CHAPTER 14 LINEAR PARTIAL DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS OF 2ND ORDER

14.1 LINEAR HOMOGENEOUS PARTIAL DIFFERENTIAL EQUATIONS OF nTH ORDER WITH CONSTANT COEFFICIENTS

An equation of the type

$$a_0 \frac{\partial^n z}{\partial x^n} + a_1 \frac{\partial^n z}{\partial x^{n-1} \partial y} + ... + a_n \frac{\partial^n z}{\partial y^n} = F(x, y) \qquad ... (1)$$

is called a homogeneous linear partial differential equation of nth order with constant coefficients. It is called homogeneous because all the terms contain derivatives of the same order.

Putting $\frac{\partial}{\partial x} = D$ and $\frac{\partial}{\partial y} = D'$ in (1), we get

$$(a_0 D^n + a_1 D^{n-1} D' + ... + a_n D'^n) z = F(x, y)$$

$$\Rightarrow \qquad f(D, D') z = F(x, y)$$

14.2 RULES FOR FINDING THE COMPLEMENTARY FUNCTION

Consider the equation

$$a_0 \frac{\partial^2 z}{\partial x^2} + a_1 \frac{\partial^2 z}{\partial x \partial y} + a_2 \frac{\partial^2 z}{\partial y^2} = 0 \Rightarrow \qquad (a_0 D^2 + a_1 DD' + a_2 D'^2) z = 0$$

Step 1 : Put $D = m$ and $D' = 1$

$$a_0 m^2 + a_1 m + a_2 = 0$$

This is the auxiliary equation.

Step 2 : Solve the auxiliary equation.

Case 1. If the roots of the auxiliary equation are real and different; say m_1, m_2.

Then $\qquad \text{C.F.} = f_1(y + m_1 x) + f_2(y + m_2 x)$ ...

Theory: $(D - m_1 D')(D - m_2 D') z = 0$

(1) will be satisfied by the solution of

$$(D - m_2 D') z = 0 \quad \Rightarrow \quad p - m_2 q = 0$$

This is a Lagrange's linear equation. Its subsidiary equations are

$$\frac{dx}{1} = \frac{dy}{-m_2} = \frac{dz}{0} \Rightarrow y + m_2 x = C_1 \text{ and } z = C_2$$

$\therefore$ Solution of (2) is $\qquad z = f_2(y + m_2 x)$

Similarly the solution of $(D - m_1 D')z = 0$ is

$$z = f_1(y + m_1 x)$$

Hence the complete solution of (1) is

$$z = f_1(y + m_1 x) + f_2(y + m_2 x)$$

Case 2. If the roots are equal; say m

Then $\qquad C.F. = f_1(y + mx) + x f_2(y + mx)$

Theory:

$$(D - m D')(D - m D')z = 0 \qquad ...(1)$$

Let $\qquad (D - mD')z = u \qquad ...(2)$

(1) becomes $\qquad (D - m D')u = 0 \qquad ...(3)$

Solution of (3) is $\qquad u = f(y + mx)$

(2) becomes $\qquad (D - mD')z = f(y + mx) \quad \Rightarrow p - mq = f(y + mx)$

This is Lagrange's equation and its subsidiary equations are

$$\underset{(i)}{\frac{dx}{1}} = \underset{(ii)}{\frac{dy}{-m}} = \underset{(iii)}{\frac{dz}{f(y + mx)}}$$

From (*i*) and (*ii*), $y + mx = C_1$ and $dz = f(y + mx)\,dx$

$$dz = f(C_1)\,dx \quad \Rightarrow \quad z = f(y + mx).x + f_1(y + mx)$$

S. No.	Roots of A.E.	C.F.
1.	m_1, m_2, m_3 (different)	$f_1(y + m_1 x) + f_2(y + m_2 x) + f_3(y + m_3 x)$
2.	$m_1, m_2, m_3 \begin{bmatrix} m_2 = m_1 \\ m_3 \neq m_1 \end{bmatrix}$	$f_1(y + m_1 x) + x\, f_2(y + m_1 x) + f_3(y + m_3 x)$
3.	m_1, m_2, m_3 $m_1 = m_2 = m_3$	$f_1(y + m_1 x) + x\, f_2(y + m_1 x) + x^2 f_3(y + m_1 x)$

Example 1. *Solve* $\quad (D^3 - 4D^2 D' + 3DD'^2)z = 0.$

Solution. $\qquad (D^3 - 4D^2 D' + 3DD'^2)z = 0. \qquad [D = m,\ D' = 1]$

Its auxiliary equation is $m^3 - 4m^2 + 3m = 0$

$$m(m^2 - 4m + 3) = 0$$

$$m(m - 1)(m - 3) = 0 \qquad \Rightarrow \qquad m = 0, 1, 3$$

The required solution is

$$z = f_1(y) + f_2(y + x) + f_3(y + 3x) \qquad \text{Ans.}$$

Example 2. *Solve* $\quad \frac{\partial^2 z}{\partial x^2} - 4\frac{\partial^2 z}{\partial x \partial y} + 4\frac{\partial^2 z}{\partial y^2} = 0.$

Solution. $(D^2 - 4DD' + 4D'^2)z = 0$

Its auxiliary equation is $m^2 - 4m + 4 = 0$

$\Rightarrow$ $(m-2)^2 = 0 \Rightarrow m = 2, 2$ $[D = m, D' = 1]$

The required solution is

$$z = f_1(y+2x) + x\, f_2(y+2x)$$ **Ans.**

Example 3. *Solve the linear partial differential equation*

$$\frac{\partial^4 z}{\partial x^4} - 2\frac{\partial^4 z}{\partial x^3 \partial y} + 2\frac{\partial^4 z}{\partial x \partial y^3} - \frac{\partial^4 z}{\partial y^4} = 0.$$

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Solution. Here, we have

$$(D^4 - 2D^3D' + 2DD'^3 - D'^4)\, z = 0, \text{ where } D \equiv \frac{\partial}{\partial x} \text{ and } D' \equiv \frac{\partial}{\partial y}$$

Auxiliary equation is

$$m^4 - 2m^3 + 2m - 1 = 0$$

$$m^3(m-1) - m^2(m-1) - m(m-1) + 1(m-1) = 0$$

$\Rightarrow$ $(m^3 - m^2 - m + 1)(m-1) = 0$

$\Rightarrow$ $(m+1)(m-1)^3 = 0$

$\Rightarrow$ $m = -1, 1, 1, 1$

$\therefore$ $z = f_1(y-x) + f_2(y+x) + x f_3(y+x) + x^2 f_4(y+x)$ **Ans.**

Example 4. *Solve the linear partial differential equation* $\frac{\partial^4 z}{\partial x^4} + \frac{\partial^4 z}{\partial y^4} = 0.$

Solution. $(D^4 + D'^4)\, z = 0$

Auxiliary equation is $m^4 + 1 = 0$

$\Rightarrow$ $m^4 + 1 + 2m^2 = 2m^2$

$\Rightarrow$ $(m^2+1)^2 - (m\sqrt{2})^2 = 0$

$\Rightarrow$ $(m^2 + \sqrt{2}m + 1)(m^2 - \sqrt{2}m + 1) = 0$

so that $m^2 + \sqrt{2}m + 1 = 0$ or $m^2 - \sqrt{2}m + 1 = 0$

$\Rightarrow$ $$m = \frac{-1+i}{\sqrt{2}}, \frac{1+i}{\sqrt{2}}, \frac{-1-i}{\sqrt{2}}, \frac{1-i}{\sqrt{2}},$$

Hence $$z = f_1\left\{y + \left(\frac{-1+i}{\sqrt{2}}\right)x\right\} + f_2\left\{y + \left(\frac{1+i}{\sqrt{2}}\right)x\right\}$$

$$+ f_3\left\{y + \left(\frac{-1-i}{\sqrt{2}}\right)x\right\} + f_4\left\{y + \left(\frac{1-i}{\sqrt{2}}\right)x\right\}$$ **Ans.**

EXERCISE 14.1

Solve the following equations:

1. $\frac{\partial^2 z}{\partial x^2} + \frac{4\partial^2 z}{\partial x \partial y} - 5\frac{\partial^2 z}{\partial y^2} = 0$ **Ans.** $z = f_1(y+x) + f_2(y-5x)$

2. $2\frac{\partial^2 z}{\partial x^2} + 5\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = 0$ **Ans.** $z = f_1(2y-x) + f_2(y-2x$

3. $(D^3 - 6D^2D' + 11DD'^2 - 6D'^3)z = 0$ **Ans.** $z = f_1(y+x) + f_2(y+2x) + f_3(y+3x)$

4. $\dfrac{\partial^2 z}{\partial x^2} - 2\dfrac{\partial^2 z}{\partial x \partial y} + \dfrac{\partial^2 z}{\partial y^2} = 0$ **Ans.** $z = f_1(y+x) + x f_2(y+x)$

5. $(D^3 - 6D^2D' + 12DD'^2 - 8D'^3)z = 0$ **Ans.** $z = f_1(y+2x) + x f_2(y+2x) + x^2 f_3(y+2x)$

6. $\dfrac{\partial^4 z}{\partial x^4} - \dfrac{\partial^4 z}{\partial y^4} = 0$ **Ans.** $z = f_1(y+x) + f_2(y-x) + f_3(y+ix) + f_4(y-ix)$

7. $\dfrac{\partial^3 z}{\partial x^3} - 4\dfrac{\partial^3 z}{\partial x^2 \partial y} + 4\dfrac{\partial^3 z}{\partial x \partial y^2} = 0$ **Ans.** $z = f_1(y) + f_2(y+2x) + xf_3(y+2x)$

8. $\dfrac{\partial^3 z}{\partial x^3} - 7\dfrac{\partial^3 z}{\partial x \partial y^2} + 6\dfrac{\partial^3 z}{\partial y^3} = 0$ **Ans.** $z = f_1(y+x) + f_2(y+2x) + f_3(y-3x)$

9. $\dfrac{\partial^4 z}{\partial x^4} + \dfrac{\partial^4 z}{\partial y^4} = 2\dfrac{\partial^4 z}{\partial x^2 \partial y^2}$ **Ans.** $z = f_1(y-x) + x f_2(y-2x) + f_3(y+x) + xf_4(y+x)$

10. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$, when $u = \sin y$, $x = 0$ for all y and $u \to 0$ when $x \to \infty$. *(A.M.I.E., Summer 2000)*

14.3 GENERAL RULES FOR FINDING THE PARTICULAR INTEGRAL

Given partial differential equation is

$$f(D, D')\, z = F(x, y)$$

$$\text{P.I.} = \frac{1}{f(D, D')} F(x, y)$$

If $f(D, D')$ is a homogeneous function of D and D′ of degree n and R.H.S. function $\phi(ax+by)$, $e^{(ax+by)}$, $ax+by$, $\sin(ax+by)$. Then the particular intergral

$$\text{P.I.} = \frac{1}{F(D, D')} \phi(ax+by)$$

P.I. of $\dfrac{1}{F(D, D')} F(x, y) = \dfrac{1}{F(a, b)} \iiint \cdots \int \phi(u)\, du\, du \ldots du$ (n times), where $u = ax + by$

GENERAL RULE

Integrate $\phi(u)$ w.r.t. u, n times and after integration replace u by $ax + by$.

Case of failure:

To find P.I.

The given equation is

$$F(D, D')\, z = \phi(ax+by) \text{ and } F(a, b) = 0$$

Procedure: Let $F(D, D')$ is a homogeneous function of degree n.

Differentiating $F(D, D')$ partially w.r.t D and multiply L.H.S by x, we g

$$x \frac{1}{\dfrac{\partial}{\partial D} F(D, D')} \phi(ax+by)$$

If F (a, b) is again zero.

Differentiate it second time and multiply by x to get $x^2 \dfrac{1}{\dfrac{\partial^2}{\partial D^2} F(D, D')} \phi(ax+by)$