CHAPTER

APPLICATIONS OF PARTI DIFFERENTIAL EQUATIO

15.1 INTRODUCTION

In applied mathematics, the partial differential equations generally arise from the mathema formulation of physical problems. Subject to certain given conditions, called boundary condi solving such an equation is known as solving a boundary value problem.

The method of solution of such equations differs from that used in the case of ordi differential equations. We first find out the general solution of the ordinary differential equ and determine the particular solution with the help of given conditions. Here, from the start, w to find particular solutions of the partial differential equations which satisfy all the boun conditions. Method of separation of variables is employed to solve the applied partial differe equation.

15.2 METHOD OF SEPARATION OF VARIABLES

(*U.P., II Semester, June 2*

In this method, we assume that the dependent variable is the product of two functions, ea which involves only one of the independent variables. So two ordinary differential equation formed.

Example 1. *Solve* $\frac{\partial u}{\partial x} = 3\frac{\partial u}{\partial t}$ *using method of separation of variables.*

(*GBTU II Sem., Jan. 2*

Solution. Here, we have

$$\frac{\partial u}{\partial x} - 3\frac{\partial u}{\partial t} = 0 \quad ..$$

Let $$u = X(x)\, T(t) \quad ..$$

Where X is a function x only and T is a function of t only.

On differentiating (2) partially w.r.t x, we get

$$\frac{\partial u}{\partial x} = \frac{\partial X}{\partial x}\cdot T \quad ..$$

On differentiating (2) partially w.r.t. 't', we get

$$\frac{\partial u}{\partial t} = X\cdot\frac{\partial T}{\partial t} \quad ..$$

Putting the values of $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial t}$ from (3) and (4) in (1), we get

$$\frac{\partial X}{\partial x}\cdot T - 3\cdot X\frac{\partial T}{\partial t} = 0 \quad \text{... (5)}$$

Dividing (5) by XT, we get

$$\frac{1}{X}\cdot\frac{\partial X}{\partial x} - \frac{3}{T}\cdot\frac{\partial T}{\partial t} = 0$$

$$\Rightarrow \quad \frac{1}{x}\cdot\frac{\partial X}{\partial x} = \frac{3}{T}\frac{\partial T}{\partial t} = K$$

$$\Rightarrow \quad \frac{1}{X}\cdot\frac{\partial X}{\partial x} = K \text{ and } \frac{3}{T}\frac{\partial T}{\partial t} = K$$

$$\Rightarrow \quad \frac{\partial X}{X} = K\partial x \text{ and } \frac{\partial T}{T} = \frac{K}{3}\partial t$$

$$\Rightarrow \quad \log X = Kx + C_1 \text{ and } \log T = \frac{K}{3}t + C_2$$

$$\Rightarrow \quad X = e^{Kx + C_1} \text{ and } T = e^{\frac{K}{3}t + C_2}$$

Putting the values of X and T in (2), we get

$$u = e^{Kx+C_1}\cdot e^{\frac{K}{3}t+C_2} = e^{K\left(x+\frac{t}{3}\right)+C_1+C_2} = e^{k\left(x+\frac{t}{3}\right)}\cdot e^{C_1+C_2}$$

Hence, $u = A\cdot e^{K\left(x+\frac{t}{3}\right)}$, where $A = e^{C_1+C_2}$ **Ans.**

Example 2. *Solve the following equation* $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$ *by the method of separation of variables.* *(U.P., II Semester, Summer 2009, 2005)*

Solution. Given equation is

$$\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0 \quad \text{... (1)}$$

Let $\quad z = X(x)\,Y(y) \quad$ ... (2)

where X is a function of x only and Y is a function of y only.

$$\frac{\partial z}{\partial x} = Y\frac{dX}{dx}, \qquad \frac{\partial^2 z}{\partial x^2} = Y\frac{d^2 X}{dx^2}$$

$$\frac{\partial z}{\partial y} = X\frac{dY}{dy}$$

Putting all values in equation (1), we get

$$Y\frac{d^2X}{dx^2}-2Y\frac{dX}{dx}+X\frac{dY}{dy}=0$$

Dividing by XY, we have

$$\frac{1}{X}\frac{d^2X}{dx^2}-\frac{2}{X}\frac{dX}{dx}+\frac{1}{Y}\frac{dY}{dy}=0$$

Separating the variables, we have

$$\frac{1}{X}\frac{d^2X}{dx^2}-\frac{2}{X}\frac{dX}{dx}=-\frac{1}{Y}\frac{dY}{dy}=K\ (let)$$

where K is a constant.

$$\frac{1}{X}\frac{d^2X}{dx^2}-\frac{2}{X}\frac{dX}{dx}=K$$

$$\frac{d^2X}{dx^2}-2\frac{dX}{dx}=KX$$

$\Rightarrow$ $$(D^2-2D-K)X=0$$

A. E. is

$$m^2-2m-K=0$$

$$m=\frac{2\pm\sqrt{4+4K}}{2}$$

$$m=1\pm\sqrt{1+K}$$

$$-\frac{1}{Y}\frac{dY}{dy}=K$$

$$\frac{dY}{dy}+KY=0$$

$$(D+K)Y=0$$

A.E. is $m+K=0 \quad\Rightarrow\quad m=-K$

$$\Rightarrow\quad Y=C_3e^{-Ky}\quad ...(4)$$

Thus $$X=C_1e^{(1+\sqrt{1+K})x}+C_2e^{(1-\sqrt{1+K})x} \quad ...(3)$$

Putting the values of X and Y from (3) and (4) in (2), we get

$$Z=\left\{C_1e^{(1+\sqrt{1+K})x}+C_2e^{(1-\sqrt{1+K})x}\right\}C_3e^{-Ky} \quad \textbf{Ans.}$$

Example 3. *Using the method of separation of variables, solve*

$$\frac{\partial u}{\partial x}=2\frac{\partial u}{\partial t}+u$$

where $u(x,0)=6e^{-3x}$ *(U.P. II Semester summer 2006, A.M.I.E.T.E., Summer 2002)*

Solution. $$\frac{\partial u}{\partial x}=2\frac{\partial u}{\partial t}+u \quad ...(1)$$

Let $u=X(x).T(t)$ ... (2)

where X is a function of x only and T is a function of t only.

Putting the value of u in (1), we get

$$\frac{\partial(X.T)}{\partial x}=2\frac{\partial}{\partial t}(X.T)+X.T$$

$$T\frac{dX}{dx}=2X\frac{dT}{dt}+X.T$$

On separating the variables, we get

$$\frac{1}{X}\frac{dX}{dx}=\frac{2}{T}\frac{dT}{dt}+1=C \quad \text{[On dividing by } XT]$$

$$\frac{1}{X}\frac{dX}{dx} = C$$

$$\Rightarrow \quad \frac{dX}{dx} = CX$$

$$\Rightarrow \quad DX - CX = 0$$

$$\Rightarrow \quad (D - C)\,X = 0$$

A.E. is $m - C = 0 \Rightarrow m = C$

$$\Rightarrow \quad X = ae^{cx}$$

$$\frac{2}{T}\frac{dT}{dt} + 1 = C$$

$$\Rightarrow \quad \frac{dT}{dt} + \frac{T}{2} = \frac{CT}{2}$$

$$\Rightarrow \quad DT - \left(\frac{C}{2} - \frac{1}{2}\right)T = 0$$

A.E. is $m - \left(\frac{C}{2} - \frac{1}{2}\right) = 0$

$$\Rightarrow \quad m = \frac{1}{2}(C-1)$$

$$\Rightarrow \quad T = be^{\frac{1}{2}(c-1)t}$$

Putting the values of X and T in (2), we have $u = ae^{cx}.be^{\frac{1}{2}(c-1)t}$

$$\Rightarrow \quad u = abe^{cx+\frac{1}{2}(c-1)t}$$

On putting $t = 0$ and $u = 6e^{-3x}$ in (3), we get

$$6e^{-3x} = abe^{cx} \qquad \Rightarrow ab = 6 \text{ and } c = -3$$

Putting the values of ab and c in (3), we have

$$u = 6e^{-3x+\frac{1}{2}(-3-1)t}$$

$$u = 6e^{-3x-2t}$$

which is the required solution.

Example 4. *Use the method of separation of variables to solve the equatio*

Again comparing b_3, we get

$$2C_1\, C_3\, \sqrt{k} = 1, \Rightarrow C_1\, C_3 = \frac{1}{2\sqrt{k}} = \frac{1}{2\sqrt{-1}} = \frac{1}{2\,i} \qquad (\because k = -1)$$

Hence, $u\,(x,y) = \dfrac{1}{2\sqrt{2}}\,(e^{\sqrt{2}x} - e^{-\sqrt{2}x}) + \dfrac{1}{2i}\,(e^{ix} - e^{-ix})\, e^{-3y}$

$$u\,(x,y) = \frac{1}{\sqrt{2}}\sinh\sqrt{2}\;x + e^{-3y}\,\sin x. \qquad \textbf{Ans.}$$

EXERCISE 30.1

Using the method of separation of variables, find the solution of the following equations

1. $2x\dfrac{\partial z}{\partial x} - 3y\dfrac{\partial z}{\partial y} = 0$ **Ans.** $z = cx^{\frac{k}{2}}y^{\frac{k}{3}}$

2. $\dfrac{\partial u}{\partial x} + u = \dfrac{\partial u}{\partial t}$ if $u = 4e^{-3x}$ when $t = 0$ **Ans.** $u = 4e^{-3x-2t}$

3. $4\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} = 3u$ and $u = e^{-5y}$ when $x = 0$ **Ans.** $u = e^{2x-5y}$

4. $4\dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial x} = 3u$, $u = 3e^{-x} - e^{-5x}$ at $t = 0$ *(A.M.I.E.T.E., Winter 2000)* **Ans.** $u = 3e^{t-x} - e^{2t-5x}$

5. $3\dfrac{\partial u}{\partial x} + 2\dfrac{\partial u}{\partial y} = 0$; $u(x, 0) = 4e^{-x}$ *(A.M.I.E.T.E., Summer 2000)* **Ans.** $u = 4e^{-x+3/2y}$

6. $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} = 2(x + y)u$ **Ans.** $u = ce^{x^2+y^2+k(x-y)}$

7. $\dfrac{\partial u}{\partial t} = 4\dfrac{\partial^2 u}{\partial x^2}$ if $u(x, 0) = 4x - \dfrac{1}{2}x^2$ **Ans.** $u = \left(4x - \dfrac{x^2}{2}\right)e^{-p^2 t}$

8. $\dfrac{\partial^2 u}{\partial x^2} = 2\dfrac{\partial u}{\partial t}$ if $u(x,0) = x(4 - x)$ **Ans.** $u = x(4 - x)e^{\frac{p^2 t}{2}}$

9. $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$ if $u(x,0) = 2x$ when $0 \le x \le \dfrac{l}{2}$

   $= 2(l - x)$ when $\dfrac{l}{2} \le x \le l$

   Ans. $u = 2xe^{-h^2 t}$ for $0 \le x \le \dfrac{l}{2}$, $u = 2(l - x)e^{-h^2 t}$ for $\dfrac{l}{2} \le x \le l.$

10. $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$ if $u(x, 0) = \sin \pi x$ **Ans.** $u = \sin \pi x.e^{-pl}$

11. $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}$ if $u(x,0) = x^2(25 - x^2)$ **Ans.** $u = x^2(25 - x^2)e^{-p^2 t}$

12. $\dfrac{\partial^2 z}{\partial x^2} - 2\dfrac{\partial z}{\partial x} - \dfrac{\partial z}{\partial y} = 0$ **Ans.** $z = c_1 e^{[1+\sqrt{(1+p)}]x+p^2 y} + c_2 e^{[1-\sqrt{1+P}]x+p^2 y}$

13. $\dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial^2 u}{\partial x^2}$ If $u(x,0) = \dfrac{1}{2}x(1 - x)$ **Ans.** $u = \dfrac{x}{2}(1 - x)\cos pt + c_2 \sin pt(c_3 \cos px + c_4 \sin px)$

14. $16\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2}$ if $u(x,0) = x^2(5 - x)$ **Ans.** $u = x^2(5 - x)\cos pt + c_4 \sin pt\left(c_1 \cos\dfrac{px}{4} + c_2 \sin\dfrac{px}{4}\right)$