Cauchy Euler Homogeneous Linear diff. eq^n →

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + \cdots \; a_0 y = \phi(x)$$

$a_0, a_1, a_2 \; - - - \; a_n$ are constants.

Put $\quad x = e^z \quad , \quad \log x = z$

$$x\frac{dy}{dx} = Dy$$

$$x^2 \frac{d^2y}{dx^2} = D(D-1)y$$

$$x^3 \frac{d^3y}{dx^3} = D(D-1)(D-2)y$$

eg: $\quad x^2 y'' - 4xy' + 4y = 4x^2 - 6x^3$

$$D(D-1)y - 4Dy + 4y = 4e^{2z} - 6e^{3z}$$

$$(D^2 - 5D + 4)y = 4e^{2z} - 6e^{3z}$$

A.E $\quad m^2 - 5m + 4 = 0$

$$m = 1, 4$$

C.F. $\quad c_1 e^z + c_2 e^{4z} \qquad = c_1 x + c_2 x^4$

P.I. $\quad \frac{1}{(D^2 - 5D + 4)} (4e^{2z} - 6e^{3z})$

$$\frac{4e^{2z}}{(D^2 - 5D + 4)} - \frac{6e^{3z}}{(D^2 - 5D + 4)}$$

$$\frac{4e^{2z}}{4 - 10 + 4} - \frac{6e^{3z}}{9 - 15 + 4}$$

$$= -2e^{2z} + 3e^{3z} = -2x^2 + 3x^3$$

$y =$ C.F. + P.I.

Q2. $(x^3D^3 + x^2D^2 - 2)y = x - \frac{1}{x^3}$

$$[D(D-1)(D-2) + D(D-1) - 2]y = e^z + e^{-3z}$$

$$(D^3 - 2D^2 + D - 2)y = e^z + e^{-3z}$$

A.E. $m^3 - 2m^2 + m - 2 = 0$

$$m^2(m-2) + 1(m-2) = 0$$

$$(m-2)(m^2+1) = 0$$

$$m = 2, \pm i$$

C.F. $c_1 e^{2z} + c_2 \cos z + c_3 \sin z$

P.I. $\dfrac{e^z}{D^3 - 2D^2 + D - 2} + \dfrac{e^{-3z}}{(D^3 - 2D^2 + D - 2)}$

$$\frac{e^z}{1-2+1-2} + \frac{e^{-3z}}{-27-2(9)-3-2}$$

$$\frac{e^z}{-2} + \frac{e^{-3z}}{(-50)}$$

$y =$ C.F. + P.I.

$$= c_1 e^{2z} + c_2 \cos z + c_3 \sin z - \frac{e^z}{2} - \frac{e^{-3z}}{50}$$

$$= c_1 x^2 + c_2 \cos(\log x) + c_3 \sin(\log x) - \frac{x}{2} - \frac{1}{50x^3}$$

eg. $(x^2D^2 + xD + 1)y = \sin(\log x^2)$

$\Rightarrow$ A.E. is $m^3 - 2m^2 + m - 2 = 0$

i.e. $(m-2)(m^2+1) = 0$ i.e. $m = 2,$

$$\text{C.F.} = C_1 e^{2z} + C_2 \cos z + C_3 \sin z$$

$$\text{P.I.} = \frac{1}{D_1^3 - 2D_1^2 + D_1 - 2} e^z + \frac{1}{D_1^3 - 2D_1^2 + D_1 - 2} e^{-3z}$$

$$= \frac{1}{1-2+1-2} e^z + \frac{1}{-27-18-3-2} e^{-3z}$$

$$= -\frac{e^z}{2} - \frac{1}{50} e^{-3z}$$

$\therefore$
$$y = C_1 e^{2z} + C_2 \cos z + C_3 \sin z - \frac{1}{2} e^z - \frac{1}{50} e^{-3z}$$

$$= C_1 x^2 + C_2 \cos(\log x) + C_3 \sin(\log x) - \frac{1}{2} x - \frac{1}{50} \cdot \frac{1}{x^3}$$

Example 49. *Solve* $x^2 \dfrac{d^2y}{dx^2} + x \dfrac{dy}{dx} + y = \sin(\log x^2)$ *(Nagpur University, Summ*

Solution. We have, $\quad x^2 \dfrac{d^2y}{dx^2} + x \dfrac{dy}{dx} + y = \sin(\log x^2)$

Let $x = e^z$, so that $z = \log x$, $\quad D = \dfrac{d}{dz}$

(1) becomes

$$D(D-1)y + Dy + y = \sin 2z \Rightarrow (D^2+1)y = \text{si}$$

A.E. is $m^2 + 1 = 0 \Rightarrow m = \pm i$

$$\text{C.F.} = C_1 \cos z + C_2 \sin z$$

$$\text{P.I.} = \frac{1}{D^2+1} \sin 2z = \frac{1}{-4+1} \sin 2z = -\frac{1}{3} \sin 2z$$

Hence complete solution is $\quad y = \text{C.F.} + \text{P.I.} = C_1 \cos z + C_2 \sin z - \dfrac{1}{3} \sin 2z$

$\Rightarrow$
$$y = C_1 \cos(\log x) + C_2 \sin(\log x) - \frac{1}{3} \sin(\log x^2)$$

Example 50. *Solve* $x^3 \dfrac{d^3y}{dx^3} + 3x^2 \cdot \dfrac{d^2y}{dx^2} + x \dfrac{dy}{dx} + y = x + \log x$

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Solution. Putting $x = e^z$ or $z = \log x$ and denoting $\frac{d}{dz}$ by D the equation becomes

$$[D(D-1)(D-2) + 3D(D-1) + D + 1]\, y = e^z + z$$

$$\Rightarrow \qquad [D^3 + 1]\, y = e^z + z$$

$\therefore$ A.E. is $\qquad m^3 + 1 = 0$

$$\Rightarrow \qquad (m+1)(m^2 - m + 1) = 0 \qquad \Rightarrow \qquad m = -1, \frac{1 \pm i\sqrt{3}}{2}$$

$$\text{C.F.} = C_1 e^{-z} + e^{\frac{1}{2}z} \left\{ C_2 \cos \frac{\sqrt{3}}{2} z + C_3 \sin \frac{\sqrt{3}}{2} z \right\}$$

$$\text{P.I.} = \frac{1}{D^3 + 1} \{e^z + z\}$$

$$= \frac{1}{D^3+1} e^z + \frac{1}{D^3+1} z = \frac{e^z}{1+1} + (1 + D^3)^{-1} z$$

$$= \frac{1}{2} e^z + (1 - D^3 + \ldots)\, z = \frac{1}{2} e^z + z.$$

$\therefore$ Complete solution is

$$y = C_1 e^{-z} + e^{\frac{z}{2}} \left\{ C_2 \cos \frac{\sqrt{3}}{2} z + C_3 \sin \frac{\sqrt{3}}{2} z \right\} + \frac{1}{2} e^z + z$$

$$\Rightarrow \qquad y = C_1 x^{-1} + \sqrt{x} \left\{ C_2 \cos\left(\frac{\sqrt{3}}{2} \log x\right) + C_3 \sin\left(\frac{\sqrt{3}}{2} \log x\right) \right\} + \frac{1}{2} x + \log x \qquad \textbf{Ans.}$$

Example 51. *Solve:* $x^2 \frac{d^3y}{dx^3} + 3x \frac{d^2y}{dx^2} + \frac{dy}{dx} = x^2 \log x$ (*Nagpur University, Summer* 2003)

Solution. We have, $x^3 \frac{d^3y}{dx^3} + 3x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} = x^3 \log x$

Let $x = e^z$ so that $z = \log x$, $D = \frac{d}{dz}$

The equation becomes after substitution

$$[D(D-1)(D-2) + 3D(D-1) + D]\, y = z\, e^{3z} \qquad \Rightarrow \qquad D^3 y = z e^{3z}$$

Auxiliary equation is $m^3 = 0 \quad \Rightarrow \quad m = 0, 0, 0.$

C.F. $= C_1 + C_2 z + C_3 z^2 = C_1 + C_2 \log x + C_3 (\log x)^2$

$$\text{P.I.} = \frac{1}{D^3} \cdot z e^{3z} = e^{3z} \cdot \frac{1}{(D+3)^3} \cdot z$$

$$= e^{3z} \frac{1}{27} \left(1 + \frac{D}{3}\right)^{-3} z = \frac{e^{3z}}{27} (1 - D)\, z = \frac{e^{3z}}{27} (z - 1) = \frac{x^3}{27} (\log x - 1)$$

Complete solution is $\qquad y = C_1 + C_2 \log x + C_3 (\log x)^2 + \frac{x^3}{27} (\log x - 1) \qquad$ **Ans.**