Stochastic Processes Homework 2

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Exercise 1.12

a)
We express the CDF of M_+ and fix this rv's value to be p:

$$Pr(M_+ \leq p) = F_{M_+}(p) = Pr(X_1 \leq p, ..., X_n \leq p) = Pr(X_1 \leq p) \cdot ... \cdot Pr(X_n \leq p) = \prod_{i=1}^{n} Pr(X_i \leq p) = (F_X(p))^n$$

As such we have that $F_{M_+}(p) = (F_X(p))^n$

b)
Same process as above; we express the CDF of M_- and fix this rv's value to be m:

$$Pr(M_- \leq m) = F_{M_-}(m) = Pr(X_1 > m, ..., X_n > m) = 1 - Pr(X_1 \leq m) \cdot ... \cdot 1 - Pr(X_n \leq m) = \prod_{i=1}^{n}(1 - Pr(X_i \leq m)) = (1 - F_X(m))^n$$

As such we have that $F_{M_-}(m) = (1 - F_X(m))^n$

c)
Here the answer is almost already given in the hint.

$$\begin{aligned} P(M_+ = X_1, R \leq r | X_1 = x) &= \prod_{j=2}^{n} Pr(x - r < X_j \leq x) \\ &= [Pr(x - r < X_j \leq x)]^{n-1} \\ &= [F_X(x) - F_X(x - r)]^{n-1} \end{aligned}$$

We then remove the conditioning by averaging over $X_1 = x$ with pdf: $f_X(x)$, using the definition of the expectation.

$$P(M_+ = X_1, R \leq r) = \int_{-\infty}^{\infty} x f_X(x)[F_X(x) - F_X(x - r)]^{n-1} dx$$

I went to office hours to ask about where the inner n comes from, although I still do not understand, since we are not differentiating? Please enlighten me.

Exercise 1.14

It is assumed that we are required to prove the rules of expectation and variance

a) - linearity of expectation
We prove this by induction assuming discrete rv, although for the continuous case we can just substitute the sums with integrals.

Base case:

$$\begin{aligned}
E[X_1 + X_2] &= \sum_{x_1}\sum_{x_2}(x_1 + x_2)Pr(X_1 = x_1, X_2 = x_2)\\
&= \sum_{x_1}\sum_{x_2} x_1 Pr(X_1 = x_1, X_2 = x_2) + \sum_{x_1}\sum_{x_2} x_2 Pr(X_1 = x_1, X_2 = x_2)\\
&= \sum_{x_1} x_1 \sum_{x_2} Pr(X_1 = x_1, X_2 = x_2) + \sum_{x_2} x_2 \sum_{x_1} Pr(X_1 = x_1, X_2 = x_2)\\
&= \sum_{x_1} x_1 Pr(X_1 = x_1) + \sum_{x_2} x_2 Pr(X_2 = x_2) \qquad \text{(marginal pmf)}\\
&= E[X_1] + E[X_2]\\
&= \bar{X}_1 + \bar{X}_2
\end{aligned}$$

Induction step:

Assuming that $E[X_1 + ... + X_n] = \bar{X}_1 + ... + \bar{X}_n$, we want to prove that $E[X_1 + ... + X_n + X_{n+1}] = \bar{X}_1 + ... + \bar{X}_n + \bar{X}_{n+1}$:

$$\begin{aligned}
E[X_1 + ... + X_n + X_{n+1}] &= E[X_1 + ... + X_n] + E[X_{n+1}]\\
&= \bar{X}_1 + ... + \bar{X}_n + \bar{X}_{n+1}
\end{aligned}$$

In line 1, we use the base case and assume that $X_1 + ... + X_n$ can be treated as a rv.

b) - independent multiplication

Again we prove this by induction.

Base case:

$$\begin{aligned}
E[X_1 \cdot X_2] &= \sum_{x_1}\sum_{x_2}(x_1 \cdot x_2)Pr(X_1 = x_1, X_2 = x_2) && (1)\\
&= \sum_{x_1}\sum_{x_2}(x_1 \cdot x_2)Pr(X_1 = x_1) \cdot Pr(X_2 = x_2) && (2)\\
&= \sum_{x_1}\sum_{x_2} x_1 Pr(X_1 = x_1) \cdot x_2 Pr(X_2 = x_2) && (3)\\
&= \sum_{x_1} x_1 Pr(X_1 = x_1) \cdot \sum_{x_2} x_2 Pr(X_2 = x_2) && (4)\\
&= E[X_1] \cdot E[X_2] && (5)
\end{aligned}$$

(2) X_1 and X_2 are independent

Induction step:

Assuming that $E[X_1 \cdot ... \cdot X_n] = \bar{X}_1 \cdot ... \cdot \bar{X}_n$, we want to prove that $E[X_1 \cdot ... \cdot X_n \cdot X_{n+1}] = \bar{X}_1 \cdot ... \cdot \bar{X}_n \cdot \bar{X}_{n+1}$:

$$\begin{aligned}
E[X_1 \cdot ... \cdot X_n \cdot X_{n+1}] &= E[X_1 \cdot ... \cdot X_n] \cdot E[X_{n+1}]\\
&= \bar{X}_1 \cdot ... \cdot \bar{X}_n \cdot \bar{X}_{n+1}
\end{aligned}$$

c) - independent variance

Similarly, we prove this by induction. V for variance:

Base case:

$$\begin{aligned}
V[X_1 + X_2] &= E[(X_1 + X_2)^2] - E[X_1 + X_2]^2\\
&= E[X_1^2 + X_2^2 + 2X_1X_2] - (E[X_1] + E[X_2])^2\\
&= E[X_1^2] + E[X_2^2] + 2E[X_1X_2] - (E[X_1]^2 + E[X_2]^2 + 2E[X_1]E[X_2])\\
&= E[X_1^2] + E[X_2^2] + 2E[X_1X_2] - E[X_1]^2 - E[X_2]^2 - 2E[X_1]E[X_2]\\
&= E[X_1^2] + E[X_2^2] + 2E[X_1X_2] - E[X_1]^2 - E[X_2]^2 - 2E[X_1X_2]\\
&= (E[X_1^2] - E[X_1]^2) + (E[X_2^2] - E[X_2]^2)\\
&= V[X_1] + V[X_2]
\end{aligned}$$

Induction step:
Assuming that $V[X_1+...+X_n] = \bar{X}_1+...+\bar{X}_n$, we want to prove that $V[X_1+...+X_n+X_{n+1}] = \bar{X}_1+...+\bar{X}_n+\bar{X}_{n+1}$:

$$\begin{aligned} V[X_1 + ... + X_n + X_{n+1}] &= V[X_1 + ... + X_n] + V[X_{n+1}] \\ &= \bar{X}_1 + ... + \bar{X}_n + \bar{X}_{n+1} \end{aligned}$$

In line 1, we use the base case and assume that $X_1 + ... + X_n$ can be treated as one rv.

Exercise 1.20

a)

$$\begin{aligned} E[Y] &= \sum_{y\geq 0}^{\infty} Pr(Y > y) \\ &= \sum_{y\geq 0}^{\infty} 1 - F_Y(y) \\ &= \sum_{y\geq 0}^{\infty} 1 - (1 - \frac{2}{(y+1)(y+2)}) \\ &= \sum_{y\geq 0}^{\infty} \frac{2}{(y+1)(y+2)} \\ &= 2 \underbrace{\sum_{y\geq 0}^{\infty} \frac{1}{(y+1)(y+2)}}_{=1} \end{aligned}$$

Last line was checked with Wolfram Alpha[1]

b)
We find the pmf of Y and use it to check for EY

$$\begin{aligned} p_Y(y) &= F_Y(y) - F_Y(y-1) \\ &= 1 - \frac{2}{(y+1)(y+2)} - (1 - \frac{2}{((y-1)+1)((y-1)+2)}) \\ &= \frac{2}{y(y+1)} - \frac{2}{(y+1)(y+2)} \end{aligned}$$

$$\begin{aligned} E[Y] &= \sum_{0}^{\infty} y p_Y(y) \\ &= \sum_{0}^{\infty} y\left(\frac{2}{y(y+1)} - \frac{2}{(y+1)(y+2)}\right) \\ &= 2 \end{aligned}$$

Checked with Wolfram Alpha[2]. Using this we set the summation starting from 1 since multiplying by zero yields zero anyways.

[1] https://www.wolframalpha.com/input/?i=%5Csum+1%2F%28%28n%2B1%29%28n%2B2%29%29

[2] https://www.wolframalpha.com/input/?i=+%5Csum_1%5E%7B%5Cinfty%7Dy%28+-+2%2F%28%281+%2B+y%29%282+%2B+y%29%29+%2B+2%2F%28y%28y%2B1%29%29%29

c)
We find

$$\begin{aligned} E[X|Y=y] &= \sum_{x=1}^{y} x p_{X|Y}(x|y) \\ &= \sum_{x=1}^{y} x \frac{1}{y} \\ &= \frac{y+1}{2} \end{aligned}$$

$$\begin{aligned} E[X] &= E[E[X|Y]] \\ &= \sum_{y=1}^{\infty} p_Y(y) E[X|Y=y] \\ &= \sum_{y=1}^{\infty} \left(\frac{2}{y(y+1)} - \frac{2}{(y+1)(y+2)} \right) \frac{y+1}{2} \\ &= \sum_{y=1}^{\infty} \frac{1}{y} - \frac{1}{y+2} \\ &= 3/2 \end{aligned}$$

Using Wolfram Alpha ofr $E[X|Y=y]$[3] and for $E[X]$[4]. Alternative method $E[X] = \sum_x x p_X(x)$ not explored.

d)
We find

$$\begin{aligned} E[Z|Y=y] &= \sum_{z=1}^{y^2} z p_{Z|Y}(z|y) \\ &= \sum_{z=1}^{y^2} z \frac{1}{y^2} \\ &= \frac{y^2+1}{2} \end{aligned}$$

Using Wolfram Alpha[5].

$$\begin{aligned} E[Z] &= E[E[Z|Y]] \\ &= \sum_{y=1}^{\infty} p_Y(y) E[Z|Y=y] \\ &= \sum_{y=1}^{\infty} \left(\frac{2}{y(y+1)} - \frac{2}{(y+1)(y+2)} \right) \frac{y^2+1}{2} \\ &= \sum_{y=1}^{\infty} \left(\frac{y^2+1}{y(y+1)} - \frac{y^2+1}{(y+1)(y+2)} \right) \\ &= \infty \text{ (sum diverges)} \end{aligned}$$

Checked using wolfram alpha[6]

[3] https://www.wolframalpha.com/input/?i=%5Csum%28x%2Fy%29%2C+x%3D1+to+y
[4] https://www.wolframalpha.com/input/?i=sum+1%2Fy+-+1%2F%28y%2B2%29%2C+y%3D1+to+infinity
[5] https://www.wolframalpha.com/input/?i=sum+z%2F%28y%5E2%29%2C+z%3D1+to+y%5E2
[6] https://www.wolframalpha.com/input/?i=sum+z%2F%28y%5E2%29%2C+z%3D1+to+%28y%5E2%29

Exercise 1.28

$$\begin{aligned}
F_Y(y) &= Pr(Y \le y) && (6)\\
&= Pr(F_X(X) < y) && (7)\\
&= Pr(X < F_X^{-1}(y)) && (8)\\
&= F_X(F_X^{-1}(y)) && (9)\\
&= y && (10)
\end{aligned}$$

In line 3 we make use of `https://en.wikipedia.org/wiki/Cumulative_distribution_function#Inverse_distribution_function_(quantile_function)`. This is the cdf of Y over the interval 0 to 1.

Exercise 1.32

a)

$$\begin{aligned}
Pr(X \ge b) &\le \frac{\sigma^2}{\sigma^2 + b^2}\\
\beta &\le \frac{\sigma^2}{\sigma^2 + b^2}\\
\beta &\le \frac{\int_{-\infty}^{b^-} x^2 f_X(x)dx + \int_b^{\infty} x^2 f_X(x)dx}{\int_{-\infty}^{b^-} x^2 f_X(x)dx + \int_b^{\infty} x^2 f_X(x)dx + b^2}\\
b^2\beta &\le b^2 \frac{\int_{-\infty}^{b^-} x^2 f_X(x)dx + \int_b^{\infty} x^2 f_X(x)dx}{\int_{-\infty}^{b^-} x^2 f_X(x)dx + \int_b^{\infty} x^2 f_X(x)dx + b^2}
\end{aligned}$$

b), **c)**, **d)**
Not done

Exercise 1.43

We are assuming that we have to prove the Chebyshev inequality. To do this, we use the markov inequality. For non-negative rv Z_n and $\epsilon > 0$, then $P(Z_n \ge \epsilon) \le \frac{E[Z_n]}{\epsilon}$:

$$\begin{aligned}
P(|Z_n - \alpha| \ge \epsilon) &= P((Z_n - \alpha)^2 \ge \epsilon^2)\\
&\le \frac{E[(x-\mu)^2]}{\epsilon^2}
\end{aligned}$$

At line 1, we can treat $(Z_n - \alpha)^2$ as a non-negative rv and then apply the markov inequality.

Exercise 1.48

Not done

Exercise 8

Not done

Exercise 9

Not done

Excuse the unfinished exercises, I have not looked into the other exercises that much, since I am mostly focusing on another final project.