Chapter 26

Network Flow

Maximum Flow and Minimum Cut

Max flow and min cut.

Two very rich algorithmic problems.

Cornerstone problems in combinatorial optimization.

Beautiful mathematical duality.

Nontrivial applications / reductions.

Data mining.

Open-pit mining.

Project selection.

Airline scheduling.

Bipartite matching.

Baseball elimination.

Image segmentation.

Network connectivity.

Network reliability.

Distributed computing.

Egalitarian stable matching.

Security of statistical data.

Network intrusion detection.

Multi-camera scene reconstruction.

Many many more . . .

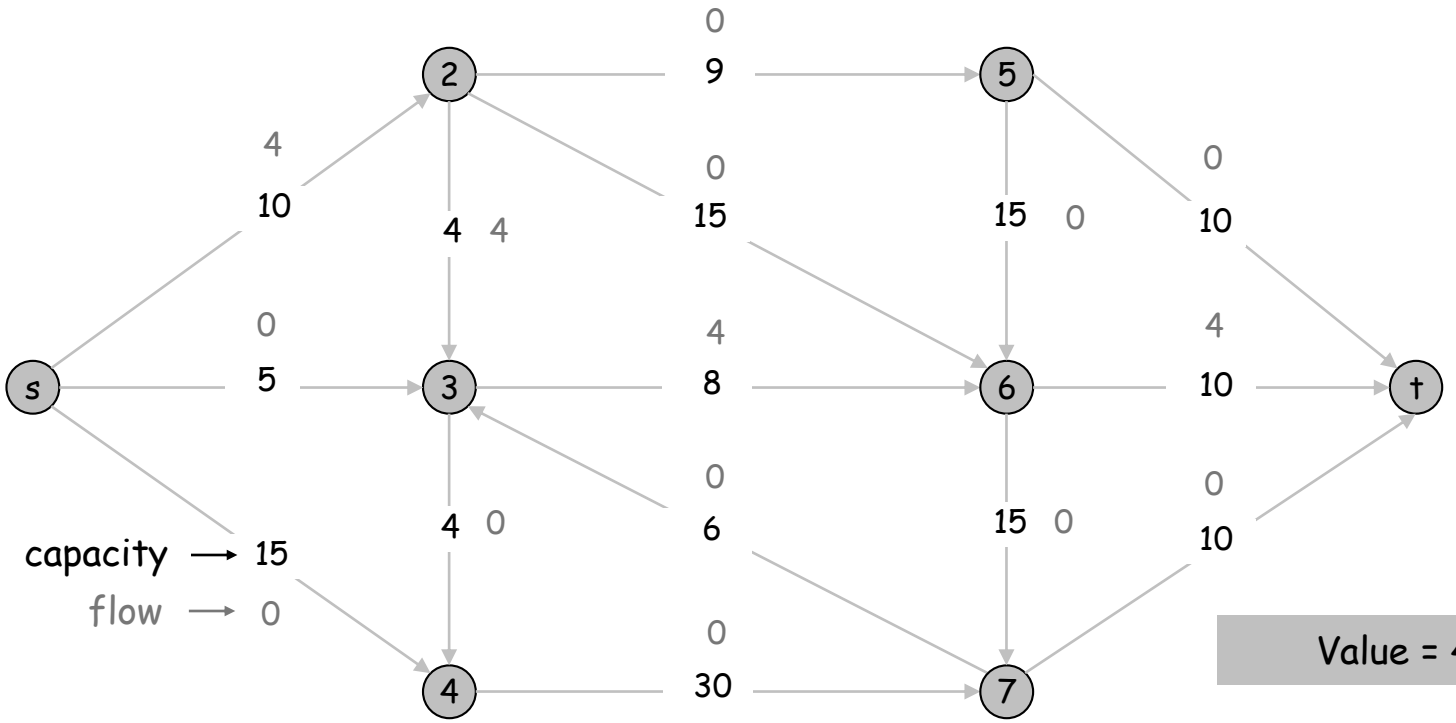
Flows

Def. An s-t flow is a function that satisfies:

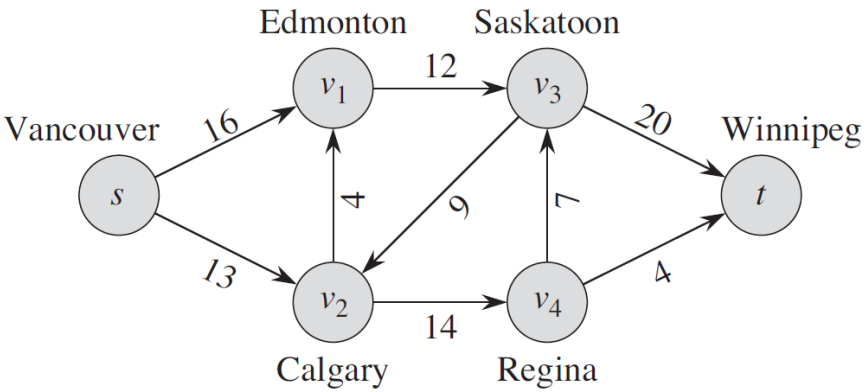
For each $e \in E$: $0 \le f(e) \le c(e)$ (capacity)

For each $v \in V - \{s, t\}$: $\sum_{e \text{ in to } v} f(e) = \sum_{e \text{ out of } v} f(e)$ (conservation)

Def. The value of a flow f is: $v(f) = \sum_{e \text{ out of } s} f(e)$.



Value = 4



Capacity constraint: For all $u, v \in V$, we require $0 \leq f(u, v) \leq c(u, v)$.

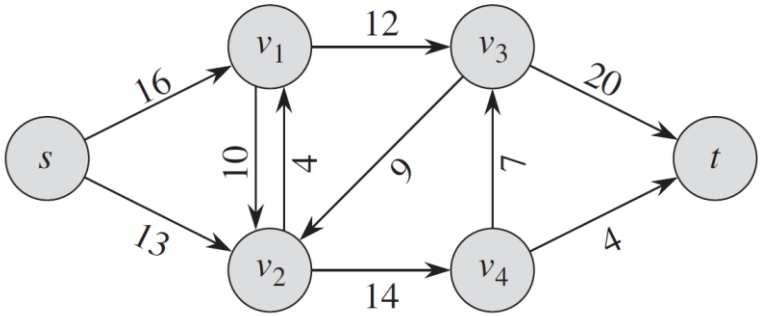
Flow conservation: For all $u \in V - \{s, t\}$, we require

$$\sum_{v \in V} f(v, u) = \sum_{v \in V} f(u, v) \ .$$

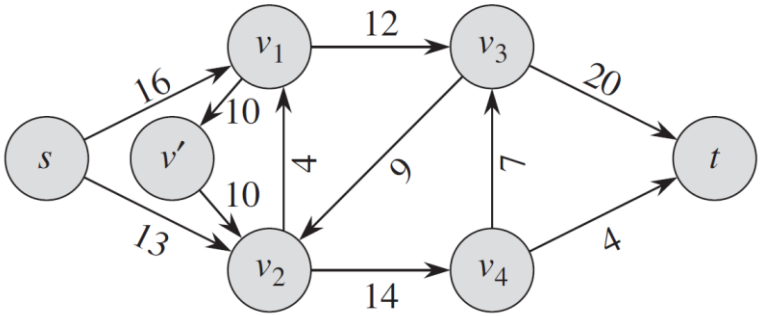
When $(u, v) \notin E$, there can be no flow from u to v, and $f(u, v) = 0$.

value $|f|$ of a flow f is defined as

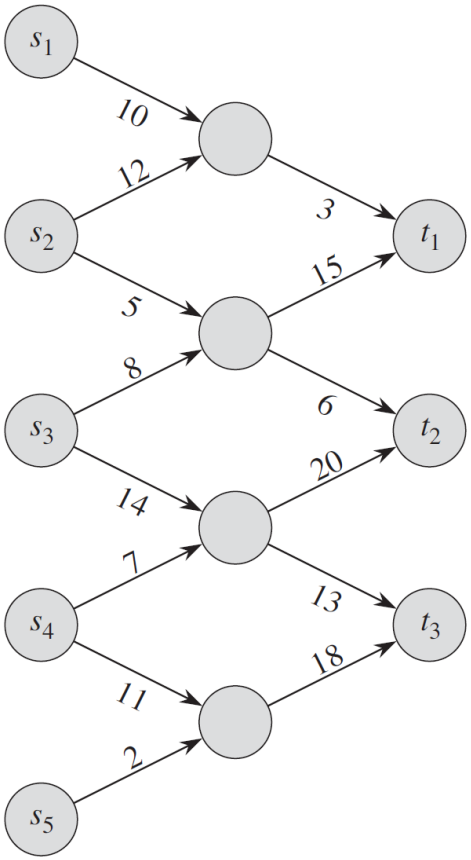
$$|f| = \sum_{v \in V} f(s, v) - \sum_{v \in V} f(v, s) \ ,$$


s
v1
v2
v3
v4
t
16
13
12
14
10
4
9
7
20
4

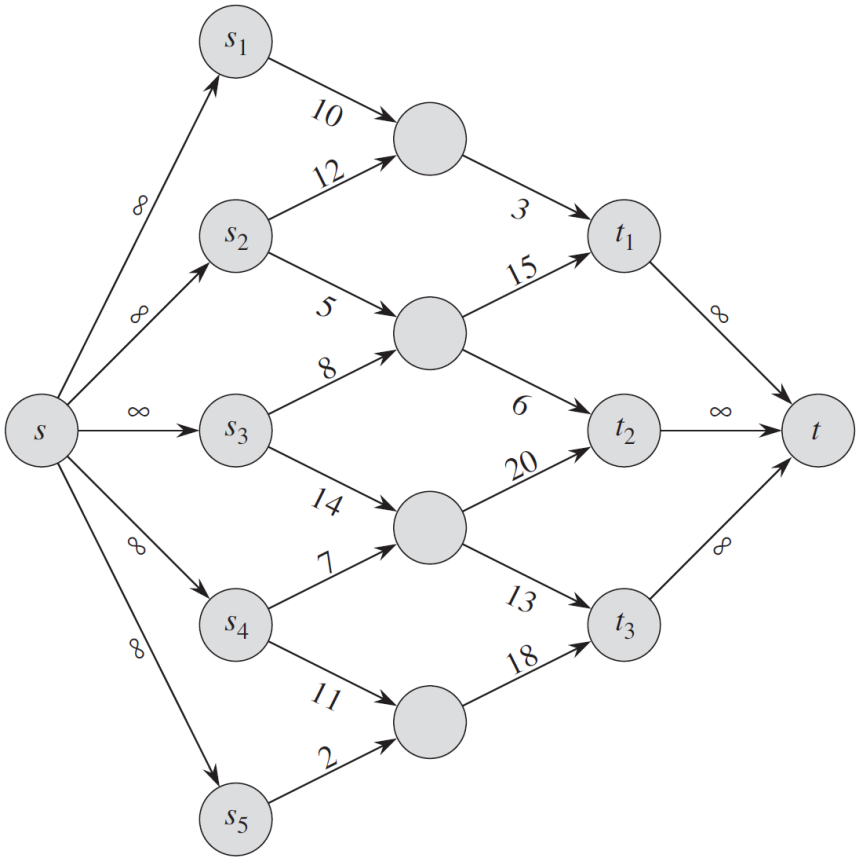
(a)


s
v1
v'
v2
v3
v4
t
16
13
12
10
10
14
4
9
7
20
4

(b)


s1
s2
s3
s4
s5
t1
t2
t3
10
12
3
15
5
8
6
20
14
7
13
18
11
2

(a)


s
s1
s2
s3
s4
s5
t1
t2
t3
t
∞
10
12
3
15
5
8
6
20
14
7
13
18
11
2

(b)

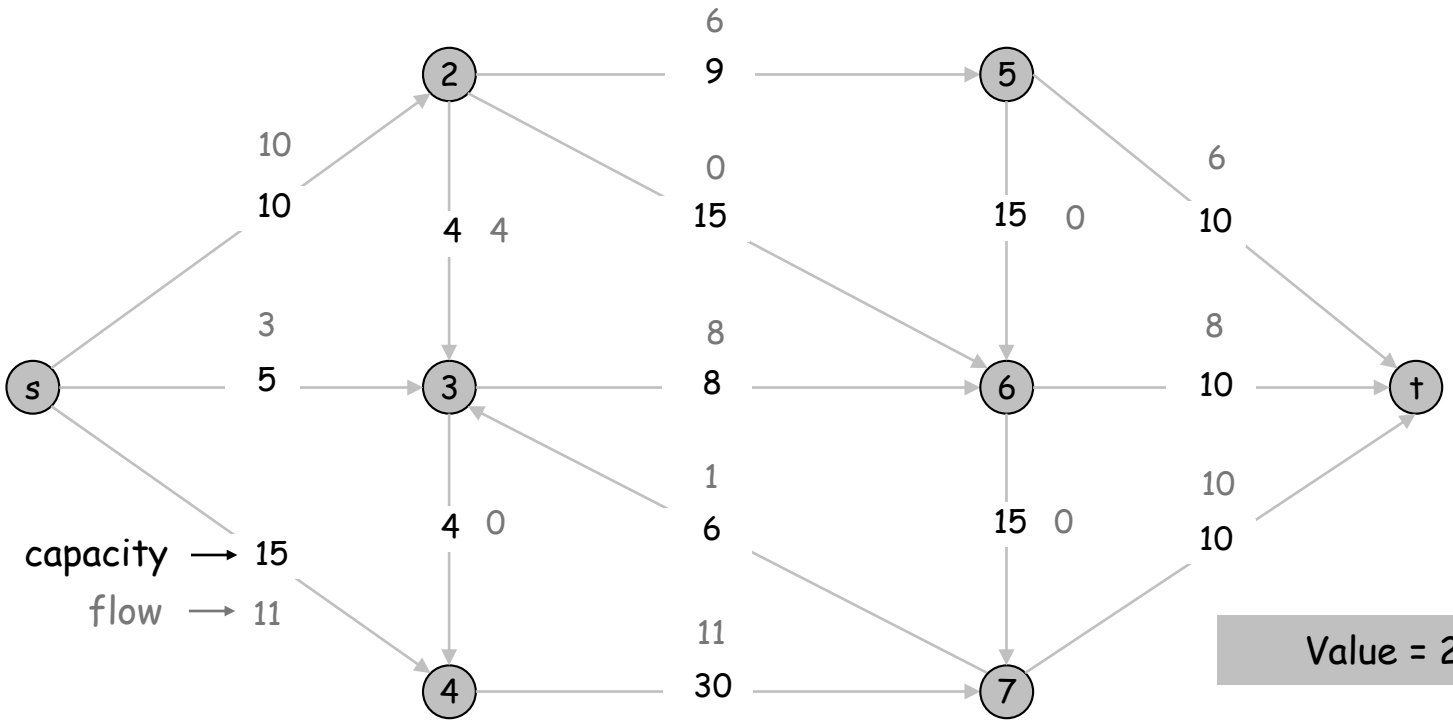
Flows

Def. An s-t flow is a function that satisfies:

For each e ∈ E: $0 \le f(e) \le c(e)$ (capacity)

For each v ∈ V – {s, t}: $\sum_{e \text{ in to } v} f(e) = \sum_{e \text{ out of } v} f(e)$ (conservation)

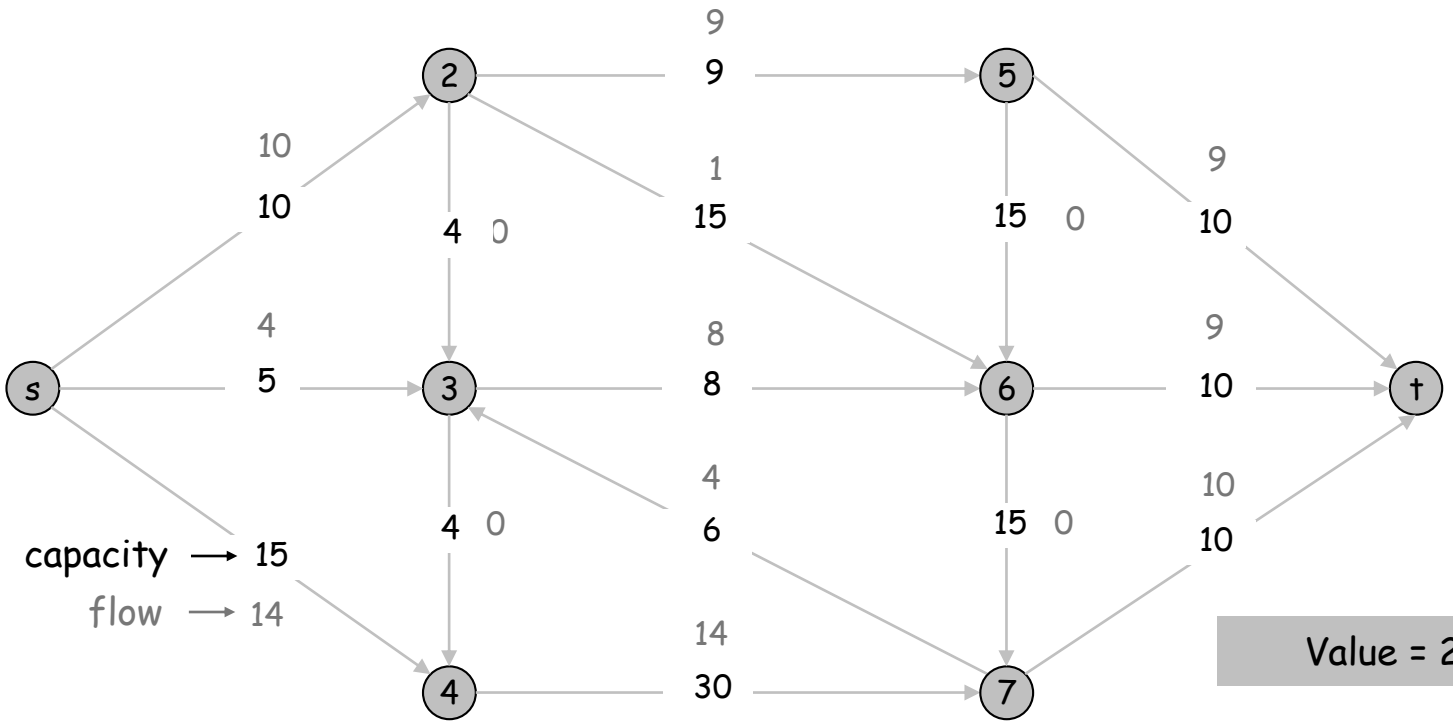
Def. The value of a flow f is: $v(f) = \sum_{e \text{ out of } s} f(e)$.



Value = 24

Maximum Flow Problem

Max flow problem. Find s-t flow of maximum value.



Minimum Cut Problem

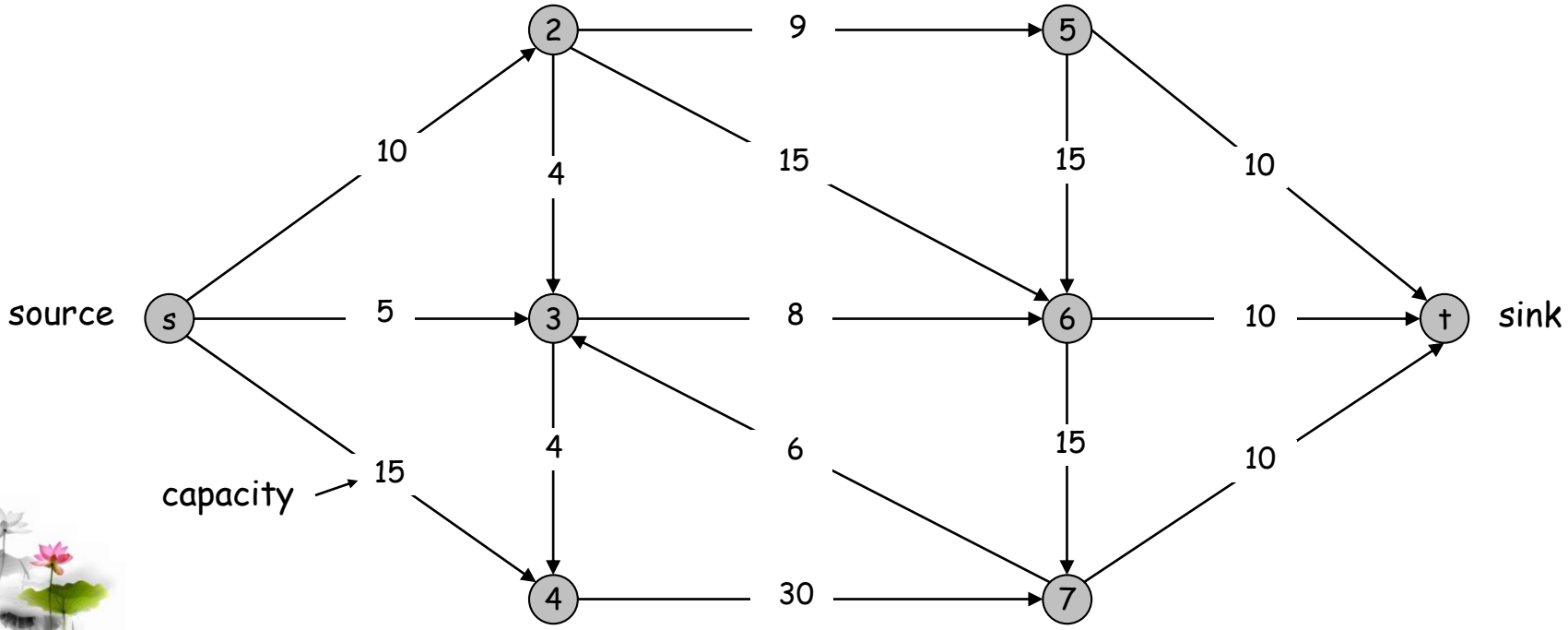
Flow network.

Abstraction for material flowing through the edges.

$G = (V, E)$ = directed graph, no parallel edges.

Two distinguished nodes: s = source, t = sink.

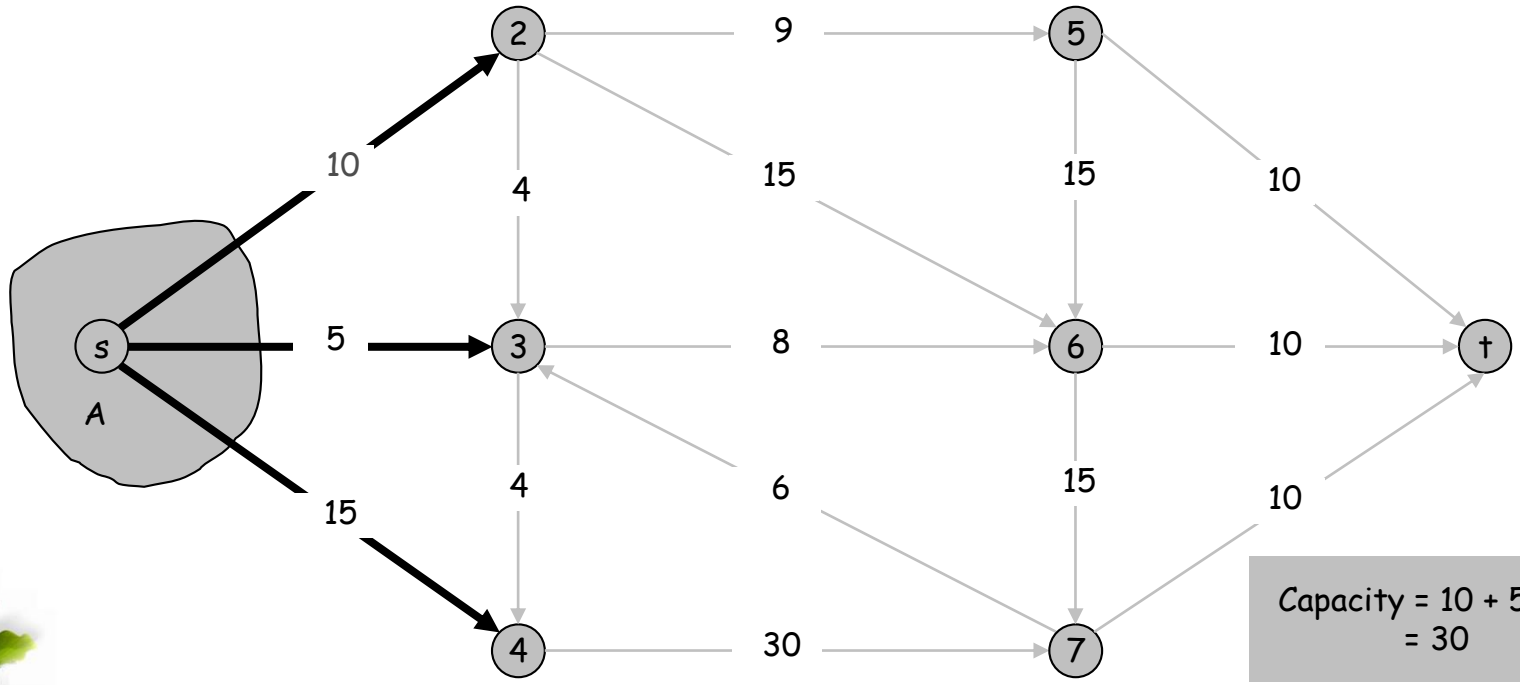
$c(e)$ = capacity of edge e.



Cuts

Def. An s-t cut is a partition (A, B) of V with $s \in A$ and $t \in B$.

Def. The capacity of a cut (A, B) is: $cap(A, B) = \sum_{e \text{ out of } A} c(e)$

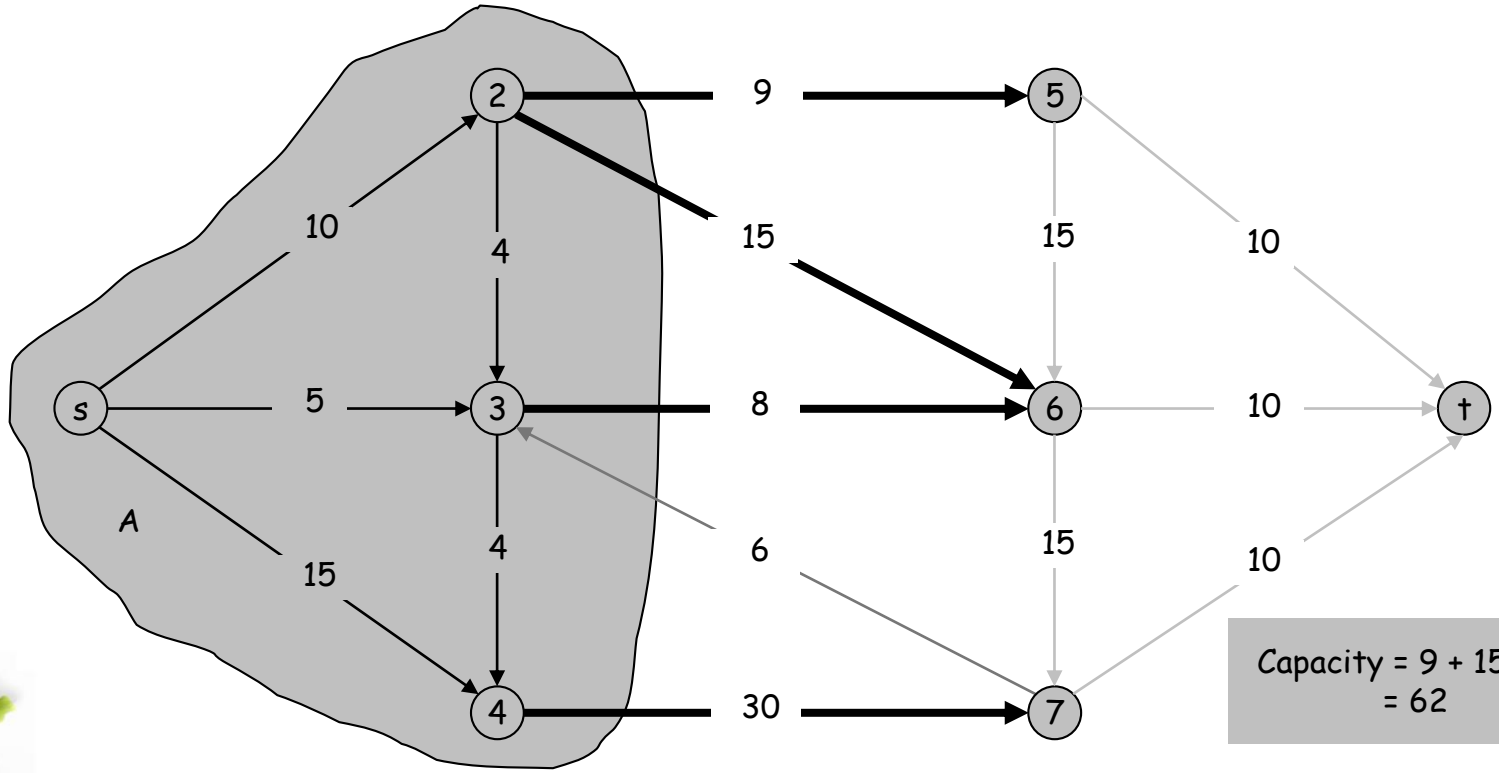


Capacity = 10 + 5 + 15
= 30

Cuts

Def. An s-t cut is a partition (A, B) of V with $s \in A$ and $t \in B$.

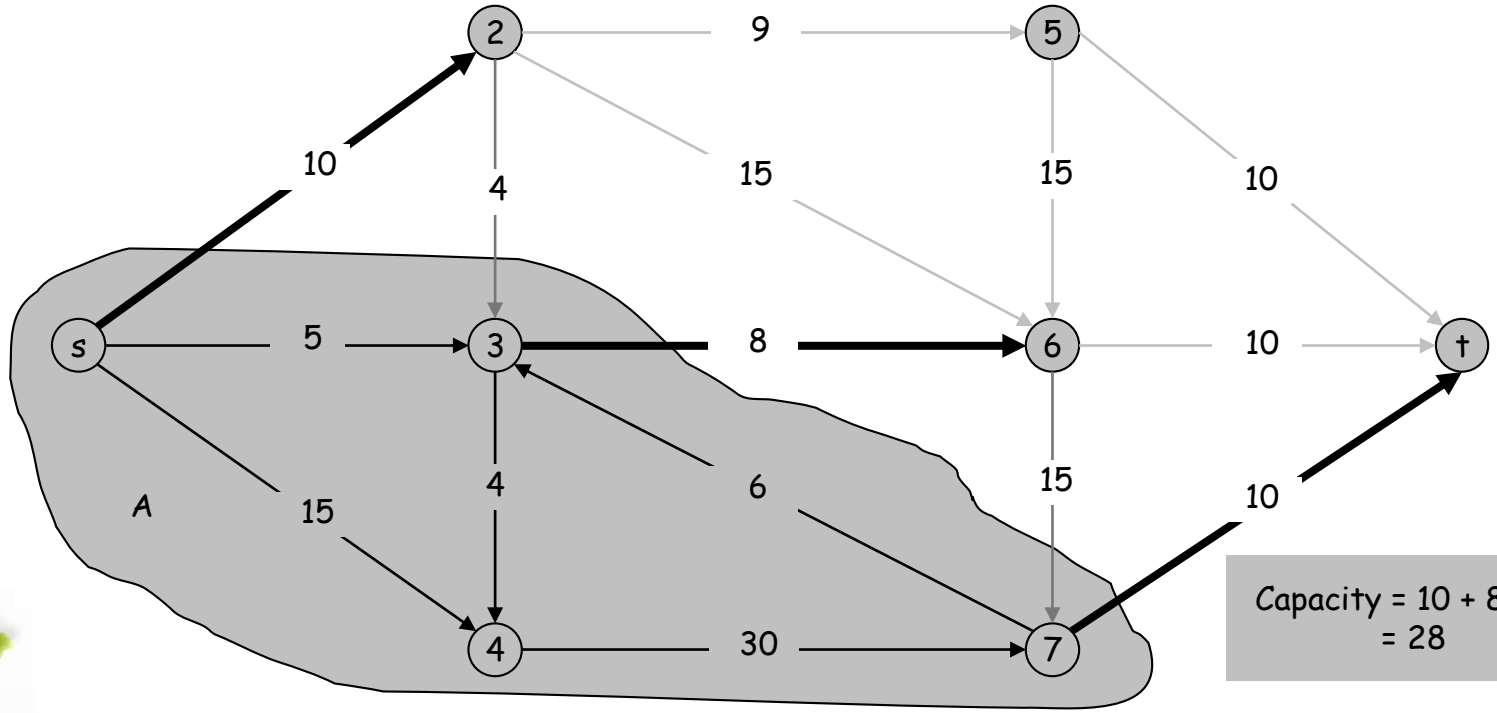
Def. The capacity of a cut (A, B) is: $cap(A, B) = \sum_{e \text{ out of } A} c(e)$



Capacity = 9 + 15 + 8 + 30
= 62

Minimum Cut Problem

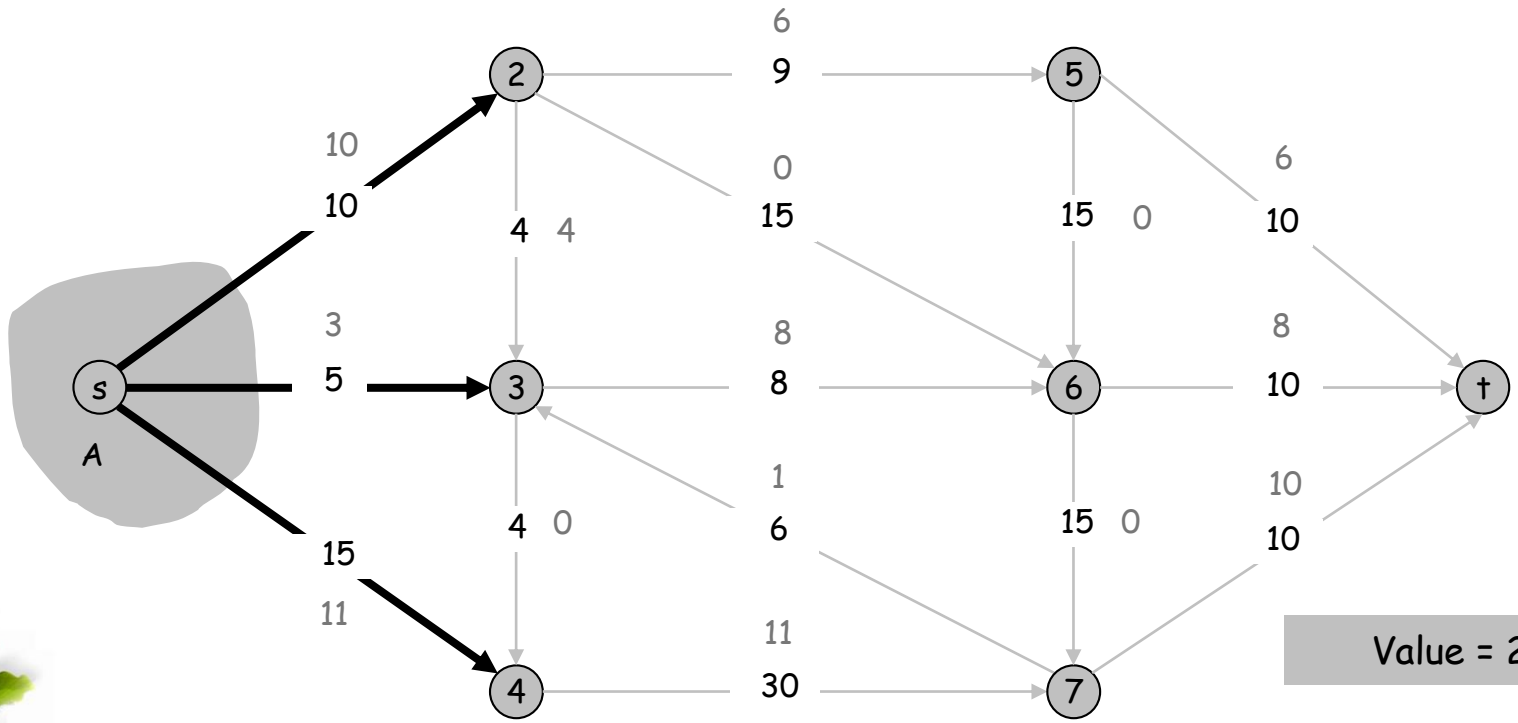
Min s-t cut problem. Find an s-t cut of minimum capacity.



Flows and Cuts

Flow value lemma. Let f be any flow, and let (A, B) be any s-t cut. Then, the net flow sent across the cut is equal to the amount leaving s.

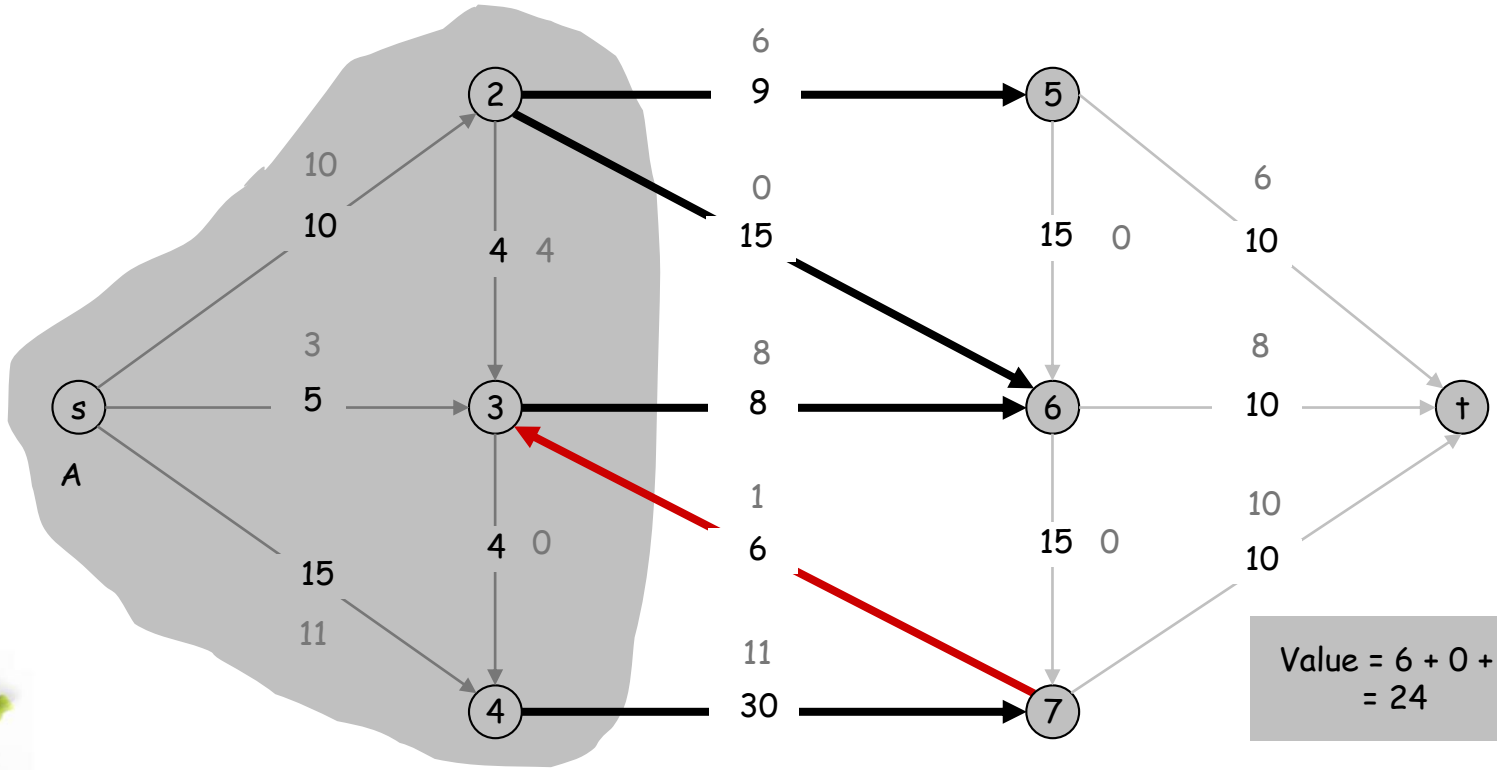
$$\sum_{e \text{ out of } A} f(e) \; - \sum_{e \text{ in to A}} f(e) \; = \; v(f)$$



Flows and Cuts

Flow value lemma. Let f be any flow, and let (A, B) be any s-t cut. Then, the net flow sent across the cut is equal to the amount leaving s.

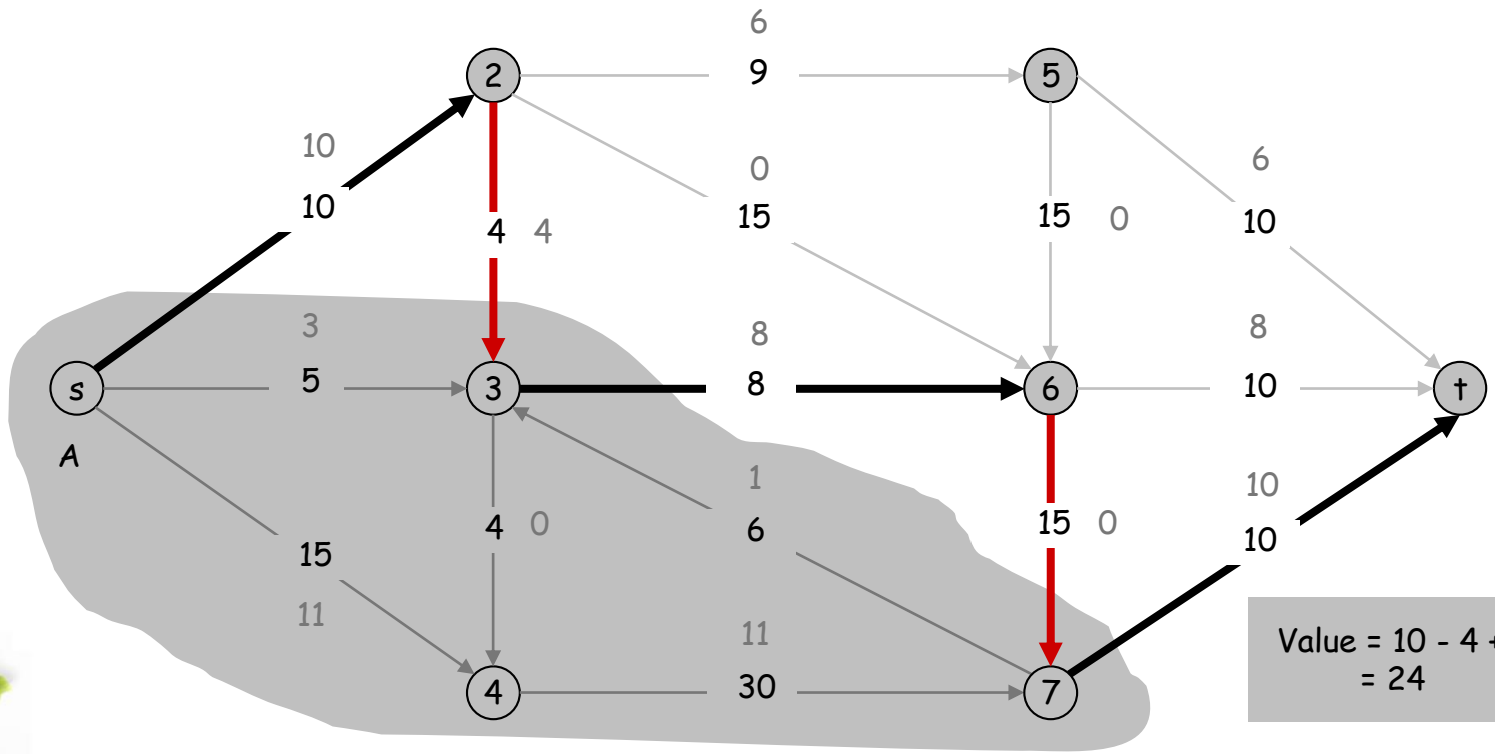
$$\sum_{e \text{ out of } A} f(e) \; - \sum_{e \text{ in to A}} f(e) \; = \; v(f)$$



Flows and Cuts

Flow value lemma. Let f be any flow, and let (A, B) be any s-t cut. Then, the net flow sent across the cut is equal to the amount leaving s.

$$\sum_{e \text{ out of } A} f(e) \; - \sum_{e \text{ in to A}} f(e) \; = \; v(f)$$



Flows and Cuts

Flow value lemma. Let f be any flow, and let (A, B) be any s-t cut. Then

$$\sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e) = v(f).$$

Pf.

$$v(f) = \sum_{e \text{ out of } s} f(e)$$

by flow conservation, all terms except flow into v = s are 0 $\longrightarrow$

$$= \sum_{v \in A} \left(\sum_{e \text{ out of } v} f(e) - \sum_{e \text{ in to v}} f(e) \right)$$

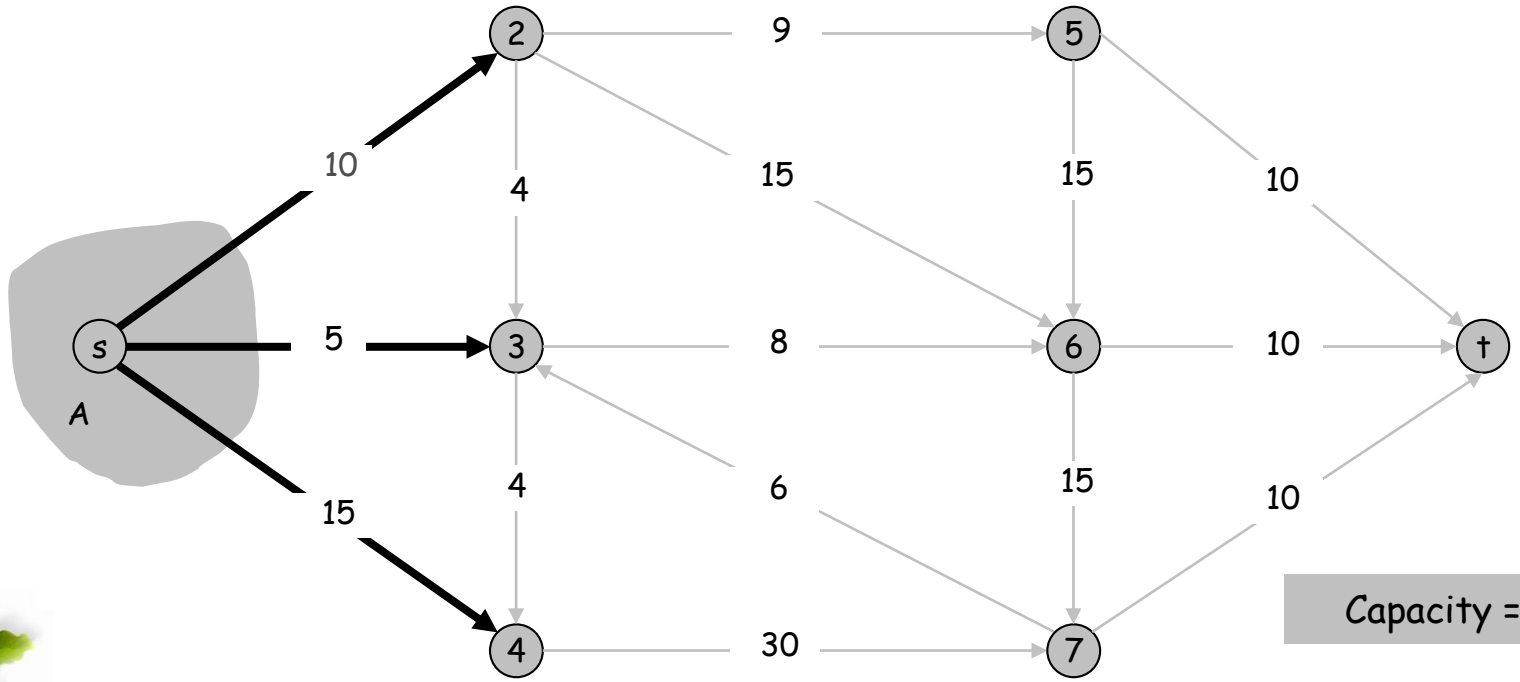
Notice that t is not in A

$$= \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to A}} f(e).$$

Flows and Cuts

Weak duality. Let f be any flow, and let (A, B) be any s-t cut. Then the value of the flow is at most the capacity of the cut.

Cut capacity = 30 $\Rightarrow$ Flow value $\leq$ 30

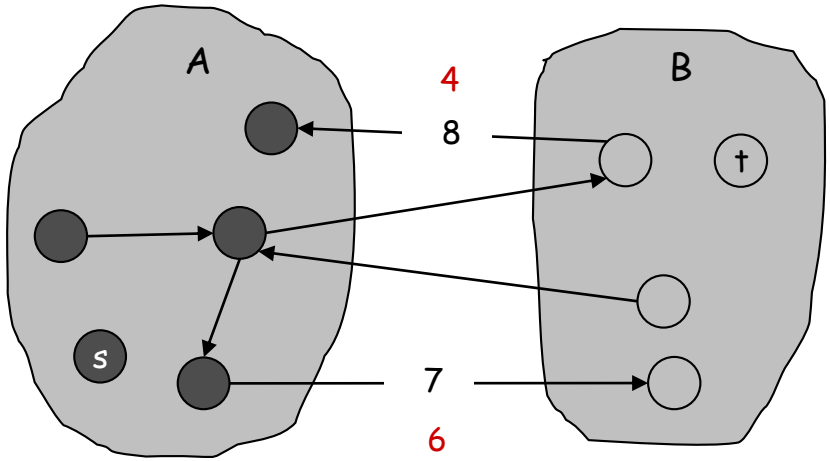


Flows and Cuts

Weak duality. Let f be any flow. Then, for any s-t cut (A, B) we have $v(f) \leq \text{cap}(A, B)$.

Pf.

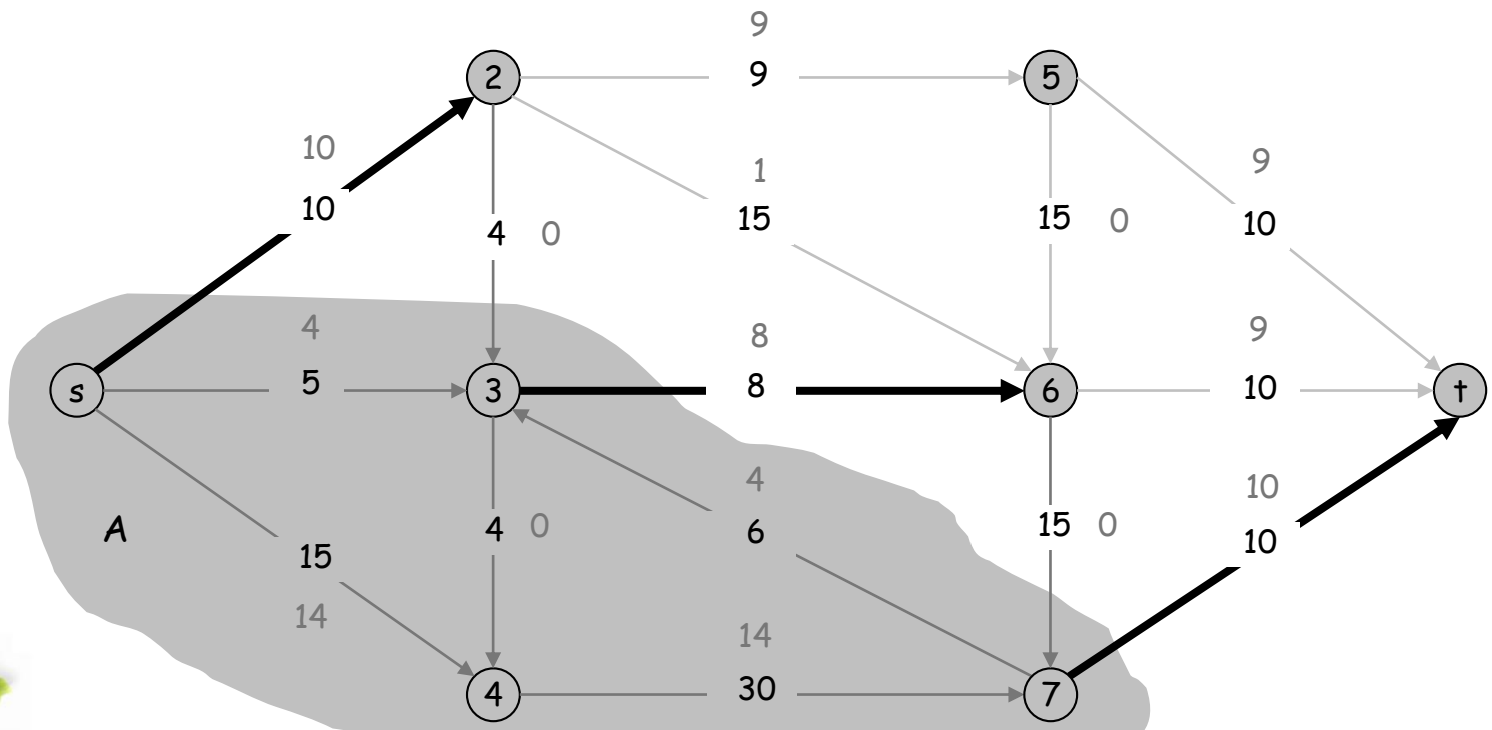
$$
\begin{aligned}
v(f) &= \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to } A} f(e) \\
&\leq \sum_{e \text{ out of } A} f(e) \\
&\leq \sum_{e \text{ out of } A} c(e) \\
&= \text{cap}(A, B) \quad \blacksquare
\end{aligned}
$$



Certificate of Optimality

Corollary. Let f be any flow, and let (A, B) be any cut.
If v(f) = cap(A, B), then f is a max flow and (A, B) is a min cut.

Value of flow = 28
Cut capacity = 28 $\Rightarrow$ Flow value $\leq$ 28



Towards a Max Flow Algorithm

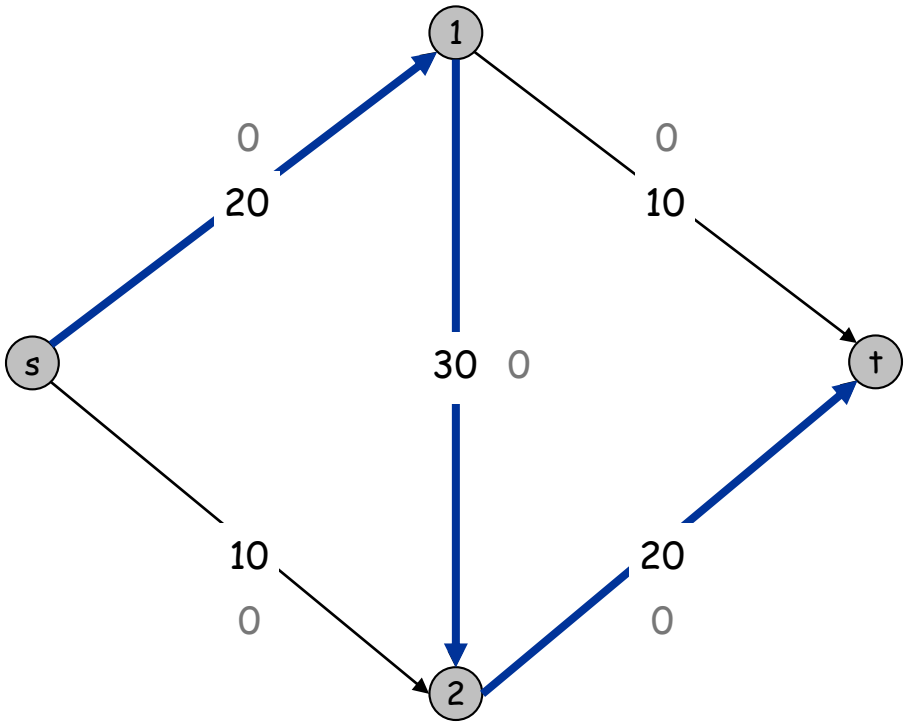
Greedy algorithm.

Start with f(e) = 0 for all edge e ∈ E.

Find an s-t path P where each edge has f(e) < c(e).

Augment flow along path P.

Repeat until you get stuck.



Flow value = 0

Towards a Max Flow Algorithm

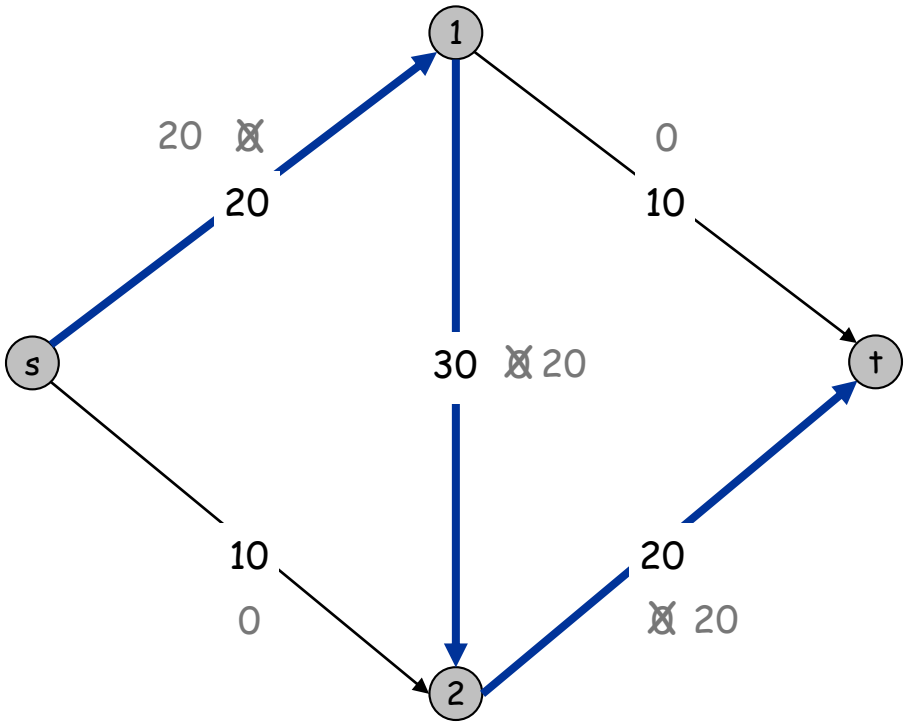
Greedy algorithm.

Start with f(e) = 0 for all edge e ∈ E.

Find an s-t path P where each edge has f(e) < c(e).

Augment flow along path P.

Repeat until you get stuck.



Flow value = 20

Towards a Max Flow Algorithm

Greedy algorithm.

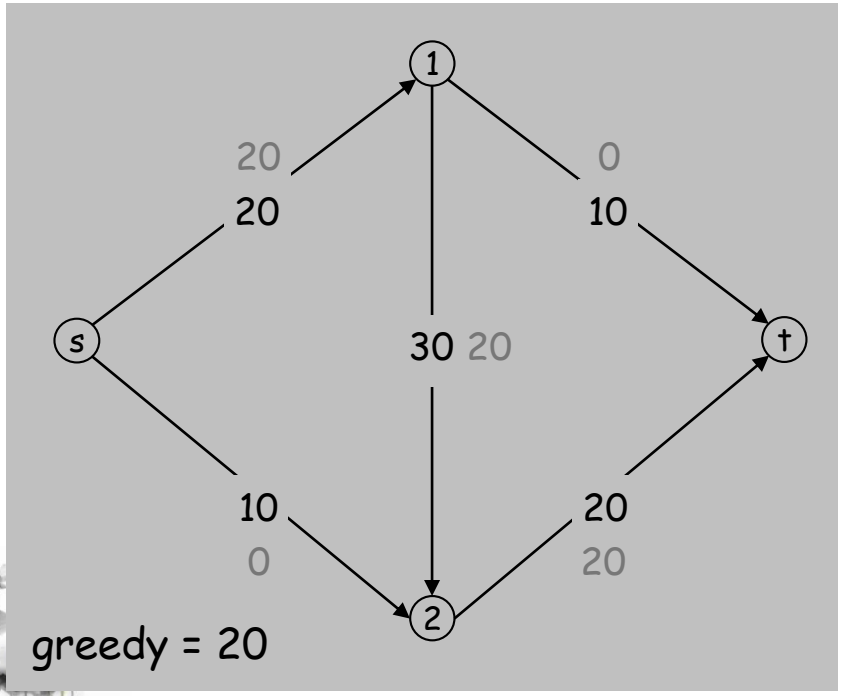
Start with f(e) = 0 for all edge e ∈ E.

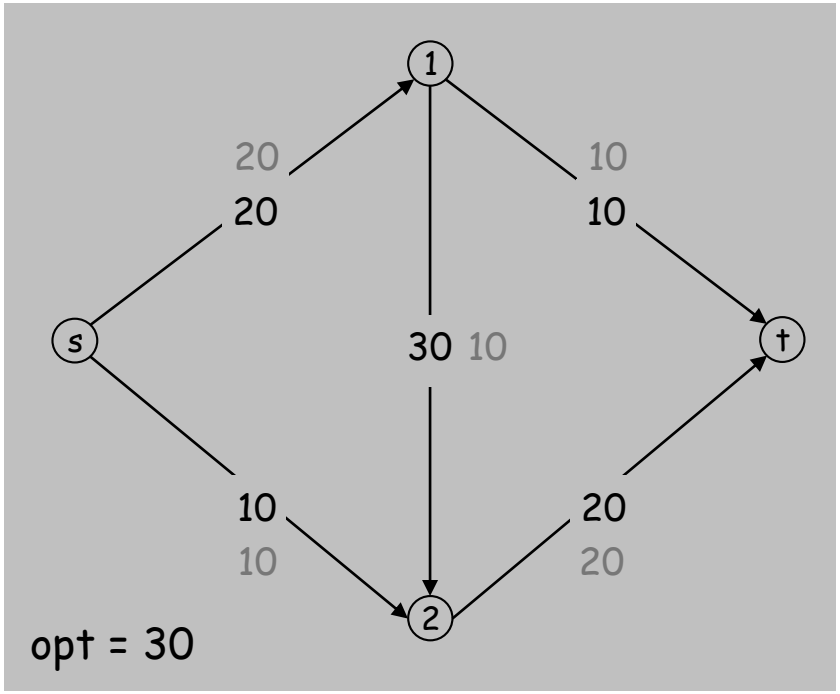
Find an s-t path P where each edge has f(e) < c(e).

Augment flow along path P.

Repeat until you get stuck.

↖ locally optimality ⇏ global optimality

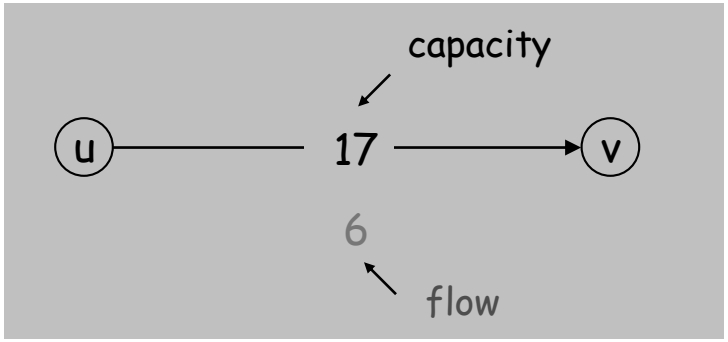




Residual Graph

Original edge: $e = (u, v) \in E$.

Flow f(e), capacity c(e).



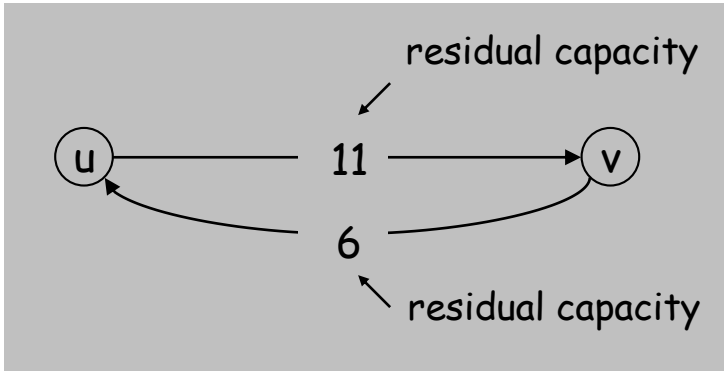
Residual edge.

"Undo" flow sent.

$e = (u, v)$ and $e^R = (v, u)$.

Residual capacity:

$$c_f(e) = \begin{cases} c(e) - f(e) & \text{if } e \in E \\ f(e) & \text{if } e^R \in E \end{cases}$$

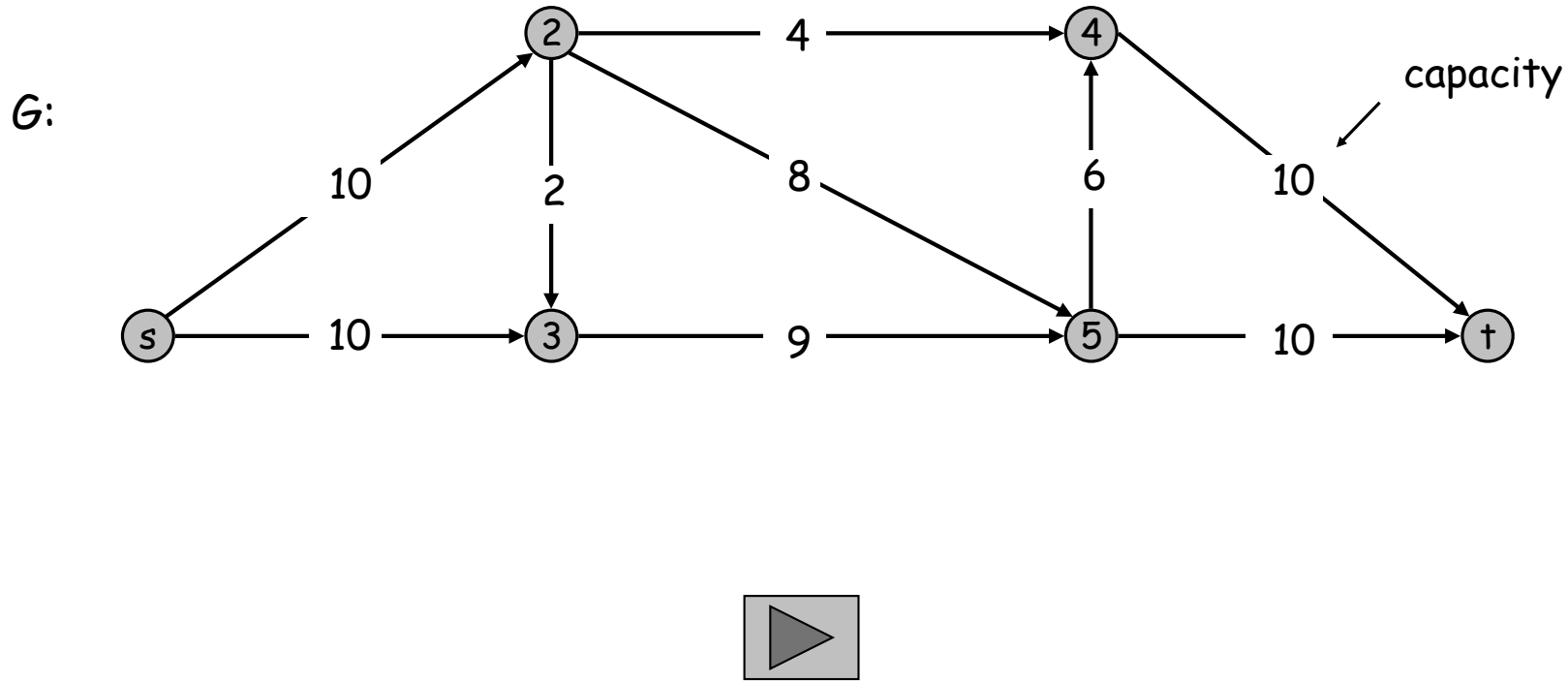


Residual graph: $G_f = (V, E_f)$.

Residual edges with positive residual capacity.

$E_f = \{e : f(e) < c(e)\} \cup \{e^R : c(e) > 0\}$.

Ford-Fulkerson Algorithm



Augmenting Path Algorithm

```
Augment(f, c, P) {
   b ← bottleneck(P)
   foreach e ∈ P {
      if (e ∈ E) f(e) ← f(e) + b      forward edge
      else       f(e^R) ← f(e) - b    reverse edge
   }
   return f
}
```

```
Ford-Fulkerson(G, s, t, c) {
   foreach e ∈ E  f(e) ← 0
   G_f ← residual graph

   while (there exists augmenting path P) {
      f ← Augment(f, c, P)
      update G_f
   }
   return f
}
```

Max-Flow Min-Cut Theorem

Augmenting path theorem. Flow f is a max flow iff there are no augmenting paths.

Max-flow min-cut theorem. [Ford-Fulkerson 1956] The value of the max flow is equal to the value of the min cut.

Proof strategy. We prove both simultaneously by showing the TFAE:

(i) There exists a cut (A, B) such that v(f) = cap(A, B).
(ii) Flow f is a max flow.
(iii) There is no augmenting path relative to f.

(i) $\Rightarrow$ (ii) This was the corollary to weak duality lemma.

(ii) $\Rightarrow$ (iii) We show contrapositive.

Let f be a flow. If there exists an augmenting path, then we can improve f by sending flow along path.

Proof of Max-Flow Min-Cut Theorem

(iii) $\Rightarrow$ (i)

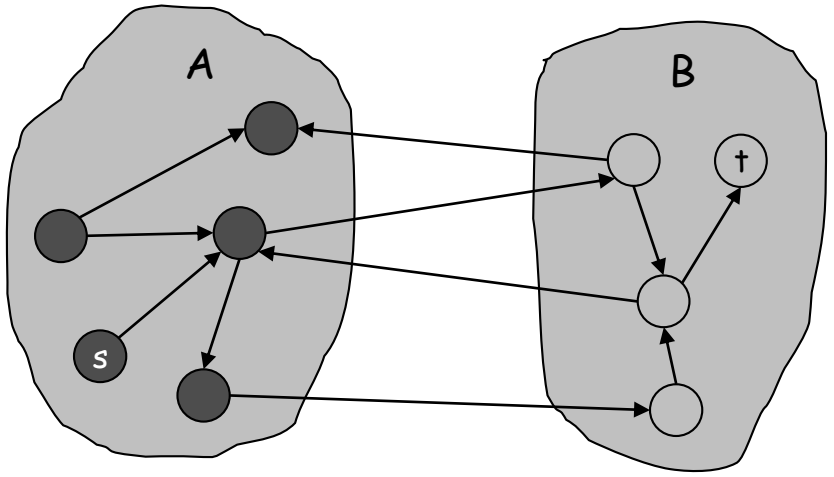
Let f be a flow with no augmenting paths.

Let A be set of vertices reachable from s in residual graph.

By definition of A, $s \in A$.

By definition of f, $t \notin A$.

$$
\begin{aligned}
v(f) &= \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to A}} f(e) \\
&= \sum_{e \text{ out of } A} c(e) \\
&= cap(A, B) \quad \blacksquare
\end{aligned}
$$



original network

Any incoming edge with positive flow will introduce an outgoing residual edge with positive capacity.

Running Time

Assumption. All capacities are integers between 1 and C.

Invariant. Every flow value $f(e)$ and every residual capacities $c_f(e)$ remains an integer throughout the algorithm.

Theorem. The algorithm terminates in at most $v(f^*) \leq nC$ iterations.
Pf. Each augmentation increase value by at least 1. ▪

Corollary. If $C = 1$, Ford-Fulkerson runs in $O(mn)$ time.

Integrality theorem. If all capacities are integers, then there exists a max flow f for which every flow value $f(e)$ is an integer.
Pf. Since algorithm terminates, theorem follows from invariant. ▪

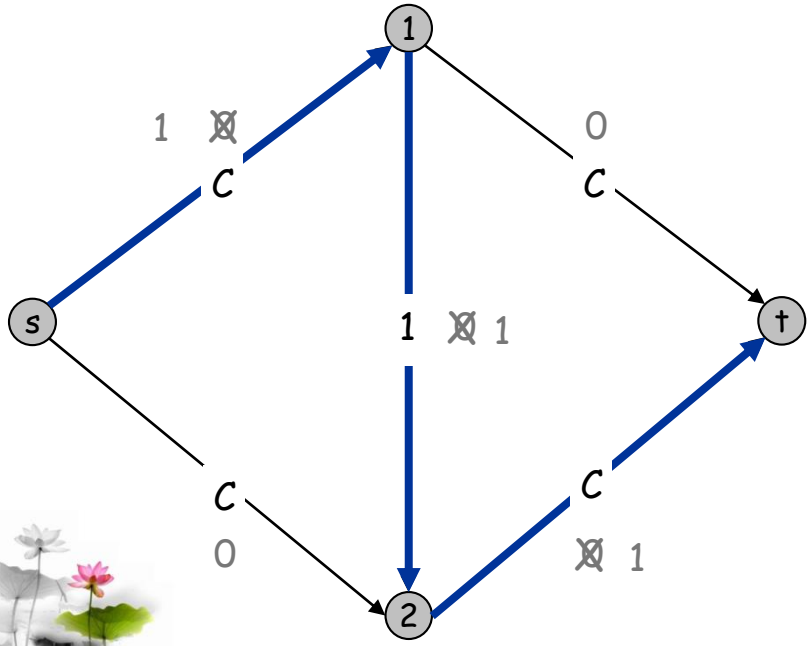
7.3 Choosing Good Augmenting Paths

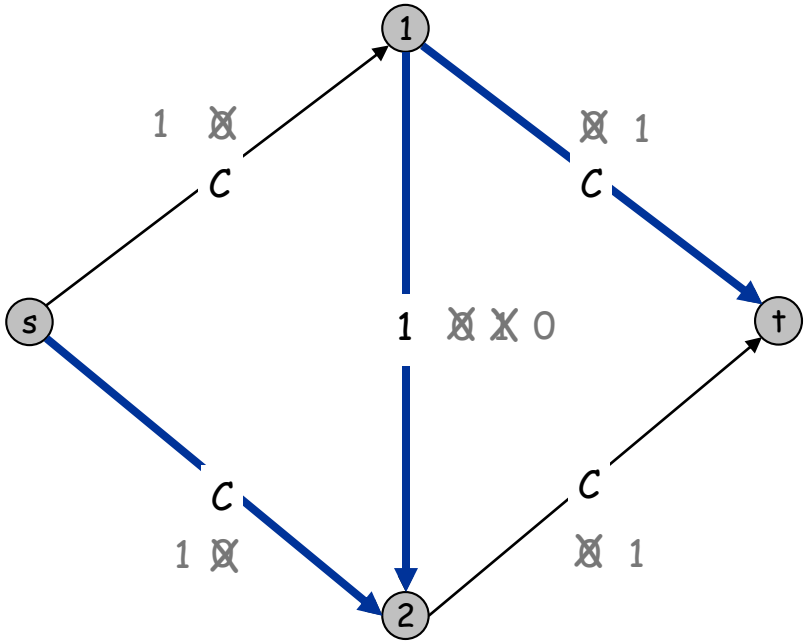
Ford-Fulkerson: Exponential Number of Augmentations

Q. Is generic Ford-Fulkerson algorithm polynomial in input size?

m, n, and log C

A. No. If max capacity is C, then algorithm can take C iterations.





Choosing Good Augmenting Paths

Use care when selecting augmenting paths.

- Some choices lead to exponential algorithms.
- Clever choices lead to polynomial algorithms.
- If capacities are irrational, algorithm not guaranteed to terminate!

Goal: choose augmenting paths so that:

- Can find augmenting paths efficiently.
- Few iterations.

Choose augmenting paths with: [Edmonds-Karp 1972, Dinitz 1970]

- Max bottleneck capacity.
- Sufficiently large bottleneck capacity.
- Fewest number of edges.

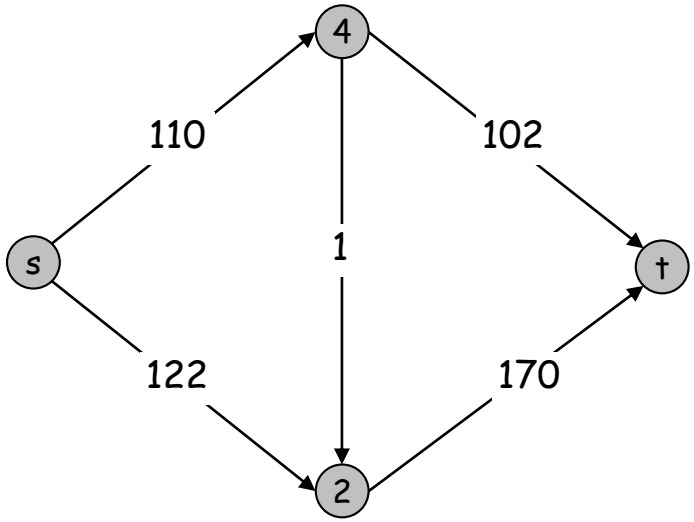
Capacity Scaling

Intuition. Choosing path with highest bottleneck capacity increases flow by max possible amount.

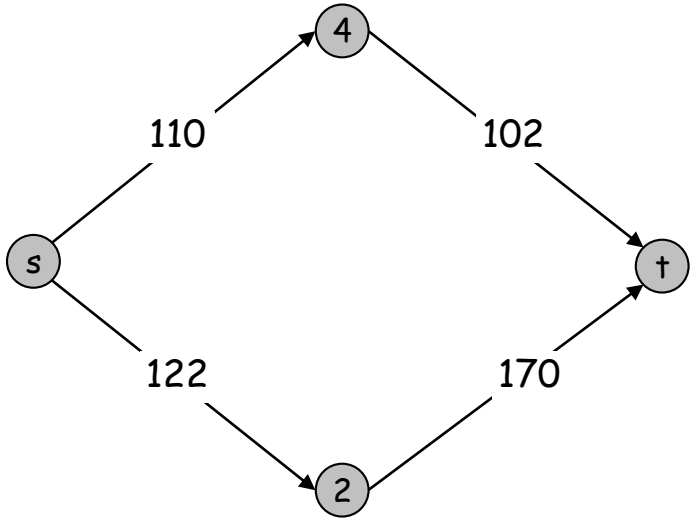
Don't worry about finding exact highest bottleneck path.

Maintain scaling parameter Δ.

Let $G_f(\Delta)$ be the subgraph of the residual graph consisting of only arcs with capacity at least Δ.



G_f



$G_f(100)$

Capacity Scaling

```
Scaling-Max-Flow(G, s, t, c) {
    foreach e ∈ E  f(e) ← 0
    Δ ← smallest power of 2 greater than or equal to C
    G_f ← residual graph

    while (Δ ≥ 1) {
        G_f(Δ) ← Δ-residual graph
        while (there exists augmenting path P in G_f(Δ)) {
            f ← augment(f, c, P)
            update G_f(Δ)
        }
        Δ ← Δ / 2
    }
    return f
}
```

Capacity Scaling: Correctness

Assumption. All edge capacities are integers between 1 and C.

Integrality invariant. All flow and residual capacity values are integral.

Correctness. If the algorithm terminates, then f is a max flow.

Pf.

By integrality invariant, when $\Delta = 1 \Rightarrow G_f(\Delta) = G_f$.

Upon termination of $\Delta = 1$ phase, there are no augmenting paths. ▪

Capacity Scaling: Running Time

Lemma 1. The outer while loop repeats $1 + \lceil \log_2 C \rceil$ times.

Pf. Initially $C \leq \Delta < 2C$. Δ decreases by a factor of 2 each iteration. ▪

Lemma 2. Let f be the flow at the end of a Δ-scaling phase. Then the value of the maximum flow is at most $v(f) + m\,\Delta$. ← proof on next slide

Lemma 3. There are at most 2m augmentations per scaling phase.

Let f be the flow at the end of the previous scaling phase.

L2 $\Rightarrow$ $v(f^*) \leq v(f) + m\,(2\Delta)$.

Each augmentation in a Δ-phase increases $v(f)$ by at least Δ. ▪

Theorem. The scaling max-flow algorithm finds a max flow in $O(m \log C)$ augmentations. It can be implemented to run in $O(m^2 \log C)$ time. ▪

Capacity Scaling: Running Time

Lemma 2. Let f be the flow at the end of a Δ-scaling phase. Then value of the maximum flow is at most $v(f) + m\Delta$.

Pf. (almost identical to proof of max-flow min-cut theorem)

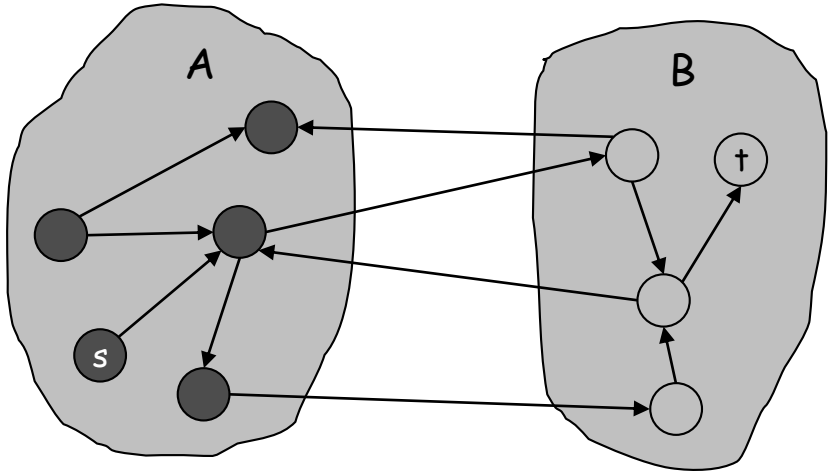
We show that at the end of a Δ-phase, there exists a cut (A, B) such that $cap(A, B) \leq v(f) + m\Delta$.

Choose A to be the set of nodes reachable from s in $G_f(\Delta)$.

By definition of A, $s \in A$.

By definition of f, $t \notin A$.

$$
\begin{aligned}
v(f) &= \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ in to A}} f(e) \\
&\geq \sum_{e \text{ out of } A} (c(e) - \Delta) - \sum_{e \text{ in to } A} \Delta \\
&= \sum_{e \text{ out of } A} c(e) - \sum_{e \text{ out of } A} \Delta - \sum_{e \text{ in to } A} \Delta \\
&\geq cap(A, B) - m\Delta \qquad \blacksquare
\end{aligned}
$$



original network

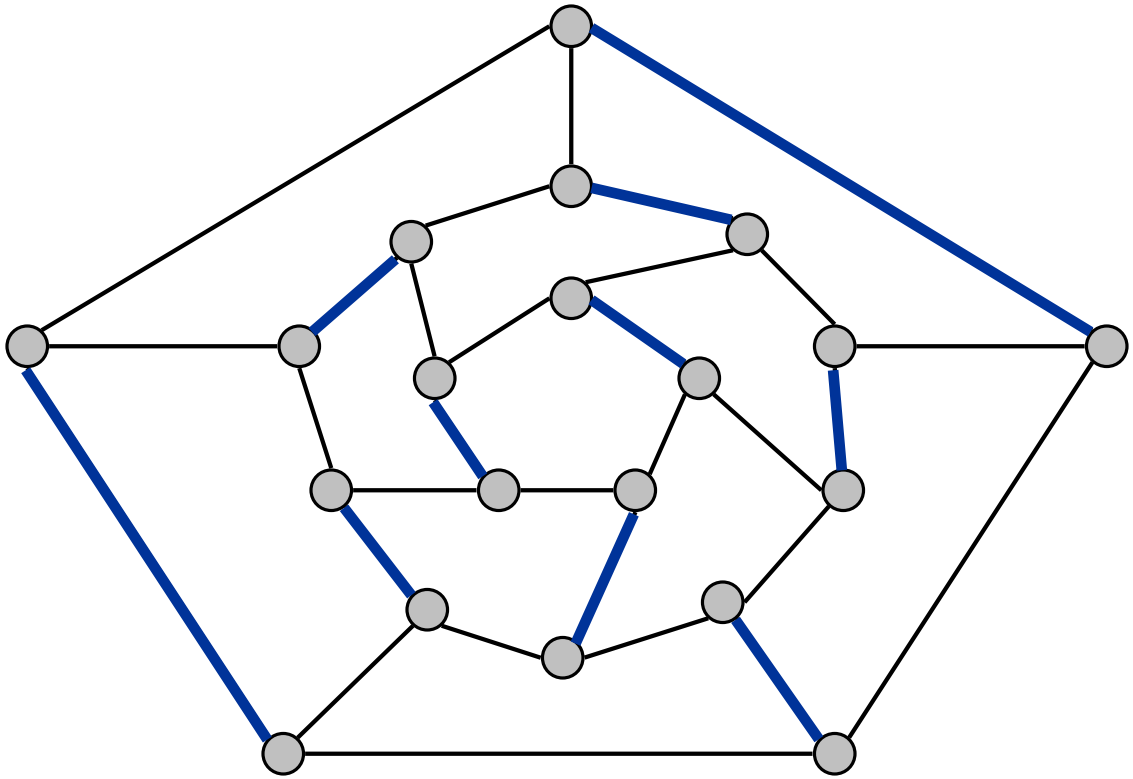
Matching

Matching.

Input: undirected graph $G = (V, E)$.

$M \subseteq E$ is a matching if each node appears in at most one edge in M.

Max matching: find a max cardinality matching.



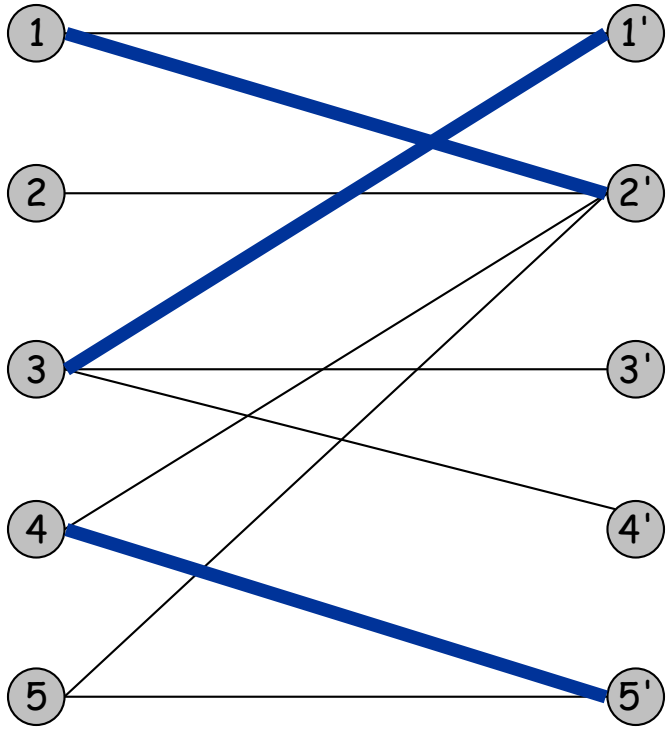
Bipartite Matching

Bipartite matching.

Input: undirected, bipartite graph $G = (L \cup R, E)$.

$M \subseteq E$ is a matching if each node appears in at most edge in M.

Max matching: find a max cardinality matching.



matching

1-2', 3-1', 4-5'

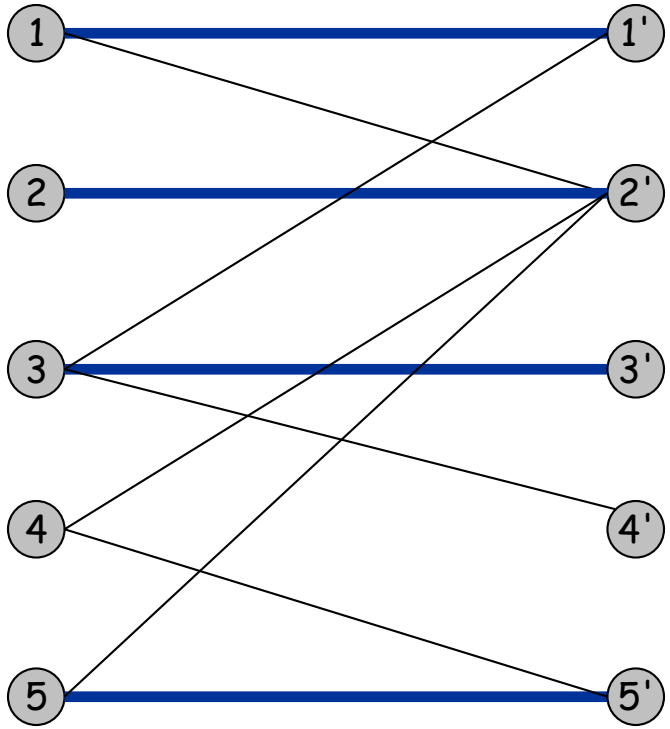
Bipartite Matching

Bipartite matching.

Input: undirected, bipartite graph $G = (L \cup R, E)$.

$M \subseteq E$ is a matching if each node appears in at most edge in M.

Max matching: find a max cardinality matching.



max matching

1-1', 2-2', 3-3' 4-4'

Bipartite Matching

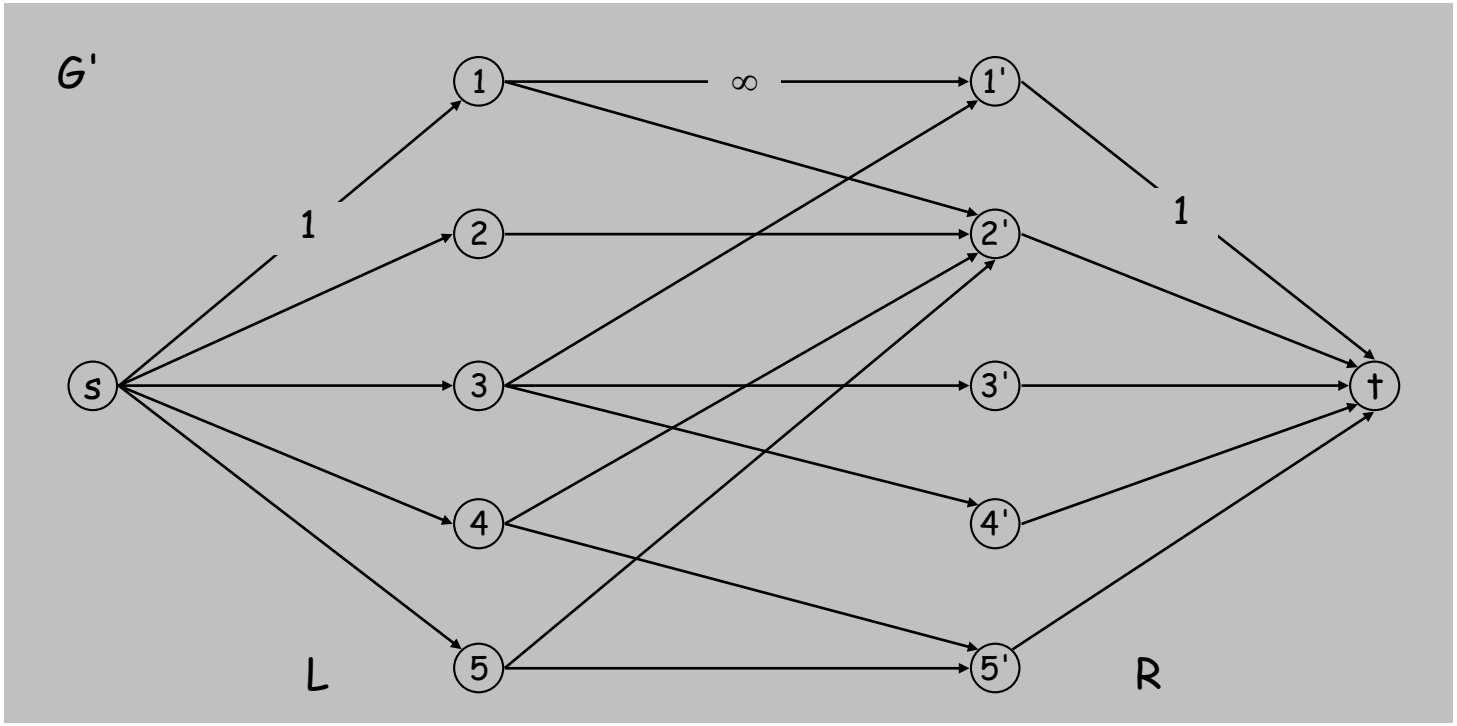
Max flow formulation.

Create digraph G' = (L ∪ R ∪ {s, t}, E').

Direct all edges from L to R, and assign infinite (or unit) capacity.

Add source s, and unit capacity edges from s to each node in L.

Add sink t, and unit capacity edges from each node in R to t.



Bipartite Matching: Proof of Correctness

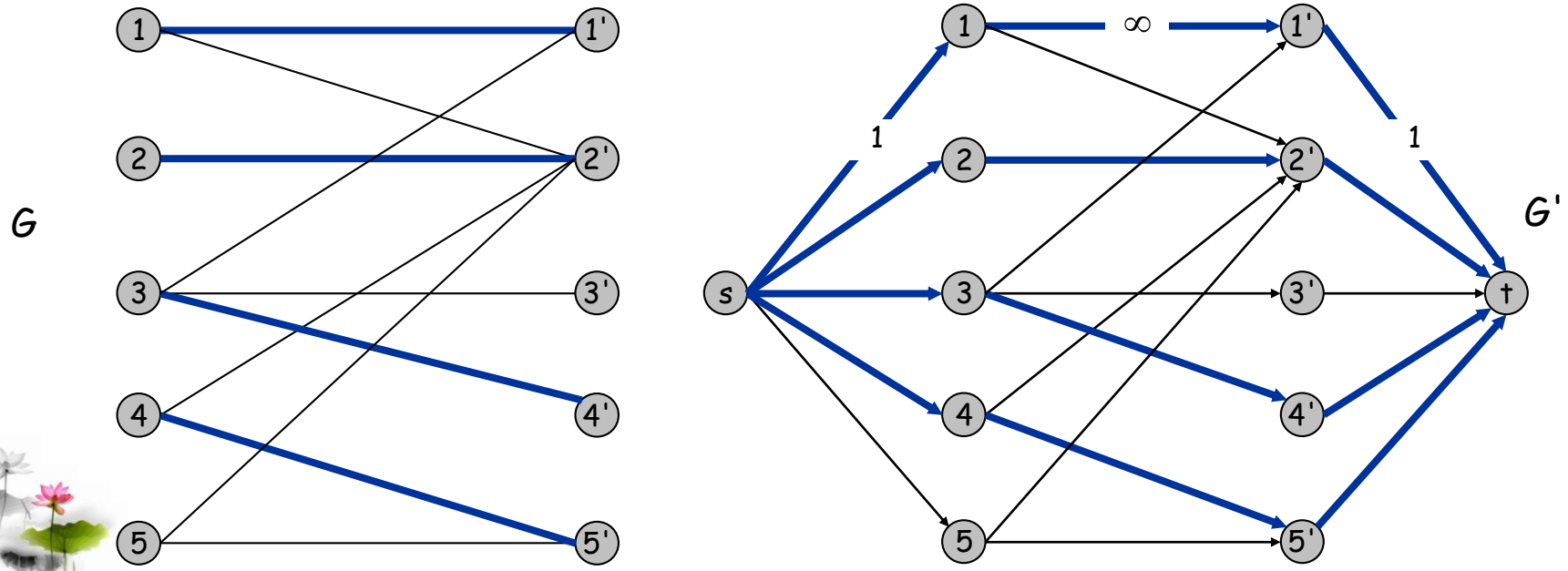
Theorem. Max cardinality matching in G = value of max flow in G'.

Pf. $\leq$

Given max matching M of cardinality k.

Consider flow f that sends 1 unit along each of k paths.

f is a flow, and has cardinality k. ▪



Bipartite Matching: Proof of Correctness

Theorem. Max cardinality matching in G = value of max flow in G'.

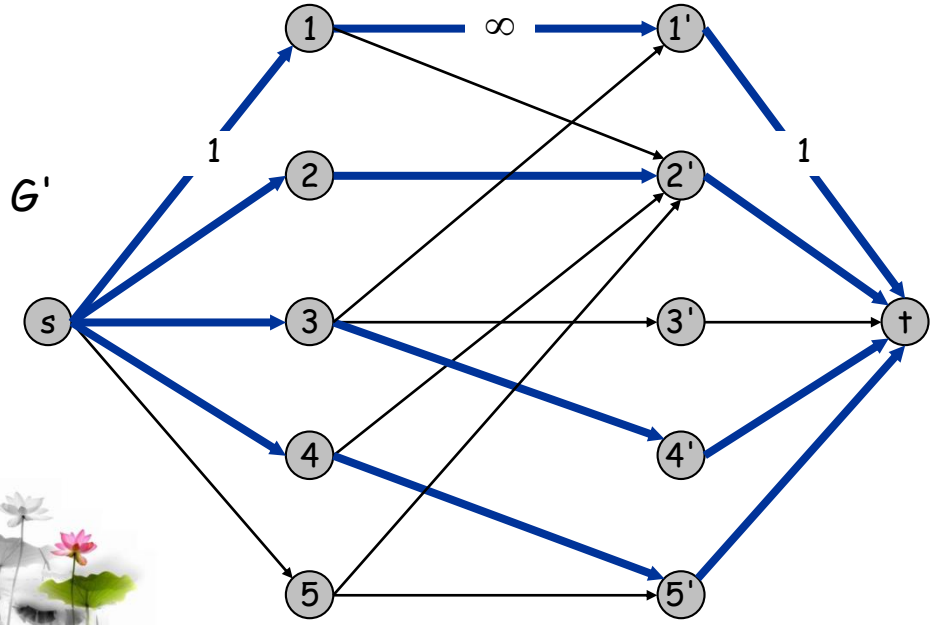
Pf. $\geq$

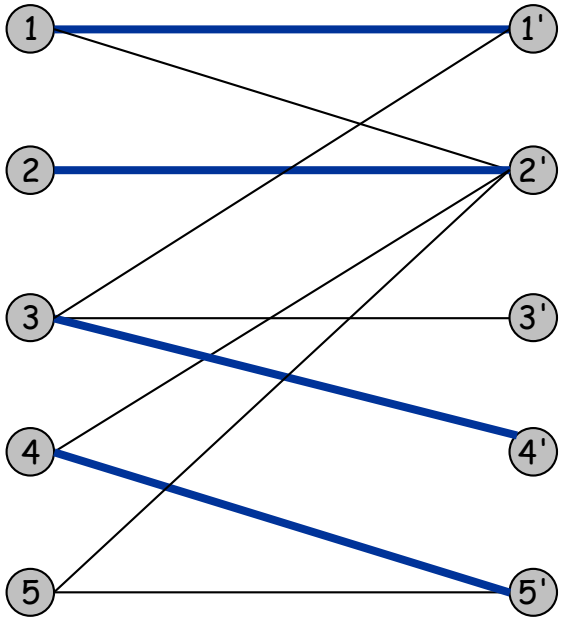
Let f be a max flow in G' of value k.

Integrality theorem $\Rightarrow$ k is integral and can assume f is 0-1.

Consider M = set of edges from L to R with $f(e) = 1$.

- each node in L and R participates in at most one edge in M
- $|M| = k$: consider cut $(L \cup s, R \cup t)$ ▪





G