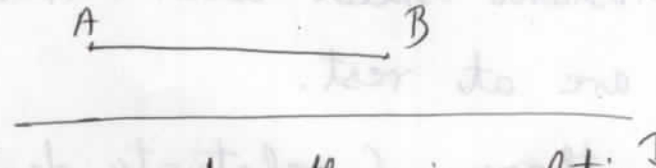
Relativistic Mechanics

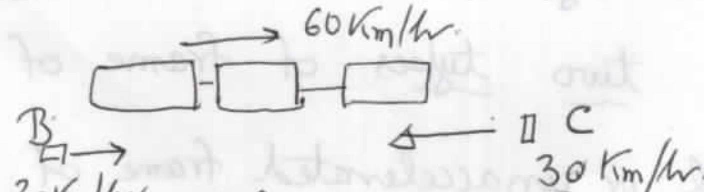
The three fundamental quantity of mechanics length, mass & time are absolute according to Newtonian mech. But acc. to relativistic mech. they all are relative & not absolute.



Length is relative.

What is the time now?

The answer depends upon the location because the time now in India is different from the time in America. Thus time is relative.



The vel. of train would be different to diff. observers. A will observe speed of train = 60 Km/hr.

B → 60 − 20 = 40 Km/hr & C = 60 + 30 = 90 Km/hr

So motion is relative & not absolute

P.T.O.

The theory which deals with the relativity of motion & rest is called the theory of relativity.
→ Special theory of relativity
→ General " " ".

Special theory of relativity: It deals with objects or systems which are either moving at a constant speed with respect to each other or are at rest.

General theory of relativity deals with systems which are accelerating or retarding with respect to each other.

Frame of Reference

The position & motion of the body is described accurately with respect to some well defined cartesian co-ordinate system is known as frame of reference.

There are two types of frame of reference

(i) Inertial or unaccelerated frame of reference.

(ii) Non-inertial or accelerated " " ".

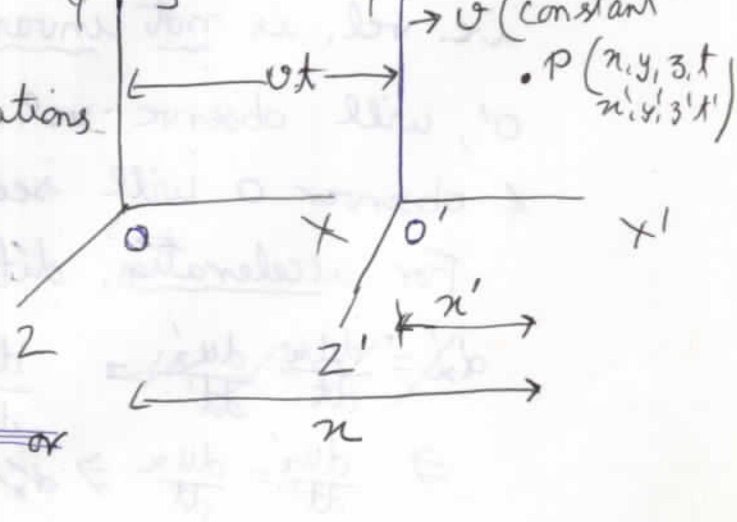
(i) Inertial frame of ref: The frame of reference which is either at rest or moving with uniform velocity is called Inertial frame of ref. i.e.

unaccelerated frame of reference are inertial frame of ref. In this frame, bodies obey Newton's law of inertia & other laws of Newtonian mech. In inertial frame, a body not acted upon by external force, is at rest or moves with a constant velocity. The laws of Physics will be same for all observers in this frame of ref.

(ii) Non-inertial frame: An accelerated or retarded frame of ref. is called non-inertial frame of ref. In this frame, the Newton's laws are not valid and a body, not acted upon by an external force is accelerated or retarded.

Galilean Transformations:

The equations relating the two sets of observations of an event in two different frames of ref. are called transformation eqns.. ~~In classical mech. or~~



The relevant transformation eqns in Classical Mech. or Newtonian mech. where the speed of the observer or object is very small compared to the speed of light. ~~these relevant transformation eqns.~~ are called Galilean transformation eqns.

From Fig. we see that

$$x' = x - vt;\ y' = y,\ z' = z,\ \&\ t' = t \quad \text{——} (1)$$

& Inverse of G.T. eqns are obtained by replacing primed by unprime & unprime by prime & v is replaced by $-v$ i.e.

$$x = x' + vt';\ y = y',\ z = z',\ t = t' \quad \text{——} (2)$$

The velocity of particle is of Interest.

$$x' = x - vt$$

$$\therefore u_x' = \frac{dx'}{dt'} = \frac{d}{dt}(x - vt) \quad [\because t = t',\ \&\ \text{using eqn}(1)]$$

$$= \frac{dx}{dt} - v \quad \text{or, } \boxed{u_x' = u_x - v},\ u_y' = u_y\ \&\ u_z' = u_z \quad (3)$$

i.e. vel. is not invariant under G.T., observer O', will observe vel. of particle P to be $u_x' = u_x - v$, & observer O will see it as u_x.

For acceleration, diff. eqns (3) we get

$$\cancel{a_x' = \frac{du_x}{dt}\ \frac{du_x'}{dt'}} = \frac{du_x'}{dt'} = \frac{du_x}{dt} - 0\ ;\ \frac{du_y'}{dt'} = \frac{du_y}{dt}\ ;\ \frac{du_z'}{dt'} = \frac{du_z}{dt}$$

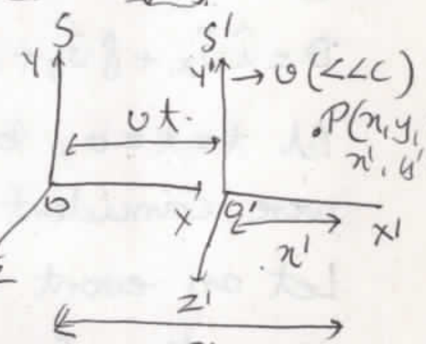
$$\Rightarrow \frac{du_x'}{dt'} = \frac{du_x}{dt} \Rightarrow \cancel{a_x' = a_x;\ a_y' = a_y,\ a_z' = a_z} \Rightarrow a' = a$$

i.e. Acceleration is invariant under G.T. or, $a_x' = a_x$

Galilean Transformation

The equations relating the co-ordinates of a particle in two inertial frames, ~~&~~ when the relative vel. between the frames is negligible in comparision to speed of light. Then the equations are called as Galilean transformation eqns.

Consider two frames S & S', S being at rest & S' moving with a constant speed $v(<<c)$ w.r.t. frame S. Let there be two observers O & O' observing an event at Point P from S & S' simultaneously.



Case (ii) S' is moving with vel $v(<<c)$ along common positive X direction

Let at time $t = t' = 0$, both frames were coincident with each other. When time starts, frame S' starts moving with constant vel. v.* After time t, it would traverse a distance 'vt'. As there is no relative motion along the Y & Z direction, therefore from fig we have

$$\left.\begin{aligned} x' &= x - vt \\ y' &= y \\ z' &= z \\ \& \; t' &= t \end{aligned}\right\} \quad (1)$$

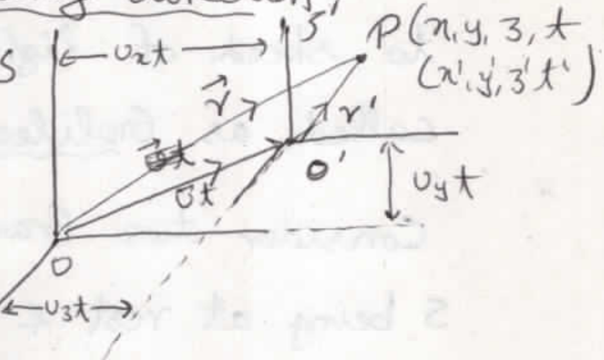
*[Let the co-ord. of point P be (x, y, z, t) to observer O & it be (x', y', z', t') to observer O'.] ~~Then from fig we have~~ ←

P.T.O.

These eqns (1) are called Galilean transformation eqns.

Case(ii): The frame S' is moving along a straight line relative to S along any direction.

Let S' be moving relative to S with vel. v such that $\vec{v} = \hat{i}\vec{v}_x + \hat{j}\vec{v}_y + \hat{k}\vec{v}_z$. ~~...~~



At $t = t' = 0$, both frames were coincident with each other.

Let an event occurs at point P at any instant, when the frame S' is separated from S by distance $v_x t$, $v_y t$ & $v_z t$ along X, Y & Z axes resp. From fig we have

$$x' = x - v_x t,\ y' = y - v_y t,\ z' = z - v_z t,\ t' = t \Big\} \quad (2)$$

By triangle law of addition in $\Delta POO'$ we have ~~from $\Delta POO'$ we have~~

$$\vec{r} = \vec{v}t + r' \quad \text{———} (3)$$

or $\vec{r}' = \vec{r} - \vec{v}t$ ——— (4)

This is general eqn. of Galilean transformation eqn. (G.T eqn.)

velocity $r' = r - vt \quad \therefore \frac{dr'}{dt'} = \frac{dr}{dt} - v$

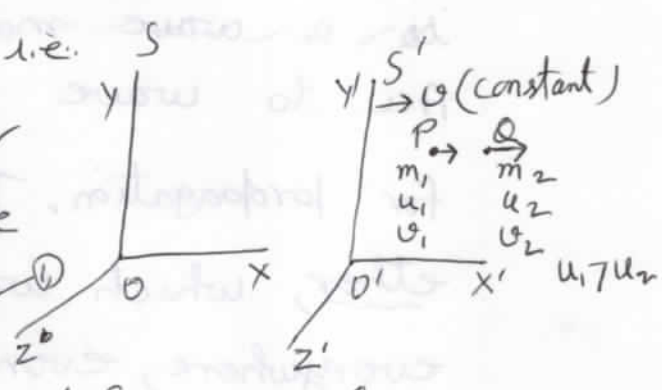
Let $\frac{dr'}{dt'} = u'$ & $\frac{dr}{dt} = u$

$\therefore u' = u - v$ ——— (5) $\Rightarrow$ vel. is not invariant i.e. variant under G.T.

Acceleration: $\frac{d^2r'}{dt'^2} = \frac{d^2r}{dt^2}$ or, $\boxed{a' = a}$ $\Rightarrow$ acceleration is invariant under G.T.

It means that Newton's 2nd law is invariant under G. T's..

The form of conservation of momentum & energy remains invariant under G. T's, i.e.



By conservation of linear mom. in S-frame we have

$$m_1 u_1 + m_2 u_2 = m_1 v_1 + m_2 v_2 \quad (1)$$

Using G. T's eqns. for the S' frame we have

$$u_1 = u_1' + v; \quad u_2 = u_2' + v; \quad v_1 = v_1' + v \;\&\; v_2 = v_2' + v \quad (2)$$

∴ In S' frame we have from eqns. (1) & (2)

$$m_1(u_1'+v) + m_2(u_2'+v) = m_1(v_1'+v) + m_2(v_2'+v)$$

$$\Rightarrow \boxed{m_1 u_1' + m_2 u_2' = m_1 v_1' + m_2 v_2'}$$

[Note: The numerical values of mom. are different in the two frames but its form is same, so the Form of momentum is conserved.]

Similarly the form of conservation of Kinetic energy is also ~~conserved~~ invariant i.e. in frame S we have

$$\frac{1}{2} m_1 u_1^2 + \frac{1}{2} m_2 u_2^2 = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2$$

& in frame S' we have

$$\frac{1}{2} m_1 u_1'^2 + \frac{1}{2} m_2 u_2'^2 = \frac{1}{2} m_1 v_1'^2 + \frac{1}{2} m_2 v_2'^2.$$

P. T. O.

The Ether Hypothesis:

The phenomena of interference, diffraction, & polarisation lead to establish that light ~~is a wave motion~~ behaves as a wave.

Acc. to wave theory, light requires a medium for propagation. The hypothetical medium is called the ether, which was assumed to be present everywhere, even in free space. The velocity of wave through a medium (ether) is given by $v=\sqrt{\frac{E}{\rho}}$, where E & ρ are elasticity & density respectively of the medium. In case of light (e.m. wave), the vel. is 3×10^{8} m/sec. So the medium should have high elasticity (very rigid) & very low density (massless).

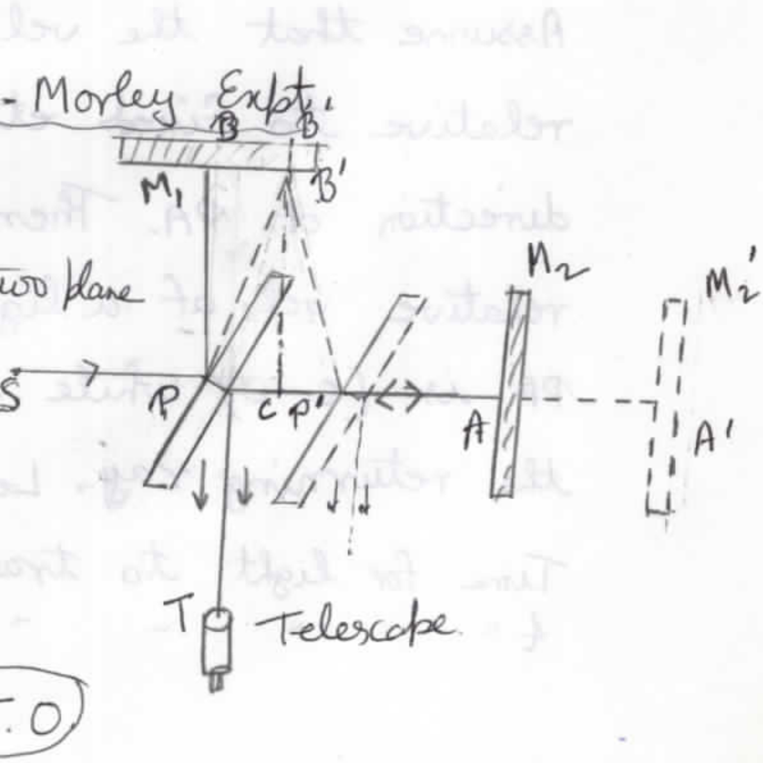
So, the unusual properties of ether are supposed to be invisible, massless, perfectly transparent, ~~low density~~ massless, very high rigidity (same as that of steel) & yet planets as well as other objects of necessity moved through the ether freely without disturbing it. In this way, ether provides a fixed or absolute frame of reference.

An interesting question is whether relative motion between the earth and the ether can be detected. If it is detected, we can choose a fixed frame of reference in a stationary ether (absolute frame of reference). Then we can express all motion w.r.t. this absolute ether frame of reference

The experiments of Michelson & Morley cannot be explained by the stationary ether hypothesis. Despite best efforts, the presence of ether could not be detected. Hence the concept of absolute frame of reference failed.

Michelson-Morley Expt.

The apparatus used in this expt. are two plane Mirrors M_1 & M_2, half silvered plate P & a telescope. They are arranged accordingly as shown in fig.



(P.T.O)

The reflected ray (PB) & transmitted ray (PA) ~~returned~~ reflected back to P are directed towards telescope T, here the two beams interfare & form interferance fringes that can be observed by the telescope T.

The two mirrors M_1 & M_2 are at equal distance 'd'. If the apparatus is at rest in ether then the two rays (reflected & transmitted) would take equal time to return to P.

The whole apparatus is floated ~~in~~ on Mercury. One arm (PA) was pointed in the direction of earth's motion round the sun & the other (PB) was pointed $\perp$ to this motion. Assume that the vel. of the apparatus (or earth) relative to fixed ether is 'v' in the direction of PA. Then acc. to Galilean Transformation relative vel. of a light ray w.r.t. ~~stationary ether~~ earth travelling along PA is $(c-v)$ while it would be $(c+v)$ for the returning ray. Let $PA = PB = d$

Time for light to travel from P to A $= d/(c-v)$

& " " " " " " A to P $= \dfrac{d}{(c+v)}$

$\therefore$ Total time $t = \frac{d}{(c-v)} + \frac{d}{(c+v)} = \frac{2cd}{(c^2-v^2)}$

$$= \frac{2cd}{c^2(1-\frac{v^2}{c^2})} = \frac{2d}{c}\left(1-\frac{v^2}{c^2}\right)^{-1}_{\downarrow} = \frac{2d}{c}\left(1+\frac{v^2}{c^2}+\cdots\right) \quad (1)$$

$\left(\text{as } \frac{v^2}{c^2} < 1\right)$

Now, consider the ray moving upwards from P to B. It will strike the mirror M_1 not at B but at B′ due to the motion of the earth. If t_1 is the time taken by the ray starting from P to reach M_1, then

$PB' = ct_1$ & $BB' = vt_1$, Now

$(PB')^2 = PC^2 + CB'^2 = (BB')^2 + (PB)^2$

or, $c^2t_1^2 = v^2t_1^2 + d^2$ or, $t_1^2(c^2-v^2) = d^2$

$\therefore t_1 = \frac{d}{\sqrt{c^2-v^2}}$ $\therefore$ Total time $t' = 2t_1 = \frac{2d}{\sqrt{c^2-v^2}}$

or, $t' = \frac{2d}{c\sqrt{1-v^2/c^2}} = \frac{2d}{c}\left(1+\frac{v^2}{2c^2}+\cdots\right) \quad (2)$

Here $t > t' \Rightarrow PM_2 > PM_1$

The time difference

$\Delta t = t - t' = \frac{2d}{c}\left[1+\frac{v^2}{c^2}\right] - \frac{2d}{c}\left[1+\frac{v^2}{2c^2}\right] = \frac{dv^2}{c^3}$

The distance travelled by light in time Δt $= c\times\Delta t = \frac{dv^2}{c^2}$. This is path difference bet. the two rays [reflected & transmitted rays].

(P. T. O.)

On rotating the apparatus through 90°, the path PM_1 became $> PM_2$ by the same amount $\left(\frac{dv^2}{c^2}\right)$ but in opposite direction. Therefore total path diff. bet. the two rays along earth's motion direction $= \frac{2dv^2}{c^2}$.

We know, path diff 'λ' corresponds to one fringe shift.

$\therefore$ path diff. $\frac{2dv^2}{c^2}$ corresponds to

$$\Delta n = \frac{2dv^2}{\lambda c^2}.$$

From experimental data we have

$d = 1 \times 10^3$ cm; $\lambda = 5.0 \times 10^{-5}$ cm, $v = 3 \times 10^6$ cm/sec, $c = 3 \times 10^{10}$ cm/sec

$\Delta n = 0.4$ of a fringe shift.

Michelson & Morley believed that they could detect a shift as small as 0.01 of a fringe shift. But in the expt. no displacement of the fringes was observed. This negative result suggests that it is impossible to measure the speed of the earth relative to the fixed ether. Thus all attempts to make ether as a fixed or absolute frame of reference failed.

Explanation of negative result

(1) Michelson himself proposed that the earth dragged along with it the ether in its immediate neighbourhood if any, so that there is no relative motion bet. the earth & ether.

(2) Lorentz suggested that there was contraction of bodies along the direction of motion through the ether. In the experiment the distance ~~PB~~ perp. to motion i.e $PB = d$ will remain unchanged but distance PA along the relative motion taken place will get shortened & would become $d\sqrt{(1-v^2/c^2)}$.

(3) The proper explanation was given by Einstein. He concluded that the vel. of light in free space is a universal constant c. The vel. of light is c rather than $|c+v|$ or $|c-v|$ in any frame of reference.

Einstein's postulates of special theory of relativity.

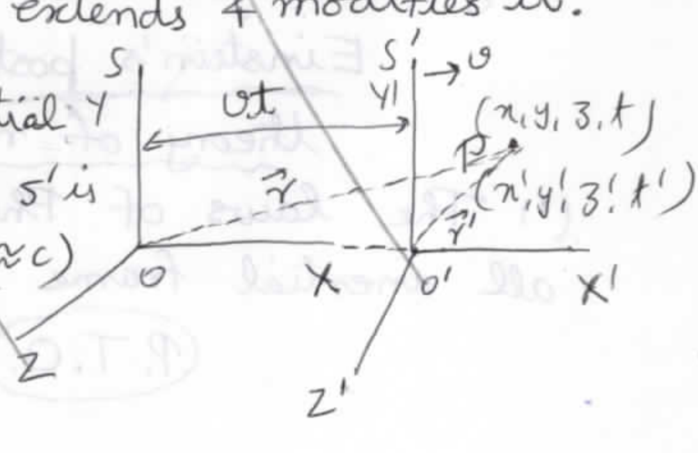
(1) The laws of Physics are the same in all inertial frame of reference.

(2) The velocity of light in free space is a universal constant c. It is independent of the relative motion between the source & the observer.

Lorentz Transformation Eqns ~~for~~ of space & time

The Galilean transformation (G.T) eqns are not suitable under the new concept of special theory of relativity, where the speed of the object or observer is comparable with the vel. of light, therefore, G.T. eqns are replaced by Lorentz transformations eqn (L.T. eqn) of space & time which is based on spl. theory of relativity. For speeds much smaller than, the vel. of c, the L.T. eqns reduce to G.T. eqns. Therefore, the Einstein's theory of relativity does not overthrow, the classical theory, but rather extends & modifies it.

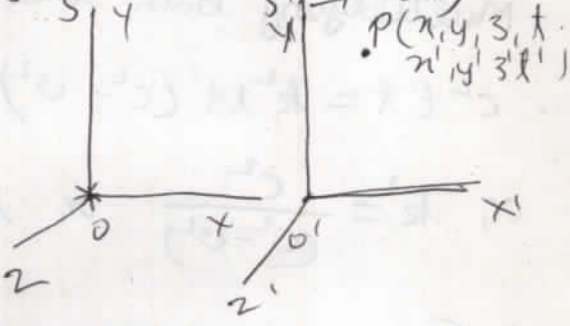
· Consider two inertial frames S & S' & system S' is moving with high vel ($\approx c$) v in the positive x direction.



Lorentz Space time Transformation Eqn. (L.T. eqn)

The Galilean transformation eqns are not suitable under the new concepts of special theory of relativity, where the speed of the observer or object is comparable with the velocity of light. The new transformation eqns were known as Lorentz space time transformation eqn. based on the two postulates of Einsteins special theory of relativity

Let at time $t = t' = o$, both frames were coincident with each other. Let an event takes place at P



Let us take a transformation eqn. as

$x' = k(x - vt)$ —— (1), where k is a constant

Acc. to 1st postulate of special theory of relativity eqn. for x has same form except v is replaced by $-v$ i.e.

$x = k(x' + vt')$ —— (2) or, $\frac{x}{k} = x' + vt'$ —— (2a)

Putting expression of x' from eqn (1) to eqn (2a) we get

$\frac{x}{k} = [k(x - vt) + vt']$ or, $\frac{x}{k} = k(x - vt) + vt'$

$\therefore t' = \frac{x}{kv} - \frac{kx}{v} + kt$

or, $t' = kt - \frac{kx}{v}\left(1 - \frac{1}{k^2}\right)$ —— (3)

~~Similarly~~ Acc. to 1st Postulate $t = kt' + \frac{kx'}{v}\left(1 - \frac{1}{k^2}\right)$ —— (4)

Now, Let a flash light is emitted from the common origin of S & S' at time $t = t' = o$. Acc. to second

P.T.O.

postulate of Einstein's theory, light flash will move with vel. of c for both the observers at o & o'. After some time the position of flash light will be

$x = ct$ & $x' = ct'$ for o & o' resp.

Putting the ~~value~~ expression of x & x' in eqn(1) & eqn(2), we have

$ct' = k(ct - vt)$ & $ct = k(ct' + vt')$

or, $ct' = kt(c-v)$ or, $ct = kt'(c+v)$

Multiplying both the eqns with each other we get

$c^2t't = k^2tt'(c^2 - v^2)$ or, $c^2 = k^2(c^2 - v^2)$

or, $k^2 = \dfrac{c^2}{(c^2 - v^2)}$ or, $k^2 = \dfrac{1}{\left(1 - \frac{v^2}{c^2}\right)}$ ∴ $k = \dfrac{1}{\sqrt{1 - v^2/c^2}}$ —(5)

∴ From eqns(1) & (5) we have

$$\boxed{x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}} \quad \text{—— (6)} \quad \text{or,} \quad \boxed{x' = \frac{x - vt}{\sqrt{1 - \beta^2}}} \quad \text{where } \beta = \frac{v}{c};$$

Squaring eqn(5) we have

$k^2 = \dfrac{1}{\left(1 - \frac{v^2}{c^2}\right)}$ or, $\dfrac{1}{k^2} = \left(1 - \dfrac{v^2}{c^2}\right)$ or, $\left(1 - \dfrac{1}{k^2}\right) = \dfrac{v^2}{c^2}$

By putting the value of $\left(1 - \frac{1}{k^2}\right) = \frac{v^2}{c^2}$ in eqn(3) we get

$$t' = kt - \frac{kx}{v}\left(\frac{v^2}{c^2}\right) = k\left(t - \frac{x}{v} \cdot \frac{v^2}{c^2}\right) = k\left(t - \frac{vx}{c^2}\right)$$

$$\text{or, } t' = \frac{(t - vx/c^2)}{\sqrt{1 - v^2/c^2}} \quad \text{or, } \boxed{t' = \frac{(t - vx/c^2)}{\sqrt{1 - \beta^2}}} \text{ —(7)} \quad \text{where } \beta = \frac{v}{c}$$

As frame S' is moving along common +ve x direction therefore

$$\left.\begin{aligned} y' &= y, \\ z' &= z. \end{aligned}\right\} \text{ —— (8)}$$

Eqns (6), (7) & (8) are known as Lorentz space time transformation eqn.

And Inverse L.T. eqn is $x = \dfrac{x' + vt'}{\sqrt{1 - \beta^2}}$, $t = \dfrac{t' + vx'/c^2}{\sqrt{1 - \beta^2}}$, $y = y'$ & $z = z'$.

These eqns. convert measurements made in frame S' into those in frame S.

It can be seen that for values $v \ll c$, Lorentz transformation reduces to the G.T.

i.e. if $v \ll c$, $\frac{v}{c} \to 0$. then $\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \approx 1$

$\therefore$ Above L.T. eqns becomes $x' = x - vt$; $y' = y$, $z' = z$, $t' = t$, which are G.T. equations.

<u>Invariance of L.T. & its significance</u>:

The quantities like $x^2 + y^2 + z^2 = c^2t^2$; $x^2 - c^2$~~...~~, $E^2 - c^2p^2$; $x^2 - c^2t^2$; $c^2B^2 - E^2$; $\vec{E} \cdot \vec{B}$ (Maxwell's eqn), charge, rest mass m_0; c; four dim. volume element $dx\,dy\,dz\,dt$ are invariant under L.T. i.e they retain the same form & have same value in both frame of refs.

<u>Significance</u>: If a quantity is invariant under L.T. then we are not bothered whether it is in rest frame or in moving frame because that quantity has same form ~~& same value~~ in both frames.

P.T.O.

Q1. Show that the quantity $x^2-c^2t^2$ is invariant under Lorentz transformation.

Soln. Let us consider the expression

$$x'^2-c^2t'^2=\frac{(x-vt)^2}{1-\frac{v^2}{c^2}}-c^2\frac{\left[t-\frac{vx}{c^2}\right]^2}{1-\frac{v^2}{c^2}}$$

or, $\frac{c^2(x-vt)^2}{(c^2-v^2)}-\frac{c^4}{(c^2-v^2)}\left[t-\frac{vx}{c^2}\right]^2$

or, $\frac{c^2}{(c^2-v^2)}\left[x^2-2xvt+v^2t^2-c^2t^2+2xvt-\frac{v^2x^2}{c^2}\right]$

or, $\frac{\cancel{c^2}}{(c^2-v^2)}\left[\frac{x^2c^2+v^2t^2c^2-c^4t^2-v^2x^2}{\cancel{c^2}}\right]$.

or, $\frac{1}{(c^2-v^2)}\left[c^2(x^2-c^2t^2)-v^2(x^2-c^2t^2)\right]$.

or, $\frac{1}{\cancel{(c^2-v^2)}}\cancel{(c^2-v^2)}(x^2-c^2t^2)=x^2-c^2t^2$.

$\Rightarrow x'^2-c^2t'^2=x^2-c^2t^2$ i.e. $x^2-c^2t^2$ is invariant under L.T.

Q2. Prove that 3 dimensional volume element $dx\,dy\,dz$ is not invariant, but 4 dimensional volume element $dx\,dy\,dz\,dt$ is invariant under L.T.

Soln. Acc. to Lorentz contraction, we have

$dx'=dx\sqrt{1-\beta^2}$; $dy'=dy$, $dz'=dz$ where $\beta=v/c$

$\therefore$ 3 dim. vol. element in frame S'

$=dx'dy'dz'=dx\sqrt{1-\beta^2}\,dy\,dz=dx\,dy\,dz\sqrt{1-\beta^2}$.

Thus 3 dimensional volume element is not invariant under L.T's

Acc. to time dilation

$$dt' = \frac{dt}{\sqrt{1-\beta^2}}$$

$\therefore$ 4 dimensional vol. element in system

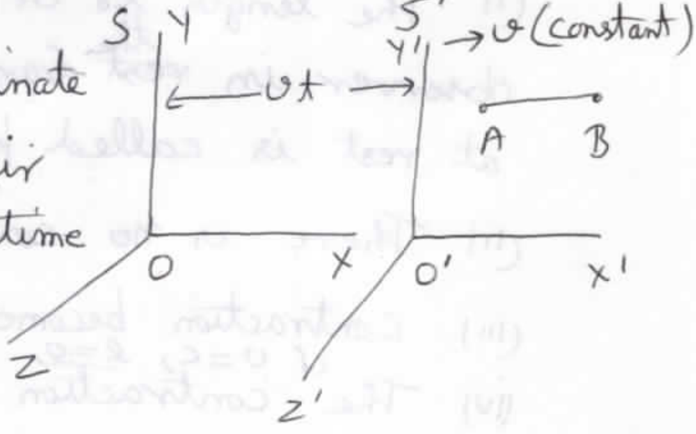
$$S' = dx'\,dy'\,dz'\,dt' = dx\,\sqrt{1-\beta^2}\,dy\,dz\,\frac{dt}{\sqrt{1-\beta^2}} = dx\,dy\,dz\,dt$$

= 4 dim. vol. element in system S.

Hence 4 dim. vol. element is invariant under L.T.

Length contraction:

Lorentz-Fitzgerald, proposed that the length of a moving body (moving with vel. $\approx c$) measured by a stationary observer appears to be contracted in the direction of motion. "The appeared decrease in the length of the body along the direction of motion is called length contraction."



Consider two co-ordinate systems S & S' with their X-axes coinciding at time $t=0$. Imagine a rod AB at rest relative to S'.

(P.T.O.)

$l_0 = x_2' - x_1'$ ——— (1), is proper length of rod.

Let x_1 & x_2 be the co-ordinates of the ends of the rod at the same instant of time in S. Then

$l = x_2 - x_1$ ——— (2), where l is the length of the rod, measured relative to S. Acc. to L.T

$$x_2' = \frac{x_2 - vt}{\sqrt{1-\beta^2}} \quad \& \quad x_1' = \frac{x_1 - vt}{\sqrt{1-\beta^2}}, \text{ where } \beta = v/c$$

$$\therefore x_2' - x_1' = \frac{x_2 - x_1}{\sqrt{1-\beta^2}} \quad \text{or, } l_0 = \frac{l}{\sqrt{1-\beta^2}}$$

$$\text{or, } \boxed{l = l_0\sqrt{1-\beta^2}} \text{ ——— (3) or, } l = l_0\sqrt{1 - v^2/c^2}$$

As $v/c < 1$, so $\boxed{l < l_0}$. Therefore, to the observer in S it would appear that the length of the rod placed in S' has contracted by the factor $\sqrt{1-v^2/c^2}$ or, $\sqrt{1-\beta^2}$.

Important Points:

(i) The length l_0 of the rod measured by an observer in ~~rest~~ the frame in which the rod is at rest is called proper length or actual length.

(ii) There is no contraction along $\perp$ dir. of mot.

(iii) contraction becomes appreciable only when $v \approx c$.

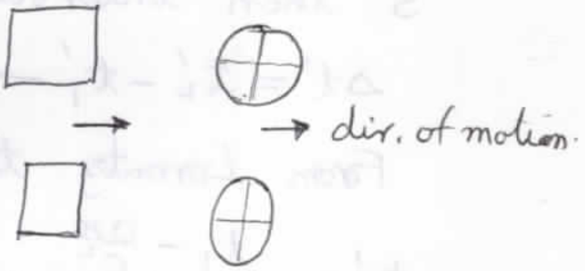
if $v = c$, $l = 0$.

(iv) The contraction is reciprocal i.e. if two identical rods are at rest, one in frame S & the other in frame S'; each observer in its own frame observes that the other rod is shorter than the rod of his own frame by the factor of $\sqrt{1-\beta^2}$, where $\beta = v/c$.

(V) A square & a circle in one frame of ref. appear to the observer in the other frame of reference to be a rectangle & an ellipse resp.

Ref. frame at rest

Ref. frame in motion

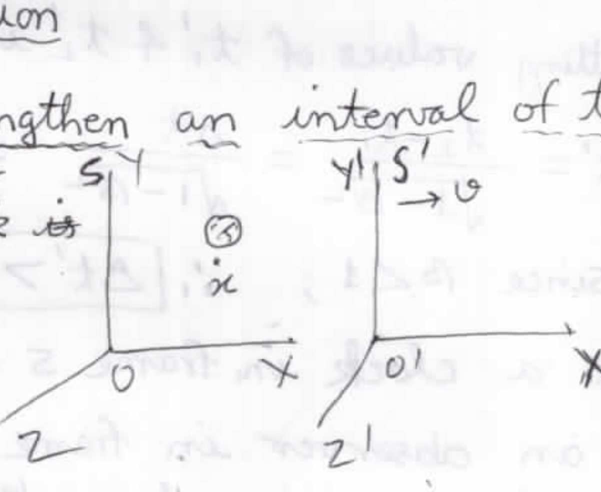


(shortened along the direction of motion).

Time dilation

Time Dilation means lengthen an interval of time.

Suppose an ~~clock is~~ event has taken place in ~~situated in the~~ system S at position x & gives signals of intervals Δt i.e.



$\Delta t = t_2 - t_1$ ——— (1)

P. T. O.

A clock in a moving frame of Ref. (say S') measures a longer time interval bet. two events that occur in a stationary frame of ref. ~~than the same time interval measured by the clock in stationary frame~~ (S). This is known as time dilation.

In S frame

$\Delta t = t_2 - t_1$ —— (1). ~~In frame S~~

If this is observed by an observer in system S' then interval $\Delta t'$ will be given by

$\Delta t' = t_2' - t_1'$ —— (2)

From Lorentz transformation, time in position x is given as ~~we have~~

$$t_1' = \frac{t_1 - \frac{vx}{c^2}}{\sqrt{1-\cancel{\frac{v^2}{c^2}}\beta^2}} \quad \text{—— (3)} \qquad t_2' = \frac{t_2 - \frac{vx}{c^2}}{\sqrt{1-\beta^2}} \quad \text{—— (4)}$$

Putting values of t_1' & t_2' in eqn. (2) we get

$$\Delta t' = \frac{t_2 - t_1}{\sqrt{1-\beta^2}} = \frac{\Delta t}{\sqrt{1-\beta^2}} \Rightarrow \boxed{\Delta t' = \frac{\Delta t}{\sqrt{1-\beta^2}}} \quad \text{—— (5)}$$

Since $\beta < 1$; $\therefore \boxed{\Delta t' > \Delta t}$

i.e. a clock in frame S appears to go slow to an observer in frame S',

Similarly it can be shown that $\boxed{\Delta t = \frac{\Delta t'}{\sqrt{1-\beta^2}}}$ i.e.

The observer in stationary frame feels that the clock in moving frame is slow as compared to his own. Thus the consequence of time dilation is reciprocal.

Experimental verification of Time dilation

Time dilation is real effect!

Example: Meson decay

The direct experimental confirmation of time dilation is found in an experiment on cosmic ray particles called mesons. μ mesons are created at high altitudes in the earth atmos. approx. 10 Km above the earth's surface & reaches the earth's surface. A μ meson are unstable & decays into an electron in an average time of 2×10^{-6} sec. Such mesons have a speed of $2.994\times10^{8}\,\mathrm{m\,sec^{-1}}$ & can travel a distance of only $(2.994\times10^{8})(2\times10^{-6}) \simeq 600\,\mathrm{m} \simeq 0.6\,\mathrm{Km}$.

Now the question arises how μ mesons travel a distance of $\approx 10\,\mathrm{Km}$ to reach earth's surface. This is possible because of time dilation effect. In fact μ mesons have an average life time $t_0 = 2\times10^{-6}$ sec in its own frame of ref. In the observer's frame of ref. on earth surface, the life time of the μ-meson is lengthened (dilated) as

$$t = \frac{t_0}{\sqrt{1-v^2/c^2}} = \frac{2.0\times10^{-6}}{\sqrt{1-(0.998)^2}} = 3.17\times10^{-5}\,\mathrm{sec}.$$

$\therefore$ μ-meson can travel a distance $= (2.994\times10^{8})(3.17\times10^{-5})$
$= 9500\,\mathrm{m} \simeq 9.5\,\mathrm{Km}$.

This explains the presence of μ-mesons on earth's surface.

P.T.O.

Concept of simultaneity of events

Consider two events, the explosion of a pair of time bombs that occur at the same time to an observer O in a reference frame S. Let the two events occur at different locations x_1 & x_2. Consider another observer O′ in S′ [Fig see below] moving with a uniform relative speed v w.r.t. S in the positive x-direction.

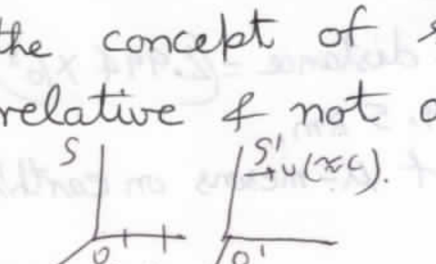
To observer O′, the explosion at x_1 & t occurs at the time

$$t_1' = \frac{t - \frac{v x_1}{c^2}}{\sqrt{1-\beta^2}}$$ & at x_2 & t occurs at time

$$t_2' = \frac{t - \frac{v x_2}{c^2}}{\sqrt{1-\beta^2}}.$$ Therefore, the time interval bet. the two events as observed by O′

$$= t' = t_2' - t_1' = \frac{\frac{v}{c^2}(x_1 - x_2)}{\sqrt{1-\beta^2}} \neq 0 \text{ (As } x_1 \neq x_2).$$

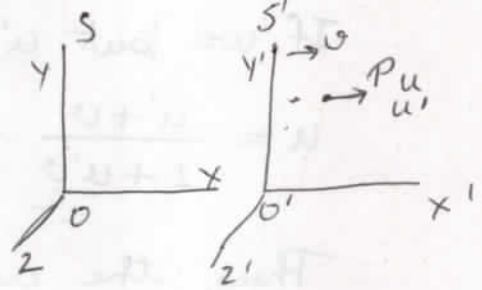
This is not zero. This indicates that two events at x_1 & x_2 which are simultaneous to the observer in S do not appear so to the observer in S′. Therefore, the concept of simultaneity has only a relative & not an absolute meaning.



Relativistic Addition of velocities

Let us consider that a particle P is moving relative to both frames S & S'. An observer in S measures its 3 velocity component to be

$u_x = \frac{dx}{dt}$, $u_y = \frac{dy}{dt}$, & $u_z = \frac{dz}{dt}$



while to an observer in S' they are

$u_x' = \frac{dx'}{dt'}$, $u_y' = \frac{dy'}{dt'}$ & $u_z' = \frac{dz'}{dt'}$

using inverse of L. T. eqn we have

$$x = \frac{x' + vt'}{\sqrt{1-\beta^2}} \quad \& \quad t = \frac{t' + \frac{vx'}{c^2}}{\sqrt{1-\beta^2}} \quad \text{where } \beta = \frac{v}{c}$$

by differentiating above eqns. we get

$$dx = \frac{dx' + v\,dt'}{\sqrt{1-\beta^2}} \quad (1) \quad \& \quad dt = \frac{dt' + \frac{v}{c^2}dx'}{\sqrt{1-\beta^2}} \quad (2)$$

$$\therefore \frac{dx}{dt} = \frac{dx' + v\,dt'}{dt' + \frac{v}{c^2}dx'} \quad (3)$$

Dividing R.H.S. of eqn(3) by dt' to both numerator & denominator we have

$$\frac{dx}{dt} = \frac{\frac{dx'}{dt'} + v}{1 + \frac{v}{c^2}\frac{dx'}{dt'}} \quad \text{or,} \quad \frac{dx}{dt} = \frac{u' + v}{1 + \frac{u'v}{c^2}} \quad \text{or,} \quad \boxed{u_x = \frac{u_x' + v}{1 + \frac{u_x'v}{c^2}}}$$

Now, $y = y'$, $z = z'$ $\therefore dy = dy'$ & $dz = dz'$

$$\therefore u_y = \frac{dy}{dt} = \frac{dy'\sqrt{1-\beta^2}}{dt' + \frac{v}{c^2}dx'} \quad \text{(using eqn. 2)}$$

by dividing numerator & denominator by dt' we have

(P. T. O.)

$$\boxed{u_y = \frac{u_y'\sqrt{1-\beta^2}}{1+\frac{u_x'v}{c^2}}}$$ similarly

$$\boxed{u_z = \frac{u_z'\sqrt{1-\beta^2}}{1+\frac{u_x'v}{c^2}}}$$

$$u' = \frac{u-v}{1-\frac{uv}{c^2}}$$

If $u = c$

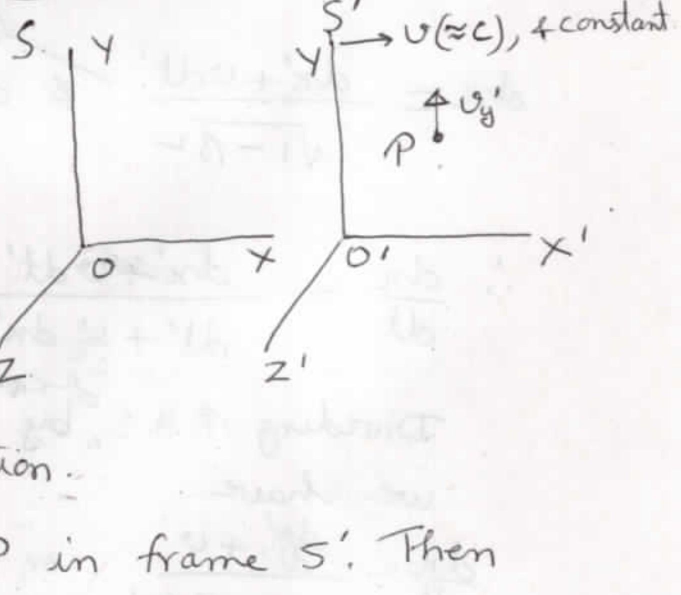
$$\therefore u' = \frac{c-v}{1-\frac{cv}{c^2}} = \frac{c(c-v)}{(c-v)} = c$$

If we put $u' = c$ & $v = c$, then

$$u = \frac{u'+v}{1+\frac{u'v}{c^2}} = \frac{c+c}{1+\frac{c.c}{c^2}} = \frac{2c}{1+1} = \frac{2c}{2} = c$$

Thus, the addition or subtraction of any velocity to the velocity of light c merely reproduces the velocity of light itself.
Hence, the velocity of light c is the maximum attainable velocity.

Variation of mass with velocity [R. P. Goyal]



Consider two inertial frames S & S' & frame S' is moving with velocity $v(\approx c)$ along common positive x direction.

Consider any particle P in frame S'. Then particle is stationary in frame S' & rest mass of particle P in frame S' is m_0, when this mass is observed by the observer in frame S, then its mass is 'm'

Let the particle P starts moving non-relativistically with constant vel. along Y' axis. (13)

If the displacement of the particle relative to the frame S' in time $\Delta t'$ is $\Delta y'$ along the y' axis, then the velocity of particle along the y' axis in frame S' is

$v'_y = \left(\frac{\Delta y'}{\Delta t'}\right)$ & its momentum is

$$p'_y = m_0 v'_y = m_0\left(\frac{\Delta y'}{\Delta t'}\right) \text{ ——— } (1)$$

In frame S, if same time interval measured to be Δt & the displacement of particle is Δy along Y axis, the vel. in S frame is

$v_y = \frac{\Delta y}{\Delta t}$ & its momentum

$$p_y = m v_y = m\left(\frac{\Delta y}{\Delta t}\right) \text{ ——— } (2)$$

From Lorentz transformation, $y = y'$ & hence $\Delta y = \Delta y'$ & acc. to time dilation

$$\Delta t = \frac{\Delta t'}{\sqrt{1-v^2/c^2}}; \ \Delta t' = \text{proper time}$$

$$\therefore p_y = m\left(\frac{\Delta y}{\Delta t}\right) = \frac{m(\Delta y')}{\frac{\Delta t'}{\left(\sqrt{1-v^2/c^2}\right)}} = \left(m\sqrt{1-\frac{v^2}{c^2}}\right)\left(\frac{\Delta y'}{\Delta t'}\right) \text{ — } (3)$$

Since momentum of the particle is invariant along y axis,

$\therefore p_y = p'_y \Rightarrow$ (from eqns (1) & (3))

$$m_0\left(\frac{\Delta y'}{\Delta t'}\right) = m\left(\sqrt{1-\frac{v^2}{c^2}}\right)\left(\frac{\Delta y'}{\Delta t'}\right) \text{ or, } \boxed{m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}} \text{ — } (4)$$

P. T. O.

This is relativistic formula for the variation of mass ~~&~~ with velocity.

If we put $v \to c$ in eqn(4) we have $m \to \infty$, ~~we have~~ i.e. an object travelling at the speed of light would have infinite mass, i.e. mass of a moving body increases.

<u>Note</u>: [It is found that electrons & protons accelerated in ~~these~~ particle accelerators machines, to velocities close to the vel. of light acquire increased masses, exactly as predicted.].

Mass energy equivalence

~~Force~~ Acc. to classical mechanics, the energy is defined in terms of work (Force $\times$ distance) & the force is the rate of change of momentum, hence,

$$F = \frac{d}{dt}(mv) \quad \text{——} \quad (1)$$

Acc. to the theory of relativity, both mass & vel. are variable. Therefore,

$$F = \frac{d}{dt}(mv) = m\frac{dv}{dt} + v\frac{dm}{dt} \quad \text{——} \quad (2)$$

Let the force F displace the body through a distance dx. Then the increase in Kinetic energy (dE_K) of the body is equal to the work done ($F.dx$)

Hence, $dE_K = F dx = m\frac{dv}{dt}dx + v\frac{dm}{dt}dx$

or, $dE_K = mv\,dv + v^2 dm$ ——— (3)

Acc. to the law of variation of mass with vel.

$m = \frac{m_0}{\sqrt{1-v^2/c^2}}$ or, $m^2 = \frac{m_0^2}{1-v^2/c^2}$ or, $m_0^2c^2 = m^2c^2 - m^2v^2$

or, $m^2c^2 = m_0^2c^2 + m^2v^2$

Differentiating both sides we get

$\cancel{2}\cancel{m}c^2 dm = 0 + (\cancel{2}v m^{\cancel{2}} dv + \cancel{2}\cancel{m}v^2 dm)$

or, $c^2 dm = mv\,dv + v^2 dm$ ——— (5)

From eqns. (3) & (5) we have

$dE_K = c^2 dm$ ——— (6)

Now consider that the body is at rest initially & by the application of force it acquires a vel. v. The mass of the body increases from m_0 to m. The total K.E. acquired by the body is given by

$E_K = \int_0^{E_K} dE_K = c^2\int_{m_0}^{m} dm = c^2(m - m_0)$

$\therefore \boxed{E_K = mc^2 - m_0c^2}$ ——— (7)

The total energy (E) is K.E + rest mass energy (m_0c^2)

$\therefore E = E_K + m_0c^2 = mc^2 - \cancel{m_0c^2} + \cancel{m_0c^2} = mc^2$

$\therefore \boxed{E = mc^2}$ This is Einstein's mass energy relation.

This eqn. signifies that mass may appear as energy & energy as mass.

P.T.O.

Relativistic relation between energy & momentum

Total relativistic energy of a particle is

$$E = mc^2 = m_0c^2/\sqrt{1-v^2/c^2} \quad \text{—— (1)}$$

its momentum $p = mv$ or, $v = p/m$ —— (2)

From eqns (1) & (2) we have

$$E = \frac{m_0c^2}{\sqrt{\left(1-\frac{p^2}{c^2m^2}\right)}} = \frac{m_0c^2}{\sqrt{1-\frac{c^2p^2}{m^2c^4}}} = \frac{m_0c^2}{\sqrt{\left(1-\frac{c^2p^2}{E^2}\right)}}$$

or, $E^2 = \dfrac{m_0^2c^4}{1-\frac{c^2p^2}{E^2}}$ or, $E^2 - c^2p^2 = m_0^2c^4$

or, $\boxed{E^2 = c^2p^2 + m_0^2c^4}$.

Mass of Photon:/ Massless particles.

A particle which has zero rest mass (m_0) is called a massless particle. In classical physics, the existence of massless particle is impossible. However, in relativistic mech., a particle with zero rest mass can exist.

The relativistic total Energy of a particle of rest mass m_0 & mom. p. may be expressed as

$E = \sqrt{c^2p^2 + m_0^2c^4}$, for massless particle $m_0 = 0$

$\therefore E = cp$ or $p = E/c$. Also $E = mc^2$ ~~& cp~~ or, $m = \frac{E}{c^2}$.

~~Also~~ We know $p = mv$ & $E = mc^2$ ∴ $mv = \frac{mc^2}{c}$ ⇒ $\boxed{v = c}$

$p = mv = E/c$ ⇒ $mv = \frac{mc^2}{c}$ or, $\boxed{v = c}$

i.e. vel. of massless particle is same that of light in free space.

Conclusion:: Every massless particle has energy cp & mom. E/c, & moves with vel. of light. A massless particle has mass so long as it is in motion with vel. c. On being stopped they cease to exist - they are either absorbed completely or are changed into heat at the surface. E.g. Photons, neutrinos, gravitons etc are massless particles.