Frobenius Method → (when $x=0$ is a regular singular pt.)

step I. Assume the solⁿ of series is

$$y = \sum_{n=0}^{\infty} a_n x^{m+n} \quad \& \text{ find } \frac{dy}{dx}, \frac{d^2y}{dx^2}.$$

step II. subsidute the value of y, y', y'' in given eqⁿ.

step III Put the coefficient of lowest power of x to zero. we will find an eqⁿ in terms of m called as 'indical Eqⁿ'.

step IV. Find the values/roots of indical Eqⁿ on the basis of values of m, it is further categorized in 3 subcases:-

case I. $m_1 \neq m_2$ & $m_1 - m_2 \neq$ integer

case II $m_1 = m_2$ (m_1 & m_2 are same)

Case III $m_1 \neq m_2$ & $m_1 - m_2 =$ integer

step V. By applying indexing, reduce all power of x to x^{m+n}.

step VI Equate the coefficient of x^{m+n}, it will give a recurrance relation & find the values of a_1, a_2, a_3 - - .

step VII substitute the values of a_1, a_2 - - in assumed solⁿ to get one of the solⁿ. The complete solⁿ will depends on the ~~value~~ nature of values of m.

Case I: when roots are distinct & $m_1 - m_2 \neq$ integer

Trial solⁿ: $y = \sum_{n=0}^{\infty} a_n x^{m+n}$

complete solⁿ $y = c_1 (y)_{m_1} + c_2 (y)_{m_2}$

eg. $2x(1-x)\frac{d^2y}{dx^2} + (1-x)\frac{dy}{dx} + 3y = 0$

about the regular singular pt. $x = 0$.

sol: Here $P_0(x) = 2x(1-x)$, $P_1(x) = (1-x)$, $P_2 = 3$

Now at Pt. $x=0$, $P_0(0) = 0 \rightarrow$ singular pt.

Now. $\lim_{x \to a} (x-a)\frac{P_1(x)}{P_0(x)} = \lim_{x \to 0} x\frac{(1-x)}{2x(1-x)} = \frac{1}{2}$

$\lim_{x \to a} (x-a)^2 \frac{P_2(x)}{P_0(x)} = \lim_{x \to 0} (x)^2 \frac{3}{2x(1-x)} = 0$

i.e. it is regular singular pt. $(x=0)$

Now by using Frobenius Method →

Let $y = \sum_{n=0}^{\infty} a_n x^{m+n}$ (m can have any value)

$\frac{dy}{dx} = \sum_{n=0}^{\infty} a_n (m+n) x^{m+n-1}$

$y'' = \sum_{n=0}^{\infty} a_n (m+n)(m+n-1) x^{m+n-2}$

Now Put the values of y, y', y'' in given diff. eqⁿ.

we get.

$$2x(1-x)\left[\sum_{n=0}^{\infty} a_n (m+n)(m+n-1) x^{m+n-2}\right] + (1-x)\left[\sum_{n=0}^{\infty} (m+n) x^{m+n-1}\right. + 3\left[\sum_{n=0}^{\infty} a_n x^{m+n}\right] = 0$$

$$(2x-2x^2)\left[\sum (m+n)(m+n-1)a_n x^{m+n-2}\right] + (1-x)\left[\sum (m+n)a_n x^{m+n-1}\right]$$
$$+ 3\left[\sum a_n x^{m+n}\right] = 0$$

$$\sum 2(m+n)(m+n-1)a_n x^{m+n-1} - \sum 2(m+n)(m+n-1)a_n x^{m+n} +$$
$$\sum (m+n)a_n x^{m+n-1} - \sum (m+n)a_n x^{m+n} + \sum 3a_n x^{m+n} = 0$$

$$= \sum a_n x^{m+n-1}(m+n)\{2(m+n-1)+1\} -$$
$$\sum\left[2(m+n)(m+n-1)+(m+n)-3\right]a_n x^{m+n} = 0$$

$$= \sum a_n(m+n)(2m+2n-1)x^{m+n-1} - \sum\left[(m+n)\{2(m+n-1)+1\}-3\right]a_n x^{m+n} = 0$$

$$= \sum a_n(2m+2n-1)(m+n)x^{m+n-1} - \sum\left[(2m+2n-1)(m+n)-3\right]a_n x^{m+n} = 0$$

$$\sum (2m+2n-1)(m+n)a_n x^{m+n-1} - \sum\left[\{2(m+n)-1\}(m+n)-3\right]a_n x^{m+n} = 0$$

$$\sum (2m+2n-1)(m+n)a_n x^{m+n-1} - \sum\left[2(m+n)^2-(m+n)-3\right]a_n x^{m+n} = 0$$

$$- \sum\left[2(m+n)^2-3(m+n)+2(m+n)-3\right]a_n x^{m+n} = 0$$

$$- \sum\left[(2m+2n-3)(m+n-1)\right]a_n x^{m+n} = 0$$

Now put the coefficient of lowest power of x i.e. x^{m-1} to zero -

$$(2m-1)(m+0)a_0 = 0$$

which is called indical Eqn.

$(2m-1)\ m = 0$

$\boxed{m = 0,\ \tfrac{1}{2}}$ Both are distinct.

case-I - $m_1 - m_2 = \tfrac{1}{2} - 0 = \tfrac{1}{2} \neq 0$

$m_1 \neq m_2$

From Eq^n (i) -

↓ indexing

$$\sum[2m+2(n+1)-1](m+n+1)\,a_{n+1}\,x^{m+n} - \sum[(2m+2n-3)(m+n+1)a_n\,x^{m+n}] = 0$$

Note

$$\sum_{n=0}^{\infty}(2m+2n+1)(m+n+1)\,a_{n+1}x^{m+n} - \sum(2m+2n-3)(m+n+1)a_n x^{m+n} = 0$$

$$\sum_{n=0}^{\infty}\left[(2m+2n+1)(m+n+1)\,a_{n+1} - (2m+2n-3)(m+n+1)\,a_n\right]x^{m+n} = 0$$

Now by Equating coefficient of x^{m+n} to zero, we get -

$$(2m+2n+1)(m+n+1)\,a_{n+1} - (2m+2n-3)(m+n+1)\,a_n = 0$$

$$a_{n+1} = \frac{(2m+2n-3)\cancel{(m+n+1)}}{(2m+2n+1)\cancel{(m+n+1)}}\,a_n$$

which is the recurrance relation.

Put $n=0$, $\boxed{a_1 = \frac{2m-3}{2m+1}a_0}$

Put $n=1$, $a_2 = \frac{2m-1}{2m+3}a_1$

$$a_2 = \frac{(2m-1)(2m-3)}{(2m+3)(2m+1)}a_0$$

similarly $a_3, a_4 \ldots$

~~Now as we~~ know that complete sol^n of case-I

$$y = c_1(y)_{m=m_1} + c_2(y)_{m=m_2}$$

Trial solⁿ $y = \sum_{n=0}^{\infty} a_n x^{m+n}$

$= a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2} + \cdots$

Now put the values of $a_1, a_2, a_3 \ldots$ in trial solⁿ.

we get,

$$y = a_0 x^m + \frac{(2m-3)}{(2m+1)} a_0 \, x^{m+1} + \frac{(2m-1)(2m-3)}{(2m+3)(2m+1)} a_0 x^{m+2} + \cdots$$

Now at $m = 1/2$

$$(y)_{1/2} = a_0 \sqrt{x} + x^{3/2} \cdot a_0 \left(\frac{1-3}{1+1}\right) + \frac{(1-1)(1-3)}{(1+3)(1+1)} a_0 x^{5/2} + \cdots$$

$$(y)_{1/2} = a_0 \sqrt{x} - a_0 x^{3/2} + 0 - 0 \cdots$$

$$\boxed{(y)_{1/2} = (x^{1/2} - x^{3/2}) a_0}$$

Now at $m=0$,

$$(y)_{m=0} = a_0 x^0 + \left(\frac{-3}{1}\right) a_0 x^1 + \frac{(-1)(-3)}{(3)(1)} x^2 a_0 + \cdots$$

$$(y)_{m=0} = a_0 \left[1 - 3x + x^2 + \frac{1}{5} x^3 \cdots \infty\right]$$

$$y = c_1 (y)_{m=1/2} + c_2 (y)_{m=0}$$

$$\boxed{y = A(x^{1/2} - x^{3/2}) + B\left(1 - 3x + x^2 + \frac{x^3}{5} + \cdots \infty\right)}$$

this is complete solⁿ.

Q $xy'' + y' + x^2 y = 0$ solve the diff. eq^n in series.

Sol: $P_0(x) = x$, $P_1(x) = 1$ $P_2(x) = x^2$

at $x = 0$ →

$\lim_{x \to a} (x-a) \frac{P_1(x)}{P_0(x)}$ $\lim_{x \to a} (x-a)^2 \frac{P_2(x)}{P_0(x)}$

$\lim_{x \to 0} x \cdot \frac{1}{x} = \boxed{1}$ $\lim_{x \to 0} x^2 \frac{x^2}{x} = \boxed{0}$

both are finite.

so $x = 0$ is a regular singular pt. of given D.E.

By Frobenius Method.

$y = \sum a_n x^{m+n}$ (m can have any value)

$\frac{dy}{dx} = \sum a_n (m+n) x^{m+n-1}$

$\frac{d^2y}{dx^2} = \sum a_n (m+n)(m+n-1) x^{m+n-2}$

Now put the value of y, y', y'' in given eq^n.

$x \left[\sum a_n (m+n)(m+n-1) x^{m+n-2} \right] + \left[\sum (m+n) a_n x^{m+n-1} \right] + x^2 \left[\sum_{n=0}^{\infty} a_n x^{m+n} \right]$

$= \sum_{n=0}^{\infty} (m+n)(m+n-1+1) a_n x^{m+n-1} + \sum_{n=0}^{\infty} a_n x^{m+n+2} = 0$ —(i)

Now by equating least power of x equal to zero.

$(m^2) a_0 = 0$

$a_0 \neq 0$, $m = 0, 0$ [this is indicial eq^n]

so roots of indicial eq^n are same $(m_1 = m_2)$ [case II]

From Eqⁿ (1), [appling indexing]

$$\sum (m+n+1)^2 a_{n+1} x^{m+n} + \sum a_{n-2} x^{m+n} = 0$$

$$\sum_{n=0}^{\infty} \left[(m+n+1)^2 a_{n+1} + a_{n-2}\right] x^{m+n} = 0$$

Now put the coefficient of x^{m+n} equal to zero.

$$(m+n+1)^2 a_{n+1} + a_{n-2} = 0$$

$$\boxed{a_{n+1} = -\frac{1}{(m+n+1)^2} a_{n-2}} \quad \forall n \geq 2 \quad \text{---(2)}$$

which is the required recurrance relation.

Now put $n=2$, $a_3 = \frac{-1}{(n+3)^2} a_0$

Now by equating coeff. of (x^m) both sides in eqⁿ (1),

$$(m+1)^2 a_1 = 0$$

$$\boxed{a_1 = 0}$$

Now by equating coeff. of x^{m+1} both sides

$$(m+2)^2 a_2 = 0$$

$$\boxed{a_2 = 0}$$

Now from eqⁿ (2) we can find $a_4, a_5 \ldots$

$$a_4 = 0$$

$$a_5 = 0$$

$$a_6 = \frac{1}{(m+6)^2 (m+3)^2} \cdot a_0$$

$$y = \Sigma a_n x^{m+n}$$

$$= a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2} - - -$$

$$= x^m \left[a_0 - \frac{1}{(m+3)^2} a_0 x^3 \cdots \right]$$

Now complete solⁿ of case II is

$$y = c_1 (y)_{m_1=0} + c_2 \left(\frac{\partial y}{\partial m} \right)_{m_1=0} \quad \text{—— (3)}$$

$$(y)_{m_1=0} = a_0 x^0 \left[1 - \frac{1}{3^2} \cdot x^3 + \frac{1}{3^2 6^2} \cdot x^6 + - - \infty \right] \longrightarrow (4)$$

Now Partially diff. eqⁿ (4) w.r.t. m, we get

$$\left(\frac{\partial y}{\partial m} \right)_{m=0} = a_0 \log x \left[1 - \frac{x^3}{3^2} + \frac{x^6}{3^2 . 6^2} + - - \infty \right] +$$

$$a_0 \left[\frac{2x^3}{3^3} - \frac{2x^6}{3^3 . 6^2} + - - - \right]$$

Now Put these value in eqⁿ (3)