \end{aligned}$$

Also, $g'(x) = ((b) - (a)) \cdot' (x)$. Plugging into (1), we get

$$\begin{aligned} |(b) - (a)|^2_{R^k} &= |((b) - (a)) \cdot' (x)(b - a)|_R \\ &= |((b) - (a)) \cdot (x)|(b - a) \\ \text{C.S.} &\leq |(b) - (a)|_{R^k} \cdot |'(x)|_{R^k}(b - a), \end{aligned}$$

which implies $|(b) - (a)| \leq (b - a)|(x)|_{R^k}$.

# § 3 Riemann-Stieltjes Integral

✓ Riemann Integral (Integral Calculus)

✓✓ Riemann-Stieltjes integral (more general than Riemann Integral)

(∗) Lebesgue integral (measure theory)

Assume: $f : [a, b]R$ is bounded $\implies$ $\exists m, n \in R$ such that $m := \inf_{[a,b]} f(x)$ and $M := \sup_{[a,b]} f(x)$ $\implies$ $m(b-a) \leq M(b-a)$. Partition $P$ of $[a, b]$: $a = x_0 \leq$ $n-1$ partition points$x_1 \leq \cdots \leq x_{n-1} \leq x_n = b$ with $\boxed{\Delta x_i = x_i - x_{i-1} \geq 0}$. $f$ bounded on $[a, b] \implies f$ bounded on $[x_{j-1}, x_j]$; $j = 1, \ldots, n$.

Define $m_j := \inf_{[x_{j-1},x_j]} f(x)$ and $M_j := \sup_{[x_{j-1},x_j]} f(x)$. Clearly $m_j \leq M_j$ for each $j = 1, \ldots, n$.

Then for given $P$ and $f$ on $[a, b]$,

$L(P, f) = \sum_{i=1}^{n} m_i \Delta x_i$ – lower Riemann sum,

$U(P, f) = \sum_{i=1}^{n} M_i \Delta x_i$ – upper Riemann sum.

Clearly, $\boxed{L(P, f) \leq U(P, f)}$.

Uppwer and lower Riemann integrals:

$_{-}\int_a^b f\,dx := \sup_{P\in P} L(P, f)$ – lower Riemann integral,

$^{-}\int_a^b f\,dx := \inf_{P\in P} U(P, f)$ – uppper Riemann integral,

where sup and inf are taken over all partitions $P$ of $[a, b]$ and $P :=$ set of all partitions of $[a, b]$.

**Lemma.**—$_{-}\int_a^b f\,dx \leq {}^{-}\int_a^b f\,dx$.

**Proof.** Let $P$ and $P_1$ be any two partitions of $[a, b]$. Then $m(b-a)✓ \leq L(P_1, f)✓ \leq U(P, f)✓ \leq M(b-a)$. Since $P_1$ is arbitrary, $m(b-a) \leq L(P_1, f) \leq \inf_{P\in P} U(P, f)$. Repeating the argument,

$$m(b-a) \leq \sup_{P\in P} L(P, f) \leq \inf_{P\in P} U(P, f) \leq M(b-a).$$

Claim follows by definition.

**Definition** (Riemann integral)**.**—$f : [a, b]R$ bounded is Riemann integrable if $_{-}\int_a^b f\,dx = {}^{-}\int_a^b f\,dx$. Notation: $R =$ set of all Riemann integrable functions.

$$L^1([a, b]) = f : [a, b]Rf \in R \text{ and } ||f||_{L^1} := \int_a^b |f|\,dx..$$

Riemann-Stieltjes integral:

✓ $f : [a, b]R$ bounded,

$\alpha : [a, b]R$ monotonically increasing ($\alpha(x) = x$ is the case for Riemann integral),

$P : a = x_0 \leq x_1 \leq \cdots \leq x_n = b$ a partition of $[a, b]$.

Then $\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1}) \geq 0$ for all $i = 1, \ldots, n$. Consider,

$$\implies L(P, f, \alpha) \leq U(P, f, \alpha)$$

$$U(P, f, \alpha) := \sum_{i=1}^{n} M_i \Delta\alpha_i L(P, f, \alpha) := \sum_{i=1}^{n} m_i \Delta\alpha_i \implies L(P, f, \alpha) \leq U(P, f, \alpha)$$

since $M_i \geq m_i$, $\Delta\alpha_i \geq 0$; $i = 1, \ldots, n$. Define $_-\int_a^b f\,d\alpha := \sup_{P \in P} L(P, f, \alpha)$ and $^-\int_a^b f\,d\alpha := \inf_{P \in P} U(P, f, \alpha)$.

**Definition** (Riemann Stieltjes integral)**.**—$f$ is Riemann-Stieltjes integral if $_-\int_a^b f\,d\alpha = {}^-\int_a^b f\,d\alpha$, where $R(\alpha) =$ set of all Riemann-Stieltjes integrable functions.

**Remark 44.**—For a uniform partition, $\Delta x = \frac{b-a}{n}$ and $x_k = a + k \cdot \Delta x$.

Refinement of a partition $P : a = x_0 \leq \cdots \leq x_n = b$ is another partition $P^*$ such that $P^* \supset P$. Common refinement: $P^*$ is a common refinement of two partitions $P_1$ and $P_2$ if $P^* = P_1 \cup P_2$.

**Theorem.** *If $P^*$ is a refinement of $P$, then*

$L(P, f, \alpha) \leq L(P^*, f, \alpha)$ — *(1)*

$U(P, f, \alpha) \geq U(P^*, f, \alpha)$ — *(2)*

**Proof.** Let $P : a = x_0 \leq \cdots \leq x_n = b$ be a partition of $[a, b]$ and $P^*$ be a refinement of $P$ with just one more point, say $x^*$. Assume $x_{j-1} < x^* < x_j$ for some $j = 1, \ldots, n$. Then for $x_{j-1} < x^* < x_j$ we have $w_1 := \inf_{[x_{j-1}, x^*]} f(x)$ and $w_2 := \inf_{[x^*, x_j]} f(x)$ and $m_j = \inf_{[x_{j-1}, x_j]} f(x)$ imply $m_j \leq w_1$ and $m_j \leq w_2$, so that

$$\begin{aligned}
L(P^*, f, \alpha) &= m_1 \Delta\alpha_1 + \cdots + m_{j-1}\Delta\alpha_{j-1} \\
&\quad + j^{\text{th}} w_1 \Delta\alpha^* + w_2 \Delta\alpha_j \\
&\quad + m_{j+1}\Delta\alpha_{j+1} + \cdots + m_n \Delta\alpha_n \\
L(P, f, \alpha) &= (m_1 \Delta\alpha_1 + \cdots + m_n \Delta\alpha_n) \\
L(P^*, f, \alpha) - L(P, f, \alpha) &= w_1(\alpha(x^*) - \alpha(x_{j-1})) + w_2(\alpha(x_j) - \alpha(x^*)) \\
&\quad - m_j(\Delta\alpha_j \alpha(x_j) - \alpha(x_{j-1})) \\
&= (\geq 0 w_1 - m_j)(\geq 0 \alpha(x^*) - \alpha(x_{j-1})) \\
&\quad + (\geq 0 w_2 - m_j)(\geq 0 \alpha(x_j) - \alpha(x^*)) \geq 0
\end{aligned}$$

Repeating this process we get $L(P^*, f, \alpha) \geq L(P, f, \alpha)$. Similarly (2).

**Recall 79.**—$M_i = \sup_{[x_{i-1},x_i]} f(x)$, $m_i = \inf_{[x_{i-1},x_i]} f(x)$, then

$$U(p,f,\alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i L(P,f,\alpha) = \sum_{i=1}^{n} m_i \Delta\alpha_i \Delta\alpha i = \alpha(x_i) - \alpha(x_{i-1}) \geq 0$$

With $_-\int_a^b f\,d\alpha = \sup_{p\in P} L(P,f,\alpha)$ and $^-\int_a^b f\,d\alpha = \inf_{p\in P} U(P,f,\alpha)$. Then $f \in R(\alpha)$ if and only if

$$\boxed{_-\int_a^b f\,d\alpha = {}^-\int_a^b f\,d\alpha.}$$

**Theorem.** $_-\int_a^b f\,d\alpha \leq {}^-\int_a^b f\,d\alpha$.

**Proof.** Let $P_1, P_2$ be any two partitions of $[a,b]$. Let $P^* = P_1 \cup P_2$ (common refinement). Then

$$L(P_1,f,\alpha)\text{refinement} \leq L(P^*,f,\alpha) \leq U(P^*,f,\alpha)\text{refinement} \leq U(P_2,f,\alpha).$$

Fix $x \in P_2$. Then $U(P_2,f,\alpha)$ is an upper bound for $L(P,f,\alpha)$ for any $P \in P$. Then

$$_-\int_a^b f\,d\alpha \sup_{P\in P} L(P,f,\alpha) \leq U(P_2,f,\alpha).$$

Since $_-\int_a^b f\,d\alpha \leq U(P_2,f,\alpha)$ and $P_2 \in P$ is arbitrary, $_-\int_a^b f\,d\alpha \leq \inf_{P\in P} U(P,f,\alpha) = {}^-\int_a^b f\,d\alpha$ implies $_-\int_a^b f\,d\alpha \leq^- \int_a^b f\,d\alpha$.

**Theorem Integrability criteria.** $f \in R(\alpha) \iff \forall\epsilon > 0$, $\exists$ *a partition* $P = P(\epsilon)$ *of* $[a,b]$ *such that*

$$U(P,f,\alpha) - L(P,f,\alpha) < \epsilon.$$

**Proof.** $\Longrightarrow$ Suppose $_-\int_a^b f\,d\alpha = {}^-\int_a^b f\,d\alpha f \in R(\alpha)$ and let $\epsilon > 0$ be fixed. Since $_-\int_a^b f\,d\alpha = \sup_{P\in P} L(P,f,\alpha)$ and $^-\int_a^b f\,d\alpha = \inf_{P\in P} U(P,f,\alpha)$. Then $\exists P_1, P_2 \in P$ such that

$$_-\int_a^b f\,d\alpha - L(P_1,f,\alpha) < \tfrac{\epsilon}{2} \text{ — (1)},$$

$$U(P_2,f,\alpha) - {}^-\int_a^b f\,d\alpha < \tfrac{\epsilon}{2} \text{ — (2)}.$$

Let $P^* = P_1 \cup P_2$. Then $U(P^*,f,\alpha) \leq U(P_2,f,\alpha)(7) < {}^-\int_a^b f\,d\alpha + \frac{\epsilon}{2} f \in R(\alpha) = {}_-\int_a^b f\,d\alpha + \frac{\epsilon}{2}(1) < L(P_1,f,\alpha) + \frac{\epsilon}{2} + \frac{\epsilon}{2} \leq L(P^*,f,\alpha) + \epsilon$. This implies $U(P^*,f,\alpha) - L(P^*,f,\alpha) < \epsilon$ ✓

NTS: $_-\int_a^b f\,d\alpha = {}^-\int_a^b f\,d\alpha$. It is enough to show: for any $\epsilon > 0$, $0 ✓ \leq {}^-\int_a^b f\,d\alpha - {}_-\int_a^b f\,d\alpha < \epsilon$. Let $\epsilon > 0$ be fixed. Then by hypothesis, $\exists$ a partition $P$ of $[a,b]$ such that $U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$. But $L(P,f,\alpha) \leq_- \int_a^b f d\alpha \leq {}^-\int_a^b f d\alpha \leq U(P,f,\alpha)$ implies $^-\int_a^b f d\alpha - {}_-\int_a^b f d\alpha \leq U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$.

**Theorem.** *Let $\epsilon > 0$ be fixed and let $P$ be such that $U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$. — (1) Then*

*(a) (1) holds true for any refinement $P^*$ of $P$.*

*(b) For any $s_i, t_i \in [x_{i-1}, x_i]$,*

$$\sum_{i=1}^{n} |f(s_i) - f(t_i)|\Delta\alpha_i < \epsilon.$$

*(c) For any $t_i \in [x_{i-1}, x_i]$ for $i = 1, \ldots, n$*

$$\int_a^b f\, d\alpha - \sum_{i=1}^{n} f(t_i)\Delta\alpha_i < \epsilon.$$

**Proof.** (a) Let $P^*$ be a refinement of $P$. Then since $L(P, f, \alpha) \leq L(P^*, f, \alpha)$ and $U(P, f, \alpha) \geq U(P^*, f, \alpha)$ and $U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$, we get $U(P^*, f, \alpha) - L(P^*, f, \alpha) < \epsilon$.

(b) Clearly $f(s_i), f(t_i) \in [m_i, M_i]$ for $i = 1, \ldots, n$. That means that $|f(s_i) - f(t_i)| \leq M_i - m_i$ implies

$$\begin{aligned}
\sum_{i=1}^{n} |f(s_i) - f(t_i)|\Delta\alpha_i &\leq \sum_{i=1}^{n}(M_i - m_i)\Delta\alpha_i \\
&= \sum_{i=1}^{n} M_i\Delta\alpha_i - \sum_{i=1}^{n} m_i\Delta\alpha_i \\
&= U(P, f, \alpha) - L(P, f, \alpha)(1) < \epsilon.
\end{aligned}$$

(c) Let $t_i \in [x_{i-1}, x_i]$ for $i = 1, \ldots, n$. Then $m_i \leq f(t_i) \leq M_i$ for $i = 1, \ldots, n$ implies

$$L(P, f, \alpha)\sum_{i=1}^{n} m_i\Delta\alpha_i \leq \sum_{i=1}^{n} f(t_i)\Delta\alpha_i \leq U(P, f, \alpha)\sum_{i=1}^{n} M_i\Delta\alpha_i.$$

Also, $L(P, f, \alpha) \leq \int_a^b f\, d\alpha \leq U(P, f, \alpha)(1) \implies \int_a^b f\, d\alpha - \sum_{i=1}^{n} f(t_i)\Delta\alpha_i < \epsilon$.

Which $f \in R(\alpha)$?

**Theorem.** *Let $f : [a, b]R$ be continuous. Then $f \in R(\alpha)$ on $[a, b]$.*

**Proof.** Let $\epsilon > 0$ be fixed. NTS: $\exists P \in P$ of $[a, b]$ such that $U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$.

**Note 96.—** (i) $\alpha : [a, b]R$ monotonically increasing ( $\implies$ $\alpha$ bounded), so $\exists\eta > 0$ such that $[\alpha(b) - \alpha(a)]\eta < \epsilon$.

(ii) $f$ continuous on $[a,b] \implies f$ uniformly continuous on $[a,b]$. So $\exists\delta > 0$ such that $\forall x,t \in [a,b]$ with $|x-t| < \delta$ implies $|f(x)-f(t)| < \eta$. Let $P$ be a partition such that $\Delta x_i = x_i - x_{i-1} < \delta$ for $i = 1,\dots,n$. Then $|f(x_i) - f(x_{i-1})| \leq M_i - m_i < \eta$ because $M_i, m_i$ are achieved on $[x_{i-1}, x_i]$. Then

$$
\begin{aligned}
U(P,f,\alpha) - L(P,f,\alpha) &= \sum_{i=1}^{n} M_i \Delta\alpha_i - \sum_{i=1}^{n} m_i \Delta\alpha_i = \sum_{i=1}^{n}(< \eta M_i - m_i)\Delta\alpha_i \\
&< \eta \sum_{i=1}^{n} \Delta\alpha_i = \eta[\alpha(x_1) - = \alpha(a)\alpha(x_0) + \cdots + = \alpha(b)\alpha(x_n) - \alpha(x_{n-1})] \\
&= \eta[\alpha(b) - \alpha(a)] < \epsilon.
\end{aligned}
$$

This implies $f \in R(\alpha)$ on $[a,b]$.

**Example 338.**—Show $\lim_{h0} \frac{f(x+h)+f(x-h)-2\cdot f(x)}{h^2} = f''(x)$.

**Proof.** Since $f''$ exists, both $f''(x) = \lim_{h0} \frac{f'(x+h)-f'(x)}{h}$ and $f''(x) = \lim_{h0} \frac{f'(x)-f'(x-h)}{h}$ exist. Then

$$
\begin{aligned}
f''(x) &= \frac{1}{2}\lim_{h0} \frac{f'(x+h)-f'(x)}{h} + \frac{1}{2}\lim_{h0} \frac{f'(x)-f'(x-h)}{h} \\
&= \lim_{h0} \frac{f'(x+h)-f'(x)+f'(x)-f'(x-h)}{2h} \\
&= \lim_{h0} \frac{f'(x+h)-f'(x-h)}{2h}.
\end{aligned}
$$

Observe $\lim_{h0} \boxed{\frac{f(x+h)+f(x-h)-2\cdot f(x)}{h^2}} = \lim_{h0} \frac{f'(x+h)-f'(x-h)}{2h}$ by L'hospital rule. So

$$
f''(x) = \lim_{h0} \frac{f(x+h)-f(x-h)-2f(x)}{h^2}.
$$

**Example 339.**—$f(x) = 3x|x| \implies f(x) = \begin{cases} -3x^2, & x < 0 \\ 3x^2, & x \geq 0 \end{cases}$. So $f'(x) = \begin{cases} -6x, & x < 0 \\ 6x, & x \geq 0 \end{cases}$. Then $f''(x) = \begin{cases} -6, & x < 0 \\ 6, & x \geq 0 \end{cases}$. $\lim_{h0} \frac{f(h)=f(-h)-2\cdot f(0)}{h^2} = \lim_{h0} \frac{3h|h|-3h|h|-0}{h^2} = 0$

**Example 340.**—$f(x) = \begin{cases} |x|^a \sin(|x|^{-c}), & x \neq 0 \\ 0 & x = 0 \end{cases}$, $f : [-1,1]R$, $\sin x$ is differentiable. Suppose $a > 0$, show $f$ is continuous.

**Example 341.**—Prove if $f$ is continuous, then $a > 0$. Suppose $f$ is continous on $[-1,1]$. BWOC suppose $a \leq 0$. Now consider the sequence $x_n = \frac{1}{\frac{\pi}{2}+2\pi n}^{1/c}$. Note as $n\infty$, $x_n 0$ and since $f$ is continuous $f(x_n) f(0) = 0$. But $f(x_n) = \frac{1}{\frac{\pi}{2}+2\pi n}^{a/c} \sin\frac{\pi}{2} + 2\pi n = \frac{\pi}{2} + 2\pi n^{-a/c}$. If $a = 0$, then as $n\infty$, $f(x_n)1 \neq 0$ and if $a < 0$, then $f(x_n)\infty \neq 0$. Therefore we have a contradiction. Thus $a > 0$.

**Example 342.**—Prove $f'(0)$ exists if and only if $a > 1$.

( $\implies$ ) Suppose $f'(0)$ exists, WTS $a > 1$. BWOC, suppose $a \leq 1$. If $a < 1$, $\lim_{h0} \frac{f'(0+h)-f(0)}{h} = \lim_{h0} \frac{|h|^a \sin(|h|^{-c})}{h} \lim_{h0} |h|^{a-1} \sin(|h|^{-c})$ implies $a-1 < 0$, so $\lim_{h0} |h|^{a-1}\infty$ is a contradiction. If $a = 1$,

$$\begin{aligned}\lim_{h0} \frac{f(0+h)-f(0)}{h} &= \lim_{h0} \frac{|h|\sin(|h|^{-c})}{h} \\ &= \lim_{h0} \sin(|h|^{-c})\end{aligned}$$

does not exist. Contradiction, therefore $a > 1$.

() Suppose $a > 1$, WTS $f'(0)$ exists.

$$\lim_{h0} \frac{f(0+h)-f(0)}{h} = \lim_{h0} \frac{|h|^a \sin(|h|^{-c})}{h} = \lim_{h0} |h|^{a-1} \sin(|h|^{-c}) = 0 \implies f'(0) = 0$$

.

**Example 343** (3).—By Chain rule & product rule

$$f'(x) = \frac{|x|}{x} a|x|^{a-1} \sin(|x|^{-c}) - c|x|^{a-c-1} \cos(|x|^{-c})$$

for $x \neq 0$. WTS: by contradiction: $f'(x)$ is not bounded if $a < 1+c$. Let $x_n = (2n\pi + \frac{\pi}{4})^{\frac{-1}{c}}$ for $n \in N$. Then $f'(x_n) = \frac{\sqrt{2}}{2}(ax_n^{a-1} - cx_n^{a-c-1})$ when $a < 1+c$, $f'(x_n) - \infty$.

WTS: $f'(x)$ is bounded if $a > 1+c$. $|f'(x)| \leq |a|\,|x|^{a-1} + c|x|^{a-c-1}$ is bounded on $x \in [-1, 1]$ if $a - 1 \geq 0$ and $a - c - 1 \geq 0$, $a - c - 1 \geq 0$ impliels $a \geq 1 + c$.

When $f \in R(\alpha)$ on $[a, b]$?

✓ $f$ continuous on $[a, b]$Thm 6.8 $\implies$ $f \in R(\alpha)$ on $[a, b]$.

**Recall 80.**—$f \in R(\alpha)$ on $[a, b]$ if and only if $\forall \epsilon > 0$, $a = x_0 \leq \cdots \leq x_n = b \exists P = P(\epsilon)$ such that $U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$.

$$U(P, f, \alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i L(P, f, \alpha) = \sum_{i=1}^{n} m_i \Delta\alpha_i \begin{cases} M_i = \sup_{[x_{i-1}, x_i]} f(x) \\ m_i = \inf_{[x_{i-1}, x_i]} f(x) \end{cases},$$

where $\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1}) \geq 0$ because $\alpha$ monotonically increasing on $[a, b]$.

**Theorem.** *Suppose*

- $f$ *monotonic on* $[a, b]$,

$*$ $\alpha$ *continuous on* $[a, b]$ *(*$\alpha \uparrow$ *on* $[a, b]$*).*

*Then* $f \in R(\alpha)$ *on* $[a, b]$.

**Proof.** Let $\epsilon > 0$ be fixed. NTS: $\exists$ a partition $P : x_0 \le x_1 \le \cdots \le x_n = b$ s.t. $U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$. WLOG assume $f$ monotonically increasing on $[a, b]$.

$$(1) \begin{cases} \text{Let } N \in N \text{ such that} \\ [\alpha(b) - \alpha(a)][f(b) - f(a)] < N\epsilon \end{cases}.$$

$\alpha$ continuous + monotonically increasing on $[a, b] \implies \exists a = x_0 \le \cdots \le x_N = b$ such that $\Delta\alpha_j = \alpha(x_j) - \alpha(x_{j-1}) = \frac{\alpha(b)-\alpha(a)}{N}$ for all $j = 1, \ldots, N$.

**Note 97.**—$f \uparrow \implies M_j = f(x_j)$ and $m_j = f(x_{j-1})$ for all $j = 1, \ldots, N$. Then

$$\begin{aligned} U(P, f, \alpha) - L(P, f, \alpha) &= \sum_{j=1}^{N} (M_j - m_j)\Delta\alpha_j \\ &= \sum_{j=1}^{N} [f(x_j) - f(x_{j-1})]\frac{[\alpha(b) - \alpha(a)]}{N} \\ &= [f(b) - f(a)]\frac{[\alpha(b) - \alpha(a)]}{N}(1) < \epsilon. \end{aligned}$$

$\implies f \in R(\alpha)$ on $[a, b]$.

**Theorem.** *Suppose*

- *$f$ is bounded on $[a, b]$,*
- *$f$ not continuous at $\theta_1, \ldots, \theta_k$ in $[a, b]$.*

* *$\alpha$ is continuous at $\theta_i$ for all $i = 1, \ldots, k$ ($\alpha \uparrow$).*

*Then $f \in R(\alpha)$ on $[a, b]$.*

**Proof.** Let $\epsilon > 0$ be fixed. NTS: $\exists P$ such that $U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$. $\alpha$ continuous at $\theta_i$; $i = 1, \ldots, k \implies \exists u_i, v_i \in [a, b]$ such that

- $\theta_i \in [u_i, v_i]$ for all $i = 1, \ldots, k$, and
- $\sum_{i=1}^{k} \alpha(v_i) - \alpha(u_i) < \epsilon_1 = \frac{\epsilon}{2M+\alpha(b)-\alpha(a)}$. — $(*)$

Now work with $f$: Define $G := [a, b] \bigcup_{i=1}^{k} (u_i, v_i) = [a, u_1] \cup [v_1, u_1] \cup \cdots \cup [v_k, b]$ – closed (finite union of closed sets) $\implies$ $G$ is compact (closed + bounded subset of $R$). This implies $f$ is uniformly continuous on $G$. So, $\exists \delta > 0$ such that $\forall x, y \in G$ with $|x - y| < \delta$, $|f(x) - f(y)| < \epsilon_1$.

Construct $P$: Let $n > 2k$ and $a = x_0 \le \cdots \le x_n = b$ such that

- $u_i, v_i \in P$ for all $i = 1, \ldots, k$

- $t \in (u_i, v_i) \implies t \notin P$ for all $i = 1, \ldots, k$
- if for fixed $i$ $x_{i-1} \neq u_j$ for all $j = 1, \ldots, k$, then $\Delta x_i < \delta$
  (choose $x_i < u_j$ if $x_{i-1} < u_j$, choose $x_i > v_j$ if $x_{i-1} \geq v_j$)

$f$ bounded $\implies$ $M = \sup |f(x)| < \infty$. Then $\boxed{M_i - m_i \leq 2M}$ for all $i = 1, \ldots, n$. Also, if $\boxed{x_{i-1} \neq u_j}$ for some $i$ and $\forall j$ then $\boxed{M_i - m_i < \epsilon_1}$ (using uniform continuity since $\Delta x_i < \delta$). Then

$$\begin{aligned} U(P, f, \alpha) - L(P, f, \alpha) &= \underbrace{\sum_{i=1}^{n} (\overbrace{2M}^{M_i - m_i}) \Delta\alpha_i}_{u_{i-1} = u_j \text{ for some } j} + \underbrace{\sum_{i=1}^{n} (\overbrace{\epsilon_1}^{M_i - m_i}) \Delta\alpha_i}_{x_{i-1} \neq u_j \ \forall j = 1, \ldots, k} \\ &\leq 2M \underbrace{\sum_{i=1}^{n} \Delta\alpha_i}_{x_{i-1} = u_j} + \epsilon_1 \underbrace{\sum_{i=1}^{n} \Delta\alpha_i}_{x_{i-1} \neq u_j} \\ &\overset{(*)}{<} 2M\epsilon_1 + \epsilon_1(\alpha(b) - \alpha(a)) \\ &= \underline{\underline{\epsilon_1}}[2M + \alpha(b) - \alpha(a)] = \epsilon > 0 \end{aligned}$$

since $\epsilon > 0$ is arbitraty, this proves $f \in R(\alpha)$ on $[a, b]$.

**Theorem.** *Suppose*

- $f \in R(\alpha)$ *on* $[a, b]$,
- $m \leq f(x) \leq M$ *on* $[a, b]$,
- $\phi : [m, M] R$ *continuous.*

*Then* $h = \phi \circ f \in R(\alpha)$ *on* $[a, b]$. *(Assume* $\alpha \uparrow$ *on* $[a, b]$.*)*

**Proof.** Let $\epsilon > 0$ be fixed. NTS: $\exists P := x_0 \leq \cdots \leq x_n = b$ such that $U(P, h, \alpha) - L(P, h, \alpha) < \epsilon$.

1. $\phi$ uniformly continuous on $[m, M]$ $\implies$ $\exists \delta > 0$ such that $x, y \in [m, M]$ with $|x - y| < \delta \implies |\phi(x) - \phi(y)| < \epsilon$.
2. $f \in R(\alpha)$ on $[a, b]$ $\implies$ $\exists P := x_0 \leq \cdots \leq x_n = b$ such that $U(P, f, \alpha) - L(P, f, \alpha) <$ $\ldots$.

Define $m_i^* = \inf_{[x_{i-1}, x_i]} h(x)$ and $M_i^* = \sup_{[x_{i-1}, x_i]} h(x)$. $f$ bounded + $\phi$ continuous $\implies$ $h = \phi \circ f$ is bounded on $[a, b]$ $\implies$ $m_i^*, M_i^*$ are well defined. $\ldots$ to be continued.

**Recall 81.**—Suppose

- $f \in R(\alpha)$ on $[a, b]$ and $m \leq f(x) \leq M$.

- $\phi : [m, M]R$ continuous.

Then $h = \phi \circ f \in R(\alpha)$ on $[a, b]$.

**Proof.** Let $\epsilon > 0$ be fixed. NTS: $\exists P$ such that $U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$.

**Note 98.**—:

$$(1) \begin{cases} \phi : [m, M]R \text{ continuous } \implies \phi \text{ uniformly continuous} \\ \implies \exists \delta > 0 \text{ such that } \forall x, y \in [m, M] \text{ such that } |x - y| < \delta \\ \text{implies } |\phi(x) - \phi(y)| < \epsilon. \end{cases}$$

$$(2) \begin{cases} f \in R(\alpha) \text{ on } [a, b] \implies \exists P : a = x_0 \leq \cdots \leq x_n = b \\ \text{such that } U(P, f, \alpha) - L(P, f, \alpha) < \delta^2. \end{cases}$$

Define $m_i, M_i$ usual way. Define $m_i^* = \inf_{[x_{i-1}, x_i]} h(x), M_i^* = \sup_{[x_{i-1}, x_i]} h(x)$. $m_i^*$ and $M_i^*$ are well defined since $\phi$ continuous and $f$ bounded. Let $i \in A \subset 1, \ldots, n$ if $M_i - m_i < \delta$, and $i \in B \subset 1, \ldots, n$ if $M_i - m_i \geq \delta$. If $i \in A$, then $M_i - m_i < \delta$, so $\boxed{M_i^* - m_i^* < \epsilon}$. If $i \in B$, then $M_i^* - m_i^* \leq 2K$, where $K = \sup_{[a,b]} |h(x)| < \infty$. In this case, claim:

$$\sum_{i \in B} \Delta\alpha_i < \delta.$$

**Proof.** $\delta \sum_{i \in B} \Delta\alpha_i \leq \sum_{i \in B} (M_i - m_i) \Delta\alpha_i \leq U(P, f, \alpha) - L(P, f, \alpha) < \delta^2 \implies \sum_{i \in B} \Delta\alpha_i < \delta$.

Then

$$\begin{aligned} U(P, h, \alpha) - L(P, h, \alpha) &= \sum_{i \in A} (< \epsilon M_i^* - m_i^*) < \alpha(b) - \alpha(a) \Delta\alpha_i + \sum_{i \in B} (\leq 2K 2M_i^* - m_i^*) < \delta\Delta\alpha_i \\ &< \epsilon[\alpha(b) - \alpha(a)] + \delta 2K \\ &< \epsilon[\alpha(b) - \alpha(a) + 2K] \text{ if } \delta < \epsilon \end{aligned}$$

Since $\epsilon > 0$ is arbitraty, $U(P, h, \alpha) - L(P, h, \alpha) < \epsilon'$ for any $\epsilon' > 0$. Hence $h \in R(\alpha)$ on $[a, b]$.

**Theorem.** *Let $f, g \in R(\alpha)$ on $[a, b]$ and $c \in R$. Then*

*(a) $f + g$, $cf \in R(\alpha)$ on $[a, b]$, and $\int_a^b [f + g]\, d\alpha = \int_a^b f\, d\alpha + \int_a^b g\, d\alpha$, $\int_a^b cf\, d\alpha = c \int_a^b f d\alpha$.*

*(b) $f \leq g$ on $[a, b]$ then $\in_a^b f\, d\alpha \leq \int_a^b g\, d\alpha$.*

*(c) Let $a < c < b$. Then $f \in R(\alpha)$ on $[a, c]$ and on $[c, b]$, $\int_a^b f\, d\alpha = \int_a^c f\, d\alpha + \int_c^b f\, d\alpha$.*

*(d) If $|f| \leq M$ on $[a, b]$ then $|\int_a^b f\, d\alpha| \leq M[\alpha(b) - \alpha(a)]$.*

*(e) $f \in R(\alpha_1)$ and $f \in R(\alpha_2)$ on $[a, b]$, then*

- *$f \in R(\alpha_1+\alpha_2)$ on $[a,b]$, $\int_a^b f\,d(\alpha_1+\alpha_2) = \int_a^b f\,d\alpha_1 + \int_a^b f\,d\alpha_2$*
- *$\int_a^b f\,d(c\alpha) = c\int_a^b f\,d\alpha$ for $c>0$ constant.*

**Theorem.** *Suppose $f, g \in R(\alpha)$ on $[a,b]$. Then*

*1) $fg \in R(\alpha)$ on $[a,b]$.*

*2) $|f| \in R(\alpha)$ on $[a,b]$ and*

$$\int_a^b f\,d\alpha \le \int_a^b |f|\,d\alpha.$$

**Recall 82.**—$fg = \frac{(f+g)^2-(f-g)^2}{4}$

**Proof.** Since $f, g$ bounded on $[a,b]$, $\exists M_1, M_2 > 0$ such that $|f| \le M_1$ and $|g| \le M_2$ on $[a,b]$.
$\implies |f+g|, |f-g| \le M_1 + M_2$
$\implies -(M_1+M_2) \le (f+g), (f-g) \le M_1+M_2$ on $[a,b]$.
Then $\phi(t) = t^2$ continuous on $[-(M_1+M_2), M_1+M_2]$. So $h(t) = \phi\circ(f+g) = (f+g)^2 \in R(\alpha)$ and $\phi\circ(f-g) = (f-g)^2 \in R(\alpha)$ on $[a,b]$. Then $fg = \frac{(f+g)-(f-g)^2}{4} \in R(\alpha)$ on $[a,b]$.

*2).* $\phi(t) = |t|$ continuous on $[-M_1, M_1]$. So $h = \phi\circ f \in R(\alpha)$ on $[a,b]$. Now, $\int_a^b f\,d\alpha = c\int_a^b f\,d\alpha$, where $c = \int_a^b f\,d\alpha$.

$$\int_a^b f\,d\alpha = c\int_a^b f\,d\alpha = \int_a^b cf\,d\alpha \le \int_a^b |f|\,d\alpha.$$

Evaluating $f \in R(\alpha)$:

**Definition** (Heaviside function)**.**—$I(x) := \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \le 0 \end{cases}$. $I(x)$ is monotonically increasing. $s \in R \implies I(x-s)$ is shifted Heaviside function.

**Theorem.** *Suppose*

- *$f$ bounded on $[a,b]$,*
- *$\exists s \in (a,b)$ such that $f$ continuous at $s$, and*
- *$\alpha(x) = I(x-s)$.*

*Then $f \in R(\alpha)$ on $[a,b]$ and $\int_a^b f\,d\alpha = f(s)$.*

**Proof.** $f \in R(\alpha)$, since $\alpha$ is not continuous at $s$ but $f$ is continuous at $s$. Let $P : a = x_0 \leq x_i \leq x_1 < x_2 < x_3 = b$, where $s = x_1$. Then $U(P, f, \alpha) = \sum_{i=1}^{3} M_i \Delta\alpha_i = M_2$. Since

$$\begin{aligned}
\Delta\alpha_1 &= = 0\alpha(x_1) - = 0\alpha(x_0) = 0 \\
\Delta\alpha_2 &= = 1\alpha(x_2) - = 0\alpha(x_1) = 1 \\
\Delta\alpha_3 &= = 1\alpha(x_3) - = 1\alpha(x_2) = 0
\end{aligned}$$

Similarly, $L(P, f, \alpha) = m_2$. Let $\epsilon > 0$ be fixed. Then $f$ continuous at $s \implies f(x_2)f(s)$ as $x_2 s$. This means $M_2 - m_2 < \epsilon$ by letting $x_2 s$. So $U(P, f, \alpha) - L(P, f, \alpha) = M_2 - m_2 < \epsilon$. This implies $\int_a^b f\, d\alpha = f(s)$.

**Theorem.** *Suppose*

- $\sum c_n$ *converges,* $c_n \geq 0$.
- $\exists s_n$ *of distinct points in* $(a, b)$ *such that* $\alpha(x) = \sum_n c_n I(x - s_n)$
- $f$ *continuous on* $[a, b]$.

*Then* $f \in R(\alpha)$ *and* $\int_a^b f\, d\alpha = \sum c_n f(s_n)$.

**Proof.** Step 1: Show $f \in R(\alpha)$ on $[a, b]$. Enough to show $\alpha$ is well defined and $\alpha$ is monotonically increasing.

- $\alpha$ is well defined by comparison test, since $c_n I(x - s_n) \leq c_n$ for all $n$ ($I(x - s_n) \leq 1$) and $\sum c_n$ converges.
- Let $x_1 < x_2$. Then $x_1 - s_n < x_2 < s_n \implies I(x_1 - x_n) \leq I(x_2 - s_n)$ implies

$$\alpha(x_1) = \sum c_n I(x_1 - s_n) \leq \sum c_n I(x_2 - s_n) = \alpha(x_2).$$

This implies $\alpha \uparrow \implies f \in R(\alpha)$.

*Step 2.* $\int_a^b f\, d\alpha = \sum c_n f(s_n)$.

Let $\epsilon > 0$. NTS: $\exists N \in N$ such that $\int_a^b f\, d\alpha - \sum_{k=1}^{N} c_k f(s_k) < \epsilon$. $\sum c_n$ converges $\implies \exists N \in N$ such that $\sum_{k=N+1}^{\infty} c_k < \frac{\epsilon}{M}$, where $M = \sup_{[a,b]} |f(x)|$. Define

$$\begin{aligned}
\alpha_1(x) &= \sum_{k=1}^{N} c_k I(x - s_k) \qquad \uparrow \\
\alpha_2(x) &= \sum_{k=N+1}^{\infty} c_k I(x - s_k) \qquad \uparrow
\end{aligned}$$

$f$ continuous on $[a, b] \implies f \in R(\alpha_1)$ and $f \in R(\alpha_2)$. But

$$\alpha_1(x) = \beta_1 c_1 I(x - s_1) + \cdots + \beta_N c_N I(x - s_N) = \beta_1(x) + \cdots + \beta_N(x),$$

so

$$\int_a^b f\,d\alpha_1 = \int_a^b f\,d(\beta_1 + \cdots + \beta_N)$$
$$= \int_a^b f\,d\beta_1 + \cdots + \int_a^b f\,d\beta_N$$
$$\text{Thm } 259 = c_1 f(s_1) + \cdots + c_N f(s_N)$$

Now, estimate $\int_a^b f\,d\alpha_2$. Note

$$\alpha_2(a) = \sum_{k=N+1}^{\infty} c_k = 0 I(< 0a - s_k) = 0$$
$$\alpha_2(b) = \sum_{k=N+1}^{\infty} c_k = 1 I(> 0b - s_k) = \sum_{k=N+1}^{\infty} c_k < \frac{\epsilon}{M}.$$

So $\int_a^b f\,d\alpha_2 \text{Thm } 258 \leq \int_a^b |f|\,d\alpha_2 \leq M[\alpha_2(b) - = 0\alpha_2(a)] < M \cdot \frac{\epsilon}{M} = \epsilon$. Therefore,

$$\int_a^b f\,d\alpha - \sum_{k=1}^{N} c_k f(s_k) = \int_a^b f\,d\alpha_1 + \int_a^b f\,d\alpha_2 - \sum_{k=1}^{N} c_k f(s_k) = \int_a^b f\,d\alpha_2 < \epsilon,$$

so $\int_a^b f\,d\alpha = \sum c_n f(s_n)$.

Connection between $R$ and $R(\alpha)$ on $[a, b]$.

**Theorem.** *Suppose*

- ✓ $\alpha \uparrow$ *on* $[a, b]$,
- $*$ $\alpha' = \frac{d}{dx}\alpha \in R$ *on* $[a, b]$,
- ✓ $f$ *bounded on* $[a, b]$.

*Then* $f \in R(\alpha)$ *on* $[a, b] \iff f\alpha' \in R$ *on* $[a, b]$ *and* $\int_a^b f\,d\alpha = \int_a^b f\alpha'\,dx$.

**Proof.** Enough to show $\int_a^b f\,d\alpha = \int_a^b f\alpha'\,dx$. We will show: $^{-}\int_a^b f\,d\alpha =^{-} \int_a^b f\alpha'\,dx$ — (1) and $_{-}\int_a^b f\,d\alpha = {}_{-}\int_a^b f\alpha'\,dx$ — (2). We'll prove (1), proof of (2) is similar. Let $\epsilon > 0$. We'll show $^{-}\int_a^b f\,d\alpha -^{-} \int_a^b f\alpha'\,dx < \epsilon$. For this we'll show: $\exists$ a partition $P$ such that $|U(P, f, \alpha) - U(P, f\alpha')| < \epsilon$. Let $\epsilon > 0$ be fixed. $\alpha' \in R$ on $[a, b] \implies \exists P : x_0 \leq x_1 \leq \cdots \leq x_n = b$ such that $U(P, \alpha') - L(P, \alpha') < \frac{\epsilon}{2M}$, $|f(x)| \leq M$, $x \in [a, b]$. $\alpha'$ exists $\implies$ by MVT, $\exists t_i \in (x_{i-1}, x_i)$ such that

$$\alpha'(t_i) = \frac{\alpha(x_i) - \alpha(x_{i-1})}{x_i - x_{i-1}} = \frac{\Delta\alpha_i}{\Delta x_i}$$

$\implies \boxed{\Delta\alpha_i = \alpha'(t_i)\Delta x_i}$. If $s_i \in [x_{i-1}, x_i]$ then by Theorem 252(b) $\sum_{i=1}^{n} |\alpha'(s_i) - \alpha'(t_i)|\Delta x_i < \frac{\epsilon}{2M}$ — (3). Then

$$\sum_{i=1}^{n} f(s_i)\alpha'(t_i)\Delta x_i \Delta\alpha_i - \sum_{i=1}^{n} f(s_i)\alpha'(s_i)\Delta x_i = \sum_{i=1}^{n} f(s_i)\alpha'(t_i) - \alpha'(s_i)\Delta x_i$$

$$\leq M \sum_{i=1}^{n} \alpha'(t_i) - \alpha'(s_i)\Delta x_i < M \cdot \frac{\epsilon}{2M} = \frac{\epsilon}{2} \qquad \text{— (4)}$$

This means $\sum_{i=1}^{n} f(s_i)\Delta\alpha_i < \sum_{i=1}^{n} f(s_i)\alpha'(s_i)\Delta x_i + \frac{\epsilon}{2}$ for any $s_i \in [x_{i-1}, x_i] \implies U(P, f, \alpha) < U(P, f\alpha') + \frac{\epsilon}{2}$. Switching $f$ and $f\alpha'$ and repeating, $U(P, f\alpha') < U(P, f, \alpha) + \frac{\epsilon}{2} \implies U(P, f, \alpha) - U(P, f\alpha') \leq \frac{\epsilon}{2} < \epsilon$. Repeating the argument with $P^*$ of $P$ and that $U(P^*, \alpha') - L(P^*, \alpha') < \frac{\epsilon}{2M}$, we get $^{-}\int_a^b f\, d\alpha = {}^{-}\int_a^b f\alpha'\, dx \leq \frac{\epsilon}{2} < \epsilon. \implies {}^{-}\int_a^b f\, d\alpha = {}^{-}\int_a^b f\alpha'\, dx$. Similarly $_{-}\int_a^b f\, d\alpha = {}_{-}\int_a^b f\alpha'\, dx \implies \int_a^b f\, d\alpha = \int_a^b f\alpha'\, dx$.

**Theorem Change of variable.** *Suppose*

- $f \in R(\alpha)$ *on* $[a, b]$,
- $\alpha \uparrow$ *on* $[a, b]$,
- $\phi : [A, B][a, b]$ *strictly increasing and continuous*

*Then* $g \in R(\beta)$ *and* $\int_A^B g\, d\beta = \int_a^b f\, d\alpha$, *where* $g = f \circ \phi$ *and* $\beta = \alpha \circ \phi$.

**Note 99.**—Take $\alpha(x) = x$. Then $\alpha' \in R$ and $\alpha \uparrow$. Then $\beta = \alpha \circ \phi = \phi$. Suppose $\phi' \in R$ on $[A, B]$. Then

$$\int_a^b f\, d\alpha = \int_a^b f\, dx \text{Thm } 262 = \int_A^B g\, d\beta = \int_A^B (f \circ \phi)\, d\phi d\beta \text{Thm } 261 = \int_A^B f(\phi(y))\phi'(y)\, dy.$$

**Proof.** Let $\epsilon > 0$. NTS: $\exists$ partition $Q$ of $[A, B]$ such that $U(Q, g, \beta) - L(Q, g, \beta) < \epsilon$, and $\int_A^B g\, d\beta = \int_a^b f\, d\alpha$. $f \in R(\alpha)$ on $[a, b] \implies \exists P : x_0 \leq \cdots \leq x_n$ of $[a, b]$ such that $U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$ — $(*)$. $\phi$ strictly increasing on $[a, b] \implies \phi$ is injective $\implies \exists$ a $y_0 \leq \cdots \leq y_n$partition $Q$ of $[A, B]$ such that $x_i = \phi(y_i)$; $i = 1, \ldots, n$. Also,

$$x \in [x_{i-1}, x_i] f(x) = y \in [y_{i-1}, y_i] f(\phi(y)) = y \in [y_{i-1}, y_i](f \circ \phi)(y) = y \in [y_{i-1}, y_i] g(y) \implies$$

$U(P, f, \alpha) = U(Q, g, \beta)$ and $L(P, f, \alpha) = L(Q, g, \beta)$. So $(*) \implies U(Q, g, \beta) - L(Q, g, \beta) < \epsilon \implies g \in R(\beta)$. Clearly $\int_A^B g\, d\beta = \int_a^b f\, d\alpha$.

**Theorem Mean Value Theorem for Integrals.** *Suppose*

- $f$ *continuous on* $(a, b)$

*Then* $\exists x_0 \in [a, b]$ *such that* $\int_a^b f(x)\, dx = f(x_0)(b - a)$ *or* $\frac{1}{b-a}\int_a^b f(x)\, dx = f(x_0)$.

**Proof.** $f$ continuous on $[a,b] \implies f$ bounded on $[a,b] \implies \exists m, M \in R$ such that $m \leq f(x) \leq M$; $x \in [a,b]$. Also, $f$ continuous $\implies f \in R$ on $[a,b]$. Then $m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$. Let $A := \frac{1}{b-a}\int_a^b f(x)\,dx$. Then $m \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq M$ or $m \leq A \leq M$. By IVT, $\exists x_0 \in (a,b)$ such that $f(x_0) = A$ ✓.

**Remark 45.**—$m = \min f(x)$, $M = \max f(x)$.

**Theorem.** *Let $f \in R$ on $[a,b]$. Define $F(x) := \int_a^x f(t)\,dt$ accumulation function; $x \in [a,b]$. Then*

1. *$F$ is continuous on $[a,b]$.*
2. *If in addition, $f$ continuous at $x_0 \in (a,b)$, then $F$ is differentiable at $x_0$ and $F'(x_0) = f(x_0)$.*

**Note 100.**—In particular, if $f$ is continuous on $[a,b]$ then $F$ is differentiable on $(a,b)$.

**Proof.** 1. Let $\epsilon > 0$ be fixed. Let $x, y \in [a,b]$ with $x \neq y$. Then

$$|F(x) - F(y)| = \int_a^x f(t)\,dt - \int_a^y f(t)\,dt = \int_x^y f(t)\,dt \leq M|y-x|$$

since $f \in R$ on $[a,b] \implies \exists M > 0$ such that $|f(x)| \leq M$ on $[a,b]$. Then if $|y-x| < \frac{\epsilon}{M} = \delta$, then $|F(x) - F(y)| < \epsilon$. So $F$ is uniformly continuous on $[a,b]$.

2. Need to show: $\lim_{tx_0} \frac{F(t)-F(x_0)}{t-x_0} = f(x_0)$. Fix $\epsilon > 0$. Enough to show: $\exists \delta > 0$ such that $\forall s,t \in [a,b]$ with $s,t \in (x_0-\delta, x_0+\delta)$ and $s < t$ (WLOG)

$$\frac{F(t)-F(s)}{t-s} - f(x_0) < \epsilon.$$

$f$ continuous at $x_0 \implies \exists \delta > 0$ such that $\forall z \in [a,b]$ with $|z-x_0| < \delta$, $|f(z)-f(x_0)| < \epsilon$. Now,

$$\begin{aligned} \frac{F(t)-F(s)}{t-s} - f(x_0) &= \frac{1}{t-s}\int_t^s f(u)\,du - f(x_0) \\ \text{MVT(I)} &= f(x^*) - f(x_0) < \epsilon \qquad \text{for some } x^* \in (s,t) \end{aligned}$$

since $|x^* - x_0| < \delta$.

**Theorem Fundamental Theorem of Calculus.** *Suppose*

- *$f \in R$ on $[a,b]$,*
- *$\exists$ a differentiable function $F$ such that $F' = f$.*

*Then $\int_a^b f\,dx = F(b) - F(a)$.*

**Proof.** Fix $\epsilon > 0$. NTS: $\int_a^b f(x)\,dx - F(b) + F(a) < \epsilon$. $f \in R \implies \exists P : x_0 \leq x_1 \leq \cdots \leq x_n = b$ such that $U(P,f) - L(P,f) < \epsilon$. $F$ differentiable on $(a,b) \implies$ by MVT $\exists t_i \in (x_{i-1}, x_i)$ such that $F(x_i) - F(x_{i-1}) = F'(t_i)\Delta x_i$ for all $i = 1, \ldots, n$. By Theorem 252(c) implies

$$\begin{aligned}
&\int_a^b f\,dx - \sum_{i=1}^n F(x_i) - F(x_{i-1}) f(t_i)\Delta x_i < \epsilon \\
\implies &\int_a^b f\,dx - \sum_{i=1}^n F(x_i) - F(x_{i-1}) < \epsilon \\
\implies &\int_a^b f\,dm - F(b) + F(a) < \epsilon.
\end{aligned}$$

**Theorem Integration by parts formula.** *Suppose*

- $F, G : [a,b]R$ *are differentiable functions on* $(a,b)$*,*
- $F', G' \in R$ *on* $[a,b]$*.*

*Then* $\int_a^b FG'\,dx = F(b)G(b) - F(a)G(a) - \int_a^b GF'\,dx$*.*

**Proof.** Take $H := FG \implies H' = F{\in}\, RG' + G{\in}\, RF' \in R$ since $F, G \in R$ (being continuous) and product of integrable functions is integrable. Applying Theorem 265 to $H'$ (for $f$), we get

$$\begin{aligned}
\int_a^b H'\,dx = H(b) - H(a) \implies &\int_a^b FG' + GF'\,dx = F(b)G(b) - F(a)G(a) \\
\implies &\int_a^b FG'\,dx = F(b)G(b) - F(a)G(a) - \int_a^b GF'\,dx.
\end{aligned}$$

Vector-valued : $[a,b]R^k$ as $(t) = (f_1(t), \ldots, f_k(t))$. Define $\int_a^b (t)\,dt = (\int_a^b f_1(t)\,dt, \ldots, \int_a^b f_k(t)\,dt)$. Most theorems for scalar-valued functions hold also for .

# § 4 Sequence and series of functions

Let $f_n : E^{\text{any set}} R$ (or $C$) for each $n \in N$

**Definition** (Pointwise convergence)**.**—A sequence of functions $f_n$ defined on $E$ (any set) converges pointwise to a function $f$ (defined on $E$) if and only if $\forall x \in E, \forall \epsilon > 0$, $\exists N = N(\epsilon, x) \in N$ such that $\forall x \in E$

$$|f_n(x) - f(x)| < \epsilon \qquad \text{for } n > N.$$

**Remark 46.**—$\forall x \in E$, $f_n(x) f(x)$.

**Definition** (Uniform convergene)**.**—A sequence of functions $f_n$ converges uniformly to $f$ on $E$ if and only if $\forall \epsilon > 0$, $\exists N \in N$, $N = N(\epsilon)$ such that $|f_N(x) - f(x)| < \epsilon$ for $n \geq N$ and $\forall x \in E$.

When can we expect properties of $f_n$ to be preserved in the limiting process?

**Example 344.**—Pointwise convergence does not preserve continuity. $f_n(x) = \frac{n^2 x}{1+n^2 x}$; $x \in R$, $\boxed{n \geq 2}$. If $x \neq 0$, $\lim_{n\infty} f_n(x) = \frac{n^2}{\frac{1}{x}+n^2} = 11$ as $n\infty$. For $x = 0$, $f_n(0) = 00$ as $n\infty$. $f(x) = \begin{cases} 1 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ – not continuous at $x = 0$.

Two criterions for uniform convergence:

**Theorem Cauchy Criterion.** *$f_n : ER$ converges uniformly on $E$ if and only if $\forall \epsilon > 0$, $\exists N \in N$, $N = N(\epsilon)$ such that $|f_n(x) - f_m(x)| < \epsilon$ for all $m, n > N$.*

**Recall 83.**—Criterion for uniform convergence

1. Cauch-criterion $f_n \subset (E, d)$ converges uniformly on $E$ if and only if $\forall \epsilon > 0$ $\exists N \in N$ such that $|f_n(t) - f_m(t)| < \epsilon$ for all $m, n > N$ and $\forall t \in E$.

2. For each $n$, define $\epsilon_n := \sup_{x \in E} |f_n(x) - f(x)|$, where $f_n f f$ pointwise.

Then $f_n f$ uniformly if and only if $\epsilon_n 0$ as $n\infty$.

**Theorem.** *Suppose $f_n, f$ real-valued functions defined on a metric space $(E, d)$ such that*

- *$f_n f$ uniformly on $E$,*
- *$x$ a limit point of $E$ and $\lim_{tx} f_n(t) = A_n$ for all $n \in N$.*

*Then*

1. *$A_n$ converges in $R$ to $A$, say*
2. *$\lim_{tx} f(t) = A$.*

**Proof.** (1) Let $\epsilon > 0$. We'll show that $A_n \subset R$ is Cauchy. Since $f_n f$ uniformly on $E$, $\exists N \in N$ such that $|f_n(t) - f_m(t)| < \frac{\epsilon}{3}$ for all $t \in E$, $n, m > N$. Since $\lim_{tx} f_k(t) = A_k$ for all $k \in N$, $\exists \delta > 0$ such that $\forall t \in (N_\delta(x)x) \cap E$, $|f_k(t) - A_k| < \frac{\epsilon}{3}$. Then for $m, n > N$ and $\forall t \in (N_\delta(x)x) \cap E$,

$$\begin{aligned} |A_n - A_m| &= |A_n - f_n(t) + f_n(t) - A_m + f_m(t) - f_m(t)| \\ \Delta &\leq |A_n - f_n(t)| + |f_m(t) - A_m| + |f_n(t) - f_m(t)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \implies A_n \text{ is Cauchy,} \end{aligned}$$

which implies $\exists A \in R$ such that $A_n A$.

(2) NTS: $\lim_{tx} f(t) = A$. NTS: $\exists \delta > 0$ such that $\forall t \in (N_\delta(x)x) \cap E$, $|f(t) - A| < \epsilon$ using $|f(t) - A| = |f(t) - f_n(t) + f_n(t) + A_n - A_n - A|$. $A_n A \implies \exists N \in N$ such that $|A_n - A| < \frac{\epsilon}{3}$, $n > N$. So $f_n(t) A_n$ as $tx \implies \exists \delta > 0$ such that $\forall t \in (N_\delta(x)x) \cap E$, $|f_n(t) - A_n| < \frac{\epsilon}{3}$. Then for $n > N$ and for $t \in (N_\delta(x)x) \cap E$,

$$|f(t) - A| \leq < \frac{\epsilon}{3} \text{ since } f_n f \text{ unif.} |f(t) - f_n(t)| + |f_n(t) - A_n| + |A_n - A| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$

implies $\lim_{tx} f(t) = A$.

*Corollary.*—If $f_n$ is a sequence of continuous functions on $E$ such that $f_n f$ uniformly, then $f$ is continuous on $E$. (Uniform limit of continuous functions is continuous).

**Remark 47.**—For each $x \in E$, $\lim_{tx} f_n(T) = f_n(x)$ and $f_n(x) f(x)$. Then $\lim_{tx} f(t) = f(x)$ or

$$\lim_{tx} \lim_{n\infty} = \lim_{n\infty} f_n(x) = \lim_{n\infty} \lim_{tx} f_n(T)$$

i.e., the limit composition communtes when there is uniform convergence.

Converse of corollary 32 is not necessarily true.

**Example 345.**—Take $f_n(x) = \frac{1}{nx}$; $x \neq 0$. Then $cts f_n cts 0 = f$ pointwise on $(0, 1]$. But $f_n \not\emptyset$ uniformly.

Take $\boxed{\epsilon = \frac{1}{2}}$ and $\boxed{x_n = \frac{1}{n+1}} \in (0, 1]$. Then $|f_n(x) - 0| = \frac{1}{n\frac{1}{n+1}} = \frac{n+1}{n} = 1 + \frac{1}{n} > \frac{1}{2} = \epsilon$.

**Theorem.** *Suppose $K \subset (E, d)$ is compact, and*

- *$f_n$ sequence of functions on $K$,*
- *$f_n f$ pointwise on $K$,*
- *$f$ continuous on $K$,*
- *$f_n(x) \geq f_{n+1}(x)$ for all $n \in N$ and $\forall x \in K$.*

*Then $f_n f$ uniformly on $K$.*

**Remark 48.**—Uses compact nested sets in proof.

**Theorem Uniform convergence and integrability.** *Suppose*

- $\alpha$ *monotonically increasing on* $[a, b]$,
- $f_n \in R(\alpha)$ *for alll* $n \in N$ *on* $[a, b]$,
- $f_n f$ *uniformly on* $[a, b]$.

*Then*

- $f \in R(\alpha)$ *on* $[a, b]$, *and*
- $\int_a^b f\, d\alpha = \lim_{n\infty} \int_a^b f_n\, d\alpha$

*or,* $\int_a^b \lim_{n\infty} f_n\, d\alpha = \lim_{n\infty} \int_a^b f_n\, d\alpha$.

**Proof.** Define $\epsilon_n := \sup_{x\in[a,b]} |f_n(x) - f(x)|\ \forall n \in N$. By (2) for uniform convergence $\epsilon_n 0$ as $n\infty$. $(*)$

Then $|f_n(x) - f(x)| \leq \epsilon_n$ for all $x \in [a, b]$, $\forall n \in N$. This implies

$$(**) \qquad \epsilon_n \leq f(x) - f_n(x) \leq \epsilon_n \implies f_n(x) - \epsilon_n \leq f(x) \leq \epsilon_n + f_n(x).$$

So $_{-}\int_a^b f_n(x) - \epsilon_n\, d\alpha \leq\ _{-}\int_a^b f\, d\alpha \leq\ ^{-}\int_a^b f\, d\alpha \leq\ ^{-}\int_a^b \epsilon_n + f_n(x)\, d\alpha$.

$$\implies 0 \leq\ ^{-}\int_a^b f\, d\alpha -\ _{-}\int_a^b f\, d\alpha \leq \int_a^b \epsilon_n + f_n - f_n + \epsilon_n\, d\alpha = 2\epsilon_n \int_a^b d\alpha$$

$$= 2 \downarrow 0 \text{ as } n\infty \epsilon_n[\alpha(b) - \alpha(a)]$$

.

Fix $\epsilon > 0$. Then $\exists N \in N$ such that $0 \leq\ ^{-}\int_a^b f\, d\alpha -\ _{-}\int_a^b f\, d\alpha < \epsilon$ for $n > N$.

This implies $^{-}\int_a^b f\, d\alpha =_{-} \int_a^b f\, d\alpha = \int_a^b f\, d\alpha \implies f \in R(\alpha)$. Now,

$$\int_a^b f\, d\alpha - \int_a^b f_n\, d\alpha \leq \downarrow 0 \epsilon_n \alpha(b) - \alpha(a) \int_a^b d\alpha \qquad \text{using } (**)$$

For fixed $\epsilon > 0$, $\exists N \in N$ such that $\int_a^b f\, d\alpha - \int_a^b f_n\, d\alpha < \epsilon \implies \int_a^b f\, d\alpha = \lim_{n\infty} \int_a^b f_n\, d\alpha$.

**Recall 84.**—$\in R(\alpha) f_n f$ uniformly on $[a, b] \implies f \in R(\alpha)$ and $\lim_{n\infty} \int_a^b f_n\, d\alpha = \int_a^b f\, d\alpha = \int_a^b \lim_{n\infty} f_n\, d\alpha$.

## § 4.1 Series of functions

We say $\sum_{n=1}^{\infty} f_n(x)$ converges pointwise (uniform) on $(E, d)$ if and only if the sequence of partial sums $s_n$ converges pointwise (uniform) on $(E, d)$.

*Corollary.*—Suppose

- $f_n \in R(\alpha)$ on $[a, b]$, $\forall n$
- $\sum_{n=1}^{\infty} f_n(x) = f(x)$ uniformly on $[a, b]$.

Then $\int_a^b f\, d\alpha = \sum_{n=1}^{\infty} \int_a^b f_n\, d\alpha$ or, $\int_a^b \sum_{n=1}^{\infty} f_n\, d\alpha = \sum_{n=1}^{\infty} \int_a^b f_n\, d\alpha$.

**Proof.** Let $s_n := \sum_{k=1}^{n} f_k$ for each $n \in N$. $f_n \in R(\alpha)$ for each $n \implies s_n \in R(\alpha)$ for each $n \in N$. $\sum f_n$ converges uniformly to $f \implies s_n f$ uniformly. By Theorem 270, $\lim_{n\infty} \int_a^b s_n\, d\alpha = \int_a^b f\, d\alpha$. That is, $\lim_{n\infty} \int_a^b \sum_{k=1}^{n} f_k\, d\alpha = \int_a^b \sum_{n=1}^{\infty} f_n\, d\alpha \implies \lim_{n\infty} \sum_{k=1}^{n} \int_a^b f_k\, d\alpha = \int_a^b \sum_{n=1}^{\infty} f_n\, d\alpha \implies \sum_{n=1}^{\infty} \int_a^b f_n\, d\alpha = \int_a^b \lim_{n\infty} f_n\, d\alpha$.

## § 4.2 Uniform convergence and differentiability

**Note 101.**—Uniform convergence does not necessarily preserve differentiability.

**Recall 85.**—$\epsilon_n := \sup_{x\in E} |f_n(x) - f(x)|$, $f_n f$ uniformly on $E$ if and only if $\epsilon_n 0$ as $n\infty$.

**Example 346.**—Define $f_n(x) = \frac{\sin(nx)}{\sqrt{n}}$; $n \in N$, $x \in [-1, 1]$. Then $f_n 0$ pointwise on $[-1, 1]$. $f_n 0$ uniformly? Define $\epsilon_n := \sup_{x\in[-1,1]} |f_n(x) - 0| = \sup_{x\in[-1,1]} \frac{|\sin(nx)|}{\sqrt{n}} 0$ as $n\infty$. Now, $f(x) \equiv 0$ on $[-1, 1] \implies f'(x) = 0$ ✓. Also, $f_n'(x) = \sqrt{n}\cos(nx)$. In particular, $f_n'(0) = \sqrt{n} \not 0 = f'(0)$ implies that

$$f_n'(x) \not f'(x).$$

**Theorem.** *Suppose*

1. $f_n : [a, b] R$ *differentiable,*
2. $\exists x_0 \in (a, b)$ *such that* $f_n(x_0)_n \subset R$ *converges,*
3. $f_n'$ *converges uniformly.*

*Then*

a) $f_n$ *converges uniformly to a function, say* $f$

b) $\lim_{n\infty} f_n'(x) = f'(x)$

**Proof.** Cases:

a) We'll use Cauchy criterion to show $f_n$ converges uniformly. Let $\epsilon > 0$ be fixed. $f_n(x_0) \subset R$ converges $\implies \exists N_1 \in N$ such that $|f_n(x_0) - f_m(x_0)| < \epsilon/2$ for $n, m > N_1$. Next, $f'_n$ converges uniformly $\implies \exists N_2 \in N$ such that $|f'_n(t) - f'_m(t)| < \frac{\epsilon}{2(b-a)}$ (Not usable in This form!). Since $f_n - f_m$ is differentiable, by MVT, for any $[s_1, s_2] \subset [a, b]$, $\exists t_0 \in (s_1, s_2)$ such that

$$f'_n(t_0) - f'_m(t_0) = \frac{(f_n - f_m)(s_1) - (f_n - f_m)(s_2)}{s_1 - s_2}.$$

Then

$$\begin{aligned}|f_n(s_1) - f_m(s_1) - f_n(s_2) + f_m(s_2)| &= |f'_n(t_0) - f'_m(t_0)|(s_1 - s_2)\\ &< \frac{\epsilon}{2(b-a)}|s_1 - s_2| < \frac{\epsilon}{2}(b-a)(b-a) = \frac{\epsilon}{2}\end{aligned}$$

For $n, m > \max N_1, N_2$ and for all $t \in [a, b]$

$$\begin{aligned}|f_n(t) - f_m(t)| &= |f_n(t) - f_n(x_0) + f_n(x_0) - f_m(x_0) + f_m(x_0) - f_m(t)|\\ &\le |f_n(x_0) - f_m(x_0)| + |f_n(t) - f_n(x_0) + f_m(x_0) - f_m(t)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon\end{aligned}$$

$\implies f_n$ is uniformly Cauchy $\implies f_n$ converges to some $f$.

b) Remains to show: Fix $x \in (a, b)$, $\lim_{n\infty} \lim_{t\neq xtx} \frac{f_n(t)-f_n(x)}{t-x} f'_n(x) = \lim_{t\neq xtx} \frac{f(t)-f(x)}{t-x} f'(x)$.
Define $\phi_n(t) := \frac{f_n(t)-f_n(x)}{t-x}$, $\phi(t) = \frac{f(t)-f(x)}{t-x}$; $t \neq x$. Claim: $\phi_n \phi$ uniformly on $[a, b]x$. Fix $\epsilon > 0$. For $m, n > N_2$ (same $N_2$ as above), $|f'_n(t) - f'_m(t)| < \frac{\epsilon}{2(b-a)}$. Using definition of $\phi_n$, letting $st$, ...

**Recall 86.**—Suppose

- $f_n : [a, b]R$ differentiable
- $\exists x_0 \in (a, b)$ such that $f_n(x_0)$ converges
- $f'_n$ converges uniformly

Then

(I) $f_n$ converges uniformly to, say $f$, ✓✓ and

(II) $\lim_{n\infty} f'_n(x) = f'(x)$; $a < x < b$.

**Proof.** (II) Fix $x \in (a, b)$. NTS: $\lim_{n\infty} f'_n(x) = f'(x)$. Define $\phi_n(t) = \frac{f_n(t)-f_n(x)}{t-x}$ and $\phi(t) = \frac{f(t)-f(x)}{t-x}$; $f_n$ converges uniformly $\implies \phi_n$ converges uniformly on $[a, b]x$. In particular, $\phi_n \phi$ uniformly on $[a, b]x$. Applying Theorem, 268

$$\lim_{n\infty} f'_n(x) \lim_{tx} \phi_n(t) = \lim_{tx} \phi(t) \lim_{n\infty} \phi_n(t)$$

implies $\lim_{n\infty} f'_n(x) = f'(x)$.

**Theorem not in book, stronger assumption.** *Suppose*

- $f_n : [a, b]R$ *such that* $f_n f$ *pointwise on* $[a, b]$*,*
- $f_n'$ *is continuous for each* $n$
- $f_n' g$ *uniformly.*

*Then* $g$ *is continuous* ✓✓ *and* $g = f'$ *on* $[a, b]$*.*

**Proof.** $f_n'$ continuous + $f_n'$unif$g \implies g$ continuous ✓✓. Let $x \in (a, b)$. $f_n'$ continuous on $[a, b] \implies f_n' \in R$ on $[a, b]$. For any $c \leq x$ such that $[c, x] \subset [a, b]$, since $f_n'$unif$g$

$$\int_c^x \lim_{n\infty} f_n'(t) g(t)\, dt \text{Thm } 270 = \lim_{n\infty} \int_c^x f_n'(t)\, dt$$

$$\implies \int_c^x g(t)\, dt = \lim_{n\infty} \int_c^x f_n'(t)\, dt \text{FTC} = \lim_{n\infty} [\downarrow f(x) f_n(x) - \downarrow f(c) f_n(c)] = f(x) - f(c)$$

implies $g(x) = f'(x)$.

## § 4.3 Space of functions

$(X, d)$ – metric space

$C(X) = f : XRf$ continuous + bounded – space of continuous bounded functions

If $X$ is compact, $C(X) = f : XRf$ continuous

✓ Vector Space with + and $\cdot$ defined as

$$(f_1 + f_2)(x) = f_1(x) + f_2(x)(cf)(x) = cf(x) \forall x \in X, c \in R.$$

Define norm on $C(X)$ by: $||f||_\infty := \sup_{x \in X} |f(x)|$ satisfies:

1. $||f||_\infty \geq 0$ for all $f \in C(X)$ and $||f||_\infty = 0$ if and only if $f \equiv 0$.
2. $||cf||_\infty = |c|\, ||f||_\infty$; $c \in R,\ f \in C(X)$
3. $||f + g||_\infty \leq ||f||_\infty + ||g||_\infty$ – triangle inequality

   $(\sup_{x \in X} |(f + g)(x)| \leq \sup_{x \in X} |f(x)| + \sup_{x \in X} |g(x)|)$

Metric on $C(X)$ induced by $||f||_\infty$ (supremum norm) $d(f, g) = ||f - g||_\infty$. $C(X)$ is complete (every Cauchy sequence converges) with respect to $|| \cdot ||_\infty$. That is, if $f_n$ is a Cauchy sequence in $C(X)$, then $\forall \epsilon > 0$, $\exists N \in N$ such that for $n, m > N$ $||f_n - f_m||_\infty < \epsilon$. – Such spaces are called Banach spaces: A complete normed space. Hilbert space – A complete normed space, where norm is induced by inner product. Inner product (Dot product in $R^k$) $, = x_1 y_1 + \cdots + x_k y_k$. Then $||||_{R^k} = \sqrt{x, x}$. For function space, replace by $f$.

**Note 102.**—On $C([a,b])$, we can define

$$||f||_\infty = \sup_{[a,b]} |f(x)|, ||f||_1 = \int_a^b |f(t)|\, dt \text{define norms}$$

$C([a,b])$ is complete with $||f||_\infty$ but not complete with $||f||_1$.

**Example 347** ($C[0,1]$)**.**—Take $f_n(x) = \begin{cases} nx; & 0 \le x \le \frac{1}{n} \\ 1; & x > \frac{1}{n} \end{cases}$ on $[0,1]$. $f_n f(x) = \begin{cases} 1; & 0 < x \le 1 \\ 0; & x = 0 \end{cases}$ not continuous.

Extra credit problems chapter 7, exercises 1,2,3.
Final exam: Fri 3:30-6:30 (Here)
Definitions (10 pts)
True/False (15 pts) – Relating theorems and examples
Ch 7 (5 pts) – (1,2,3,7 + definitions)
7 problems to choose from 9 – must do 2 from each3 from each section

**Example 348** (Test 3, 1b)**.**—$\alpha : [a,b]R \uparrow +$ continuous at $x_0$, $f(x) = \begin{cases} 1 & x \ne x_0 \\ 0 & x = x_0 \end{cases}$. Prove $f \in R(\alpha)$.

**Proof.** Since $f$ is bounded + discontinuous at $x_0$ and $\alpha$ is continuous at $x_0$, then $f \in R(\alpha)$.
$\int_a^b f\, d\alpha = 0$: Ever $P$ contains $x \ne x_0$, $\inf_{[x_{i-1},x_i]} f(x) = 0$ therefore

$$\int_a^b f\, d\alpha = \sup_P (L(P,f,\alpha)) = 0$$

.

**Proof.** Define $f^*(x) = \begin{cases} f(1) & 0 \le x < 1 \\ f(x) & x \ge 1 \end{cases}$. Let $P : 0 \le 1 \le \cdots \le n-1 \le n$ be a partition of $[0,m]$.

$$U(P,f) = \sum_{i=1}^{n} f^*(i)$$
$$L(P,f) = \sum_{i=0}^{n-1} f^*(i)$$

$f \in R$ on $[1,m] \implies f^* \in R$ on $[0,m]$, so

$$L(P,f^*) = \sum_{i=0}^{n-1} \le \int_0^n f^*(x) \le \sum_{i=1}^{n} f^*(i) = U(P,f).$$

( $\implies$ ) Suppose $\int_1^\infty f(x)\,dx$ converges. Then

$$\lim_{n\infty}\int_1^n f(x)\,dx < \infty \implies \lim_{n\infty}\int_0^m f^*(x)\,dx < \infty.$$

$$\sum_{i=0}^{n-1} f^*(i) \le \int_0^\infty f^*(x)\,dx < \infty$$

$\implies$ $s_n$ is bounded Thm 3.23 $\implies$ $\sum_{i=0}^\infty f^*(i)$ converges.

() Suppose by contrapositive $\int_1^\infty f(x)\,dx$ diverges, then

$$\begin{aligned} &\implies \lim_{n\infty}\sum_{i=1}^n f(x) < \infty \\ \implies \infty = \int_1^n f(x)\,dx \le \sum_{i=1}^n f(i) &\implies = \infty \\ &\implies \sum_{i=1}^\infty f(i) = \infty. \end{aligned}$$

*Test 3, 4.* BWOC, suppose $f(x_0) = c > 0$ for some $x_0 \in [a,b]$. By continuity, there is a $\delta$-neighborhood $(x_0-\delta, x_0+\delta)$ such that $|f(x_0) - f(x)| < \frac{c}{2} \implies f(x) > \frac{c}{2}$ for all $x \in (x_0-\delta, x_0+\delta)$.

$$\begin{aligned} -\frac{c}{2} &< f(x_0) - f(x) < \frac{c}{2} \\ -\frac{c}{2} &< \boxed{c - f(x) < \tfrac{c}{2}} \\ \frac{c}{2} &< f(x) \end{aligned}$$

Let $P$ be a partition such that $x_{j-1} = x_0 - \delta$ and $x_j = x_0 + \delta$. Then $m_j \ge \frac{c}{2}$ and $M_j \ge f(x_0) = c$. So...

- $L(P,f) \ge \frac{c}{2}$
- $U(P,f) \ge c$

In particular, we know $\int_a^b f(x)\,dx \ge L(P,f) \ge \frac{c}{2} > 0$. But $\int_a^b f(x)\,dx = 0$, a contradiction. Thus $f(x) = 0$ for all $x \in [a,b]$.

**Example 349** (Counterexample)**.**—$f(x) = \begin{cases} 1 & \text{if } x \in Q \\ -1 & \text{if } x \in RQ \cap [a,b] \end{cases}$. Clearly, $f^2 \in R$, since $f(x)^2 = 1$ for all $x \in [a,b]$. However, by the density of $Q$ and $RQ$ in $[a,b]$, we know that for any partition $P$, $m_j = -1$, $M_j = 1$ for all $j \in [1,n]$. Thus $_-\int f(x)\,dx = \sup_P \sum_i (-1)\Delta x_i = -(b-a)$ and $^-\int f(x)\,dx = \inf_P \sum_i \Delta x_i = b-a$. Thus, since $b \ne a$, we have $_-\int f(x)\,dx \ne ^-\int f(x)\,dx$ and so $f \notin R$.