Discrete Mathematics
BCSC 0010
Module 2

Graph Theory

Connectivity in graphs

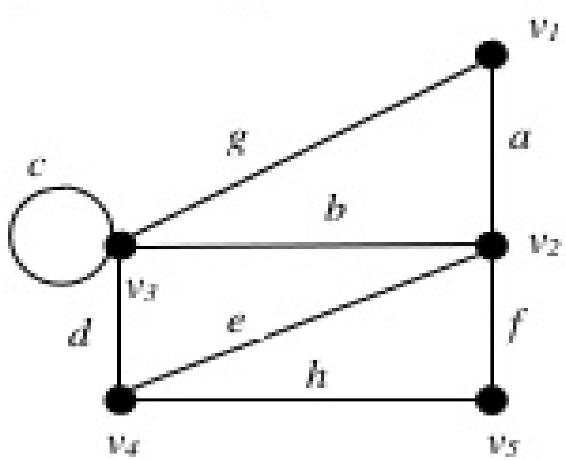
Introduction

- A **path** of length n from u to v in G is a sequence of n edges e1, ... ,en of G such that e1 is associated with {xo, x1}, e2 is associated with {x1, x2} and so on, with en associated with {xn-1, xn}, where xo = u and xn = v.

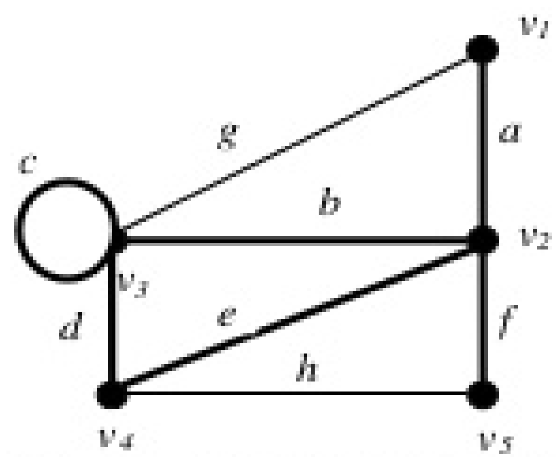
- The path is a **cycle** if it begins and ends at the same vertex, that is, if u = v, and has length greater than zero.

Terminologies

- A **walk** is defined to be an alternating sequence of vertices and edges of a graph, that is, *vo, e1, v1, e2, ... , Vn-1, en, Vn, where Vi-1 and Vi are the endpoints of ei for i = l, 2, ... , n.*



Graph G:

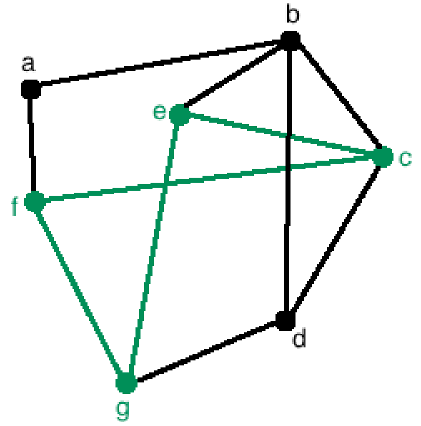


v_1 a v_2 b v_3 c v_3 d v_4 e v_2 f v_5 is a Open walk.

Terminologies

- **Closed walk** *is a walk that* begins and ends at the same vertex.

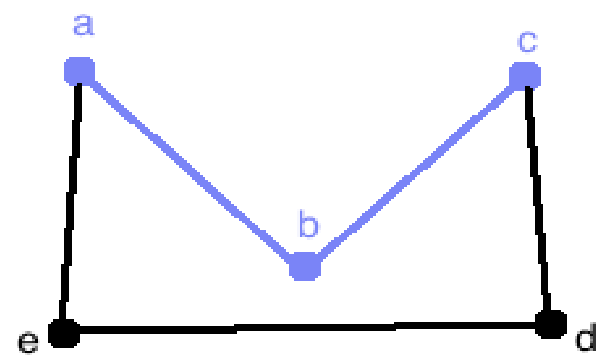
- Here, walk can be defined by *cegfc*, and the start and end vertices of the walk is *c*. Hence this walk is closed.



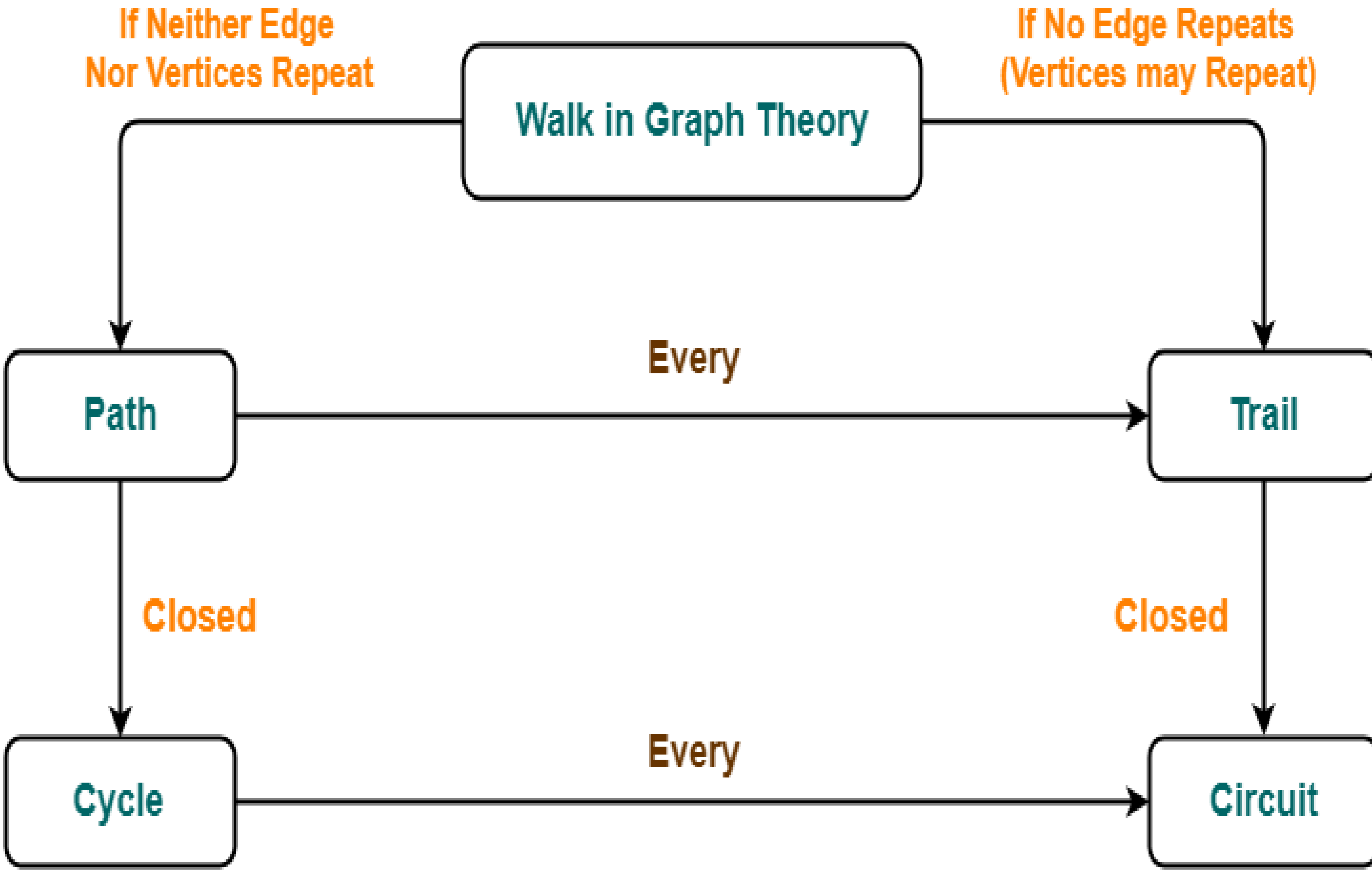
- **Circuit** is a **closed walk** where vertices can repeat, but not edges.
- **Cycle** is a **closed walk** where neither vertices nor edges can repeat.

Terminologies

- **Trail** is used to denote a walk that has no repeated edge



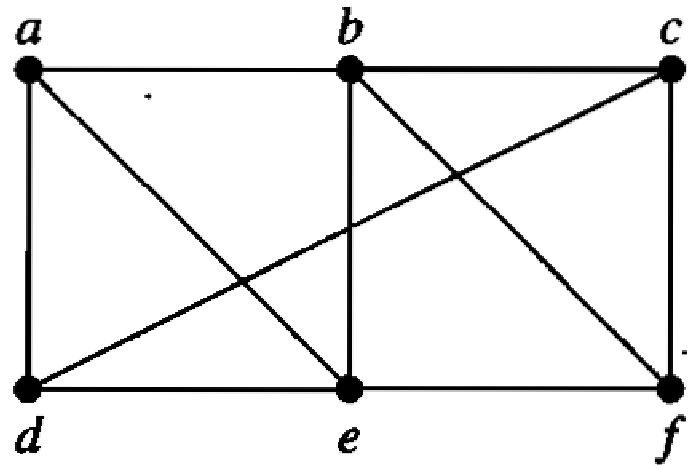
- Here, walk can be defined as *abc*. Since, there are no repeated edges so this walk is also a **trail.**



Important Chart to Remember

Example

In the simple graph shown in Figure



•*a, d, c, f, e is a simple path of length 4 because {a, d},{d, c}, {c, f} and {f,e} are all edges.*

•*However, d, e, c, a is not a path, because {e, c} is not an* edge.

•Note that *b, c, f, e, b is a circuit of length 4 because {b, c}, {c, f}, {f, e}, and {e, b} are* edges, and this path begins and ends at *b.*

•*The path a, b, e, d, a, b, which is of length 5, is not* simple because it contains the edge *{a, b} twice.*

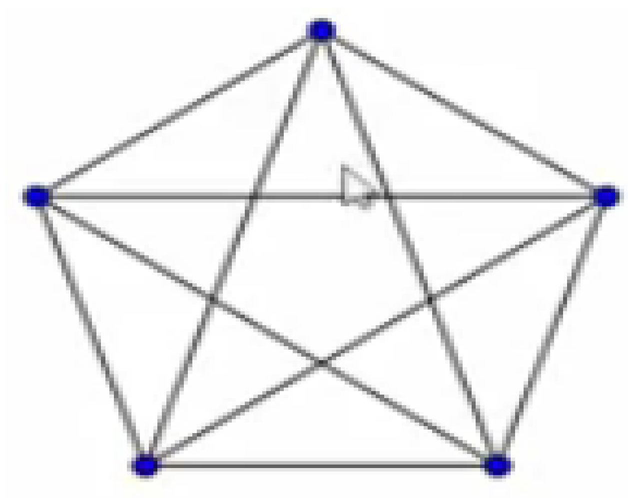
Euler path and circuit

- An Euler Path visits every **edge** once.
- An Euler Circuit (or 'cycle') visits every **edge** once AND begins and ends at the same vertex.

- An Euler Circuit is possible if every vertex has even degree.
- An Euler Path is possible, but an Euler Circuit is not possible, if exactly two vertices have odd degree.
- No Euler Path is possible if more than two vertices have odd degree.

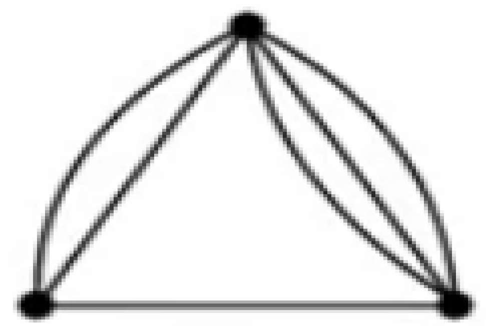
Example

- First we need to check the degree of all vertices
- This has an Euler circuit



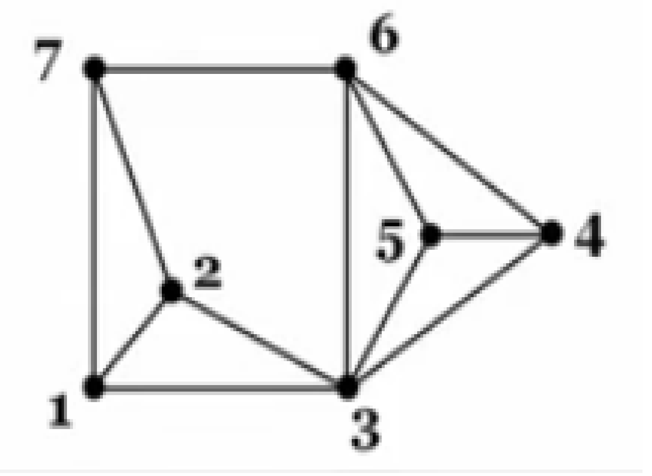
Example

- This has an Euler path, but no circuit



Example

- This has no Euler Path



Necessary and sufficient conditions for Euler circuits and paths

- A connected multigraph with at least two vertices has an Euler circuit if and only if each of its vertices has even degree.

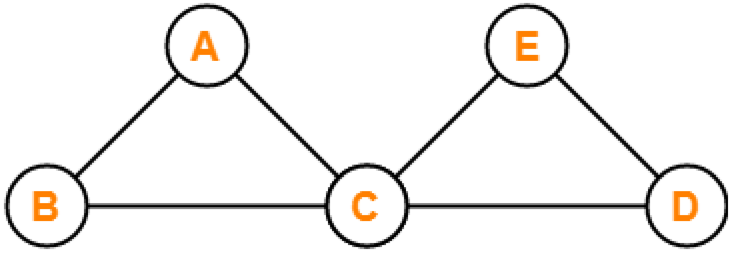
- A connected multigraph has an Euler path but not an Euler circuit if and only if it has exactly two vertices of odd degree.

Euler Graph

- Any connected graph is called as an Euler Graph if and only if all its vertices are of even degree.

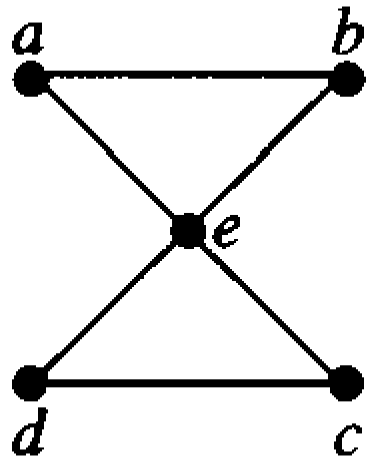
OR

- An Euler Graph is a connected graph that contains an Euler Circuit.

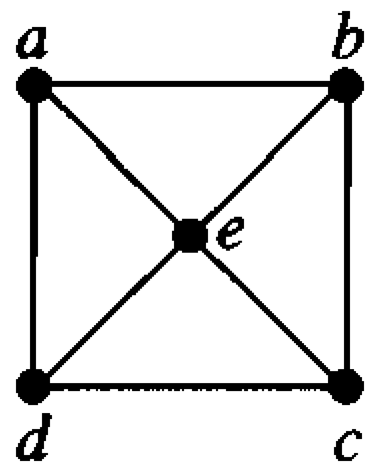


Example of Euler Graph

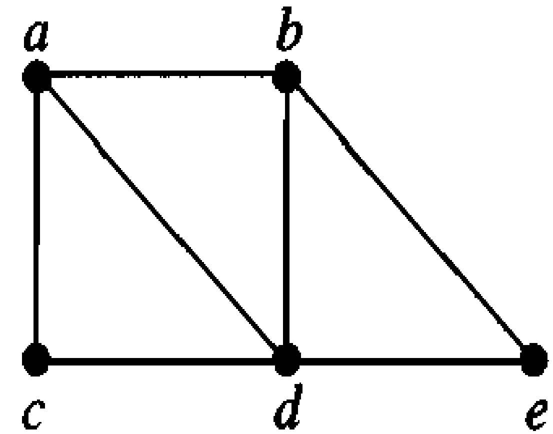
Determine whether Euler paths and circuits exists in the following graphs



G_1



G_2



G_3

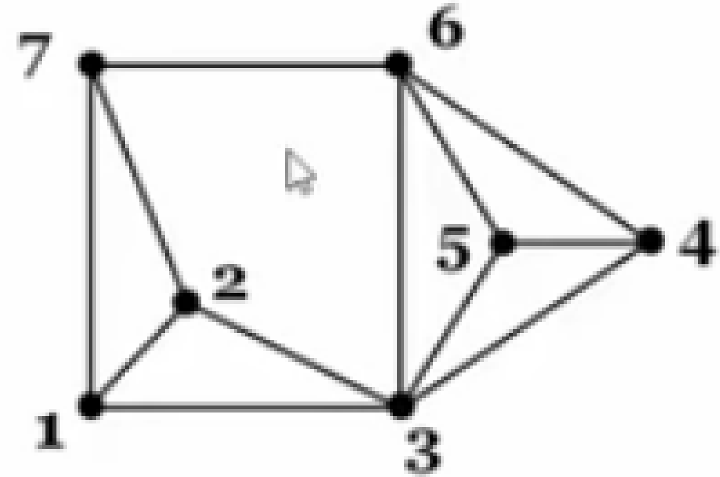
Hamiltonian path and circuit

- An Hamiltonian Path visits every **vertex** once.
- An Hamiltonian Circuit (or 'cycle') visits every **vertex** once AND begins and ends at the same vertex.

- Circuits are harder to achieve than paths.
- If you have a circuit, then you have a path. To make the path you just remove an edge!
- Unlike the situation with (Euler) edge-visiting, there is generally no theorem that tells if a Hamiltonian path or circuit is possible. You just have to try.

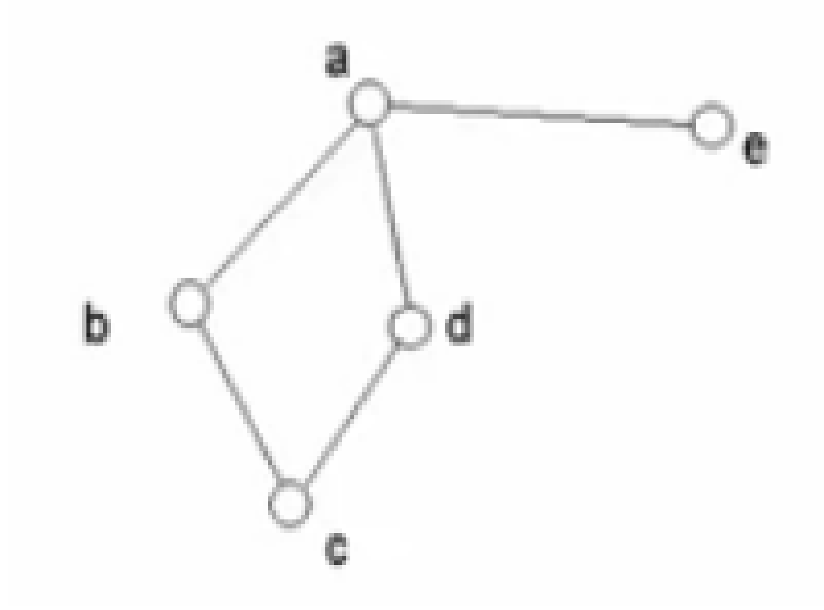
Example

- This has a Hamiltonian path and a Hamiltonian circuit



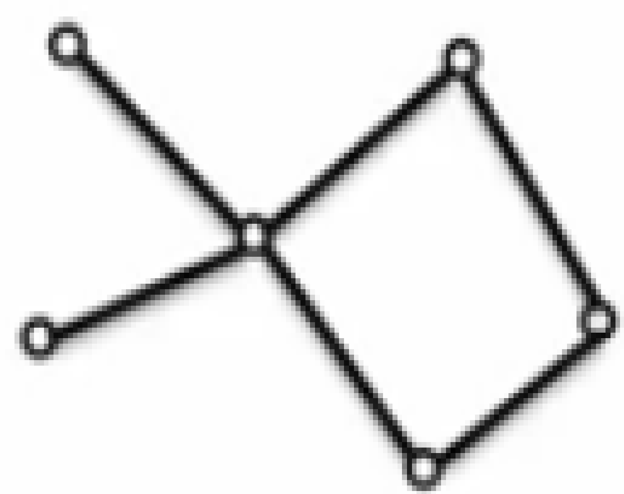
Example

- This has a Hamiltonian path but does not have Hamiltonian circuit



Example

- This has no Hamiltonian Path

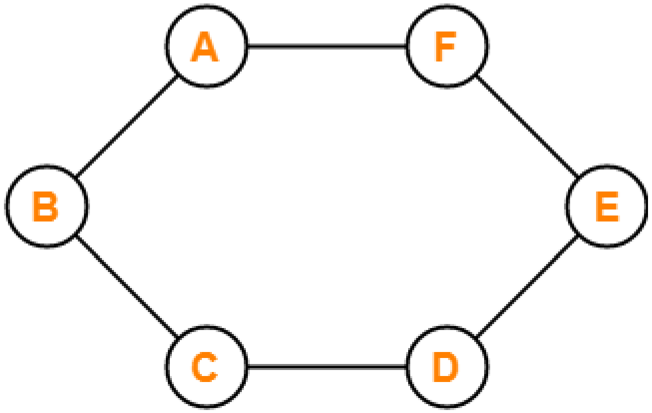


Hamiltonian Graph

- If there exists a **closed walk** in the connected graph that visits every vertex of the graph exactly once (except starting vertex) is called as a **Hamiltonian graph.**

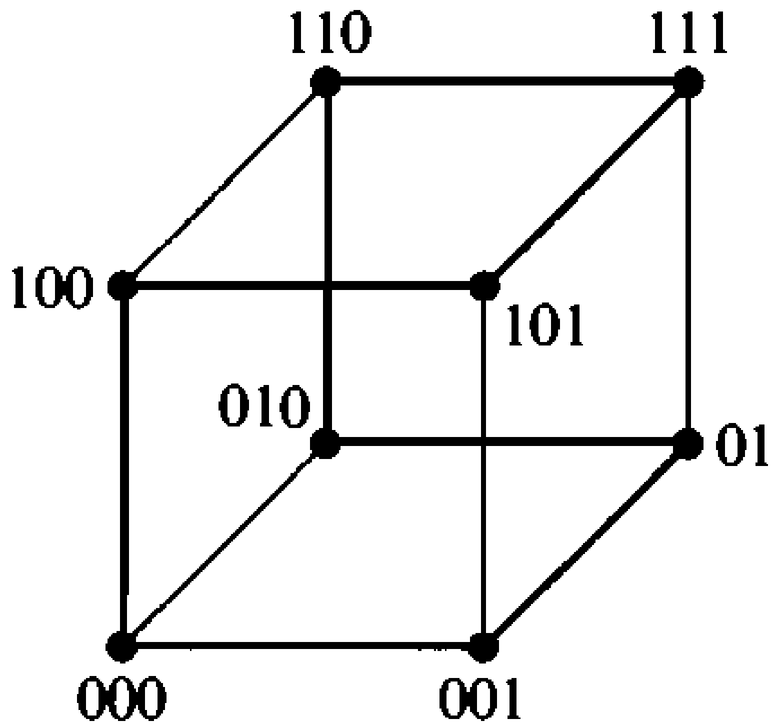
OR

- Any connected graph that contains a Hamiltonian circuit is called as a **Hamiltonian Graph.**



Example of Hamiltonian Graph

Example



A Hamiltonian Circuit for *Q3.*

Determine whether Hamiltonian paths and circuits exists in the following graphs