Classification of linear PDE of Second order

Consider the Differential Eq^n of second order in two independent variables x & y as

$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2}+f\left(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right)=0 \quad (1)$$

where A, B, C are Constants or Continuous functions of x and y Possessing Continuous Partial derivatives and A is Positive.

Now Eq^n (1) is

(i) elliptic if $B^2-4AC<0$

(ii) Hyperbolic if $B^2-4AC>0$

(iii) Parabolic if $B^2-4AC=0$.

Ex (1) $z_{xx}=z_{tt}$ (Wave equation)

Sol^n $z_{xx}-z_{tt}=0$

Compare with Eq (1), we have

$A=1,\ B=0,\ C=-1$

Now $B^2-4AC=0+4=4>0$

Hence wave equation is a hyperbolic.

Ex (2) $u_t=u_{xx}$ (Heat flow eq^n)

Sol^n $u_{xx}-u_t=0$

Here $A=1,\ B=0,\ C=0$

Now $B^2-4AC=0$

Hence Heat flow eq^n is Parabolic.

Ex 3 $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 u}{\partial y^2}$

Sol^n Here $A=1,\ B=1,\ C=1$

$\therefore B^2-4AC = 1^2 - 4\times1\times1 = -3<0$

Hence the given Eq^n is elliptic.

Ex 4 $4\frac{\partial^2 u}{\partial x^2} + 4\frac{\partial^2 u}{\partial x\partial t} + \frac{\partial^2 u}{\partial t^2} = 0$

Sol^n Here $A=4,\ B=4,\ C=1$

$\therefore B^2-4AC = 16 - 4\times4\times1 = 0$

Hence Given Eq^n is Parabolic

Ex 5 $5\frac{\partial^2 u}{\partial x^2} - 9\frac{\partial^2 u}{\partial x\partial t} + 4\frac{\partial^2 u}{\partial t^2} = 0$

Sol^n Here $A=5,\ B=-9,\ C=4$

$\therefore B^2-4AC = 81-4\times5\times4 = 1>0$

Hence the given Eq^n is hyperbolic

Ex-6 Find whether the following operators are hyperbolic, Parabolic and elliptic.

(i) $x^2\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} + u.$

Sol^n Here $A=x^2,\ B=0,\ C=-1$

$\therefore B^2-4AC = 0-4x^2(-1) = 4x^2$

The operator is

(a) Parabolic if $4x^2=0 \Rightarrow x=0$

(b) hyperbolic if $4x^2>0 \Rightarrow x\neq0$

(ii) $t\frac{\partial^2 u}{\partial t^2} + 2\frac{\partial^2 u}{\partial x\partial t} + x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}$

Sol^n Here $A=t^2,\ B=2,\ C=x.$

$\therefore B^2 - 4AC = 4 - 4tx$

The given operator is

(a) Parabolic if $4 - 4tx = 0 \Rightarrow 4 = 4tx$
$\Rightarrow tx = 1$

(b) hyperbolic if $4 - 4tx > 0 \Rightarrow 4 > 4tx$
$\Rightarrow tx < 1$

(c) Elliptic if $4 - 4tx < 0 \Rightarrow 4 < 4tx$
$\Rightarrow tx > 1.$

(iii) $x \dfrac{\partial^2 u}{\partial x^2} + t \dfrac{\partial^2 u}{\partial x \partial t} + \dfrac{\partial^2 u}{\partial t^2}$

Here $A = x$, $B = t$, $C = 1$

$\therefore B^2 - 4AC = t^2 - 4x$

The given operator is

(a) Parabolic if $t^2 - 4x = 0 \Rightarrow t^2 = 4x$

(b) hyperbolic if $t^2 - 4x > 0 \Rightarrow t^2 > 4x$

(c) Elliptic if $t^2 - 4x < 0 \Rightarrow t^2 < 4x$

Ex-7 $\sqrt{y^2+x^2}\, u_{xx} + 2(x-y) u_{xy} + \sqrt{y^2+x^2}\, u_{yy} = 0$

Here $A = \sqrt{y^2+x^2}$, $B = 2(x-y)$, $C = \sqrt{y^2+x^2}$

$\therefore B^2 - 4AC = 4(x-y)^2 - 4\sqrt{y^2+x^2} \cdot \sqrt{y^2+x^2}$
$= 4(x^2+y^2-2xy) - 4(y^2+x^2)$
$= -8xy$

The given eq^n is

Case (1) ~~P~~ If $x > 0, y > 0$ or $x < 0, y < 0$
then $B^2 - 4AC < 0$
$\Rightarrow$ Elliptic

Case-ii if $x>0, y<0$ OR $x<0, y>0$

then $B^2-4AC>0$

$\Rightarrow$ hyperbolic

Case III if $x=0$ OR $y=0$

then $B^2-4AC=0$

$\Rightarrow$ Parabolic

Ex-8 classify the PDE

$$2\frac{\partial^2 u}{\partial t^2}+4\frac{\partial^2 u}{\partial x\partial t}+3\frac{\partial^2 u}{\partial x^2}=0$$

Here $A=2, B=4, C=3$

$\therefore B^2-4AC=16-4\times2\times3=-8<0$

Hence the Eq^n is Elliptic

Ex-9 show that the Eq^n

$$\frac{\partial^2 u}{\partial t^2}=c^2\frac{\partial^2 u}{\partial x^2} \text{ is hyperbolic}$$

Soln the given Eq^n is

$$\frac{\partial^2 u}{\partial t^2}-c^2\frac{\partial^2 u}{\partial x^2}=0$$

Here $A=1, B=0, C=-c^2$

$\therefore B^2-4AC=0-4\times1\times(-c^2)=4c^2>0$

Hence the Eq^n is hyperbolic

Ex-10 ~~Show that~~ classify

$$u_{xx}+xu_{yy}+u_y=0$$

Here $A=1, B=0, C=x$

$\therefore B^2 - 4AC = \cancel{x^2 - 4 \times 1 \times 0} \; 0 - 4 \times 1 \times x$
$= -4x$

The Eq^n is
(i) elliptic if $-4x < 0 \Rightarrow x > 0$
(ii) hyperbolic if $-4x > 0 \Rightarrow x < 0$
(iii) Parabolic if $-4x = 0 \Rightarrow x = 0$

Ex - 11 Classify $z_{xx} + 2x z_{xy} + (1-y^2) z_{yy} = 0$

Here $A = 1$, $B = 2x$, $C = (1-y^2)$

$\therefore B^2 - 4AC = 4x^2 - 4 \times 1 (1-y^2)$
$= 4(x^2 + y^2 - 1)$

the Eq^n is
(i) Parabolic if $4(x^2+y^2-1) = 0$
$\Rightarrow x^2 + y^2 = 1$
(ii) hyperbolic if $4(x^2+y^2-1) > 0$
$\Rightarrow x^2 + y^2 > 1$
(iii) elliptic if $4(x^2+y^2-1) < 0$
$\Rightarrow x^2 + y^2 < 1$

Homework

(i) classify $\frac{\partial^2 u}{\partial t^2} + t \frac{\partial^2 u}{\partial x \partial t} + x \frac{\partial^2 u}{\partial x^2} + 2 \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + 6u = 0$

(ii) classify the PDE
$x^2 \frac{\partial^2 u}{\partial t^2} + 3 \frac{\partial^2 u}{\partial x \partial t} + x \frac{\partial^2 u}{\partial x^2} + 17 \frac{\partial u}{\partial t} = 100u$

(iii) Classify the following
(a) $z_{xx} = z_{yy}$ (b) $z_{xx} + z_{yy} = 0$ (c) $u_{xx} - 2u_{xy} + u_{yy} = 0$

(iv) classify

$$(1-x^2)\frac{\partial^2 z}{\partial x^2} - 2xy\frac{\partial^2 z}{\partial x \partial y} + (1-y^2)\frac{\partial^2 z}{\partial y^2} + x\frac{\partial z}{\partial x} + 3x^2y\frac{\partial z}{\partial y} - 2z = 0$$

(v) classify the Eq^n

$$y^2 r - 2xy s + x^2 t = \frac{y^2}{z} p + \frac{x^2}{y} q$$