Case III. $Q = x^n$

P.I. $\frac{x^n}{f(D)} = [f(D)]^{-1} . x^n$

Expand $[f(D)]^{-1}$ by the binomial theorem in ascending powers of D as far as the result of operation on x^n is zero.

Q1. $(D^2-1)y = x^2$

A.E $m^2-1 = 0$

$m = \pm 1$

C.F. $= C_1 e^x + C_2 e^{-x}$

P.I. $= \frac{x^2}{(D^2-1)}$

$= -(1-D^2)^{-1} . x^2$

$= -[1+D^2 \ldots]. x^2$

$= -[x^2 + D^2 x^2 \ldots]$

$= -[x^2+2]$

$y =$ C.F. + P.I.

Q 1. $(D^2+2D+1)y = x$

2. $D^2 y = x^2+2x-1$

Q1. $(D^2 + 2D + 1)y = x$

A.E $m^2 + 2m + 1 = 0$

$m = -1, -1$

C.F. : $(c_1 + c_2 x)e^{-x}$

P.I. $\dfrac{x}{(1 + 2D + D^2)}$

$= [1 + (2D + D^2)]^{-1} x$

$= [1 - (2D + D^2) \cdots] x$

$= x - (2D + D^2).x + \cdots$

$= x - (2Dx + D^2 x) + \cdots$

$= x - (2)$

$y = C.F. + P.I.$

$= (c_1 + c_2 x)e^{-x} + x - 2$

Q2. $(D^2 - 4D + 3)y = x^3$

C.F. $= c_1 e^{x} + c_2 e^{3x}$

P.I. $\dfrac{x^3}{(3 - 4D + D^2)} = \dfrac{1}{3[1 - \frac{4}{3}D + \frac{D^2}{3}]} . x^3$

$= \frac{1}{3}[1 + (-\frac{4}{3}D + \frac{D^2}{3})]^{-1} x^3$

$= \frac{1}{3}[1 - (\frac{-4}{3}D + \frac{D^2}{3}) + \frac{(-1)(-1-1)}{2!}(\frac{-4D}{3} + \frac{D^2}{3})^2 \cdots] x^3$

$+ \frac{(-1)(-1-1)(-1-2)}{3!}(\frac{-4D}{3} + \frac{D^2}{3})^3 \cdots$

$$= \frac{1}{3}\left[x^3 - \left(-\frac{4}{3}D + \frac{D^2}{3}\right)x^3 + \left(-\frac{4D}{3} + \frac{D^2}{3}\right)^2 x^3 + \frac{64}{27}D^3x^3 \right]$$

$$= \frac{1}{3}\left[x^3 - \left(-\frac{4}{3}.(3x^2) + \frac{1}{3}(6x)\right) + \right.$$

$$\left. \left(\frac{16}{9}D^2 + \frac{D^4}{9} - \frac{8}{9}D^3\right)x^3 + \frac{64}{27}\times 6 \right]$$

$$= \frac{1}{3}\left[x^3 - \left(-4x^2 + 2x\right) + \left(\frac{16}{9}\times 6x + 0 - \frac{8}{9}\times 6\right) + \frac{64\times 2}{9} \right]$$

$$= \frac{1}{3}\left[x^3 + 4x^2 - 2x + \frac{32}{3}x - \frac{16}{3} + \frac{128}{9} \right]$$

$$y = C.F. + P.I.$$

Q3. $(D^2 - 6D + 9)\,y = x$

$$C.F. = (C_1 + C_2 x)\,e^{3x}$$

P.I. $\dfrac{x}{(9 - 6D + D^2)} = \dfrac{1}{9}\left[1 + (-6D + D^2)\right]^{-1}.\,x$

$$= \frac{1}{9}\left[1 - (-6D + D^2) + \frac{(-1)(-1-1)}{2!}(-6D + D^2)^2 \dots\right]x$$

$$= \frac{1}{9}\left[x - (-6)\right] = \frac{1}{9}(x + 6)$$

$$y = C.F. + P.I.$$

On putting $y = 0$, ...

$$0 = C_1 + \frac{R}{p}l \quad \Rightarrow \quad C_1 = -\frac{Rl}{p}$$

On differentiating (1), we get

$$\frac{dy}{dx} = -a\,C_1 \sin ax + a\,C_2 \cos ax - \frac{R}{p}$$

On putting $\frac{dy}{dx} = 0$ and $x = 0$ in (2), we get

$$0 = a\,C_2 - \frac{R}{p} \quad \Rightarrow \quad C_2 = \frac{R}{a.p}$$

On putting the values of C_1 and C_2 in (1), we get

$$y = -\frac{R}{p} l \cos ax + \frac{R}{a.p} \sin ax + \frac{R}{p}(l - x)$$

$$y = \frac{R}{p}\left[\frac{\sin ax}{a} - l \cos ax + l - x\right]$$

EXERCISE 2.4

1. Find the P.I. of $\frac{d^2y}{dx^2} = x^2 + 2x - 1$ (*GBTU, II Sem. Jan. 2013*) **Ans.** $\frac{x^4}{12} + \frac{x^3}{3} \ldots$

Solve the following equations :

2. $(D^2 + 5D + 4)\, y = 3 - 2x$ **Ans.** $C_1 e^{-x} + C_2 e^{-4x} + \frac{1}{8}(11 - 4x)$

3. $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = x$ **Ans.** $(C_1 + C_2 x)\, e^{-x} + x - 2$

4. $(2D^2 + 3D + 4)\, y = x^2 - 2x$ **Ans.** $e^{-\frac{3}{4}x}\left[A \cos \frac{\sqrt{23}}{4} x + B \sin \frac{\sqrt{23}}{4} x\right] + \frac{1}{32}[8x^2 - 28x + \ldots$

5. $(D^2 - 4D + 3)\, y = x^3$ **Ans.** $C_1 e^{x} + C_2 e^{3x} + \frac{1}{27}(9x^3 + 36x^2 + 78x + 80)$.

6. $\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} - 6\frac{dy}{dx} = 1 + x^2$. **Ans.** $A + Be^{-2x} + C e^{3x} - \frac{1}{36}\left(2x^3 - x^2 + \frac{25}{3}x\right)$

7. $\frac{d^4y}{dx^4} + 4y = x^4$ **Ans.** $e^{x}(C_1 \cos x + C_2 \sin x) + e^{-x}(C_3 \cos x + C_4 \sin x) + \frac{1}{4}\ldots$

8. $\frac{d^2y}{dx^2} + 2P\frac{dy}{dx} + (p^2 + q^2)y = e^{cx} + p.q\, x^2$

Ans. $e^{-px}[C_1 \cos qx + C_2 \sin qx] + \frac{e^{Cx}}{(p + C)^2 + q^2} + \frac{pq}{p^2 + q^2}\left[x^2 - \frac{4px}{p^2 + q^2} + \frac{6p^2 \ldots}{(p^2 \ldots}\right.$

9. $D^2(D^2 + 4)\, y = 96\, x^2$ **Ans.** $C_1 + C_2 x + C_3 \cos 2x + C_4 \sin 2x + 2x^2 (x^2 - \ldots$

Case. IV

$$\frac{1}{f(D)} \cdot e^{ax} \cdot \phi(x) = e^{ax} \cdot \frac{1}{f(D+a)} \cdot \phi(x)$$

$=$

Q1. $(D^2 - 2D + 2)\, y = e^x \cos x$

A.E. $m^2 - 2m + 2 = 0$

$m = 1 \pm i$

C.F. $e^x (C_1 \cos x + C_2 \sin x)$

P.I. $\dfrac{e^x \cos x}{(D^2 - 2D + 2)}$

$$= e^x \cdot \left[\frac{1}{(D+1)^2 - 2(D+1) + 2} \cos x \right]$$

$$= e^x \cdot \left[\frac{1}{(D^2+1)} \cos x \right]$$

$$= e^x \left[x \cdot \frac{1}{2D} \cdot \cos x \right]$$

$$= e^x \cdot \left[\frac{x}{2} \sin x \right]$$

$y = \text{C.F.} + \text{P.I.}$

Q2. $(D^2 - 4D + 4)\, y = x^3 e^{2x}$

A.E. $m^2 - 4m + 4 = 0$

$m = 2, 2$

$$3\left[\ \ldots \left(\frac{-1}{3}D + \frac{D}{3}\right) x^{\ldots} + \left(-\frac{4D}{3} + \frac{D^2}{3}\right)\right.$$

C.F. $\quad (c_1 + c_2 x)\, e^{2x}$

P.I. $\quad = \dfrac{x^3 . e^{2x}}{D^2 - 4D + 4}$

$$= e^{2x} . \frac{1}{(D+2)^2 - 4(D+2) + 4}\, x^3$$

$$= e^{2x} . \frac{1}{D^2} . x^3$$

$$= e^{2x} . \frac{1}{D}\left(\frac{x^4}{4}\right) = e^{2x} . \frac{x^5}{20}$$

Q3. $\quad (D^2 - 2D)\, y = e^x \sin x$

A.E. $\quad m^2 - 2m = 0$

$$m = 0, 2.$$

C.F. $\quad c_1 + c_2 e^{2x}$

P.I. $\quad \dfrac{e^x \sin x}{(D^2 - 2D)}$

$$= e^x . \left[\frac{1}{(D+1)^2 - 2(D+1)} \sin x\right]$$

$$= e^x . \left[\frac{1}{D^2 + 2D + 1 - 2D - 2} . \sin x\right]$$

$$= e^x . \left[\frac{1}{D^2 - 1} . \sin x\right]$$

$$= e^x . \frac{1}{(-2)} \sin x$$

$y =$ C.F. + P.I.

$$= e^x \frac{}{(D+1)^2 - 2(D+1) + 2}$$

$$= \frac{1}{2} x e^x \sin x \quad \left[\text{If } f(-a^2) = 0, \text{ then } \frac{1}{f(D^2)}\phi(x) = x \frac{1}{f'(D)}\right.$$

$y =$ C.F. + P.I.

$$= e^x (A \cos x + B \sin x) + \frac{1}{2}(x+1) + \frac{1}{2} x e^x \sin x.$$

Example 29. *Solve* : $(D^2 - 4D + 4)\, y = x^3 e^{2x}$

Solution. We have, $(D^2 - 4D + 4)\, y = x^3 e^{2x}$

A.E. is $m^2 - 4m + 4 = 0 \Rightarrow (m-2)^2 = 0 \Rightarrow m = 2, 2$

$$\text{C.F.} = (C_1 + C_2 x)\, e^{2x}$$

$$\text{P.I.} = \frac{1}{D^2 - 4D + 4} x^3 \cdot e^{2x} = e^{2x} \frac{1}{(D+2)^2 - 4(D+2) + 4} x^3$$

$$= e^{2x} \frac{1}{D^2} x^3 = e^{2x} \cdot \frac{1}{D}\left(\frac{x^4}{4}\right) = e^{2x} \cdot \frac{x^5}{20}$$

The complete solution is $y = (C_1 + C_2 x) e^{2x} + e^{2x} \cdot \frac{x^5}{20}$

Example 30. Solve $(D^4 - 1)\, y = e^x \cos x$ *(Uttarakhand II Seme*

Solution. Here, we have

$$(D^4 - 1)\, y = e^x \cos x$$

A.E. is $m^4 - 1 = 0 \Rightarrow (m+1)(m-1)(m^2+1) = 0$

$\Rightarrow m = -1, 1, +i, -i$

$$\text{C.F.} = C_1 e^{-x} + C_2 e^x + (C_3 \cos x + C_4 \sin x)$$

$$\text{P.I.} = \frac{1}{D^4 - 1} e^x \cos x$$

$$= e^x \frac{1}{(D+1)^4 - 1} \cos x = e^x \frac{1}{D^4 + 6D^3 + 4D^2 + 6D} \cos x$$

$$= e^x \frac{1}{(-1)^2 + 6(-1)D + 4(-1) + 6D} \cos x$$

$$= e^x \frac{1}{1 - 6D - 4 + 6D} \cos x = -\frac{e^x \cos x}{3}$$

Complete solution is y = C.F. + P.I.

$$\Rightarrow y = C_1 e^{-x} + C_2 e^x + (C_3 \cos x + C_4 \sin x) - \frac{e^x \cos x}{3}$$

Example 31. *Solve* : $\frac{d^4 y}{dx^4} - y = \cos x . \cosh x$ *(Nagpur University, Sum*

Solution. We have, $(D^4 - 1)\, y = \cos x \cosh$

A.E. is : m^4 ...

$$\text{P.I.} = \frac{1}{D^4-1}\cos x\cosh x = \frac{1}{D^4-1}\cos x\left(\frac{e^x+e^{-x}}{2}\right)$$

$$= \frac{1}{2}\left[\frac{1}{D^4-1}e^x\cos x + \frac{1}{D^4-1}e^{-x}\cos x\right]$$

$$= \frac{1}{2}\left[e^x\frac{1}{(D+1)^4-1}\cos x + e^{-x}\frac{1}{(D-1)^4-1}\cos x\right]$$

$$= \frac{1}{2}\left[e^x\frac{1}{D^4+4D^3+6D^2+4D}\cos x + e^{-x}\frac{1}{D^4-4D^3+6D^2-4D}\cos x\right]$$

$$= \frac{1}{2}\left[e^x\frac{1}{(-1)^2+4D(-1)+6(-1)+4D}\cos x + e^{-x}\frac{1}{(-1)^2-4D(-1)+6(-1)-4D}\cos x\right]$$

$$= \frac{1}{2}\left[e^x\frac{1}{-5}\cos x + e^{-x}\frac{1}{-5}\cos x\right]$$

$$= -\frac{1}{5}\left(\frac{e^x+e^{-x}}{2}\right)\cos x = -\frac{1}{5}\cosh x\cos x$$

Hence, the complete solution is

$$y = C_1e^x + C_2e^{-x} + C_3\cos x + C_4\sin x - \frac{1}{5}\cos x\cosh x$$ **Ans.**

Example 32. *Solve the differential equation :*

$$\frac{d^3y}{dx^3} - 7\frac{d^2y}{dx^2} + 10\frac{dy}{dx} = e^{2x}\sin x$$ *(Nagpur University, Summer* 2005)

Solution. We have, $\frac{d^3y}{dx^3} - 7\frac{d^2y}{dx^2} + 10\frac{dy}{dx} = e^{2x}\sin x$

$\Rightarrow$ $D^3y - 7D^2y + 10\,Dy = e^{2x}\sin x \Rightarrow (D^3 - 7D^2 + 10D)\,y = e^{2x}\sin x$

A.E. is $m^3 - 7m^2 + 10\,m = 0 \Rightarrow (m-2)(m^2-5m) = 0$

$\Rightarrow$ $m(m-2)(m-5) = 0 \Rightarrow m = 0, 2, 5$

$$\text{C.F} = C_1e^{0x} + C_2e^{2x} + C_3e^{5x}$$

$$\text{P.I.} = \frac{1}{D^3-7D^2+10D}e^{2x}\sin x = e^{2x}\frac{1}{(D+2)^3-7(D+2)^2+10(D+2)}.\sin x$$

$$= e^{2x}\frac{1}{D^3+6D^2+12D+8-7D^2-28D-28+10D+20}.\sin x$$

$$= e^{2x}\frac{1}{D^3-D^2-6D}\sin x = e^{2x}\frac{1}{(-1^2)D-(-1^2)-6D}\sin x$$

$$= e^{2x}\frac{1}{-D+1-6D}\sin x = e^{2x}\frac{1}{1-7D}\sin x$$

$$= e^{2x}\frac{1+7D}{1-49D^2}\sin x = e^{2x}\frac{1+7D}{1-49(-1^2)}\sin x$$

$$= e^{2x}\,\frac{1+7D}{50}\sin x \;=\; \frac{e^{2x}}{50}(\sin x + 7\cos x)$$

Complete solution is

$$y = \text{C.F.} + \text{P.I.}$$

$$\Rightarrow \qquad y = C_1 + C_2\,e^{2x} + C_3\,e^{5x} + \frac{e^{2x}}{50}(\sin x + 7\cos x)$$

Example 33. *Solve* $(D^2 + 2)\,y = e^x \cos x + x^2 e^{2x}$ (M.U. II Sem

Solution. Here, we have

$$(D^2 + 2)y = e^x \cos x + x^2 e^{2x}$$

A.E. is $m^2 + 2 = 0 \quad \Rightarrow \quad m = \pm i\sqrt{2}$

$$\text{C.F.} = C_1 \cos\sqrt{2}\,x + C_2 \sin\sqrt{2}\,x$$

$$\text{P.I.} = \frac{1}{D^2+2}(e^x \cos x + x^2 e^{2x}) = \frac{1}{D^2+2}e^x\cos x + \frac{1}{D^2+2}x^2 e^{2x}$$

$$= e^x\frac{1}{(D+1)^2+2}\cos x + e^{2x}\frac{1}{(D+2)^2+2}x^2$$

$$= e^x\frac{1}{D^2+2D+3}\cos x + e^{2x}\frac{1}{D^2+4D+6}x^2$$

$$= e^x\frac{1}{-1+2D+3}\cos x + \frac{e^{2x}}{6}\,\frac{1}{1+\frac{4D}{6}+\frac{D^2}{6}}x^2$$

$$= e^x\frac{1}{2D+2}\cos x + \frac{e^{2x}}{6}\left[1+\frac{4D}{6}+\frac{D^2}{6}\right]^{-1}x^2$$

$$= \frac{e^x}{2}\,\frac{D-1}{D^2-1}\cos x + \frac{e^{2x}}{6}\left[1-\frac{4D}{6}-\frac{D^2}{6}+\frac{4}{9}D^2+\ldots\right]x^2$$

$$= \frac{e^x}{2}\,\frac{D-1}{-1-1}\cos x + \frac{e^{2x}}{6}\left[x^2-\frac{2}{3}(2x)-\frac{1}{6}(2)+\frac{4}{9}(2)\right]$$

$$= -\frac{e^x}{4}(D-1)\cos x + \frac{e^{2x}}{6}\left[x^2-\frac{4x}{3}-\frac{1}{3}+\frac{8}{9}\right]$$

$$= -\frac{e^x}{4}(D\cos x - \cos x) + \frac{e^{2x}}{6}\left[x^2-\frac{4x}{3}+\frac{5}{9}\right]$$

$$= -\frac{e^x}{4}(-\sin x - \cos x) + \frac{e^{2x}}{6}\left[x^2-\frac{4x}{3}+\frac{5}{9}\right]$$

$$= \frac{e^x}{4}(\sin x + \cos x) + \frac{e^{2x}}{6}\left[x^2-\frac{4x}{3}+\frac{5}{9}\right]$$

Complete solution is $y = \text{C.F.} + \text{P.I.}$

$$y = C_1\cos\sqrt{2}\,x + C_2\sin\sqrt{2}x + \frac{e^x}{4}(\sin x + \cos x) + \frac{e^{2x}}{6}\left(x^2 - \frac{4x}{3} + \ldots\right.$$

Example 34. *A body executes damped forced vibrations given by the equation*

$$\frac{d^2x}{}$$

$$= e^{3x}.\left(\frac{x^2}{2}-\frac{x}{2}+\frac{1}{4}\right)-\frac{3e^x}{16}(2\cos 2x+2\sin 2x)$$

The complete solution is

$$y = C_1 e^x + C_2 e^{3x} + e^{3x}\left(\frac{x^2}{2}-\frac{x}{2}\right)-\frac{3e^x}{8}(\cos 2x+\sin 2x)$$ **Ans.**

The term $\frac{e^{3x}}{4}$ has been omitted from the P.I., since C_2e^{3x} is present in the C.F.

Example 38. *Find the complete solution of* $\frac{d^2y}{dx^2}-3\frac{dy}{dx}+2y = xe^{3x}+\sin 2x$

(*U.P. II Semester* 2003)

Solution. The auxiliary equation is

$$m^2 - 3m + 2 = 0 \quad\Rightarrow\quad m^2 - 2m - m + 2 = 0$$

$$\Rightarrow \quad (m-2)(m-1) = 0 \Rightarrow \quad m = 1, 2$$

$$\text{C.F.} = C_1e^x + C_2e^{2x}$$

$$\text{P.I.} = \frac{1}{D^2-3D+2}(xe^{3x}+\sin 2x)$$

$$= \frac{1}{D^2-3D+2}xe^{3x}+\frac{1}{D^2-3D+2}\sin 2x$$

$$= e^{3x}\frac{1}{(D+3)^2-3(D+3)+2}x+\frac{1}{-4-3D+2}\sin 2x$$

$$= e^{3x}\frac{1}{D^2+6D+9-3D-9+2}x+\frac{1}{-3D-2}\sin 2x$$

$$= e^{3x}\frac{1}{D^2+3D+2}x-\frac{1}{3D+2}\sin 2x = \frac{e^{3x}}{2}\left[1+\left(\frac{3D+D^2}{2}\right)\right]^{-1}x-\frac{(3D-2)}{9D^2-4}\sin 2x$$

$$= \frac{e^{3x}}{2}\left[1-\left(\frac{3D+D^2}{2}\right)+\ldots\right]x-\frac{(3D-2)}{9(-4)-4}\sin 2x$$

$$= \frac{e^{3x}}{2}\left[x-\left(\frac{3D+D^2}{2}\right)x+\ldots\right]-\frac{(3D-2)}{-36-4}\sin 2x = \frac{e^{3x}}{2}\left[x-\frac{3}{2}\right]+\frac{3D-2}{40}\sin 2x$$

$$\Rightarrow \text{P.I.} = \frac{e^{3x}}{4}(2x-3)+\frac{1}{40}(6\cos 2x-2\sin 2x) = \frac{e^{3x}}{4}(2x-3)+\frac{3}{20}\cos 2x-\frac{1}{20}\sin 2x$$

The complete solution is

$$y = \text{C.F.} + \text{P.I.}$$

$$\Rightarrow \quad y = C_1e^x + C_2e^{2x}+\frac{e^{3x}}{4}(2x-3)+\frac{3}{20}\cos 2x-\frac{1}{20}\sin 2x$$ **Ans.**

Example 39. *Solve* : $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+y = xe^x\cos x$

(*U.P. II Semester 2009, K. U., 2009, M.U. II Semester, 2009*)

Solution. Here we have $(D^2-2D+1)\,y = x\,e^x\cos x$