Regularization and Sparsity: From Lasso to Structured Penalties

1 Motivation

In high-dimensional settings, classical estimation methods such as ordinary least squares (OLS) often suffer from overfitting, instability, and poor interpretability. Regularization addresses these challenges by introducing additional constraints or penalties that control model complexity. In this lecture, we focus on sparsity-inducing regularization methods, beginning with lasso and extending to structured sparsity penalties.

2 Linear Regression and Regularization

Consider the linear regression model

$$y_i = x_i^\top \beta + \varepsilon_i, \quad i = 1, \ldots, n,$$

where $x_i \in \mathbb{R}^p$ and $\beta \in \mathbb{R}^p$.

The ordinary least squares estimator solves

$$\hat{\beta}_{\text{OLS}} = \arg\min_\beta \sum_{i=1}^n (y_i - x_i^\top \beta)^2.$$

When p is large or predictors are highly correlated, OLS becomes unstable.

2.1 Ridge Regression

Ridge regression introduces an ℓ_2 penalty:

$$\hat{\beta}_{\text{ridge}} = \arg\min_\beta \left\{ \sum_{i=1}^n (y_i - x_i^\top \beta)^2 + \lambda \sum_{j=1}^p \beta_j^2 \right\}.$$

Ridge regression shrinks coefficients toward zero but does not produce exact zeros.

2.2 Lasso

The lasso replaces the ℓ_2 penalty with an ℓ_1 penalty:

$$\hat{\beta}_{\text{lasso}} = \arg\min_{\beta} \left\{ \sum_{i=1}^{n} (y_i - x_i^\top \beta)^2 + \lambda \sum_{j=1}^{p} |\beta_j| \right\}.$$

The lasso performs variable selection by producing exact zero coefficients.

3 Geometric Interpretation of Sparsity

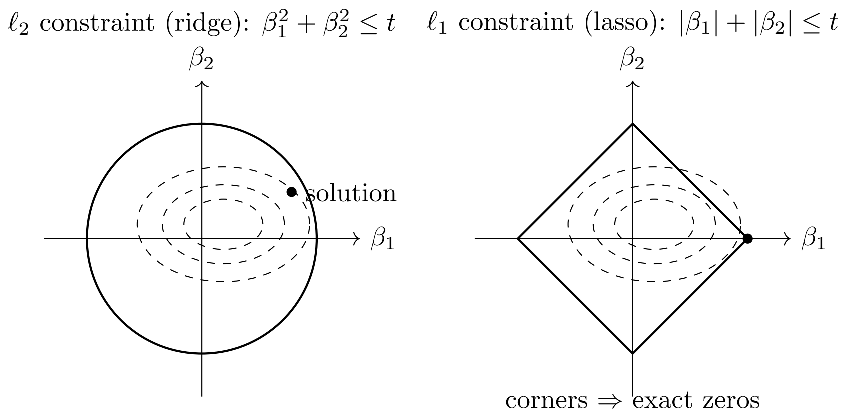

Figure 1: Geometry explains sparsity: the ℓ_1 ball has corners aligned with coordinate axes, making exact zeros more likely than under ℓ_2 regularization.

Regularized estimators can be written in the general form

$$\min_{\beta} \; \mathcal{L}(\beta) + \lambda \, \Omega(\beta),$$

or equivalently,

$$\min_{\beta} \; \mathcal{L}(\beta) \quad \text{subject to} \quad \Omega(\beta) \le t.$$

3.1 ℓ_2 Geometry

The ridge constraint $\sum_j \beta_j^2 \le t$ defines a spherical constraint region with a smooth boundary. Loss contours typically intersect this region away from the coordinate axes, leading to shrinkage without sparsity.

3.2 ℓ_1 Geometry

The lasso constraint $\sum_j |\beta_j| \le t$ defines a diamond-shaped region with sharp corners aligned with the coordinate axes. Loss contours are likely to intersect the constraint at these corners, resulting in exact zeros.

4 Elastic Net

Elastic Net combines ℓ_1 and ℓ_2 penalties:

$$\min_{\beta} \ \mathcal{L}(\beta) + \lambda_1 \sum_{j=1}^{p} |\beta_j| + \lambda_2 \sum_{j=1}^{p} \beta_j^2.$$

Elastic Net was developed to address two limitations of lasso:

- Instability under highly correlated predictors
- The tendency of lasso to select at most n variables when $p \gg n$

The ℓ_2 component encourages correlated predictors to enter the model together (the *grouping effect*), while the ℓ_1 component preserves sparsity.

5 From Individual to Structured Sparsity

5.1 Group Lasso

In many applications, predictors naturally form groups. Let

$$\beta = (\beta_{(1)}, \ldots, \beta_{(G)}),$$

where $\beta_{(g)} \in \mathbb{R}^{p_g}$ represents the coefficients in group g.

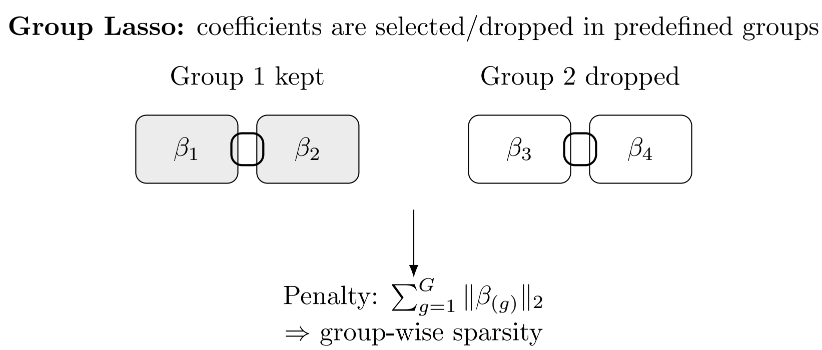

Figure 2: Group lasso promotes sparsity at the group level: entire coefficient blocks are either set to zero or kept.

The group lasso solves

$$\min_{\beta} \ \mathcal{L}(\beta) + \lambda \sum_{g=1}^{G} \|\beta_{(g)}\|_2.$$

This penalty enforces sparsity at the group level: entire groups are either selected or excluded.

Examples

- Dummy variables for categorical predictors
- Basis expansions (splines, wavelets)
- Brain regions or functional networks in neuroimaging

5.2 Fused Lasso

Group lasso assumes known groups. When predictors are ordered, it may be more appropriate to penalize differences between neighboring coefficients.

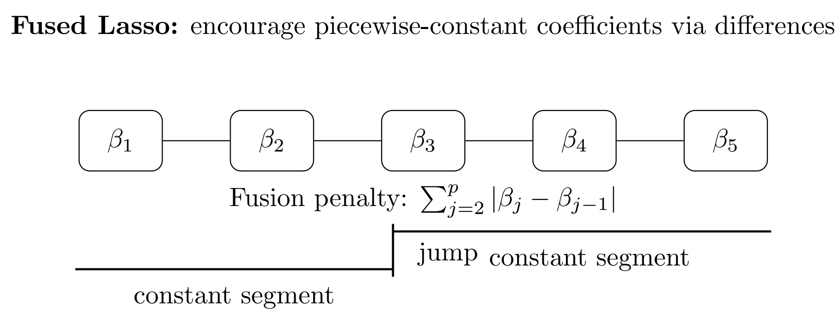

Figure 3: Fused lasso penalizes successive differences, encouraging piecewise-constant patterns with a small number of change points.

The fused lasso solves

$$\min_{\beta} \ \mathcal{L}(\beta) + \lambda_1 \sum_{j=1}^{p} |\beta_j| + \lambda_2 \sum_{j=2}^{p} |\beta_j - \beta_{j-1}|.$$

This penalty encourages both sparsity and piecewise-constant structure.

Applications

- Time series regression
- Genomic data
- Spatial and imaging data

5.3 Sparse Group Lasso

Sparse group lasso combines group-level and within-group sparsity:

$$\min_{\beta} \ \mathcal{L}(\beta) + \lambda_1 \sum_{j=1}^{p} |\beta_j| + \lambda_2 \sum_{g=1}^{G} \|\beta_{(g)}\|_2.$$

This allows selection of important groups while retaining flexibility within groups.

5.4 Graph-Guided Lasso

Fused lasso can be generalized to arbitrary graphs. Let $G = (V, E)$ be a graph over predictors. The graph-guided lasso uses

$$\lambda \sum_{(j,k)\in E} w_{jk}|\beta_j - \beta_k|.$$

This encourages smoothness over graph-connected variables.

5.5 Multitask Lasso

In multitask learning, we observe responses

$$Y^{(t)} = X\beta^{(t)}, \quad t = 1, \ldots, T.$$

The multitask lasso solves

$$\min_{\{\beta^{(t)}\}} \sum_{t=1}^{T} \mathcal{L}^{(t)}(\beta^{(t)}) + \lambda \sum_{j=1}^{p} \left\| (\beta_j^{(1)}, \ldots, \beta_j^{(T)}) \right\|_2.$$

This enforces shared sparsity across tasks.

5.6 Collaborative Learning for Multi-View Data (Multi-View Lasso)

In many modern applications, each subject is measured using multiple *views* (or modalities). For example, in multimodal neuroimaging we may have structural MRI, diffusion MRI, and resting-state fMRI. Multi-view learning aims to integrate these feature sets while borrowing strength across views.

Multi-view setup. Suppose each subject has V views of predictors:

$$X = \left(X^{(1)}, X^{(2)}, \ldots, X^{(V)}\right),$$

where $X^{(v)} \in \mathbb{R}^{n \times p_v}$, and we model the outcome using an additive multi-view regression model:

$$y = \sum_{v=1}^{V} X^{(v)}\beta^{(v)} + \varepsilon,$$

with view-specific coefficients $\beta^{(v)} \in \mathbb{R}^{p_v}$.

Collaborative (multi-view) lasso. A natural extension of the multitask penalty is to encourage *shared sparsity across views*. One common formulation is

$$\min_{\{\beta^{(v)}\}} \mathcal{L}\left(y, \sum_{v=1}^{V} X^{(v)}\beta^{(v)}\right) + \lambda \sum_{j=1}^{p^\star} \left\| \left(\beta_j^{(1)}, \beta_j^{(2)}, \dots, \beta_j^{(V)}\right) \right\|_2,$$

where $p^\star$ indexes *aligned features* across views (e.g., the same region-of-interest measured in different modalities). The ℓ_2-norm couples the coefficients across views, while the outer sum induces sparsity at the feature level.

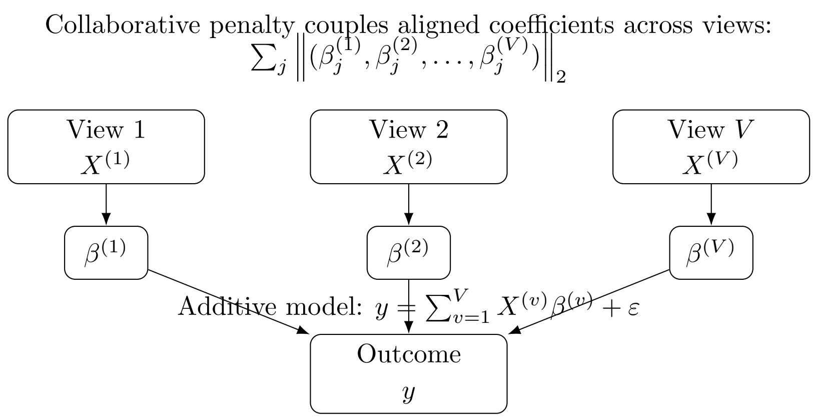

Figure 4: Collaborative (multi-view) learning: multiple feature views contribute to one outcome, with a penalty that encourages shared sparsity across views.

Interpretation. This penalty encourages a feature to be either:

- selected jointly across views (the coefficient vector across views is nonzero), or
- excluded across views (the coefficient vector across views is exactly zero).

At the same time, it allows view-specific effect sizes when a feature is selected.

Relationship to multitask lasso. Multitask lasso enforces shared sparsity across *tasks* (multiple responses). Multi-view lasso enforces shared sparsity across *views* (multiple feature sets) for a single response. Both are instances of structured sparsity using group norms.

6 Unifying Perspective

All methods discussed can be written as

$$\min_{\beta} \mathcal{L}(\beta) + \lambda\,\Omega(\beta),$$

where the penalty $\Omega(\beta)$ encodes prior structural assumptions.

Key takeaway:

Regularization is a mechanism for encoding structure and prior knowledge into learning algorithms.

References

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  (Elastic Net: bridges ridge and lasso; stabilizes variable selection under collinearity.)

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  (Group Lasso: enforces sparsity at the group level.)

- Tibshirani, R., Saunders, M., Rosset, S., Zhu, J., and Knight, K. (2005). *Sparsity and smoothness via the fused lasso.* Journal of the Royal Statistical Society: Series B, 67(1), 91–108.
  (Fused Lasso: sparsity + smoothness through penalties on coefficient differences.)

- Jacob, L., Obozinski, G., and Vert, J.-P. (2009). *Group lasso with overlap and graph lasso.* Proceedings of the 26th International Conference on Machine Learning (ICML).
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- Obozinski, G., Wainwright, M. J., and Jordan, M. I. (2011). *Support union recovery in high-dimensional multivariate regression.* The Annals of Statistics, 39(1), 1–47.
  (Multitask Lasso: shared sparsity across related tasks.)

- Gross, S. M. and Tibshirani, R. (2015). *Collaborative regression.* Biostatistics, 16(2), 326–338.
  (Collaborative learning: joint sparsity across multiple data views for the same outcome.)

Extended Reading

- Ding, D. Y., Li, S., Narasimhan, B., and Tibshirani, R. (2022). *Cooperative learning for multiview analysis.* Proceedings of the National Academy of Sciences, 119(38), e2202113119.
  (Cooperative learning: modern framework for multiview integration extending collaborative regression.)