1. $(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$ [Legendre's Diff

2. Bessel's Diff. Eq.

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2-n^2)y = 0$$

3. Chebyshev Diff. eq^n.

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + n^2y = 0$$

4. $x(x-2)^2y'' + 2(x-2)y' + (x+3)y = 0$

5. $2x^2y'' + xy' - (x+1)y = 0$

6. $2x^2y'' + 3xy' + (x^2-4)y = 0$

7. $3xy'' + 2y' + y = 0$

$$2a_2 - a_0 = 0 \quad \Rightarrow \quad a_2 = \frac{1}{2}a_0$$

$$6a_3 = 0 \quad \Rightarrow \quad a_3 = 0 \qquad \text{(Constant term)}$$

$$\text{(Coefficient of } x)$$

$$12a_4 + 3a_2 = 0 \quad \Rightarrow \quad a_4 = \frac{-1}{4}a_2 = -\frac{1}{8}a_0 \qquad \text{(Coefficient of } x^2)$$

$$20a_5 + 8a_3 = 0 \quad \Rightarrow \quad a_5 = -\frac{2}{5}a_3 = 0 \qquad \text{(Coefficient of } x^3)$$

So solution is
$$y = a_0\left[1 + \frac{x^2}{2} - \frac{x^4}{8} \dots\right] + a_1 x$$
Ans.

7.7 SINGULAR POINTS ABOUT $x = a$

Definition. Consider the equation

$$y'' + P(x)\,y' + Q(x)\,y = 0 \qquad \dots (1)$$

and assume that functions P and Q are not analytic ($P = \infty$ or $Q = \infty$) at $x = a$, so that $x = a$ is not *ordinary point* but a *singular point of* (1).

There are two types of singular points. (1) Regular singular point, (2) Irregular singular points.

1. Regular Singular Point:

If $(x - a)\,P$ and $(x - a)^2\,Q$ are not infinite at $x = a$, then $x = a$ is a regular singular point

2. Irregular Singular Point:

If $(x - a)\,P$ and $(x - a)^2\,Q$ are infinite at $x = a$, then $x = a$ is an irregular singular point

Example 5. *Solve the differential equation*

$$y'' + (x-1)^2\,y' - 4\,(x-1)\,y = 0$$

in series about the ordinary point $x = 1$.

Solution. Put $\qquad x = t + 1 \qquad$ (or $x - 1 = t$)

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{dy}{dt} \qquad \left(\therefore \frac{dt}{dx} = 1\right)$$

$$\Rightarrow \qquad \frac{d}{dx} \equiv \frac{d}{dt}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d^2y}{dt^2}$$

$\therefore$ The given equation becomes,

$$\frac{d^2y}{dt^2} + t^2 y' - 4t\,y = 0$$

Now, $t = 0$ is an ordinary point

Assu...

Solve the following differential equation by power series method :

1. $\dfrac{d^2y}{dx^2}+xy=0$ **Ans.** $y=a_0\left(1-\dfrac{x^3}{3!}+\dfrac{4x^6}{6!}-\dfrac{28x^9}{9!}+\ldots\right)+a_1\left(x-\dfrac{2x^4}{4!}+\dfrac{10x^7}{7!}+\ldots\right)$

2. $y''-xy'+x^2y=0$ **Ans.** $y=a_0\left(1-\dfrac{1}{12}x^4-\ldots\right)+a_1\left(x+\dfrac{1}{6}x^3-\dfrac{1}{40}x^5\ldots\right)$

3. $(x^2+1)y''+xy'-xy=0$ (U.P. (C.O.) 2008)

Ans. $y=a_0\left(1+\dfrac{x^3}{6}-\dfrac{3}{40}x^5+\ldots\right)+a_1\left(x-\dfrac{x^3}{6}+\dfrac{x^4}{12}+\dfrac{3}{40}x^5\ldots\right)$

4. $y''-2x^2y'+4xy=x^2+2x+4$

Ans. $y=a_0\left(1-\dfrac{2}{3}x^2-\dfrac{2}{45}x^6-\dfrac{2}{405}x^9\ldots\right)+a_1\left(x-\dfrac{1}{6}x^4-\dfrac{1}{63}x^7-\dfrac{1}{567}x^{10}\ldots\right)$

$+2x^2+\dfrac{1}{3}x^3+\dfrac{1}{12}x^4+\dfrac{1}{45}x^6+\dfrac{1}{126}x^7+\dfrac{1}{405}x^9+\dfrac{1}{1134}x^{10}+\ldots$

5. $(x^2+2)y''+xy'-(1+xy)=0$ **Ans.** $y=a_0\left(1+\dfrac{1}{4}x^2+\dfrac{1}{12}x^3-\dfrac{1}{32}x^4\ldots\right)+a_1\left(x+\dfrac{1}{24}x^4+\ldots\right.$

7.8 SINGULAR POINT ABOUT $x = 0$.

Consider the differential equation

$$\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=0$$

If $(x-0)P$ and $(x-0)^2Q$ are not infinite at $x=0$, then $x=0$ is a regular singular po[int]
Otherwise it is an irregular singular point.

Note: In this section we will solve those differential equation where x_0 is a regular singu[lar] point.

Example 6. *Find regular singular points of the differential equation.*

$$2x^2\frac{d^2y}{dx^2}+3x\frac{dy}{dx}+(x^2-4)y=0 \quad \ldots$$

Solution. We have,
$$\frac{d^2y}{dx^2}+\frac{3}{2x}\frac{dy}{dx}+\frac{x^2-4}{2x^2}y=0$$

$$P=\frac{3}{2x} \quad \text{and} \quad Q=\frac{x^2-4}{2x^2}$$

P and Q are not analytic (infinity) at $x = 0$. So, $x = 0$ is not ordinary point but as $(x-0)P$ and $(x-0)^2 Q$ are analytic (not infinite) so $x = 0$ is a regular singular point

$$x \cdot P = x\left(\frac{3}{2x}\right) = \frac{3}{2} \neq \infty \text{ at } x = 0.$$

$$x^2 Q = x^2 \cdot \frac{x^2-4}{2x^2} = \frac{1}{2}(x^2-4) \neq \infty \text{ at } x = 0$$

Example 7. *Find regular singular points of the differential equation:*

$$x(x-2)^2 y'' + 2(x-2)y' + (x+3)y = 0$$

Solution. Here, we have $\quad P = \dfrac{2(x-2)}{x\,(x-2)^2} = \dfrac{2}{x\,(x-2)}$ and $Q = \dfrac{x+3}{x\,(x-2)^2}$

P and Q are not analytic ($P = \infty$, $Q = \infty$) at $x = 0$ and $x = 2$.

Hence both these points are singular points of (1).

(*i*) At $x = 0$

$$xP = x \cdot \frac{2}{x\,(x-2)} = \frac{2}{x-2} \neq \infty \quad \text{at } x = 0$$

$$x^2 Q = x^2 \cdot \frac{x+3}{x\,(x-2)^2} = \frac{x\,(x+3)}{(x-2)^2} \neq \infty \quad \text{at } x = 0$$

Hence, $x\,P$ and $x^2\,Q$ are analytic ($xP \neq \infty$, $x^2 Q \neq \infty$) at $x = 0$. So $x = 0$ is a regular singular point.

(*ii*) At $x = 2$

$$(x-2)P = (x-2) \cdot \frac{2}{x(x-2)} = \frac{2}{x} \neq \infty \text{ at } x = 2$$

$$(x-2)^2 Q = (x-2)^2 \frac{(x+3)}{x(x-2)^2} = \frac{x+3}{x} \neq \infty \text{ at } x = 2.$$

Since both $(x-2)\,P$ and $(x-2)^2 Q$ are analytic ($(x-2)\,P \neq \infty$, $(x-2)^2 Q \neq 0$) at $x = 2$, $x = 2$ is a regular singular point.

The solution of a differential equation about a regular singular point can be obtained.

The cases of irregular singular points are beyond the scope of this book.

7.9 FROBENIUS METHOD

If $x = 0$ is a regular singularity of the equation:

$$\frac{d^2 y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)\,y = 0$$

then the series solution is

$[P_0(0) = 0]$ ...(1)