Discrete Mathematics
BCSC 0010
Module 2

Graph Theory

Introduction to Graphs

Introduction

- Graphs are data structures consisting of vertices and edges that connect these vertices.

- Before going in details first of all we must know the applications of graph

Applications of Graph

- to represent who influences whom in an organization
- to model acquaintanceships between people
- to model roadmaps
- to represent the competition of different species in an ecological niche

Applications of Graph

Using graph models, we can determine:

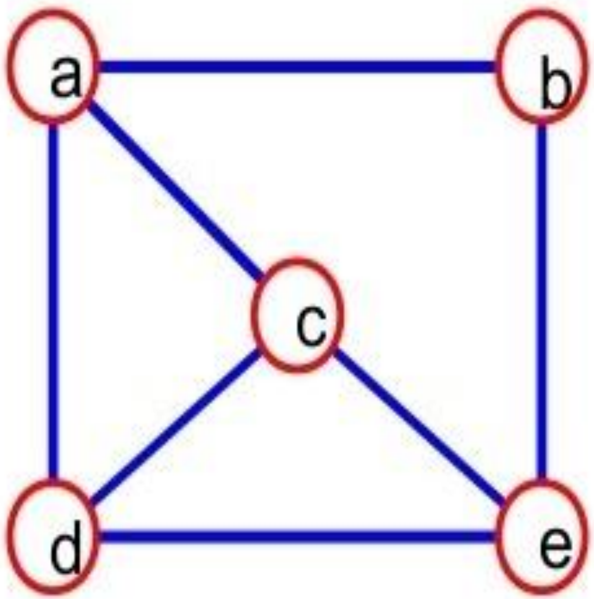
- whether it is possible to walk down all the streets in a city without going down a street twice
- the number of colors needed to color the regions of a map
- whether two computers are connected by a communications link
- the shortest path between two cities in a transportation network

Graph

- A graph G = (V,E) is composed of:

  V: set of vertices

  E: set of edges connecting the vertices in V
- An edge e = (u,v) is a pair of vertices
- Example:

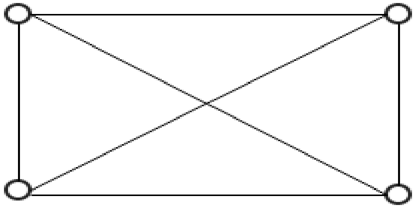


V= {a,b,c,d,e}

E= {(a,b),(a,c),
(a,d),
(b,e),(c,d),(c,e),
(d,e)}

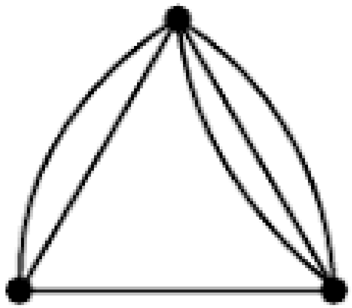
Terminologies

- A graph in which each edge connects two different vertices and where no two edges connect the same pair of vertices is called a **simple graph.**



Simple Graph

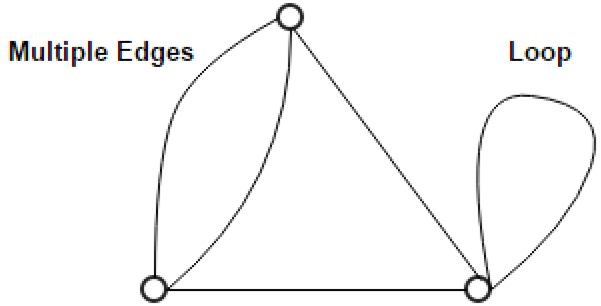
- **Multiple edges** means more than one edge connecting the same pair of vertices. These are called **parallel edges**.
- Graphs that contains some **parallel edges** are called **multigraphs.**



multigraph

Terminologies

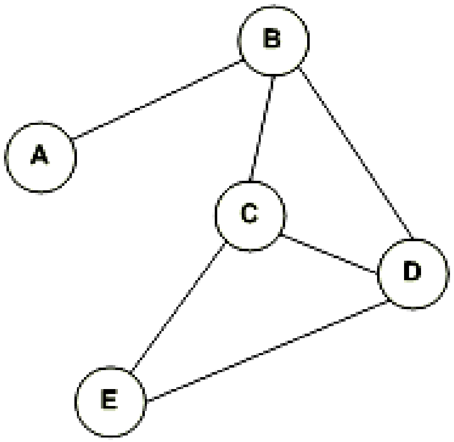
- Edges that connect a vertex to itself are called **loops.**
- Graphs that may include loops and multiple edges are sometimes called **pseudographs.**



Terminologies

An edge which is associated with an unordered pair of vertices is called an **undirected edge**

An **undirected graph** is a graph whose all edges are undirected.



Eg.

(A,B) is same as (B,A)

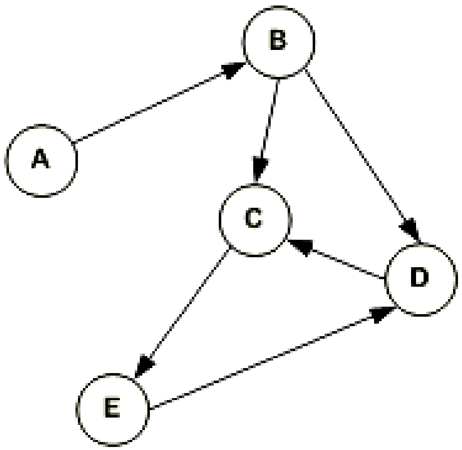
It is called unordered pair as there is no direction between vertex A and vertex B

Terminologies

An edge which is associated with an ordered pair of vertices is called a **directed edge**

A **directed graph** is a graph whose all edges are directed. Directed graph is also known as **digraph**.

Edges are directed by arrows.



Eg.

Since there is a direction between

Vertices. Vertices pairs are called ordered

pairs.

(A,B) is an ordered pair

(B,A) does not exist in this graph

Terminologies

- When a directed graph has no loops and has no multiple directed edges, it is called as **simple directed graph.**

- Directed graphs that may have multiple directed edges from a vertex to a second (possibly the same) vertex are called **directed multigraphs.**

Terminologies

- When there are *m directed edges, each associated to an* ordered pair of vertices *(u, v), we say that (u, v) is an edge of* ***multiplicity*** *m.*

- A graph with both directed and undirected edges is called a **mixed graph**

Terminologies

TABLE 1 Graph Terminology.

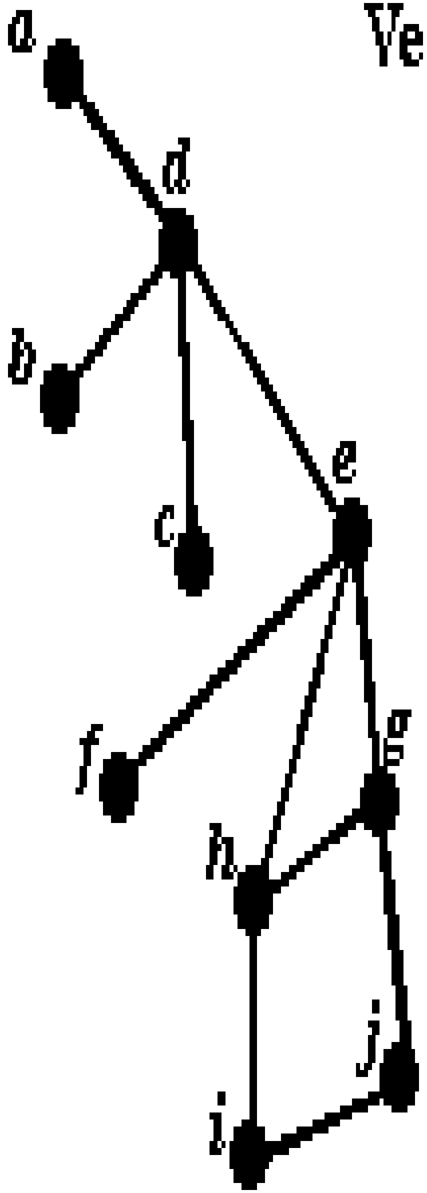
Type	*Edges*	*Multiple Edges Allowed?*	*Loops Allowed?*
Simple graph	Undirected	No	No
Multigraph	Undirected	Yes	No
Pseudograph	Undirected	Yes	Yes
Simple directed graph	Directed	No	No
Directed multigraph	Directed	Yes	Yes
Mixed graph	Directed and undirected	Yes	Yes

Terminologies

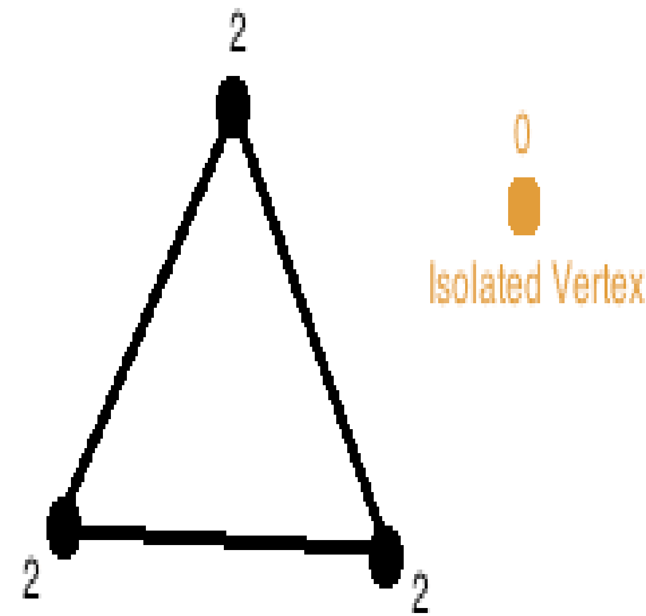
- Two vertices *u and v in an undirected graph G are called* ***adjacent*** *(or neighbors) in G if u* and *v are endpoints of an edge of G.*
- *If e is associated with {u, v}, the edge e is called* ***incident*** *with the vertices u and v.*
- *The vertices u and v are* called ***endpoints*** *of an edge associated with {u, v}.*

Terminologies

- The ***degree*** *of a vertex in an undirected graph is the number of edges incident with it, except* that a loop at a vertex contributes twice to the degree of that vertex.
- The degree of the vertex *v is denoted by deg(v).*
- A vertex of degree zero is called **isolated.**
- A vertex is **pendant** if and only if it has degree one.

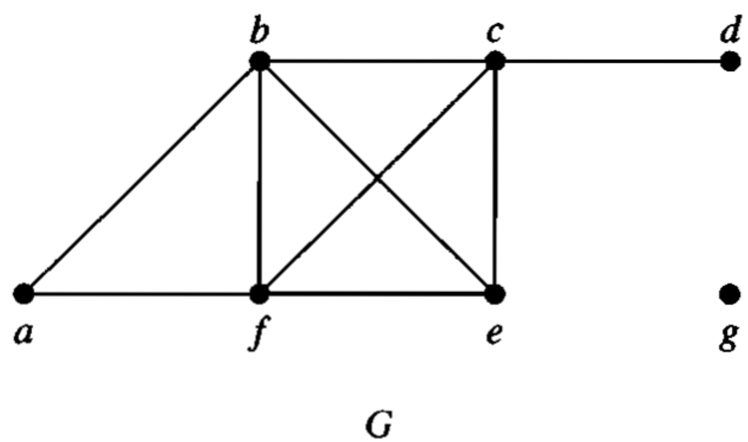


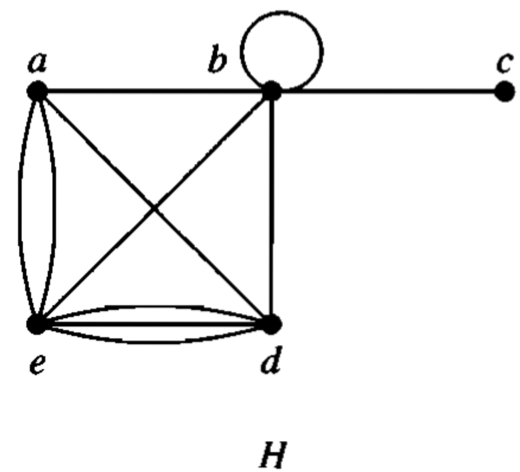
Vertex	Degree
a	1
b	1
c	1
d	4
e	4
f	1
g	3
h	3
i	2
j	2



Problem

- What are the degrees of the vertices in the graphs *G and H ?*





THE HANDSHAKING THEOREM

- Let G = *(V, E) be an undirected graph with e* edges. Then

$$2e = \sum_{v \in V} \deg(v).$$

Example

How many edges are there in a graph with 10 vertices each of degree six?

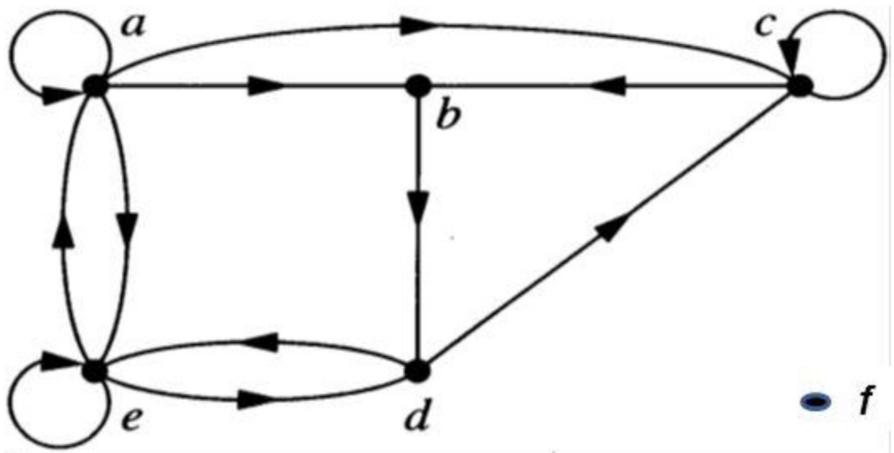
Solution: *Because the sum of the degrees of the vertices is 6· lO = 60, it follows that 2e = 60.*
Therefore, *e = 30.*

Terminologies

In a graph with directed edges

- the in-degree *of a vertex v, denoted by* $\deg^-(v)$ *is the number* of edges with *v as their terminal vertex.*
- *The out-degree of v, denoted by* $\deg^+(v)$ *is the* number of edges with *v as their initial vertex. (Note that a loop at a vertex contributes to* both the in-degree and the out-degree of this vertex.)

In-degree and Out-degree: Example



$deg^{-}(a) = 2,\quad deg^{-}(b) = 2$
$deg^{-}(c) = 3,\quad deg^{-}(d) = 2$
$deg^{-}(e) = 3,\quad deg^{-}(f) = 0$

$deg^{+}(a) = 4,\ deg^{+}(b) = 1$
$deg^{+}(c) = 2,\ deg^{+}(d) = 2$
$deg^{+}(e) = 3,\ deg^{+}(f) = 0$

Theorem

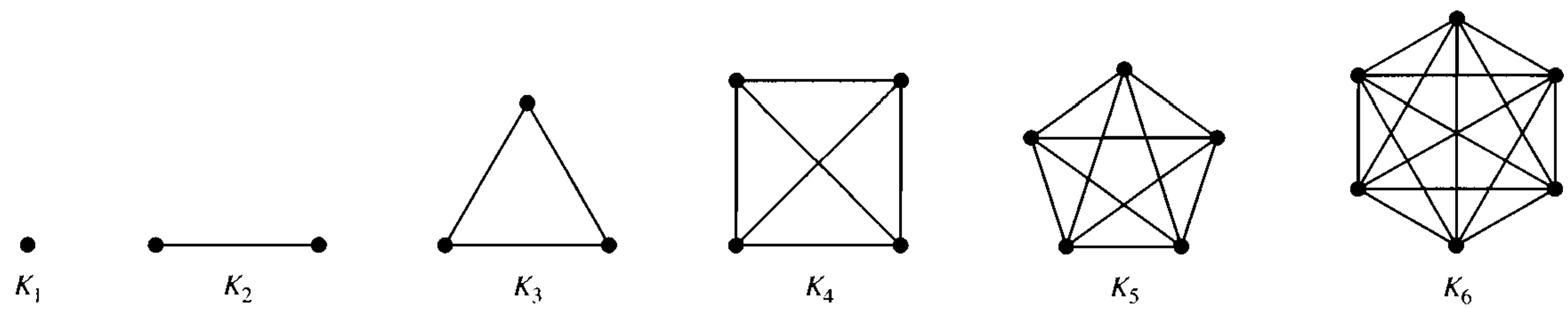
- Let G = *(V, E) be a graph with directed edges. Then*

$$\sum_{v \in V} \deg^-(v) = \sum_{v \in V} \deg^+(v) = |E|.$$

The sum of the in-degrees and the sum of the out-degrees of all vertices in a graph with directed edges are the same. Both of these sums are the number of edges in the graph.

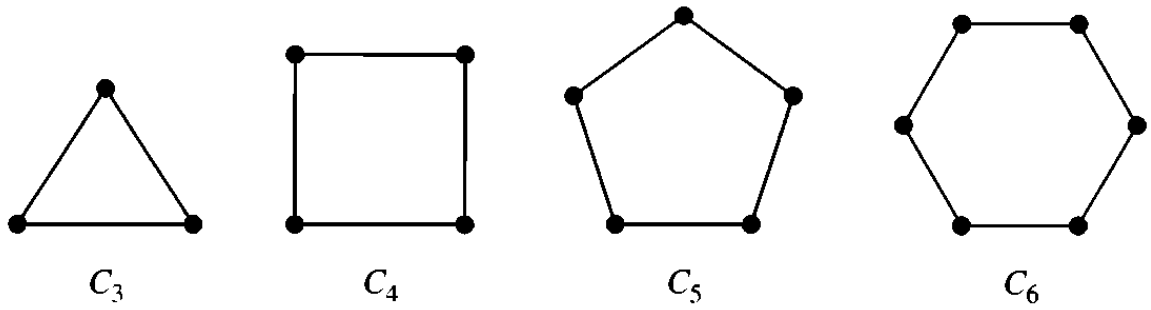
Some Special Simple Graphs

- **Complete Graphs** The complete graph on *n vertices, denoted by Kn, is the simple graph* that contains exactly one edge between each pair of distinct vertices. The graphs *Kn, for n =* 1,2, 3, 4, 5, 6, are as follows:



Some Special Simple Graphs

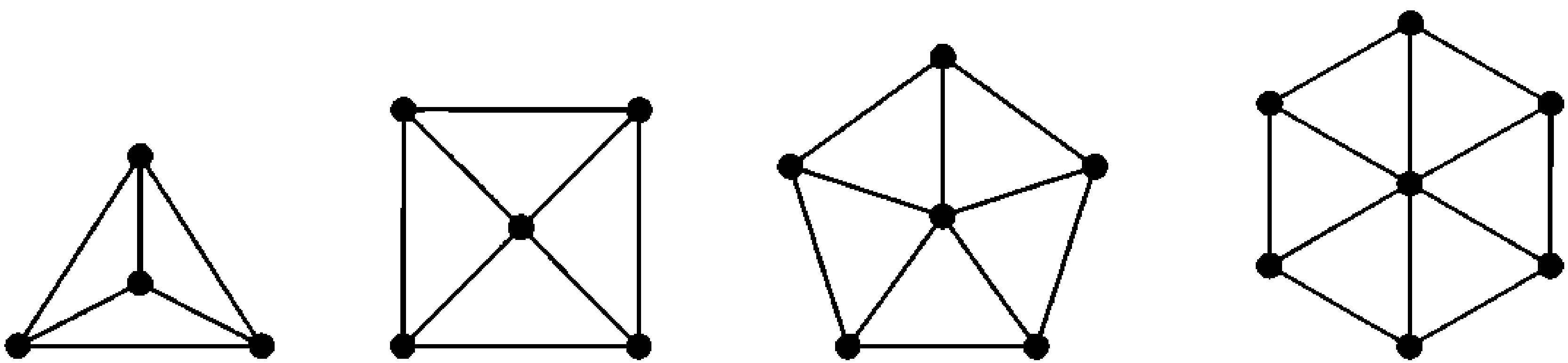
- **Cycles** The cycle *Cn, n >= 3, consists of n vertices v1, v2,...,vn and edges {v1, v2}, {v2,v3},... ,{vn-1,vn}, and {vn,v1}. The cycles C3, C4 , C5, and C6 are as follows:*



Some Special Simple Graphs

- **Wheels** We obtain the wheel *Wn, when we add an additional vertex to the cycle Cn, for*

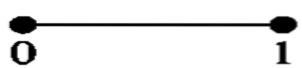
  n >=3, and connect this new vertex to each of the *n vertices in Cn, by new edges.The wheels W3, W4,W5, and W6 are displayed as follows:*



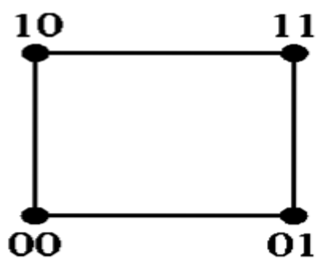
W_3 W_4 W_5 W_6

Some Special Simple Graphs

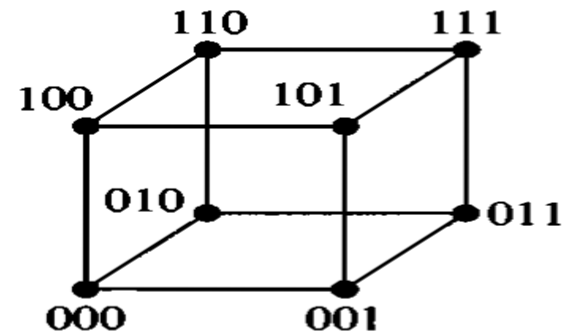
- **n-Cubes** The n-dimensional hypercube, or n-cube, denoted by *Qn, is the graph that has* vertices representing the *2n bit strings of length n. Two vertices are adjacent if and only if the* bit strings that they represent differ in exactly one bit position. The graphs *Q1, Q2, and Q3 are* displayed as follows:



Q_1



Q_2



Q_3

n-Cubes continued..

- Note that you can construct the *(n + l)-cube Qn+1 from the n-cube Qn* by making two copies of *Qn, prefacing the labels on the vertices with a 0 in one copy of Qn* and with a 1 in the other copy of *Qn,*
- *and adding edges connecting two vertices that have labels* differing only in the first bit.
- *Eg. Q3 is constructed from Q2 by drawing two copies* of *Q2 as the top and bottom faces of Q3, adding 0 at the beginning of the label of each vertex* in the bottom face and 1 at the beginning of the label of each vertex in the top face.