Case - III $\quad F(x,y) = \phi(ax+by)$

P.I. $\quad \dfrac{1}{F(D,D')} F(x,y) = \dfrac{1}{F(a,b)} \iiint \cdots \int \phi(u)\, du\, du \cdots du$

where $ax+by=u$

eg: $(D^2 + 3DD' + 2D'^2)z = x+y$

C.F. $\quad f_1(y-x) + f_2(y-2x)$

P.I. $\quad \dfrac{(x+y)}{(D^2+3DD'+2D'^2)} = \dfrac{1}{1+3(1)(1)+2(1)^2} \iint u\, du\, du$

$$= \frac{1}{6}\,\frac{u^3}{6} = \frac{(x+y)^3}{36}$$

$z = $ C.F. + P.I.

eg: $(D^2 + DD' - 6D'^2)z = x+y$ $\qquad (-3, 2)$

P.I: $\quad \dfrac{x+y}{(D^2+DD'-6D'^2)} = \dfrac{1}{1+1-6} \iint u\, du\, du$

$$= -\frac{u^3}{24} = -\frac{(x+y)^3}{24}$$

$z = $ C.f. + P.I.

eg. $(D^2 - D'^2)z = x-y$

P.I. $\quad \dfrac{x-y}{(D-D')(D+D')} = x\,\dfrac{1}{2D}(x-y)$

$$= \frac{x}{2}\int u\, du = \frac{x}{2}\cdot\frac{u^2}{2}$$

$$= \frac{x}{4}(x-y)^2$$

$z = $ C.F. + P.I.

A.E. is $m^2 - 1 = 0$

$\Rightarrow$ $$m^2 = 1 \Rightarrow m = 1, -1$$

$\therefore$ $$\text{C.F.} = f_1(y + x) + f_2(y - x)$$

$$\text{P.I.} = \frac{1}{D^2 - D'^2}(x - y)$$

$$\text{P.I.} = \frac{1}{(D - D')(D + D')}(x - y)$$

$$= x\frac{1}{D - D'}(x - y) \qquad \text{(case of failure)}$$

$$= x\frac{1}{1-(-1)}\int u\,du \qquad \text{where } u = x - y$$

$$= \frac{x}{2}\frac{u^2}{2}$$

$$= \frac{x}{4}(x - y)^2$$

Complete solution is $z = \text{C.F.} + \text{P.I.} = f_1(y + x) + f_2(y - x) + \frac{x}{4}(x - y)^2$ **Ans.**

Example 17. *Solve :* $(D^2 + 2DD' - 8D'^2)z = \sqrt{2x + 3y}$

Solution. Here, we have

$$(D^2 + 2DD' - 8D'^2)z = \sqrt{2x + 3y}$$

A.E. is $m^2 + 2m - 8 = 0 \Rightarrow (m + 4)(m - 2) = 0 \Rightarrow m = 2, m = -4$

$$C.F. = f_1(y + 2x) + f_2(y - 4x)$$

$$P.I. = \frac{1}{D^2 + 2DD' - 8D'^2}\sqrt{2x + 3y}$$

$$= \frac{1}{D^2 + 2DD' - 8D'^2}(2x + 3y)^{\frac{1}{2}}$$

$$= \frac{1}{(2)^2 + 2(2)(3) - 8(3)^2}\iint u^{\frac{1}{2}}du\,du, \text{ where } u = 2x + 3y$$

$$= \frac{1}{-56}\frac{u^{5/2}}{\frac{3}{2}\cdot\frac{5}{2}}$$

$$= -\frac{1}{56}\left[\frac{4}{15}(2x + 3y)^{\frac{5}{2}}\right]$$

$$= -\frac{1}{210}(2x + 3y)^{\frac{5}{2}}$$

Hence, the complete solution = *C.F.* + *P.I.*

$$= f_1(y + 2x) + f_2(y - 4x) - \frac{1}{210}(2x + 3y)^{\frac{5}{2}}$$ **Ans.**

Example 18. *Solve* $(D^3 - 3D^2D' - 4DD'^2 + 12D'^3)z = \sin(y + 2x)$.

Solution. Here, we have

$$(D^3 - 3D^2D' - 4DD'^2 + 12D'^3)z = 0 \qquad \ldots(1)$$

Putting $D = m$ and $D' = 1$ in (1) ; we get

A.E. is $m^3 - 3m^2 - 4m + 12 = 0$

$$\Rightarrow \quad m^2(m-3) - 4(m-3) = 0$$

$$\Rightarrow \quad (m^2 - 4)(m - 3) = 0 \Rightarrow m = \pm 2, 3$$

$$\therefore \quad \text{C.F.} = f_1(y + 2x) + f_2(y - 2x) + f_3(y + 3x)$$

$$P.I. = \frac{1}{D^3 - 3D^2D' - 4DD'^2 + 12D'^3}\sin(y + 2x)$$

$$= \frac{1}{2^3 - 3(2)^2(1) - 4(2)(1)^2 + 12(1)^3}\iiint \sin u\, du\, du\, du \qquad \begin{pmatrix}\text{where } u = y + 2x\\ \text{case of failure}\end{pmatrix}$$

$$= x\frac{1}{3D^2 - 6DD' - 4D'^2}\iint \sin u\, du\, du$$

$$= \frac{x}{3(2)^2 - 6(2)(1) - 4(1)^2}(-\sin u) = \frac{x}{4}\sin(y + 2x)$$

Complete solution is C.F. + P.I. $z = f_1(y + 2x) + f_2(y - 2x) + f_3(y + 3x) + \frac{x}{4}\sin(y + 2x)$

Ans.

Example 19. *Solve* : $(4D^2 - 4DD' + D'^2)z = 16\log(x + 2y)$

Solution. Here, we have

$$(4D^2 - 4DD' + D'^2)z = 16\log(x + 2y)$$

Auxilliary equation is

$$4m^2 - 4m + 1 = 0$$

$$\Rightarrow \quad (2m - 1)^2 = 0 \quad \Rightarrow \quad m = \frac{1}{2}, \frac{1}{2}$$

$$C.F. = f_1\left(y + \frac{x}{2}\right) + x\, f_2\left(y + \frac{x}{2}\right)$$

$$\text{P.I.} = \frac{1}{4D^2 - 4DD' + D'^2}16\log(x + 2y)$$

$$= \frac{1}{4(1)^2 - 4(1)(2) + (2)^2}16\iint \log u\, du\, du, \text{ where } u = x + 2y \quad \text{(case of failur}$$

$$= x\frac{1}{8D - 4D'}16\int \log u\, du \qquad \text{(case of fail}$$

$$= x\frac{1}{8(1) - 4(2)}16\log u$$

$$= 16x^2\left(\frac{1}{8}\right)\log u = 16\frac{x^2}{8}\log(x + 2y) = 2x^2\log(x + 2y)$$

The complete solution $= C.F. + P.I.$

$$= f_1\left(y + \frac{x}{2}\right) + xf_2\left(y + \frac{x}{2}\right) + 2x^2\log(x + 2y)$$

EXERCISE 14.4

Solve the following equations

1. $\dfrac{\partial^2 z}{\partial x^2} - \dfrac{\partial^2 z}{\partial y^2} = x - y$ **Ans.** $z = f_1(y-x) + f_2(y+x) + \dfrac{x^3}{6} - \dfrac{x^2 y}{2}$

2. $\dfrac{\partial^2 z}{\partial x^2} + 3\dfrac{\partial^2 z}{\partial x \partial y} - 4\dfrac{\partial^2 z}{\partial y^2} = x + \sin y$ **Ans.** $z = f_1(y+x) + f_2(y-4x) + \dfrac{x^3}{6} + \dfrac{1}{4}\sin y$

3. $\dfrac{\partial^3 z}{\partial x^2 \partial y} - 2\dfrac{\partial^3 z}{\partial x \partial y^2} + \dfrac{\partial^3 y}{\partial y^3} = \dfrac{1}{x^2}$ **Ans.** $z = f_1(x) + f_2(y+x) + xf_3(y+x) - y \log x$

4. $(D^3 - 4D^2D' + 5DD'^2 - 2D'^3)z = e^{y+2x} + (y+x)^{\frac{1}{2}}$

Ans. $z = f_1(y+x) + xf_2(y+x) + f_3(y+2x) + xe^{y+x} - \dfrac{x^2}{3}(y+x)^{\frac{3}{2}}$

5. $\dfrac{\partial^3 z}{\partial x^3} - 4\dfrac{\partial^3 z}{\partial x^2 \partial y} + 4\dfrac{\partial^3 z}{\partial x \partial y^2} = 2\sin(3x + 2y)$ **Ans.** $z = f_1(y+2x) + f_2(2y+x) - \dfrac{5}{3}x\cos(2x+y)$

6. $\dfrac{\partial^2 z}{\partial x^2} - 4\dfrac{\partial^2 z}{\partial x \partial y} + 3\dfrac{\partial^2 z}{\partial y^2} = \sqrt{x+3y}$ **Ans.** $z = f_1(y+x) + f_2(y+3x) + \dfrac{1}{60}(x+3y)^{\frac{5}{2}}$

Case IV. *When* $F(x, y) = x^m y^n$

$$\boxed{P.I. = \frac{1}{f(D, D')} x^m y^n = [f(D, D')]^{-1} x^m y^n}$$

(*a*) If $m > n$, then $\dfrac{1}{f(D, D')}$ is expanded in the powers of $\dfrac{D'}{D}$.

(*b*) If $m < n$, then $\dfrac{1}{f(D, D')}$ is expanded in the powers of $\dfrac{D}{D'}$.

Example 20. *Solve* : $(D^2 + D'^2)z = x^2y^2$

Solution. Here, we have

$$(D^2 + D'^2)z = x^2y^2$$

Putting $D = m$ and $D' = 1$, we get the A.E. as

$$m^2 + 1 = 0 \quad \Rightarrow \quad m^2 = -1 \quad \Rightarrow \quad m = \pm i.$$

$\therefore \quad C.F. = f_1(y + ix) + f_2(y - ix)$

$$P.I. = \frac{1}{D^2 + D'^2}(x^2y^2) = \frac{1}{D^2} \cdot \frac{1}{\left(1 + \dfrac{D'^2}{D^2}\right)}(x^2y^2)$$

$$= \frac{1}{D^2}\left(1 + \frac{D'^2}{D^2}\right)^{-1}(x^2y^2)$$

$$= \frac{1}{D^2}\left(1 - \frac{D'^2}{D^2}\right)(x^2y^2)$$

$$= \frac{1}{D^2}(x^2y^2) - \frac{D'^2}{D^4}(x^2y^2)$$

Case IV $\quad \frac{1}{f(D,D')} x^m y^n = [f(D,D')]^{-1} x^m y^n$

(a) If $m > n$, then $\frac{1}{f(D,D')}$ is expanded in power of D'/D.

(b) If $m < n$, then $\frac{1}{f(D,D')}$ is expanded in power of D/D'.

eg. $(D^2 + D'^2)y = x^2y^2$

A.E. $\quad m^2 + 1 = 0$

$m = \pm i$

$C.F. = f_1(y+ix) + f_2(y-ix)$

P.I. $\quad \frac{1}{D^2+D'^2}(x^2y^2)$

$= \frac{1}{D^2}\left[1 + \frac{D'^2}{D^2}\right]^{-1} x^2y^2$

$= \frac{1}{D^2}\left[1 - \frac{D'^2}{D^2} \cdots\right] x^2y^2$

$= \frac{1}{D^2} x^2y^2 - \frac{D'^2}{D^4}(x^2y^2)$

$= \frac{x^4}{12}y^2 - \frac{2x^2}{D^4} = \frac{1}{180}(15x^4y^2 - x^6)$

$Z = C.F. + P.I.$

eg. $(D^3 - D'^3)Z = x^3y^3$

P.I. $\quad \frac{x^6y^3}{120} + \frac{x^9}{10080}$

$Z = C.F. + P.I$

$$= \frac{x^4}{12} y^2 - \frac{1}{D^4}(2x^2)$$

$$= \frac{x^4}{12} y^2 - 2 \cdot \frac{x^6}{3 \cdot 4 \cdot 5 \cdot 6}$$

$$= \frac{1}{180}(15x^4 y^2 - x^6)$$

Thus, the complete solution is

$$z = f_1(y + ix) + f_2(y - ix) + \frac{1}{180}(15x^4 y^2 - x^6)$$

An

Example 21. *Solve* : $\frac{\partial^3 z}{\partial x^3} - \frac{\partial^3 z}{\partial y^3} = x^3 y^3$

Solution. Here, we have (*Q. Bank U.P.* 2002

$$\frac{\partial^3 z}{\partial x^3} - \frac{\partial^3 z}{\partial y^3} = x^3 y^3$$

$$\Rightarrow \quad (D^3 - D'^3)z = x^3 y^3$$

Putting $D = m$ and $D' = 1$ in above, we have

$A.E.$ is $m^3 - 1 = 0 \Rightarrow m = 1, w, w^2$

Where w is one of the cube roots of unity.

$$\therefore \quad C.F. = f_1(y + x) + f_2(y + wx) + f_3(y + w^2 x)$$

$$P.I. = \frac{1}{D^3 - D'^3}(x^3 y^3) = \frac{1}{D^3\left(1 - \frac{D'^3}{D^3}\right)} x^3 y^3$$

$$= \frac{1}{D^3} \cdot \left(1 - \frac{D'^3}{D^3}\right)^{-1} (x^3 y^3) = \frac{1}{D^3}\left(1 + \frac{D'^3}{D^3}\right)(x^3 y^3)$$

$$= \frac{1}{D^3}\left[x^3 y^3 + \frac{1}{D^3} D'^3 (x^3 y^3)\right] = \frac{1}{D^3}\left[x^3 y^3 + \frac{1}{D^3}(6x^3)\right]$$

$$= \frac{1}{D^3}(x^3 y^3) + \frac{1}{D^6}(6x^3) = \frac{x^6 y^3}{6 \cdot 5 \cdot 4} + \frac{6x^9}{4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9}$$

$$= \frac{x^6 y^3}{120} + \frac{x^9}{10080}$$

Hence, the complete solution is

$$z = C.F. + P.I. = f_1(y + x) + f_2(y + wx) + f_3(y + w^2 x) + \frac{x^6 y^3}{120} + \frac{x^9}{10080}$$

A

Example 22. *Solve* $\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = x^2 + xy + y^2$

Solution. Here, we have

$$\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = x^2 + xy + y^2$$

A.E. is $m^2 + 2m + 1 = 0 \Rightarrow (m+1)^2 = 0 \Rightarrow m = -1, -1$

$$C.F. = f_1(y-x) + xf_2(y-x)$$

$$P.I. = \frac{1}{D^2 + 2DD' + D'^2}(x^2 + xy + y^2)$$

$$= \frac{1}{D^2}\frac{1}{\left(1 + \frac{2D'}{D} + \frac{D'^2}{D^2}\right)}(x^2 + xy + y^2)$$

$$= \frac{1}{D^2}\left(1 + \frac{2D'}{D} + \frac{D'^2}{D^2}\right)^{-1}(x^2 + xy + y^2)$$

$$= \frac{1}{D^2}\left(1 - \frac{2D'}{D} - \frac{D'^2}{D^2} + \frac{4D'^2}{D^2} + \ldots\right)(x^2 + xy + y^2)$$

$$= \left(\frac{1}{D^2} - \frac{2D'}{D^3} + \frac{3D'^2}{D^4}\right)(x^2 + xy + y^2)$$

$$= \frac{1}{D^2}(x^2 + xy + y^2) - \frac{2D'}{D^3}(x^2 + xy + y^2) + \frac{3D'^2}{D^4}(x^2 + xy + y^2)$$

$$= \left(\frac{x^4}{12} + \frac{x^3y}{6} + \frac{x^2y^2}{2}\right) - \left(\frac{2}{D^3}\right)(x + 2y) + \frac{3}{D^4}(2)$$

$$= \frac{x^4}{12} + \frac{x^3y}{6} + \frac{x^2y^2}{2} - \frac{x^4}{12} - \frac{2x^3y}{3} + \frac{6x^4}{2\cdot 3\cdot 4}$$

$$= \frac{x^4}{12} + \frac{x^3y}{6} + \frac{x^2y^2}{2} - \frac{x^4}{12} - \frac{2x^3y}{3} + \frac{x^4}{4}$$

$$= \frac{x^4}{4} - \frac{1}{2}x^3y + \frac{x^2y^2}{2}$$

Hence, the complete solution is

$$z = f_1(y-x) + xf_2(y-x) + \frac{x^4}{4} - \frac{x^3y}{2} + \frac{x^2y^2}{2}$$

Example 23. *Solve* $\frac{\partial^3 z}{\partial x^3} - 2\frac{\partial^3 z}{\partial x^2 \partial y} = 2e^{2x} + 3x^2y$

Solution. We have, $\frac{\partial^3 z}{\partial x^3} - 2\frac{\partial^3 z}{\partial x^2 \partial y} = 2e^{2x} + 3x^2y$

$$(D^3 - 2D^2D')z = 2e^{2x} + 3x^2y$$

Its auxiliary equation is

$$m^3 - 2m^2 = 0$$

$$m^2(m-2) = 0$$

$$m = 0, 0, 2.$$

$$\text{C.F.} = f_1(y) + x\,f_2(y) + f_3(y+2x)$$

$$\text{P.I.} = \frac{1}{D^3 - 2D^2D'}(2e^{2x} + 3x^2y)$$

$$= \frac{1}{D^3 - 2D^2D'} 2e^{2x} + \frac{1}{D^3 - 2D^2D'} 3x^2y$$

$$= 2\frac{e^{2x}}{(2)^3 - 2(2)^2(0)} + 3\frac{1}{D^3\left(1 - \frac{2D'}{D}\right)} x^2y = \frac{2e^{2x}}{8} + \frac{3}{D^3}(1 - \frac{2D'}{D})^{-1} x^2y$$

$$= \frac{e^{2x}}{4} + \frac{3}{D^3}\left(1 + \frac{2D'}{D} \ldots\right)x^2y = \frac{e^{2x}}{4} + \frac{3}{D^3}\left[x^2y + \frac{2}{D}x^2\right] = \frac{e^{2x}}{4} + \frac{3}{D^3}\left(x^2y + \frac{2x^3}{3}\right)$$

$$= \frac{e^{2x}}{4} + 3y\frac{1}{D^3}x^2 + \frac{2}{D^3}x^3 = \frac{e^{2x}}{4} + 3y\frac{x^5}{3.4.5} + 2\frac{x^6}{4.5.6} = \frac{e^{2x}}{4} + \frac{x^5y}{20} + \frac{x^6}{60}$$

$$= \frac{1}{60}(15e^{2x} + 3x^5y + x^6)$$

Hence, the complete solution is

$$z = f_1(y) + x f_2(y) + f_3(y + 2x) + \frac{1}{60}(15e^{2x} + 3x^5y + x^6)$$ **Ans.**

Example 24. *Solve* $(D^2 + 4D'^2)z = x^2y^4$

Solution. Here, we have

$$(D^2 + 4D'^2)z = x^2y^4$$

A.E. is $m^2 + 4 = 0 \quad \Rightarrow \quad (m + 2i)(m - 2i) = 0 \quad \Rightarrow \quad m = 2i, -2i$

$$C.F. = f_1(y + 2ix) + f_2(y - 2ix)$$

$$P.I. = \frac{1}{D^2 + 4D'^2} x^2y^4$$

$$= \frac{1}{4D'^2\left(1 + \frac{D^2}{4D'^2}\right)} x^2y^4 \qquad \left(\begin{array}{l}\text{Here the power of } y > \\ \text{the power of } x \text{ in } x^2y^4 \\ \text{so we take } 4D'^2 \text{ as common}\end{array}\right)$$

$$= \frac{1}{4D'^2}\left[1 + \frac{D^2}{4D'^2}\right]^{-1} x^2y^4$$

$$= \frac{1}{4D'^2}\left[1 - \frac{D^2}{4D'^2} + \ldots\ldots\right] x^2y^4$$

$$= \frac{1}{4D'^2}\left[x^2y^4 - \frac{D^2}{4D'^2}(x^2y^4) + \ldots\ldots\right]$$

$$= \frac{1}{4D'^2}\left[x^2y^4 - \frac{2}{4D'^2}(y^4) + \ldots\ldots\right]$$

$$= \frac{1}{4D'^2}\left[x^2y^4 - \frac{1}{2}\frac{y^6}{5 \times 6}\right]$$

$$= \frac{1}{4}\left[\frac{x^2y^6}{5 \times 6} - \frac{1}{2}\frac{y^8}{5 \times 6 \times 7 \times 8}\right]$$

$$= \frac{1}{4 \times 5 \times 6 \times 14 \times 8}[14 \times 8\, x^2y^6 - y^8]$$

$$= \frac{1}{13440}[112\,x^2y^6 - y^8]$$

The complete solution $= C.F. + P.I.$

$$= f_1(y+2ix) + f_2(y-2ix) + \frac{1}{13440}(112\,x^2y^6 - y^8)$$ **Ans.**

EXERCISE 14.5

Solve the following equations :

1. $\dfrac{\partial^2 z}{\partial x^2} + 3\dfrac{\partial^2 z}{\partial x\,\partial y} + 2\dfrac{\partial^2 z}{\partial y^2} = 12xy$ *(A.M.I.E., Winter 2001)*

   Ans. $z = f_1(y-x) + f_2(y-2x) + 2x^3y - \dfrac{3x^4}{2}$

2. $\dfrac{\partial^2 z}{\partial x^2} - \dfrac{\partial^2 z}{\partial x\partial y} - 6\dfrac{\partial^2 z}{dy^2} = xy$ **Ans.** $z = f_1(y-2x) + f_2(y+3x) + \dfrac{x^3y}{6} + \dfrac{x^4}{24}$

3. $(D^3 - 3D^2D')z = x^2y$ **Ans.** $z = \phi_1(y+x) + \phi_2(y-x) + \dfrac{1}{12}e^{2x-y} - xe^{x+y} - \dfrac{1}{3}\cos(x+2y)$

4. $\dfrac{\partial^3 z}{\partial x^3} - 2\dfrac{\partial^3 z}{\partial x^2\partial y} = 2e^{2x} + 3x^2y$ **Ans.** $z = f_1(y) + xf_2(y) + f_3(y+2x) + \dfrac{1}{60}(15e^{2x} + 3x^5y + x^6)$

5. $(D^2 - 6DD' + 9D'^2)z = 12x^2 + 36xy$ **Ans.** $z = f_1(y+3x) + xf_2(y+3x) + 6x^3y + 10x^4$

6. $(D^2 - D'^2 + D + 3D' - 2)z = e^{x-y} - x^2y$

   Ans. $z = e^{-2x}f_1(y+x) + e^x f_2(y-x) - \dfrac{1}{4}e^{x-y} + \dfrac{1}{2}\left(x^2y + xy + \dfrac{3}{2}x^2 + \dfrac{3y}{2} + 3x + \dfrac{21}{4}\right)$

14.5 P.I. OF ANY FUNCTION

If the function on the R.H.S. of the P.D.E. is not of the form, given in previous cases. Then

$$\text{P.I.} = \frac{1}{F(D, D')}\phi(x, y)$$

$F(D, D')$ is factorized to get

$$F(D, D') = (D - m_1D')\,(D - m_2D')\,......\,(D - m_nD')$$

$$\text{P.I.} = \frac{1}{(D - m_1D)\,(D - m_2D')\ldots(D - m_nD')}\phi(x, y)$$

Let us consider

$$\text{P.I.} = \frac{1}{D - m_1D'}\phi\,(x, y) \qquad \text{(Taking only one term)}$$

$$\Rightarrow \qquad p - m_1q = \phi\,(x, y)$$

Subsidiary equations are (Lagrange's equations)

$$\frac{dx}{1} = \frac{dy}{-m_1} = \frac{dz}{\phi(x, y)}$$