Chapter 6

Series Solutions of Differential Equations

6.1 Introduction

In the previous two chapters, we have studied the methods of solving general linear first order differential equations, constant coefficient second and higher order differential equations and a special case of a variable coefficient differential equation (Euler-Cauchy type equation). The solutions of these equations were all closed form solutions in terms of standard functions. However, it is often not possible to express the solutions of variable coefficient equations in closed form using the standard functions. In such cases, we seek the solution as an infinite series in terms of the independent variable. Many of the important physical problems can be described by second order variable coefficient equations. Solutions of such equations can be obtained in terms of infinite series. The series solution methods can be classified into two categories: power series method and general series solution method (*Frobenius method*). In the following sections we discuss the application of these methods.

6.2 Ordinary and Singular Points of an Equation

Consider the variable coefficient second order, linear homogeneous equation

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0, \; a_0(x) \neq 0. \tag{6.1}$$

Dividing the equation by $a_0(x)$, we can write it in the standard form (*normal form* or *canonical form*) as

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{6.2}$$

where $p(x) = a_1(x)/a_0(x)$ and $q(x) = a_2(x)/a_0(x)$. Let x_0 be a point in the interval I and $a_1(x)$, $a_2(x)$ be analytic at x_0 (that is $a_1(x)$, $a_2(x)$ are differentiable at x_0 and at every point in its neighborhood). We define the following.

Ordinary point A point $x_0 \in I$ is said to be an *ordinary point* of Eq. (6.1) if $a_0(x_0) \neq 0$. Ordinary point is also called a regular point of the equation.

Singular point A point $x_0 \in I$ is said to be a *singular point* of Eq. (6.1) if $a_0(x_0) = 0$.

Example 6.1 Find the regular and singular points of the differential equations

(i) $(1 - x^2)\, y'' - 2xy' + n(n + 1)\, y = 0$,

(ii) $x^2y'' + axy' + by = 0$, a, b are constants.

Solution

(i) Setting $a_0(x) = 1 - x^2 = 0$, we get $x = \pm 1$. Hence, $x = \pm 1$ are the singular points of the equation, while all others points are regular points. Note that $a_1(x) = -2x$ and $a_2(x) = n(n + 1)$ are analytic at $x = \pm 1$.

(ii) Setting $a_0(x) = x^2 = 0$, we get $x = 0$, Hence, $x = 0$ is the singular point of the equation, while all other points are regular points.

Let $x_0 \in I$ be a singular point of Eq. (6.1), where $a_1(x)$ and $a_2(x)$ are analytic at x_0. Consider now the standard form of Eq. (6.1), which is given by

$$y'' + p(x)\, y' + q(x)\, y = 0$$

where $$p(x) = a_1(x)/a_0(x) \quad \text{and} \quad q(x) = a_2(x)/a_0(x).$$

Now, write this equation as

$$y'' + \frac{p_1(x)}{x - x_0}\, y' + \frac{q_1(x)}{(x - x_0)^2}\, y = 0 \tag{6.3}$$

where $p_1(x) = (x - x_0)\, p(x) = \dfrac{(x - x_0)a_1(x)}{a_0(x)}$, and $q_1(x) = (x - x_0)^2\, q(x) = \dfrac{(x - x_0)^2 a_2(x)}{a_0(x)}$.

We define the following.

Regular singular point A singular point x_0 of Eq. (6.1) is said to be a *regular singular point* if and only if the functions $p_1(x)$ and $q_1(x)$ defined in Eq. (6.3) have removable discontinuities at x_0 and become analytic when these discontinuities are removed. In other words, after the discontinuity at x_0 is removed, the functions $p_1(x)$ and $q_1(x)$ have Taylor series expansions about the point $x = x_0$. What is intended in the above is that the definitions of $p_1(x)$, $q_1(x)$ given in Eq. (6.3) may be continuously extended to include the point x_0 and these continuous extensions will lead to analytic functions at the point x_0.

Irregular singular point A singular point x_0 of Eq. (6.1) is said to be an *irregular singular point* if and only if x_0 is not a regular singular point. In other words, if either $p_1(x)$ or $q_1(x)$ or both $p_1(x)$ and $q_1(x)$ do not have Taylor series expansions about the point $x = x_0$, then x_0 is an irregular singular point.

Example 6.2 Classify the singular points of the following equations

(i) $x^2y'' + axy' + by = 0$, a, b constants,

(ii) $x^2y'' + xy' + (x^2 - n^2)\, y = 0$, n constant,

(iii) $(1 - x^2)\, y'' - 2xy' + n(n + 1)\, y = 0$, n constant,

(iv) $x^3(x - 2)\, y'' + x^3y' + 6y = 0$.

Solution

(i) Setting $a_0(x) = x^2 = 0$, we get the singular point of the equation as $x = 0$. Dividing the equation by x^2, we get

$$y'' + \frac{ax}{x^2}y' + \frac{b}{x^2}y = 0, \text{ or } y'' + \frac{a}{x}y' + \frac{b}{x^2}y = 0, \quad x \neq 0. \tag{6.4}$$

Comparing with Eq. (6.3), we get $p_1(x) = a$ and $q_1(x) = b$. (Note that $x = 0$ is a removable discontinuity of $p_1(x) = (ax^2)/x^2$, and $q_1(x) = (bx^2)/x^2$.) Since a and b are constants, the expressions of $p_1(x)$ and $q_1(x)$ are the Taylor series expansions. Hence, $x = 0$ is a regular singular point of the given equation.

(ii) The singular point of the equation is $x = 0$. Dividing by x^2, we get

$$y'' + \frac{x}{x^2}y' + \frac{1}{x^2}(x^2 - n^2)\, y = 0, \quad \text{or} \quad y'' + \frac{1}{x}y' + \frac{1}{x^2}(x^2 - n^2)\, y = 0, \; x \neq 0.$$

Comparing with Eq. (6.3), we get $p_1(x) = 1$ and $q_1(x) = x^2 - n^2$. (Note that $x = 0$ is a removable discontinuity of $p_1(x) = x^2/x^2$, and $q_1(x) = x^2(x^2 - n^2)/x^2$). The expressions of $p_1(x)$ and $q_1(x)$ are the Taylor series expansions about $x = 0$. Hence, $x = 0$ is a regular singular point of the given equation.

(ii) Setting $a_0(x) = 1 - x^2 = 0$, we get the singular points of the equation as $x = \pm 1$. Dividing the equation by $(1 - x^2)$ and writing it in the required form, we get

$$y'' - \frac{2x}{1 - x^2}y' + \frac{n(n+1)}{1 - x^2}\, y = 0, \; x \neq \pm 1 \tag{6.5}$$

or $$y'' - \frac{1}{(1 - x^2)}\left[\frac{2x}{1 + x^2}\right]y' + \frac{1}{(1 + x)^2}\left[\frac{(1 - x)n(n+1)}{1 + x}\right]y = 0$$

where the singularity at $x = 1$ is being considered. Now,

$$p_1(x) = -\frac{2x}{1 + x}, \text{ and } q_1(x) = \frac{(1 - x)n(n+1)}{1 + x}$$

are both analytic at $x = 1$ and hence Taylor series expansions of these functions about $x = 1$ exist. For example,

$$p_1(x) = -\frac{2(x - 1 + 1)}{(x - 1 + 2)} = -[1 + (x - 1)]\left[1 + \frac{(1 + 1)}{2}\right]^{-1}$$

$$= -[1 + (x - 1)]\left[1 - \frac{(x-1)}{2} + \frac{(x-1)^2}{4} - \ldots\right] = -\left[1 + \frac{(x-1)}{2} - \frac{(x-1)^2}{4} + \ldots\right].$$

Similarly, Taylor series expansion for $q_1(x)$ can be written.

Write now, the Eq. (6.5) as

$$y'' - \frac{1}{1 + x}\left[\frac{2x}{1 - x}\right] + \frac{1}{(1 + x)^2}\left[\frac{(1 + x)\, n(n+1)}{1 - x}\right]y = 0$$

where the singularity at $x = -1$ is being considered. Now

$$p_1(x) = -\frac{2x}{1 + x}, \text{ and } q_1(x) = \frac{(1 + x)\, n(n+1)}{1 - x}$$

are both analytic at $x = -1$ and hence have Taylor series expansions about $x = -1$. Therefore, $x = \pm 1$ are regular singular points of the given equation.

(iv) Setting $a_0(x) = x^3(x-2) = 0$, we get the singular points as $x = 0, 2$. Dividing the equation throughout by $x^3(x-2)$, we get

$$y'' + \frac{x^3}{x^3(x-2)}y' + \frac{6}{x^3(x-2)}y = 0, \quad x \neq 0, 2. \tag{6.6}$$

Consider now, the singularity at $x = 2$.

Comparing Eq. (6.6) with

$$y'' + \frac{p_1(x)}{x-2}y' + \frac{q_1(x)}{(x-2)^2}y = 0$$

we get $p_1(x) = 1$, and $q_1(x) = 6(x-2)/x^3$. Since $p_1(x)$ and $q_1(x)$ are both analytic at $x = 2$, their Taylor series expansions about $x = 2$ exist. Hence, $x = 2$ is a regular singular point of the given equation. Consider now, the singularity at $x = 0$.

Comparing Eq. (6.6) with

$$y'' + \frac{p_1(x)}{x}y' + \frac{q_1(x)}{x^2}y = 0,$$

we get $p_1(x) = x/(x-2)$, and $q_1(x) = 6/[x(x-2)]$. Now, $q_1(x)$ is not analytic at $x = 0$, and Taylor series expansion of $q_1(x)$ about $x = 0$ does not exist. Hence, $x = 0$ is an irregular singular point of the equation.

Exercise 6.1

Find the singular points of the following differential equations and classify them.

1. $x^2y'' + (x + x^2)y' - y = 0.$
2. $x^2y'' + 2xy' + (x^2 - n^2)y = 0$, n constant.
3. $x^2y'' - 5y' + 3x^2y = 0.$
4. $xy'' + y' + xy = 0.$
5. $x^2y'' + (\sin x)\, y' + (\cos x)\, y = 0.$
6. $x^3(x^2 - 1)\, y'' - x(x + 1)\, y' - (x - 1)\, y = 0.$
7. $(x^2 + x - 2)^2y'' + 3(x + 2)y' + (x - 1)\, y = 0.$
8. $x^2y'' + 4xy' + (x^2 + 2)\, y = 0.$
9. $x^4y'' + 4x^3y' + y = 0.$
10. $x^3y'' + 3xy' + 6y = 0.$