☆ Function of Complex Variable :→

Introduction :→

Complex No. :→ It is denoted by z and given as $z = x + iy = (x, y)$, where x is the Real part and y is the Imaginary Part of z.

☆ Polar form of Complex No. : Put $x = r\cos\theta$
$y = r\sin\theta$

$$x^2 + y^2 = r^2$$

Argument $\tan\theta = \frac{y}{x} \Rightarrow \boxed{\theta = \tan^{-1}(y/x)}$

$z = r\cos\theta + ir\sin\theta$.

$z = r(\cos\theta + i\sin\theta) = re^{i\theta}$

$$\boxed{z = re^{i\theta}}$$

Conjugate if z is denoted by $\bar{z}$ and given as :

$$\boxed{\bar{z} = x - iy = re^{-i\theta}}$$

Modulus of z is denoted by $|z|$ and given as :

$$|z| = |x+iy| = \sqrt{(R.P)^2 + (I.P)^2}$$

$$\boxed{|z| = \sqrt{x^2+y^2}}$$

$$z.\bar{z} = (x+iy)(x-iy) = x^2 - i^2y^2.$$

$$z\bar{z} = x^2+y^2 = |z|^2.$$

$$\Rightarrow \boxed{|z|^2 = z.\bar{z}}$$

$|z| = r$

We have $z = x+iy$

$$|x+iy| = r$$

$$\sqrt{x^2+y^2} = r$$

Squaring both sides

$$\boxed{x^2+y^2 = r^2}$$, eq^n of circle having centre at origin and radius r.

(2) : $|z - z_0| = r$

where $z = x+iy$, $z_0 = x_0 + iy_0$.

$$|(x+iy) - (x_0+iy_0)| = r$$

$$\Rightarrow |(x-x_0) + i(y-y_0)| = r$$

$$\Rightarrow \sqrt{(x-x_0)^2 + (y-y_0)^2} = r$$

Squaring both sides :

$$(x-x_0)^2 + (y-y_0)^2 = r^2$$

Eq^n of Circle centre at (x_0, y_0) and having radius r.

Function of Complex Variable :-

It is denoted by ω or $f(z)$ and defined as,

$$w = f(z) = u(x,y) + iv(x,y)$$

u and v are real & imaginary part of $f(z)$.

Separate into Real and Imaginary Part of the following fn. :

(1) $f(z) = z^2 - 2z + 3$

(Sol): $\because f(z) = (x-iy)^2 - 2(x+iy) + 3$

$\Rightarrow x^2 + i^2y^2 - 2ixy - 2x - 2iy + 3$

$\Rightarrow u(x,y) = x^2 - y^2 - 2x + 3$

$\Rightarrow v(x,y) = -2xy - 2y$

(i) $\sin ix = i\sinh x$.

(ii) $\cos ix = \cosh x$.

(iii) $e^{ix} = \cos x + i\sin x$.

(iv) $e^{-ix} = \cos x - i\sin x$.

~~(Sol): $\Rightarrow f($~~

(Q): $f(z) = \sin z$.

(Sol) $f(z) = \sin(x+iy)$

$f(z) = \sin x\cos iy + \cos x\sin iy$.

$f(z) = \sin x\cosh y + i\cos x\sinh y = u + iv$

$$\boxed{u = \sin x\cosh y,\ v = \cos x\sinh y}$$

(Q) $f(z) = \sin \bar{z}$.

(Sol): $f(z) = \sin(x-iy)$

$f(z) = \sin x\cos iy - \cos x\sin iy$

$f(z) = \sin x\cosh y - i\cos x\sinh y = u + iv$

$$\boxed{u = \sin x\cosh y,\ v = -\cos x\sinh y}$$

(Q) : $f(z) = \cos z$

(Sol) :→ $f(z) = \cos(x+iy)$

$f(z) = \cos x \cos iy - \sin x \sin iy$.

$f(z) = \cos x \cosh y - i \sin x \sinh y = u + iv$.

$u = \cos x \cosh y$, $v = -\sin x \sinh y$.

(Q) :→ $f(z) = e^z$

$f(z) = e^{x+iy} = e^x \cdot e^{iy}$.

$f(z) = e^x (\cos y + i \sin y) = u + iv$.

$$\boxed{\begin{array}{l} u = e^x \cos y \\ \hline v = e^x \sin y \end{array}}$$ (Ans)

(Q) : → $f(z) = \log z$

(Sol) : → $f(z) = \log(x+iy)$

$\log(m+n)$, $\log mn \to \log m + \log n$.

We have ;

$z = re^{i\theta}$ (Polar form)

$r = \sqrt{x^2+y^2}$

$\theta = \tan^{-1} y/x$.

$f(z) = \log z = \log re^{i\theta} = \log r + \log e^{i\theta}$.

$f(z) = \log r + i\theta \Rightarrow \log(x^2+y^2)^{1/2} + i\tan^{-1}(y/$

$= u + iv$

$\cancel{= \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}(}$

$u = \frac{1}{2}\log(x^2+y^2)$

$v = \tan^{-1} y/x$. (Ans)

Single valued function :→

For each value of z, there corresponds a one value of $f(z)$, then $f(z)$ is said to be single valued function otherwise multivalued function.

Singular Point :→

$$f(z) = \frac{z^2}{(z+2)(z-4)}$$

Single Point of $f(z)$ are :

$$(z+2)(z-4) = 0$$

$$\Rightarrow \boxed{z = -2, 4}$$

Every exponential fn., every algebric, trignometric are diff. in its domain

Differentiability of $f(z)$:→

~~A single valued function $f(z)$~~

The differentiability of a single valued fn. is defined as :

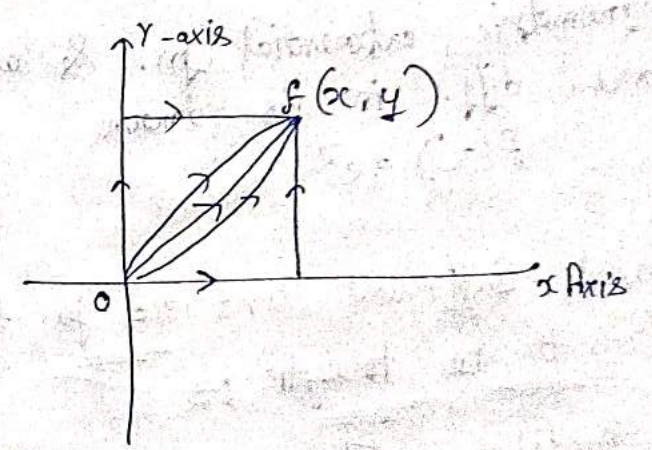
$$f'(z) = \lim_{\delta z \to 0} \left[\frac{f(z+\delta z) - f(z)}{\delta z} \right]$$

(delta)

where : $z = x + iy$.

$\delta z = \delta x + i\delta y$.

Provided limit exist and Independent of the path along which $\delta z \to 0$.



Infinite path are possible :

Analytic Function :→ (Holomorphic or Regular fn)

A single valued fn. $f(z)$ is said to be Analytic at the point z_0 if it is differentia not only at z_0, but in the neighbourhood of z_0.

$|z - z_0| < \varepsilon$ [Epsilon very small but greater than 0]

* $f(z)$ is analytic in the region R, then it is differentiable at every point of R.

* Every trignometric, exponential fn. & algebraic function are diff. in its domain
$f(z) = e^z$

⇒ Necessary and Sufficient Conditions for the function $f(z)$ to be Analytic :⇒

(1) Necessary Condition :⇒ The necessary condition for the fn. $f(z)$ to be Analytic in region R are :

(i) $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ are continuous fn. in region R.

(ii) :⇒ $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.
(Cauchy Reiman equations)
(C-R equations).

(2) Sufficient Conditions :

Let $f(z) = u + iv$, be a single-valued function, Possessing Partial derivatives
$\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ at each point of Region R and satisfying C-R ; ~~C-CN~~

then $f'(z)$ must exist.
($f(z)$ must be differentiable)

~~$f(z)$~~

→ Diff. coeff. of $f(z)$ is given as

$$\boxed{\frac{df}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}}$$

or

$$\boxed{\frac{df}{dz} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}}$$

Tute :

Analytic function :-

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

$$w = f(z) = u + iv$$

$$\frac{dw}{dz} = \frac{df}{dz} = \frac{\partial u}{\partial x} + \frac{i\partial v}{\partial x}.$$

(Q) : Show that $f(z) = z + 2\bar{z}$ is not analytic anywhere in the complex plane.

(Sol) :→ $f(z) = (x+iy) + 2(x-iy) = u + iv.$

$$u = 3x \quad , v = -y ,$$

$$\frac{\partial u}{\partial x} = 3 , \quad \frac{\partial u}{\partial y} = 0 .$$

$$\frac{\partial v}{\partial x} = 0 , \quad \frac{\partial v}{\partial y} = -1 .$$

C-R eq^n :

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} \Rightarrow 3 \neq -1$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \Rightarrow 0 = 0.$$

C-R eqn. are not satisfied so the given fn is not analytic anywhere.

(Q): $f(z) = z|z|$.

(Sol): $\Rightarrow f(z) = (x+iy)\sqrt{x^2+y^2} = u + iv$.

$$u = x\sqrt{x^2+y^2}, \quad v = y\sqrt{x^2+y^2}$$

$$\frac{\partial u}{\partial x} = \frac{2x^2+y^2}{\sqrt{x^2+y^2}}; \quad \frac{\partial u}{\partial y} = \frac{xy}{\sqrt{x^2+y^2}}$$

$$\frac{\partial v}{\partial x} = \frac{xy}{\sqrt{x^2+y^2}}, \quad \frac{\partial v}{\partial y} = \frac{x^2+2y^2}{\sqrt{x^2+y^2}}$$

Using C-R eqn: $\Rightarrow$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\frac{2x^2+y^2}{\sqrt{x^2+y^2}} = \frac{x^2+2y^2}{\sqrt{x^2+y^2}}$$

$$\frac{xy}{\sqrt{x^2+y^2}} = -\frac{xy}{\sqrt{x^2+y^2}}$$

C-R eqn. are not satisfied, so this is not Analytic.

(Q): Show that the fn. $f(z) = \log z$ is analytic everywhere in the complex frame except at the origin and its derivative is $\frac{1}{z}$.

(Sol): $\Rightarrow f(z) = \log(x+iy)$

$$= \log(re^{i\theta})$$

$$= \log r + \log e^{i\theta}$$

$$= \log r + i\theta \log e$$

$$= \log r + i\theta$$

$$= \log(x^2+y^2)^{1/2} + i\tan^{-1} y/x$$

$$u = \frac{1}{2}\log(x^2+y^2); \quad v = \tan^{-1} y/x.$$

$$\frac{\partial v}{\partial x} = \cancel{\frac{2x}{\sqrt{x^2+y^2}}} = \frac{x}{x^2+y^2}$$

$$\frac{\partial v}{\partial y} = \cancel{\frac{2y}{\sqrt{x^2+y^2}}} = \frac{y}{x^2+y^2}$$

$$\frac{\partial v}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}} \times \left(\frac{-y}{x^2}\right) = \frac{-y}{x^2+y^2}$$

$$\frac{\partial v}{\partial y} = \frac{1}{1+\frac{y^2}{x^2}} \times \left(\frac{1}{x}\right) = \frac{x}{x^2+y^2}$$

C.R eqn.:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Hence, C.R eqn. are satisfied except at the origin $(x=0, y=0)$, so the eqn. is analytic everywhere.

$$\frac{dw}{dz} = \frac{d}{dz}(\log z) = \frac{1}{z}$$

(Q): Find the value of a and b such that fn. $f(z) = x^2 + ay^2 - 2xy + i(bx^2 - y^2 + 2xy)$ is Analytic, find also $f'(z)$.

(Sol):

$$u = x^2 + ay^2 - 2xy$$

$$v = bx^2 - y^2 + 2xy$$

$$\frac{\partial u}{\partial x} = 2x - 2y, \quad \frac{\partial u}{\partial y} = 2ay - 2x.$$

$$\frac{\partial v}{\partial x} = 2bx + 2y, \quad \frac{\partial v}{\partial y} = -2y + 2x.$$

C-R eqn:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

$$2x - 2y = -2y + 2x \quad \text{— (1)}$$

$$2ay - 2x = -2bx - 2y \quad \text{— (2)}$$

$-2 = -2b \Rightarrow \boxed{b = 1}$

$2a = -2 \Rightarrow \boxed{a = -1}$.

Now;

$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$.

$= 2x - 2y + i(2x + 2y)$

$= f'(z) = 2\left((x+iy) + i\left(x - \frac{i}{i \cdot i} y\right)\right)$

$f'(z) = 2z(1+i)$

$f'(z) = 2(1+i)z$.

(Q): $\Rightarrow$ $f(z) = -x^2 + xy + y^2 + i(ax^2 + bxy + cy^2)$ is Analytic, express $f(z)$ in terms of z and also find $f'(z)$.

(Sol): $\Rightarrow$ $u = -x^2 + xy + y^2$

$v = ax^2 + bxy + cy^2$.

$\frac{\partial u}{\partial x} = -2x + y$

$\frac{\partial u}{\partial y} = x + 2y$

$\frac{\partial v}{\partial x} = 2xa + by + 0$.

$\frac{\partial v}{\partial y} = 0 + bx + 2yc$.

$a = -\frac{1}{2}$, $b = -2$, $c = \frac{1}{2}$

(Q): $f(z) = -x^2 + xy + y^2 + i\left(-\frac{1}{2}x^2 - 2xy + \frac{1}{2}y^2\right)$

(Sol): $\Rightarrow f(z) = -x^2 + yx + y^2 - \frac{i}{2}x^2 - i2xy + \frac{i}{2}y^2$

$-x^2 + y^2 - 2ixy = (ix - y)^2 =$

$i^2\left(x - \frac{i}{i \cdot i}y\right)^2 = -(x + iy)^2$

$= \frac{i}{2}\left(-x^2 + y^2 + \frac{2ixy}{i \times i}\right)$

$= \frac{i}{2}\left(-x^2 + y^2 - 2ixy\right) = \frac{i}{2}(ix - y)^2.$

$f(z) = -x^2 + xy + y^2 - \frac{i}{2}x^2 - 2ixy + \frac{i}{2}y^2 \Rightarrow$

$-z^2 - \frac{i}{2} \cdot z^2 = -\left(\frac{2+i}{2}\right)z^2.$

continue:

☆☆ Partial Derivative at Origin:

(1) $\frac{\partial u}{\partial x} = \lim_{x \to 0}\left[\frac{u(x,0) - u(0,0)}{x}\right]$

(2): $\frac{\partial v}{\partial y} = \lim_{y \to 0}\left[\frac{u(0,y) - u(0,0)}{y}\right]$

(3) $\frac{\partial v}{\partial x} = \lim_{x \to 0}\left[\frac{v(x,0) - v(0,0)}{x}\right]$

(4) $\frac{\partial v}{\partial y} = \lim_{y \to 0}\left[\frac{v(0,y) - v(0,0)}{y}\right]$

☆ Differentiability at origin:

$f'(0) = \lim_{z \to 0}\left[\frac{f(z) - f(0)}{z}\right]$

$f(0) = 0$ | $f(z) = u(x,y) + iv(x,y)$

$u(x,y) = 0$

At $x = 0,\ y = 0$ | $v(x,y) = 0$, $v(0,0) = 0$.

~~Provided~~

Provided limit exist & Independent of the path along which $z \to 0$.

(Q) :→ Verify $f(z) = \dfrac{xy^2(x+iy)}{x^2+y^4}$, $z \neq 0$

$f(0) = 0$ is analytic or not at origin verify

~~(Sol)~~: $f(z) = \begin{cases} \dfrac{xy^2(x+iy)}{x^2+y^4}, & z \neq 0 \\ 0, & z \neq 0 \end{cases}$

(Sol) :→ $f(z) = \dfrac{xy^2(x+iy)}{x^2+y^4}$

$u(x,y) = \dfrac{x^2y^2}{x^2+y^4}$

$v(x,y) = \dfrac{xy^3}{x^2+y^4}$

$f(0) = 0 \Rightarrow u(0,0) = 0, \; v(0,0) = 0$

Partial Derivatives at origin

$$\frac{\partial u}{\partial x} = \lim_{x \to 0} \left[\frac{u(x,0) - u(0,0)}{x} \right]$$

$$\frac{\partial u}{\partial x} = \lim_{x \to 0} \left(\frac{0}{x^3} \right) = 0$$

$$\frac{\partial u}{\partial y} = \lim_{y \to 0} \left[\frac{u(0,y) - u(0,0)}{y} \right]$$

$$= \lim_{y \to 0} \left(\frac{0}{y^5} \right) = 0$$

$$\frac{\partial v}{\partial x} = \lim_{x \to 0} \left[\frac{v(x,0) - v(0,0)}{x} \right]$$

$$\frac{\partial v}{\partial x} = \lim_{x \to 0} \left(\frac{0}{x^3} \right) = 0$$

$$\frac{\partial v}{\partial y} = \lim_{y \to 0} \left(\frac{0}{y^5} \right) = 0$$

C-R eqn. are satisfied at origin.

Diff. at origin :→

$$f'(0) = \lim_{z \to 0} \left[\frac{f(z) - f(0)}{z} \right]$$

$$f'(0) = \lim_{z \to 0} \left[\frac{f(z)}{z} \right]$$

$$f'(0) = \lim_{z \to 0} \left[\frac{xy^2 (x+iy)}{(x^2+y^4) z} \right]$$

$$f'(0) = \lim_{\substack{x \to 0 \\ y \to 0}} \left[\frac{xy^2}{x^2+y^4} \right] \quad - (1)$$

(1) Along Real Axis i.e : $y = 0$,

$$f'(0) = \lim_{x \to 0} \left(\frac{0}{x^2} \right) = 0.$$

(2) Along Imaginary Axis i.e: $x = 0$

$$f'(0) = \lim_{y \to 0} \left(\frac{0}{y^4} \right) = 0.$$

(iii): Along the curve $y^2 = x$ (parabola).

$$f'(0) = \lim_{y \to 0} \left(\frac{y^4}{y^4+y^4} \right) = \frac{1}{2}$$

The function is not differentiable at origin, so the given fn. is not analytic at origin

(Q): Prove that the fn. $f(z) = \frac{x^3(1+i) - y^3(1-}{x^2+y^2}$ if $z \neq 0$; and, $f(0) = 0$, is continous & C-R eqn are satisfied at origin, yet $f'(0)$ does not exist.

Harmonic Functions :→

Any which satisfy laplace eqn ; i.e $\nabla^2 = 0$ $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 0 \right)$ is called Harmonic function.

→ If $f(z) = u + iv$ is Analytic fn, then Prove that its real & Imaginary part both are Harmonic func.

Proof :→ : $f(z) = u + iv$ be an analytic func.
so :

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{— (1)}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{— (2)}$$

Partially diff. eqn (1) w.r.t x and (2) w.r.to y,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \quad \text{— (3)}$$

$$\frac{\partial^2 v}{\partial y^2} = -\frac{\partial^2 v}{\partial y \, \partial x} \quad \text{--- (4)}$$

(3) + (4)

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 .$$

Hence it is Harmonic.

(Q) : Show that the ~~f(z)~~ fn. $v = x^3 - 3xy^2$ is Harmonic.

(Sol): $\Rightarrow$ $\frac{\partial v}{\partial x} = 3x^2 - 3y^2$, $\therefore \frac{\partial^2 v}{\partial x^2} = 6x$,

$\frac{\partial v}{\partial y} = -6xy$; $\frac{\partial^2 v}{\partial y^2} = -6x$.

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 6x - 6x = 0 .$$

So $v(x, y)$ is Harmonic func.

☆ $f(z) = u + iv$, is an Analytic fn.

then u and v both are Harmonic conjugate. i.e :$\Rightarrow$ Harmonic Conjugate of $u(x, y)$, is $v(x, y)$ and Harmonic Conjugate of $v(x, y)$ is $u(x, y)$.

(V.I Exam Purpose)

Construct the Analytic fn $f(z)$ by milne's thomson method :→

1. Case : If real part of $f(z)$ is given i.e : u is given :→

We have :

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \quad \text{—— (i)}$$

Using C-R eqn. :

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \quad \text{—— (2)}$$

Let $\frac{\partial u}{\partial x} = \phi_1(x,y)$ and $\frac{\partial u}{\partial y} = \phi_2(x,y)$

$$f'(z) = \phi_1(x,y) - i\phi_2(x,y) \quad \text{—— (3)}$$

Replace x by z and y by zero in eqn (3)

$$f'(z) = \phi_1(z,0) - i\phi_2(z,0) \quad \text{— (4)}$$

On Integration :

$$f(z) = \int \left[\phi_1(z,0) - i\phi_2(z,0)\right] dz + \text{Constant} \quad (5)$$

2. Case :→ : If Imag. part of $f(z)$ is given, i.e v is given :→

$$f'(z) = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}$$

Let $\frac{\partial v}{\partial y} = \psi_1(x,y)$, $\frac{\partial v}{\partial x} = \psi_2(x,y)$

$$f'(z) = \psi_1(x,y) + i\psi_2(x,y)$$

Replace x by z and y be zero

$$f'(z) = \psi_1(z,0) + i\psi_2(z,0)$$

on Int, then :

$$f(z) = \int \left[\psi_1(z,0) + i\psi_2(z,0)\right] dz + \text{Constant}$$

(0) :→ Prove that $u = e^x(x\cos y - y\sin y)$ is a harmonic fn. find an Analytic func. $f(z) = u + iv$ such that $f(1) = e$ and also find its harmonic conjugate.

(Sol) :→ $u = e^x(x\cos y - y\sin y)$

$$\frac{\partial u}{\partial x} = e^x(\cos y) + e^x(x\cos y - y\sin y)$$

$$\frac{\partial u}{\partial x} = e^x(\cos y + x\cos y - y\sin y) = \phi_1(x,y)$$

$$\frac{\partial^2 u}{\partial x^2} = e^x(\cos y) + e^x(\cos y + x\cos y - y\sin y)$$

$$\frac{\partial^2 u}{\partial x^2} = e^x(2\cos y + x\cos y - y\sin y) \quad \text{—(1)}$$

$$\frac{\partial u}{\partial y} = e^x\left[-x\sin y - (y\cos y + \sin y)\right] = \phi_2(x,y)$$

$$\frac{\partial^2 u}{\partial y^2} = e^x\left[-x\cos y - (-y\sin y + \cos y +\right.$$

$$\frac{\partial^2 u}{\partial y^2} = e^x\left[-x\cos y + y\sin y - 2\cos y\right] \quad \text{—(2)}$$

(1) + (2)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Hence $u(x,y)$ is harmonic.

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}.$$

$$f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = \phi_1(x,y) - i\phi_2(x,y)$$

Now By Milne's thomson method:

$$f(z) = \int \left[\phi_1(z,0) - i\phi_2(z,0)\right] dz + c_1$$

$$\phi_1(z,0) = e^z(1+z)$$

$$\phi_2(z,0) = 0$$

$$f(z) = \int e^z(1+z)\,dz + \text{constant}$$

[Int by di rule

Using $f(1) = e$

$f(1) = e + c_1 \Rightarrow e = e + c_1 \Rightarrow c_1 = 0$

$\Rightarrow c_1 = 0$.

So; $\boxed{f(z) = ze^{z}}$.

Harmonic conjugate of u is v:

$$\boxed{v = e^{x}(y\cos y + x\sin y)}.$$

$f(z) = (x+iy)e^{x}(\cos y + i\sin y)$

$u + iv = (xe^{x}\cos y - e^{x}y\sin y) + i[ye^{x}\cos y + xe^{x}\sin y]$

(Q): Find Analytic fun. $f(z)$ if imaginary part is $\cos x \cosh y$.

(Sol): $v = \cos x \cosh y$

$\frac{\partial v}{\partial x} = -\sin x \cosh y = \psi_0(x,y) = \psi_0(z,0)$

$\frac{\partial v}{\partial y} = \cos x \sinh y = \psi_1(x,y) = \psi_1(z,0)$

$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$.

$f'(z) = \underbrace{\left(\frac{\partial v}{\partial y}\right)}_{\psi_0} + i\underbrace{\left(\frac{\partial v}{\partial x}\right)}_{\psi_2}$

By milne's thomson method:

$f(z) = \int [\psi_1(z,0) + i\psi_2(z,0)]\,dz$.

$$\boxed{f(z) = i\cos z + c_1}$$

Recap:

$\left.\begin{matrix} u \\ v \end{matrix}\right] \rightarrow f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \rightarrow \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$

$\downarrow$

$\phi_1(x,y) - \phi_2(x,y)$

$f(z) = \left[\phi_1(z,0) - i\phi_2(z,0)\right]dz + c_1$

Case: 3rd If $u+v$ is given:

We have:

$f(z) = u + iv$ — (1)

$if(z) = iu - v$ — (2)

(1) + (2)

$f(z) + if(z) = (u-v) + i(u+v)$

$f(z)\cdot(1+i) = (u-v) + i(u+v)$

$F(z) = U + iV$ — (3)

where $F(z) = (1+i)f(z)$

$U = u-v$, $V = u+v$

According Case II:

$F(z) = \int\left[\varphi_1(z,0) + i\varphi_2(z,0)\right]dz + c_1$

where $\varphi_1(z,0) = \left(\frac{\partial v}{\partial y}\right)_{(z,0)}$

$\varphi_2(z,0) = \left(\frac{\partial v}{\partial x}\right)_{(z,0)}$

and $F(z) = (1+i)f(z)$

$\Rightarrow f(z) = \frac{F(z)}{1+i}$

Case: 4] If $u-v$ is given:

$f(z) = u + iv$ — (1)

$if(z) = iu - v$ — (2)

(1) + (2)

$f(z) + if(z) = (u-v) + i(u+v)$

$f(z)[1+i] = (u-v) + i(u+v)$

$F(z) = U + iV$ — (3)

where $f(z) = (1+i)f(z)$

$U = u - v$, $V = u + v$.

According to 1 case

$$F(z) = \int \left[\phi_1(z,0) - i\phi_2(z,0) \right] dz + c_1$$

where $\phi_1(z,0) = \left(\frac{\partial u}{\partial x}\right)_{(z,0)}$

$$\phi_2(z,0) = \left(\frac{\partial u}{\partial y}\right)_{(z,0)}$$

and $F(z) = (1+i) f(z)$

$$\Rightarrow f(z) = \frac{F(z)}{1+i}$$

Q: Construct $f(z) = u + iv$ in terms of z if $u + v = \frac{x}{x^2+y^2}$ and $f(1) = 1$.

(Sol): → We have :→

$f(z) = u + iv$ — (1)

$if(z) = iu - v$ — (2)

(1) + (2)

$f(z) + if(z) = (u - v) + i(u + v)$

$F(z) = U + iV$ — (3)

where $F(z) = (1+i) f(z)$

$U = u - v$, $V = u + v$.

By Milne Jn. method ; $\cancel{F'(z) = \frac{\partial u}{\partial x} + \partial}$

$$F(z) = \int \left[\phi_1(z,0) + i\phi_2(z,0) \right] dz + c_1 - (4)$$

$$u + v = V = \frac{x}{x^2+y^2}$$

$$\frac{\partial v}{\partial x} = \frac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)^2} = \frac{y^2 - x^2}{(x^2+y^2)^2}$$

$$= \phi_2(x,y)$$

$$\frac{\partial v}{\partial y} = \frac{x(-2y)}{(x^2+y^2)^2} = \frac{-2xy}{(x^2+y^2)^2} = \phi_1(x,y).$$

$$\phi_1(z,0) = 0$$

$$\phi_2(z,0) = -\frac{z^2}{z^4} = -\frac{1}{z^2}$$

$$F(z) = \int -i\,\frac{1}{z^2}\,dz + C_1$$

$$F(z) = i\cdot\frac{1}{z} + C_1.$$

$$= (1+i)f(z) = \frac{i}{z} + C_1$$

Using $f(1) = 1$.

$$= (1+i) = i + C_1 \Rightarrow C_1 = 1$$

$$(i+1)\,f(z) = \frac{i}{z} + 1$$

$$f(z) = \left(\frac{i}{1+i}\right)\cdot\frac{1}{z} + \frac{1}{1+i}$$

$$f(z) = \frac{(1-i)i}{(1+i)(1-i)}\cdot\frac{1}{z} + \frac{(1-i)}{(1+i)(1-i)}$$

$$\boxed{f(z) = \left(\frac{1+i}{z}\right)\frac{1}{z} + \frac{1-i}{z}}$$

(Q) :→ Construct $f(z) = u + iv$ in terms of z is $u - v = e^{x}(\cos y - \sin y)$.

(Ans) :→ $f(z) = u + iv$ — (1)

$if(z) = iu - v$ — (2)

$$f(z) + if(z) = (u-v) + i(u+v)$$

$$f(z)[1+i] = (u-v) + i(u+v)$$

$$F(z) = U + iV.$$

$U = u - v$, $V = u + v$.

Ac. to Case 1 :→

$$F(z) = \int \left[\phi_1(z,0) - i\phi_2(z,0)\right] dz + C_1$$

where $\phi_1(z,0) = \frac{\partial u}{\partial x}(z,0)$

$$\phi_2(z,0) = \frac{\partial u}{\partial y}(z,0)$$

$$u - v = 0 = e^{x}(\cos y - \sin y)$$

$$\frac{\partial u}{\partial x} = e^{x}(\cos y - \sin y) = \phi_1(x,y)$$

$$\frac{\partial u}{\partial y} = (-\sin y - \cos y)e^{x} = \phi_2(x,y)$$

By milne's thomson theorem:

$$F(z) = \int \phi_1(z,0) - i\phi_2(z,0)]\,dz + C$$

$$F(z) = \int [e^{z} + ie^{z}]\,dz + C$$

$$F(z) = e^{z} + ie^{z} + C$$

$$\cancel{f(z) = e^{z}}$$

$$(1+i)f(z) = e^{z} + ie^{z} + c$$

$$f(z) = \frac{e^{z}(1+i)}{1+i} + \frac{C}{1+i}$$

(Q):⇒ Show that the fn.

$$f(z) = \frac{x^3 y^5 (x+iy)}{x^6+y^{10}}, \quad z \neq 0 \text{ and } f(0) = 0$$

is not Analytic at origin, Although it satisfies C-R eqn at origin :

(Sol): It is given that the fn. $f(z)$ satisfies C-R eqn. at origin.

Diff. at origin :

$$f'(0) = \lim_{z \to 0} \left[\frac{f(z) - f(0)}{z} \right]$$

$$f'(0) = \lim_{z \to 0} \left(\frac{f(z)}{z} \right)$$

$$f'(0) = \frac{x^3 y^5 (x+iy)}{(x^6+y^{10}) z}$$

$$= \lim_{\substack{x \to 0 \\ y \to 0}} \frac{x^3 y^5}{(x^6+y^{10})} \quad \text{——— (1)}$$

(1) Along the curve $y = x^{3/5} \Rightarrow y^5 = x^3$.

$$f'(0) = \lim_{x \to 0} \frac{x^3 \cdot x^3}{x^6 + x^6} = \frac{1}{2}.$$

(2): Along Real Axis : $(y = 0)$

$$f'(0) = \lim_{x \to 0} \left(\frac{0}{x^6} \right)$$

$$f'(0) = 0.$$

~~So~~ $f(z)$ is not diff. at origin so the given fn. is not analytic at origin.

(Q) :⇒ Construct the Analytic fn. $f(z)$ if $u = \frac{1}{2}\log(x^2$ and also find its Harmonic conjugate.

(Sol):⇒ $f(z) = \log z$.

$$u + iv = \log re^{i\theta} = \log r + i\theta.$$

$$v = \tan^{-1} y/x.$$

(Q.10): Show that the func.,

$$u(x,y) = x^4 - 6x^2y^2 + y^4 \text{ is Harmonic.}$$

Also find the Analytic func. $f(z)$.

(Que.) If $u - v = (x-y)(x^2+4xy+y^2)$ find $f(z)$ in terms of z.

(Sol):- (1)

$$u = x^4 - 6x^2y^2 + y^4$$

$$\frac{\partial u}{\partial x} = 4x^3 - 12xy^2 \; ; \; \frac{\partial^2 u}{\partial x^2} = 12x^2 - 12y^2$$

$$\frac{\partial u}{\partial y} = 0 - 12x^2y + 4y^3$$

$$\frac{\partial u}{\partial y} = 4y^3 - 12x^2y.$$

$$\frac{\partial^2 u}{\partial y^2} = 12y^2 - 12x^2.$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 12x^2 - 12y^2 + 12y^2 - 12x^2$$

$$= 0$$

So $u(x,y)$ is Harmonic function.

$$\cancel{u - v = x^3 + x^2y + xy^2 - yx^2}$$

$$f(z) = u + iv \; , \; if(z) = -iu - v$$

$$① + ②$$

$$f(z) + if(z) = u + iv - iu - v$$

$$\cancel{f(z) + if}$$

$$f(z)[1+i] = (u - v) + i(u + v)$$

$$U = u - v$$

$$V = u + v.$$

$$f(z)[1+i] = U + iV.$$

$$F(z) = \int [\varphi_1(z,0) - i\varphi_2(z,0)]\,dz + C_1$$

where $\varphi_1(z,0) = \frac{\partial u}{\partial x}(z,0)$

$$\varphi_2(z,0) = \frac{\partial u}{\partial x}(z,0)$$

$$u - v = U = (x-y)(x^4 + 4xy + y^2$$

☆☆ CR eq^n. in Polar Form (r, θ) :-

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{dv}{d\theta} \quad \text{and} \quad \frac{\partial u}{\partial \theta} = -r\frac{dv}{dr}.$$

$$f'(z) = \frac{dw}{dz} = (\cos\theta - i\sin\theta)\frac{\partial w}{\partial r}.$$

☆ Complex Integration :→

$$\int_{y=x^2}^{y=x} f(z)\,dz$$

$z_1 \Rightarrow y = x$

$z_2 \Rightarrow y = x^2$

Infinite curve & path are possible in complex.

(Q) :⇒ Evaluate $\int_0^{2+i} (\bar{z})^2\,dz$, along the real axis, from $z=0$ to $z=2$ and then along a line parallel to y-axis from $z=2$ to $z=2+i$

(Ans) :⇒

$$I = \int_0^{2+i} (\bar{z})^2\,dz$$

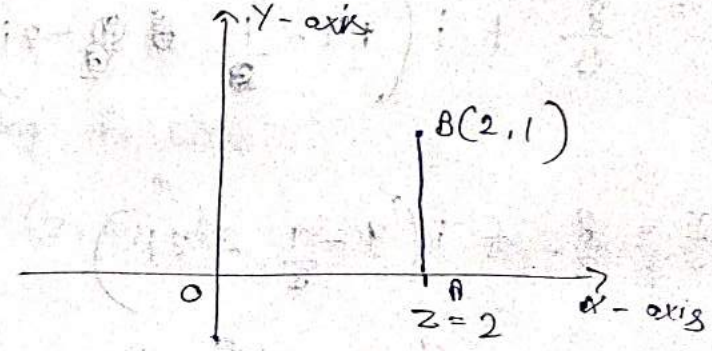
$z = 0 \Rightarrow (0,0)$

$z = 2+i \Rightarrow (2,1)$

$z = x + iy \Rightarrow dz = dx + i\,dy$.

$\bar{z} = x - iy \Rightarrow \bar{z}^2 = x^2 - y^2 - 2ixy$.

$$I = \int_{(0,0)}^{(2,1)} (x^2 - y^2 - 2ixy)(dx + i\,dy) \quad \text{—— (1)}$$



$z = 0 \Rightarrow (0,0)$

$z = 2 \Rightarrow (2,0)$

$z = 2+i = (2,1)$

$$I = \int_{OA} (x^2 - y^2 - 2ixy)(dx + i\,dy) + \int_{AB} (x^2 - y^2 - 2ixy)(dx + i\,dy)$$

Along OA, $y = 0 \Rightarrow dy = 0$, x varies from 0 to 2.

Along AB, $x = 2 \Rightarrow dx = 0$, y varies from 0 to 1

$$I = \int_0^2 x^2 dx + \int_0^1 (4 - y^2 - 4iy)(i\,dy)$$

$$I = \left(\frac{x^3}{3}\right)_0^2 + i\left(-4y - \frac{y^3}{3} - 4i\frac{y^3}{2}\right)_0^1$$

$$I = \frac{8}{3} + i\left(+4 - \frac{1}{3} - 2i\right)$$

$$I = \frac{8}{3} + i\left(4 - \frac{1}{3} - 2i\right)$$

$$I = \frac{8}{3} + i\left(\frac{12-1}{3} - 2i\right)$$

$$I = \frac{8}{3} + \left(\frac{11}{3} - 2i\right)i$$

$$I = \frac{8}{3} + \left(\frac{11 - 6i}{3}\right)i$$

$$I = \frac{8}{3} + \frac{11}{3}i + \frac{6}{3}$$

$$\cancel{I = \frac{8}{3} + 6 + \frac{11}{3}i = \frac{8+18}{3} + \frac{11}{3}i}$$

$$\cancel{I = \frac{26}{3} + \frac{11}{3}i}$$

$$\cancel{I = \frac{1}{3}(26 + 11i)}$$

$$I = \frac{14}{3} + \frac{11}{3}i.$$

(Ques): Prove that :

(i) $\oint_C \frac{dz}{z-a} = 2\pi i.$

↳ (Along the curve C)

(ii) $\oint_C (z-a)^n dz = 0$ (n is an Integer $\neq -1$)

where curve C is $|z-a| = r$

(Sol: i): ⇒ Let ;

$$I = \oint_C \frac{dz}{z-a} \quad \text{—(1)}$$

given that

$$|z-a| = r \Rightarrow z - a = re^{i\theta}.$$

$$\frac{dz}{d\theta} = ire^{i\theta} \Rightarrow dz = ire^{i\theta}\, d\theta, \quad (r \text{ fixed})$$

θ varies from 0 to 2π.

$$I = \int_0^{2\pi} \frac{ire^{i\theta} \cdot d\theta}{re^{i\theta}}$$

$$I = i(\theta)_0^{2\pi} = 2\pi i$$

(Sol: ii):→ Let

$$I = \oint_C (z-a)^n\, dz = 0$$

given that

$$|z-a| = r \Rightarrow z - a = re^{i\theta}$$

$$\frac{dz}{d\theta} = \cancel{ire^{i\theta}}\ ire^{i\theta} \Rightarrow dz = ire^{i\theta}\, d\theta$$

$$\cancel{I = \int_0^{2\pi}}$$

$$I = \oint_C (z-a)^n\, dz$$

$$I = \int_0^{2\pi} r^n e^{in\theta}\, ire^{i\theta}\, d\theta$$

$$I = ir^{n+1} \int_0^{2\pi} e^{i(n+1)\theta}\, d\theta$$

$$I = ir^{n+1} \left[\frac{e^{i(n+1)\theta}}{i(n+1)} \right]_0^{2\pi}$$

$$I = \frac{r^{n+1}}{n+1} \left[e^{i(n+1)2\pi} - 1 \right]$$

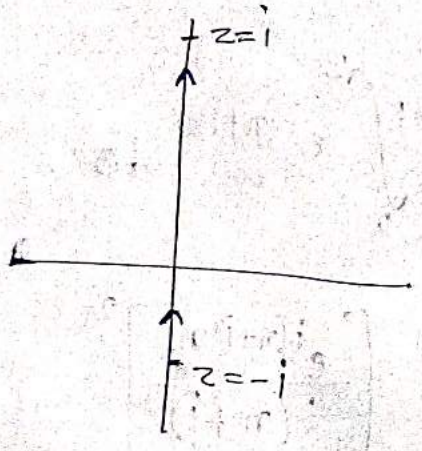
$$I = \frac{r^{n+1}}{n+1} \left[\cos(n+1)2\pi + i\sin(n+1)2\pi - 1 \right]$$

$$I = 0$$

(Q): ⇒ Evaluate the Integral $\int_c |z|\,dz$

where curve c is the straight line from $z = -i$ to $z = i$.

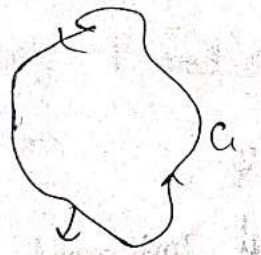
(Sol) :→



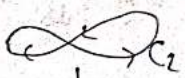
(S

☆☆ Simply and Multiply Connected Region :⇒

A curve is called simple closed curve if it does not intersect ~~itself~~ itself anywhere. A curve intersect itself is called a multiple curve.



(Simple closed curve)
↓
(Simply Connected Region)



↓
(Multiple curve)
↓
(Multiply Connected Region)

☆☆ Cauchy Integral Theorem :→

If $f(z)$ is an Analytic function and $f'(z)$ is Continuous at each point within and on a simple closed curve c, then

$$\int_C f(z)\,dz = 0.$$

(Q): Evaluate :

$$\int_C \frac{z}{(z-3)(z+2)}\,dz$$

where curve C is $|z-1| = 1/2$.

(Ans):→ Singular Point of ~~Integral~~ Integrand are

$$(z-3)(z+2) = 0$$

$$\Rightarrow z = 3, -2.$$

$$z = 3 \Rightarrow |3-1| = 2 > \frac{1}{2} \quad (z = 3 \text{ lie outside})$$

Now;

$$z = -2 \Rightarrow |-2-1| = 3 > \frac{1}{2} \quad (\text{Outside})$$

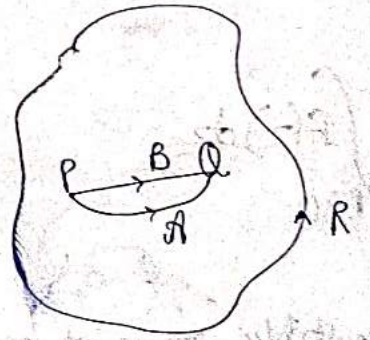
By Cauchy Integral Theorem :

$$\int_C \frac{z}{(z-3)(z+2)}\,dz = 0$$

Corollary :

If $f(z)$ is Analytic in a Region R and P and Q are two points in Region R, then $\int_P^Q f(z)\,dz$ is Independent of the Path Joining P and Q and lying entirely in R

(Proof):→



Let PAQ and PBQ be any two path joining point P and Q ;

By Cauchy Integral theorem

$$\int_{PAQBP} f(z) = 0.$$

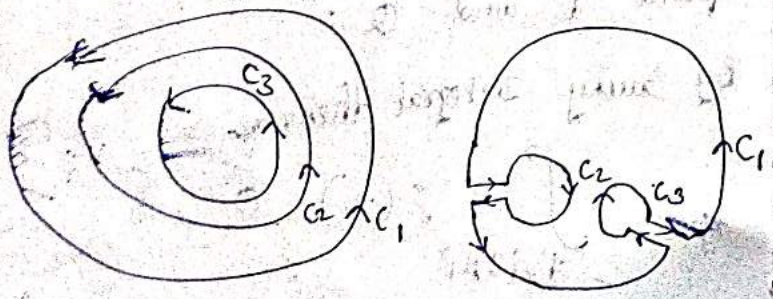
This implies that :

$$\Rightarrow \int_{PAQ} f(z)\,dz + \int_{QBP} f(z)\,dz = 0$$

$$\Rightarrow \int_{PAQ} f(z)\,dz - \int_{PBQ} f(z)\,dz = 0.$$

$$\Rightarrow \int_{PAQ} f(z)\,dz = \int_{PBQ} f(z)\,dz$$

Remark :→ A multiply connected region can be convert into a Simply Connected region to give it one or more cuts.

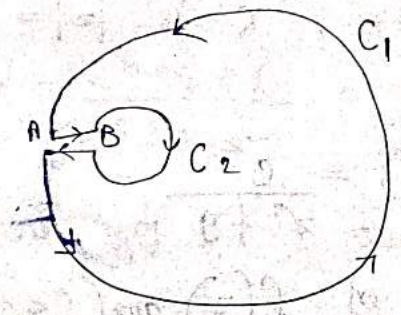


Extension of C.I.T to multiply Connected region :→

Statement :→ If $f(z)$ is Analytic in the region R, b/w two simple closed curve C_1 and C_2 then

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz.$$

(Proof) :→



By C.I.T :→

$$\int_C f(z)\,dz = 0$$

$$\Rightarrow \int_{C_1} f(z)\,dz + \int_{AB} f(z)\,dz - \int_{C_2} f(z)\,dz$$

$$+ \int_{BA} f(z)\,dz = 0.$$

$$\Rightarrow \int_{C_1} f(z)\,dz + \cancel{\int_{AB} f(z)\,dz} - \int_{C_2} f(z)\,dz$$

$$\cancel{-\int_{AB} f(z)\,dz} = 0$$

(Q): $\Rightarrow$ Evaluate :

$$\oint_C \frac{e^z}{z^2+9}\,dz, \quad C \text{ is } |z| = 2.$$

(Sol) :$\Rightarrow$ let $f(z) = \dfrac{e^z}{z^2+9}$

Singular Points of $f(z)$ are

$z^2 + 9 = 0 \Rightarrow z^2 = -9 \Rightarrow z = \pm 3i$.

$|z| = 2$

$z = 3i \Rightarrow |3i| = 3 > 2$ (outside)

$z = -3i \Rightarrow |-3i| = 3 > 2$ (outside)

By C.I.T

$$\int_C \frac{e^z}{z^2+9}\,dz = 0.$$

(Q) : $\Rightarrow$ Evaluate :

$$\oint_C \frac{z^3+z+1}{z^2-3z+2}\,dz, \text{ where curve } C \text{ is } 4x^2 + 9y^2 = 1.$$

(Sol) :$\Rightarrow$ Let

$$f(z) = \frac{z^3+z+1}{z^2-3z+2} = \frac{z^3+z+1}{(z-1)(z-2)}$$

Singular Point of $f(z)$ are

$(z-1)(z-2) = 0 \Rightarrow z = 1, 2$.

$$\frac{x^2}{(1/2)^2} + \frac{y^2}{(1/3)^2} = 1$$

$z = 1 \Rightarrow (1, 0)$

$z = 1 \Rightarrow (1,0) \Rightarrow 4 > 1$ (outside)

$z = 2 \Rightarrow (2,0) \Rightarrow 16 > 1$ (outside)

By C.I.T : $\oint_C f(z)\,dz = 0$.

(Pute)

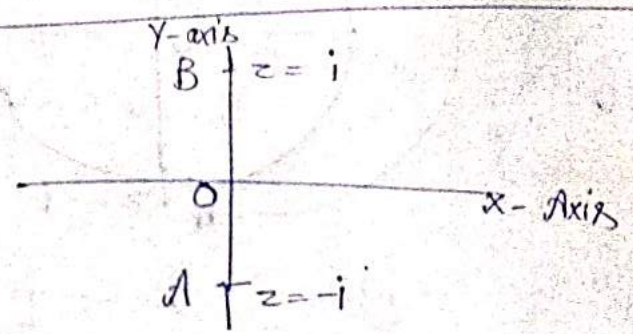
(Q): Evaluate the Integral $\int_C |z|\,dz$, where curve C is

(i) The sit line from $z=-i$ to $z=i$.

(ii) The left half of the unit circle, $|z|=1$ from $z=-i$ to $z=i$.

(Sol):→ Let $I = \int_C |z|\,dz$ — (1)

Note: → If $\oint$ this integration was given then it is cauchy Integral theorem, otherwise do line integral



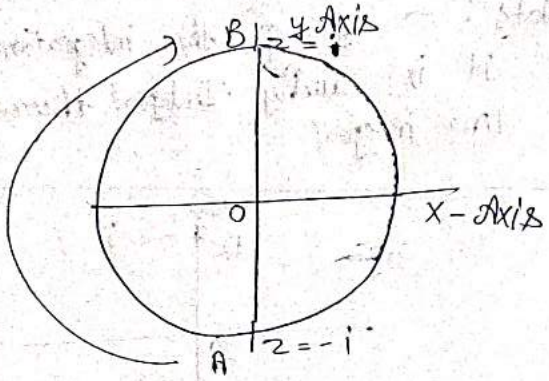
$$I = \int_{AB} |z|\,dz = \int_{AB} |x+iy|\,(dx+i\,dy)$$

Along AB, $x=0 \Rightarrow dx=0$.

$$\Rightarrow \int_{AB}^{C} |iy|\, i\, dy$$

$$\Rightarrow i\int_{-1}^{0} -y\, dy + i\int_{0}^{1} y\, dy =$$

(ii) : Let $I = \int_C |z|\, dz$ —— (1)



$|z| = 1 \Rightarrow z = e^{i\theta}$ (P.F)

$\Rightarrow dz = ie^{i\theta}\, d\theta$.

$$I = \int ie^{i\theta}\, d\theta.$$

$$I = \int_{\frac{3\pi}{2}}^{\frac{\pi}{2}} ie^{i\theta}\, d\theta = i\left(\frac{e^{i\theta}}{i}\right)_{\frac{3\pi}{2}}^{\pi/2}$$

$$= e^{i\pi/2} - e^{i3\pi/2}$$

$$= \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} - \cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right)$$

(Q) : Evaluate : $\int_0^{3+i} z^2\, dz$ along

(i) The line $y = \frac{x}{3}$

(ii) The real axis to 3 then vertically to $3+i$.

(iii) The parabola $x = 3y^2$.

(Ans) Let $I = \int_{(0,0)}^{(3,1)} z^2\, dz = \int_{(0,0)}^{(3,1)} (x+iy)^2 (dx + i\,dy)$ —— (1)

Along $y = \frac{x}{3} \Rightarrow dy = \frac{1}{3}\, dx$

$$I = \int_0^3 \left(x + i\frac{x}{3}\right)^2 \left(dx + \frac{i}{3}\, dx\right)$$

$$I = \int_0^3 \left(x^2 - \frac{x^2}{9} + \frac{2}{3}x^2 i\right)\left(1 + \frac{i}{3}\right) dx$$

$$I = \left(\frac{8}{9} + \frac{2}{3}i\right)\left(1 + \frac{i}{3}\right)\int_0^3 x^2\, dx.$$

$$I = \left(\frac{8}{9} + \frac{2}{3}i\right)\left(1 + \frac{i}{3}\right)\left[\frac{x^3}{3}\right]_0^3$$

$$= \left(\frac{8}{9} + \frac{2}{3}i + \frac{i8}{27} - \frac{2}{9}\right) 9$$

$$= 8 + 6i + \frac{8i}{3} - 2$$

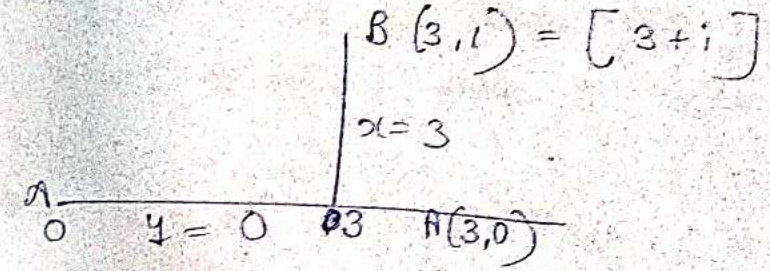
$$= 6 + \left(6 + \frac{8}{3}\right) i$$

$$= 6 + \left(\frac{18+8}{3}\right) i$$

$$= 6 + \left(\frac{26}{3}\right) i$$

(ii) Along OA, $y = 0 \Rightarrow dy = 0$

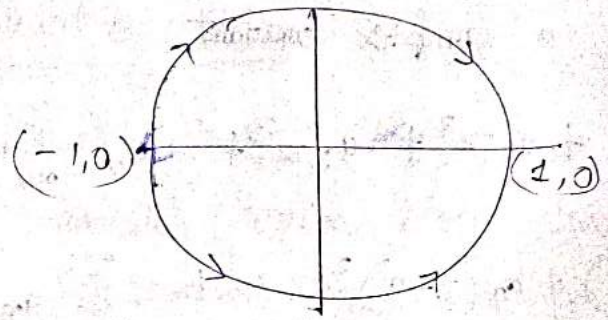
Along AB, $x = 3 \Rightarrow dx = 0$



Same as (i) (answer)

(Q): → Prove that $\int_C \frac{1}{z}\, dz = -\pi i$ or πi according as C is the semicircular arc $|z|=1$ from $z=-1$ to $z=1$ above or below the real axis.

(Ans) :→ $z = e^{i\theta}$, $dz = ie^{i\theta} d\theta$,



π से 0
π से 2π } limits.

(Q) : Log z question in P.D.F

☆☆ Power Series representation of $f(z)$:→

A series of the form $\sum_{n=0}^{\infty} a_n z^n$ or $\sum_{n=0}^{\infty} a_n (z-a)^n$, whose terms are variable is called Power Series, where z is a complex variable and a_n is a complex constant.

$$a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \dots = \sum_{n=0}^{\infty} a_n z^n$$

$$a_0 + a_1 (z-a) + a_2 (z-a)^2 \dots = \sum_{n=0}^{\infty} a_n (z-a)^n$$

☆☆ Taylor's Series:

If $f(z)$ is analytic inside a circle c with centre at the point a then for all z inside the circle c $f(z) = f(a) + (z-a) f'(a) + \frac{(z-a)^2 f''(a)}{\lfloor 2} + \frac{(z-a)^n f^n(a)}{\lfloor n} \dots \infty$

(OR)

$$f(z) = a_0 + a_1 (z-a) + a_2 (z-a)^2 + \dots a_n (z-a)^n + \dots \infty$$

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n \; ; \text{ where } a_n = \frac{f^n(a)}{\lfloor n}$$

☆☆ Laurent's Series :→

If $f(z)$ is Analytic inside and on the boundary of the annulus (region b/w two circles) of region R bounded by two cocentric circles C_1 & C_2 of Radius r_1 and r_2 $(r_1 > r_2)$ respectively, Having Center at the point $z = a$ then for all z in region R, $f(z)$

$$f(z) = \underbrace{a_0 + a_1 (z-a) + a_2 (z-a)^2 + a_n (z-a)^n + \dots \infty}_{\text{(P.P of Taylor's series)}} + \underbrace{\frac{b_1}{(z-a)} + \frac{b_2}{(z-a)^2} + \frac{b_3}{(z-a)^3} + \dots + \frac{b_n}{(z-a)^n} + \dots \infty}_{\text{Principle part of Laurents series}}$$

(OR)

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}$$

where $a_n = \frac{1}{2\pi i} \int_{C_1} \frac{f(w)}{(w-a)^{n+1}} \, dw$ $\quad n = 0, 1, 2 \ldots$

$$b_n = \frac{1}{2\pi i} \int_{C_2} \frac{f(w)}{(w-a)^{-n+1}} \, dw \quad n = 1, 2, 3 \ldots$$

(Q) :→ $(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots$

(2) $(1+x)^{-1} = 1 - x + x^2 - x^3 + \ldots$ If $|x| < 1$

(3) $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots$

(4) $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3$

(Q) :→ Expand $\frac{1}{z^2 - 3z + 2}$ in the region

(i) $|z| < 1$ (ii) $1 < |z| < 2$

(iii) $|z| > 2$ (iv) $0 < |z-1| < 1$

(Ans): Let $f(z) = \frac{1}{z^2 - 3z + 2} = \frac{1}{(z-1)(z-2)}$

$$f(z) = -\frac{1}{z-1} + \frac{1}{z-2} \quad \text{--- (1)}$$

(1) $|z| < 1$

$$f(z) = (1-z)^{-1} - \frac{1}{2}\left(1 - \frac{z}{2}\right)^{-1}$$

(To greater hoga use common denge with sign & then binomial theorem of expand करेंगे)

$$f(z) = 1 + z + z^2 + z^3 + \ldots z^n + \ldots$$
$$- \frac{1}{2}\left[1 + \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{2}\right)^3 + \ldots + \left(\frac{z}{2}\right)^n + \ldots\right]$$

$$f(z) = \sum_{n=0}^{\infty} z^n - \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n$$

Note : It is ~~two~~ Taylor's series expansion

(ii) $1 < |z| < 2$.

$$f(z) = -\frac{1}{z}\left(1-\frac{1}{z}\right)^{-1} - \frac{1}{z}\left(1-\frac{z}{2}\right)^{-1}$$

We will take z common from $\left(-\frac{1}{z-1}\right)$ bcoz z is greater & in 2nd part we will take z common

Expanding it by Binomial:

$$f(z) = -\frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^n}+\dots\infty\right)$$

$$-\frac{1}{z}\left(1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\left(\frac{z}{2}\right)^3+\dots\infty\right)$$

$$f(z) = -\frac{1}{z}\sum_{n=0}^{\infty}\frac{1}{z^n} - \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n$$

(It is laurent's series expansion)

(iii)

(iv) $0 < |z-1| < 1$

$$f(z) = -\frac{1}{(z-1)} + \frac{1}{(z-1)-1} = -\frac{1}{(z-1)} - \left[1-(z-1)\right]^{-1}$$

$$f(z) = -\frac{1}{z-1} - \left[1+(z-1)+(z-1)^2+(z-1)^n+\dots\right]$$

$$f(z) = -\frac{1}{z-1} - \sum_{n=0}^{\infty}(z-1)^n$$

It is laurent's series expansion.

(ii) $1 < |z| < 2$

$$f(z) = -\frac{1}{z}\left(1-\frac{1}{z}\right)^{-1} - \frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}$$

We will take z common from $\left(-\frac{1}{z-1}\right)$ bcoz z is greater & in 2nd part we will take 2 common

Expanding it by Binomial:

$$f(z) = -\frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^n}+\dots\infty\right)$$

$$-\frac{1}{2}\left(1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\left(\frac{z}{2}\right)^3+\dots\infty\right)$$

$$f(z) = -\frac{1}{z}\sum_{n=0}^{\infty}\frac{1}{z^n} - \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n$$

(It is laurent's series expansion)

(iii)

(iv) $0 < |z-1| < 1$

$$f(z) = -\frac{1}{(z-1)} + \frac{1}{(z-1)-1} = -\frac{1}{(z-1)} - \frac{1}{\left(1-(z-1)\right)}$$

$$f(z) = -\frac{1}{z-1} - \left[1+(z-1)+(z-1)^2+(z-1)^n+\dots\right]$$

$$f(z) = -\frac{1}{z-1} - \sum_{n=0}^{\infty}(z-1)^n$$

It is laurent's series expansion

1 (a): Find the taylor's & laurents series which represent the fn. $f(z) = \frac{z^2 - 1}{(z+2)(z+3)}$

(i) $|z| < 2$ (ii) $2 < |z| < 3$

(iii) $|z| > 3$.

(Ans). → For Partial Power of denr. should be greater than power of num.

$$z^2 + 5z + 6 \overline{\big)\, z^2 - 1 \,\big(\, 1}$$

$$\underline{-z^2 + 5z + 6}$$

$$-5z - 7$$

1 $$f(z) = 1 - \frac{5z + 7}{(z+2)(z+3)}$$

(Q): Expand the following fns:

(1) $f(z) = \dfrac{e^z}{(z-1)^2}$ at $z = 1$.

(Sol): Let $z - 1 = t \Rightarrow z = 1 + t$.

$$f(z) = \frac{e^{1+t}}{t^2} = e\left(\frac{e^t}{t^2}\right)$$

$$f(z) = \frac{e}{t^2}\left(1 + t + \frac{t^2}{\lfloor 2} + \frac{t^3}{\lfloor 3} + \frac{t^4}{\lfloor 4} + \dots\right)$$

$$f(z) = e\left(\frac{1}{t^2} + \frac{1}{t} + \frac{1}{2} + \frac{t}{6} + \dots\right)$$

$$f(z) = e\left[\frac{1}{(z-1)^2} + \frac{1}{z-1} + \frac{1}{2} + \frac{z-1}{6} + \dots\right]$$

(Laurent's series expansion)

(Q) : Expand the foll. function :→

(1) $f(z) = \dfrac{\sin z}{z-\pi}$, about $z = \pi$.

(Sol) :→ → (Singular point value in den. so always these are laurent series)

~~Let $f(z) = \sin(\pi + t$~~

Let : $z - \pi = t \Rightarrow z = t + \pi$.

$$f(z) = \frac{\sin(t+\pi)}{t} = -\frac{\sin t}{t} .$$

$$f(z) = -\frac{1}{t}\left(t - \frac{t^3}{\lfloor 3} + \frac{t^5}{\lfloor 5} \cdots\right)$$

$$f(z) = -1 + \frac{(z-\pi)^2}{\lfloor 3} - \frac{(z-\pi)^4}{\lfloor 5} + \cdots$$

This is <u>taylor</u> series expansion, & $(z-\pi)$ is non-singular point .

(Q) : $\dfrac{1-\cos z}{z^3}$, about $z = 0$.

(Sol) : $\Rightarrow \cos z = 1 - \dfrac{z^2}{\lfloor 2} + \dfrac{z^4}{\lfloor 4} \cdots$

(Q): Using Taylor's series, show that:

$$\log z = (z-1) - \frac{(z-1)^2}{2} + \frac{(z-1)^3}{3} + \dots$$

where $|z-1| < 1$

Ac. to Taylor's series :→

(Sol) :→ $f(z) = f(a) + (z-a)\frac{f'(a)}{\lfloor 1} + (z-a)^2\frac{f''(a)}{\lfloor 2} + \dots$

Let $f(z) = \log z$, and $z = 1$.

$f'(z) = \frac{1}{z}$, $f''(z) = -\frac{1}{z^2}$, $f'''(z) = \frac{2}{z^3}$

$f(1) = \log 1 = 0$, $f'(1) = 1$, $f''(1) = -1$, $f'''(1) = 2$

So by Taylor's series :→

$$f(z) = (z-1) - \frac{(z-1)^2}{2} + (z-1)^3 \times \frac{2}{3\times 2} + \dots$$

(Q): Obtain the Taylor's series expansion of $f(z) = \frac{1}{z^2-4z+3}$ about the point $z=4$.

Find its region of Convergence.

(Sol) :→ $f(z) = \frac{1}{z^2-4z+3} = \frac{1}{(z-1)(z-3)}$

(I^{st} Method)

$$f(z) = f(4) + (z-4)\frac{f'(4)}{\lfloor 1} + (z-4)^2\frac{f''(4)}{\lfloor 2} + \dots$$

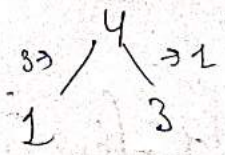
IInd Method :→

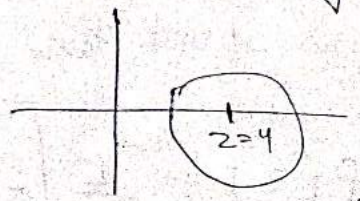
Singular Points of $f(z)$ are

$z^2 - 4z + 2 = 0 \Rightarrow (z-1)(z-3) = 0$

$[z = 1, 3]$

Note :→ Singular point outside region is region of Convergence.





→ Min. distance would be smaller radius (1)

Region of Convergence would be :→

$|z-4| < 1$

Now do partial fraction :→ $\frac{1}{(z-1)(z-3)}$

$$f(z) = -\frac{1}{2(z-1)} + \frac{1}{2(z-3)}$$

$$f(z) = \frac{1}{2}\left[-\frac{1}{(z-4)+3} + \frac{1}{(z-4)+1}\right]$$

$$f(z) = \frac{1}{2}\left[-\frac{1}{3}\left\{1+\frac{(z-4)}{3}\right\}^{-1} + \left\{1+(z-4)\right\}^{-1}\right]$$

Now expand this eqn:

$$f(z) = \frac{1}{2}\left[-\frac{1}{3}\left\{1-\left(\frac{z-4}{3}\right)+\left(\frac{z-4}{3}\right)^2 - \left(\frac{z-4}{3}\right)^3 + \dots \infty\right\} + \left\{1-(z-4)+(z-4)^2-(z-4)^3 + \dots \infty\right\}\right]$$

$$\cancel{f(z) = \frac{1}{2}\left[-\frac{1}{3}\sum_{n=}^{\infty}\left(\frac{z-4}{3}\right)^n + \sum_{n=}^{\infty}(z-4)^n\right]}$$

$$f(z) = \frac{1}{2}\left[-\frac{1}{3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z-4}{3}\right)^n + \sum_{n=0}^{\infty}(-1)^n(z-4)^n\right]$$

Singularity :→

A singularity of fn. $f(z)$ is a point at which the fnc. ceases to be Analytic.

Isolated and Non-Isolated Singularity :→

If $z = a$ is a singularity of $f(z)$ and if their is no other singularity with a of small circle surrounding the point $z = a$, then $z = a$ is said to be isolated singularity of the fnc. $f(z)$ otherwise non-isolated singularity.

(Ex): $f(z) = \dfrac{z}{(z+1)(z-2)}$

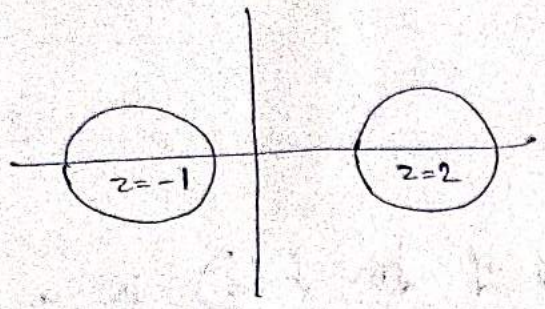
Singular Point, of $f(z)$ are $(z+1)(z-2) = 0 \Rightarrow z = -1, 2$

$|z - 2| = \varepsilon$ → radius

(centre point)

$\varepsilon = 0.00001$

$|z+1| = \varepsilon_2$



Both the singular point $(z=-1, 2)$ are the isolated singular point

(Q) :→ $f(z) = \dfrac{1}{\sin(\pi/z)}$

Sol) :→ Singular Points of $f(z)$ are

$$\sin\left(\frac{\pi}{z}\right) = 0 = \sin n\pi$$

$$\Rightarrow \frac{\pi}{z} = n\pi \Rightarrow z = \frac{1}{n} \text{ and } n \in I.$$

$$z = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5} \ldots 0$$

(Biggest Singular Point) (Smallist Sing. Point)

Singular Point is density higher is at point 0,

~~z = 0 (Not~~

Thus $z = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \ldots 0$, are the singularities of the fn. $(F(z))$ all these singular Point except $z=0$, are isolated and $z=0$ is non isolated singular point.

☆ Types of Singularity :→

$$f(z) = a_0 + a_1(z-a) + \ldots a_0(z-a)^n + \ldots \infty$$

$$\boxed{+ \frac{b_1}{z-a} + \frac{b_2}{(z-a)^2} + \frac{b_n}{(z-a)^n} + \ldots \infty}$$

↓ Princ. Part of Laurents expansion

(1) Removable : If the principle part of Lorent singularity expansion contains no terms

At the point $z = 0$

(Ex) :→ $f(z) = \dfrac{\sin(z-a)}{z-a}$ at $z = a$

(Sol) :→ $f(z) = \frac{1}{(z-a)}\left[(z-a) + \frac{(z-a)^3}{\lfloor 3} + \frac{(z-a)^5}{\lfloor 5} + \dots\right]$

$$f(z) = 1 + \frac{(z-a)^2}{\lfloor 3} + \frac{(z-a)^4}{\lfloor 5} + \dots$$

At $z = a$ it is removable singularity.

(2) Essential Singularity :

If the principle part of Lorentz expansion contains infinite no. of terms at the point $z = a$.

(Ex) :→ $\cancel{f(z) = \frac{1}{z-a} + \frac{1}{\lfloor 3 (z-a)^3}}$

(Ex) $f(z) = \sin\left(\frac{1}{(z-a)}\right)$ at $z = a$.

(Sol) :→ $f(z) = \frac{1}{(z-a)} + \frac{1}{\lfloor 3 \cdot (z-a)^3} + \frac{1}{\lfloor 5 \cdot (z-a)^5} + \dots$

(3) : Poll Singularity :→

. If the principle part of Lorentz expansion contains finite no. of terms.

(Ex) :→ $f(z) = \frac{1}{(z-2)\cdot(z-3)^2(z+4)^{10}}$

at $z = 2$ (Simple pole or Pole of I^{st} order)

(Sol) :→ $z = 3$ (Pole of order II)

$z = -4$ (Pole of order 10)

+ Limit Point of Zeros are Isolated Essential Singularity :

* Limit point of Poles are non-Isolated essential Singularity.

(Q) :→ Find out the zeros & discuss the nature of singularity of $f(z) = \frac{z-2}{z^2} \cdot \sin\left(\frac{1}{z-1}\right)$

(Sol) ⇒ zero at $f(z)$ are :⇒

$$\Rightarrow (z-2)\sin\left(\frac{1}{z-1}\right) = 0$$

$$\Rightarrow z = 2 \quad (\text{I order zero})$$

$$\Rightarrow \sin\left(\frac{1}{z-1}\right) = 0 = \sin n\pi.$$

$$\Rightarrow \frac{1}{z-1} = n\pi$$

$$\Rightarrow z = 1 + \frac{1}{n\pi}$$

$z = 1$ is the limit point of zeroes,
So at $z = 1$, Isolated essential Singularity

(Q) :⇒ Discuss Singularity of $\frac{1}{1-e^z}$ at $z = 2\pi i$.

(Sol) :⇒ $1 - e^z = 0 \Rightarrow e^z = 1$.

$$\Rightarrow e^z = e^{2n\pi i} \quad (n \in I)$$

$$\Rightarrow z = 2n\pi i \quad (\text{Simple Pole})$$

$z = 0, 2\pi i, 4\pi i, 6\pi i$.

(Q) :⇒ Discuss Singularity of $\frac{\cot \pi z}{(z-a)^2}$ at $z = a$ and $z = \infty$.

(Sol) :⇒ $f(z) = \frac{\cos \pi z}{\sin \pi z \cdot (z-a)^2}$

If we are not talking about zeros then put den. $= 0$.

$$\sin \pi z \cdot (z-a)^2 = 0$$

$$z = a, n \quad (\text{II}^{nd} \text{ Order})$$

Non Isolated essential at $z = n$ and IInd order of Pole at $z = a$.

(Tute) :→

(Q) : Find the $f(z) = \dfrac{7z-2}{z^3-z^2-2z}$

in the given region.

(i) $0 < |z+1| < 1$

(ii) $1 < |z+1| < 3$

(iii) $|z+1| > 3$

(Sol) : $f(z) = \dfrac{7z-2}{z(z^2-z-2)}$

$= z(z^2-z-2) = 0$

$z(z-2)(z+1)$

$f(z) = \dfrac{7z-2}{z(z-2)(z+1)}$

$\cancel{f(z) = 7z-2}$

$f(z) = \dfrac{-3}{z+1} + \dfrac{1}{z} + \dfrac{2}{z-2}$

$f(z) = \dfrac{-3}{z+1} + \dfrac{1}{(z+1)-1} + \dfrac{2}{(z+1)-3}$ — (1)

(i)

$f(z) = \dfrac{-3}{z+1} - \left[1-(z+1)\right]^{-1} - \dfrac{2}{3}\left[1-\dfrac{(z+1)}{3}\right]^{-1}$

[Taking greater value common]

$(1-x)^{-1} = 1 + x + x^2 + x^3 + \dots x^n + \dots \infty$

$f(z) = \dfrac{-3}{z+1} - \left[1 + (z+1) + (z+1)^2 + \dots (z+1)^n + \dots \infty\right]$

$- \dfrac{2}{3}\left[1 + \left(\dfrac{z+1}{3}\right) + \left(\dfrac{z+1}{3}\right)^2 + \dots \left(\dfrac{z+1}{3}\right)^n \dots \infty\right]$

$f(z) = \dfrac{-3}{z+1} - \sum_{n=0}^{\infty} (z+1)^n - \dfrac{2}{3}\sum_{n=0}^{\infty}\left(\dfrac{z+1}{3}\right)^n$

Laurentz series expansion.

(ii) Using (i) eqn :

$$f(z) = -\frac{3}{z+1} + \frac{1}{(z+1)-1} + \frac{2}{(z+1)-3} \quad \text{—— (1)}$$

$$f(z) = -\frac{3}{z+1} + \frac{1}{z+1}\left[1 - \frac{1}{z+1}\right]^{-1} - \frac{2}{3}\left[1 - \left(\frac{z+1}{3}\right)\right]^{-1}$$

$$f(z) = -\frac{3}{z+1} + \frac{1}{z+1}\left[1 + \frac{1}{z+1} + \left(\frac{1}{z+1}\right)^2 + \left(\frac{1}{z+1}\right)^3 + \cdots \left(\frac{1}{z+1}\right)^n + \cdots \infty\right] - \frac{2}{3}\left[1 + \frac{z+1}{3} + \left(\frac{z+1}{3}\right)^2 + \left(\frac{z+1}{3}\right)^3 \cdots \left(\frac{z+1}{3}\right)^n \cdots \infty\right]$$

$$\cancel{f(z) = -\frac{3}{z+1} - \sum_{n=0}^{\infty}\left(\frac{1}{z+1}\right)^n}$$

$$f(z) = -\frac{3}{z+1} + \frac{1}{z+1}\sum_{n=0}^{\infty}\left(\frac{1}{z+1}\right)^n - \frac{2}{3}\left(\frac{z+1}{3}\right)^n$$

(iii) : $f(z) = -\frac{3}{z+1} + \frac{1}{z+1}\left[1 - \frac{1}{z+1}\right]^{-1} + \frac{2}{z+1}\left[1 - \frac{3}{z+1}\right]^{-1}$

$\frac{1}{z+1}$

$\frac{1}{z+1}(1-$

$1 - \frac{z+1}{3}$

$\left(\frac{3-z+1}{3}\right)$

$\frac{2}{3}\left(\frac{-3}{1-z}\right)$

$\frac{2}{z}$

$z+1-1$

(Q). :> if $|z+1| < 1$, then prove that $z^{-2} = 1 + \sum_{n=1}^{\infty} (n+1)(z+1)^n$

(Sol) :> Let :

$$f(z) = z^{-2} = (z+1-1)^{-2}$$

$$f(z) = [1-(z+1)]^{-2}$$

$$\cancel{(1-x)^{-2} = 1 + 2x + 3x^2 + \ldots + \infty = \sum_{n=0}^{\infty} (n+1)x^n}$$

$$f(z) = [1-(z+1)]^{-2} = 1 + [2(z+1) + 3(z+1)^2 + (n+1)(z+1)^n + \ldots \infty]$$

$$f(z) = 1 + \sum_{n=1}^{\infty} (n+1)(z+1)^n.$$

(Q) : Find Taylor's & Laurent's series which represent the fn $f(z)$:

$$f(z) = \frac{z^2-1}{(z+2)(z+3)}$$ when

(i) $|z| < 2$

(ii) $2 < |z| < 3$

(iii) $|z| > 3$

(Sol)

$$\begin{array}{r|l} z^2+5z+6 & z^2-1 \quad (1 \\ & z^2+5z+6 \\ & -\quad -\quad - \\ \hline & -5z-7 \end{array}$$

$$f(z) = 1 - \left[\frac{5z+7}{(z+2)(z+3)}\right]$$

$$f(z) = 1 + \frac{3}{z+2} - \frac{8}{z+3} \quad \text{--- (1)}$$

Now take greater value common acc. to region & then expand with binomial.

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + \ldots$$

$f(z) =$

$$f(z) = 1 + \frac{3}{2}\left[\frac{z}{2}+1\right]^{-1} - \frac{8}{3}\left[\frac{z}{3}+1\right]^{-1}$$

→ Both degrees are same so we will divide

$$f(z) = 1 + \frac{3}{2}\left(1 - \frac{z}{2} + \left(\frac{z}{2}\right)^2 - \left(\frac{z}{3}\right)^3 + \cdots \left(\frac{z}{3}\right)^n \cdots \infty\right)$$

$$- \frac{8}{3}\left(1 - \frac{z}{3} + \left(\frac{z}{3}\right)^2 - \left(\frac{z}{3}\right)^3 + \cdots \left(\frac{z}{3}\right)^n + \cdots \infty\right)$$

$$= 1 + \frac{3}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n (-1)^n - \frac{8}{3}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n (-1)^n$$

Taylor's ~~theory~~ series expansion

(ii) $f(z) = 1 + \frac{3}{z+2} - \frac{8}{z+3}$

$= 1 + \frac{3}{z} \cdot ($

$\left(\frac{z}{2}+1\right)$

$\frac{z+2}{2}$

$\left(\frac{z}{2+2}\right)\frac{3}{2}$

$\frac{3}{}$

(Imp)

Residue :→

Def. of residue at the Pole :→

At $z = a$, be the pole of order n of a single valued fnc. $f(z)$ and C_1 be any circle with center at $z = a$ and having radius r. Which does not contain any other singularity except at $z = a$, then $f(z)$ is analytic within the annulus $r < |z-a| < R$, hence it can be expanded by Laurentz series in the form $f(z) =$

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \dots a_n(z-a)^n + \dots$$

$$\dots \frac{b_1}{z-a} + \frac{b_2}{(z-a)^2} + \frac{b_3}{(z-a)^3} + \dots \frac{b_n}{(z-a)^n} + \dots \infty$$

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}$$

where $a_n = \frac{1}{2\pi i} \int \frac{f(z)}{(z-a)^{n+1}} dz$

$$b_n = \frac{1}{2\pi i} \int \frac{f(z)}{(z-a)^{-n+1}} dz.$$

Particularly, the coefficient :

$$b_1 = \frac{1}{2\pi i} \int_{C_1} f(z)\, dz$$ is called residue at $z = a$.

Methods of finding Residue :→

(1) if $z = a$ is a pole of order n of the function $f(z)$ then residue at $z = a$ is given as :

$$R = \frac{1}{\lfloor n-1} \lim_{z \to a} \frac{d^{n-1}}{dz^{n-1}} (z-a)^n f(z)$$

(a) if $n = 1$ ($z = a$, simple pole ~~as~~ or I order pole)

$$R_1 = \lim_{z \to a} (z-a) f(z)$$

(b) If $n \to 2$ ($z = a$, Pole of order II)

$$R_2 = \lim_{z \to a} \frac{d}{dz}\left[(z-a)^2 \cdot f(z)\right]$$

Q: Determine Poles and residue at each Pole of the func.

$$f(z) = \frac{z^2}{(z-1)^2(z+3)}$$

(Sol): ⇒ Poles of $f(z)$ are

$(z-1)^2(z+3)$

$z = 1$ (2nd order pole)

$z = -3$ (simple or I order pole)

Residue of $z = 1$:

$$R_1 = \lim_{z \to 1} \frac{d}{dz}\left[(z-1)^2 \cdot \frac{z^2}{(z-1)^2(z+3)}\right]$$

$$= \lim_{z \to 1} \frac{d}{dz}\left(\frac{z^2}{z+3}\right)$$

$$R_1 = \lim_{z \to 1} \left(\frac{(z+3)(2z) - z^2}{(z+3)^2}\right) = \frac{7}{16}$$

(ii) $$R_2 = \lim_{z \to a} \cancel{(z+3)} \left(\frac{z^2}{\cancel{(z-1)^2(z+3)}}\right)$$

$$= \cancel{\lim_{z \to a} \left(\frac{(-3)^2}{(-3+3)}\right)}$$

$$R_2 \Rightarrow \cancel{\lim_{z \to -3}} = \left(\frac{9}{16}\right)$$

☆☆ Cauchy - residue theorem :→

Let $f(z)$ be an analytic fn. within and on a simple closed curve C except at finite no. of poles $z_1, z_2, z_3, \ldots z_n$, and residue at these poles is $R_1, R_2, R_3, \ldots R_n$ then

$$\int_C f(z)\,dz = 2\pi i\,(R_1 + R_2 + \ldots R_n).$$

Note: Cauchy residue theorem is used for inside points.

(Q): Evaluate $\displaystyle\int_C \frac{12z-7}{(z-1)^2(2z+3)}\,dz$

(Sol): → where curve C is :

(i) $|z| = 2$, (ii) $|z+i| = \sqrt{3}$

(Sol): → Let

$$f(z) = \frac{12z-7}{(z-1)^2(2z+3)}$$

Poles of $f(z)$ are

$(z-1)^2(2z+3) = 0$

$\Rightarrow z = 1$ (II order)

$\Rightarrow z = \frac{-3}{2}$ (I order)

(i) $|z| = 2$, $z = 1 \Rightarrow |1| = 1 < 2$,

$z = 1$ (Inside), $z = -\frac{3}{2} \rightarrow \left|-\frac{3}{2}\right| = \frac{3}{2} < 2$

$z = -\frac{3}{2}$ (Inside)

(a) Residue at $z = 1$.

$$R_1 = \lim_{z\to 1} \frac{d}{dz}\left[(z-1)^2 \frac{(12z-7)}{(z-1)^2(2z+3)}\right]$$

$$R_1 = \lim_{z\to 1}\left[\frac{(2z+3)(12) - (12z-7)(2)}{(2z+3)^2}\right] = \frac{50}{25}$$

$$\boxed{R_1 = 2}$$

(b) Residue at $z = -3/2$

$$R_2 = \lim_{z \to -3/2} \left[\frac{(z+3/2)(12z-7)}{(z-1)^2(2z+3)} \right]$$

$$R_2 = \lim_{z \to -3/2} \left[\frac{12z-7}{2(z-1)^2} \right]$$

$$\begin{array}{r} 6.25 \\ \times\ 2 \\ \hline 12.50 \end{array} \qquad \begin{array}{r} 25 \\ 25 \\ \hline 5 \end{array}$$

$$R_2 = \frac{-25}{2\left(-\frac{3}{2}-1\right)^2} = \frac{-25}{2(-2.5)^2} = \frac{-25}{2(6.25)}$$

$$= \frac{-25}{12.50}$$

$$R_2 = -2.$$

By Cauchy - residue theorem :

$$\int_C \frac{12z-7}{(z-1)^2(2z+3)}\,dz = 2\pi i\,(R_1 + R_2)$$

$$= 2\pi i\,(2-2) = 0$$

(ii) $|z+i| = \sqrt{3}$, $z = 1 \Rightarrow |1+i| = \sqrt{2} < \sqrt{3}$

$$z = -\frac{3}{2} \rightarrow \left|-\frac{3}{2}+i\right| = \sqrt{\frac{9}{4}+1} = \sqrt{\frac{13}{4}}$$

$$\left|\frac{13}{4}\right| > \sqrt{3}$$

(By Cauchy Residue theorem)

$$\int_C \frac{12z-7}{(z-1)^2(2z+3)}\,dz = 2\pi i\,(2) = 4\pi i$$