DS6410: Homework 1

Due: February 9, 2026

1 ESI EX 2.5 (a)

Suppose $Y = X^T\beta + \epsilon$, where $\epsilon \sim N(0, \sigma^2)$ and we fit the model by least squares to the training data. For an arbitrary test point x_0, we have $\hat{y}_0 = x_0^T\hat{\beta}$, which can be written as $\hat{y}_0 = x_0^T\beta + \sum_{i=1}^N l_i(x_0)\epsilon_i$, where $l_i(x_0)$ is the ith element of $X(X^TX)^{-1}x_0$. Please derive the equation (2.27)

$$\begin{aligned} EPE(x_0) &= \mathbb{E}_{y_0|x_0}\mathbb{E}_T(y_0 - \hat{y}_0)^2 \\ &= \sigma^2 + \mathbb{E}_T x_0^T(X^TX)^{-1}x_0\sigma^2. \end{aligned}$$

You need to use $Var(\hat{\beta}_0) = (X^TX)^{-1}\sigma^2$.

2 The first two Bartlett identities

Let X be a random variable with probability density (or mass) function $f(x;\theta)$, where $\theta \in \Theta \subset \mathbb{R}$ is an unknown parameter. Define the log-likelihood function

$$\ell(\theta) = \log f(X;\theta),$$

and assume that the usual regularity conditions hold so that differentiation with respect to θ can be interchanged with expectation.

(a) Show that

$$\mathbb{E}\left[\frac{\partial}{\partial\theta}\ell(\theta)\right] = 0.$$

This result is known as the *first Bartlett identity*.

(b) Show that

$$\mathbb{E}\left[\left(\frac{\partial}{\partial\theta}\ell(\theta)\right)^2\right] = -\mathbb{E}\left[\frac{\partial^2}{\partial\theta^2}\ell(\theta)\right].$$

This result is known as the *second Bartlett identity*.

3 Gamma GLM

Let $Y \sim \text{Gamma}(\alpha, \beta)$ with shape $\alpha > 0$ and scale $\beta > 0$. Recall that

$$\mathbb{E}[Y] = \mu = \alpha\beta, \qquad f(y; \alpha, \beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} y^{\alpha-1} e^{-y/\beta}, \quad y > 0.$$

In this problem, you will express the Gamma distribution in *exponential dispersion family* form and derive (i) the **canonical link** and (ii) the **variance function**.

(a) Rewrite the Gamma density in the exponential dispersion family form

$$f(y; \theta, \phi) = \exp\left\{ \frac{1}{\phi}\big(y\theta - b(\theta)\big) + c(y, \phi) \right\},$$

and identify θ, ϕ, $b(\theta)$, and $c(y, \phi)$.

(b) Show that the mean satisfies

$$\mu = b'(\theta),$$

and hence the **canonical link** is

$$g(\mu) = \theta = (b')^{-1}(\mu).$$

Derive an explicit expression for $g(\mu)$.

(c) The variance function for an exponential dispersion family is defined by

$$\text{Var}(Y \mid \mu) = \phi\, V(\mu), \qquad V(\mu) = b''\big((b')^{-1}(\mu)\big).$$

Using your results from parts (a)–(b), derive the variance function $V(\mu)$ for the Gamma family.

Notes/Hint: You may find it helpful to express θ as a function of $\mu = \alpha\beta$ (not of α and β separately), and to use the identities $\mu = b'(\theta)$ and $V(\mu) = b''(\theta(\mu))$.

4 Binomial Proportion GLM

Suppose

$$Y \sim \frac{1}{m}\,\mathrm{Bin}(m, \pi), \qquad 0 < \pi < 1,$$

so that $mY \sim \mathrm{Bin}(m, \pi)$ and $\mathbb{E}[Y] = \mu = \pi$. The probability mass function of Y is

$$f(y) = \binom{m}{my} \pi^{my} (1-\pi)^{m-my}, \qquad y \in \left\{0, \frac{1}{m}, \ldots, 1\right\}.$$

In this problem, you will express the binomial proportion model in the *exponential dispersion family* form and derive the **canonical link** and the **variance function**.

(a) Rewrite the density of Y in the exponential dispersion family form

$$f(y; \theta, \phi) = \exp\left\{\frac{1}{\phi}\big(y\theta - b(\theta)\big) + c(y, \phi)\right\},$$

and identify the natural parameter θ, the dispersion parameter ϕ, and the function $b(\theta)$.

(b) Show that the mean satisfies

$$\mu = b'(\theta),$$

and hence derive the **canonical link function**

$$g(\mu) = \theta = (b')^{-1}(\mu).$$

(c) Recall that for an exponential dispersion family,

$$\mathrm{Var}(Y \mid \mu) = \phi\, V(\mu), \qquad V(\mu) = b''\big((b')^{-1}(\mu)\big).$$

Using your results from parts (a)–(b), derive the variance function $V(\mu)$ for the binomial proportion model.

(d) Briefly explain how this formulation leads to logistic regression as a special case of a binomial GLM.