⇒ Methods in recurrence relation:

→ Characteristic root method
→ Generating method
→ Iteration method
} IMP

CHARACTERISTIC ROOT METHOD:

Q. $c_0 a_n + c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k} = f(n)$

⇒ Linear recurrence relation ($\because$ power is 1)

$c_0, c_1, \dots, c_k \rightarrow$ constants

$a_n, a_{n-1}, \dots, a_{n-k} \rightarrow$ recursive terms

$f(n) \longrightarrow$ recursive function

If, $f(n) = 0 \longrightarrow$ Homogenous

$f(n) \neq 0 \longrightarrow$ Non-homogenous

Degree = 1 ($\because$ power (max) = 1)

Order = (highest subscript ~ lowest subscript)
$= n - (n-k) = k$

For, $a_n = 2a_{n-1} + 3a_{n-2} \Rightarrow a_n - 2a_{n-1} - 3a_{n-2} = 0$

Here $f(n) = 0$, order $= n-(n-2) = 2$, degree = 1

$\therefore$ It's a homogenous recurrence relation of order = 2 & degree = 1.

Solution
- Homogenous $a_n^{(h)}$
- Particular $a_n^{(p)}$

$$\boxed{a_n = a_n^{(h)} + a_n^{(p)}}$$

CHARACTERISTIC ROOT METHOD:

$\Rightarrow C_0 a_n + C_1 a_{n-1} + C_2 a_{n-2} + \dots + C_k a_{n-k} = 0$

$\Rightarrow$ Assume: $a_n = A\alpha^{n}$

$C_0 A\alpha^{n} + C_1 A\alpha^{n-1} + C_2 A\alpha^{n-2} + \dots + C_k A\alpha^{n-k} = 0$

$\Rightarrow A\alpha^{n-k}\left(C_0\alpha^{k} + C_1\alpha^{k-1} + C_2\alpha^{k-2} + \dots + C_k\right) = 0$

$\therefore C_0\alpha^{k} + C_1\alpha^{k-1} + C_2\alpha^{k-2} + \dots + C_k = 0$ (solⁿ)

[$\because A\alpha^{n-k}$ is zero.]

Q. $a_n - 4a_{n-1} + 5a_{(n-2)} = 0$

$\Rightarrow$ Order $= n - n + 2 = 2$

Char. eqⁿ:

$$\boxed{\alpha^2 - 4\alpha + 5 = 0}$$

Q. $a_n - 4a_{n-1} = 0$

$\Rightarrow$ Order $= 1$

Char. eqⁿ: $\boxed{\alpha - 4 = 0}$

Degree $= 1 =$ No. of roots

$\Rightarrow \boxed{\alpha_1 = 4}$

* Cases of roots:

Case I: If roots are real & distinct.
$\alpha_1, \alpha_2, \dots, \alpha_k$

$\therefore$ Solⁿ: $a_n^{(h)} = A_1\alpha_1^{n} + A_2\alpha_2^{n} + \dots + A_k\alpha_k^{n}$

$A_1, A_2, A_3 \rightarrow$ constants that can be determined by using boundary conditions.

Case II: Multiple roots

$(\alpha - 4)^2 = 0$

$\alpha = 4, 4$

$\alpha = 4$ with multiplicity $(m) = 2$.

Multiple roots, multiplicity m.

$\longrightarrow a_n^{(h)} = (A_1 n^{m-1} + A_2 n^{m-2} + A_3 n^{m-3} + \dots + A_m)\alpha^n$

Case III : Mixed

$\longrightarrow a_n = A_1 (3)^n + (A_2 n + A_3) 2^n$

Q. $a_n - a_{n-1} - 2a_{n-2} = 0$ | $a_0 = 0$, $a_1 = 1$

$\Rightarrow$ Qu $\alpha^2 - \alpha - 2 = 0$

$\alpha_1 = 2, \ \alpha_2 = -1$ (Distinct roots)

Case I:

$a_n^{(h)} = A_1 \alpha_1^n + A_2 \alpha_2^n$

$a_n = A_1 (2)^n + A_2 (-1)^n$ —— (i)

Now,

$a_0 = A_1 + A_2 \Rightarrow A_1 + A_2 = 0$ —— (ii)

$a_1 = 2A_1 - A_2 \Rightarrow 2A_1 - A_2 = 1$ —— (iii)

$\Rightarrow A_1 = 1/3$, $A_2 = -1/3$

Put in eqn (i)

$$\boxed{a_n = 2^n/3 - \frac{(-1)^n}{3}}$$

Q. $a_n - a_{n-1} - a_{n-2} = 0$; $a_0 = 1$; $a_1 = 1$
Solve the recurrence rel.

$\Rightarrow$ Order $= n-(n-2) = 2$

$\therefore$ Char. root eq^n :

$$\alpha^2 - \alpha - 1 = 0$$

$$\text{Roots} = \frac{-b \pm \sqrt{b^2+4ac}}{2a}$$

$$= \frac{1 \pm \sqrt{1+4}}{2}$$

$$= \frac{1 \pm \sqrt{5}}{2} = \boxed{\alpha_1 = 1.618} \;\&\; \boxed{\alpha_2 = -0.618}$$

Case I :

$$a_n^{(h)} = A_1 \alpha_1^n + A_2 \alpha_2^n$$

$$a_n^{(h)} = (1.618)^n A_1 + (-0.618)^n A_2 \quad \text{— (i)}$$

Now,

$a_0 = A_1 + A_2 \Rightarrow A_1 + A_2 = 1$ — (ii)

$a_1 = 1.618 A_1 - 0.618 A_2$ — ~~(iii)~~

$\Rightarrow 1.618 A_1 - 0.618 A_2 = 1$ — (iii)

$A_1 = 0.723$ $\qquad A_2 = 0.276$

$\therefore$

$$\therefore \boxed{a_n^{(h)} = (1.618)^n \cdot 0.723 + (-0.618)^n \, 0.267}$$

Q $\Rightarrow$ $3a_n - 3a_{n-1} + 5a_{n-2} = 0$

$\Rightarrow$ Order $= n-n+2 = 2$

Ch. eq^n $\Rightarrow$ $3\alpha^2 - 3\alpha + 5 = 0$

Q. $a_n - 6a_{n-2} = 0$

$\Rightarrow$ Order $= n-n+2 = 2$

Ch. eq^n $=$ $\alpha^2 - 6\alpha - 2 = 0$

$a_n = 4(a_{n-1} - a_{n-2})$; $a_0 = a_1 = 1$

$\Rightarrow$ Ch. eq^n :

$$\alpha^2 - 4\alpha + 1 = 0$$

Roots = $(2, 2)$ $\Rightarrow$ $\alpha_1 = 2$; $\alpha_2 = 2$

Case II :

$$a_n^{(h)} = (A_1 n + A_2) 2^n \quad \text{— (i)}$$

Now,

$$a_0 = (A_2)2 \Rightarrow \boxed{A_2 = \frac{1}{2}} \quad \text{— (ii)} \qquad \cancel{A_1 = 0 \text{ — (ii)}}$$

$$a_1 = (A_1 + A_2)2 \Rightarrow$$

$$\therefore \boxed{\cancel{a_n^{(h)} = 1}}$$

$$1 = \left(A_1 + \frac{1}{2}\right) 2$$

$$1 = 2A_1 + 1 \Rightarrow \boxed{A_1 = \frac{-1}{2}} \quad \text{— (iii)}$$

$$\therefore a_n^{(h)} = \left(-\frac{1}{2} n + 1\right) 2$$

$$\boxed{a_n^{(h)} = -n + 2}$$

* PARTICULAR SOL^N * depends on $f(n)$

Case I: $f(n)$ is of polynomial type.

$f(n) = F_1 n^t + F_2 n^{t-1} + F_3 n^{t-2} + \dots + F_{t+1}$

Then the sol^n will be:

$a_n = P_1 n^t + P_2 n^{t-1} + P_3 n^{t-2} + \dots + P_{t+1}$

where t is the degree.

$P_1, P_2, P_3 \dots$ are constants.

Case II: $f(n) = (\text{Polynomial}) \times \beta^n$

(i) If β is not char. root of given rec. rel.

$a_n = (\text{sol}^n \text{ of polynomial}) \times \beta^n$

(ii) If β is a char. root with multiplicity $(m-1)$

$a_n = n^{m-1} (\text{sol}^n \text{ of polynomial}) \times \beta^n$

Q. $a_n + 5a_{n-1} + 6a_{n-2} = 3n^2 - 2n + 1$ —— (i)

$\Rightarrow f(n) = 3n^2 - 2n + 1$ (everything apart from recursive terms)

$t = 2$

$\therefore$ The sol^n will be,

$a_n = P_1 n^2 + P_2 n + P_3$ —— (ii)

Put (ii) in (i):

$$\Rightarrow (P_1 n^2 + P_2 n + P_3) + 5[P_1(n-1)^2 + P_2(n-1) + P_3] + 6[P_1(n-2)^2 + P_2(n-2) + P_3] = 3n^2 - 2n + 1$$

$$\Rightarrow P_1 n^2 + P_2 n + P_3 + 5P_1 n^2 + 5P_1 - 10P_1 n + 5P_2 n - 5P_2 + 5P_3 + 6P_1 n^2 + 24P_1 - 24P_1 + 6P_2 - 12P_2 + 6P_3 = 3n^2 - 2n + 1$$

$\Rightarrow (12P_1)n^2 + (12P_2 - 34P_1)n + (29P_1 + 17P_2 + 12P_3) = 3n^2 - 2n + 1$

Now, compare L.H.S with R.H.S.

$\Rightarrow 12P_1 = 3 \Rightarrow P_1 = \frac{3}{12} = \boxed{\frac{1}{4}}$

$\Rightarrow 12P_2 - 34P_1 = -2$

$12P_2 - \cancel{34}^{17} \times \frac{1}{\cancel{4}_2} = -2$

$24P_2 - 17 = -4$

$P_2 = \boxed{\frac{13}{24}}$

$\Rightarrow 29P_1 - 17P_2 + 12P_3 = 1 \Rightarrow P_3 = \boxed{71/288}$

Put the values of P_1, P_2, P_3 in (ii)

$$\boxed{a_n = \frac{1}{4}n^2 + \frac{13}{24}n + \frac{71}{288}}$$

Q. $a_n + 5a_{n-1} + 6a_{n-2} = (42)4^n$ — (i)

$\Rightarrow f(n) = (42)4^n$

To find whether β is root or not.

Char. eqn: $a_n + 5a_{n-1} + 6a_{n-2} = 0$

$\alpha^2 + 5\alpha + 6 = 0$

$\Rightarrow \alpha_1 = -2$ & $\alpha_2 = -3$

$\therefore \beta$ is not a char. root of given rel. $\because \beta = 4$

$\Rightarrow$ ~~a_n~~ $\boxed{t = 0}$

~~$\therefore a_n = (P_1 n^0)4^n = (P_1)4^n$~~

$\therefore a^n = n^{t-1}(P_1 n^0)4^n$

$a_n = P_1 4^n$ — (ii)

Put (ii) in (i)

$P_1 4^n + (5P_1)4^{n-1} + (6P_1)4^{n-2} = 42(4)^n$

$\Rightarrow 4^{n-2}(16P_1 + 20P_1 + 6P_1) = 42(4)^n$

$\Rightarrow (42P_1)4^{n-2} = (42)4^n$

$\Rightarrow \frac{42P_1}{16}$

$\therefore$ $P_1 = \boxed{16}$, on comparing L.H.S & R.H.S

Put P_1 in eq^n (ii)

$$\boxed{\begin{aligned} a_n &= 16(4)^n \\ \Rightarrow a_n &= (4)^{n+2} \end{aligned}}$$

Q. $a_n - 5a_{n-1} + 6a_{n-2} = 1$ — (i) Find Part. sol^n.

$\Rightarrow f(n) = 1 \quad (t=0)$

$\therefore$ Sol^n:

$a_n = P_1 n^0 = P_1 = P$ — (ii)

Putting (ii) in (i)

$P - 5P + 6P = 1$

$2P = 1 \Rightarrow P = \boxed{1/2}$

Putting P in (ii)

$$\boxed{a_n = 1/2}$$

Q. $a_n - 2a_{n-1} = (3)2^n$ — (i)

$\Rightarrow f(n) = (3)2^n \longrightarrow \boxed{\beta = 2}$

char eq^n:

$\alpha - 2 = 0 \Rightarrow \boxed{\alpha = 2}$

$\therefore$ β is a char. root given eq^n, with $(m-1) = 1$

$\Rightarrow$ The sol^n of polynomial:

$a_n = n^{(m-1)}(\text{sol}^n \text{ poly}) \times \beta^n$

$a_n = n(P)2^n$ — (ii)

Putting (ii) in (i)

$n(P)2^n - 2((n-1)(P)2^{n-1}) = (3)2^n$

$n(P)2^n - (n-1)P2^n = (3)2^n$

$(n) P - (n) 2P + P = 3 \Rightarrow \boxed{P = 3}$

$\Rightarrow$ Putting P in (ii) :

$$a_n = n(3) 2^n$$

$$\boxed{a_n = 3n \cdot 2^n}$$

Q. $a_{n+2} - 5a_{n+1} + 6a_n = 2^n$ — (i)

$\Rightarrow f(n) = 2^n \longrightarrow \beta = 2$

char. eqⁿ :

$$\alpha^2 - 5\alpha + 6 = 0$$

$$\cancel{\alpha^2 + \alpha - 6\alpha + 6 = 0}$$

$$\cancel{\alpha(\alpha+1) - 6(\alpha+1) = 0}$$

$$\Rightarrow \alpha = 2 \quad \& \quad \alpha = 3$$

$\therefore$ β is a char. eqⁿ root with $(m-1) = 1$

The solⁿ:

$$a_n = n P 2^n \quad \text{— (ii)}$$

eqⁿ (ii) in (i) :

$$\left((n+2) P 2^{n+2}\right) - 5\left((n+1) P 2^{n+1}\right) + 6 n P 2^n = 2^n$$

$$4nP + 8P - 5(2nP + 2P) + 6nP = 1$$

$$4nP + 8P - 10nP - 10P + 6nP = 1$$

$$-2P = 1 \Rightarrow \cancel{\boxed{P = -1/6}} \Rightarrow \boxed{P = -\frac{1}{2}}$$

$\Rightarrow$ Putting P in (ii)

$$a_n = n \left(-\frac{1}{2}\right) 2^n$$

$$\boxed{a_n = -n \cdot 2^{n-1}}$$

* GENERATING FUNCTION *

1 → Closed form of any infinite series

2 → Let $a_0, a_1, a_2, a_3, \ldots$ as be a ∞ series.

3 → Then power series of x will be

$$g(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$$

$$\boxed{g(x) = \sum_{n=0}^{\infty} a_n x^n}$$ → generating func. of given ∞ series

Q. Given series : $1, -1, 1, -1, 1, \ldots$

$\Rightarrow a_0 = 1, \; a_1 = -1, \; a_2 = 1, \; a_3 = -1, \ldots$

Power series in x :

$$g(x) = a_0 + a_1(x) + a_2(x)^2 + a_3(x)^3$$

$$g(x) = 1 - x + x^2 - x^3 + x^4 \ldots$$

[$\because$ sum of ∞ terms of G.P. $= \frac{a}{1-r}$

$a \to$ first term

$r \to$ common ratio]

$a = 1 \; ; \; r = -x$

$$\boxed{g(x) = \frac{1}{1+x}}$$ → Gen. func. of series

Q. Given series : $1, 1, 1, \ldots$

$\Rightarrow g(x) = a_0 + a_1(x) + a_2(x)^2 + \ldots$

$= 1 + x + x^2 + \ldots$

$a = 1 \; ; \; r = x$

$$\boxed{g(x) = \frac{1}{1-x}}$$

* NOTE :

n^{th} term	gen func
$a_n = (-1)^n$	$1/1+x$
$a_n = 1$	$1/1-x$

Q. $a_n = n+1$

$\Rightarrow a_0 = 1 \; ; \; a_1 = 2 \; ; \; a_2 = 3 \; ; \; a_3 = 4 \; ; \ldots$

Power series :

$$g(x) = 1 + 2x + 3x^2 + 4x^3 + \ldots \quad \text{—(i)}$$

(Multiply both side by x)

$$x g(x) = x + 2x^2 + 3x^3 + 4x^4 + \ldots \quad \text{—(ii)}$$

Now, (i) - (ii)

$$g(x)(1-x) = 1 + x + x^2 + x^3 + \ldots$$

$$g(x)(1-x) = \frac{1}{1-x}$$

$$\boxed{g(x) = \frac{1}{(1-x)^2}}$$

Q. $a_n = 3^n$

$\Rightarrow a_0 = 1 \; , \; a_1 = 3 \; , \; a_2 = 9 \; , \; a_3 = 27 \ldots$

Power series

$$g(x) = 1 + 3x + 9x^2 + 27x^3 + \ldots$$

$[a = 1 \quad r = 3x]$

$$\boxed{g(x) = \frac{1}{1-3x}}$$

Q. $a_n = n$

$\Rightarrow a_0 = 0 \; , \; a_1 = 1 \; , \; a_2 = 2 \; , \; a_3 = 3 \; , \ldots$

Power series :

$$g(x) = \cancel{0 + 1x + 2x + 3x} + 0 + x + 2x^2 + 3x^3 + \ldots \quad \text{—(i)}$$

$[\cancel{a = x} \quad \cancel{r =}$

Divide x both sides

$$\frac{g(x)}{x} = 1 + 2x + 3x^2 + 4x^3 + \ldots \quad \text{—(ii)}$$

Now, eq^n (ii) - (i)

$$\left(\frac{1-x}{x}\right) g(x) = 1 + x + x^2 + x^3 + \ldots$$

$$\left(\frac{1-x}{x}\right) g(x) = \frac{1}{1-x}$$

$$\boxed{g(x) = \frac{x}{(1-x)^2}}$$

Q. $a_n = 3^n$

$\Rightarrow a_0 = 1, \ a_1 = 3, \ a_2 = 3^2, \ a_3 = 3^3, \ldots$

Power series:

$$g(x) = a_0 + a_1 x + a_2 x^2 + \ldots$$

$$g(x) = 1 + 3x + (3x)^2 + (3x)^3 + \ldots$$

$$[a = 1 \ ; \ x = 3x]$$

$$\boxed{g(x) = \frac{1}{1-3x}}$$

* n^{th} term — generating function:

n^{th} term	generating function
$\rightarrow 1$	$\frac{1}{1-x}$
$\rightarrow (-1)^n$	$\frac{1}{1+x}$
$\rightarrow (n+1)$	$\frac{1}{(1-x)^2}$
$\rightarrow n$	$\frac{x}{(1-x)^2}$
$\rightarrow a^n$	$\frac{1}{1-ax}$
$\rightarrow (-a)^n$	$\frac{1}{1+ax}$
$\rightarrow n(n+1)$	$\frac{2x}{(1-x)^3}$
$\rightarrow (n+1)(n+2)$	$\frac{2}{(1-x)^3}$
$\rightarrow \frac{1}{n!}$	e^x

Q. Solve the rec. rel. using gen. func.:

① $a_n - 2a_{n-1} - 3a_{n-2} = 0$; $a_0 = 3$; $a_1 = 1$

⇒ Multiply both sides by x^n:

$$a_n x^n - 2a_{n-1} x^n - 3a_{n-2} x^n = 0$$

[Min. value of n = lowest subscript = 0
∴ $n-2 = 0 \Rightarrow n = 2$]

② Summing $\forall n$, $n \geq 2$

$$\therefore \sum_{n=2}^{\infty} a_n x^n - 2\sum_{n=2}^{\infty} a_{n-1} x^n - 3\sum_{n=2}^{\infty} a_{n-2} x^n = 0 \quad \text{--- } (\alpha)$$

③ I Term:

$$\sum_{n=2}^{\infty} a_n x^n = a_2 x^2 + a_3 x^3 + \ldots \quad \text{--- (i)}$$

Power series:

$$g(x) = a_0 + a_1 x + a_2 x^2 + \ldots$$

$$g(x) - a_0 - a_1 x = a_2 x^2 + a_3 x^3 + \ldots \quad \text{--- (ii)}$$

By (i) & (ii)

$$\Rightarrow \boxed{\sum_{n=2}^{\infty} a_n x^n = g(x) - a_0 - a_1 x} \quad \text{--- (A)}$$

④ II Term:

$$\sum_{n=2}^{\infty} a_{n-1} x^n = a_1 x^2 + a_2 x^3 + a_3 x^4 + \ldots \quad \text{--- (iii)}$$

Power series:

$$g(x) = a_0 + a_1 x + a_2 x^2 + \ldots$$

(Multiply x both sides)

$$x(g(x) - a_0) = a_1 x^2 + a_2 x^3 + a_3 x^4 + \ldots \quad \text{--- (iv)}$$

By (iii) & (iv)

$$\boxed{\sum_{n=2}^{\infty} a_{n-1} x^n = x(g(x) - a_0)} \quad \text{--- (B)}$$

III. Term:

(5) $\sum_{n=3}^{\infty} a_{n-2} x^n = a_0 x^2 + a_1 x^3 + a_2 x^4 + \dots$ — (v)

Power series!

$g(x) = a_0 + a_1 x + a_2 x^2 + \dots$

(Multiply x^2 both sides)

$x^2 \cdot g(x) = a_0 x^2 + a_1 x^3 + a_2 x^4 + \dots$ — (vi)

By (v) & (vi)

$$\boxed{\sum_{n=2}^{\infty} a_{n-2} x^n = \cancel{a_0 x}\ x^2 \cdot g x} \quad \text{— (c)}$$

(6) Put A, B & C in eqn α:

$\Rightarrow [g(x) - a_0 - a_1 x] - 2[x(g(x) - a_0)] - 3[x^2 \cdot g x] = 0$

[Put a_0 & a_1]

$\Rightarrow [g(x) - 3 - x] - 2x[g(x) - 3] - 3x^2[g(x)] = 0$

$\Rightarrow g(x) - 3 - x - 2x g(x) + 6x - 3x^2 \cdot g(x) = 0$

$\Rightarrow g(x)(1 - 2x - 3x^2) + 5x - 3 = 0$

$\Rightarrow g(x) = \dfrac{3 - 5x}{1 - 2x - 3x^2}$

$\Rightarrow g(x) = \dfrac{3 - 5x}{(1 - 3x)(1 + x)}$ — (β)

(7) Solve RHS of β using partial fraction:

$\dfrac{3 - 5x}{(1 - 3x)(1 + x)} = \dfrac{A}{(1 - 3x)} + \dfrac{B}{(1 + x)}$ — (γ)

(Make $A = 0$ & $B = 0$ once)

$3-5x = A(1+x) + B(1-3x)$

Put $x=-1$:

$3+5 = B \times 4 \Rightarrow B = \frac{8}{4} \Rightarrow \boxed{B=2}$

Put $x = 1/3$:

$3 - 5/3 = A(1+\frac{1}{3}) \Rightarrow \frac{4}{3} = A \cdot \frac{4}{3} \Rightarrow \boxed{A=1}$

(8) Put A, B in eqn (Y)

$$\Rightarrow \frac{3-5x}{(1-3x)(1+x)} = \frac{1}{1-3x} + \frac{2}{1+x}$$

Put this in β_2 :

$$g(x) = \frac{1}{1-3x} + \frac{2}{1+x}$$

(9) [Write corr. n^{th} term of gen. func.]

converted to a_n

$\Rightarrow g(x) = 3^n + 2(-1)^n$

$\Rightarrow \boxed{a_n = 3^n + 2\cdot(-1)^n}$

Rec. Rel. Summary

Obj: To solve the rec. rel.

(i) Char. Root Method:

$a_n = a_n^{(h)} + a_n^{(p)}$

$a_n^{(h)}$ → roots
- → Real / distinct
- → Multiple

$a_n^{(p)}$ → $f(n)$
- Poly
- (Poly) β^n
  - → (i) β is root
  - → (ii) " isn't "

→ Boundary conditions are must

(ii) Gen. func. method:

→ Multiply by x^n

→ Summation eqn

→ solve terms sep.

→ Find $g(x) = $ ___ → Partial fraction

Q. $a_{n+2} - 5a_{n+1} + 6a_n = 2^n$; $a_0 = 2$; $a_1 = 1$

Ans → Multiplying by x^n ;

$\therefore a_{n+2}x^n - 5a_{n+1}x^n + 6a_n x^n = 2^n$

→ Put lowest subscript $= 0$

$\therefore n = 0$ (Min. limit value)

Summation $\forall$ $n \geq 0$

$$\therefore \sum_{n=0}^{\infty} a_{n+2}x^n - 5\sum_{n=0}^{\infty} a_{n+1}x^n + 6\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} 2^n \quad \text{— } (\alpha)$$

Ist term: $\sum_{n=0}^{\infty} a_{n+2}x^n = a_2 + a_3 x + a_4 x^2 + \ldots$ — (i)

Power series:

$g(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$

$\therefore g(x) - a_0 - a_1 x = x^2(a_2 + a_3 x + a_4 x^2 + \ldots)$ — (ii)

$\therefore$ From (i) & (ii)

$$\sum_{n=0}^{\infty} a_{n+2}x^n = \frac{g(x) - a_0 - a_1 x}{x^2} \quad \text{— (A)}$$

IInd term: $\sum_{n=0}^{\infty} a_{n+1}x^n = a_1 + a_2 x + a_3 x^2 + \ldots$ — (iii)

Power series:

$g(x) = a_0 + a_1 x + a_2 x^2 + \ldots$

$\frac{g(x) - a_0}{x} = a_1 + a_2 x + a_3 x^2 + \ldots$ — (iv)

$\therefore$ From (iii) & (iv)

$$\sum_{n=0}^{\infty} a_{n+1}x^n = \frac{g(x) - a_0}{x} \quad \text{— (B)}$$

III Term:

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 = g(x) \text{ --- (C)}$$

IV Term:

$$\sum_{n=0}^{\infty} 2^n = 1 + 2x + 4x^2 \ldots = \frac{1}{1-2x} \text{ --- (D)}$$

Now, put A, B, C & D in (α):

$$\Rightarrow \frac{g(x) - a_0 - a_1 x}{x^2} - 5\left(\frac{g(x) - a_0}{x}\right) + 6g(x) = \frac{1}{1-2x}$$

$$\Rightarrow g(x) - a_0 - a_1 x - 5x\,g(x) + 5x a_0 + 6x^2 g(x) = \frac{x^2}{1-2x}$$

$$\Rightarrow g(x)\left[1 - 5x + 6x^2\right] - a_0\left[1 - 5x\right] - a_1\left[x\right] = \frac{x^2}{1-2x}$$

$$\Rightarrow \quad '' \quad - 2 + 10x - x = \frac{x^2}{1-2x}$$

$$\Rightarrow \quad '' \quad - 2 + 9x = \frac{x^2}{1-2x}$$

$$\Rightarrow \quad '' \quad = \frac{x^2}{1-2x} + 2 - 9x$$

$$\Rightarrow g(x) = \frac{x^2 + 2 - 4x - 9x + 18x^2}{(1-2x)^2(1-3x)\cancel{(1-2x)}}$$

$$'' = \frac{19x^2 - 13x + 2}{(1-3x)(1-2x)^2} \text{ --- (B)}$$

Solve RHS of B using partial fractions:

$$\frac{19x^2 - 13x + 2}{(1-3x)(1-2x)^2} = \frac{A}{(1-3x)} + \frac{B}{(1-2x)} + \frac{C}{(1-2x)^2}$$

$$\Rightarrow A(1-2x)^2 + B(1-2x)(1-3x) + C(1-3x)$$

Put ~~x = 1/2~~

$$\cancel{19x^2 - 13x + 2 = C\left(1 - \frac{3}{2}\right)}$$

Put $x = 1/3$

$$\frac{19}{9} - \frac{13}{3} + 2 = -\frac{C}{2}$$

$$\frac{19 - 26 + 8}{\cancel{4}\,2} = -\frac{C}{\cancel{2}} \Rightarrow C = -\frac{1}{2}$$

SETS

Q. Power set of $\{0,1,2\}$ $= 2^n = 2^3 = 8$

$\Rightarrow \{\{0\},\{1\},\{2\},\{0,1\},\{0,2\},\{1,2\},\{0,1,2\},\phi\}$

* Cardinality $|A|$ = No. of elements in set.

* Cartesian Product:

$A \times B = \{\{a,b\} \mid a \in A \text{ and } b \in B\}$

↳ Ordered pair

Q. Cartesian product of:

$A = \{0,1\}$, $B = \{1,2\}$, $C = \{1,2,3\}$

$\Rightarrow A \times B \times C = \{(0,1,1), (0,1,2), (0,1,3), (1,1,1), (1,1,2), (1,1,3), (0,2,1), (0,2,2), (0,2,3), (1,2,3), (1,2,2), (1,2,1)\}$

* PIGEONHOLE PRINCIPLE *

If there are n pigeonholes & $(kn+1)$ pigeons or more, then there will be atleast 1 pigeonhole with 2 pigeons. (k → +ve int)

k+1	1	1	1	1

$\Rightarrow kn+1 \rightarrow$ pigeonholes (min) $\quad k+1 \rightarrow$ atleast (hold)

$n \rightarrow$ pigeons.

Q. Find min. no. of stu. in a class to be sure that 3 of them are born in same month

$\Rightarrow n = 12$ months $\quad \Rightarrow k+1 = 3 \Rightarrow k = 2$

$\Rightarrow kn+1 = 12 \times 2 + 1 = \boxed{25 \text{ students}}$

Q. If 9 books are to kept 4 selfes. Then there must be atleast 1 shelf that'll contain atleast 3 books:

⇒ $n = 4 \rightarrow$ No. of shelves

$k+1 = 3 \rightarrow$ Atleast 3 books ⇒ $k = 2$

$kn+1 = 4 \times 2 + 1 = 9$ = 9 books PROVED

Q. Min. no. of stu. req. in a class to be sure that atleast 6 of them will receive same grades, of 5 grades (A, B, C, D, F)

⇒ $n = 5 \rightarrow$ grades

$k+1 = 6 \rightarrow$ atleast ⇒ $k = 5$

$kn+1 = 5 \times 5 + 1 = \boxed{26 \text{ students}}$

Q. What is the min no. of people req. to ensure that 3 were born on the same day of week.

⇒ $n = 7 \rightarrow$ no. of days in week

$k+1 = 3 \rightarrow k = 2$

$kn+1 = \boxed{15 \text{ people}}$

* Difference of sets:

$A - B = \{x : x \in A \text{ \& } x \notin B\}$

* Complement of set: $\bar{A} = \{x : x \notin A\} = U - A$

* Disjoint sets: $A \cap B = \phi$

* Cardinality of union:

→ $(A \cup B) = A + B - (A \cap B)$

→ $(A \cup B \cup C) = A + B + C - (A \cap B) - (A \cap C) - (B \cap C) + (A \cap B \cap C)$

Q. Total patients = 50 = U
Pneumonia = 25 = $n(P)$
Bronchitis = 30 = $n(B)$
$n(P \cap B) = 10$.

(i) $n(P \cup B) = 25 + 30 - 10 = \boxed{45}$

(ii) Not diagnosed = $50 - 45 = \boxed{5} = n(P \cup B)'$

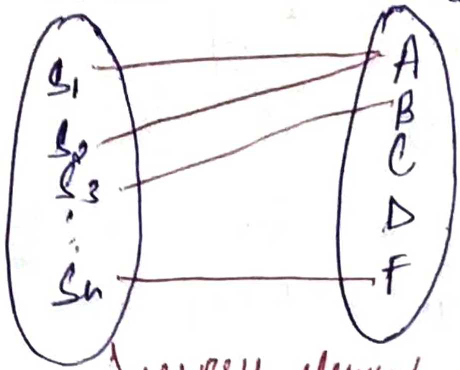
Q. $n(U) = 100$
$n(J) = 45$, $n(C) = 30$; $n(P) = 20$
$n(J \cap C) = 6$, $n(J \cap P) = 1$; $n(C \cap P) = 5$
$n(J \cap C \cap P) = 1$

$\Rightarrow n(J \cup C \cup P)' = n(U) - n(J \cup C \cup P)$
$= 100 - [45 + 30 + 20 - 6 - 1 - 5 + 1]$
$= 100 - 96 = \boxed{4}$

* FUNCTIONS *

→ For 2 non-empty sets A & B, func. f is assignment of exactly one element of B to element of A.

Domain　　　　Co-domain



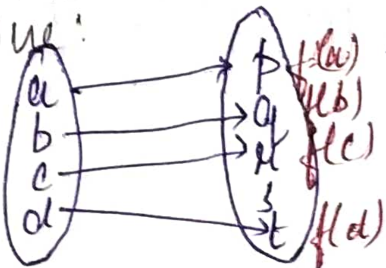
every element should be mapped.

$$\boxed{\begin{array}{l} f: A \to B \\ f(a) = b \end{array}}$$

Range: {A, B, F}

TYPES of FUNCTION:

(i) one-to-one: (Injective)

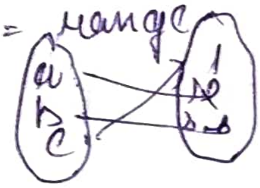


never assign same value to 2 diff. domain.

Q. $f(x) = x^2$ $(z \to z)$, one-to-one?

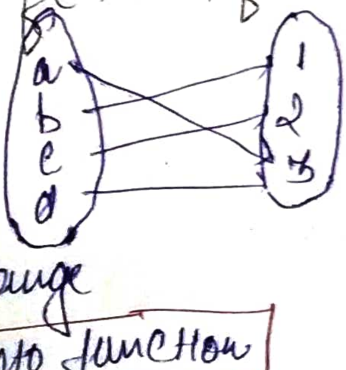
⇒ For $f(2) = 4 = f(-2)$
but $2 \neq 2$ ∴ $\boxed{\text{Not one-to-one.}}$

(ii) onto: (surjective)　　　co-domain = range



$\forall b \exists a (f(a) = b)$

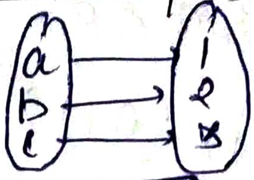
Q: $f: \{a, b, c, d\} \to \{1, 2, 3\}$
$f(a) = 3$; $f(b) = 1$; $f(c) = 2$; $f(d) = 3$, is it onto.



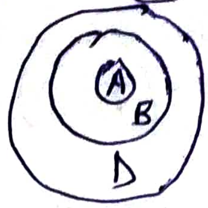
Domain = range

∴ is $\boxed{\text{onto function}}$

(iii). One-to-one correspondence: if its both one-to-one & onto.
(Bijective)



* $A \subset B \subset D$:



* RELATIONS *

Let A & B be to non-empty sets:

$A = \{1, 2, 3\}$ $B = \{1, 2, 4\}$

$A \times B = \{(1,1), (1,2), (1,4), (2,1), (2,2), (2,4), (3,1), (3,2), (3,4)\}$

(aRb)

R = "equal to" = $\{(2,2), (1,1)\}$

R = " ~~greater~~ smaller than" = $\{(1,2), (1,4), (2,4), (3,4)\}$

$R = \{(a, b) : a \in A, b \in B \text{ and } aRb\}$

* Complement of a relation:

$A = \{1, 2\}$ $B = \{3, 4\}$

$A \times B = \{(1,3), (1,4), (2,3), (2,4)\}$

$R = \{(1, 3), (2, 4)\}$

$\bar{R} = \{(1,4), (2,3)\}$

$R^{-1} = \{(y, x) : y \in B, x \in A, (x, y) \in R\}$

$R^{-1} = \{(3,1), (4,2)\}$

Reflexive: $(a,a) \in R$ If every element

$R = \{(1,1), (2,2), (1,2)\}$

If $A = \{1, 2\}$

Irreflexive: opposite of reflexive

$(a,a) \notin R \ \forall \ a \in A$

SYMMETRIC RELATION:

$\forall a \forall b ((a,b) \in R \rightarrow (b,a) \in R)$

$A = \{1,2\}$

$R = \{(1,1), (1,2), (2,1)\}$

TRANSITIVE:

$\forall a \forall b \forall c ((a,b) \in R$ and $(b,c) \in R \rightarrow (a,c) \in R)$

$A = \{1,2,3\}$

$R = \{(1,1), (1,2), (2,3), (3,3), (1,3)\}$

equivalence Relation:

If a relation is reflexive, symmetric & transitive.

Q. Let $(a-b) \in R \Rightarrow (a-b) \% 3 == 0$

$\Rightarrow$ For a,
$(a-a) = 0 \% 3 == 0$ ∴ reflexive

$(a-b) = 3q$
$\Rightarrow (b-a) = -3q$ ∴ symmetric

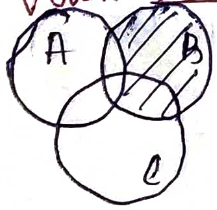
$(a-b) = 3q$
$(b-c) = 3p$
$(a-c) = 3(q+p)$ ∴ transitive.

∴ equivalence

Q. $(B-A) \cup (C-A) = (B \cup C) - A$ [If $A \subseteq B$ & $B \subseteq A$ then $A = B$]

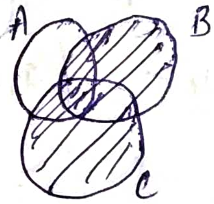
By Venn Dia:

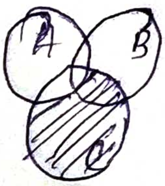
$\Rightarrow$



= B-A

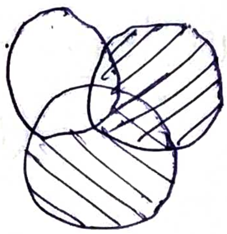
B∪C =



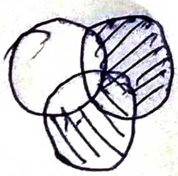


= C-A

(B∪C) - A =



∴



= (B-A) ∪ (C-A)

L.H.S = R.H.S.

By set-builder method:

L.H.S

$\Rightarrow x \in (B-A) \cup (C-A)$

$\Rightarrow x \in (B-A)$ or $x \in (C-A)$

$\Rightarrow (x \in B$ and $x \notin A)$ or $(x \in C$ and $x \notin A)$

$\Rightarrow (x \in B$ or $x \in C)$ and $(x \notin A)$

$\Rightarrow (x \in B \cup C)$ and $(x \notin A)$

$\Rightarrow x \in (B \cup C) - A =$ R.H.S Proved

Q. $A \times (B \cup C) = (A \times B) \cup (A \times C)$

$\Rightarrow$ L.H.S

Let (x, y) be an ordered pair, $\Rightarrow (x \in A)$ and $y \in (B \cup C)$

$\Rightarrow (x \in A)$ and $(y \in B$ or $y \in C)$

$\Rightarrow (x \in A$ and $y \in B)$ or $(x \in A$ and $y \in C)$

$\Rightarrow (A \times B)$ or $(A \times C) = (A \times B) \cup (A \times C) =$ R.H.S Prove

R.H.S

Let (x, y) & (x, z) be 2 ordered pair

$\Rightarrow (x, y) \in (A \times B)$ or $(x, z) \in (A \times C)$

$\Rightarrow (x \in A$ and $y \in B)$ or $(x \in A$ and $z \in C)$

$\Rightarrow (x \in A)$ and $(y \in B$ or $z \in C)$

$\Rightarrow$ ~~Cox~~ $A \times (B \cup C) =$ L.H.S Proved

* MATHEMATIC INDUCTION *

Basic step: $P(1) \longrightarrow$ Verify

Induction step: $P(K) \longrightarrow$ Assume

$P(K+1) \longrightarrow$ Prove

Q. $P(n): 1+2+3+\ldots+n = \frac{n(n+1)}{2}$

Basic step:

$\Rightarrow$ $n=1$:

$$1 = \frac{1(1+1)}{2} = \frac{2}{2} = 1 \Rightarrow 1=1$$

Proved for $n=1$.

Inductive step:

Let $P(K)$ is true,

$$1+2+3+\ldots+K = \frac{K(K+1)}{2} \quad \text{— (1)}$$

To prove:

$$\boxed{1+2+3+\ldots+(K+1) = \frac{(K+1)(K+2)}{2}}$$

$$\Rightarrow \quad '' = 1+2+3+\ldots+K+(K+1)$$

$$'' = \frac{K(K+1)}{2} + (K+1) \quad \text{[By eq}^n\text{ (1)]}$$

$$= \frac{(K+1)(K+2)}{2}$$

Thus shows that $P(K)+1$ is true under the assumption $P(K)$ is true.

Since completed inductive & basis step,

$\therefore$ by Mathematical Induction, we say that $\boxed{P(N) \text{ is true for all integer.}}$

Q. Construct a formula for the sum of first n (+ve) odd integers. And prove it with MDI.

$\Rightarrow \quad a = 1, \; d = 2 \quad = \frac{n}{2}\left[2 + (n-1)2\right]$

$= n\left[1 + n - 1\right] = \underline{\underline{n^2}}$

Basis step:

$P(n) = 1 + 3 + 5 + \dots + (2n-1) = n^2$

For $n=1$:

$1 = 1^2 = 1 \Rightarrow 1 = 1$

Proved for $n=1$.

Inductive step:

Let $P(K)$ be true,

$1 + 3 + 5 + \dots + (2K-1) = K^2 \quad \text{—} (1)$

To prove:

$1 + 3 + 5 + \dots + (2(K+1)-1) = (K+1)^2$

Now,

$1 + 3 + 5 + \dots + (2(K+1)-1)$

$= 1 + 3 + 5 + \dots + (2K-1) + (2K+1)$

$= K^2 + (2K+1) \quad \because (a^2 + b + 2ab = (a+b)^2)$

$= (K+1)^2$

This shows that $P(K+1)$ is true under the assumption $P(K)$ is true.

we've completed inductive & basis step,

$\therefore$ by M.I. we say that $\boxed{P(N) \text{ is true } \forall I}$

Q. P.T., $n^3 - n$ is divisible by 3.

Basis step:

$n=1$ $\quad 1^3 - 1 = 0$, 0 is div. by 3.

Proved for $n=1$

Inductive step:

$n=k$ $\quad (k^3 - k)$ is div. by 3.

To proved:

$n = k+1$ $\quad (k+1)^3 - (k+1)$

$= k^3 + \cancel{1} + 3k^2 + 3k - k - \cancel{1}$

$= \cancel{k^3 + 3k^2} + (k^3 - k) + 3(k^2 + k)$

$\because (k^3 - k)$ is divisible by 3

$\therefore (k^3 - k) + 3(k^2 + k)$ is also div. by 3.

Proved

Q. Use M.I. to prove that, $2^n < n!$ $\forall n \geq 4$

Basis step:

$n=4$ $\quad 2^4 = 16$ & $4! = 24$

$16 < 24$

Proved for $n=4$

Inductive step:

$n=k$ $\quad 2^k < k!$

To prove: $\cancel{2^{k+1} = (k+1)!}$ $\quad 2^{(k+1)} < (k+1)!$

$2^{k+1} = 2 \cdot 2^k$

$2^{k+1} < 2 \cdot k!$ $\quad [\because k \geq 4]$

$\Rightarrow 2^{k+1} < (k+1)\, k!$ $\quad \therefore [2 < k+1]$

$\Rightarrow 2^{(k+1)} < (k+1)!$

$\therefore$ Proved for $n = k+1$

Q. There are 48 diff. time period during which, classes at uni. can be scheduled, if there are 687 diff. classes. Find out no. of diff. rooms.

$\Rightarrow n = 48 \qquad kn+1 = 687$

$48k+1 = 687 \Rightarrow k = \frac{686}{48} = 14.29$

$\therefore \quad k+1 = \boxed{15 \text{ rooms}}$

* COUNTING PROBLEMS:

Q1 Suppose a stu. can choose 1 computer proj. from one of the given 3 list

List 1 → 23 | List 2 → 15 | List 3 → 18

How many possible ways to choose a project.

Q2 Suppose a clg. has 3 diff. history, 2 diff. sociology & 4 diff. science courses, then find a way a student can choose one of each kind of course.

[Product rule → one from each
Sum rule → one from all]

A1 → $23 + 15 + 18 = \boxed{56}$ projects

A2 → $3 \times 2 \times 4 = \boxed{24}$ ways

Q3 → In how way we can select 3 stu. from 5 stu. to stand line.

$\Rightarrow \underset{1^{st}}{5} \times \underset{2^{nd}\text{ pos}^n}{4} \times \underset{3^{rd}}{3} = \boxed{60}$ ways

→ 5 stu.

→ $5 \times 4 \times 3 \times 2 \times 1 = \boxed{120}$ ways

$\Rightarrow \boxed{\text{Possible ways} = {}^nP_r} = \frac{n!}{(n-r)!}$ without repetition

Q4. Find the no. of 3 letter words using only the 6 letters without repitition.

$\Rightarrow$ $^6P_3 = 6 \times 5 \times 4 = \boxed{120 \text{ words}}$

Permutation with repitition:

Let $P(n; n_1, n_2, \dots, n_n)$ denote the no. of permutation of n objs out of which n_1 are alike, n_2 are alike and so on.

So,

$$\boxed{P(n; n_1, n_2, \dots, n_n) = \frac{n!}{n_1!\, n_2!\, n_3! \dots n_n!}}$$

Q. Find the no. of 7 letter words that can be formed using BENZENE.

$\Rightarrow$ $n = 7$, $n_1 = 3$, $n_2 = 2$

$P(7; 3, 2) = \frac{7!}{3! \times 2!} = \frac{7 \times 5 \times 6 \times \cancel{4}^{2}}{\cancel{2}} = \boxed{420}$

Ordered samples:

→ Samples with replacement $= n^n$

→ Samples without replacement $= n \times (n-1) \times \dots \times (n-r)$

Q. How many ways are there to select 1st, 2nd & 3rd prize winner from 100 diff. people who participated in a contest.

$\Rightarrow$ $100 \times 99 \times 98 = 970200$

Q. How many permutation of the letter A, B, C, D, E, F, G, H, if every string contains abc.

$\Rightarrow$ $\underbrace{ABC}_{1}, \underbrace{DEFGH}_{5}$ $\therefore n = 6$

$\Rightarrow 6! = \boxed{720}$

Q. Find the no. of distinct permutation that can be formed from all the letters of each words.
(i) THOSE (ii) UNUSUAL (iii) SOCIOLOGICAL

→ (i) $\frac{5!}{} = 120$

(ii) $\frac{7!}{3!} = 840$

(iii) $\frac{12!}{3! \times 2! \times 2! \times 2!}$

$= \boxed{\cancel{3326400}}$

$= 19958400$

Q. All vowels always come together 'TUESDAY'

⇒ $\underbrace{UEA}_{1}$ TSDY
2 3 4 5

$= {}^5P_5 \times {}^3P_3 = 5! \times 3! = 120 \times 6 = \boxed{720}$

→ vowels & consonants are alternate:

⇒ _ T U S E D AY _

⇒ ${}^4P_4 \times {}^3P_3 = 4! \times 3! = \cancel{18x}\ \boxed{144}$

Q. No. of ways in which letters of the word 'COTTON' can be arranged so that 2 T's don't come together

⇒ Total $= \frac{6!}{2!\,2!} = \boxed{180}$

T's together $= \frac{{}^5P_5}{2!} \times \frac{{}^2P_2}{2!} = \frac{120}{2} \times \frac{2!}{2!} = \boxed{60}$

T's don't come together $= 180 - 60 = \boxed{120}$

Q. A

COMBINATION:

$$\boxed{{}^nC_r = \frac{n!}{r!(n-r)!}}$$

Q. A farmer buys 3 cows, 2 goats & 4 hens from a man who has 6 cows, 5 goats & 8 hens. Find the no.s of choice he has.

⇒ Cows: ${}^6C_3 = \frac{6!}{3!\,3!} = \boxed{20}$

Goat: ${}^5C_2 = \frac{5!}{2!\,3!} = \boxed{10}$

Hens: ${}^8C_4 = \frac{8!}{4!\,4!} = \boxed{70}$

∴ Total choices = $70 \times 20 \times 10 = \boxed{14000}$

Q. A box contains 8 blue socks & 6 red socks can be drawn, Find the no. of ways 2 socks can be of any colour, such that they

⇒ ${}^{14}C_2 = \boxed{91}$

→ They must be of same colour.

⇒ $\underbrace{{}^8C_2}_{\text{Blue}} + \underbrace{{}^6C_2}_{\text{Red}}$ ⇒ $\frac{8!}{6!\,2!} + \frac{6!}{4!\times 2!} = 28 + 15 = \boxed{43}$

Q. The Indian cricket team consist of 16 players, 2 → WK, B → 5, 4 → B. How many ways can you select team of 1 → WC & atleast

$\cancel{{}^{16}C_{11}} \times {}^2C_1 \times ({}^5C_5 + {}^5C_4)$

⇒ Case 1:

⇒ ${}^2C_1 \times {}^5C_4 \times {}^9C_6$

⇒ $2 \times 5 \times 84$

⇒ 840

Case 2:

⇒ ${}^2C_1 \times {}^5C_5 \times {}^9C_5$

⇒ $2 \times 1 \times 126$

⇒ 252

⇒ Total = $\boxed{1092}$

Q. Solve using gen. func.: $a_0 = 0$, $a_1 = 1$

$\Rightarrow a_n - 3a_{n-1} + 6a_{n-2} = 0$

$\Rightarrow$ Multiply by x^n $\quad$ [$n-2=0$, $n=2$ (Min value)

$a_n x^n - 3a_{n-1}x^n + 6a_{n-2}x^n = 0$

summing $\forall n$; $n \geq 2$

$\sum_{n=2}^{\infty} a_n x^n - 3\sum_{n=2}^{\infty} a_{n-1} x^n - 6\sum_{n=2}^{\infty} a_{n-2}x^n = 0$ —— (I)

I term:

$\sum_{2}^{\infty} a_n x_n = a_2 x^2 + a_3 x^3 + a_4 x^4 + \dots$ (i)

Power series:

$g(x) = a_0 + a_1 x + a_2 x^2 + \dots$

$g(x) - a_0 - a_1 x = a_2 x^2 + a_3 x^3 + \dots$ —— (ii)

From (i) & (ii)

$\boxed{\sum_{n=2}^{\infty} a_n x_n = g(x) - a_0 - a_1 x}$ —— (A)

II term: $\sum_{n=2}^{\infty} a_{n-1} x^n = a_1 x^2 + a_2 x^3$ —— (iii)

From pow. series:

$g(x) = a_0 + a_1(x) + a_2 x^2$

$x(g(x)) = a_0 x + a_1 x^2$ —— (iv)

$\therefore \boxed{\sum_{n=2}^{\infty} a_{n-1} x^n = x(g(x) - a_0)}$ —— (B)

III term:

$\sum_{n=2}^{\infty} a_{n-2} x^n = a_0 x^2 + a_1 x^3 + \dots$

$\therefore \boxed{\sum_{n=2}^{\infty} a_{n-2} x^n = x^2 g(x)}$ —— (C)

Ⓐ, Ⓑ & Ⓒ in (α):

$g(x) - a_0 - a_1 x - 3x(g(x) - a_0) - 6x^2 g(x) = 0$

$g(x) - 2 - x - 3x(gx - 2) - 6x^2 g(x) = 0$

$g(x) - 2 - x - 3gx + 6x - 6x^2 g(x) = 0$

$g(x)[1 - 3x - 6x^2] = \frac{2-5x}{1}$

$\therefore g(x) = \frac{2-5x}{(1-2x)(1-x)}$

R.H.C:

$\frac{2-5x}{(1-2x)(1-x)} = \frac{A}{(1-2x)} + \frac{B}{(1-x)}$

$2-5x = A(1-x) + B(1-2x)$

Put $x = 1$

$2 - 5 = -B \Rightarrow \boxed{B = 3}$

Put $x = 1/2$

$2 - 5/2 = A/2 \Rightarrow \boxed{A = -1}$

$\therefore \frac{2-5x}{(1-2x)(1-x)} = \frac{-1}{(1-2x)} + \frac{3}{(1-x)}$

$\therefore \boxed{g(x) = -2^n + 3}$

$\Rightarrow \boxed{a_n = -2^n + 3}$

Q. $a_n - 3a_{n-1} = 4 \cdot 2^n$ —— (α)

Char eqn:

$\alpha - 3 = 0 \longrightarrow \boxed{\alpha = 3}$

$\boxed{a_n^{(h)} = A_1 3^n} \longrightarrow$ Homogenous:

Now, $f(n) = 4 \cdot 2^n$

$t = 0$

$\Rightarrow a_n = (P) \cdot 2^n$ —— (i) ($\because \beta$ is not a char. root)

Put in α

$P2^n - 3P2^{n-1} = 4 \cdot 2^n$

$P\left(2^n - \frac{3 \cdot 2^n}{2}\right) = 4 \cdot 2^n$

$2^n\left(P - \frac{3P}{2}\right) = 4 \cdot 2^n \Rightarrow 2P - 3P = 8$

$\boxed{P = -8}$

$\Rightarrow P$ in (i)

$\boxed{a_n^{(P)} = -8 \cdot 2^n} \longrightarrow$ Particular solⁿ.

$\Rightarrow a_n = a_n^{(h)} + a_n^{(P)}$

$\therefore \boxed{a_n = A_1 3^n - 8 \cdot 2^n}$ Ans

Q. $f(n) = (4n) 2^n$

$\Rightarrow t = 1 \quad (m-1) = 2$

$a_n = n^2 (P_1 n + P_2) \cdot 2^n$

Q. $a_n - 3a_{n-1} + 2a_{n-2} = 0$ $\qquad a_0 = 2 \quad a_1 = 3$

⇒ Multiply by x^n.

$$a_n x^n - 3a_{n-1} x^n + 2a_{n-2} x^n = 0 \qquad (n \geq 2)$$

⇒ Summation,

$$\sum_{n=2}^{\infty} a_n x^n - 3\sum_{2}^{\infty} a_{n-1} x^n + \sum_{2}^{\infty} 2a_{n-2} x^n = 0 \quad \text{—} \quad (\alpha)$$

I term:

$$\sum_{n=2}^{\infty} a_n x^n = a_2 x^2 + a_3 x^3 + \dots$$

$$g(x) = a_0 + a_1 x + a_2 x^2$$

$$\therefore \sum_{n=2}^{\infty} a_n = g(x) - a_0 - a_1 x \quad \text{—} \quad (A)$$

II term: $\sum_{2}^{\infty} a_{n-1} x^n = x g(x) - a_0 x$ — (B)

III term: $\sum_{2}^{\infty} a_{n-2} x^n = x^2 g(x)$ — (C)

Put in (α)

$$\Rightarrow g(x) - a_0 - a_1 x - 3x g(x) - 3a_0 x + 2x^2 g(x) = 0$$

$$\Rightarrow g(x) - 2 - 3x - 3x g(x) + 6x + 2x^2 g(x) = 0$$

$$\Rightarrow g(x)(2x^2 - 3x + 1) = 9x + 2$$

$$\Rightarrow g(x) = \frac{\cancel{9x+2}\; 2 - 3x}{(x-1)(2x-1)} \quad \text{—} \quad (\beta)$$

R.H.S |

$$\frac{\cancel{9x+2}\; 2 - 3x}{(x-1)(2x-1)} = \frac{A}{(x-1)} + \frac{B}{(2x-1)}$$

$$2 - 3x \;\cancel{9x+2} = (2x-1)A + (x-1)B$$

Let $x = \frac{1}{2}$

$\therefore \frac{9}{2} + 2 = \frac{1}{2} B \Rightarrow \boxed{13 = B}$

Let $x = 1$

$\boxed{11 = A}$

$\therefore \frac{9x+1}{(x-1)(2x-1)} = \frac{11}{(x-1)} + \frac{13}{(2x-1)}$

$\therefore$ Put in B

$\Rightarrow g(x) = 11 - 13 \cdot 2^n$

$\Rightarrow \boxed{a_n = 11 - 13 \cdot 2^n}$

Let $x = 1/2$

$2 - \frac{3}{2} = -\frac{1}{2} B \Rightarrow \boxed{B = -1}$

Let $x = 1$

$\boxed{1 = A}$

$\therefore \frac{2-3x}{(x-1)(2x-1)} = \frac{1}{(x-1)} - \frac{1}{(2x-1)}$

$\therefore g(x) = 1 + 2^n$

$\Rightarrow \boxed{a_n = 1 + 2^n}$

MODULE - 2

PREPOSITIONAL LOGIC :

Any declarative stament is a preposition, if it either true or false.
→ True not both

→ Delhi is capital of India. True
→ 2 + 3 = 4 False
→ Do your homework. Not a preposition.

* Prepositional variables : Used to denote preposition. ie., P, Q, R, p, q, r.

→ The truth value of a preposition is denoted by T (True) or F (False).

* compound preposition : Joining two or more preposition.

P = Lucknow is in UP. It is capital of UP.
→ combining 2 or more premitive preposition.
→ We can combine them using logical operators (connectives) like ∧ (and), ∨ (or).

eg: John is smart or he studies every night.
Roses are red and violets are blue.

* Connectives :

1. AND ⟶ conjuction ∧
2. OR ⟶ disjunction ∨
3. NOT ⟶ Negation ¬ or ~
4. if...then → Implication → (conditional)
5. if and only if ⟶ Biimplication ↔ (equivalence)

1. AND (
P ∧
o
P →
Q →
P ∧
2. OR
3.
4.

1. AND (Conj

P	Q	P ∧ Q
0	0	0
0	1	0
1	0	0
1	1	1

P	Q	P ∧ Q
T	T	T
T	F	F
F	T	F
F	F	F

P → Today is Sunday

Q → It is Sunny morning

P ∧ Q → Today is Sunday & it is Sunny morning.

2. OR (Disjunction):

P	Q	P ∨ Q
T	T	T
T	F	T
F	T	T
F	F	F

3. NOT (Negation)

P	¬P
T	F
F	T

4. Conditional or Implication: (If P then Q)

P	Q	P → Q
T	T	T
T	F	F
F	T	T →(नही आया पर मिली)
F	F	T

P → First आया

Q → Bike मिली

IMPLICATION

P	Q	$P \leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

$P \leftrightarrow Q \equiv P \to Q \wedge Q \to P$

$\cancel{P \wedge Q}$

* CONDITIONAL STATEMENT :

- → if P then Q
- → if P, Q
- → Q when P
- → Q whenever P
- → P only-if Q

How to identify?

$\Rightarrow \boxed{P \to Q}$

Q. Let P & Q be the statements:

P : Harsh learns Discrete Maths.

Q : Harsh will find a good job.

Find its conditional statement.

⇒ If Harsh learns Discrete Maths then he will find a good job.

NOTE:

⇒ Converse of $\boxed{P \to Q : Q \to P}$

⇒ Contrapositive of $\boxed{P \to Q : \sim Q \to \sim P}$

⇒ Inverse of $\boxed{P \to Q : \sim P \to \sim Q}$

Q. What are the converse, contrapositive & inverse of :

P→Q: The home team wins whenever its raining.

(Q: The home team wins; P: whenever its raining)

⇒ Q→P: ~~Whenever~~ If it rains, then the home team wins.

~Q→~P: ~~Whenever~~ If it doesn't rain, then the home team doesn't win / lose.

~P→~Q: ~~The home team lose / doesn't win whenever it doesn't rain.~~ If its not raining, the home team doesn't win.

* Priority of connectives :

$\neg$

$\left.\begin{matrix}\wedge \\ \vee\end{matrix}\right\}$ $\downarrow$

$\rightarrow$

$\leftrightarrow$

Q. Write Truth table of $\neg(P \wedge \neg Q)$.

P	Q	¬Q	P ∧ ¬Q	¬(P ∧ ¬Q)
T	T	F	F	T
T	F	T	T	F
F	T	F	F	T
F	F	T	F	T

Q. Write TT of $(P \vee \sim Q) \rightarrow (P \wedge Q)$

P	Q	~Q	P ∨ ~Q	P ∧ Q	(P ∨ ~Q) → (P ∧ Q)
T	T	F	T	T	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	T	F	F

Write TT of XOR:

P	Q	$P \oplus Q$
T	T	F
T	F	T
F	T	T
F	F	F

* <u>Tautology</u> : All true
* <u>Contradiction</u> : All false
* Contin<u>gency</u> : Mix

Q. $(P \rightarrow Q) \longleftrightarrow (\sim Q \rightarrow \sim P)$. Tautology or not

P	Q	$\sim P$	$\sim Q$	$P \rightarrow Q$	$\sim Q \rightarrow \sim P$	Ans
T	T	F	F	T	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

$\therefore$ Tautology is there.

LOGICAL EQUIVALENCE:

- Two compound prepositions, that have sametruth values in all possible cases are called logically equivalente preposition.

In other words,
the compound preposition P & Q are logically equivalent if $P \leftrightarrow Q$. is tautalogy.

$\sim(p \vee q)$ and $\sim p \wedge \sim q$ are logically equivalent.

p	q	$\sim p$	$\sim q$	$\sim(p \vee q)$ P	$\sim p \wedge \sim q$ Q	$P \leftrightarrow Q$
T	T	F	F	F	F	T
T	F	F	T	F	F	T
F	T	T	F	F	F	T
F	F	T	T	T	T	T

$\because P \leftrightarrow Q$, is tautology.

$\therefore \sim(p \vee q)$ & $\sim p \wedge \sim q$ $\boxed{\text{are logically equi.}}$

Q. P : Today is wednesday

Q : It is hot.

R : It is sunny.

$$(P \vee Q) \rightarrow (\sim P \wedge \sim Q)$$

$\Rightarrow$ If today is wednesday or it is hot then today is not wednesday & it is hot.

Q. Const. the TT for $(p \rightarrow (q \rightarrow r)) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r))$

p	q	r	$p \rightarrow q$	$p \rightarrow r$	$q \rightarrow r$	$(p \rightarrow (q \rightarrow r))$	$((p \rightarrow q) \rightarrow (p \rightarrow r))$	Ans
T	T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F	T
T	F	T	F	T	T	T	T	T
T	F	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T	T
F	T	F	T	T	F	T	T	T
F	F	T	T	T	T	T	T	T
F	F	F	T	T	T	T	T	T

Q. If today is Holi then tommorrow is Monday
(today is Holi = P, tommorrow is Monday = Q)

$\sim Q \to \sim P \Rightarrow$ If ~~Today~~ tomorrow is not Monday then today is not Holi.

* By using LOGICAL EQUIVALENCES :

Q. $\sim(p \vee (\sim p \wedge q))$ and $\sim p \wedge \sim q$

$\equiv \sim(p \vee (\sim p \wedge q))$

$\equiv \sim p \wedge \sim(\sim p \wedge q)$ (DeMorgan's law)

$\equiv \sim p \wedge (\sim(\sim p) \vee \sim q)$ (")

$\equiv \sim p \wedge (p \vee \sim q)$ (Double Negation law)

$\equiv (\sim p \wedge p) \vee (\sim p \wedge \sim q)$ (Distributive law)

$\equiv F \vee (\sim p \wedge \sim q)$ (Negation law)

$\equiv (\sim p \wedge \sim q) \equiv \boxed{RHS}$ (by Identity law)

Q. $(p \wedge q) \to (p \vee q)$ is tautology.

$\equiv \sim(p \wedge q) \vee (p \vee q)$ (Conditional logical equi.)

$\equiv (\sim p \vee \sim q) \vee (p \vee q)$ (Demorgan's law)

$\equiv \sim p \vee p \vee \sim q \vee q$

$\equiv T \vee T \equiv T$ (Negation law) (Disjun

$\therefore$ $\boxed{\text{equi. is tautology.}}$

Q. Use DeMorgan's law to express the negation of :

"John will go to the concert (P) $\boxed{\text{or}}$ Steve (Q) will go to the concert."

$\Rightarrow P \vee Q$ (Given)
$\Rightarrow \sim(P \vee Q)$ (Find)
$\equiv \sim P \wedge \sim Q$ (Demorgans law)
$\therefore$ "John will not go to the concert and Steve will not go to the concert".

* VALIDITY of ARGUMENTS:
An argument in prepositional logic is ~~pro~~ sequence of preposition.
All but the final preposition in the argument are called premises or hypothesis.
And the final preposition is called conclusion.

$$\text{Premises or Hypothesis} \begin{cases} P_1 \\ P_2 \\ \vdots \\ P_{n-1} \end{cases}$$
$$\text{Conclusion} \leftarrow \quad \therefore P_n$$

Q. $\overset{P_1:}{\text{"If}}$ you have a current password, you can log on to the network",
$\overset{P_2:}{\text{"You}}$ have current password therfore you can logon to the network".

$\Rightarrow$ 2 premises $\longrightarrow$ 1 conclusion.

$\Rightarrow$ [An argument is valid, when,
$(P_1 \wedge P_2 \wedge P_3 \ldots \wedge P_n \rightarrow Q)$
is tautology.]

P = Password ; Q = Login

Symbolic form: $P_1 = P \rightarrow Q$
$$\begin{array}{l} P \rightarrow Q \\ P \\ \hline \therefore Q \end{array}$$

Q. State which rule of inference is the basis of argument:

"It is below freezing & raining now".

∴ "It is below freezing now".

$\Rightarrow$ $\dfrac{P \wedge Q}{\therefore P}$ → $\boxed{\text{Simplification rule}}$

Q. $(P \wedge Q) \rightarrow P$. Const. table.

P	Q	$(P \wedge Q)$	$(P \wedge Q) \rightarrow P$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

Yes, it is tautology.

Q. State which rule of inference, is used in the foll. argument.

"If it rains today, then we will not have BBQ today. If we do not have BBQ today, then we will have a BBQ tomorrow. ∴, if it rains today, then we will have a BBQ tomm."

$\Rightarrow$ $P \Rightarrow$ rains

$Q \Rightarrow$ BBQ not today

$R \Rightarrow$ BBQ tomorrow.

∴ $P \rightarrow Q$

$Q \rightarrow R$

$\therefore P \rightarrow R$

Hypothetical syllogism rule.

$(P \to Q) \wedge (Q \to R) \to (P \to R)$, must be tautology.

Q. " It is not sunny this afternoon & its colder than yesterday", " You go swimming only if it is sunny", " If we do not go swimming, than we will take a canoe trip", & if we take canoe trip, then we will be home by sunset".

Conclusion → " We will be home by sunset.

⇒ P → It is sunny this afternoon.
Q → Colder than yesterday
R → Swimming
S → Canoe trip
T → Home by sunset.

$$\begin{array}{l} \sim P \wedge Q \\ R \to P \\ \sim R \to S \\ S \to T \\ \hline \therefore T \end{array}$$

$\Rightarrow ((\sim P \wedge Q) \wedge (R \to P) \wedge (\sim R \to S) \wedge (S \to T)) \to T$

1. $\sim P \wedge Q$ (given)
2. $\sim P$ (simplification) on 1.
3. $R \to P$ (Given)
4. $\sim R$ (Modus tollens on 2. & 3.)
5. $\sim R \to S$ (Given)
6. S (Modus ponens on 4. & 5.)
7. $S \to T$ (Given)
8. T (Modus ponens on 6. & 7.)

VALIDITY of ARGUMENT:

Q: $\sim q$
$p \to q$
$\therefore \sim p$

$(p_1 \wedge p_2 \wedge \dots \wedge p_n) \to c$ is a tautology.

$\therefore$ compound statement:
$((\sim q) \wedge (p \to q)) \to \sim p$

p	q	$\sim p$	$\sim q$	$p \to q$	$(\sim q) \wedge (p \to q)$	
T	T	F	F	T	F	T
T	F	F	T	F	F	T
F	T	T	F	T	F	T
F	F	T	T	T	T	T

Yes, it is tautology.

Q. "If you send me an e-mail message then I will finish writing the prog.",
" If .... then I will wake up feeling refreshed."

$\Rightarrow$ P → Send e-mail
Q → Finish writing email.
R → Sleep early
S → Wake up refreshed.

$\therefore$ compound st:
$(P \to Q) \wedge (\sim P \to R) \wedge (R \to S) \to (\sim Q \to S)$

Proof:

1. $\cancel{P \to Q}$ (Given)
2. $\cancel{(\sim P \to R)}$ (Given)
3. $\cancel{(R \to S)}$ (")
4. $\cancel{(\sim P \to S)}$ (H.S on 2. & 3.)
5. $\cancel{(\sim Q \to S)}$ (Given)
6. $\cancel{(\sim Q \to \sim P)}$ (Contrapositive on 1.)
7. $\cancel{(\sim Q \to S)}$ (H.S

1. $P \to Q$ (Given)
2. $\sim Q \to \sim P$ (Contrapositive of 1.)
3. $\sim P \to R$ (Given)
4. $\sim Q \to R$ (H.S on 2. & 3.)
5. $R \to S$ (Given)
6. $\sim Q \to S$ (H.S on 4. & 5.)

Q. $\sim t \to \sim r$

$\sim s$

$t \to w$

$u \vee s$

$\therefore w$

Compound statement:

$(\sim t \to \sim r) \wedge (\sim s) \wedge (t \to w) \wedge (u \wedge s) \to w$

$\Rightarrow$ 1. $(\sim t \to \sim r)$ (Given)

$\Rightarrow$ 2. $(u \to t)$ (Contrapositive of 1.)

$\Rightarrow$ 3. $(t \to w)$ (Given)

$\Rightarrow$ 4. $(u \to w)$ (H.S.) on 2 & 3

⇒ 5. $(r \vee s)$ (Given)

6. $(s \vee r)$ (Commutative)

7. $(\neg s)$ (Given)

8. r (Disjunctive syll. on 6 & 7)

9. w (Modus ponens on 4 & 8)

Q. $p \rightarrow (q \rightarrow r)$

$$\frac{p \wedge q}{\therefore \; r}$$

⇒ 1. $(p \wedge q)$ (Given)

2. p (simplification)

3. $p \rightarrow (q \rightarrow r)$ (Given)

4. $q \rightarrow r$ (Modus ponens on 2 & 3)

5. q (simplification on 1)

6. r (Modus ponens on 5 & 4)

Q. $(p \rightarrow q) \rightarrow q \equiv (p \vee q)$

p	q	$p \rightarrow q$	$(p \rightarrow q) \rightarrow q$	$(p \vee q)$
T	T	T	T	T
T	F	F	T	T
F	T	T	T	T
F	F	T	F	F

$\Rightarrow$ L.H.S

$\Rightarrow (p \to q) \to q$ (Logical equi.)

$\Rightarrow \sim(p \to q) \vee q$ (")

$\Rightarrow \sim(\sim p \vee q) \vee q$ (DeMorgan's law)

$\Rightarrow (p \wedge \sim q) \vee q$ (Distributive)

$\Rightarrow (p \vee q) \wedge (\sim q \vee q)$ (Negation)

$\Rightarrow (p \vee q) \wedge T \Rightarrow (p \vee q)$ (Identity)

Q. $p \to (q \vee r) \equiv (p \to q) \vee (p \to r)$

$\Rightarrow (p \to q) \vee (p \to r)$

* VALIDITY of ARGUMENT by RESOLUTION PRINCIPLE

* Literals : Prepositional variables
  i.e., $p, q, \sim p, \sim q, \ldots$
* Clause : Disjunction of literals
  $p \vee q$, $\sim p \vee \sim q$, etc.

* Example :

$$\begin{array}{l} P \\ P \to Q \\ \hline \therefore Q \end{array}$$

clause form :

C_1 : P

C_2 : $\sim P \to Q$ (By L.E.)

C_3 : $\sim Q$

C_4 : Q — By applying Resol. Prin. on C_1 & C_2

C_5 : $\square$ — C_3 & C_4

↳ empty clause

$\therefore$ The argument is valid.

Q.

$$\begin{array}{l} \sim t \to \sim r \\ \sim s \\ t \to w \\ r \vee s \\ \hline \therefore w \end{array}$$

clause form :

C_1 : $t \vee \sim r$ (by L.E.)

C_2 : $\sim s$

C_3 : $\sim t \vee w$ (by L.E.)

C_4 : $r \vee s$

C_5 : $\sim w$

C_6 : $\sim r \vee w$ (C_1 & C_3)

C_7 : r (C_2 & C_4)

C_8 : w (C_6 & C_7)

C_9 : $\square$ (C_5 & C_8)

Q. $p \to q$

$q \to r$

$r \to s$

$\sim s$

$p \vee t$

$\therefore t$

⇒ Clause form:

$C_1 : \sim p \vee q$

$C_2 : \sim q \vee r$

$C_3 : \sim r \vee s$

$C_4 : \sim s$

$C_5 : p \vee t$

$C_6 : \sim t$

$C_7 : \sim p \vee r$ (C_1 & C_2)

$C_8 : \sim p \vee s$ (C_3 & C_7)

$C_9 : \sim p$ (C_8 & C_4)

$C_{10} : t$ (C_5 & C_9)

$C_{11} : \square$ (C_{10} & C_6)

NORMAL FORMS:

⇒ Product / conjunction / $\wedge$

⇒ Sum / disjunction / $\vee$

1. Disjunctive Normal Form (DNF)
   Sum of elementary ~~sum~~ products
   $(\wedge) \vee (\wedge)$

2. Conjunctive Normal Form (CNF)
   Product of elementary sums.
   $(\vee) \wedge (\vee)$

* elementary product:
$(p \wedge q) \vee (\sim p \wedge \sim q) \vee (p) \vee (\sim q) \vee (p \wedge \sim p)$

* elementary sum:
$(p \vee q) \wedge (\sim p \vee \sim q) \wedge (p \vee \sim p) \wedge (p) \wedge (\sim q$

* Steps:

1. Replace conditional $(\rightarrow)$ or biconditional $(\leftrightarrow$ by $\wedge, \vee, \sim$.
2. Manipulate to get the required form.

Q. $(P \rightarrow Q) \wedge \sim Q$. Find its DNF.

$\equiv P \rightarrow Q$ (Given)

$\equiv (\sim P \vee Q) \wedge \sim Q$ (by L.E)

$\equiv (\sim P \wedge \sim Q) \vee (Q \wedge \sim Q)$ (Distributive)

Q. $p \vee (\sim p \rightarrow (q \vee (q \rightarrow \sim r)))$. Find its DNF

$\equiv p \vee (\sim p \rightarrow (q \vee (\sim q \vee \sim r))))$ [L.E]

$\equiv p \vee (\sim p \rightarrow ((q \vee \sim q) \vee (q \vee \sim r)))$ [D

$\equiv p \vee (\sim p \rightarrow (q \vee \sim r))$ [Negation]

$\equiv p \vee (p \vee (q \vee (\sim q \vee \sim r)))$ [by L.E.]

$\equiv p \vee p \vee q \vee \sim q \vee \sim r$

MINTERMS

Let p & q be the 2 propositional variables, all the possible formulas which consists of product of p & ~p ($p \wedge \sim p$) OR ($q \wedge \sim q$), but should not contain both the variable & its negation in one formula are called minterms of p & q.

* (p or ~p) and (q or ~q) shouldn't have p or ~p together.

∴ Minterms:

$(p \wedge q)$
$(p \wedge \sim q)$
$(\sim p \wedge q)$
$(\sim p \wedge \sim q)$

MAXTERM:

* same as MINTERM, just the product ($\wedge$) is replaced by ($\vee$).

∴ $(p \vee q)$
$(p \vee \sim q)$
$(\sim p \vee q)$
$(\sim p \vee \sim q)$

Principle DNF (PDNF):

sum of minterms.

Product CNF (PCNF)

Product of maxterms.

* Method to find PDNF & PCNF:

* in the digi. notes *

Q. Find PDNF & PCNF of $(P \to Q)$.

⇒ TT:

P	Q	$P \to Q$
T	T	T
T	F	F
F	T	T
F	F	T

PDNF: $(P \wedge Q) \vee (\sim P \wedge Q) \vee (\sim P \wedge \sim Q)$

PCNF: $\sim P \vee Q$

Q. PDNF & PCNF of $(P \leftrightarrow Q)$

P	Q	$P \leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

PDNF: $(P \wedge Q) \vee (\sim P \wedge \sim Q)$

PCNF: $(\sim P \vee Q) \wedge (P \vee \sim Q)$

Q. $(P \wedge Q) \vee (\sim P \wedge R) \vee (Q \wedge R)$

P	Q	R	$P \wedge Q$	$(\sim P \wedge R)$	$(Q \wedge R)$	Q...

* PREDICATE LOGIC *

$P(x) \rightarrow x > 3$; $Q(x,y) \rightarrow x + y = 5$

↳ variables → Predicate

→ If we give value to predicate, it becomes preposition.

$P(1) \rightarrow 1 > 3 \rightarrow$ False.

* **Quantifier**: Domain of variable,
(कितनी values के लिए true है)

$\forall x P(x)$ → Universal: $(\forall)$ true for all value of domain.

$\exists x P(x)$ → Existential: $(\exists)$ true for certain values of values.

↳ existential & universal quantification.

Q. $P(x) \rightarrow x+1 > x, x \in R$

$\Rightarrow \forall x P(x) \rightarrow$ True

$\exists x P(x) \rightarrow$ False

Q. $P(x)$ $x < 2$; $x \in R$

$\Rightarrow \forall x P(x) \longrightarrow$ False

$\exists x P(x) \longrightarrow$ True

Q. $P(x)$ $x^2 > 0$; $x \in Z$

$\Rightarrow \forall x P(x) \rightarrow$ False

$\exists x P(x) \rightarrow$ True

Translating English sentences into logical expr:

Q. every student in this class has studied maths.

⇒ Introduce x:

* $\boxed{\text{Domain} \to \text{class}}$

→ For every student x, x has studied Maths.

$\forall x \quad M(x)$

* $\boxed{\text{Domain} \to \text{GLA}}$

→ For every student x, if x is student of this class then x, has studied Maths.

$\forall x \quad S(x) \quad M(x)$

→ For some student x, x is a student of this class and x has studied Maths.

Q. "Every student in this class has visited either Canada or Mexico."

⇒ $\boxed{\text{Domain} \to \text{Class}}$

→ For every student x, x has visited either Canada or Mexico. $\forall x((C(x) \vee M(x)))$

⇒ $\boxed{\text{Domain} \to \text{GLA}}$

→ For every student x, if x is student of this class then x has visited either Canada or Mexico. $\forall x(S(x) \to ((C(x) \vee M(x)))$

* GROUP THEORY *

Algebraic structure:

$x + y = t$

To define: $(S, *)$
$S \downarrow$ set
$* \hookrightarrow$ operation
combination of values.

An algebra has the foll. components sets & operation. It is represented by $(S, *)$

It may be $+, -, /, *, +_n, \times_n$.

→ How to define operation table for given combination.

$*$	1	2	3
1	1	2	3
2	2	4	6
3	3	6	9

Closure Property: $(\{1, 2, 3\}, *)$

It states that when you perform an operation on any 2 no.s in a set, the result of the computation is another no. in the same set.

Suppose A is a set with an operation $*$ then A is stated to be closed under $*$ if,

$$\boxed{a * b \in A \quad \forall \; a, b \in A}$$

→ Addition of real no.s gives real no., ∴ it satisfy the closure property.

Q. $(Z+, +)$ closure prop. ∀ combination
$(Z+, -)$
$(Z+, *)$
$(Z+, /)$

$\Rightarrow 2+3, 5 \in Z+$ ✓
$5-8 ; -3 \notin Z+$ ✗
$Z+, *, 2 \times 3 = 6 \in Z+$ ✓
$2/4 = 1/2 \notin Z+$ ✗

Associative property:

An operation $*$ on a set A is said to be associative, if for any elements, a, b & c in A, we have $\boxed{(a*b)*c = a*(b*c)}$.

Q. Check if associative:
$(Z+, +) = 5+(2+3) = (5+2)+2$ ✓
$(Z+, -) = (5-2)-3 \neq 5-(2-3)$ ✗
$(Z+, *) = (1 \times 2) \times 3 = 1 \times (2 \times 3)$ ✓
$(Z+, /) = (4/2)/8 \neq 4/(2/8)$ ✗

Commutative property:

$$\boxed{a*b = b*a}$$

$\Rightarrow (Z+, +)$ ✓
$\Rightarrow (Z+, -)$ ✗
$\Rightarrow (Z+, *)$ ✓
$\Rightarrow (Z+, /)$ ✗

Identity law:

$$\boxed{\begin{array}{l} a * e = a \\ e * a = a \end{array}}$$

→ Multiplicative Identity:
$$\boxed{a * 1 = a}$$
→ Additive Identity:
$$\boxed{a + 0 = a}$$

Inverse property:

$$\boxed{\begin{array}{l} \text{If, } a * b = e \\ \text{then, } b = a^{-1} \end{array}}$$

* Order of properties / Algebraic structures $(s, *)$

Properties	Algebraic structures
→ Closure, → Associative	Semigroup
→ Closure, → Associative, → Identity	Monoid
→ Closure, → Associative, → Identity, → Inverse	Group
→ Closure, → Associative, → Identity, → Inverse, → Commutative	Abelian Group

Q. Show that set N is a semi-group & monoid w.r.t $*$.

⇒ ~~Let a, b ∈ N~~
~~For a~~

⇒ Here, $N = \{1, 2, 3, 4, \ldots\}$

1. Closure: We know, product of 2 N is a N.

i.e., $a * b \in N \ \forall \ a, b \in N$

∴ Multiplication is closure prop.

2. Associativity: $*$ of N is associa

i.e., $(a * b) * c = a * (b * c)$
$\forall \ a, b, c \in N$

∴ N is a semigroup w.r.t $*$.

3. Identity: We have , $1 \in N$,
$a * 1 = 1 * a = a \quad \forall a \in N$.

$\therefore$ N is a monoid wrt $*$.

Q. Show that Z is an Abelian group wrt +

$\Rightarrow$ Here $Z = \{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\}$

1. Closure: We know sum of 2 Z is an Z
i.e., $a+b \in Z \;\forall\; a, b \in Z$
$\therefore$ + is closure prop.

2. Associativity: + of Z is associative,
i.e., $(a+b)+c = a+(b+c) \;\forall\; a, b, c \in Z$
$\therefore$ + is associative.

3. Identity. We have , $0 \in Z$
$a+0 = 0+a = a, \;\forall\; a \in Z$
$\therefore$ + is identity.

4. Inverse:
$a+b = 0, \;\forall\; a, b \in Z$
$b = -a$
$\therefore$ + is inverse.

5. Commutative: + of Z is commutative
i.e., $a+b = b+a \;\forall\; a, b \in Z$
$\therefore$ + is commutative

$\therefore$ Z is Abelian group wrt +.

* $Z_m =$ 0 to $m-1$ $\quad Z_5 = \{0, 1, 2, 3, 4\}$

* $Z_m^* =$ 1 to $m-1$ $\quad Z_5^* = \{1, 2, 3, 4\}$

MODULO SYSTEM:

1. Addition Modulo ($+_m$):

⇒ Let m is a +ve integer for any 2 +ve int a & b.

I case:

$a +_m b = a + b$; if $a + b < m$

II case:

$a +_m b = r$; if $a + b \geq m$

where r is remainder obtained by dividing $a+b$ by m.

Q. $2 +_6 3 = 5$

Q. $2 +_6 5 = 1$ $(\because 7 \% 6 = 1)$

2. Multiplication Modulo ($\times_m$):

I case:

$a \times_m b = a \times b$; if $a + b < m$

II case:

$a \times_m b = r$; if $a \times b \geq m$

Q. $4 \times_5 5 = 0$

Q. $3 \times_4 2 = 2$

Q. $3 \times_5 4 = 2$

Q. $2 \times_5 2 = 4$

Q.

$+_4$	1	2	3
1	2	3	0
2	3	0	1
3	0	1	2

Q. Show that $(Z_6, +_6)$ is abelian group.

⇒

$+_6$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

Closure: ∵ all the entries in the table belongs to Z_6. ∴ satisfies closure prop.

Associative: $2 +_6 (3 +_6 4) = (2 +_6 3) +_6 4$

$2 +_6 1 \qquad 5 +_6 4$

3 ∴ it associative.

Identity: The first row is identical to the top ∴ the element heading first row, i.e., 0 is the identity element.

Inverse: ∵ The inverse of 0, 1, 2, 3, 4, 5 are 0, 5, 4, 3, 2, 1 resp.
∴ It satisfies inverse law.

Commutative: ∵ The corres. rows & column of the composition table are identical, ∴ it is commutative.

∴ ABELIAN GROUP

Q. Show that $(Z_6^*, \times_6)$. is abelian group.

⇒

$\times_6$	1	2	3	4	5
1	1	2	3	4	5
2	2	4	0	2	4
3	3	0	3	0	3
4	4	2	0	4	2
5	5	4	3	2	1

Closure: 0 doesn't belong to Z_6^* ∴ " satisfy closure prop.

∴ NOT ABELIAN GROUP

Q. Show that $(Z_6, \times_6)$ is an abelian group.

⇒

$\times_6$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

Closure : ∵ All the entries in the composition table belongs to Z_6 ∴ Closure.

Associative: $2 \times_6 (3 \times_6 4)$ $\quad$ $(2 \times_6 3) \times_6 4$

$2 \times_6 0$ $\quad$ $0 \times_6 4$

0 $\quad$ 0

∴ associative

Identity : II row is sim. to top
∴ identity element = 1

Inverse: Inverse of 0, 1, 2, 3, 4, 5 is 0 not present.

∴ MONOID GROUP

Q. $(Z^+, *)$, $[a * b = a + b + 2]$
Show that it is commutative as well as associative.

⇒ Commutative:

$a * b = a + b + 2$

$= b + a + 2$ (Sum of int is comm)

$= b * a$

∴ It is commutative.

Associative:

$(a*b)*c = \cancel{a*(b*c) \ \forall\ a, b}$

$(a+b+2)*c$

$a+b+c+4$

$a*(b*c) = a*(b+c+2)$

$= a+b+c+4$

$\because (a*b)*c = a*(b*c)$

$\therefore$ It is associative.

Q. $M(x)$: x is a mammal

$A(x)$: x is a animal

$W(x)$: x is a warm blooded.

Translate into formula:

"Every mammal is warm blooded."

$\Rightarrow$ ~~For Domain $\Rightarrow$ Mammal~~.

$\forall x\,(M(x) \rightarrow W(x))$

$\Rightarrow$ For Domain $\Rightarrow$ Animal + Humans

For every animal x, if x is mamm, x is warm blooded.

$\forall x\,(\overset{A(x)\ \wedge}{M(x)} \rightarrow W(x))$

Q. $\exists x\,(A(x) \wedge \sim M(x))$

$\Rightarrow$ For ~~every~~ some living beings x, if x is an animal & it is not a mammal.

$\Rightarrow$ "Some animals are not mammals".

Q. If it rains, then they do not drive the car,

$\Rightarrow$ $P \to$ It rains $\Rightarrow P \to \sim Q$ (given)

$Q \to$ Drive the car

Now,

$P \to Q \equiv \sim Q \to \sim P$

$\therefore P \to \sim Q \equiv Q \to \sim P$ (contrapositive)

$\Rightarrow$ "If they drive the car, then it is not raining."

Also

$P \to \sim Q \equiv (\sim P \vee \sim Q)$ (By L.E.)

$\equiv \sim(\sim P \vee \sim Q)$ (Take negative)

$\equiv P \wedge Q$ (By De Morgan's law)

RING $(R, +, \cdot)$.

1. $(R, +)$ is an abelian group.
2. $(R, \cdot)$ is a semi- group.
3. Distributive law,

$$a \cdot (b + c) = a \cdot b + a \cdot c$$

* Ring with Unity: If R is a ring & it contains multiplicative identity 1, then its called Ring with unity.

* Commutative Ring: If R is a ring & it satisfies the commutative law w.r.t multiplication, then R is called commutative ring.

* Ring with zero divisor:

$ab = 0$
$a \neq 0$ & $b \neq 0$

$2 \times 3 = 6 \neq 0$
$2 \times 6\,3 = 0$

Here a & b are zero divisors.

Ring R is Ring with zero divisors, if there exists non- zero elements $a, b \in R$ such that, $ab = 0$, where $a \neq 0$ & $b \neq 0$

* Integral Domain: Commutative Ring R, is a Integral Domain if R has no zero divisor, such that,

$ab = 0$, $a = 0$ or $b = 0$

* Field: Commutative ring R, with an identity element 1, is a field if every non- zero element has a multiplicative inverse.

$a \times In = I$
$a + In = I$

Q. $(Z_6, +_6, \times_6)$

⇒

$+_6$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

$\times_6$	0	1	2	3	4	
0	0	0	0	0	0	
1	0	1	2	3	4	
2	0	2	4	0	2	
3	0	3	0	3	0	
4	0	4	2	0	4	
5	0	5	4	3	2	1

⇒ 1. $(R_1, +_6) \rightarrow$ Abelian group

2. $(R_1, \times_6) \rightarrow$ semigroup

$$2 \times_6 (3 \times_6 4) = (2 \times_6 3) \times_6 4$$

3. Distributive

$$2 \times_6 (3 +_6 4) = (2 \times_6 3) + (2 \times_6 4)$$

$$2 \times_6 1 \qquad\qquad 0 +_6 2$$

$$2 \qquad\qquad 2$$

4. Field: Mul. Identity = 1

$1^{-1} = 1$, $5^{-1} = 5$ } Inverse of 2, 3, 4 doesn't exist.

ORDER of a GROUP:

The no. of elements in a group G, is called order of G.

It is denoted by $|G|$.

* Order of an element of a group:

If G is a group & any element $a \in G$. Then, the +ve smallest integer (n) is said to be an order of element a i.e. $O(a)$.

If $a^n = e$,
where $e \rightarrow$ identity element of a group.

eg: $a * a = e \Rightarrow O(a) = 2$

$b * b * b = e \Rightarrow O(b) = 3$

Q. $G = \{1, -1, i, -i\}$, it's a group under multiplication.
Find the order of each element of the group.

$\Rightarrow e = 1$

$O(1) \Rightarrow 1^1 = 1 \Rightarrow \boxed{O(1) = 1}$

$O(-1) \Rightarrow (-1) \times (-1) \Rightarrow \boxed{O(-1) = 2}$

$O(i) \Rightarrow i \times i \times i \times i = i^4 = 1 \Rightarrow \boxed{O(i) = 4}$

$O(-i) = -i^4 = 1 \Rightarrow \boxed{O(-i) = 4}$

Q. $G = (\{0, 1, 2, 3\}, +_4)$
Find $O(2)$ & $O(3)$

$\Rightarrow e = 0$

$O(2) = 2 +_4 2 = 0 \Rightarrow \boxed{O(2) = 2}$

$O(3) = 3 +_4 3 +_4 3 +_4 3 = 0 \Rightarrow \boxed{O(3) = 4}$

$+_4$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

$\Rightarrow e = 0$

Cyclic Group: A group G is said to be cyclic, if all its elements can be generated by a single elements a where $a \in G$.

Q. $G = (\{1, -1, i, -i\}, \times)$, cyclic or not.

$\Rightarrow$ $o(1) = 1$ $\quad$ $o(-1) = 2$

$o(i) = o(-i) = 4$

$|G| = 4$

$\Rightarrow$ The elements with order equal to group's order can be cyclic elements

$\therefore$ $i = i$ $\quad$ & $\quad$ $-i = -i$

$i^2 = -1$ $\quad$ $-i^2 = -1$

$i^3 = -i$ $\quad$ $-i^3 = i$

$i^4 = 1$ $\quad$ $i^4 = 1$

$\because$ all the elements are generated

$\therefore$ it is a cyclic group.

Q. $(Z_5, +_5)$ cyclic or not?

How many generators if cyclic.

$+_5$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

(e)

$o(0) = 1$ $\quad$ $o(3) = 5$

$o(1) = 5$ $\quad$ $o(4) = 5$

$o(2) = 5$

Now,

$1^1 = 1$	$2^1 = 2$
$1^2 = 2$	$2^2 = 4$
$1^3 = 3$	$2^3 = 1$
$1^4 = 4$	$2^4 = 3$
$1^5 = 0$	$2^5 = 0$

&,

$3^1 = 3$	$4^1 = 4$
$3^2 = 1$	$4^2 = 3$
$3^3 = 4$	$4^3 = 2$
$3^4 = 2$	$4^4 = 1$
$3^5 = 0$	$4^5 = 0$

$\because$ all the elements are generated,

$\therefore$ it is cyclic with 4 generators,

i.e. $\{1, 2, 3, 4\}$.

SUBGROUP

group : $(\{a, b, c, d, e, f\}, *)$

$(\{b, c, d\}, *)$ subgroup if it follow the property of group.

* coset : $(G, *)$ $(H, *)$ $a \in G$

$Ha = \{ha : h \in H\}$ right coset of H on G.

$aH = \{ah : h \in H\}$ left " " " .

Q. $H = \{h_1, h_2, h_3, h_4\}$

$\Rightarrow \{h_1a, h_2a, h_3a, h_4a\}$ (right coset)

Ha

Q. $(G = \{e, a, b, ab\}, *)$ $(H = \{e, a\}, *)$

All right cosets.

$\Rightarrow He = \{ee, ae\} = \{e, a\} = H$

$Ha = \{ea, aa\} = \{a, ae\} = H$

$Hb = \{eb, ab\} = \{b, ab\}$

$Hab = \{eab, aab\} = \{ab, b\}$

$a^2 = b^2 = e$, where e is identity element

2 distinct cosets, i.e., He, Hb.

* [Union of all disjoint cosets = group]

Q. $(Z, +)$.

$Z = \{\ldots, -4, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

$H = \{\ldots -9, -6, -3, 0, 3, 6, 9, \ldots\}$

All right cosets,

$\Rightarrow$ $H+0 = \{\ldots -9, -6, -3, 0, 3, 6, 9, \ldots\}$

$H+1 = \{\ldots -8, -5, -2, 1, 4, 7, 10, \ldots\}$

$H+2 = \{\ldots -7, -4, -1, 2, 5, 8, 11, \ldots\}$

$H+3 = \{\ldots -6, -3, 0, 3, 6, 9, \ldots\} = H+0$

Repetitions started.

$\therefore$ 3 distinct cosets, i.e., $H+0$, $H+1$, $H+2$

Theorem

If H be a subgroup of a finite group G, then, the order of H divides the order of G.

i.e., if order of $(G) = n$

& $O(H) = m$,

then $\boxed{\frac{n}{m} = k}$ OR $\boxed{n = km}$

$k \to$ index

Proof: $H = \{h_1, h_2, \ldots, h_m\}$

Let $a_1, a_2 \in G$

$Ha_1 = \{h_1 a_1, h_2 a_1, \ldots, h_m a_1\} \Rightarrow O(Ha_1) = m$

$Ha_2 = \{h_1 a_2, h_2 a_2, \ldots, h_m a_2\} \Rightarrow O(Ha_2) = m$

Let, $Ha_1, Ha_2, \ldots, Ha_k$, are k disjoint cosets

then,

$$G = Ha_1 \cup Ha_2 \cup \ldots \cup Ha_k.$$

$$O(G) = O(Ha_1) + O(Ha_2) + \ldots + O(Ha_k)$$

$$O(G) = m + m + \ldots + m \text{ (k times)}$$

$$\boxed{n = k \cdot m}$$

Q. $(\mathbb{Z}, *)$

$a * b = a + b + 1$, Abelian or not?

$\Rightarrow$ Closure: Sum of all integers is a integer. i.e., $a * b \in \mathbb{Z}$ (if $a, b \in \mathbb{Z}$)

Associative:

$$(a*b)*c = a*(b*c)$$

$$(a+b+1)+c+1 = a+(b+c+1)+1$$

$$a+b+c+2 = a+b+c+2$$

$\therefore$ it is associative.

Identity:

We have $-1 \in Z$ $\therefore e = -1$

$\forall a \in Z$

$a * (-1) = a+1-1 = a$

Inverse:

$a * a^{-1} = e$

$a + a^{-1} + 1 = -1$

$a^{-1} = -2-a \in Z$

Commutative:

$a*b = b*a$

$a+b+1 = b+a+1$

Integer follow commutative law,

$a+b+1 = a+b+1$

$\therefore$ It is an Abelian group

$\because$ It follows all 5 properties.

Rational no.

Q. $(Q^+, *)$, $a*b = ab/2$, $a, b \in Q^+$

Abelian or not.

$\Rightarrow$ Closure:

WRt,

$a*b \in Q^+$

$ab/2 \in Q^+$ $\forall a, b \in Q^+$

Associative:

$(a*b)*c = a*(b*c)$

$ab/2 * c = a * bc/2$

$\frac{abc}{4} = \frac{abc}{4}$ $\forall a, b, c \in Q^+$

Identity:

$a * e = a \Rightarrow ae/2 = a \Rightarrow \boxed{e = 2} \in Q^+$

$\therefore a * (2) = a \quad \forall\, a \in Q^+$

Inverse:

$a * (a^{-1}) = e$

$\frac{a \cdot a^{-1}}{2} = 2 \Rightarrow a^{-1} = 4/a \in Q^+$

$\forall\, a, a^{-1} \in Q^+$

Commutative:

$a * b = ab/2$

$b * a = ba/2 = ab/2$

$\therefore a * b = b * a \quad \forall\, a, b \in Q^+$

$\therefore$ It is an Abelian Group

$\because$ It follows all 5 properties.

Q. $\left(M = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} & \\ & \end{bmatrix}, \ldots \right\}, * \right)$

$a, b, c, d \in R$. Abelian or not?

$\Rightarrow$ Closure:

$A = \begin{bmatrix} 2 & 3 \\ 4 & 2 \end{bmatrix} \in R \qquad B = \begin{bmatrix} 1 & 2 \\ 8 & 4 \end{bmatrix} \in R$

$A * B = \begin{bmatrix} 11 & 16 \\ 10 & 16 \end{bmatrix} \in M, \quad \forall\, A, B \in R$

Associative: $C = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$

$(A * B) * C = a * (b * C)$

$$\begin{bmatrix} 11 & 16 \\ 10 & 16 \end{bmatrix} * \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 4 & 2 \end{bmatrix} * \begin{bmatrix} 5 & 4 \\ 11 & 10 \end{bmatrix}$$

$$\begin{bmatrix} 43 & 38 \\ 42 & 36 \end{bmatrix} = \begin{bmatrix} 43 & 38 \\ 42 & 36 \end{bmatrix}$$

$\forall A, B, C \in M$

Identity: $e = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$$A * e = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} * \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = A$$

$$a * \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = a \quad \forall a \in M$$

Commutative: $a * b \neq b * a$

$$\begin{bmatrix} 2 & 3 \\ 4 & 2 \end{bmatrix} * \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \neq \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 11 & 16 \\ 10 & 16 \end{bmatrix} \neq \begin{bmatrix} 10 & 7 \\ 22 & 17 \end{bmatrix}$$

$a * b \neq b * a$

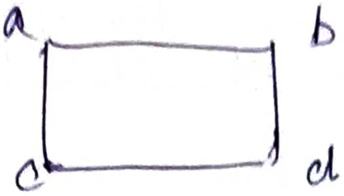
So, it is an abelian

Inverse:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

* GRAPHS *

→ Graphs are data structures consisting of vertices & edges that connect these vertices.

→ 

$G = (V, E)$
V: set of vertices
E: set of edges

→ simple graph: 1 edge b/w 2 vertices

→ Multiple graph: Multiple edges b/w 2 vertices.

→ Loops: Edges that connects a vertex to itself (+2 for degree)

→ Pseudographs: Graph containing loops & vertices.

→ Undirected graph: An edge associated with unordered pair.

→ Directed " : Ordered pair

* Multiplicity: No. of edges b/w ordered pair.

* Degree of vertex: No. of edges connected to it.

→ If degree = 0; isolated node
degree = 1; pendent node.

The HANDSHAKING THEOREM

Let $G = (V, E)$ be an undirected graph with e edges. Then,

$$\boxed{2e = \sum_{v \in V} \deg(v)}$$

Q. Total vertices = 10
Degree of each vertice = 6

$\Rightarrow$ 2e = sum of degree of each vertice

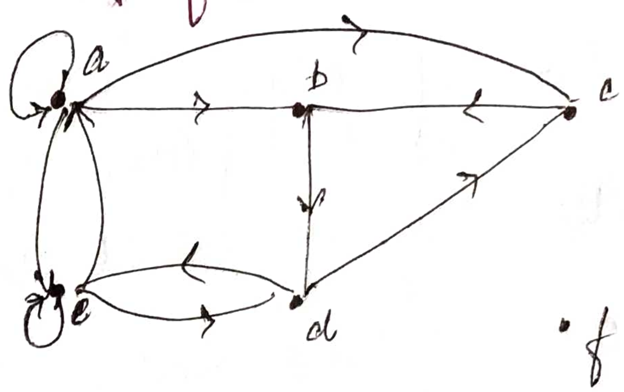
$2e = (10 \times 6)$

$\boxed{e = 30}$

* For Directed Graph:

Indegree: No. of edges coming to a vertex
Out degree: No. " " " out of a "
Loop falls under both ∴ +2 for loop.

Q.



$\Rightarrow$ $\deg^- a = 2$ $\qquad$ $\deg^+ a = 4$

$\deg^- b = 2$ $\qquad$ $\deg^+ b = 1$

$\deg^- c = 3$ $\qquad$ $\deg^+ c = 2$

$\deg^- d = 2$ $\qquad$ $\deg^+ d = 2$

$\deg^- e = 3$ $\qquad$ $\deg^+ e = 3$

$\deg^- f = 0$ $\qquad$ $\deg^+ f = 0$

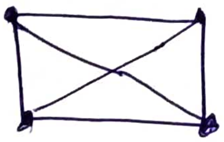
THEOREM: Let $G = (V, E)$ be a graph with directed edges, then

$$\boxed{\sum_{v \in V} deg^-(v) = \sum deg^+(v) = |E|}$$

The sum of in-degrees & the sum of out-degrees of all vertices in a graph with directed edges are the same. Both of these sums are edges of graph.

* complete graph: Exactly one edge for each ordered pair.

$n = 4$



K_4

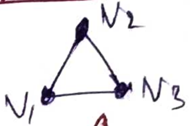
$n = 5$

K_5

* cycle: For cycle C_n, $n >= 3$, there should be $v_1, v_2, \ldots, v_n$ vertices & edges $\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}$

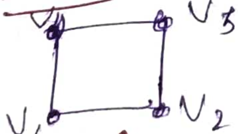
eg.

$n = 3$



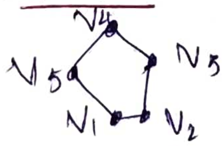
C_3

$n = 4$



C_4

$n = 5$



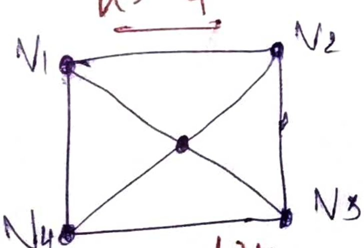
C_5

* wheel: We obtain wheel when we add an additional vertex to cycle C_n, for $n >= 3$, connect new vertex to all other vertex.

$n = 3$

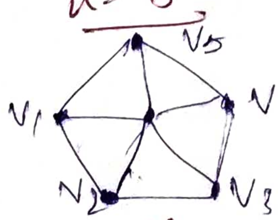
W_3

$n = 4$



W_4

$n = 5$

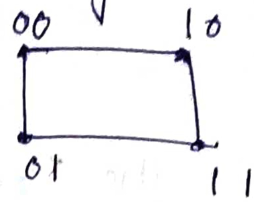


W_5

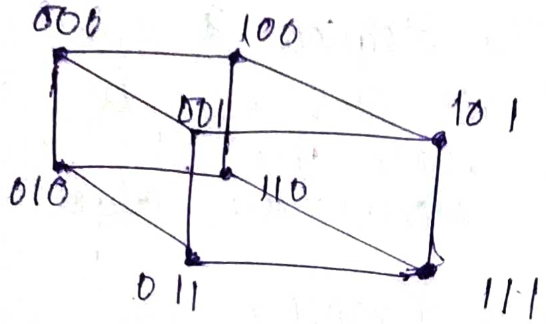
* n- Cubes: Graph that has vertices representing the 2n bit strings of length n.

Q_1 :

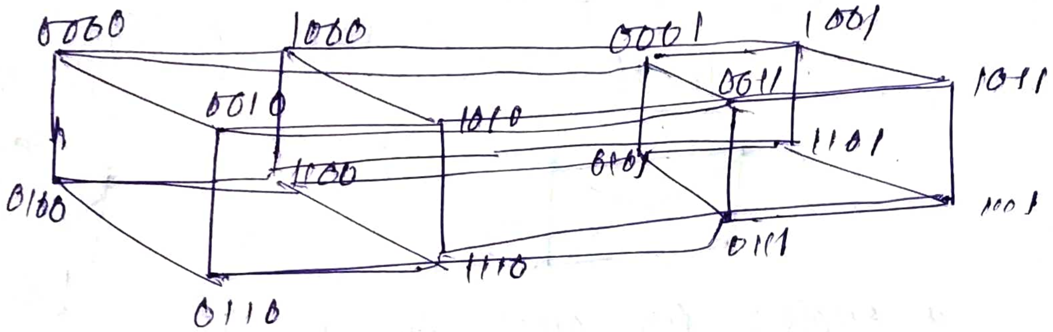
Q_2 :



Q_3 :



Q_4 :



* There is a pair of group being

HOMOMORPHISM of GROUP:

Let $(G, *)$ & $(H, \cdot)$ are the two groups

then a func. $f: G \to H$ is called a group homomorphism, if $\forall\ a, b \in G$, it holds that

$$\boxed{f(a * b) = f(a) \cdot f(b)}$$

Q. $(R, +)$ & $(R^+, *)$, $f(x) = 2^x$.

Find whether it is Homomorphism.

$\Rightarrow$ Let $x, y \in R$.

To Prove:

$$f(x+y) = f(x) * f(y)$$

$$f(x+y) = 2^{x+y} = 2^x * 2^y = f(x) * f(y)$$

$$\Rightarrow f(x+y) = f(x) * f(y)$$

$\therefore$ It is Homomorphism.

Q. $(Z_4, +_4)$ $(S, \times_4)$, $S = \{1, 3, 7, 9\}$

$$f: Z_4 \to S$$

$f(0) = 1$; $f(1) = 3$; $f(2) = 9$; $f(3) = 7$

$\Rightarrow$

$+_4$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

$\times_4$	1	3	7	9
1	1	3	3	1
3	3	1	1	3
7	3	1	1	3
9	1	3	3	1

For $(1, 2) \in Z_4$,

$$f(1 +_4 2) \neq f(1) \times_4 f(2)$$

$$f(3) = 7 \qquad 3 \times_4 9 = 3$$

$\therefore$ Not homomorphism

Q. Do the above question replacing $(S, \times_4)$ by $(S, \times_{10})$

$\times_{10}$	1	3	7	9
1	1	3	7	9
3	3	9	1	7
7	7	1	9	3
9	9	7	~~6~~ 3	1

* Properties of Homomorphism:

1. f maps the identity element e_G of G to the identity element e_H of H.

$$\boxed{f(e_G) = e_H}$$

2. It also maps inverses to inverses such that,

$$\boxed{f(a^{-1}) = (f(a))^{-1}} ; \forall a \in G$$

1. Let $(G, *)$ & $(H, \cdot)$ are the two groups & a function $f : G \to H$ is group homomorphism.

Let $a \in G \Rightarrow f(a) \in H$

$\Rightarrow f(a) \cdot e_H = f(a)$ ($\because e_H$ is identity element of H.)

$\Rightarrow f(a) \cdot e_H = f(a * e_G)$ ($\because f$ is homomorphism)

$\Rightarrow e_H = f(e_G)$ (by left cancellation law in H)

$$\therefore \boxed{f(e_G) = e_H}$$

2. Let $(G, *)$ & $(H, \cdot)$ are the 2 groups & a func. $f: G \to H$ is group homomorphism.

Let $a \in G$ & $a^{-1} \in G$.

$\Rightarrow f(a) \in H$ & $f(a^{-1}) \in H$

$e_H = f(e_G) = f(a * a^{-1}) = f(a) \cdot f(a^{-1})$ — (i)

$e_H = f(e_G) = f(a^{-1} * a) = f(a^{-1}) \cdot f(a)$ — (ii)

From (i) & (ii)

$f(a) \cdot f(a^{-1}) = f(a^{-1}) \cdot f(a) = e^H$

$\therefore \boxed{f(a^{-1}) = \{f(a)\}^{-1}} \quad \forall a \in G$

* POSETS *

Any relation R on set A is called POSET, if it satisfies :-

Reflexive (a, a) aRa

Antisymmetric aRb the bRa, if $a = b$ $\forall a, b \in A$

Transitive

* $A = \{1, 2, 3\}$

$R = \{\underbrace{(1,1), (2,2), (3,3)}_{\text{Anti-symmetric}}, \underbrace{(1,2), (2,1)}_{\text{Not antisymmetric}}\}$

Q. $A = \{1, 2, 3\}$ $R_1 = \{(1,2), (1,3), (1,1)\}$ but anti-sym. & transitive

$\Rightarrow$ Not reflexive $\therefore$ NOT POSET

$R_2 = \{(1,2)(1,3)\}$

$\Rightarrow$ Antisymmetric, transitive, reflexive.

$R_3 = \{(1,1), (2,2), (3,3), (1,3), (2,3)\}$

$\Rightarrow$ Reflexive, antisymmetric & transitive

$\therefore$ It's POSET.

Q $R = \{(a,b) \mid a,b \in \mathbb{Z},\ a < b\}$

⇒ $(3,3)$, but $3 < 3$ ∴ (Not reflexive)

Q. $R = \{(a,b) \mid a,b \in \mathbb{Z},\ a \leq b\}$

⇒ Reflexive, antisymmetric & transitive.

∴ It's a POSET.

Q. $R = \{(a,b) \mid a,b \in \mathbb{Z}^+,\ b/a \in \mathbb{Z}^+\}$

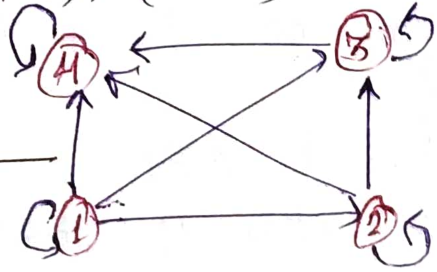
⇒ Reflexive ∵ $a/a = 1 \in \mathbb{Z}^+ \ \forall\ a \in \mathbb{Z}^+$

Antisymmetric & transitive ∴ It's a POSET.

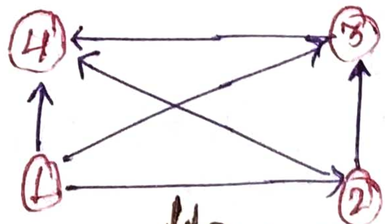
HASSE DIAGRAM: (POSET DIAGRAM)

$A = \{1,2,3\}$

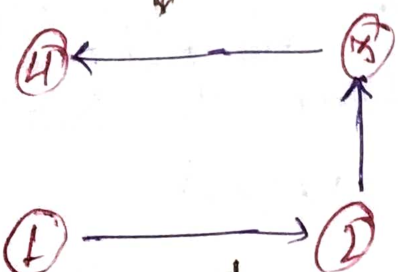
$R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)\}$



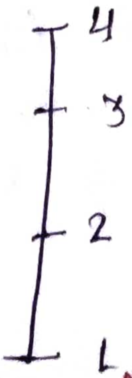
(Remove the loops)



(Remove transitive edges)



(Arrange in heirarchial order)

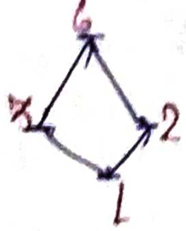


Hasse Dia.

Q. $A = \{1, 2, 3, 6\}$

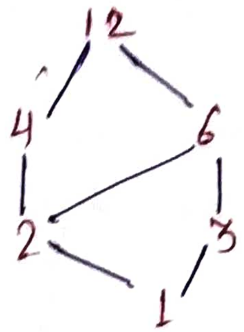
$R = \{(1,1), (1,2), (1,3), (1,6), (2,2), (2,6), (3,3), (3,6), (6,6)\}$

$\Rightarrow$

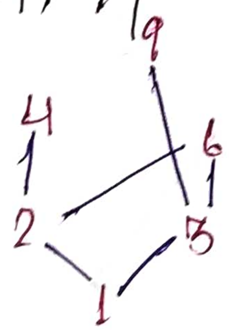


Q. A $[\{1, 2, 3, 4, 6, 12\}, /]$ b/a [A.K]

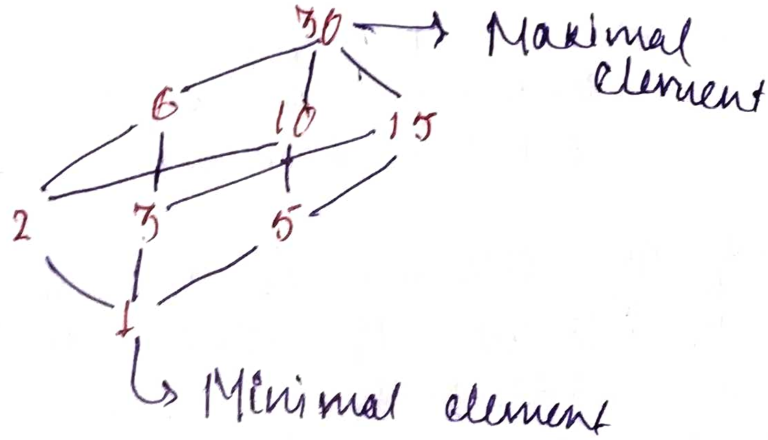
$\Rightarrow$



Q. $[\{1, 2, 3, 4, 6, 9\}, /]$

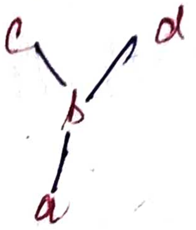


Q. $[\{1, 2, 3, 5, 6, 10, 15, 30\}, /]$



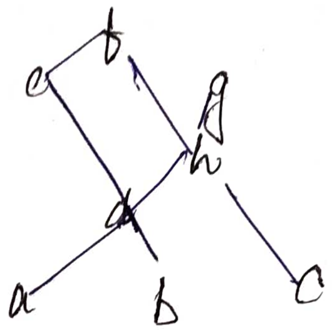
MINIMAL ELEMENT: If no element is related to an element then that element is minimal element of a poset.

MAXIMAL ELEMENT: If an element is not related to any other element then that element is maximal element of poset.



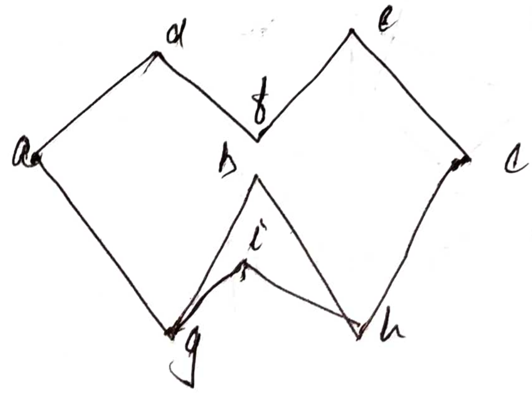
c, d ⇒ Maximal
a ⇒ Minimal.

Q.



a, b, c ⇒ Minimal
f, g ⇒ Maximal.

Q.



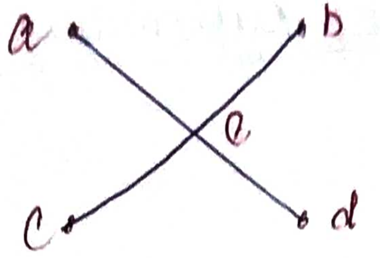
g, h, f ⇒ Minimal
d, e, b, i ⇒ Maximal

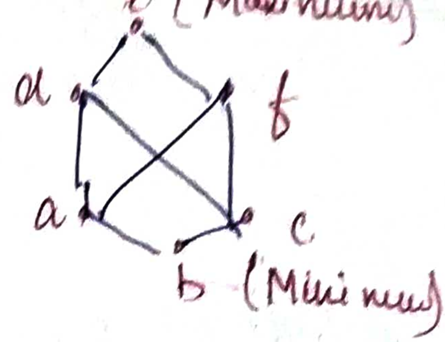
Q. .a .b .c

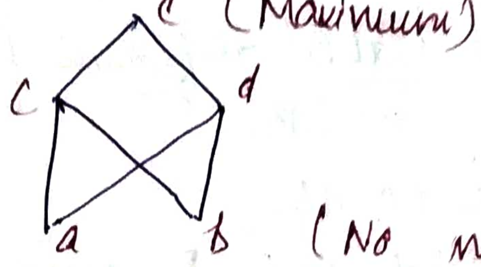
a, b, c ⇒ Minimal
a, b, c ⇒ Maximal.

* If an element is maximal & every element is related to it its called maximum.

* If an element is minimal & it is related every other element is related, then its called minimum.

Q. 



Q. 

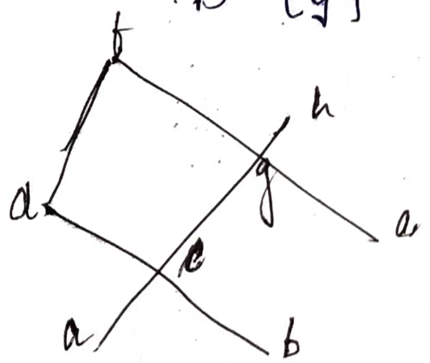
(No minimum)

* Let B be the subset of A, then,

<u>Upper Bound</u> (wrt subset) an element $x \in A$, is in upperbound of B if $(y, x) \in$ poset for $\forall y \in B$.

<u>Lower Bound</u>: an element $x \in A$ is in lowerbound of B if $(x, y) \in$ poset, $\forall y \in B$

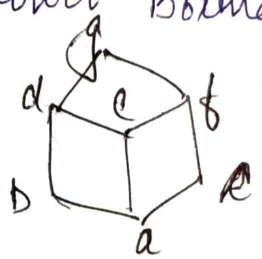
$B = \{y\}$; $A = \{$elements of Hasse Diagram$\}$

Q. 

$B = \{e, c\}$

$A = \{a, b, c, d, e, f, g, h\}$

$\Rightarrow$ Upper Bound $= \{f, g, h\}$
Lower Bound $= \{\phi\}$

Q. 

$B = \{e, f\}$

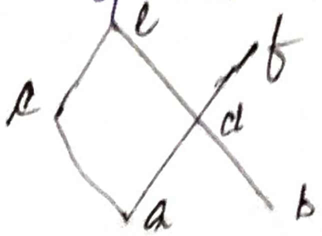
$\Rightarrow UB(B) = \{g, f\}$
$LB(B) = \{c, a\}$

Least Upper Bound / LUB / Supremum / Join / ∨

Least element in UB.

Greatest Lower Bound / GLB / Infimum / Meet / ∧

Greatest element in LB.

Q. 

$B = \{c, d\}$

Find join & meet.

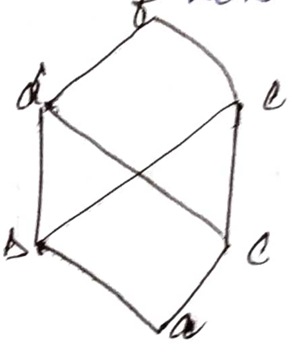
⇒ $UB(B) = \{e\}$ $LB(B) = \{a\}$

$LUB = e$ $GLB = a$

if $B = \{e, f\}$

⇒ $UB(B) = \{\phi\}$ $LB(B) = \{a, b, d\}$

$LUB = \phi$ $GLB = d$

Q. 

$B = \{d, e\}$

$UB(B) = \{f\}$

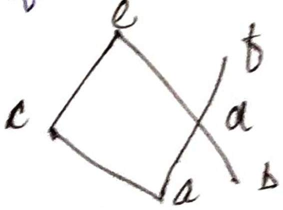
$LUB = f$

$LB(B) = \{b, c, a\}$

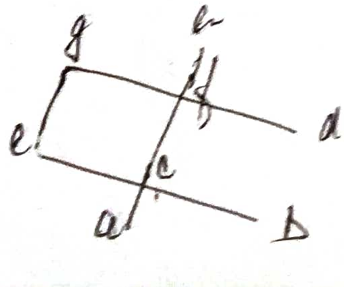
$GLB = \phi$ ∵ (single element was not found)

Join Semi Lattice (JSL):

In a poset if join exist for every pair of element then the poset is called join semi-lattice.

Q. 

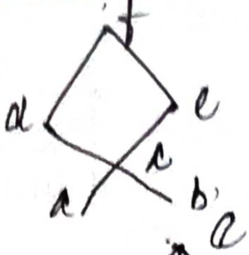
(Here e ∨ f doesn't exist then the pair is not a JSL)

Q. 

Here g ∨ h does not exist

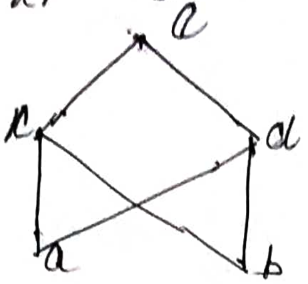
∴ not JSL

Q.



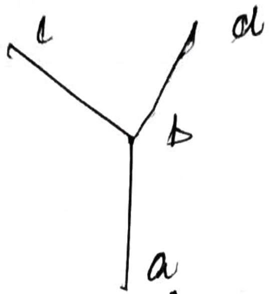
It is JSL.

Q.



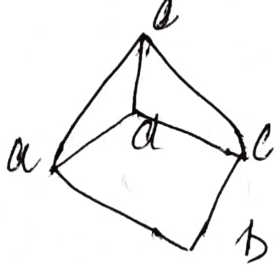
It is not a JSL.
∵ a∧b doesn't exist.

Q.



Here each element is in meet.
∴ It is MSL.

Q.



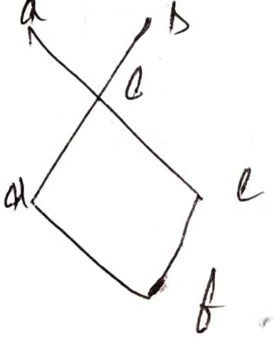
It is MSL.

LATTICE : If a Hasse Diagram (Poset) is both JSL & MSL, then it is called lattice

Represented as: [A, ∨, ∧]

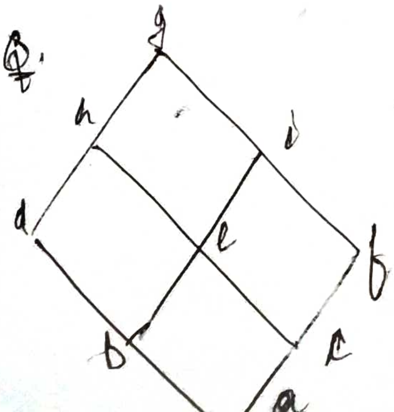
→ It has single top & single bottom.

Q.

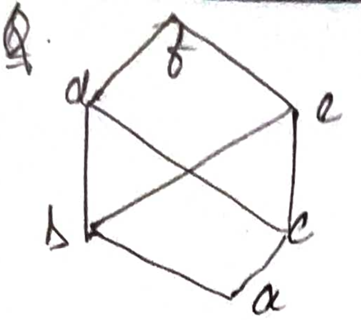


⇒ It is JSL
Not MSL ⇒ Not Lattice

Q.



⇒ It is JSL & MSL ⇒ Lattice

Q. 

$\Rightarrow$ TSL doesn't exist, MSL doesn't exist $\rightarrow$ $\boxed{\text{Not lattice}}$

$d \wedge e$, $b \vee c$

BOUNDED LATTICE : A lattice which has both the min. & max. element is called bounded lattice.

Representation:

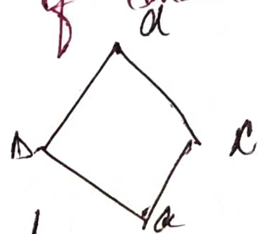
Minimum: O 0

Maximum: I 1

* Complement of element in a lattice:

In a bounded lattice L for any element $a \in L$, if there exists an element $b \in L$ such that, $a \vee b = I$ & $a \wedge b = O$ then a & b are called complement of each other.

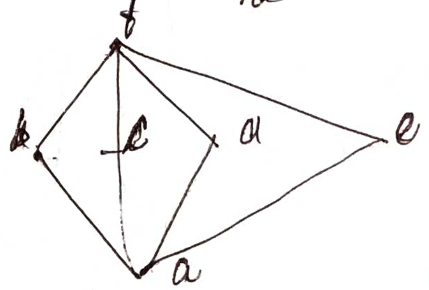
→ Max. & min. element are complement of each other.



$a^c = d$ $a \vee d = d$ (Max I)

$d^c = a$ $a \wedge d = a$ (Min. O)

(Also mention max & min separately)

Q. 

$a \in L$, $b \in L$

Max (I) = f

Min (O) = a

$a^c = f$

$b^c = c, d, e$

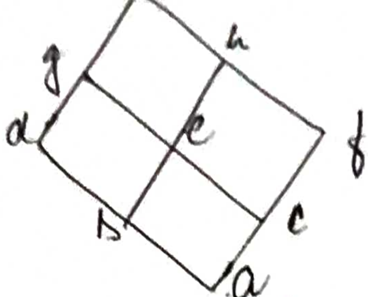
$c^c = b, d, e$

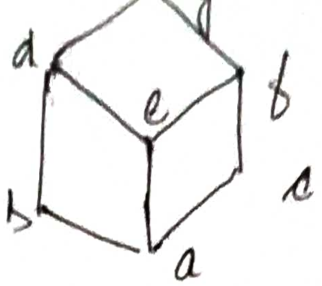
$d^c = b, c, e$

$e^c = b, d, c$

$f^c = a$

Q.





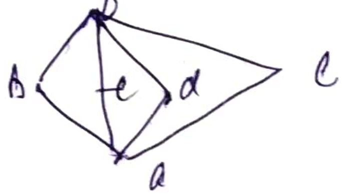
⇒ $a^c = i$ Max = i
Min = a
$b^c =$ ~~a~~ no
$c^c =$ ~~b~~ no
$d^c =$ ~~c~~ f
$e^c =$ ~~d, b, f~~ no
$f^c =$ d, ~~b~~
$g^c =$ ~~b~~ no
$h^c =$ ~~g~~ no
$i^c = a$

$a^c = g$
$b^c = c$
$c^c = b$
$d^c = f$
$e^c =$
$f^c = d$
$g^c = a$

→ 2 element पकड़ो और ऊपर जहाँ मिले वो JOIN
→ " " " नीचे " " " MEET

Complemented Lattice: Complemented a lattice L is said to be complemented, if every element a ∈ L, must have atleast 1 complement.

eg.:



Distributive/ Modular lattice: A lattice L is said to be distributive if,

∀ a, b, c ∈ L, it satisfies that

$$\boxed{a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)}$$

~~and~~ or

$$\boxed{a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)}$$

(atmost 1 complement)