$\frac{E}{H} = \mu_0 c$

also $\frac{E}{H} = \frac{1}{\epsilon_0 c}$

Wave Impedance (Z_0)

The ratio of ~~E~~ magnitude of E to H is called impedance (Z_0). i.e.

$\epsilon_0 = 8.85 \times 10^{-12} \, C^2/(N\text{-}m^2) \;$ F/m

$\mu_0 = 4\pi \times 10^{-7} \, N/A^2 = \text{Henry/m}$

$$Z_0 = \left|\frac{E}{H}\right| = \frac{E_0}{H_0} = \mu_0 c = \sqrt{\frac{\mu_0}{\epsilon_0}} \quad \left(\because c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}\right)$$

$$= \sqrt{\frac{4\pi \times 10^{-7}}{8.85 \times 10^{-12}}} = 376.6 \text{ ohm} \approx 377\,\Omega.$$

(see next page)

Since the ratio $Z_0 = \frac{E_0}{H_0}$ is real & positive this implies that field vectors $\vec{E}$ & $\vec{H}$ are in the same phase i.e. they have the same relative magnitude at all points at all times.

The Poynting Vector, which is the energy flow per unit area per unit time for a plane e.m. wave is given by

$$\vec{S} = \vec{E} \times \vec{H} = \vec{E} \times \frac{\hat{n} \times \vec{E}}{\mu_0 c} \quad [\text{using eqn. 9}].$$

$$= \frac{1}{\mu_0 c} E \times (\hat{n} \times E) = \frac{1}{\mu_0 c}\left[(\bar{E}.\bar{E})\hat{n} - (\bar{E}.\hat{n})E\right].$$

Since $\bar{E}.\hat{n} = 0$ because both being perp.

$$\therefore \vec{S} = \frac{1}{\mu_0 c} \bar{E}^2 \hat{n} = \frac{\bar{E}^2}{Z_0} \hat{n} \quad [\because Z_0 = \mu_0 c].$$

For a plane e.m.w. of ang. freq ω, the average value of S over a complete cycle is given by

$$\langle S \rangle = \frac{1}{Z_0} \langle E^2 \rangle \hat{n} = \frac{1}{Z_0} \langle (E_0 e^{i\vec{K}.\vec{r} - i\omega t})^2 \rangle_{\text{real}} \hat{n}$$

$$= \frac{1}{Z_0} E_0^2 \langle \cos^2(\omega t - \vec{K}.\vec{r}) \rangle \hat{n}$$

$$= \frac{1}{Z_0} \frac{E_0^2}{2} \hat{n} \quad \left[\because \langle \cos^2 \omega t - K.r \rangle = \frac{1}{2}\right].$$

$$= \frac{1}{Z_0} E_{rms}^2 \hat{n} \quad \text{—— (1)} \quad \text{Since } E_{rms} = \frac{E_0}{\sqrt{2}}.$$

Plane e.m. waves in Conducting medium

In conducting medium

$D = \epsilon E$, $B = \mu H$, $J = \sigma E$; $\rho = 0 =$ Charge density.

(σ = conductivity)

[$\rho = 0$, $\because$ there is no net charge within a conductor because the charge resides on the surface of the conductor]. Therefore Maxwell's eqn. becomes

$$\vec{\nabla} \cdot \vec{D} = \nabla \cdot E = 0 \quad \text{——} (1)$$

$$\vec{\nabla} \cdot \vec{B} = 0 \quad \text{——} (2)$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial B}{\partial t} \quad \text{or} \quad \nabla \times E = -\mu \frac{\partial H}{\partial t} \quad \text{——} (3).$$

$$\vec{\nabla} \times \vec{H} = J + \frac{\partial D}{\partial t} \quad \text{or,} \quad \nabla \times H = \sigma E + \epsilon \frac{\partial E}{\partial t} \quad \text{——} (4).$$

Taking curl of eqn (3) we get

$$\nabla \times (\nabla \times E) = -\mu \frac{\partial}{\partial t}(\nabla \times H)$$

Putting the value of $(\nabla \times H)$ from eqn. (4) we get

$$\nabla \times (\nabla \times E) = -\mu \frac{\partial}{\partial t}\left(\sigma E + \epsilon \frac{\partial E}{\partial t}\right)$$

$$\text{or,} \quad \nabla \times (\nabla \times E) = -\mu\sigma \frac{\partial E}{\partial t} - \mu\epsilon \frac{\partial^2 E}{\partial t^2} \quad \text{——} (5)$$

Now $\nabla \times (\nabla \times E) = \nabla(\nabla \cdot E) - \nabla^2 E$

But $\vec{\nabla} \cdot \vec{E} = 0$ from eqn. (1)

$\therefore$ Eqn. (5) becomes

$$\boxed{\begin{aligned} &\nabla^2 E - \mu\sigma \frac{\partial E}{\partial t} - \mu\epsilon \frac{\partial^2 E}{\partial t^2} = 0 \quad \text{——} (6(a)) \\ \text{or,} \quad &\frac{\partial^2 E}{\partial x^2} - \mu\sigma \frac{\partial E}{\partial t} - \mu\epsilon \frac{\partial^2 E}{\partial t^2} = 0 \quad \text{——} (6(b)) \end{aligned}}$$

In one dimension.

Similarly by taking Curl of eqn.(4) & then using eqn (3) we get

$$\boxed{\nabla^2 H - \mu\sigma \frac{\partial H}{\partial t} - \mu\epsilon \frac{\partial^2 H}{\partial t^2} = 0} \quad \text{—— (7(a))}$$

or In one dimension

$$\boxed{\frac{\partial^2 H}{\partial x^2} - \mu\sigma \frac{\partial H}{\partial t} - \mu\epsilon \frac{\partial^2 H}{\partial t^2} = 0} \quad \text{—— (7(b))}$$

Eqn.(6(b)) & (7(b)) are plane e.m. wave eqns. in conducting medium. An e.m. wave is rapidly attenuated in a conducting medium. In fact in a good conductor the attenuation is so rapid that at high radio frequencies the wave penetrates the conductor to a very small depth.

Plane - Wave

The plane waves are those waves which are travelling in one direction, i.e., say, in the Z direction, and have no x & y dependency and their amplitudes is same at any point in a plane ⊥ to the direction of propagation.

Plane e.m. wave in conducting medium:

The three dimensional wave eqn. for a conducting medium is given by

$$\nabla^2 E - \mu\epsilon \frac{\partial^2 E}{\partial t^2} - \mu\sigma \frac{\partial E}{\partial t} = 0 \quad \text{or,} \quad \nabla^2 H - \mu\epsilon \frac{\partial^2 H}{\partial t^2} - \mu\sigma \frac{\partial H}{\partial t} = 0$$

or

$$\text{or, } \frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial y^2} + \frac{\partial^2 E}{\partial z^2} - \mu\epsilon \frac{\partial^2 E}{\partial t^2} - \mu\sigma \frac{\partial E}{\partial t} = 0 \quad \text{—— (1)}$$

P.T.O.

In one dimension we have

$$\frac{\partial^2 E}{\partial x^2} - \mu\sigma\frac{\partial E}{\partial t} - \mu\epsilon\frac{\partial^2 E}{\partial t^2} = 0 \quad\text{——— } 1(a)$$

or

$$\frac{\partial^2 H}{\partial x^2} - \mu\sigma\frac{\partial H}{\partial t} - \mu\epsilon\frac{\partial^2 H}{\partial t^2} = 0 \quad\text{——— } 1(b)$$

Let us consider a plane e.m. wave propagating in z direction with $\vec{E}$ in x direction & vector $\vec{H}$ in Y direction. Then

$$\frac{\partial(E_x \text{ or } H_y)}{\partial x} = \frac{\partial(E_x \text{ or } H_y)}{\partial y} = 0 \quad \& \; E = E_x\hat{a}_x \;\&\; H = H_y\hat{a}_y$$

Therefore eqn (1) can be written as

$$\frac{\partial^2 E_x}{\partial z^2} = \mu\epsilon\frac{\partial^2 E_x}{\partial t^2} + \mu\sigma\frac{\partial E_x}{\partial t} \quad\text{——— } (2)$$

The solution of eqn (2) may be expressed as

$$E = E_0 e^{i(K.z-\omega t)} \text{ or } H = H_0 e^{i(K.z-\omega t)} \quad\text{——— } (3)$$

where K = propagation vector $= \frac{2\pi}{\lambda}\hat{n}$; $\hat{n}$ = unit vector along propagation (Z) direction.

Now, $\frac{\partial E}{\partial z} = iKE$ & $\frac{\partial^2 E}{\partial z^2} = (iK)^2 E = -K^2 E$

$\frac{\partial E}{\partial t} = -i\omega E$ & $\frac{\partial^2 E}{\partial t^2} = -\omega^2 E$

Putting these values in eqn $\cancel{1}$(2) we get

See page 12(i) first

$$(-K^2 + \mu\epsilon\omega^2 + i\mu\sigma\omega)E = 0$$

Since E is arbitrary so,

$$(-K^2 + \mu\epsilon\omega^2 + i\mu\sigma\omega) = 0;$$

$$\cancel{\frac{\partial^2 E_x}{\partial z^2} = -K^2 E_x} \quad \cancel{(1*)}$$

& $$\frac{\partial^2 E_x}{\partial z^2} = \mu\epsilon(-\omega^2 E) + \mu\sigma(-i\omega E_x)$$

$$\Rightarrow \frac{\partial^2 E_x}{\partial z^2} = -[\mu\epsilon\omega^2 + i\mu\sigma\omega]E_x \quad \text{——— } (2*)$$

$$\Rightarrow \partial^2 E_x/\partial z^2 = -K^2 E_x$$

~~Comparing eqns (1* & 2*) we get~~ where

$$K^2 = \mu\epsilon\omega^2 + i\mu\sigma\omega. \quad \text{——— } (3*)$$

Let $K = \alpha + i\beta$ ——— $(4*)$

$\because K^2 = \alpha^2 - \beta^2 + 2i\alpha\beta$

On solving above eqns we get expressions of α & β.

$\therefore$ The required solution is

$$E = E_0 e^{i(\alpha + i\beta)\hat{n}\cdot z - i\omega t}$$

or, $$E = E_0 e^{-\beta\hat{n}\cdot z} \cdot e^{i(\alpha\hat{n}\cdot z - \omega t)} \quad \text{——— } (5*)$$

From eqn $(5*)$, it is obvious that field amplitudes are attenuated due to the presence of the term $e^{-\beta\hat{n}\cdot\bar{z}}$. The quantity β is a measure of attenuation & is known as absorption co-efficient. Greater the value of β greater is the attenuation. The term $1/\beta$, measures the depth at which electric field of wave entering a conductor is damped to $[e = 2.718]$ $1/e = 0.367$ times of its initial amplitude at the surface. This depth is called skin depth or penetration depth denoted by δ.

$$\delta = \frac{1}{\beta} = \sqrt{\frac{2}{\mu\sigma\omega}}$$ for good conductor. For perfect conductor $\sigma = \infty$ & $\delta = 0$.

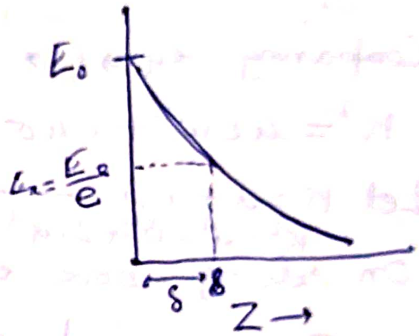
(P. T. O.)

If δ be the penetration depth or skin depth

$\Rightarrow E_0 e^{-\beta z} = \frac{E_0}{e}$

$\Rightarrow e^{-\beta z} = e^{-1} \Rightarrow \beta z = 1$ or, $z = \frac{1}{\beta}$ or, $\beta = \frac{1}{z} = \frac{1}{\delta}$

$$\boxed{\beta = \frac{1}{z} = \frac{1}{\delta}}$$



For good conductor $\alpha = \beta$

$\alpha = \beta = \sqrt{\frac{\mu\sigma\omega}{2}}$

$$\therefore \boxed{\delta = \frac{1}{\beta} = \sqrt{\frac{2}{\mu\sigma\omega}}}$$

$\therefore$ The required solution is

$$\left.\begin{aligned} E &= E_0 e^{i(\alpha+i\beta)\hat{n}.z - i\omega t} \\ &\quad \& \\ H &= H_0 e^{i(\alpha+i\beta)\hat{n}.z - i\omega t} \end{aligned}\right] \text{ or } \left.\begin{aligned} E &= E_0 e^{-\beta\hat{n}.\bar{z}} e^{i(\alpha\hat{n}.\bar{z}-\omega t)} \\ H &= H_0 e^{-\beta\hat{n}.z} e^{i(\alpha\hat{n}.\bar{z}-\omega t)} \end{aligned}\right] \quad (7)$$

From eqn(7), it is obvious that field amplitudes are attenuated due to the presence of the term $e^{-\beta\hat{n}.\bar{z}}$. The quantity β is a measure of attenuation & is known as absorption co-efficient.

For poor conductor $\frac{\sigma}{\omega\epsilon} \ll 1$ & for good conductor $\frac{\sigma}{\omega\epsilon} \gg 1$

$\therefore$ For good conductor

$$\alpha = \beta = \frac{\omega\sqrt{\mu\epsilon}}{\sqrt{2}}\left[\frac{\sigma}{\omega\epsilon}\right]^{1/2} = \sqrt{\frac{\omega\mu\sigma}{2}}.$$

Greater the value of β, greater is the attenuation. The term $\frac{1}{\beta}$, measures the depth at which electric field of wave entering a conductor is damped to $1/e = 0.367$ times of its initial amplitude at the surface. This depth is called skin depth or penetration depth denoted by δ [e=2.718]

$$\boxed{\delta = \frac{1}{\beta} = \sqrt{\frac{2}{\mu\omega\sigma}}}$$ for good conductor. For perfect conductor $\sigma = \infty$ (infinity) & $\delta = 0$

For poor conductor or good di-electrics

$\frac{\sigma}{\omega\epsilon} \ll 1$, Now $\beta = \omega\sqrt{\frac{\mu\epsilon}{2}}\left[\left(1+\frac{\sigma^2}{\omega^2\epsilon^2}\right)^{1/2} - 1\right]^{1/2}$ —(8)

As $\frac{\sigma}{\omega\epsilon} < 1$, $\therefore$ By Binomial expansion we have

$$\left(1+\frac{\sigma^2}{\omega^2\epsilon^2}\right)^{1/2} = 1 + \frac{\sigma^2}{2\omega^2\epsilon^2} + \ldots\ldots$$

$\therefore$ Eqn (8) becomes

$$\beta = \omega\sqrt{\frac{\mu\epsilon}{2}}\left[\cancel{1} + \frac{\sigma^2}{2\omega^2\epsilon^2} + \ldots\ldots - \cancel{1}\right]^{1/2} = \omega\sqrt{\frac{\mu\epsilon}{2}}\left(\frac{\sigma}{\sqrt{2}\,\omega\epsilon}\right)$$

$\therefore \boxed{\beta = \frac{\sigma}{2}\sqrt{\frac{\mu}{\epsilon}}}$ $\therefore \boxed{\delta = \frac{1}{\beta} = \frac{2}{\sigma}\sqrt{\frac{\epsilon}{\mu}}}$ → Skin depth

For di-electric, the skin depth is independent of frequency.

Poynting vector ($\vec{S}$)

Poynting vector ($\vec{S}$) is the energy flow per unit area per unit time for a plane e.m. wave, & is given by $\vec{S} = \vec{E} \times \vec{H}$, its unit is Watts/$m^2$.

Poynting ~~Theorem~~ Vector

At any point in electromagnetic field, the product of electric field intensity $\bar{E}$ & mag. field intensity $\vec{H}$ is a measure of rate of energy flow at that point. Mathematically, $\bar{S} = \bar{E} \times \bar{H}$, the direction of energy flow is perpendicular to $\bar{E}$ & $\bar{H}$ i.e. along the direction of propagation of e.m wave.

Proof: By Maxwell's 3rd & 4th eqn. we have

$$\bar{\nabla} \times \bar{E} = -\mu \frac{\partial \bar{H}}{\partial t} \quad \text{——(1)}$$

$$\bar{\nabla} \times \bar{H} = \bar{J} + \epsilon \frac{\partial \bar{E}}{\partial t} \quad \text{——(2)}$$

by taking dot product with H on both sides of eqn(1)
& " " " " E " " " " eqn(2), we get

$$\vec{H} \cdot (\vec{\nabla} \times \vec{E}) = -\mu \, \vec{H} \cdot \frac{\partial \vec{H}}{\partial t} \quad \text{——(3)}$$

$$\vec{E} \cdot (\vec{\nabla} \times \vec{H}) = \vec{E} \cdot \vec{J} + \epsilon \, \vec{E} \cdot \frac{\partial \vec{E}}{\partial t} \quad \text{——(4)}$$

Subtracting eqn.(4) from eqn(3) we get:

$$H \cdot (\nabla \times E) - E \cdot (\nabla \times H) = -\mu H \cdot \frac{\partial H}{\partial t} - E \cdot J - \epsilon E \frac{\partial E}{\partial t} \quad \text{——(5)}$$

By Vector identity we know that

$$H \cdot (\nabla \times E) - E \cdot (\nabla \times H) = \nabla \cdot (E \times H).$$

∴ Eqn.(5) becomes

$$\nabla \cdot (E \times H) = -E \cdot J - \mu H \cdot \frac{\partial H}{\partial t} - \epsilon E \frac{\partial E}{\partial t} \quad \text{——(6)}$$

P.T.O.

By Calculus we know that

$$H\cdot\frac{\partial H}{\partial t}=\frac{1}{2}\frac{\partial}{\partial t}(H^2)=\frac{\partial}{\partial t}\left(\frac{1}{2}H^2\right),\quad\therefore\ \mu\left(H\cdot\frac{\partial H}{\partial t}\right)=\frac{\partial}{\partial t}\left(\frac{1}{2}\mu H^2\right)$$

Similarly $\epsilon\left(E\cdot\frac{\partial E}{\partial t}\right)=\frac{\partial}{\partial t}\left(\frac{1}{2}\epsilon E^2\right)$

$\therefore$ Eqn (6) becomes

$$\nabla\cdot(E\times H)=-E\cdot J-\frac{\partial}{\partial t}\left(\frac{1}{2}\epsilon E^2+\frac{1}{2}\mu H^2\right) \quad \text{—— (7)}$$

By taking volume integral of both sides we get

$$\oint_V \nabla\cdot(E\times H)\,dv=-\oint_V E\cdot J\,dv-\frac{\partial}{\partial t}\oint_V\left(\frac{1}{2}\epsilon E^2+\frac{1}{2}\mu H^2\right)dv \quad \text{—— (8)}$$

Using Divergence theorem on L.H.S. of eqn (8) we have

$$\oint_S(\bar{E}\times\bar{H})\,ds+\oint_V\bar{E}\cdot\bar{J}\,dv=-\frac{\partial}{\partial t}\oint_V\left(\frac{1}{2}\epsilon E^2+\frac{1}{2}\mu H^2\right)dv \quad \text{—— (9)}$$

Let us discuss each terms

(i) The term $\oint_V(E\cdot J)\,dv$ represent instantaneous power dissipated in volume V $\left[E=\frac{\text{voltage}}{\text{distance}}=\frac{V}{d}\ \&\ J=\frac{I}{A}\right.$ $\therefore E\cdot J=\frac{VI}{Ad}=$ power loss in unit volume $\left.=I^2R\right]$

(ii) $-\frac{\partial}{\partial t}\oint_V\left(\underbrace{\frac{1}{2}\epsilon E^2}_{\text{Electric energy density}}+\underbrace{\frac{1}{2}\mu H^2}_{\text{Mag. energy density}}\right)dv=$ The rate of decrease of total stored energy in volume V.

(iii) $\oint_S(\bar{E}\times\bar{H})\,ds$, gives the rate of flow of energy outward through the surface.

The product $\bar{E}\times\bar{H}=\bar{S}$, is a measure of rate of energy flow per unit area. $\bar{S}$ is called Poynting vector. Thus the integral of $\bar{S}$ over a closed surface represent the rate at which electromagnetic energy crosses the closed surface.

Poynting theorem states that: The work done on the charge by an electromagnetic force is equal to the decrease in energy stored in the field & work done is less than the energy which flowed out the surface. It is also called conservation law in electrodynamics.