Methods of Finding P.I.

$$F(D)y = Q$$

Case (II) when $Q(x) = \sin ax \,|\, \cos ax$

$$P.I. = \frac{1}{f(D^2)} \sin ax \,|\, \cos ax$$

$$= \frac{1}{f(-a^2)} \sin ax \,|\, \cos ax \quad \text{provided } f(-a^2) \neq 0$$

Note: In that case, we replace $D^2 \to -a^2$, $D^3 \to -a^2 D$, $D^4 \to (-a^2)^2 = a^4$ ....

Ques (1) Find the P.I. of D.E. $(D^2+4)y = \sin 3x + \cos x$

Solution: $P.I. = \frac{1}{f(D)} Q(x) = \frac{1}{(D^2+4)} (\sin 3x + \cos x)$

$$= \frac{1}{D^2+4} \sin 3x + \frac{1}{D^2+4} \cos x$$

$$= \frac{1}{-(3)^2+4} \sin 3x + \frac{1}{-(1)^2+4} \cos x$$

$$= \frac{1}{-9+4} \sin 3x + \frac{1}{-1+4} \cos x = -\frac{1}{5} \sin 3x + \frac{1}{3} \cos x \quad \underline{\text{Ans}}$$

Ques (2) Find the P.I. of D.E. $(D^3+1)y = \sin(2x+1)$

Solution: $P.I. = \frac{1}{(D^3+1)} \sin(2x+1)$ $\quad (a=2)$ $\quad D^3 \to -(2)^2 D = -4D$

$$= \frac{1}{(-4D+1)} \sin(2x+1)$$

$$= \frac{1}{(1-4D)} \sin(2x+1)$$

$$= \frac{(1+4D)}{(1-4D)(1+4D)} \sin(2x+1)$$

$$= \frac{(1+4D)}{1-16D^2} \sin(2x+1)$$

$$= \frac{(1+4D)}{1-16(-4)} \sin(2x+1)$$

$$= \frac{(1+4D)(\sin(2x+1))}{65} = \frac{1}{65} (\sin(2x+1) + 4D\sin(2x+1))$$

$$= \frac{1}{65} \left(\sin(2x+1) + 4\frac{d}{dx} \sin(2x+1)\right)$$

$$P.I. = \frac{1}{65}[\sin(2x+1) + 8\cos(2x+1)] \text{ Ans}$$ ②

Case of Failure: If $f(-a^2) = 0$

$$P.I. = x \frac{1}{f'(D^2)} \sin ax | \cos ax$$

$$= x \frac{1}{f'(-a^2)} \sin ax | \cos ax \quad (\text{provided } f'(-a^2) \neq 0)$$

If $f'(-a^2) = 0$ again case of failure

$$P.I. = x^2 \frac{1}{f''(D^2)} \sin ax | \cos ax$$

$$= x^2 \frac{1}{f''(-a^2)} \sin ax | \cos ax \quad (\text{provided } f''(-a^2) \neq 0) \text{ and so on.}$$

Ques (3) Find the P.I. of $(D^2+4)y = \sin 2x$

Solution: $P.I. = \frac{1}{(D^2+4)} \sin 2x = \frac{1}{-(2)^2+4} \sin 2x$ (case of failure)

$$= x \frac{1}{2D} \sin 2x = \frac{x}{2} \frac{1}{D} \sin 2x$$

$$= \frac{x}{2} \int \sin 2x \, dx = \frac{x}{2}\left(-\frac{1}{2}\right)\cos 2x$$

$$= -\frac{x}{4} \cos 2x \text{ Ans}$$

Ques (4) Find P.I. of $(D^2-4)y = \cos^2 x$.

Solution: $P.I. = \frac{1}{(D^2-4)} \cos^2 x = \frac{1}{(D^2-4)} \frac{1}{2}(1+\cos 2x) = \frac{1}{2} \frac{1}{D^2-4}(1+\cos 2x)$

$$= \frac{1}{2}\left(\frac{1}{D^2-4} 1 + \frac{1}{D^2-4} \cos 2x\right)$$

$$= \frac{1}{2}\left(\frac{1}{0-4} 1 + \frac{1}{(-4)-4} \cos 2x\right)$$

$$= \frac{1}{2}\left(-\frac{1}{4} - \frac{1}{8} \cos 2x\right)$$

$$P.I. = -\frac{1}{8} - \frac{1}{16} \cos 2x \text{ Ans}$$

Ques (5) Solve the D.E. $(D^2+4)y = \cos x \cos 3x$.

Solution: $(D^2+4)y = 0$

A.E. $m^2+4=0 \Rightarrow m = \pm 2i \quad (\alpha = 0, \beta = 2)$

C.F. $= e^{0x}(c_1 \cos 2x + c_2 \sin 2x) = c_1 \cos 2x + c_2 \sin 2x$

$$P.I. = \frac{1}{D^2+4}\cos x\cos 3x$$

$$= \frac{1}{D^2+4}\frac{1}{2}(2\cos x\cos 3x)$$

$$= \frac{1}{2}\frac{1}{D^2+4}(\cos 4x + \cos 2x) \quad [2\cos A\cos B = \cos(A+B)+\cos(A-B)]$$

$$= \frac{1}{2}\left(\frac{1}{D^2+4}\cos 4x + \frac{1}{D^2+4}\cos 2x\right)$$

$$= \frac{1}{2}\left(\frac{1}{-16+4}\cos 4x + \frac{1}{-(2)^2+4}\cos 2x\right)$$

$\Downarrow$ (COF)

$$= \frac{1}{2}\left(-\frac{1}{12}\cos 4x + x\frac{1}{2D}\cos 2x\right)$$

$$= -\frac{1}{24}\cos 4x + \frac{1}{2}x\frac{1}{2}\frac{1}{D}\cos 2x$$

$$= -\frac{1}{24}\cos 4x + \frac{x}{4}\int \cos 2x\, dx$$

$$P.I. = -\frac{1}{24}\cos 4x + \frac{x}{4}\left(\frac{\sin 2x}{2}\right)$$

$$P.I. = -\frac{1}{24}\cos 4x + \frac{x}{8}\sin 2x$$

Hence, the solution of given D.E. is

$$y = C.F. + P.I.$$

$$\boxed{y = c_1\cos 2x + c_2\sin 2x - \frac{1}{24}\cos 4x + \frac{x}{8}\sin 2x} \text{ — Ans}$$

Case (III) When $Q(x) = x^m$ where m is +ve integer.

$$P.I. = \frac{1}{f(D)}Q$$

Procedure: Take out the lowest degree term as a common in $f(D)$ and then use the Binomial Expansion.

Note: Binomial expansion is given by

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{\lfloor 2}x^2 + \frac{n(n-1)(n-2)}{\lfloor 3}x^3 + \cdots \quad (n \in I)$$

Ques (1) Solve the D.E. $(D^2+5D+4)y = x$

Solution: $(D^2+5D+4)y = 0$

A.E. is given by $m^2+5m+4=0 \Rightarrow (m+1)(m+4)=0$

$\Rightarrow m = -1, -4$

$$C.F. = c_1 e^{-x} + c_2 e^{-4x}$$

$$P.I. = \frac{1}{D^2+5D+4}\, x$$

$$= \frac{1}{4\left[1+\frac{1}{4}(D^2+5D)\right]}\, x$$

$$= \frac{1}{4}\left[1+\frac{1}{4}(D^2+5D)\right]^{-1} x$$

$$= \frac{1}{4}\left(1 - \frac{1}{4}(D^2+5D) + \frac{1}{16}(D^2+5D)^2 + \cdots\right) x$$

$$= \frac{1}{4}\left(x - \frac{1}{4}(D^2+5D)x + 0\right)$$

$$= \frac{1}{4}\left(x - \frac{1}{4}\left(\frac{d^2}{dx^2}x + 5\frac{d}{dx}x\right)\right)$$

$$= \frac{1}{4}\left(x - \frac{1}{4}(0+5\cdot 1)\right)$$

$$= \frac{1}{4}\left(x - \frac{5}{4}\right) = \frac{x}{4} - \frac{5}{16}$$

Hence, the solution of given D.E. is

$$y = C.F. + P.I.$$

$$y = c_1 e^{-x} + c_2 e^{-4x} + \frac{x}{4} - \frac{5}{16}$$ Ans

Ques (2) Solve the D.E. $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + \frac{dy}{dx} = e^{0x} + x^2 + x$

Solution: Here, we have $(D^3+2D^2+D)y = e^{0x} + x^2 + x$

$$(D^3+2D^2+D)y = 0$$

A.E. is given by $m^3+2m^2+m=0$

$$m(m^2+2m+1)=0$$

$$m(m+1)^2 = 0 \Rightarrow m = 0,-1,-1$$

$$C.F. = c_1 e^{0x} + (c_2 + x c_3)e^{-x}$$

$$C.F. = c_1 + (c_2 + x c_3)e^{-x}$$

$$P.I. = \frac{1}{D^3+2D^2+D}\left(e^{0x} + x + x^2\right)$$

$$= \frac{1}{D^3+2D^2+D}(e^x) + \frac{1}{D^3+2D^2+D}\, x+x^2$$

$$= \frac{1}{1+2+1}\, e^x + \frac{1}{D(D^2+2D+1)}\, x+x^2$$

$$= \frac{1}{4} e^x + \frac{1}{D(D+1)^2}\, x+x^2$$

$$= \frac{1}{4} e^x + \frac{1}{D}(1+D)^{-2}(x+x^2)$$

$$= \frac{1}{4} e^x + \frac{1}{D}(1-2D+3D^2-4D^3+\cdots)(x+x^2)$$

(By Binomial Expansion)

$$= \frac{1}{4} e^x + \frac{1}{D}\left[(x+x^2) - 2D(x+x^2) + 3D^2(x+x^2) + 0 + 0 + \cdots\right]$$

$$= \frac{1}{4} e^x + \frac{1}{D}\left[x+x^2 - 2(1+2x) + 3(2)\right]$$

$$= \frac{1}{4} e^x + \frac{1}{D}(x+x^2-2-4x+6)$$

$$= \frac{1}{4} e^x + \frac{1}{D}(x^2-3x+4)$$

$$P.I. = \frac{1}{4} e^x + \frac{x^3}{3} - \frac{3x^2}{2} + 4x$$

Hence, the solution of given, D.E. is

$$y = C.F + P.I.$$

$$\boxed{y = C_1 + (C_2 + xC_3)e^{-x} + \frac{1}{4} e^x + \frac{x^3}{3} - \frac{3x^2}{2} + 4x}$$

Case (IV) When $Q = e^{ax} V$ where V is a function of x

$$P.I. = \frac{1}{f(D)} e^{ax} \cdot V = e^{ax} \frac{1}{f(D+a)} V$$

Ques: Solve the D.E. $(D^2-2D+4)y = e^x \cos x$

Solution: $(D^2-2D+4)y = 0$

A.E. $m^2-2m+4=0$

$$m = \frac{2 \pm \sqrt{4-16}}{2}$$

$$m = \frac{2 \pm \sqrt{12}i}{2} = \frac{2 \pm 2\sqrt{3}i}{2} = 1 \pm \sqrt{3}i$$

$$C.F = e^{x}(c_1\cos\sqrt{3}x + c_2\sin\sqrt{3}x)$$

$$P.I. = \frac{1}{D^2-2D+4}\, e^{x}\cos x$$

$$= e^{x}\frac{1}{(D+1)^2-2(D+1)+4}\cos x$$

$$= e^{x}\frac{1}{D^2+1+\cancel{2D}-\cancel{2D}-2+4}\cos x$$

$$= e^{x}\frac{1}{D^2+3}\cos x$$

$$P.I = e^{x}\frac{1}{-(1)^2+3}\cos x = \frac{e^{x}\cos x}{2}$$

$$y = C.F. + P.I. = e^{x}(c_1\cos\sqrt{3}x + c_2\sin\sqrt{3}x) + \frac{e^{x}}{2}\cos x$$

Ans

Ques(2) Solve the D.E. $y'' - 2y' + 2y = e^{x}\sin x$

Solution: Here, we have $(D^2-2D+2)y = e^{x}\sin x$

$$(D^2-2D+2)y = 0$$

$$A.E.\quad m^2 - 2m + 2 = 0 \Rightarrow m = \frac{2 \pm \sqrt{4-8}}{2}$$

$$m = \frac{2\pm\sqrt{-4}}{2} = \frac{2\pm 2i}{2} = 1 \pm i$$

$$C.F. = e^{x}(c_1\cos x + c_2\sin x)$$

$$P.I. = \frac{1}{D^2-2D+2}\, e^{x}\sin x$$

$$= e^{x}\frac{1}{(D+1)^2-2(D+1)+2}\sin x$$

$$= e^{x}\frac{1}{D^2+1+\cancel{2D}-\cancel{2D}-\cancel{2}+\cancel{2}}\sin x$$

$$= e^{x}\frac{1}{D^2+1}\sin x$$

$$= e^{x}\frac{1}{-1+1}\sin x \quad (C.F)$$

$$= xe^{x}\frac{1}{2D}\sin x = -\frac{1}{2}xe^{x}\cos x = P.I.$$

$$y = e^{x}(c_1\cos x + c_2\sin x) - \frac{1}{2}xe^{x}\cos x$$ Ans

Case(V) When Q is any other function of x.

$$\frac{1}{D-\alpha} Q = e^{\alpha x}\int e^{-\alpha x} Q\,dx$$

$$\frac{1}{D+\alpha} Q = e^{-\alpha x}\int e^{\alpha x} Q\,dx$$

Cauchy-Euler's Equation | Linear Differential Equation with variable coefficient | Equation Reducible to Linear Differential Equation with Constant Coefficient.

General form of CEE is given by

$$a_0 x^n \frac{d^n y}{dx^n} + a_1 x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + a_2 x^{n-2}\frac{d^{n-2}y}{dx^{n-2}} + \dots + a_n y = Q \quad -(1)$$

where $a_0, a_1, \dots a_n$ are constants and Q is a constant or function of x only.

Put $x = e^z$ in Equation (1) so that $z = \log x$ and let $\theta \equiv \frac{d}{dz}$

Replace $x\frac{d}{dx}$ by θ

$x^2\frac{d^2}{dx^2}$ by $\theta(\theta-1)$

$x^3\frac{d^3}{dx^3}$ by $\theta(\theta-1)(\theta-2)$ and so on.

Finally, Equation (1) Reduces to Linear Differential Equation with Constant coefficient as z is the independent variable. In the final Solution we put $z = \log x$.

Ques(1) $x^3\frac{d^3y}{dx^3} + 2x^2\frac{d^2y}{dx^2} + 2y = 10\left(x+\frac{1}{x}\right)$

Solution: Put $x = e^z$ so that $z = \log x$ and $\theta \equiv \frac{d}{dz}$

$$[\theta(\theta-1)(\theta-2) + 2\theta(\theta-1) + 2]y = 10(e^z + e^{-z})$$

$$(\theta(\theta^2-3\theta+2) + 2\theta^2 - 2\theta + 2)y = 10(e^z+e^{-z})$$

$$(\theta^3 - 3\theta^2 + 2\theta + 2\theta^2 - 2\theta + 2)y = 10(e^z+e^{-z})$$

$$(\theta^3 - \theta^2 + 2)y = 10(e^z + e^{-z}) \quad -(1)$$

Equation (1) is LDE with constant coefficient with z as an independent variable.

$(\theta^3-\theta^2+2)y=0$

A.E. $m^3-m^2+2=0$

$(m+1)(m^2-2m+2)=0$

$m=-1,\quad m=\frac{2\pm\sqrt{4-8}}{2}=\frac{2\pm 2i}{2}=1\pm i \Rightarrow m=-1, 1\pm i$

$C.F.=c_1e^{-z}+e^{z}(c_1\cos z+c_2\sin z)$

$P.I.=\frac{1}{10}\frac{1}{\theta^3-\theta^2+2}(e^{z}+e^{-z})$

$=\frac{1}{10}\left(\frac{1}{\theta^3-\theta^2+2}e^{z}+\frac{1}{\theta^3-\theta^2+2}e^{-z}\right)$

$=\frac{1}{10}\left(\frac{1}{1-1+2}e^{z}+\frac{1}{-1-1+2}e^{-z}\right)\rightarrow(C.F)$

$=\frac{1}{10}\left(\frac{1}{2}e^{z}+z\frac{1}{3\theta^2-2\theta}e^{-z}\right)$

$=\frac{1}{10}\left(\frac{1}{2}e^{z}+z\frac{1}{3+2}e^{-z}\right)$

$P.I.=\frac{1}{10}\left(\frac{1}{2}e^{z}+\frac{z}{5}e^{-z}\right)$

$P.I.=\frac{1}{20}e^{z}+\frac{z}{50}e^{-z}$

$y=c_1e^{-z}+e^{z}(c_1\cos z+c_2\sin z)+\frac{1}{20}e^{z}+\frac{z}{50}e^{-z}$

$y=c_1\cdot\frac{1}{x}+x\,(c_1\cos(\log x)+c_2\sin(\log x))+\frac{1}{20}x+\frac{\log x}{50}\cdot\frac{1}{x}$

$y=\frac{c_1}{x}+x(c_1\cos(\log x)+c_2\sin(\log x))+\frac{x}{20}+\frac{1}{50x}\log x$ Ans

Ques: (2) Solve the D.E. $x^2\frac{d^2y}{dx^2}+4x\frac{dy}{dx}+2y=0$

Solution: Put $x=e^{z}$, so that $z=\log x$ and $\frac{d}{dx}\equiv\theta$

$(\theta(\theta-1)+4\theta+2)y=0 \Rightarrow (\theta^2-\theta+4\theta+2)y=0$

$(\theta^2+3\theta+2)y=0$ — (1)

Equation (1) is a Linear Differential Equation with constant coefficient.

A.E. $m^2+3m+2=0 \Rightarrow (m+1)(m+2)=0 \Rightarrow m=-1,-2$

$C.F.=c_1e^{-z}+c_2e^{-2z}$

$y=c_1e^{-z}+c_2e^{-2z}\Rightarrow \boxed{y=c_1\frac{1}{x}+c_2\frac{1}{x^2}}$ Ans