Partial Differential Equation

Homogeneous PDE:-

$$F(D,D')z = \phi(x,y) \text{ of degree } n$$

Case 1st:- $\phi(x,y)$ is function of $ax+by$.

Step 1: replace D by a and D' by b in $F(D,D')$ to get $F(a,b)$.

Step 2: Put $ax+by = u$ and integrate $\phi(u)$, n times w.r.t u

i.e. $$P.I. = \frac{1}{F(a,b)} \iiint \dots \int \phi(u) \underbrace{du\,du\,du \dots du}_{n \text{ times}}$$

Step 3: replace u by $ax+by$.

<u>Note</u> If $F(a,b)=0$, then method fails.

Case of failure :- i.e $F(D,D')z = \phi(ax+by)$; $F(a,b)=0$

Step 1: differentiate $F(D,D')$ partially w.r.t to D and multiply the expansion by x.

i.e. $$P.I. = \frac{1}{F(D,D')}\phi(ax+by); \; F(a,b)=0$$

$$= x \frac{1}{\frac{\partial}{\partial D} F(D,D')}\phi(ax+by)$$

we proceed this method as long as derivative $F(D,D')$ vanishes when D is replaced by a and D' by b.

<u>Note</u> If Highest derivative term in D' in $F(D,D')$ then in case of failure:

$$P.I. = \frac{1}{F(D,D')}\phi(ax+by); \; F(a,b)=0$$

$$= y \frac{1}{\frac{\partial}{\partial D'} F(D,D')}\phi(ax+by)$$

Case ii): If $\phi(x,y) = x^m y^n$ $\quad m > n$

To obtained P.I., we expand $F(D, D')$ in an infinite series of ascending powers of D or D' or D/D' or $\frac{D'}{D}$

Example 1. $\frac{\partial^3 z}{\partial x^3} - \frac{\partial^3 z}{\partial y^3} = x^3 y^3$

The given P.D.E. $(D^3 - D'^3)z = x^3 y^3$

A.E. $m^3 - 1 = 0 \qquad D \to m, \; D' \to 1$

$(m-1)(m^2+m+1) = 0$

$\Rightarrow m = 1, \omega, \omega^2$

where ω is cube root of unity

C.F. $= f_1(y+x) + f_2(y+\omega x) + f_3(y+\omega^2 x)$

$$P.I = \frac{1}{D^3 - D'^3} \cdot (x^3 y^3)$$

$$= \frac{1}{D^3\left\{1 - \frac{D'^3}{D^3}\right\}} (x^3 y^3)$$

$$= \frac{1}{D^3}\left(1 - \frac{D'^3}{D^3}\right)^{-1} (x^3 y^3) \qquad \because (1-x)^{-1} = 1 + x + x^2 + \cdots\cdots$$

$$= \frac{1}{D^3}\left(1 + \frac{D'^3}{D^3} + \cdots \infty\right)(x^3 y^3)$$

$$= \frac{1}{D^3}\left(x^3 y^3 + \frac{1}{D^3} x^3 D'^3(y^3) + \cdots \infty\right)$$

$$= \frac{1}{D^3}\left(x^3 y^3 + \frac{1}{D^3} x^3 . 6 + 0 + 0 + \cdots\right)$$

$$= y^3 \frac{1}{D^3} x^3 + 6 . \frac{1}{D^6} x^3$$

$$= \frac{x^6 y^3}{6 \cdot 5 \cdot 4} + \frac{6 x^9}{4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9} = \frac{x^6 y^3}{120} + \frac{x^9}{10080} \text{ Ans}$$

Complete solution = C.F + P.I.

$$z = f_1(y+x) + f_2(y+\omega x) + f_3(1+\omega^2 x) + \frac{x^6 y^3}{120} + \frac{x^9}{10080}$$

Q2 $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = e^{x+2y} + x^3 + xy$

Soln A.E: $(D^2 - 2DD' + D'^2)z = e^{x+2y} + x^3 + xy$

A.E. $m^2 - 2m + 1 = 0 \Rightarrow m = 1, 1$

C.F. $= f_1(y+x) + x f_2(y+x)$

P.I. $= \frac{1}{D^2 - 2DD' + D'^2}\{e^{x+2y} + x^3 + xy\} = P_1 + P_2 + P_3$

$x + 2y = u$

$P_1 = \frac{1}{D^2 - 2DD' + D'^2} e^{x+2y}$, replace D by 1 D' by 2

$= \frac{1}{(D-D')^2} e^{x+2y} = \frac{1}{(1-2)^2}\iint e^u \, du\, du =$

$= \int e^u du = e^u = e^{x+2y}$

$P_2 = \frac{1}{(D-D')^2} x^3 =$ Here $x^3 = (1x + 0y)^3$ Let $u = 1x + 0y$
i.e $a = 1 \quad b = 0$

$= \frac{1}{(1-0)^2}\iint u^3 \, du\, du$

$= \frac{u^5}{4 \cdot 5} = \frac{x^5}{20}$

$P_3 = \frac{1}{(D-D')^2} xy = \frac{1}{D^2}\left\{1 - \frac{D'}{D}\right\}^{-2} xy$

$= \frac{1}{D^2}\left\{1 + 2\frac{D'}{D} \cdots \infty\right\} xy$ leaving higher term.

$= \frac{1}{D^2}\left\{xy + \frac{2}{D} D'(xy) + 0 + 0 + \cdots\right\}$

$= \frac{1}{D^2}\left\{xy + \frac{2}{D} x\right\} = \frac{1}{D^2}\{xy + x^2\}$

$= \frac{x^3}{2 \cdot 3} y + \frac{x^4}{3 \cdot 4} = \frac{x^3 y}{6} + \frac{x^4}{12}$

Hence Soln $z = C.F. + P.I.$

$= f_1(y-x) + x f_2(y-x) + e^{x+2y} + \frac{x^5}{20} + \frac{x^4}{12}$
$+ \frac{x^3 y}{6}$

Ans.

Q $\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = x^2 + xy + y^2$

Soln PDE $(D^2 + 2DD' + D'^2)z = x^2 + xy + y^2$

A.E $m^2 + 2m + 1 = 0$

$\Rightarrow (m+1)^2 = 0 \Rightarrow m = -1, -1$

C.F. $= f_1(y-x) + x f_2(y-x)$

P.I. $= \frac{1}{F(D,D')}\phi(x,y) = \frac{1}{(D^2+2DD'+D'^2)} \cdot x^2 + xy + y^2$

$= P_1 + P_2 + P_3$

$\Rightarrow P_1 = \frac{1}{(D^2+2DD'+D'^2)} x^2 = \frac{1}{(D+D')^2} x^2 = \frac{1}{(D+D')^2}(1x+0y)^2$

$= \frac{1}{(1+0)^2}\iint u^2\,du\,du$; Let $1x+0y = u$ replace $D \to 1$ $D' \to 0$

$= \frac{u^4}{3\cdot 4} = \frac{x^4}{12}$

$P_2 = \frac{1}{(D+D')^2} xy = \frac{1}{D^2\left(1+\frac{D'}{D}\right)^2} xy = \frac{1}{D^2}\left\{1+\frac{D'}{D}\right\}^{-2} xy$

$= \frac{1}{D^2}\left\{1 - \frac{2D'}{D} + \cdots\right\} xy$ (leaving the higher term of derivatives)

$= \frac{1}{D^2}\left\{xy - \frac{2D'xy}{D}\right\}$

$= \frac{1}{D^2}\left\{xy - \frac{2x}{D}\right\} = \frac{1}{D^2}\left\{xy - \frac{2x^2}{2}\right\}$

$= \frac{1}{D}\left(\frac{x^2y}{2} - \frac{x^3}{3}\right) = \frac{x^3y}{6} - \frac{x^4}{12}$

$P_3 = \frac{1}{(D+D')^2} y^2 = \frac{1}{(0+01)^2}\iint v^2\,dv\,dv$ Let $y = 0x + 1y = v$ $D \to 0$ $D' \to 1$

$= \frac{1}{(0+1)^2}\left\{\frac{v^4}{4\cdot 3\cdot 2}\right\} = \frac{y^4}{6\times 2} = \frac{y^4}{12}$

General Method to solve P.I.

$\because$ $P.I. = \dfrac{1}{F(D,D')}\phi(x,y)$; $F(D,D')$ is PDE of ~~degree~~ order n.

$$= \frac{1}{(D-m_1D')(D-m_2D')\cdots\cdots(D-m_nD')}\phi(x,y)$$

$$= \frac{1}{(D-m_1D')}\cdot\frac{1}{(D-m_2D')}\cdots\frac{1}{(D-m_nD')}\phi(x,y)$$

First we solve $\dfrac{1}{D-m_1D'}\phi(x,y) = \int \phi(x, c-mx)\,dx$

Here $y = c-mx$

In last again replace $c = y+mx$.

Similarly we solve P.I. for others factor $(D-m_2D')$ .... $(D-m_nD')$.

Q 1: $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial x\partial y} - \dfrac{6\partial^2 z}{\partial y^2} = y\cos x$;

or $r+s-6t = y\cos x$

$p = \dfrac{\partial z}{\partial x}$, $q = \dfrac{\partial z}{\partial y}$, $r = \dfrac{\partial^2 z}{\partial x^2}$

$s = \dfrac{\partial^2 z}{\partial x\partial y}$, $t = \dfrac{\partial^2 z}{\partial y^2}$

PDE: $(D^2+DD'-6D'^2)z = y\cos x$;

A.E. $m^2+m-6 = 0 \Rightarrow (m-2)(m+3) = 0$

$\Rightarrow m = 2, -3$

 D D'

C.F. $= f_1(y+2x) + f_2(y-3x)$ $\qquad y+mx = c$

$P.I. = \dfrac{1}{D^2+DD'-6D'^2}y\cos x =$ *

$= \dfrac{1}{(D-2D')(D+3D')}y\cos x$,

Replace $y = c-mx$, $m = -3$

$= c+3x$

$= \dfrac{1}{D-2D'}\int (c+3x)\cos x\,dx$

$= \dfrac{1}{D-2D'}\left[c\sin x + 3\{x\sin x + \cos x\}\right]$ *

$= \dfrac{1}{D-2D'}\{(c+3x)\sin x + 3\cos x\} = \dfrac{1}{D-2D'}\{y\sin x + 3\cos x\}$

$c+3x \rightarrow y$

$= \frac{1}{D-2D'}\{y\sin x+3\cos x\}$ $\because 3x+c=y$

$m=2$

$= \int(b-2x)\sin x\,dx+3\int\cos x\,dx$ $\because b-2x=y$

$= \int b\sin x\,dx - 2\int x\sin x + 3\int\cos x\,dx$ **

$= -b\cos x - 2\{-x\cos x+\sin x\} + 3\sin x$ *

$= -(b-2x)\cos x - 2\sin x + 3\sin x$

$= -y\cos x + 1\sin x$ $(b-2x=y)$

Hence complete solution

$z = C.F. + P.I.$

$= f_1(y+2x)+f_2(y-3x) - y\cos x+\sin x$

2Q $(D^2+2DD'+D'^2)z = 2\cos y - x\sin y$ $z=f(x,y)$

~~PDE~~ A.E. $m^2+2m+1=0$

$m=-1,-1 \Rightarrow$ $C.F. = f_1(y-x)+xf_2(y-x)$

$P.I. = \frac{1}{D^2+2DD'+D'^2}(2\cos y - x\sin y) = P_1 - P_2$

$P_1 = \frac{1}{(D+D')^2}2\cos y = \frac{1}{(0+1)^2}\iint 2\cos u\,du\,du$ Let $y=u=0x+1y$

$D\to 0$ $D'=1$

$= \cancel{\frac{u^3}{3}} = \cancel{\frac{y^3}{3}}$ $= -2\cos u$

$y=c-mx$

$y=(c+x)$

$= -2\cos y$

$P_2 = \frac{1}{(D+D')^2}x\sin y = \frac{1}{(D+D')}\int x\sin(c+x)\,dx$

$= \frac{1}{D+D'}\{-x\cos(c+x)+\sin(c+x)\}$ Put $c+x=y$

$= \frac{1}{D+D'}\{-x\cos y+\sin y\}$

$y=b-mx$ $m=-1$

$= \int\{-x\cos(b+x)+\sin(b+x)\}\,dx$

$= -\{x\sin(b+x)+\cos(b+x)\} - \{\cos(b+x)\}$

$= -x\sin y - 2\cos y$ Ans

Hence Complete soln

$$z = C.F. + P.I$$

$$= f_1(y-x) + x f_2(y-x) - 2\cos y - \{-x\sin y - 2\cos y\}$$

$$= f_1(y-x) + x f_2(y-x) + x\sin y \quad \text{Ans}$$

Q $(D^3 + 2D^2D' - DD'^2 - 2D'^3)z = (y+2)e^x$

Soln. A.E. $m^3 + 2m^2 - m - 2 = 0$

$m^2(m+2) - 1(m+2) = 0$

$(m+2)(m^2-1) = 0$

$(m+2)(m-1)(m+1) = 0$

$\Rightarrow m = 1, -1, -2$

C.F. $= f_1(y+x) + f_2(y-x) + f_3(y-2x)$

P.I. $= \dfrac{1}{F(D,D')}\phi(x,y) =$

$= \dfrac{1}{(D+D')(D-D')(D+2D')}(y+2)e^x$ $\quad y = c - mx$ Let $y = c+x$

$= \dfrac{1}{(D+2D')(D-D')}\int (c+x+2)e^x\,dx$

$= \dfrac{1}{(D+2D')(D-D')}\left[(c+x+2)e^x - e^x\right] = \dfrac{1}{(D-2D')(D-D')}(c+x+1)e^x$ $\quad (c+x \to y)$

$= \dfrac{1}{(D+2D')(D-D')}(y+1)e^x = \dfrac{1}{D-2D'}\int[(b-x)+1]e^x\,dx$

$= \dfrac{1}{(D+2D')}\{(b-x+1)e^x + e^x\} = \dfrac{1}{D+2D'}(b-x+2)e^x$ $\quad \{$Put $b-x=y$

$= \dfrac{1}{D+2D'}\{(y+2)e^x\} = \int[(d+2x)+2]e^x\,dx$

$= (d+2x+2)e^x - 2e^x = (d+2x)e^x$ $\quad d-mx = y \quad \therefore d+2x = y$

$= ye^x$ Ans

Non-Homogeneous linear PDE :—

If PDE; $F(D,D')z = \phi(x,y)$ ——(1) If $\phi(x,y)=0$

If $F(D,D')$ is not homogeneous in D & D'. Then PDE (1) is called Non-homogeneous linear PDE.

- If $F(D,D') = (D-m_1D'-a_1)(D-m_2D'-a_2)\cdots(D-m_nD'-a_n)=0$
  1. If all factors are distinct then
  $$C.F. = e^{a_1x}f_1(y+m_1x) + e^{a_2x}f_2(y+m_2x) + \cdots + e^{a_nx}f_n(y+m_nx)$$
  2. If factors are repeated e.g. $(D-mD'-a)^3z=0$
  Then $C.F. = e^{ax}f_1(y+mx) + xe^{ax}f_2(y+mx) + x^2 e^{ax} f_3(y+mx)$
  Complete soln: $z = C.F + P.I = C.F.$ $(P.I=0)$
- If $F(D,D')$ cannot be factorize into linear factors.
  Here we use trial solution

e.g. Let $(D-D'^2)z=0$ ——(2)

Let trial solution of (2)
$$z = Ae^{hx+ky} \quad \text{——(3)}$$
where A, h, k are constants

from (3) $Dz = \frac{\partial z}{\partial x} = hAe^{hx+ky}$ $D'z = \frac{\partial z}{\partial y} = kAe^{hx+ky}$
$$D'^2z = \frac{\partial^2 z}{\partial y^2} = k^2Ae^{hx+ky}$$

from (2); we get
$$Ahe^{hx+ky} - Ak^2e^{hx+ky} = 0$$
$$A(h-k^2)e^{hx+ky} = 0 \; ; \; A \neq 0$$
$$\Rightarrow h-k^2=0$$
$$\Rightarrow h=k^2$$

Hence ~~soln~~ from (2) $z = Ae^{k^2x+ky}$

Hence more general soln
$$z = \sum Ae^{k^2x+ky}$$

where A & K are arbitrary constants and $\sum$ denotes the no. of terms taken.

Ex 1: Solve the PDE $(D-D'-2)(D+D'+1)z=0$

Compare each factor with $(D-mD'-a)$, we get

$m_1=1 \quad a_1=2 \qquad m_2=-1 \quad a_2=-1$

$C.F. = e^{2x} f_1(y+x) + e^{-x} f_2(y-x)$

$P.I = 0 \quad \because \phi(x,y)=0$

Then complete soln $z = C.F + P.I = e^{2x} f_1(y+x) + e^{-x} f_2(y-x)$

Ex 2 Solve the PDE $DD'(D+5D'-1)(D-2D'-3)z=0$

Soln: Linear factors

$D=0 \qquad D'=0 \qquad D+5D'-1=0 \qquad D-2D'-3=0$

$\frac{\partial z}{\partial x}=0 \qquad \frac{\partial z}{\partial y}=0 \qquad m_1=5 \quad a_1=1 \qquad m_2=+2 \quad a_2=3$

Then part of C.F. are

$\Rightarrow C.F.= f_1(y) \quad C.F.= f_2(x) \quad C.F.= f_3(y-5x) \times e^{x} \quad C.F. = e^{3x} f_4(y+2x)$

Complete soln $z = f_1(y) + f_2(x) + e^{x} f_3(y-5x) + e^{3x} f_4(y+2x)$

P.I: Case i) $\phi(x,y) = e^{ax+by}$ then $P.I = \frac{1}{F(a,b)} e^{ax+by}$

Ex 3 Solve the $r+2s+t+2p+2q+z = e^{5x+3y}$. $p=\frac{\partial z}{\partial x}$

Given: $(D^2+2DD'+D'^2+2D+2D'+1)z = F(D,D')$

on factorization $\{(D+D')^2+2(D+D')+1\}z = F(D,D')$

or $(D+D'+1)^2 z = F(D,D')$

Here $F(D,D')$ having repeating factor with

$m_1=-1 \quad a_1=-1, \; m_2=-1, \; a_2=-1$

Then $C.F. = e^{-x} f_1(y-x) + x e^{-x} f_2(y-x)$

$P.I = \frac{1}{F(D,D')} \phi(ax+by) = \frac{1}{(D+D'+1)^2} e^{5x+3y}$ $\left\{ \begin{array}{l} D \to 5 \\ D' \to 3 \end{array} \right.$

$= \frac{1}{(5+3+1)^2} e^{5x+3y}$

$= \frac{1}{81} e^{5x+3y} = \frac{1}{81} e^{5x+3y}$

Hence complete soln

$z = e^{-x} f_1(y-x) + x e^{-x} f_2(y-x) + \frac{1}{81} e^{5x+3y}$ Ans

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Case IIIrd :- $\phi(x,y) = x^m y^n$, then P.I.

$$P.I. = \frac{1}{F(D,D')} x^m y^n = \frac{1}{F(D,D')} x^m y^n = [F(D,D')]^{-1} x^m y^n$$

Expanding $[F(D,D')]^{-1}$ in powers of D'/D when $(m>n)$ or $\{D/D'$ when $n>m\}$ or D or D' as per condition.

Ex1 solve the PDE $(D-D'-1)(D-D'-2)Z = e^{13x+2y} + 2x + 3y$

Soln On comparing each factor with $(D-mD'-a)$, we gets

$$m_1 = +1 \quad a_1 = 1 \qquad m_2 = +1 \quad a_2 = 2$$

$$C.F. = e^{x} f_1(y+x) + e^{2x} f_2(y+x)$$

Particular Integral $PI = P_1 + P_2 + P_3$

$$P_1 = \frac{1}{(D-D'-1)(D-D'-2)} \cdot e^{13x+2y} = \frac{1}{(13-2-1)(13-2-2)} e^{13x+2y} = \frac{1}{90} e^{13x+2y}$$

$$P_2 = \frac{1}{(-1)\{1-(D-D')\}(-2)\{1-\frac{(D-D')}{2}\}} 2x = \{1-(D-D')\}^{-1}\{1-\frac{(D-D')}{2}\}^{-1} \cdot x$$

$$= \{1+(D-D')\}\{1+\frac{D-D'}{2}\}x$$ expanding for only 2 terms and leaving Higher derivative term due to function x

$$= \{1 + \frac{D-D'}{2} + (D-D') + \frac{(D-D')^2}{2}\} x$$

$$= \{1 + \frac{3}{2}(D-D') + \frac{D^2+D'^2-2DD'}{2}\} x = \{1+\frac{3}{2}(D-D') + \frac{D^2}{2} + \frac{D'^2}{2} - DD'\} x$$

$$= \{x + \frac{3}{2}(1-0) + \frac{0}{2} + \frac{0}{2} - 0\} = (x + \frac{3}{2})$$

$$P_3 = \frac{1}{(D-D'-1)(D-D'-2)} 3y = \frac{1}{2}[1-(D-D')]^{-1}[1-\frac{(D-D')}{2}]^{-1}(3y)$$

$$= \frac{3}{2}\{1 + \frac{3(D-D')}{2} + \underbrace{\frac{D^2}{2} + \frac{D'^2}{2} - DD'}\} y = \frac{3}{2}\{y + \frac{3}{2}(0-1) + \frac{0}{2} + \frac{0}{2} - 0\}$$

$$= \frac{3}{2}\{y - 3/2\}$$

Complete soln: $z = e^{x} f_1(y+x) + e^{2x} f_2(y+x) + \frac{1}{90} e^{13x+2y}$

$$+ (x + \frac{3}{2}) + \frac{3}{2}(y - 3/2)$$

Q. Solve $(D+1)(D+D'-1) = e^{2x+3y}$

Soln $F(D,D') = (D+1)(D+D'-1)$ Here $m_1=0$ $a_1=-1$ $m_2=-1$ $a_2=1$

$C.F. = e^{-x}f_1(y) + e^{x}f_2(y-x)$

$$P.I. = \frac{1}{F(D,D')}\phi(x,y)$$

$$P_1 = \frac{1}{(D+1)(D+D'-1)}e^{2x+3y} = \frac{1}{(2+1)(2+3-1)} \times e^{2x+3y} = \frac{e^{2x+3y}}{12}$$

Complete soln $\quad Z = C.F. + P.I. = e^{-x}f_1(y) + e^{x}f_2(y-x) + \dfrac{e^{2x+3y}}{12}$

Case IInd :- $\quad F(D,D')Z = \sin(ax+by)$ or $\cos(ax+by)$

$$P.I. = \frac{1}{F(D^2, DD', D'^2)}\sin(ax+by) \text{ or } \cos(ax+by)$$

$D^2 \to -a^2$, $DD' = -ab$ $D'^2 = -b^2$

$$P.I. = \frac{1}{F(-a^2,-ab,-b^2)}\sin(ax+by) \text{ or } \cos(ax+by)$$

If $\quad F(D,D') = F(D^2, DD', D'^2, D, D')$ then P.I.

$$P.I. = \frac{1}{F[-a^2,-ab,-b^2,D,D']}\sin(ax+by) \text{ or } \cos(ax+by)$$

which can be evaluated further.

Ex: Solve $(D-D'-1)(D-D'-2)Z = \sin(2x+3y)$

Soln $F(D,D') = (D-D'-1)(D-D'-2)$ Here $\left.\begin{matrix} m_1=1, & m_2=1 \\ a_1=1 & a_2=2 \end{matrix}\right\}$

$C.F. = e^{x}f_1(y+x) + e^{2x}f_2(y+x)$

$$P.I. = \frac{1}{(D-D'-1)(D-D'-2)}\sin(2x+3y) = \frac{1}{D^2-2DD'+D'^2-3D+3D'+2}\sin(2x+3y)$$

$$= \frac{1}{-4+12-9-3D+3D'+2}\sin(2x+3y) \qquad D^2\to-4,\ D'^2=-9\ DD'\to-6$$

$$= \frac{1}{-3D+3D'+1}\sin(2x+3y) = -\frac{(3D-3D')+1}{\{\underbrace{3D-3D'}_{a}+\underbrace{1}_{b}\}\{\underbrace{3D-3D'}_{a}-\underbrace{1}_{b}\}}\sin(2x+3y)$$

$$= -\left\{\frac{(3D-3D')+1}{9D^2+9D'^2-18DD'-1}\sin(2x+3y)\right\}$$

$$= -\left\{\frac{3D-3D'+1}{-36-81+108-1}\sin(2x+3y)\right\}$$

$$= \frac{1}{10}(3D-3D'+1)\sin(2x+3y)$$

$$= \frac{1}{10}\left\{6\cos(2x+3y) - 9\cos(2x+3y) + \sin(2x+3y)\right\}$$

$$= \frac{1}{10}\left\{\sin(2x+3y) - 3\cos(2x+3y)\right\}$$

Hence

$$z = e^{x} f_1(y+x) + e^{2x} f_2(y+x) + \frac{1}{10}\left\{\sin(2x+3y) - 3\cos(2x+3y)\right\}$$

Q $D(D+D'-1)(D+3D'-2)z = x^2 - 4xy + y^2$

Soln: factors: $D=0$ $D+D'-1=0$ $D+3D'-2=0$

$\Rightarrow$ part of C.F $= f_1(y)$ $= e^{+x} f_2(y-x)$ $= e^{+2x} f_3(y-3x)$

C.F. $= f_1(y) + e^{+x} f_2(y-x) + e^{+2x} f_3(y-3x)$

$$\text{P.I.} = \frac{1}{D(D+D'-1)(D+3D'-2)} x^2 - 4xy + y^2 = P_1 + P_2 + P_3$$

$$P = \frac{1}{D(D+D'-1)(D+3D'-2)} x^2 - 4xy + y^2 = \frac{1}{2D}\left\{1-(D+D')\right\}^{-1}\left\{1-\frac{(D+3D')}{2}\right\}^{-1}(x^2 - 4xy + y^2)$$

$$\frac{1}{D(D+D'-1)(D+3D'-2)}(x^2-4xy+y^2) = \frac{1}{(-1)(-2)D}\left\{1-(D+D')\right\}^{-1}\left\{1-\left(\frac{D+3D'}{2}\right)\right\}^{-1} \times (x^2-4xy+y^2)$$

$$= \frac{1}{2D}\left\{1+(D+D')+(D+D')^2\right\}\left\{1+\left(\frac{D+3D'}{2}\right)+\left(\frac{D+3D'}{2}\right)^2\right\} \times (x^2-4xy+y^2)$$

$$= \frac{1}{2D}\left\{1+\frac{D+3D'}{2}+\frac{D^2+9D'^2+6DD'}{4}+(D+D')+\frac{D^2+4DD'+3D'^2}{2} + \frac{(D+D')(D^2+9D'^2+6DD')}{4}+(D^2+2DD'+D'^2) + \frac{(D^2+D'^2+2DD')(D+3D')}{2}+\frac{(D^2+2DD'+D'^2)(D^2+9D'^2+6DD')}{4}\right\} \times (x^2-4xy+y^2)$$

$$= \frac{1}{2D}\left\{1+\frac{3D+5D'}{2}+\frac{9D^2+19D'^2+22DD'}{4}\right\}(x^2-4xy+y^2)$$

$$= \frac{1}{2D}\left\{x^2-4xy+y^2+\frac{3}{2}(2x-4y)+\frac{5}{2}(0-4x+2y)+\frac{9}{4}(2) + \frac{19}{4}(0-0+2)+\frac{22}{4}(0-4+0)\right\}$$

$$= \frac{1}{2D}\left\{x^2-4xy+y^2+\frac{3}{2}(2x-4y)+\frac{5}{2}(-4x+2y)-22\right\}$$

$$= \frac{1}{2}\left\{\frac{x^3}{3}-\frac{4x^2y}{2}+\frac{xy^2}{1}+\frac{3}{2}(x^2-4xy)+\frac{5}{2}\left(\frac{-4x^2}{2}+2xy\right)-22x\right\}$$

$$= \frac{1}{2}\left\{\frac{x^3}{3}-2x^2y+xy^2-\frac{7}{2}x^2-xy-\frac{9}{2}x\right\}$$

Case IV :- when $\phi(x,y) = e^{ax+by} \cdot V$ $D \to D+a$, $D' = D'+b$

where V i) $e^{a_1x+b_1y}$ ii) $\sin(ax+by)$ or $\cos(ax+by)$

iii) $x^m \cdot y^n$ iv) Constant (Say 1)

Ex 1 $(D-3D'-2)^3 z = 6e^{2x} \sin(3x+y)$

Soln Repeated factors $(D-3D'-2)^3$ which gives

$m_1 = m_2 = m_3 = 3$ $a_1 = a_2 = a_3 = 2$

$$C.F. = e^{2x} f_1(y+3x) + xe^{2x} f_2(y+3x) + x^2e^{2x} f_3(y+3x)$$

$$P.I. = \frac{1}{F(D,D')} 6e^{2x} \sin(3x+y) = 6 \cdot \frac{1}{(D-3D'-2)^3} e^{2x+0y} \sin(3x+y)$$

$$= 6 \cdot \frac{e^{2x+0y}}{\{(D+2)-3(D'+0)-2\}^3} \sin(3x+y)$$

$D \to D+a(=2)$

$D' \to D'+b(=0)$

$$= 6 \cdot e^{2x} \frac{1}{(D-3D')^3} \sin(3x+y)$$ $(f(ax+by))$

$D-3D'$ Homog. (Ist case)

Now $D \to 3$ $D' \to 1$

But $D-3D' = 0$

$\Rightarrow$ Case failed.

$$= 6xe^{2x} \frac{1}{3(D-3D')^2} \sin(3x+y)$$

Again case failed

$$= \cancel{6} x^2e^{2x} \frac{1}{\cancel{6}(D-3D')} \sin(3x+y)$$

Again case failed

$$= x^3 \cdot e^{2x} \frac{1}{1-3\times 0} \sin(3x+y) = x^3e^{2x} \sin(3x+y) \text{ Ans.}$$

$$z = e^{2x} f_1(y+3x) + xe^{2x} f_2(y+3x) + x^2e^{2x} f_3(y+3x) + x^3e^{2x} \sin(3x+y) \text{ Ans}$$

Ex $(D^2 - DD' + D' - 1)Z = \cos(x+2y) + e^y$

$\because D^2 - DD' + D' - 1 = (D^2-1) - D'(D-1) = (D-1)\{D+1-D'\}$

$\Rightarrow (D-D'+1)(D-1) = 0$

$\Rightarrow (D-D'+1) = 0$

$m_1 = +1$ $a_1 = -1$

part of C.F. $= f_1(y+x) \times e^{-x}$

$D-1 = 0 \Rightarrow$ $m_2 = 0$ $a_2 = 1$

$\Rightarrow$ part of C.F. $= f_2(y)e^x$

$$C.F. = e^{-x} f_1(y+x) + f_2(y) e^x$$

$$P.I. = \frac{1}{D^2 - DD' + D' - 1}\cos(x+2y) + e^y = P_1 + P_2 \qquad D^2 \to -a^2,\ D'^2 \to -b^2,\ DD' \to -ab$$

$$P_1 = \frac{1}{D^2 - DD' + D' - 1}\cos(x+2y) = \frac{1}{-1+2+D'-1}\cos(x+2y) = \frac{1}{D'}\cos(x+2y)$$

$$= +2\,\underline{\sin(x+2y)}$$

$$P_2 = \frac{1}{D^2 - DD' + D' - 1} e^{y} = \frac{1}{D^2 - DD' + D' - 1} e^{y} \times 1 \text{ when } D \to D+0,\ D' \to D'+1$$

$$= e^{y} \frac{1}{(D+0)^2 - (D+0)(D'+1) - (D'+1) - 1} \times 1 = \frac{1}{0-0+1-1} e^{y} \text{ (case fail)}$$

$$= e^{y} \frac{1}{D^2 - DD' - D - D' - 2} = e^{y} \frac{1}{D(D-D')} \qquad e^{y} = e^{0x+1y}$$

$$P_2 = \frac{1}{D^2 - DD' + D' - 1} e^{y} = y \cdot \frac{1}{0 - D + 1 - 0} e^{y} = y \cdot \frac{e^{y}}{-0+1} = ye^{y}$$

C. Failed

Hence soln $z = e^{-x} f_1(y+x) + e^{x} f_2(y) + ye^{y}$

<u>Note</u> $P_2 = \frac{1}{D^2 - DD' + D' - 1} e^{y} = x \cdot \frac{1}{2D - D'} e^{y} = x \cdot \frac{1}{2\times 0 - 1} e^{y} = -xe^{y}$

Case failed

So $z = e^{-x} f_1(y+x) + e^{x} f_2(y) - xe^{y}$ Ans

<u>Exp:</u> $r - 4s + 4t + p - 2q = e^{x+y}$

<u>Soln</u> $(D^2 - 4DD' + 4D'^2 + D - 2D')z = e^{x+y}$

$\{(D-2D')^2 + (D-2D')\} z = e^{x+y}$

$(D - 2D' + 1)(D - 2D')z = e^{x+y}$

$m_1 = 2 \quad a_1 = -1 \qquad m_2 = 2 \quad a_2 = 0$

$C.F. = e^{-x} f_1(y+2x) + f_2(y+2x)$

$P.I. = \frac{1}{D^2 - 4DD' + 4D'^2 + D - 2D'} e^{x+y}$ Case failed $D \to 1 \quad D' \to 1$

$= x \frac{1}{2D - 4D' + 1} e^{x+y} = x \cdot \frac{1}{2-4+1} e^{x+y}$

$= -x \cdot e^{x+y}$

$z = e^{-x} f_1(y+2x) + f_2(y+2x) - x e^{x+y}$

$\phi(x,y) = e^{ax+by}$ ✓.

Ex. $(D^2 - D')Z = x e^{ax+a^2y}$; find P.I.

$D \to D+a$, $D' = D' + a^2$

$$\text{P.I.} = \frac{1}{D^2 - D'} x e^{ax+a^2y}$$

$$= e^{ax+a^2y} \frac{1}{(D+a)^2 - (D'+a^2)} x = e^{ax+a^2y} \cdot \frac{1}{D^2 + 2aD + \cancel{a^2} - D' - \cancel{a^2}} x$$

$$= e^{ax+a^2y} \cdot \frac{1}{D^2 + 2aD - D'} x$$

$$= e^{ax+a^2y} \cdot \frac{1}{2aD\left(1 + \left(\frac{D^2 - D'}{2aD}\right)\right)} \quad = \frac{e^{ax+a^2y}}{2a} \frac{1}{D}\left\{1 + \frac{D^2 - D'}{2aD}\right\}^{-1} x$$

$$= \frac{e^{ax+a^2y}}{2a} \times \frac{1}{D}\left\{1 - \frac{D^2 - D'}{2aD}\right\} x$$

leaving Higher Derivative term

$$= \frac{e^{ax+a^2y}}{2a} \times \frac{1}{D}\left\{1 - \frac{D}{2a} + \frac{D'}{2aD}\right\} x$$

$$= \frac{e^{ax+a^2y}}{2a} \times \frac{1}{D}\left\{x - \frac{1}{2a} + 0\right\}$$

$$= \frac{e^{ax+a^2y}}{2a} \times \left\{\frac{x^2}{2} - \frac{x}{2a}\right\} \checkmark$$

$$= e^{ax+a^2y}\left\{\frac{x^2}{4a} - \frac{x}{4a^2}\right\} = e^{ax+a^2y} \times \left(\frac{ax^2 - x}{4a^2}\right)$$

$z =$

$(D^2-D'^2-3D+3D')z = xy + e^{x+2y} + \sin(2x+3y)$ ✓

C.F. $(D^2-D'^2-3D+3D')z = 0$

$(D-D')(D+D') - 3(D-D') = 0 \Rightarrow (D-D')(D+D'-3) = 0$

$m_1 = 1 \quad a_1 = 0 \quad m_2 = -1 \quad a_2 = 3$

C.F. $= f_1(y+x) + e^{3x} f_2(y-x)$

P.I. $= P_1 + P_2 + P_3$

$P_1 = \dfrac{1}{(D-D')(D+D'-3)} xy = \dfrac{1}{D}\left\{1-\dfrac{D'}{D}\right\}^{-1} \dfrac{1}{(-3)}\left\{1-\dfrac{D+D'}{3}\right\}^{-1} xy$

$= \dfrac{-1}{3D}\left\{1+\dfrac{D'}{D}\right\}\left\{1+\dfrac{D+D'}{3} + \left(\dfrac{D+D'}{3}\right)^2\right\} xy = \dfrac{1}{3D}\left\{\cancel{1+\dfrac{D}{3}+\dfrac{D'}{3}+\dfrac{D'}{3}+\dfrac{D'}{D}}\right.$

$= -\dfrac{1}{3D}\left\{1 + \dfrac{D+D'}{3} + \dfrac{D^2+D'^2+2DD'}{9} + \dfrac{D'}{D} + \dfrac{1}{3}D' + \dfrac{D'^2}{3D} + \dfrac{D'}{D}\left(\dfrac{D+D'}{3}\right)^2\right\} = xy$

$D^2;\ D'^2;\ DD'$ ✓

$= -\dfrac{1}{3D}\left\{1 + \dfrac{D+D'}{3} + \dfrac{2DD'}{9} + \dfrac{D'}{D} + \dfrac{1}{3}D'\right\} xy$

$= -\dfrac{1}{3D}\left\{1 + \dfrac{D}{3} + \dfrac{2}{3}D' + \dfrac{2}{9}DD' + \dfrac{D'}{D}\right\} xy$

$= -\dfrac{1}{3D}\left\{xy + \dfrac{1}{3}y + \dfrac{2}{3}x + \dfrac{2}{9} + \dfrac{1}{D}x\right\}$

$= -\dfrac{1}{3D}\left\{xy + \dfrac{y}{3} + \dfrac{2}{3}x + \dfrac{2}{9} + \dfrac{x^2}{2}\right\}$ ✓

$= -\dfrac{1}{3}\left\{\dfrac{x^2 y}{2} + \dfrac{xy}{3} + \dfrac{1}{3}x^2 + \dfrac{2}{9}x + \dfrac{x^3}{6}\right\}$ ✓

$P_2 = \dfrac{1}{(D-D')(D+D'-3)} e^{1x+2y}$ case failed

$= x \cdot \dfrac{1}{2D-0-3+0} e^{x+2y}$

$= x \dfrac{1}{2\times1-3} e^{x+2y} = -xe^{x+2y}$ ✓

P_3 For practice.

Q $2s+t-3q = 5\cos(3x-2y) + e^{3x-2y}$

$\cancel{2D^2 +}\ (2DD' + D'^2 - 3D')z = 5\cos(3x-2y)$; $D = \frac{\partial}{\partial x}$, $D' = \frac{\partial}{\partial y}$

$D'(2D + D' - 3)z = 5\cos(3x-2y)$

$D'\left(D + \frac{D'}{2} - \frac{3}{2}\right)z = \frac{5}{2}\cos(3x-2y) + \frac{e^{3x-2y}}{2}$

$D' = 0$ $\quad\quad D + \frac{D'}{2} - \frac{3}{2} = 0$

part of comple funct. $\quad m = -\frac{1}{2} \quad a = \frac{3}{2}$

$= f_1(x)$ $\quad\quad = f_2\left(y - \frac{1}{2}x\right) \times e^{\frac{3}{2}x}$

$$C.F. = f_1(x) + f_2\left(y - \frac{1}{2}x\right) \times e^{\frac{3}{2}x}$$

$$P.I. = \frac{1}{2DD' + D'^2 - 3D'}\left\{5\cos(3x-2y) + e^{3x-2y}\right\}$$

$$= \frac{1}{2(6) + (-4) - 3D'} 5\cos(3x-2y) + \frac{1}{2(-6)+4+6} e^{3x-2y}$$

$$= 5\frac{1}{8-3D'}\cos(3x-2y) + \frac{1}{(-2)} e^{3x-2y}$$

$DD' = -6$

$$= 5 \cdot \left\{\frac{8+3D'}{(8-3D')(8+3D')}\cos(3x-2y)\right\} - \frac{1}{2}e^{3x-2y}$$

$$= 5 \cdot \left\{\frac{8+3D'}{64-9D'^2}\cos(3x-2y)\right\} - \frac{1}{2}e^{3x-2y}$$

$$= 5\left\{\frac{8+3D'}{64-9(-4)}\cos(3x-2y)\right\} - \frac{1}{2}e^{3x-2y}$$

$$= \frac{5}{100}\left\{8\cos(3x-2y) + 3(-2)\sin(3x-2y)\right\}$$
$$- \frac{1}{2}e^{3x-2y}$$

$$= \frac{1}{20}\left\{8\cos(3x-2y) + 6\sin(3x-2y)\right)$$
$$- \frac{1}{2}e^{3x-2y}.$$

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Ex $(D^2 - D')z = \cos(x-3y) + e^{3y}$

This is non-homog. Linear PDE; factorization is not possible then to find C.F., we consider

$$z = Ae^{hx+ky}$$

$$D^2 z = Ah^2 e^{hx+ky} \qquad D' = kAe^{hx+ky}$$

from given PDE; $Ah^2 e^{hx+ky} - Ake^{hx+ky} = 0$

$$A(h^2 - k)e^{hx+ky} = 0; \quad A \neq 0$$

$$\Rightarrow A \neq 0, \Rightarrow h^2 - k = 0 \Rightarrow k = h^2$$

Hence $C.F. = Ae^{hx+h^2y}$

General solⁿ CF $= \sum Ae^{hx+h^2y}$; $h^2 - k = 0$

$$P.I. = \frac{1}{D^2 - D'}\{\cos(x-3y) + e^{3y}\} \quad 0x+3y$$

$D^2 \to -a^2$

$$= \frac{1}{-1-D'}\cos(x-3y) + \frac{1}{0-3}e^{3y}$$

$$= \frac{-1(1-D')}{(1+D')(1-D')}\cos(x-3y) - \frac{1}{3}e^{3y}$$

$$= +\frac{(D'-1)}{1-D'^2}\cos(x-3y) - \frac{1}{3}e^{3y}$$

$$= \frac{+3\sin(x-3y)}{1-(-9)} - \frac{1}{3}e^{3y} - \frac{\cos(x-3y)}{10}$$

$$= \frac{3}{10}\sin(x-3y) - \frac{1}{3}e^{3y} - \frac{\cos(x-3y)}{10}$$

$$z = \sum Ae^{hx+h^2y} + \frac{3}{10}\sin(x-3y) - \frac{\cos(x-3y)}{10} - \frac{1}{3}e^{3y} \quad \text{Ans.}$$

where $h - k^2 = 0$

$D(D-2D')(D+D')Z = e^{x+2y}(x^2+4y^2)$ Non Homog $\xrightarrow{\text{solu}}$ Homog. Reverse not possible

C.F. $D=0$ $D-2D'=0$ $D+D'=0$

part of C.F.: $f_1(y)$, $f_2(y+2x)$, $f_3(y-x)$

$C.F. = F_1(y) + F_2(y+2x) + F_3(y-x)$

$$P.I. = \frac{1}{D(D-2D')(D+D')} e^{x+2y}(x^2+4y^2)$$

$D \to D+1$, $D' \to D+2$

$$= \frac{1}{(D+1)(D+1-2(D'+2))(D+1+(D'+2))} e^{x+2y}(x^2+4y^2)$$

$$= e^{x+2y} \frac{1}{(D+1)(D-2D'-3)(D+D'+3)}(x^2+4y^2)$$

$$= e^{x+2y} \cdot \{1+D\}^{-1} \times \frac{1}{(-3)}\left(1-\frac{D-2D'}{3}\right)^{-1} \times \frac{1}{3}\left(1+\frac{D+D'}{3}\right)^{-1}(x^2+4y^2)$$