Introduction to the Theory of Computation

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March 4, 2021

About the Course

The textbook is

- J. E. Hopcroft, R. Motwani, J. D. Ullman. Introduction to Automata Theory, Languages, and Computation (3rd edn), Pearson, 2007.

Two extended references are

- M. D. Davis, R. Sigal, E. J. Weyuker. Computability, Complexity, and Languages: Fundamentals of Theoretical Computer Science (2nd edn), Morgan Kaufmann, 1994.
- M. Sipser. Introduction to the Theory of Computation (3rd edn), Cengage, 2012.

Arrangement

- Preliminaries — 1 LECTURE
- Finite Automata — 6 LECTURES
  (Finite Automata, Regular Expressions, Regular Grammars)
- Grammars — 6 LECTURES
  (Context-Free Grammars and Languages, Pushdown Automata)
- Turning Machines — 3 LECTURES
  (Turing Machines)

All of them are covered by Chaps. 1–8 in the textbook.

Pre-requisites

- You have a good working knowledge of material for this course.
- It is highly recommended that you brush up on the following topics: sets, relations, functions, logic, proof techniques, graph theory, counting techniques, permutations and combinations, etc.

Evaluation

- Every student is required to submit some projects (30–40% of total marks) for developing the course.
- The final examination (60–70% of total marks) will be a two-hour examination at the end of the semester.

Contact Me

XU Ming, Associate Research Professor,

my interest is the *theoretical aspects of computing,*

to explore the **LIMITS** between solvable and unsolvable problems.

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Or visit me at Room B1005, Science Building.

该二维码7天内(3月9日前)有效，重新进入将更新

The teacher assistant is NAN Yi, `51205902074@stu.ecnu.edu.cn`, Room B218.

Outline

- Inductive Proofs
- Central Concepts of Automata Theory

Inductive Proofs

- Induction is an extremely important proof technique that can be used to prove assertions, it is used extensively to prove results about a large variety of recursively defined objects.

- Many of the most familiar inductive proofs deal with integers, but in automata theory, we also need inductive proofs about recursively defined concept as expressions of various sorts.

Induction on Integers

In order to prove a statement $S(n)$ on an integer n, one common approach is proving two facts:

1. *The basis step*—we show $S(i)$ is true for a particular integer i. (Usually, $i = 0$ or $i = 1$.)
2. *The inductive step*—we assume $n \geq i$, where i is the basis integer, and show that "if $S(n)$ then $S(n+1)$".

Intuitively, the two facts could convince us that $S(n)$ is true for all $n \geq i$.

Why Induction Is Valid?

The reason comes from **the well-ordering property**:

Every nonempty set of nonnegative integers has a least element.

Proof Suppose $S(n)$ was false for one or more of those integers. Then, by the well-ordering property, there would have to be a smallest value of n, say j, for which $S(j)$ is false, and yet $j \geq i$.

Now j could not be i, because we prove in the basis part that $S(i)$ is true. Thus, j must be greater than i. We now know that $j - 1 \geq i$, and $S(j - 1)$ is true.

However, we proved in the inductive part that if $n \geq i$, then $S(n)$ implies $S(n+1)$. Suppose $n = j - 1$. Then we know from the inductive step that $S(j - 1)$ implies $S(j)$. Since $S(j - 1)$ is true, we can infer $S(j)$.

We have assumed the negation of which we wanted to prove; that is, assumed $S(j)$ was false for some $j \geq i$. In the case, we derived a contradiction. So $S(n)$ is true for all $n \geq i$. □

Thus, we proved our logical reasoning system:

The Induction Principle

If we prove i) $S(i)$ and ii) $\forall n \geq i$, $S(n)$ implies $S(n+1)$, then we conclude $S(n)$ for all $n \geq i$.

Example

The harmonic numbers H_k, $k = 1, 2, \ldots$ are defined by

$$H_k = 1 + \frac{1}{2} + \cdots + \frac{1}{k}$$

Use induction principle to prove that

$$H_{2^n} \geq 1 + \frac{n}{2}.$$

This inequality can be used to show that the harmonic series is a divergent infinite series.

Proof   Let $S(n)$ be the proposition that $H_{2^n} \geq 1 + n/2$. We need to prove that $S(n)$ is true for $n = 0, 1, 2, \ldots$.

Basis step: $S(0)$ is true, since $H_{2^0} = H_1 = 1 \geq 1 + 0/2$.

Inductive step: Assume that $S(n)$ is true, so that $H_{2^n} \geq 1 + n/2$. It must be shown that $S(n+1)$, which states that $H_{2^{n+1}} \geq 1 + (n+1)/2$, must also be true under this assumption.

This can be done since

$$\begin{aligned} H_{2^{n+1}} &= 1 + \cdots + \frac{1}{2^n} + \frac{1}{2^n+1} + \cdots + \frac{1}{2^{n+1}} \\ &= H_{2^n} + \frac{1}{2^n+1} + \cdots + \frac{1}{2^{n+1}} \\ &\geq (1 + \frac{n}{2}) + 2^n \cdot \frac{1}{2^{n+1}} \\ &= 1 + \frac{n+1}{2}. \end{aligned}$$

This establishes the inductive step of the proof. Thus, the inequality for the harmonic numbers is valid for all nonnegative integers n. $\square$

Question What is wrong with the following “proof” that

all horses are the same color?

Let $S(n)$ be the proposition that all the horses in a set of n horses have the same color.

Basis step: $S(1)$ is true.

Inductive step: Assume that $S(n)$ is true, so that all the horses in any set of n horses have the same color. Consider any $n+1$ horses; number these horses $1, 2, \ldots, n, n+1$.

Now the first n of these horses all must have the same color, and the last n of these must have the same color.

Since the set of the first n horses and the set of the last n horses overlap, all $n+1$ must have the same color. This shows that $S(n+1)$ is true and completes the proof by induction. $\square$

Sometimes an inductive proof is made possible only by using a more general scheme than the one proposed. Two important generalizations of this scheme are:

1. Use several basis cases. i.e., we prove $S(i)$, $S(i+1)$, …, $S(j)$ for some $j \geq i$.
2. In proving $S(n+1)$, $(n \geq j)$, use the truth of all the statements $S(i)$, $S(i+1)$, …, $S(n)$ rather than just using $S(n)$.

The conclusion to be made from this basis and inductive step is that $S(n)$ is true for all $n \geq i$.

Example

Show that if n is an integer greater than 1, the n can be written as the product of primes.

Proof Let $S(n)$ be the proposition that n can be written as the product of primes.

Basis step: $S(2)$ is true, since 2 can be written as the product of one prime, itself.

Inductive step: Assume that $S(k)$ is true for all positive integers k with $2 \leq k \leq n$. To complete the inductive step, it must be shown that $S(n+1)$ is true under this assumption.

There are two cases to be considered, namely, when $n + 1$ is prime and when $n + 1$ is composite. If $n + 1$ is prime, we immediately see that $S(n + 1)$ is true. Otherwise, $n + 1$ is composite and can be written as the product of two positive integers a and b with $2 \leq a \leq b < n + 1$.

By the induction hypothesis, both a and b can be written as the products of primes. Thus, if $n + 1$ is composite, it can be written as the product of primes, namely, those primes in the factorization of a and those in factorization of b. $\square$

Example

Prove that every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps.

Proof We will prove this result using the induction principle first. Then we will present a proof using the general forms of integer induction.

Let $S(n)$ be the statement that postage of n cents can be formed using 4-cent and 5-cent stamps.

We begin by using the induction principle.

Basis step: Postage of 12 cents can be formed using three 4-cent stamps.

Inductive step: Assume that $S(n)$ is true, so that postage of n cents can be formed using 4-cent and 5-cent stamps. If at least one 4-cent stamp was used, replace it with a 5-cent stamp to form postage of $n+1$ cents. If no 4-cent stamp was used, postage of n cents was formed using just 5-cent stamps. Since $n \geq 12$, at least three 5-cent stamps were used. So replace three 5-cent stamps with four 4-cent stamps to form postage of $n+1$ cents. This completes the inductive step. $\square$

Next, we use the general forms of induction.

Basis step: We can form postage of 12, 13, 14 and 15 cents using three 4-cent stamps, two 4-cent stamps and one 5-cent stamp, one 4-cent stamp and two 5-cent stamps, and three 5-cent stamps, respectively.

Inductive step: Let $n \geq 15$. Assume that we can form postage of k cents, where $12 \leq k \leq n$. To form postage of $n + 1$ cents, use the stamps that form postage of n – 3 cents together with a 4-cent stamp. This complete the inductive step. □

Structural Inductions

In computer science theory, there are several recursively defined structures about which we need to prove statements. The familiar notions of trees and expressions are important examples.

Like inductions, all recursive definitions have a basis case, where one or more elementary structures are defined, and an inductive step, where more complex structures are defined in terms of previously defined structures.

Example

The recursive definition of a tree:

Basis step: A single node is a tree, and that node is the root of the tree.

Inductive step: If $T_1, T_2, \ldots, T_k$ are trees, then we can form a new tree as follows:

1. Begin with a new node N, which is the root of the tree.
2. Add copies of all the trees $T_1, T_2, \ldots, T_k$.
3. Add edges from N to the roots of each of the trees $T_1, T_2, \ldots, T_k$.

Example

Define expressions using the arithmetic operators $+$ and $*$, with both numbers and variables allowed as operands.

Basis step: Any number or letter is an expression.

Inductive step: If E and F are expressions, so are $E + F$, $E * F$, and (E).

For example, both 2 and x are expressions by the basis. The inductive step tell us $x + 2$, and $2 * (x + 2)$ are all expressions.

When we have a recursive definition, we can prove theorems about it using the following proof form, which is called structural induction. Let $S(X)$ be a statement about the structures X that are defined by some particular recursive definition.

1. As a basis step, prove $S(X)$ for the basis structure(s) X.
2. For the inductive step, take a structure X that the recursive definition says is formed from $Y_1, Y_2, \ldots, Y_k$. Assume that the statements $S(Y_1), S(Y_2), \ldots, S(Y_k)$ are true, and use these to prove that $S(X)$ is true.

The conclusion is that $S(X)$ is true for all X.

Example

Every tree has one more node than it has edges.

Proof The formal statement $S(T)$ is “if T is a tree with n nodes and e edges, then $n = e + 1$.”

Basis step: The basis case is when T is a single node. Then $n = 1$ and $e = 0$, so the relation $n = e + 1$ holds.

Inductive step: Let T be a tree built by the inductive step of the definition, from root node N and k smaller trees $T_1, T_2, \ldots, T_k$. Assume that the statements $S(T_i)$ hold for $i = 1, 2, \ldots, k$, i.e., let T_i have n_i nodes and e_i edges; then $n_i = e_i + 1$.

The nodes of T are node N and all the nodes of the T_i's. There are $1 + n_1 + n_2 + \cdots + n_k$ nodes in T. The edges of T are the k edges we added explicitly in the inductive definition step, plus the edges of the T_i's.

Hence, T has $k + e_1 + e_2 + \cdots + e_k$ edges. If we substitute $e_i + 1$ for n_i in the count of the number of nodes of T we find that T has $1 + (e_1 + 1) + (e_2 + 1) + \cdots + (e_k + 1) = k + 1 + e_1 + e_2 + \cdots + e_k$ nodes. This expression is exactly 1 more than last one that was given for the number of edges of T. Thus, T has one more node than its edges. □

Example

Every expression has an equal number of left and right parentheses.

Proof Formally, we prove the statement $S(G)$ about any expression G that is defined by the recursive definition: the numbers of left and right parentheses in G are the same.

Basis step: If G is defined by the basis, then G is a number or variable. These expressions have 0 left parentheses and 0 right parentheses, so the numbers are equal.

Induction step: There are three rules whereby expression G may have been constructed according to the inductive step in the definition: 1)$G = E + F$, 2)$G = E * F$, 3) $G = (E)$.

Assume that $S(E)$ and $S(F)$ are true; i.e., E has the same numbers of left and right parentheses, say n of each, and F likewise has the same numbers of left and right parentheses, say m of each.

Now compute the numbers of left and right parentheses in G for each of three cases:

1. If $G = E + F$, then G has $n + m$ left and right parentheses; n of each come from E and m of each come from F.
2. If $G = E * F$, the count of parentheses for G is again $n + m$ of each.
3. If $G = (E)$, then there are $n + 1$ left parentheses in G—one left parentheses is explicitly shown, and the other n are present in E. Likewise, there are $n + 1$ right parentheses in G.

In each of the three cases, the numbers of left and right parentheses in G are the same. This completes the proof. □

Question Why structural induction is a valid proof method?

Imagine the recursive definition establishing, on at a time, that contains structure $X_1, X_2, \ldots$ meet the definition. The basis elements come first, and the fact that X_i is in the defined set of structures can only depend on the membership in the defined set of the structures that precede X_i on the list.

Viewed this way, a structural induction is nothing but an induction on integer n of the statement $S(X_n)$. This induction may be of the generalized form, with multiple basis cases and an inductive step that uses all previous instances of the statement.

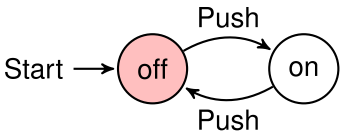
Mutual Inductions

Sometimes, we need to prove a group of statements $S_1(n), S_2(n), \ldots, S_k(n)$ together by induction on n. Automata theory provides many such situations.

In fact, proving a group of statements is not different from proving the conjunction $S_1(n) \wedge S_2(n) \wedge \cdots \wedge S_k(n)$ of all the statements. However, when there are really several independent statements to prove, it is generally less confusing to keep the statements separately and to prove them all in their own parts of the basis and inductive steps. We call this sort of proof mutual induction.

Example

Consider an on/off switch, the simplest finite automaton. The device remembers whether it is in the *on* state or the *off* state, and it allows the user to press a button whose effect is different, depending on the state of the switch.



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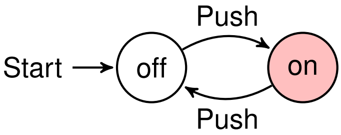

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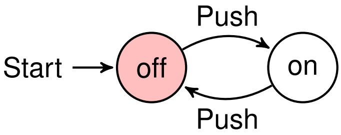

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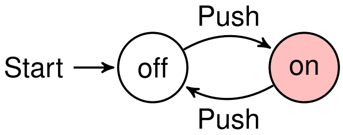


Since pushing the button switches the state between *on* and *off*, and the switch starts out in the *off* state, we expect that the following statements will together explain the operation of the switch:

$S_1(n)$: The automaton is in state off after n pushes if and only if n is even.
$S_2(n)$: The automaton is in state on after n pushes if and only if n is odd.

Proof We give the basis and inductive parts of the proofs of the statements $S_1(n)$ and $S_2(n)$ below.

Since there are two statements, each of which must be proved in both directions, there are actually four cases to the basis, and four cases to the induction as well.

Basis step: For the basis, we choose $n = 0$.

1. [$S_1(0)$; If] Since 0 is in fact even, we must show that after 0 pushes, the automaton is in state *off*. Since that is the start state, the automaton is indeed in state *off* after 0 pushes.
2. [$S_1(0)$; Only-if] The automaton is in state *off* after 0 pushes, so we must show that 0 is even. But 0 is even by definition, so there is nothing more to prove.

3. [$S_2(0)$; If] The hypothesis of the "if" part of S_2 is that 0 is odd. Since the hypothesis is false, the if-then statement is true. Thus, this part of the basis also holds.
4. [$S_2(0)$; Only-if] The hypothesis is also false, since the only way to get to state on is by following an arc labeled Push, which requires that the button be pushed at least once. Since the hypothesis is false, we can again conclude that the if-then statement is true.

Induction step: Now, we assume that $S_1(n)$ and $S_2(n)$ are true, and try to prove that $S_1(n+1)$ and $S_2(n+1)$ are true.

1. [$S_1(n+1)$; If] The hypothesis for this part is that $n+1$ is even. Thus, n is odd. The "if" part of statement $S_2(n)$ says that after n pushes, the automaton is in state on. The arc from on to *off* labeled Push tells us that the $(n+1)$st push will cause the automaton to enter state *off*. That completes the proof of the "if" part of $S_1(n+1)$.
2. [$S_1(n+1)$; Only-if] The hypothesis is that the automaton is in state off after $n+1$ pushes. Inspecting the automaton tells us that the only way to get to state *off* after one or more moves is to be in state *on* and receive an input Push. Thus, if we are in state *off* after $n+1$ pushes, we must have been in state *on* after n pushes.

Then, we may use the "only-if" part of statement $S_2(n)$ to conclude that n is odd. Consequently, $n+1$ is even, which is the desired conclusion for the only-if portion of $S_1(n+1)$.

1. $[S_2(n+1);$ If$]$ This part is essentially the same as part (1), with the roles of statements S_1 and S_2 exchanged, and with the roles of "odd" and "even" exchanged.
2. $[S_2(n+1);$ Only-if$]$ This part is essentially the same as part (2), with the roles of statements S_1 and S_2 exchanged, and with the roles of "odd" and "even" exchanged.

$\square$

We can abstract from this example the pattern for all mutual inductions:

- Each of the statements must be proved separately in the basis and in the inductive step.
- If the statements are “if and only if”, then both direction of each statement must be proved, both in the basis and in the induction.

Central Concepts of Automata Theory

Languages

We start with a finite, nonempty set Σ of symbols, called the alphabet.

Example

$\Sigma = \{0, 1\}$, the binary alphabet.

Example

$\Sigma = \{a, b, \ldots, z\}$, the set of all lower-case letters.

Example

The set of all ASCII characters, or the set of all printable ASCII characters.

From the individual symbols we construct strings, which are finite sequence of symbols from the alphabet.

Example

00011101011 is a string from the binary alphabet $\Sigma = \{0, 1\}$.

We will need to refer to the empty string, which is a string with zero occurrences of symbols from an alphabet.

The empty string is denoted ϵ.

The concatenation of strings x and y, denoted xy, is the string obtained by appending the symbols of y to the right end of x, that is, if $x = a_1 a_2 \cdots a_i$, $y = b_1 b_2 \cdots b_j$, then $xy = a_1 a_2 \cdots a_i b_1 b_2 \cdots b_j$.

Example

Let $x = 01101$, and $y = 110$. Then $xy = 01101110$.

Notice that: For any string w, $\epsilon w = w \epsilon = w$.

The length of a strings is the number of positions for symbols in the string.

The length of a string w is denoted $|w|$. For example, $|00110| = 5$, $|\epsilon| = 0$.

The power of an alphabet is a set of strings of a certain length from an alphabet.

The set of strings of length k, each of whose symbols is in Σ, is denoted Σ^k.

For example, if $\Sigma = \{0, 1\}$, then $\Sigma^0 = \{\epsilon\}$, $\Sigma^1 = \{0, 1\}$, $\Sigma^2 = \{00, 01, 10, 11\}$.

The set of all strings over an alphabet Σ is denoted Σ^*. For instance, $\{0, 1\}^* = \{\epsilon, 0, 1, 00, 10, 01, 11, 000, \ldots\}$.

The set of nonempty strings from alphabet Σ is denoted Σ^+.

1. $\Sigma^+ = \Sigma^1 \cup \Sigma^2 \cup \Sigma^3 \cup \cdots$
2. $\Sigma^* = \Sigma^+ \cup \{\epsilon\} = \Sigma^0 \cup \Sigma^1 \cup \Sigma^2 \cup \Sigma^3 \cup \cdots$

- The set Σ^* of strings over the alphabet Σ can be defined recursively:

  Basis step: $\epsilon \in \Sigma^*$.

  Inductive step: if $w \in \Sigma^*$ and $x \in \Sigma$, then $wx \in \Sigma^*$.
- The length of a string $|w|$ can be also recursively defined by

$$|\epsilon| = 0; \quad |wx| = |w| + 1 \text{ if } w \in \Sigma^* \text{ and } x \in \Sigma.$$

Question Prove $|xy| = |x| + |y|$, where x and y belong to Σ^*.

A language is defined as a subset of Σ^*, where Σ is a particular alphabet. A string in a language L will be called a sentence of L.

Example

The set of legal English words is a language over the alphabet that consists of all the letters.

Example

The set of legal JAVA programs is a language over the set of all ASCII characters.

Example

The language of all strings consisting of n 0's followed by n 1's, for some $n \geq 0$:

$$\{\epsilon, 01, 0011, 000111, \ldots\}.$$

Example

The language of strings of 0's and 1's with an equal number of each:

$$\{\epsilon, 01, 10, 0011, 0101, 1001, \ldots\}.$$

Example

The language of binary numbers whose value is a prime:

$$\{10, 11, 101, 111, 1011, \ldots\}.$$

- For any alphabet Σ, Σ^* is a language.
- $\emptyset$, the empty language, is a language over any alphabet.
- $\{\epsilon\}$, the language consisting of only the empty string, is also a language over any alphabet.

Notice that: $\emptyset \neq \{\epsilon\}$.

- It is common to describe a language using a “predicate-former”:

$$\{w \mid \text{somthing about } w\}.$$

  For example, $\{w \mid w$ consists of an equal number of 0’s and 1’s$\}$.
- It is also common to replace w by some expressions with parameters and describe the strings in the language by stating conditions on the parameters. For example, $\{0^i 1^j \mid 0 \leq i \leq j\}$ and $\{0^n 1^n \mid n \geq 1\}$.

Since languages are sets, the union, intersection, and difference of two languages are immediately defined. Let L and M are both languages, there are other two operations on languages.

- Concatenation (dot) $L.M = \{w \mid w = xy, x \in L, y \in M\}$
- Closure (star) $L^* = \bigcup_{i=0}^{\infty} L^i$,
  where $L^0 = \{\epsilon\}$, $L^1 = L$, and $L^{k+1} = L.L^k$ $(k = 1, 2, \ldots)$.

The idea of the closure of a language is somewhat tricky!

Question

- $L = \{0, 11\}$, $L^* =?$
- $L = \{\text{all strings of 0's}\}$, $L^* =?$
- $L = \emptyset$, $L^* =?$
- $L = \{\epsilon\}$, $L^* =?$

Let L, M, N are all languages, we have the following algebraic laws.

- $L.(M.N) = (L.M).N$
  Concatenation is *associative*.
- $\{\epsilon\}.L = L.\{\epsilon\} = L$
  $\{\epsilon\}$ is *identity* for concatenation.
- $\emptyset.L = L.\emptyset = \emptyset$
  $\emptyset$ is *annihilator* for concatenation.

- $L.(M \cup N) = L.M \cup L.N$
  Concatenation is left distributive over union.
- $(M \cup N).L = M.L \cup N.L$
  Concatenation is right distributive over union.
- $(L^*)^* = L^*$
  Closure is idempotent.

Question Is concatenation communicative, i.e. $L.M = M.L$?

Grammars

A grammar is a 4-tuple $G = (V, T, P, S)$, where

- V is a finite set of variables,
- T is a finite set of terminal symbols,
- P is a finite set of productions of the form $x \to y$, where $x \in (V \cup T)^+$ and $y \in (V \cup T)^*$,
- $S \in V$ is a designated variable called the start symbol.

Now we develop the notation for describing the derivations.

Let $G = (V, T, P, S)$ be a grammar, $\alpha, \beta \subset (V \cup T)^*$, and $x \to y \in P$. Then we write

$$\alpha x \beta \underset{G}{\Longrightarrow} \alpha y \beta$$

or, if G is understood

$$\alpha x \beta \Rightarrow \alpha y \beta$$

and say that $\alpha x \beta$ derives $\alpha y \beta$.

Specially, we have $x \Rightarrow y$.

We may extend the $\Rightarrow$ relationship to present zero, one, or many derivation steps.

In other words, we define $\overset{*}{\Rightarrow}$ to be the reflexive and transitive closure of $\Rightarrow$, as follows:

Basis step: Let $\alpha \in (V \cup T)^*$. Then $\alpha \overset{*}{\Rightarrow} \alpha$.

Inductive step: If $\alpha \overset{*}{\Rightarrow} \beta$, and $\beta \Rightarrow \gamma$, then $\alpha \overset{*}{\Rightarrow} \gamma$.

Use induction we can prove that if $\alpha \overset{*}{\Rightarrow} \beta$ and $\beta \overset{*}{\Rightarrow} \gamma$, then $\alpha \overset{*}{\Rightarrow} \gamma$.

Let $G(V, T, P, S)$ be a grammar. Then the set

$$L(G) = \{w \in T^* \mid S \overset{*}{\underset{G}{\Rightarrow}} w\}$$

is the language generated by G.

If $w \in L(G)$, then the sequence $S \Rightarrow w_1 \Rightarrow \cdots \Rightarrow w_n = w$ is a derivation of the sentence w. The strings $S, w_1, \ldots, w_n$, which contain variables as well as terminals, are called sentential forms of the derivation.

In general, we call $L = \{w \in T^* \mid A \overset{*}{\underset{G}{\Rightarrow}} w\}$ the language of variable A if $A \in V$.

Example

Consider the grammar $G = (V, \{a, b\}, P, S)$, with P given by

$$S \to aSb, S \to \epsilon.$$

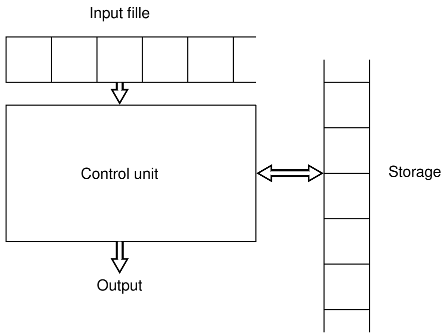
The string *aabb* is a sentence in the language generated by G.

A grammar G completely defines $L(G)$, but it may not be easy to get a very explicit description of the language from the grammar.

Here, however, the answer is fairly clear. It is not hard to conjecture that $L(G) = \{a^n b^n \mid n \geq 0\}$, and it is easy to prove it.

Automata

An automaton is an abstract model of a digital computer. Following figure shows a schematic representation of a general automaton.



This general model covers all the automata we will discuss in this course. A finite-state control will be common to all specific cases, but differences will arise from the way in which the output can be produced and the nature of the temporary storage.

It is necessary to distinguish between deterministic automata and nondeterministic automata. A deterministic automaton is one in which each move in uniquely determined by current configuration. In a nondeterministic automaton, this is not so.