1.11. LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

The equation of the form

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots\ldots + a_n y = Q \qquad \ldots(1)$$

where $a_0, a_1, a_2, \ldots\ldots, a_n$ are all constants and Q is a function of x alone is called a linear differential equation of n^{th} order with constant coefficients.

1.12. THE OPERATOR D

The part $\frac{d}{dx}$ of the symbol $\frac{dy}{dx}$ may be regarded as an operator such that when it operates on y, the result is the derivative of y. Similarly, $\frac{d^2}{dx^2}, \frac{d^3}{dx^3}, \ldots\ldots, \frac{d^n}{dx^n}$ may be regarded as operators.

For brevity, we write $\frac{d}{dx} \equiv D, \frac{d^2}{dx^2} \equiv D^2, \ldots\ldots, \frac{d^n}{dx^n} \equiv D^n$

Thus, the symbol D is a **differential operator** or simply an **operator.**

Written in symbolic form, equation (1) becomes

$$(a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots\ldots + a_{n-1} D + a_n)y = Q$$

or

$$\boxed{f(D)y = Q}$$

The operator D can be treated as an algebraic quantity. Thus,

$$D(u + v) = Du + Dv, \quad D(\lambda u) = \lambda\, Du \quad \text{and} \quad D^p D^q\, u = D^q D^p\, u = D^{p+q}\, u$$

The polynomial $f(D)$ can be factorised by ordinary rules of algebra and the factors may be written in any order.

1.13. THEOREMS

Theorem 1. *If $y = y_1, y = y_2, \ldots\ldots, y = y_n$ are n linearly independent solutions of the differential equation*

$$(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots + a_n)y = 0$$

then $u = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n$ is also its solution, where $c_1, c_2, \ldots\ldots, c_n$ are arbitrary constants.

Theorem 2. *If $y = u$ is the complete solution of the equation $f(D)y = 0$ and $y = v$ is a particular solution (containing no arbitrary constants) of the equation $f(D)y = Q$, then the complete solution of the equation*

$$f(D)y = Q \text{ is } y = u + v.$$

Note 1. The part $y = u$ is called the **complementary function (C.F.)** and the part $y = v$ is called the **particular integral (P.I.)** of the equation $f(D)\, y = Q$.

Note 2. The complete solution is $\boxed{\mathbf{y = C.F. + P.I.}}$

Thus in order to solve the equation $f(D)\, y = Q$, we first find the C.F. i.e., the complete solution of equation $f(D)\, y = 0$ and then the P.I. i.e., a particular integral (solution) of equation $f(D)\, y = Q$.

1.14. COMPLEMENTARY FUNCTION (C.F.)

Consider the differential equation

$$f(D)y = Q \qquad \text{...(1)}$$

Complementary function is actually the solution of the given differential equation (1) when its right hand side member *i.e.*, Q is replaced by zero. To find C.F., we first find auxiliary equation.

1.15. AUXILIARY EQUATION (A.E.)

Consider the differential equation $(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots\ldots + a_n)y = 0$ ...(1)

Let $y = e^{mx}$ be a solution of (i), then

$$Dy = me^{mx},\ D^2y = m^2e^{mx},\ \ldots\ldots,\ D^{n-2}y = m^{n-2}e^{mx},\ D^{n-1}y = m^{n-1}e^{mx},\ D^ny = m^ne^{mx}$$

Substituting the values of y, Dy, D^2y, ..., D^ny in (1), we get

$$(m^n + a_1m^{n-1} + a_2m^{n-2} + \ldots\ldots + a_n)\, e^{mx} = 0$$

or

$$m^n + a_1m^{n-1} + a_2m^{n-2} + \ldots\ldots + a_n = 0, \text{ since } e^{mx} \neq 0 \qquad \text{...(2)}$$

Thus $y = e^{mx}$ will be a solution of equation (1) if m satisfies equation (2).

Equation (2) is called the auxiliary equation for the differential equation (1).

1.15.1. Definition

The equation obtained by equating to zero the symbolic coefficient of y is called the **auxiliary equation,** briefly written as **A.E.**

1.15.2. Steps for Finding Auxiliary Equation

Step 1. Replace y by 1

Step 2. Replace $\dfrac{dy}{dx}$ by m

Step 3. Replace $\dfrac{d^2y}{dx^2}$ by m^2 and so on replace $\dfrac{d^ny}{dx^n}$ by m^n

Step 4. By doing so, we get an algebraic equation in m of degree n called auxiliary equation.

1.16. RULES FOR FINDING THE COMPLEMENTARY FUNCTION

Consider the equation $(D^n + a_1D^{n-1} + a_2D^{n-2} + \ldots\ldots + a_n)y = 0$ ...(1)

where all the a_i's are constant.

Its auxiliary equation is $m^n + a_1m^{n-1} + a_2\, m^{n-2} + \ldots\ldots + a_n = 0$ ...(2)

It is an algebraic equation in m of degree n. So it will give n values of m on solving.

Let $m = m_1, m_2, m_3, \ldots\ldots, m_n$ be the roots of the A.E. The C.F. of equation (i) depends upon the nature of roots of the A.E. The following cases arise.

Case I. When the roots of auxiliary equation are real and distinct

Equation (1) is equivalent to

$$(D - m_1)(D - m_2) \ldots\ldots (D - m_n)y = 0 \qquad \text{...(3)}$$

Equation (3) will be satisfied by the solutions of the equations

$$(D - m_1)y = 0,\ (D - m_2)y = 0,\ \ldots\ldots,\ (D - m_n)y = 0$$

Now, consider the equation $(D - m_1)y = 0$, *i.e.*, $\frac{dy}{dx} - m_1 y = 0$

It is a linear equation and I.F. $= e^{\int -m_1\, dx} = e^{-m_1 x}$

$\therefore$ Its solution is $y \cdot e^{-m_1 x} = \int 0 \cdot e^{-m_1 x}\, dx + c_1$ or $y = c_1 e^{m_1 x}$

Similarly, the solution of $(D - m_2)y = 0$ is $y = c_2 e^{m_2 x}$

....

the solution of $(D - m_n)y = 0$ is $y = c_n e^{m_n x}$

$$\therefore \quad \boxed{\text{C.F.} = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \ldots\ldots + c_n e^{m_n x}} \qquad \ldots(iv)$$

Case II. When the roots of auxiliary equation are equal

(*a*) When two roots of auxiliary equation are equal

Let $m_1 = m_2$

Solution of eqn. (*iii*) is (as in case I)

$$\begin{aligned} y &= \text{C.F.} + \text{P.I.} \\ &= c_1 e^{m_1 x} + c_2 e^{m_2 x} + \ldots\ldots + c_n e^{m_n x} + 0 && \mid \because \text{ P.I.} = 0 \text{ as } Q = 0 \\ &= (c_1 + c_2) e^{m_1 x} + c_3 e^{m_3 x} + \ldots\ldots + c_n e^{m_n x} && \mid \text{Here } m_1 = m_2 \\ &= c\, e^{m_1 x} + c_3 e^{m_3 x} + \ldots\ldots + c_n e^{m_n x} \end{aligned}$$

It contains $(n - 1)$ arbitrary constants and is, therefore, not the complete solution of equation (*i*).

The part of C.F. corresponding to the repeated root is the complete solution of

$$\begin{aligned} & (D - m_1)(D - m_1)y = 0 \\ \Rightarrow \quad & (D - m_1)v = 0 && \mid \text{Putting } (D - m_1)y = v \\ \Rightarrow \quad & \frac{dv}{dx} - m_1 v = 0 \end{aligned}$$

Its solution is $v = c_2 e^{m_1 x}$

$$\begin{aligned} \therefore \quad & (D - m_1)y = c_2 e^{m_1 x} \\ \Rightarrow \quad & \frac{dy}{dx} - m_1 y = c_2 e^{m_1 x} \quad \text{which is a linear equation} \\ \therefore \quad & \text{I.F.} = e^{-m_1 x} \end{aligned}$$

Its solution is

$$y e^{-m_1 x} = \int c_2 e^{m_1 x} \cdot e^{-m_1 x}\, dx + c_1 = c_2 x + c_1$$

$$\Rightarrow \quad y = (c_2 x + c_1)\, e^{m_1 x}$$

$\therefore$ Part of C.F. $= (c_1 + c_2 x)\, e^{m_1 x}$

Hence, $\boxed{\text{complete C.F.} = (c_1 + c_2 x) e^{m_1 x} + c_3\, e^{m_3 x} + \ldots\ldots + c_n\, e^{m_n x}}$

(*b*) If however, three roots of the auxiliary equation are equal say $m_1 = m_2 = m_3$ then proceeding as above,

$$\boxed{\text{C.F.} = (c_1 + c_2 x + c_3 x^2) e^{m_1 x} + c_4 e^{m_4 x} + \ldots\ldots + c_n e^{m_n x}}$$

Case III. When two roots of auxiliary equation are imaginary

Let $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$ then from (*iv*),

$$\begin{aligned}
\text{C.F.} &= c_1 e^{(\alpha + i\beta)x} + c_2 e^{(\alpha - i\beta)x} + c_3 e^{m_3 x} + \ldots\ldots + c_n e^{m_n x} \\
&= e^{\alpha x} (c_1 e^{i\beta x} + c_2 e^{-i\beta x}) + c_3 e^{m_3 x} + \ldots\ldots + c_n e^{m_n x} \\
&= e^{\alpha x} [c_1 (\cos \beta x + i \sin \beta x) + c_2 (\cos \beta x - i \sin \beta x)] + c_3 e^{m_3 x} + \ldots\ldots + c_n e^{m_n x} \\
&= e^{\alpha x} [(c_1 + c_2) \cos \beta x + i (c_1 - c_2) \sin \beta x] + c_3 e^{m_3 x} + \ldots\ldots + c_n e^{m_n x} \\
\text{C.F.} &= e^{\alpha x} (C_1 \cos \beta x + C_2 \sin \beta x) + c_3 e^{m_3 x} + \ldots\ldots + c_n e^{m_n x}
\end{aligned}$$

[taking $c_1 + c_2 = C_1$, $i(c_1 - c_2) = C_2$]

Case IV. When roots of auxiliary equation are repeated imaginary

Let $m_1 = m_2 = \alpha + i\beta$ and $m_3 = m_4 = \alpha - i\beta$ then by case II,

$$\text{C.F.} = e^{\alpha x} [(c_1 + c_2 x) \cos \beta x + (c_3 + c_4 x) \sin \beta x] + c_5 e^{m_5 x} + \ldots\ldots + c_n e^{m_n x}$$

Case V. When roots of auxiliary equation are irrational

Let $m_1 = \alpha + \sqrt{\beta}$ and $m_2 = \alpha - \sqrt{\beta}$ then

C.F. of eqn. (*i*) is given by

$$\text{C.F.} = e^{\alpha x} (c_1 \cosh \sqrt{\beta} x + c_2 \sinh \sqrt{\beta} x) + c_3 e^{m_3 x} + \ldots\ldots + c_n e^{m_n x}$$

Case VI. When roots of auxiliary equation are repeated irrational

Let $m_1 = m_2 = \alpha + \sqrt{\beta}$ and $m_3 = m_4 = \alpha - \sqrt{\beta}$ then by case II,

$$\text{C.F.} = e^{\alpha x} [(c_1 + c_2 x) \cosh \sqrt{\beta} x + (c_3 + c_4 x) \sinh \sqrt{\beta} x] + c_5 e^{m_5 x} + c_6 e^{m_6 x} + \ldots\ldots + c_n e^{m_n x}$$

ILLUSTRATIVE EXAMPLES

Example 1. *Solve :* $\dfrac{d^3 y}{dx^3} - 7 \dfrac{dy}{dx} - 6y = 0.$

Sol. The auxiliary equation is

$$m^3 - 7m - 6 = 0$$

$$\Rightarrow \quad (m + 1)(m + 2)(m - 3) = 0 \quad \Rightarrow \quad m = -1, -2, 3$$

The roots are real and distinct

$\therefore$ Complementary Function (C.F.) $= c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{3x}$

Particular Integral (P.I.) $= 0$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{3x}$$

where c_1, c_2 and c_3 are arbitrary constants of integration.

Example 2. *Solve* : $(D^3 - 3D^2 + 4)\, y = 0$, *where* $D \equiv \dfrac{d}{dx}$. (Q. Bank U.P.T.U. 2

Sol. The auxiliary equation is

$$m^3 - 3m^2 + 4 = 0$$

$$\Rightarrow \quad (m+1)(m-2)^2 = 0 \Rightarrow m = -1, 2, 2$$

$$\therefore \quad \text{C.F.} = c_1 e^{-x} + (c_2 + c_3 x)\, e^{2x}$$

$$\text{P.I.} = 0$$

$\therefore$ The complete solution is

$$y = \text{C.F.} + \text{P.I.} = c_1 e^{-x} + (c_2 + c_3 x)\, e^{2x}$$

where c_1, c_2 and c_3 are arbitrary constants of integration.

Example 3. *Solve* : $(D^4 - n^4)y = 0$, *where* $D \equiv \dfrac{d}{dx}$. (Q. Bank U.P.T.U. 20

Sol. The auxiliary equation is

$$m^4 - n^4 = 0$$

$$\Rightarrow \quad (m^2 - n^2)(m^2 + n^2) = 0$$

$$\Rightarrow \quad m = \pm n, \pm ni$$

$$\therefore \quad \text{C.F.} = c_1 e^{nx} + c_2 e^{-nx} + e^{0x}(c_3 \cos nx + c_4 \sin nx)$$

$$= c_1 e^{nx} + c_2 e^{-nx} + c_3 \cos nx + c_4 \sin nx$$

$$\text{P.I.} = 0$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = c_1 e^{nx} + c_2 e^{-nx} + c_3 \cos nx + c_4 \sin nx$$

where c_1, c_2, c_3 and c_4 are arbitrary constants of integration.

Example 4. *Solve* : $\dfrac{d^4 y}{dx^4} + 13 \dfrac{d^2 y}{dx^2} + 36y = 0.$

Sol. The auxiliary equation is

$$m^4 + 13m^2 + 36 = 0$$

$$\Rightarrow \quad (m^2 + 9)(m^2 + 4) = 0 \Rightarrow m = \pm 3i, \pm 2i$$

$$\therefore \quad \text{C.F.} = e^{0x}(c_1 \cos 3x + c_2 \sin 3x) + e^{0x}(c_3 \cos 2x + c_4 \sin 2x)$$

$$= c_1 \cos 3x + c_2 \sin 3x + c_3 \cos 2x + c_4 \sin 2x$$

$$\text{P.I.} = 0$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = c_1 \cos 3x + c_2 \sin 3x + c_3 \cos 2x + c_4 \sin 2x$$

where c_1, c_2, c_3 and c_4 are arbitrary constants of integration.

Example 5. *Solve* : $(D^2 - 2D + 4)^2\, y = 0\,;\, D \equiv \dfrac{d}{dx}$. (Q. Bank U.P.T.U. 2001

Sol. The auxiliary equation is

$$(m^2 - 2m + 4)^2 = 0$$

$$m = \frac{2 \pm \sqrt{4 - 16}}{2} \text{ (twice)}$$

$$= 1 \pm \sqrt{3}\, i,\ 1 \pm \sqrt{3}\, i$$

The roots are repeated imaginary

$$\therefore \quad \text{C.F.} = e^x [(c_1 + c_2x) \cos \sqrt{3}x + (c_3 + c_4x) \sin \sqrt{3}x]$$

$$\text{P.I.} = 0$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = e^x [(c_1 + c_2x) \cos \sqrt{3}x + (c_3 + c_4x) \sin \sqrt{3}x]$$

where c_1, c_2, c_3 and c_4 are arbitrary constants of integration.

Example 6. *Solve* : $\dfrac{d^4y}{dx^4} - 4\dfrac{d^3y}{dx^3} + 8\dfrac{d^2y}{dx^2} - 8\dfrac{dy}{dx} + 4y = 0.$

Sol. The auxiliary equation is

$$m^4 - 4m^3 + 8m^2 - 8m + 4 = 0$$

$$\Rightarrow \quad (m^2 - 2m + 2)^2 = 0$$

$$\Rightarrow \quad m = \frac{2 \pm \sqrt{4-8}}{2} \text{ (twice)}$$

$$= \frac{2 \pm 2i}{2} \text{ (twice)} = 1 \pm i, 1 \pm i$$

$$\therefore \quad \text{C.F.} = e^x [(c_1 + c_2x) \cos x + (c_3 + c_4x) \sin x]$$

$$\text{P.I.} = 0$$

The complete solution is

$$y = \text{C.F.} + \text{P.I.}$$

$$\Rightarrow \quad y = e^x [(c_1 + c_2x) \cos x + (c_3 + c_4x) \sin x]$$

where c_1, c_2, c_3 and c_4 are arbitrary constants of integration.

Example 7. *Solve* : $\dfrac{d^4y}{dx^4} + m^4 y = 0.$ (Q. Bank U.P.T.U. 2001)

Sol. The auxiliary equation is

$$\text{M}^4 + m^4 = 0$$

$$\Rightarrow \quad \text{M}^4 + m^4 + 2\text{M}^2m^2 = 2\text{M}^2m^2$$

$$\Rightarrow \quad (\text{M}^2 + m^2)^2 = 2\text{M}^2m^2$$

$$\Rightarrow \quad \text{M}^2 + m^2 = \pm \sqrt{2}\, \text{M}m.$$

Case I. Taking (+) ve sign :

$$\text{M}^2 + m^2 - \sqrt{2}\, \text{M}m = 0$$

$$\therefore \quad \text{M} = \frac{\sqrt{2}m \pm \sqrt{2m^2 - 4m^2}}{2} = \frac{\sqrt{2}m \pm i\sqrt{2}m}{2} = \frac{m}{\sqrt{2}} \pm i\frac{m}{\sqrt{2}}.$$

Case II. Taking (–) ve sign :

$$\text{M}^2 + m^2 + \sqrt{2}\, \text{M}m = 0$$

$$\therefore \quad \text{M} = \frac{-\sqrt{2}m \pm \sqrt{2m^2 - 4m^2}}{2} = \frac{-\sqrt{2}m \pm i\sqrt{2}m}{2} = -\frac{m}{\sqrt{2}} \pm i\frac{m}{\sqrt{2}}$$

$$\therefore \quad \text{M} = \frac{m}{\sqrt{2}} \pm i\frac{m}{\sqrt{2}}; -\frac{m}{\sqrt{2}} \pm i\frac{m}{\sqrt{2}}$$

$$\text{C.F.} = e^{\frac{m}{\sqrt{2}}x}\left(c_1 \cos\frac{m}{\sqrt{2}}x + c_2 \sin\frac{m}{\sqrt{2}}x\right) + e^{-\frac{m}{\sqrt{2}}x}\left(c_3 \cos\frac{m}{\sqrt{2}}x + c_4 \sin\frac{m}{\sqrt{2}}x\right)$$

$$\text{P.I.} = 0$$

$\therefore$ Complete solution is

$$y = \text{C.F.} + \text{P.I.}$$

$$= e^{\frac{m}{\sqrt{2}}x}\left(c_1 \cos\frac{m}{\sqrt{2}}x + c_2 \sin\frac{m}{\sqrt{2}}x\right) + e^{-\frac{m}{\sqrt{2}}x}\left(c_3 \cos\frac{m}{\sqrt{2}}x + c_4 \sin\frac{m}{\sqrt{2}}x\right)$$

where c_1, c_2, c_3, c_4 are arbitrary constants of integration.

Example 8. *Solve the differential equation :*

$$(D^2 + 1)^3 (D^2 + D + 1)^2 y = 0, \quad \text{where} \quad D \equiv \frac{d}{dx}.$$

Sol. Auxiliary equation is

$$(m^2 + 1)^3 (m^2 + m + 1)^2 = 0$$

$\Rightarrow$ $(m^2 + 1)^3 = 0$ gives $m = \pm i, \pm i, \pm i$

and $(m^2 + m + 1)^2 = 0$ gives $m = \dfrac{-1 \pm \sqrt{3}i}{2}, \dfrac{-1 \pm \sqrt{3}i}{2}$

Hence,

$$\text{C.F.} = e^{0x}[(c_1 + c_2x + c_3x^2)\cos x + (c_4 + c_5x + c_6x^2)\sin x]$$
$$+ e^{-x/2}\left[(c_7 + c_8x)\cos\frac{\sqrt{3}}{2}x + (c_9 + c_{10}x)\sin\frac{\sqrt{3}}{2}x\right]$$
$$= (c_1 + c_2x + c_3x^2)\cos x + (c_4 + c_5x + c_6x^2)\sin x$$
$$+ e^{-x/2}\left\{(c_7 + c_8x)\cos\frac{\sqrt{3}}{2}x + (c_9 + c_{10}x)\sin\frac{\sqrt{3}}{2}x\right\}$$

$$\text{P.I.} = 0$$

Therefore the complete solution is

$$y = \text{C.F.} + \text{P.I.}$$
$$= (c_1 + c_2x + c_3x^2)\cos x + (c_4 + c_5\, x + c_6x^2)\sin x$$
$$+ e^{-x/2}\left\{(c_7 + c_8x)\cos\frac{\sqrt{3}}{2}x + (c_9 + c_{10}x)\sin\frac{\sqrt{3}}{2}x\right\}$$

where $c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, c_9$ and c_{10} are arbitrary constants of integration.

Example 9. *Solve the differential equation*

$$\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0, \quad \text{where } R^2C = 4L \text{ and } R, C, L \text{ are constants.}$$

Sol. The given equation is

$$\left(D^2 + \frac{R}{L}D + \frac{1}{LC}\right)i = 0, \text{ where } D \equiv \frac{d}{dt}$$

Auxiliary equation is

$$m^2 + \frac{R}{L}m + \frac{1}{LC} = 0$$

$$\Rightarrow \quad m = \frac{-\frac{R}{L} \pm \sqrt{\frac{R^2}{L^2} - \frac{4}{LC}}}{2}$$

$$= -\frac{R}{2L} \pm \frac{1}{2}\sqrt{\frac{4}{LC} - \frac{4}{LC}} \qquad \Big| \because R^2 = \frac{4L}{C}$$

$$= -\frac{R}{2L}, -\frac{R}{2L}$$

Hence,

$$C.F. = (c_1 + c_2 t)\, e^{-\frac{R}{2L}t}$$

$$P.I. = 0$$

$\therefore$ The complete solution is

$$i = C.F. + P.I. = (c_1 + c_2 t)\, e^{-\frac{R}{2L}t}$$

where c_1 and c_2 are arbitrary constants of integration.

Example 10. *Solve the differential equation:*

$$\frac{d^2y}{dx^2} + y = 0 \text{ ; given that } y(0) = 2 \text{ and } y\left(\frac{\pi}{2}\right) = -2 \qquad \text{[U.P.T.U. 2008]}$$

Sol. The auxiliary equation is

$$m^2 + 1 = 0$$

$$\Rightarrow \quad m = \pm i$$

$$\therefore \quad C.F. = c_1 \cos x + c_2 \sin x$$

$$P.I. = 0$$

Hence the general solution is

$$y = C.F. + P.I.$$

$$\Rightarrow \quad y = c_1 \cos x + c_2 \sin x \qquad \text{...(1)}$$

Applying the condition $y(0) = 2$, we get

$$2 = c_1$$

Applying the condition $y\left(\frac{\pi}{2}\right) = -2$, we get

$$-2 = c_2$$

Hence from (1) the particular solution is

$$y = 2(\cos x - \sin x)$$

TEST YOUR KNOWLEDGE

Solve the differential equations :

1. $\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 12y = 0$
2. $\frac{d^2y}{dx^2} + (a+b)\frac{dy}{dx} + aby = 0$
3. $\frac{d^3y}{dx^3} + 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} + 6y = 0$
4. $\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 9x = 0$
5. $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - y = 0$
6. $\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} - \frac{dy}{dx} + y = 0$
7. $(D^4 - D^3 - 9D^2 - 11D - 4)y = 0$
8. $\frac{d^4y}{dx^4} + 8\frac{d^2y}{dx^2} + 16y = 0$
9. $\frac{d^5y}{dx^5} - \frac{d^3y}{dx^3} = 0$ (U.P.T.U. 2009)
10. $\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 8y = 0$
11. $(D^2+1)^2(D-1)y = 0$
12. $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + y = 0$
13. $(D^6 - 1)y = 0$
14. $(D^6 + 1)y = 0$
15. $\frac{d^2x}{dt^2} - 3\frac{dx}{dt} + 2x = 0$, given that when $t = 0$, $x = 0$ and $\frac{dx}{dt} = 0$

Answers

1. $y = c_1e^{3x} + c_2e^{4x}$
2. $y = c_1e^{-ax} + c_2e^{-bx}$
3. $y = c_1e^{-x} + c_2e^{-2x} + c_3e^{-3x}$
4. $x = (c_1 + c_2t)e^{-3t}$
5. $y = (c_1 + c_2x + c_3x^2)e^x$
6. $y = (c_1 + c_2x)e^x + c_3e^{-x}$
7. $y = e^{-x}(c_1 + c_2x + c_3x^2) + c_4e^{4x}$
8. $y = (c_1 + c_2x)\cos 2x + (c_3 + c_4x)\sin 2x$
9. $y = c_1 + c_2x + c_3x^2 + c_4e^{-x} + c_5e^x$
10. $y = c_1e^{2x} + c_2\cos 2x + c_3\sin 2x$
11. $y = (c_1 + c_2x)\cos x + (c_3 + c_4x)\sin x + c_5e^x$
12. $y = e^{2x}(c_1\cosh\sqrt{3}x + c_2\sinh\sqrt{3}x)$
13. $y = c_1e^x + c_2e^{-x} + e^{x/2}\left(c_3\cos\frac{\sqrt{3}}{2}x + c_4\sin\frac{\sqrt{3}}{2}x\right) + e^{-x/2}\left(c_5\cos\frac{\sqrt{3}}{2}x + c_6\sin\frac{\sqrt{3}}{2}x\right)$
14. $y = c_1\cos x + c_2\sin x + e^{\sqrt{3}x/2}\left(c_3\cos\frac{x}{2} + c_4\sin\frac{x}{2}\right) + e^{-\sqrt{3}x/2}\left(c_5\cos\frac{x}{2} + c_6\sin\frac{x}{2}\right)$.
15. $x = 0$.

1.17. THE INVERSE OPERATOR $\frac{1}{f(D)}$

$\frac{1}{f(D)}$ Q is that function of x, *free from arbitrary constants*, which when operated upon by $f(D)$ gives Q.

Thus $$f(D)\left\{\frac{1}{f(D)}Q\right\} = Q$$

$\therefore$ $f(D)$ and $\frac{1}{f(D)}$ are inverse operators.

Note 1. $\frac{1}{f(D)}$ **Q is the particular integral of** $f(D)\,y = Q$.

Note 2. $\frac{1}{D} Q = \int Q\,dx.$

Note 3. $\frac{1}{D-a} Q = e^{ax} \int Q e^{-ax}\,dx.$

1.18. RULES FOR FINDING THE PARTICULAR INTEGRAL (P.I.)

Consider the differential equation, $(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots\ldots + a_{n-1} D + a_n)y = Q$

It can be written as $f(D)y = Q$

$$\therefore \qquad \text{P.I.} = \frac{1}{f(D)} Q.$$

1.18.1. Case I. When $Q = e^{ax}$ (or e^{ax+b})

Since
$$D\,e^{ax} = a\,e^{ax}$$
$$D^2 e^{ax} = a^2 e^{ax}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$D^{n-1} e^{ax} = a^{n-1} e^{ax}$$
$$D^n e^{ax} = a^n e^{ax}$$

$$\therefore \quad (D^n + a_1 D^{n-1} + \ldots\ldots + a_{n-1} D + a_n)e^{ax} = (a^n + a_1 a^{n-1} + \ldots\ldots + a_{n-1} a + a_n)e^{ax}$$

or
$$f(D)\,e^{ax} = f(a)\,e^{ax}$$

Operating on both sides by $\frac{1}{f(D)}$;

$$\frac{1}{f(D)}\,[f(D)\,e^{ax}] = \frac{1}{f(D)}\,[f(a)\,e^{ax}]$$

or
$$e^{ax} = f(a)\,\frac{1}{f(D)}\,e^{ax}$$

Dividing both sides by $f(a)$, $\frac{1}{f(a)}\,e^{ax} = \frac{1}{f(D)}\,e^{ax}$, provided $f(a) \neq 0$

Hence
$$\boxed{\frac{1}{f(D)}\,e^{ax} = \frac{1}{f(a)}\,e^{ax}, \text{ provided } f(a) \neq 0.}$$

Case of failure : If $f(a) = 0$, the above method fails.

Since $f(a) = 0$, $D = a$ is a root of $f(D) = 0$

$\therefore$ $D - a$ is a factor of $f(D)$.

Let
$$f(D) = (D-a)\,\phi(D), \text{ where } \phi(a) \neq 0 \qquad \ldots(1)$$

Then
$$\frac{1}{f(D)}\,e^{ax} = \frac{1}{(D-a)\,\phi(D)}\,e^{ax} = \frac{1}{D-a} \cdot \frac{1}{\phi(a)}\,e^{ax}$$
$$= \frac{1}{\phi(a)} \cdot \frac{1}{D-a}\,e^{ax} = \frac{1}{\phi(a)}\,e^{ax} \int e^{ax} \cdot e^{-ax}\,dx \qquad \text{[by Note 3]}$$
$$= \frac{1}{\phi(a)} e^{ax} \int 1\,dx = x \cdot \frac{1}{\phi(a)} e^{ax} \qquad \ldots(2)$$

Differentiating both sides of (1) w.r.t. D, we get

$$f'(D) = (D-a)\,\phi'(D) + \phi(D)$$

$$\Rightarrow \qquad f'(a) = \phi(a) \qquad \text{provided } f'(a) \neq 0$$

$\therefore$ From (2), we have $\dfrac{1}{f(D)} e^{ax} = x \cdot \dfrac{1}{f'(a)} e^{ax}$

Another case of failure :

If $f'(a) = 0$, then $\dfrac{1}{f(D)} e^{ax} = x^2 \cdot \dfrac{1}{f''(a)} e^{ax}$, provided $f''(a) \neq 0$ and so on.

ILLUSTRATIVE EXAMPLES

Example 1. *Find the P.I. of $(4D^2 + 4D - 3)y = e^{2x}$.*

Sol.

$$\text{P.I.} = \frac{1}{4D^2 + 4D - 3} e^{2x} = \frac{1}{4(2)^2 + 4(2) - 3} e^{2x} \qquad \text{(Replacing D by 2)}$$

$$= \frac{1}{21} e^{2x}.$$

Example 2. *Find the P.I. of $(D^2 + 3D + 2)y = 5$.*

Sol.

$$\text{P.I.} = \frac{1}{D^2 + 3D + 2} (5e^{0x}) \qquad [\because \; e^{0x} = 1]$$

$$= 5 \cdot \frac{1}{0 + 0 + 2} e^{0x} = \frac{5}{2}. \qquad \text{(Replacing D by 0)}$$

Example 3. *Find the P.I. of $(D^3 - 3D^2 + 4)y = e^{2x}$.*

Sol.

$$\text{P.I.} = \frac{1}{D^3 - 3D^2 + 4} e^{2x}.$$

Here the denominator vanishes when D is replaced by 2. It is a case of failure.

We multiply the numerator by x and differentiate the denominator w.r.t. D.

$$\therefore \qquad \text{P.I.} = x \cdot \frac{1}{3D^2 - 6D} e^{2x}$$

It is again a case of failure. We multiply the numerator by x and differentiate the denominator w.r.t. D.

$$\therefore \qquad \text{P.I.} = x^2 \cdot \frac{1}{6D - 6} e^{2x} = x^2 \cdot \frac{1}{6(2) - 6} e^{2x} = \frac{x^2}{6} e^{2x}.$$

Example 4. *Find P.I. of $(D + 1)^3 y = e^{-x}$, where $D \equiv \dfrac{d}{dx}$.*

Sol.

$$\text{P.I.} = \frac{1}{(D+1)^3} e^{-x} = x \cdot \frac{1}{3(D+1)^2} e^{-x} \qquad | \text{ Case of failure}$$

$$= x^2 \cdot \frac{1}{3 \cdot 2 (D+1)} e^{-x} \qquad | \text{ Again case of failure}$$

$$= x^3 \cdot \frac{1}{3 \cdot 2 \cdot 1} e^{-x} = \frac{x^3}{6} e^{-x}.$$

Example 5. *Solve* : $2\frac{d^3y}{dx^3}-\frac{d^2y}{dx^2}+4\frac{dy}{dx}-2y=e^x$ (U.P.T.U. 2007)

Sol. The auxiliary equation is

$$2m^3-m^2+4m-2=0$$

$$\Rightarrow \quad 2m(m^2+2)-1(m^2+2)=0$$

$$\Rightarrow \quad (2m-1)(m^2+2)=0$$

$$\Rightarrow \quad m=\frac{1}{2}, \pm\sqrt{2}\,i$$

$$\therefore \quad \text{C.F.}=c_1e^{x/2}+c_2\cos\sqrt{2}\,x+c_3\sin\sqrt{2}\,x$$

$$\text{P.I.}=\frac{1}{2D^3-D^2+4D-2}e^x=\frac{1}{2(1)^3-(1)^2+4(1)-2}e^x=\frac{1}{3}e^x$$

Hence the complete solution is

$$y=\text{C.F.}+\text{P.I.}$$

$$\Rightarrow \quad y=c_1e^{x/2}+c_2\cos\sqrt{2}\,x+c_3\sin\sqrt{2}\,x+\frac{1}{3}e^x$$

where c_1 and c_2 are arbitrary constants of integration.

Example 6. *Solve* $(D^3-6D^2+11D-6)y=e^{-2x}+e^{-3x}$.

Sol. Auxiliary equation is

$$m^3-6m^2+11m-6=0$$

or $$(m-1)(m-2)(m-3)=0$$

whence $$m=1,2,3$$

$$\therefore \quad \text{C.F.}=c_1e^x+c_2e^{2x}+c_3e^{3x}$$

$$\text{P.I.}=\frac{1}{D^3-6D^2+11D-6}(e^{-2x}+e^{-3x})$$

$$=\frac{1}{D^3-6D^2+11D-6}e^{-2x}+\frac{1}{D^3-6D^2+11D-6}e^{-3x}$$

$$=\frac{1}{(-2)^3-6(-2)^2+11(-2)-6}e^{-2x}+\frac{1}{(-3)^3-6(-3)^2+11(-3)-6}e^{-3x}$$

$$=-\frac{1}{60}e^{-2x}-\frac{1}{120}e^{-3x}=-\frac{1}{120}(2e^{-2x}+e^{-3x})$$

Hence the complete solution is

$$y=\text{C.F.}+\text{P.I.}=c_1e^x+c_2e^{2x}+c_3e^{3x}-\frac{1}{120}(2e^{-2x}+e^{-3x})$$

where c_1, c_2 and c_3 are arbitrary constants of integration.

Example 7. *Solve* : $(D^2-a^2)y=e^{ax}-e^{-ax}$; $D\equiv\frac{d}{dx}$. (Q. Bank U.P.T.U. 2001)

Sol. Auxiliary equation is

$$m^2-a^2=0$$

$$\Rightarrow \quad m=\pm a$$

$$\therefore \quad \text{C.F.}=c_1e^{ax}+c_2e^{-ax}$$

$$\text{P.I.} = \frac{1}{D^2 - a^2}(e^{ax} - e^{-ax}) = \frac{1}{D^2 - a^2}(e^{ax}) - \frac{1}{D^2 - a^2}(e^{-ax})$$

$$= x \cdot \frac{1}{2D}(e^{ax}) - x \cdot \frac{1}{2D}(e^{-ax}) = \frac{x}{2}\frac{e^{ax}}{a} - \frac{x}{2}\left(\frac{e^{-ax}}{-a}\right)$$

$$= \frac{x}{2}\left(\frac{e^{ax} + e^{-ax}}{a}\right) = \frac{x}{a}\cosh ax$$

The complete solution is

$$y = \text{C.F.} + \text{P.I.} = c_1 e^{ax} + c_2 e^{-ax} + \frac{x}{a}\cosh ax$$

where c_1 and c_2 are arbitrary constants of integration.

Example 8. *Solve* : $(D^2 + D + 1)\, y = (1 + e^x)^2$; $D \equiv \dfrac{d}{dx}$. (Q. Bank U.P.T.U. 2002)

Sol. Auxiliary equation is

$$m^2 + m + 1 = 0$$

$$\Rightarrow \quad m = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}\, i$$

$$\text{C.F.} = e^{-\frac{1}{2}x}\left(c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right)$$

$$\text{P.I.} = \frac{1}{D^2 + D + 1}(1 + e^x)^2 = \frac{1}{D^2 + D + 1}(1 + e^{2x} + 2e^x)$$

$$= \frac{1}{D^2 + D + 1}(e^{0x}) + \frac{1}{D^2 + D + 1}(e^{2x}) + \frac{1}{D^2 + D + 1}(2e^x)$$

$$= \frac{1}{(0)^2 + (0) + 1}e^{0x} + \frac{1}{(2)^2 + (2) + 1}e^{2x} + \frac{2}{(1)^2 + (1) + 1}e^x = 1 + \frac{1}{7}e^{2x} + \frac{2}{3}e^x$$

Hence complete solution is

$$y = \text{C.F.} + \text{P.I.} = e^{-\frac{x}{2}}\left(c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right) + 1 + \frac{1}{7}e^{2x} + \frac{2}{3}e^x$$

where c_1 and c_2 are arbitrary constants of integration.

Example 9. *Solve* : $(D + 2)(D - 1)^2\, y = e^{-2x} + 2\sinh x$; $D \equiv \dfrac{d}{dx}$.

Sol. Auxiliary equation is

$(m + 2)(m - 1)^2 = 0$ so that $m = -2, 1, 1$

$$\therefore \quad \text{C.F.} = c_1 e^{-2x} + (c_2 + c_3 x)\, e^x$$

$$\text{P.I.} = \frac{1}{(D+2)(D-1)^2}(e^{-2x} + 2\sinh x)$$

$$= \frac{1}{(D+2)(D-1)^2}(e^{-2x} + e^x - e^{-x}) \qquad \left[\because \sinh x = \frac{e^x - e^{-x}}{2}\right]$$

Now $\frac{1}{(D+2)(D-1)^2}e^{-2x} = \frac{1}{D+2}\left[\frac{1}{(D-1)^2}e^{-2x}\right] = \frac{1}{D+2}\left[\frac{1}{(-2-1)^2}e^{-2x}\right]$

$$= \frac{1}{9}\cdot\frac{1}{D+2}e^{-2x} \qquad \text{| Case of failure}$$

$$= \frac{x}{9}e^{-2x}$$

$$\frac{1}{(D+2)(D-1)^2}e^{x} = \frac{1}{(D-1)^2}\left[\frac{1}{D+2}e^{x}\right] = \frac{1}{(D-1)^2}\left[\frac{1}{1+2}e^{x}\right]$$

$$= \frac{1}{3}\cdot\frac{1}{(D-1)^2}e^{x} \qquad \text{| Case of failure}$$

$$= \frac{1}{3}\cdot x\,\frac{1}{2(D-1)}e^{x} \qquad \text{| Case of failure}$$

$$= \frac{1}{3}\cdot x^2\cdot\frac{1}{2}e^{x} = \frac{1}{6}x^2e^{x}$$

$$\frac{1}{(D+2)(D-1)^2}e^{-x} = \frac{1}{(-1+2)(-1-1)^2}e^{-x} = \frac{1}{4}e^{-x}$$

$\therefore$ $$\text{P.I.} = \frac{x}{9}e^{-2x} + \frac{x^2}{6}e^{x} + \frac{1}{4}e^{-x}$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = c_1e^{-2x} + (c_2 + c_3x)e^{x} + \frac{x}{9}e^{-2x} + \frac{x^2}{6}e^{x} + \frac{1}{4}e^{-x}.$$

where c_1, c_2 and c_3 are arbitrary constants of integration.

Example 10. *Solve the differential equation*

$$\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - y = e^x + 2. \qquad \text{(Q. Bank U.P.T.U. 2001)}$$

Sol. The given equation is

$$(D^3 - 3D^2 + 3D - 1)y = e^x + 2$$

or $$(D-1)^3\,y = e^x + 2$$

Auxiliary equation is

$$(m-1)^3 = 0 \quad \Rightarrow \quad m = 1, 1, 1$$

$\therefore$ $$\text{C.F.} = (c_1 + c_2x + c_3x^2)e^x$$

$$\text{P.I.} = \frac{1}{(D-1)^3}(e^x + 2) = \frac{1}{(D-1)^3}e^x + \frac{1}{(D-1)^3}(2e^{0x})$$

$$= x\cdot\frac{1}{3(D-1)^2}e^x + \frac{1}{(0-1)^3}(2e^{0x}) = x^2\cdot\frac{1}{3.2\,(D-1)}(e^x) - 2$$

$$= x^3\cdot\frac{1}{3.2.1}(e^x) - 2 = \frac{x^3}{6}e^x - 2$$

$\therefore$ The complete solution is

$$y = \text{C.F.} + \text{P.I.} = (c_1 + c_2x + c_3x^2)e^x + \frac{x^3}{6}e^x - 2$$

where c_1, c_2 and c_3 are arbitrary constants of integration.

TEST YOUR KNOWLEDGE

Solve the following differential equations :

1. $\dfrac{d^3y}{dx^3} + y = 3 + 5e^x$
2. $\dfrac{d^2y}{dx^2} - 4y = (1 + e^x)^2$
3. $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 5y = -2\cosh x$
4. $(2D + 1)^2 y = 4e^{-x/2}$
5. $(D^2 - 2kD + k^2)\, y = e^{kx}$
6. $\dfrac{d^3y}{dx^3} + 3\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + y = e^{-x}$
7. $(D + 2)(D - 1)^3 y = e^x$
8. $\dfrac{d^2y}{dx^2} + 31\dfrac{dy}{dx} + 240y = 272\, e^{-x}$
9. $\dfrac{d^2y}{dx^2} + 2p\dfrac{dy}{dx} + (p^2 + q^2)\, y = e^{2x}$
10. $(D^4 + D^3 + D^2 - D - 2)\, y = e^x$
11. $\dfrac{d^3y}{dx^3} + y = 3 + e^{-x} + 5e^{2x}$
12. $y'' + 4y' + 13y = 18e^{-2x};\ y(0) = 0,\ y'(0) = 9$

Answers

1. $y = c_1 e^{-x} + e^{\frac{1}{2}x}\left(c_2 \cos\dfrac{\sqrt{3}}{2}x + c_3 \sin\dfrac{\sqrt{3}}{2}x\right) + 3 + \dfrac{5}{2}e^x$
2. $y = c_1 e^{2x} + c_2 e^{-2x} - \dfrac{1}{4} - \dfrac{2}{3}e^x + \dfrac{1}{4}xe^{2x}$
3. $y = e^{-2x}(c_1 \cos x + c_2 \sin x) - \dfrac{1}{10}e^x - \dfrac{1}{2}e^{-x}$
4. $y = \left(c_1 + c_2 x + \dfrac{x^2}{2}\right)e^{-x/2}$
5. $y = (c_1 + c_2 x)\, e^{kx} + \dfrac{x^2}{2}e^{kx}$
6. $y = (c_1 + c_2 x + c_3 x^2)\, e^{-x} + e^{-x} \cdot \dfrac{x^3}{6}$
7. $y = (c_1 + c_2 x + c_3 x^2)\, e^x + c_4\, e^{-2x} + \dfrac{x^3 e^x}{18}$
8. $y = c_1 e^{-15x} + c_2 e^{-16x} + \dfrac{136}{105}e^{-x}$
9. $y = e^{-px}(c_1 \cos qx + c_2 \sin qx) + \dfrac{e^{2x}}{(2 + p)^2 + q^2}$
10. $y = c_1 e^x + c_2 e^{-x} + e^{-x/2}\left[c_3 \cos\dfrac{\sqrt{7}}{2}x + c_4 \sin\dfrac{\sqrt{7}}{2}x\right] + \dfrac{1}{8}xe^x$
11. $y = c_1 e^{-x} + e^{x/2}\left(c_2 \cos\dfrac{\sqrt{3}}{2}x + c_3 \sin\dfrac{\sqrt{3}}{2}x\right) + 3 + \dfrac{5}{9}e^{2x} + \dfrac{1}{3}xe^{-x}$
12. $y = e^{-2x}(-2\cos 3x + 3\sin 3x + 2)$.

1.18.2. Case II. When Q = sin (ax + b) or cos (ax + b)

$$D \sin(ax + b) = a \cos(ax + b)$$
$$\mathbf{D^2 \sin(ax + b) = (-a^2) \sin(ax + b)}$$
$$D^3 \sin(ax + b) = -a^3 \cos(ax + b)$$
$$D^4 \sin(ax + b) = a^4 \sin(ax + b)$$

or $$\mathbf{(D^2)^2 \sin(ax + b) = (-a^2)^2 \sin(ax + b)}$$

....

....

In general, $(D^2)^n \sin(ax+b) = (-a^2)^n \sin(ax+b)$

$\therefore \quad f(D^2)\sin(ax+b) = f(-a^2)\sin(ax+b)$

Operating on both sides by $\frac{1}{f(D^2)}$,

$$\frac{1}{f(D^2)}\{f(D^2)\sin(ax+b)\} = \frac{1}{f(D^2)}\{f(-a^2)\sin(ax+b)\}$$

or
$$\sin(ax+b) = f(-a^2)\frac{1}{f(D^2)}\sin(ax+b).$$

Dividing both sides by $f(-a^2)$,

$$\mathbf{\frac{1}{f(D^2)}\sin(ax+b) = \frac{1}{f(-a^2)}\sin(ax+b),} \qquad \textbf{provided } \mathbf{f(-a^2) \neq 0}$$

Similarly, $$\mathbf{\frac{1}{f(D^2)}\cos(ax+b) = \frac{1}{f(-a^2)}\cos(ax+b),} \qquad \textbf{provided } \mathbf{f(-a^2) \neq 0}$$

Steps : When $\mathbf{Q = \sin(ax+b)}$ **or** $\mathbf{\cos(ax+b)}$**,**

1. Replace D^2 by $-a^2$,
   D^4 by a^4,
   D^6 by $-a^6$,
   D^8 by a^8 and so on.

2. By doing so, following possibilities arise :
   (*a*) If denominator reduces to a constant, it will be final step in finding P.I.
   (*b*) If denominator reduces into D only, we are then only to integrate the given function Q once.
   (*c*) If denominator reduces to a factor of the form $\alpha D + \beta$ then we operate by its conjugate $\alpha D - \beta$ on both numerator and denominator from left hand side such as

$$\frac{\alpha D - \beta}{\alpha D - \beta}\cdot\left[\frac{1}{(\alpha D + \beta)}\sin(ax+b)\right]$$

By doing so, denominator will become $\alpha^2 D^2 - \beta^2$ which in turn reduces to a constant by replacing D^2 by $-a^2$.

Now, we operate $\sin(ax+b)$ by $(\alpha D - \beta)$ and consequently, find the required particular integral.

Case of failure : If $f(-a^2) = 0$, the above method fails. Then we proceed as follows :

$$\mathbf{\frac{1}{f(D^2)}\cos(ax+b) = x\,.\,\frac{1}{f'(D^2)}\cos(ax+b),} \qquad \textbf{provided } \mathbf{f'(-a^2) \neq 0}$$

$$\mathbf{\frac{1}{f(D^2)}\sin(ax+b) = x\,.\,\frac{1}{f'(D^2)}\sin(ax+b),} \qquad \textbf{provided } \mathbf{f'(-a^2) \neq 0}$$

If $\mathbf{f'(-a^2) = 0}$**, then**

$$\mathbf{\frac{1}{f(D^2)}\sin(ax+b) = x^2\,.\,\frac{1}{f''(D^2)}\sin(ax+b),} \qquad \textbf{provided } \mathbf{f''(-a^2) \neq 0}$$

$$\mathbf{\frac{1}{f(D^2)}\cos(ax+b) = x^2\,.\,\frac{1}{f''(D^2)}\cos(ax+b),} \qquad \textbf{provided } \mathbf{f''(-a^2) \neq 0}$$

and so on.

Steps :

1. When $f(-a^2) = 0$, we differentiate the denominator w.r.t. D and multiply th expression by x simultaneously in the same step.
2. When $f'(-a^2) = 0$ (*i.e.*, step 1 fails) we again differentiate the reduced denominat in D w.r.t. D and again multiply the remaining expression by x simultaneously.
3. If there is another case of failure, above process is to be repeated again and agai until we reach a constant in the denominator or any other possibility (ies) whic we have discussed before in the same article.

ILLUSTRATIVE EXAMPLES

Example 1. *Solve the following differential equation:*

$$(D^2 + 4)y = \sin 3x + \cos 2x \quad \text{where } D \equiv \frac{d}{dx}$$ [U.P.T.U. (SUM) 2008]

Sol. Auxiliary equation is

$$m^2 + 4 = 0$$

$$\Rightarrow \quad m = \pm 2i$$

$$\therefore \quad \text{C.F.} = c_1 \cos 2x + c_2 \sin 2x$$

$$\text{P.I.} = \frac{1}{D^2 + 4}(\sin 3x) + \frac{1}{D^2 + 4}(\cos 2x)$$

$$= \frac{1}{-(3)^2 + 4}\sin 3x + x \cdot \frac{1}{2D}(\cos 2x)$$

$$= -\frac{1}{5}\sin 3x + \frac{x}{2}\left(\frac{\sin 2x}{2}\right) = -\frac{1}{5}\sin 3x + \frac{x}{4}\sin 2x$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = c_1 \cos 2x + c_2 \sin 2x - \frac{1}{5}\sin 3x + \frac{x}{4}\sin 2x$$

where c_1 and c_2 are arbitrary constants of integration.

Example 2. *Find the P.I. of* $(D^3 + 1)y = \sin(2x + 1)$.

Sol. $$\text{P.I.} = \frac{1}{D^3 + 1}\sin(2x + 1) = \frac{1}{D(-2^2) + 1}\sin(2x +$$ [Putting $D^2 = -2^2$]

$$= \frac{1}{1 - 4D}\sin(2x + 1)$$

Operating N^r and D^r by $(1 + 4D)$

$$= \frac{1 + 4D}{(1 + 4D)(1 - 4D)}\sin(2x + 1) = \frac{1 + 4D}{1 - 16D^2}\sin(2x + 1)$$

$$= \frac{1+4D}{1-16(-2^2)} \sin(2x+1) \qquad [\text{Putting } D^2 = -2^2]$$

$$= \frac{1}{65}[\sin(2x+1) + 4D\sin(2x+1)]$$

$$= \frac{1}{65}[\sin(2x+1) + 8\cos(2x+1)] \qquad \left[\because D \equiv \frac{d}{dx}\right]$$

Example 3. *Solve the following differential equations:*

(i) $\frac{d^2y}{dx^2} + a^2y = \sin ax$ (U.P.T.U. 2008)

(ii) $(D^2 - 4)y = \cos^2 x$

Sol. (i) The auxiliary equation is

$$m^2 + a^2 = 0$$

$$\Rightarrow \quad m = \pm ai$$

$$\therefore \quad \text{C.F.} = c_1 \cos ax + c_2 \sin ax$$

$$\text{P.I.} = \frac{1}{D^2 + a^2}(\sin ax) = x \cdot \frac{1}{2D}\sin ax$$

$$= \frac{x}{2}\left[\frac{-\cos ax}{a}\right] = -\frac{x}{2a}\cos ax$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = c_1 \cos ax + c_2 \sin ax - \frac{x}{2a}\cos ax$$

where c_1 and c_2 are arbitrary constants of integration.

(ii) The auxiliary equation is

$$m^2 - 4 = 0$$

$$\Rightarrow \quad m = \pm 2$$

$$\therefore \quad \text{C.F.} = c_1 e^{2x} + c_2 e^{-2x}$$

$$\text{P.I.} = \frac{1}{D^2 - 4}\cos^2 x = \frac{1}{2}\left[\frac{1}{D^2 - 4}(1 + \cos 2x)\right]$$

$$= \frac{1}{2}\left[\frac{1}{D^2 - 4}(e^{ox}) + \frac{1}{D^2 - 4}(\cos 2x)\right]$$

$$= \frac{1}{2}\left[-\frac{1}{4} + \frac{1}{(-4-4)}\cos 2x\right]$$

$$= -\frac{1}{8} - \frac{1}{16}\cos 2x$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = c_1 e^{2x} + c_2 e^{-2x} - \frac{1}{8} - \frac{1}{16} \cos 2x$$

where c_1 and c_2 are arbitrary constants of integration.

Example 4. *Solve* : $\dfrac{d^4 y}{dx^4} - m^4 y = \cos mx.$ (Q. Bank U.P.T.U. 20

Sol. Auxiliary equation is

$$M^4 - m^4 = 0$$

$$(M^2 - m^2)(M^2 + m^2) = 0$$

$$M = \pm m, \pm mi$$

$$\therefore \quad \text{C.F.} = c_1 e^{mx} + c_2 e^{-mx} + c_3 \cos mx + c_4 \sin mx$$

$$\text{P.I.} = \frac{1}{D^4 - m^4} (\cos mx) = x \cdot \frac{1}{4D^3} \cos mx$$

$$= \frac{x}{4} \cdot \frac{1}{D^2} \left(\frac{\sin mx}{m} \right) = -\frac{x}{4m^2} \left(\frac{\sin mx}{m} \right)$$

$$= -\frac{x}{4m^3} \sin mx$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = c_1 e^{mx} + c_2 e^{-mx} + c_3 \cos mx + c_4 \sin mx - \frac{x}{4m^3} \sin mx$$

where c_1, c_2, c_3 and c_4 are arbitrary constants of integration.

Example 5. *Solve* : $\dfrac{d^3 y}{dx^3} - 3 \dfrac{d^2 y}{dx^2} + 4 \dfrac{dy}{dx} - 2y = e^x + \cos x.$ [U.P.T.U. 2001, 200

Sol. Auxiliary equation is

$$m^3 - 3m^2 + 4m - 2 = 0$$

$$\Rightarrow \quad (m^2 - 2m + 2)(m - 1) = 0 \quad \Rightarrow \quad m = 1, 1 \pm i$$

$$\therefore \quad \text{C.F.} = c_1 e^x + e^x (c_2 \cos x + c_3 \sin x)$$

$$\text{P.I.} = \frac{1}{(D^3 - 3D^2 + 4D - 2)} e^x + \frac{1}{(D^3 - 3D^2 + 4D - 2)} \cos x$$

$$= x \cdot \frac{1}{3D^2 - 6D + 4} (e^x) + \frac{1}{(-D + 3 + 4D - 2)} (\cos x)$$

$$= x \cdot \frac{1}{(7-6)} e^x + \frac{1}{3D + 1} (\cos x) = x e^x + \frac{(3D - 1)}{9D^2 - 1} (\cos x)$$

$$= xe^x - \frac{1}{10} (-3 \sin x - \cos x)$$

$\therefore$ Complete solution is

$$y = \text{C.F.} + \text{P.I.} = c_1 e^x + e^x (c_2 \cos x + c_3 \sin x) + xe^x + \frac{1}{10} (3 \sin x + \cos x)$$

where c_1, c_2, c_3 are arbitrary constants of integration.

Example 6. *Solve:* $(D^2 - 4D + 1)y = \cos x \cos 2x + \sin^2 x$

Sol. Auxiliary equation is

$$m^2 - 4m + 1 = 0$$

$$\Rightarrow \quad m = \frac{4 \pm \sqrt{16-4}}{2} = 2 \pm \sqrt{3}$$

$$\therefore \quad \text{C.F.} = e^{2x}(c_1 \cosh \sqrt{3}\,x + c_2 \sinh \sqrt{3}\,x)$$

$$\text{P.I.} = \frac{1}{D^2 - 4D + 1}(\cos x \cos 2x) + \frac{1}{D^2 - 4D + 1}(\sin^2 x)$$

$$= \frac{1}{2}\left[\frac{1}{D^2 - 4D+1}(\cos 3x) + \frac{1}{D^2 - 4D+1}(\cos x)\right] + \frac{1}{D^2 - 4D+1}\left(\frac{1-\cos 2x}{2}\right)$$

$$= \frac{1}{2}(P_1 + P_2) + P_3 \qquad \ldots(1)$$

where,

$$P_1 = \frac{1}{D^2 - 4D + 1}(\cos 3x)$$

$$= \frac{1}{-9-4D+1}(\cos 3x) = -\frac{1}{4(D+2)}\cos 3x$$

$$= -\frac{1}{4}\frac{D-2}{(D^2-4)}\cos 3x = -\frac{1}{4}\frac{(D-2)}{(-9-4)}\cos 3x$$

$$= \frac{1}{52}(-3\sin 3x - 2\cos 3x)$$

$$P_2 = \frac{1}{D^2 - 4D + 1}(\cos x) = \frac{1}{-1-4D+1}\cos x = -\frac{1}{4}\sin x$$

$$P_3 = \frac{1}{2}\left[\frac{1}{D^2-4D+1}(1) - \frac{1}{D^2-4D+1}(\cos 2x)\right]$$

$$= \frac{1}{2}\left[\frac{1}{D^2-4D+1}(e^{0x}) - \frac{1}{-4-4D+1}(\cos 2x)\right]$$

$$= \frac{1}{2}\left[\frac{1}{(0)^2-4(0)+1}(e^{0x}) + \frac{1}{4D+3}(\cos 2x)\right]$$

$$= \frac{1}{2}\left[1 + \frac{4D-3}{16D^2-9}(\cos 2x)\right] = \frac{1}{2}\left[1 + \frac{4D-3}{(-64-9)}(\cos 2x)\right]$$

$$= \frac{1}{2}\left[1 - \frac{1}{73}(-8\sin 2x - 3\cos 2x)\right]$$

$$= \frac{1}{2}\left[1 + \frac{1}{73}(8\sin 2x + 3\cos 2x)\right]$$

$\therefore$ From (1),

$$\text{P.I.} = -\frac{1}{104}(3\sin 3x + 2\cos 3x) - \frac{1}{8}\sin x + \frac{1}{2} + \frac{1}{146}(8\sin 2x + 3\cos 2x)$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.}$$

$$= e^{2x}(c_1 \cosh\sqrt{3}\,x + c_2 \sinh\sqrt{3}\,x) + \frac{1}{2} - \frac{1}{8}\sin x$$

$$-\frac{1}{104}(3\sin 3x + 2\cos 3x) + \frac{1}{146}(8\sin 2x + 3\cos 2x)$$

where c_1 and c_2 are arbitrary constants of integration.

TEST YOUR KNOWLEDGE

Solve the following differential equations :

1. $\dfrac{d^3y}{dx^3} + a^2\dfrac{dy}{dx} = \sin ax$
2. $(D^2 - 4D + 3)y = \sin 3x \cos 2x$
3. $\dfrac{d^2y}{dx^2} + 4y = e^x + \sin 2x$
4. $(D^2 + 9)\,y = \cos 2x + \sin 2x$
5. $\dfrac{d^2y}{dx^2} + 2h\dfrac{dy}{dx} + h^2y = a\cos px$
6. $(D^2 - 8D + 9)\,y = 40\sin 5x$
7. $(D^2 - 4D + 4)y = e^{-4x} + 5\cos 3x$
8. $(D^4 + 2D^3 - 3D^2)y = 3e^{2x} + 4\sin x$
9. $(D^2 - 4D - 5)y = e^{2x} + 3\cos(4x + 3)$
10. $(D^2 + 5D - 6)y = \sin 4x \sin x$ (U.P.T.U. 2008)
11. $(D^2 + 4)y = \cos x \cos 3x$ [U.P.T.U. (SUM.) 2009]
12. $(D^4 + 2D^2n^2 + n^4)y = \cos mx$; $m \neq n$.

Answers

1. $y = c_1 + c_2\cos ax + c_3 \sin ax - \dfrac{x}{2a^2}\sin ax$
2. $y = c_1e^x + c_2e^{3x} + \dfrac{1}{884}(10\cos 5x - 11\sin 5x) + \dfrac{1}{20}(\sin x + 2\cos x)$
3. $y = c_1\cos 2x + c_2 \sin 2x + \dfrac{1}{5}e^x - \dfrac{x}{4}\cos 2x$
4. $y = c_1 \cos 3x + c_2 \sin 3x + \dfrac{1}{5}(\cos 2x + \sin 2x)$
5. $y = (c_1 + c_2x)\,e^{-hx} + \dfrac{a\{(h^2 - p^2)\cos px + 2hp\sin px\}}{(h^2 + p^2)^2}$
6. $y = e^{4x}(c_1\cosh\sqrt{7}\,x + c_2\sinh\sqrt{7}\,x) + \dfrac{25}{29}\cos 5x - \dfrac{10}{29}\sin 5x$
7. $y = (c_1 + c_2x)\,e^{2x} + \dfrac{1}{36}e^{-4x} - \dfrac{5}{169}(12\sin 3x + 5\cos 3x)$
8. $y = c_1 + c_2x + c_3e^x + c_4e^{-3x} + \dfrac{3}{20}e^{2x} + \dfrac{2}{5}(\cos x + 2\sin x)$

9. $y = c_1 e^{-x} + c_2 e^{5x} - \frac{1}{9} e^{2x} - \frac{3}{697} [16 \sin (4x + 3) + 21 \cos (4x + 3)]$

10. $y = c_1 e^{x} + c_2 e^{-6x} + \frac{1}{2}\left[\frac{\sin 3x - \cos 3x}{30} + \frac{31 \cos 5x - 25 \sin 5x}{1586}\right]$

11. $y = c_1 \cos 2x + c_2 \sin 2x + \frac{x}{8} \sin 2x - \frac{1}{24} \cos 4x.$

12. $y = (c_1 + c_2 x)(c_3 \cos nx + c_4 \sin nx) + \frac{1}{(n^2 - m^2)^2} \cos mx.$

1.18.3. Case III. When $Q = x^m$, m being a positive integer.

Here $$\text{P.I.} = \frac{1}{f(D)} x^m$$

Steps :

1. Take out the lowest degree term from $f(D)$ to make the first term unity (so that Binomial Theorem for a negative index is applicable). The remaining factor will be of the form $1 + \phi(D)$ or $1 - \phi(D)$.
2. Take this factor in the numerator. It takes the form $[1 + \phi(D)]^{-1}$ or $[1 - \phi(D)]^{-1}$.
3. Expand it in ascending powers of D as far as the term containing D^m, since $D^{m+1}(x^m) = 0$, $D^{m+2}(x^m) = 0$ and so on.
4. Operate on x^m term by term.

ILLUSTRATIVE EXAMPLES

Example 1. *Find the P.I. of* $(D^2 + 5D + 4)y = x^2 + 7x + 9$.

Sol. $$\text{P.I.} = \frac{1}{D^2 + 5D + 4}(x^2 + 7x + 9) = \frac{1}{4\left(1 + \frac{5D}{4} + \frac{D^2}{4}\right)}(x^2 + 7x + 9)$$

$$= \frac{1}{4}\left[1 + \left(\frac{5D}{4} + \frac{D^2}{4}\right)\right]^{-1}(x^2 + 7x + 9)$$

$$= \frac{1}{4}\left[1 - \left(\frac{5D}{4} + \frac{D^2}{4}\right) + \left(\frac{5D}{4} + \frac{D^2}{4}\right)^2 - \ldots\ldots\right](x^2 + 7x + 9)$$

$$= \frac{1}{4}\left(1 - \frac{5D}{4} - \frac{D^2}{4} + \frac{25D^2}{16} + \ldots\ldots\right)(x^2 + 7x + 9)$$

$$= \frac{1}{4}\left(1 - \frac{5D}{4} + \frac{21D^2}{16}\right)(x^2 + 7x + 9)$$ | Leaving higher powers of D

$$= \frac{1}{4}\left[(x^2+7x+9) - \frac{5}{4}D(x^2+7x+9) + \frac{21}{16}D^2(x^2+7x+9)\right]$$

$$= \frac{1}{4}\left[(x^2+7x+9) - \frac{5}{4}(2x+7) + \frac{21}{16}(2)\right] = \frac{1}{4}\left(x^2 + \frac{9}{2}x + \frac{23}{8}\right).$$

Example 2. *Find the P.I. of $y'' - 6y' + 9y = 2x^2 - x + 3$.* (Q. Bank U.P.T.U. 2001)

Sol. $$\text{P.I.} = \frac{1}{D^2 - 6D + 9}(2x^2 - x + 3) = \frac{1}{(D-3)^2}(2x^2 - x + 3)$$

$$= \frac{1}{9}\left(1 - \frac{D}{3}\right)^{-2}(2x^2 - x + 3) = \frac{1}{9}\left(1 + \frac{2D}{3} + \frac{3D^2}{9} + \ldots\right)(2x^2 - x + 3)$$

$$= \frac{1}{9}\left(1 + \frac{2D}{3} + \frac{D^2}{3}\right)(2x^2 - x + 3) \qquad | \text{ Leaving higher powers of D}$$

$$= \frac{1}{9}\left[2x^2 - x + 3 + \frac{2}{3}(4x-1) + \frac{1}{3}(4)\right] = \frac{1}{9}\left[2x^2 + \frac{5}{3}x + \frac{11}{3}\right].$$

Example 3. *Solve : $(D^3 - D^2 - 6D)\,y = 1 + x^2$; $D \equiv \frac{d}{dx}$.* (Q. Bank U.P.T.U. 2001)

Sol. Auxiliary equation is

$$m^3 - m^2 - 6m = 0$$

$$\Rightarrow \quad m(m-3)(m+2) = 0 \Rightarrow m = 0, -2, 3$$

$$\therefore \quad \text{C.F.} = c_1 + c_2e^{-2x} + c_3e^{3x}$$

$$\text{P.I.} = \frac{1}{D^3 - D^2 - 6D}(1 + x^2) = \frac{1}{-6D - D^2 + D^3}(1 + x^2)$$

$$= -\frac{1}{6D\left\{1 + \left(\frac{D - D^2}{6}\right)\right\}}(1 + x^2) = -\frac{1}{6D}\left[1 + \left(\frac{D - D^2}{6}\right)\right]^{-1}(1 + x^2)$$

$$= -\frac{1}{6D}\left[1 - \left(\frac{D - D^2}{6}\right) + \left(\frac{D - D^2}{6}\right)^2 - \ldots\right](1 + x^2)$$

$$= -\frac{1}{6D}\left[1 - \frac{D}{6} + \frac{D^2}{6} + \frac{D^2}{36}\right](1 + x^2) = -\frac{1}{6D}\left[1 + x^2 - \frac{1}{6}(2x) + \frac{7}{36}(2)\right]$$

$$= -\frac{1}{6D}\left(1 + x^2 - \frac{x}{3} + \frac{7}{18}\right) = -\frac{1}{6D}\left(x^2 - \frac{x}{3} + \frac{25}{18}\right)$$

$$= -\frac{1}{6}\left(\frac{x^3}{3} - \frac{x^2}{6} + \frac{25}{18}x\right) = -\frac{x}{18}\left(x^2 - \frac{x}{2} + \frac{25}{6}\right)$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = c_1 + c_2e^{-2x} + c_3e^{3x} - \frac{x}{18}\left(x^2 - \frac{x}{2} + \frac{25}{6}\right)$$

where c_1, c_2 and c_3 are arbitrary constants of integration.

Example 4. *Solve* $(D-2)^2 y = 8(e^{2x} + \sin 2x + x^2)$; $D = \dfrac{d}{dx}$. (Q. Bank U.P.T.U. 2001)

Sol. Auxiliary equation is

$$(m-2)^2 = 0 \Rightarrow m = 2, 2$$

$$\therefore \quad \text{C.F.} = (c_1 + c_2 x)\, e^{2x}$$

$$\text{P.I.} = \frac{1}{(D-2)^2}\,[8(e^{2x} + \sin 2x + x^2)]$$

$$= 8\left[\frac{1}{(D-2)^2}\, e^{2x} + \frac{1}{(D-2)^2} \sin 2x + \frac{1}{(D-2)^2}\, x^2\right]$$

Now $\dfrac{1}{(D-2)^2}\, e^{2x} = x \cdot \dfrac{1}{2(D-2)}\, e^{2x}$ | Case of failure

$$= x^2 \cdot \frac{1}{2}\, e^{2x}$$ | Case of failure

$$= \frac{x^2}{2}\, e^{2x}$$

$$\frac{1}{(D-2)^2} \sin 2x = \frac{1}{D^2 - 4D + 4} \sin 2x = \frac{1}{-2^2 - 4D + 4} \sin 2x$$ [Putting $D^2 = -2^2$]

$$= -\frac{1}{4D} \sin 2x = -\frac{1}{4}\int \sin 2x\, dx = -\frac{1}{4}\left(-\frac{\cos 2x}{2}\right) = \frac{1}{8}\cos 2x$$

$$\frac{1}{(D-2)^2}\, x^2 = \frac{1}{(2-D)^2}\, x^2 = \frac{1}{4\left(1-\frac{D}{2}\right)^2}\, x^2 = \frac{1}{4}\left(1-\frac{D}{2}\right)^{-2} x^2$$

$$= \frac{1}{4}\left[1 + D + \frac{3}{4}D^2 + \ldots\ldots\right] x^2 = \frac{1}{4}\left(x^2 + 2x + \frac{3}{2}\right)$$

$$\therefore \quad \text{P.I.} = 8\left[\frac{x^2}{2}\, e^{2x} + \frac{1}{8}\cos 2x + \frac{1}{4}\left(x^2 + 2x + \frac{3}{2}\right)\right]$$

$$= 4x^2 e^{2x} + \cos 2x + 2x^2 + 4x + 3$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = (c_1 + c_2 x)e^{2x} + 4x^2 e^{2x} + \cos 2x + 2x^2 + 4x + 3$$

where c_1 and c_2 are arbitrary constants of integration.

Example 5. *If* $\dfrac{d^2x}{dt^2} + \dfrac{g}{b}(x-a) = 0$ *; a, b and g are positive numbers and* $x = a'$, $\dfrac{dx}{dt} = 0$ *when* $t = 0$, *Show that* $x = a + (a' - a)\cos\sqrt{\dfrac{g}{b}}\, t$. [U.P.T.U. (SUM) 2007]

Sol. We have $\dfrac{d^2x}{dt^2} + \dfrac{g}{b}x = \dfrac{ag}{b}$

Auxiliary equation is $m^2 + \dfrac{g}{b} = 0 \Rightarrow m = \pm\sqrt{\dfrac{g}{b}}\, i$

$$\therefore \qquad \text{C.F.} = c_1 \cos\sqrt{\frac{g}{b}}\,t + c_2 \sin\sqrt{\frac{g}{b}}\,t$$

$$\text{P.I.} = \frac{1}{D^2 + \frac{g}{b}}\left(\frac{ag}{b}\right) = \frac{ag}{b}\cdot\frac{1}{\frac{g}{b}\left(1 + \frac{bD^2}{g}\right)}(1) = a\left(1 + \frac{bD^2}{g}\right)^{-1}(1) = a$$

$\therefore$ General solution is

$$x = \text{C.F.} + \text{P.I.}$$

$$\Rightarrow \qquad x = c_1 \cos\sqrt{\frac{g}{b}}\,t + c_2 \sin\sqrt{\frac{g}{b}}\,t + a \qquad ...(1)$$

At $t = 0, x = a'$

$\therefore$ From (1), $a' = c_1 + a$

$$\Rightarrow \qquad c_1 = a' - a \qquad ...(2)$$

Now, $$\frac{dx}{dt} = \sqrt{\frac{g}{b}}\left(-c_1 \sin\sqrt{\frac{g}{b}}\,t + c_2 \cos\sqrt{\frac{g}{b}}\,t\right) \qquad ...(3)$$

At $t = 0, \frac{dx}{dt} = 0$

$\therefore$ From (3), $0 = \sqrt{\frac{g}{b}}\cdot c_2$

$$\Rightarrow \qquad c_2 = 0 \qquad ...(4)$$

$\therefore$ From (1), (2) and (4), the complete solution is

$$x = (a' - a)\cos\sqrt{\frac{g}{b}}\,t + a$$

which is the required solution.

Example 6. *Find the solution of the equation $(D^2 - 1)y = 1$ which vanishes when $x = 0$ and tends to a finite limit as $x \to -\infty$ and D stands for $\frac{d}{dx}$.*

Sol. We have $(D^2 - 1)y = 1$

Auxiliary equation is

$$m^2 - 1 = 0 \Rightarrow m = \pm 1$$

$$\therefore \qquad \text{C.F.} = c_1 e^x + c_2 e^{-x}$$

$$\text{P.I.} = \frac{1}{D^2 - 1}(1)$$

$$= -(1 - D^2)^{-1}(1) = -(1 + D^2 + ...)(1) = -1$$

$\therefore$ General solution is

$$y = c_1 e^x + c_2 e^{-x} - 1 \qquad ...(1)$$

when $x = 0, y = 0$

$\therefore$ From (1), $0 = c_1 + c_2 - 1 \Rightarrow c_1 + c_2 = 1 \qquad ...(2)$

Also, y tends to a finite limit as $x \to -\infty$

This condition will be satisfied only when $c_2 = 0$

$\therefore$ From (2), $c_1 = 1$

Hence from (1), Particular solution is $y = e^x - 1$.

TEST YOUR KNOWLEDGE

Solve the following differential equations :

1. $\dfrac{d^2y}{dx^2} - 4y = x^2$
2. $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 3y = \cos x + x^2$
3. $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - 2y = x + \sin x$
4. $\dfrac{d^3y}{dx^3} + 2\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = e^{2x} + x^2 + x$
5. $(D^5 - D)y = 12e^x + 8\sin x - 2x$
6. $\dfrac{d^2y}{dx^2} + y = e^{2x} + \cosh 2x + x^3$ (*Q. Bank U.P.T.U. 2001*)
7. $(D^3 + 8)y = x^4 + 2x + 1$
8. $(D^2 + 2D + 1)y = 2x + x^2$
9. $(D^2 + D - 6)y = x$
10. $(D^3 + 3D^2 + 2D)y = x^2$
11. $(D^6 - D^4)y = x^2$
12. $(D^2 - 1)y = 2x^4 - 3x + 1$

Answers

1. $y = c_1 e^{2x} + c_2 e^{-2x} - \dfrac{1}{4}\left(x^2 + \dfrac{1}{2}\right)$
2. $y = e^x(c_1 \cos\sqrt{2}\,x + c_2 \sin\sqrt{2}\,x) + \dfrac{1}{4}(\cos x - \sin x) + \dfrac{1}{27}(9x^2 + 12x + 2)$
3. $y = c_1 e^x + c_2 e^{-2x} - \dfrac{1}{10}(\cos x + 3\sin x) - \dfrac{1}{4}(2x + 1)$
4. $y = c_1 + (c_2 + c_3 x)e^{-x} + \dfrac{1}{3}x^3 - \dfrac{3}{2}x^2 + 4x + \dfrac{1}{18}e^{2x}$
5. $y = c_1 + (c_2 + 3x)e^x + c_3 e^{-x} + c_4 \cos x + c_5 \sin x + x^2 + 2x\sin x$
6. $y = c_1 \cos x + c_2 \sin x + \dfrac{1}{5}e^{2x} + \dfrac{1}{5}\cosh 2x + x^3 - 6x$
7. $y = c_1 e^{-2x} + e^x(c_2 \cos\sqrt{3}\,x + c_3 \sin\sqrt{3}\,x) + \dfrac{1}{8}(x^4 - x + 1)$
8. $y = (c_1 + c_2 x)e^{-x} + x^2 - 2x + 2$
9. $y = c_1 e^{-3x} + c_2 e^{2x} - \dfrac{1}{6}\left(x + \dfrac{1}{6}\right)$
10. $y = c_1 + c_2 e^{-x} + c_3 e^{-2x} + \dfrac{x}{12}(2x^2 - 9x + 21)$
11. $y = c_1 + c_2 x + c_3 x^2 + c_4 x^3 + c_5 e^x + c_6 e^{-x} - \dfrac{x^4}{12} - \dfrac{x^6}{360}$
12. $y = c_1 e^x + c_2 e^{-x} - 2x^4 - 24x^2 + 3x - 49$

1.18.4. Case IV. When $Q = e^{ax}V$, where V is a function of x

Let u be a function of x, then by successive differentiation, we have

$$D(e^{ax}u) = e^{ax}Du + a\,e^{ax}u = e^{ax}(D + a)u$$

$$D^2(e^{ax}u) = D[e^{ax}(D + a)u] = e^{ax}(D^2 + aD)u + ae^{ax}(D + a)u$$

$$= e^{ax}(D^2 + 2aD + a^2)u = e^{ax}(D + a)^2 u$$

Similarly, $\quad D^3(e^{ax}u) = e^{ax}(D + a)^3 u$

In general, $\quad D^n(e^{ax}u) = e^{ax}(D + a)^n u$

$\therefore \quad f(D)(e^{ax}u) = e^{ax} f(D + a)u$

Operating on both sides by $\frac{1}{f(D)}$,

$$\frac{1}{f(D)}\left[f(D)\left(e^{ax}u\right)\right] = \frac{1}{f(D)}\left[e^{ax} f(D+a)u\right]$$

$$\Rightarrow \qquad e^{ax} u = \frac{1}{f(D)}\left[\left(e^{ax} f(D+a)\, u\right]\right. \qquad \ldots(1)$$

Now let $\quad f(D+a)\, u = V, \quad i.e., \quad u = \frac{1}{f(D+a)} V$

$\therefore$ From (1) we have $e^{ax} \frac{1}{f(D+a)} V = \frac{1}{f(D)} (e^{ax} V)$

or

$$\boxed{\frac{1}{f(D)} (e^{ax} V) = e^{ax} \frac{1}{f(D+a)} V.}$$

Thus e^{ax} which is on the right of $\frac{1}{f(D)}$ may be taken out to the left provided D is replaced by D + a.

ILLUSTRATIVE EXAMPLES

Example 1. *Obtain the general solution of the differential equation*

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 12y = (x-1)\, e^{2x}.$$

[U.P.T.U. (C.O.) 2004]

Sol. The given equation is $(D^2 + 4D - 12)\, y = (x-1)\, e^{2x}$ ...(1)

Auxiliary equation is $\quad m^2 + 4m - 12 = 0$

$\Rightarrow \qquad (m-2)(m+6) = 0$

$\Rightarrow \qquad m = 2, -6$

$\therefore \qquad$ C.F. $= c_1 e^{2x} + c_2 e^{-6x}$

$$\text{P.I.} = \frac{1}{D^2 + 4D - 12} (x-1)\, e^{2x}$$

$$= e^{2x} \cdot \frac{1}{\{(D+2)^2 + 4(D+2) - 12\}} (x-1)$$

$$= e^{2x} \cdot \frac{1}{D^2 + 8D} (x-1) = e^{2x} \cdot \frac{1}{8D}\left(1 + \frac{D}{8}\right)^{-1} (x-1)$$

$$= e^{2x} \cdot \frac{1}{8D}\left(1 - \frac{D}{8}\right)(x-1) \qquad | \text{ Leaving higher power terms}$$

$$= e^{2x} \cdot \frac{1}{8D}\left(x - 1 - \frac{1}{8}\right) = e^{2x} \cdot \frac{1}{8D}\left(x - \frac{9}{8}\right) = \frac{e^{2x}}{8}\left(\frac{x^2}{2} - \frac{9x}{8}\right)$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = c_1 e^{2x} + c_2 e^{-6x} + e^{2x}\left(\frac{x^2}{16} - \frac{9x}{64}\right)$$

where c_1 and c_2 are arbitrary constants of integration.

Example 2. *Find the complete solution of*

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = xe^{3x} + \sin 2x. \qquad \text{(U.P.T.U. 2003)}$$

Sol. Auxiliary equation is $m^2 - 3m + 2 = 0$

$\Rightarrow \qquad (m-1)(m-2) = 0 \quad \Rightarrow \quad m = 1, 2$

$\therefore \qquad \text{C.F.} = c_1e^x + c_2e^{2x}$

$$\text{P.I.} = \frac{1}{D^2 - 3D + 2}(x\,e^{3x} + \sin 2x)$$

$$= \frac{1}{D^2 - 3D + 2}(e^{3x}\cdot x) + \frac{1}{D^2 - 3D + 2}(\sin 2x)$$

$$= e^{3x}\cdot\frac{1}{\{(D+3)^2 - 3(D+3) + 2\}}(x) + \frac{1}{-4 - 3D + 2}(\sin 2x)$$

$$= e^{3x}\cdot\frac{1}{D^2 + 3D + 2}(x) + \frac{1}{-3D - 2}(\sin 2x)$$

$$= e^{3x}\cdot\frac{1}{2\left[1 + \left(\frac{3D + D^2}{2}\right)\right]}(x) - \frac{(3D-2)}{9D^2 - 4}(\sin 2x)$$

$$= \frac{e^{3x}}{2}\cdot\left[1 + \left(\frac{3D + D^2}{2}\right)\right]^{-1}(x) - \frac{(3D-2)}{(-40)}\sin 2x$$

$$= \frac{e^{3x}}{2}\left(1 - \frac{3D}{2}\right)(x) + \frac{1}{40}(6\cos 2x - 2\sin 2x)$$

$$= \frac{e^{3x}}{2}\left(x - \frac{3}{2}\right) + \frac{1}{20}(3\cos 2x - \sin 2x)$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = c_1e^x + c_2e^{2x} + \frac{e^{3x}}{2}\left(x - \frac{3}{2}\right) + \frac{1}{20}(3\cos 2x - \sin 2x)$$

where c_1 and c_2 are arbitrary constants of integration.

Example 3. *Find the P.I. of* $(D^2 - 3D + 2)y = 2e^x\cos\frac{x}{2}$. (Q. Bank U.P.T.U. 2001)

Sol. $$\text{P.I.} = \frac{1}{D^2 - 3D + 2}\left(2e^x\cos\frac{x}{2}\right)$$

$$= 2e^x\cdot\frac{1}{\{(D+1)^2 - 3(D+1) + 2\}}\cos\frac{x}{2} = 2e^x\cdot\frac{1}{D^2 - D}\cos\frac{x}{2}$$

$$= 2e^x\cdot\frac{1}{-\frac{1}{4} - D}\cos\frac{x}{2} = -2e^x\left[\frac{\left(\frac{1}{4} - D\right)}{\left(\frac{1}{4} - D\right)\left(\frac{1}{4} + D\right)}\cos\frac{x}{2}\right]$$

$$= -2e^x \frac{\left(\frac{1}{4} - D\right)}{\frac{1}{16} - D^2} \cos \frac{x}{2} = -2e^x \cdot \frac{\frac{1}{4} - D}{\left(\frac{1}{16} + \frac{1}{4}\right)} \cos \frac{x}{2}$$

$$= -\frac{32}{5} e^x \left(\frac{1}{4} \cos \frac{x}{2} + \frac{1}{2} \sin \frac{x}{2}\right) = -\frac{16}{5} e^x \left(\sin \frac{x}{2} + \frac{1}{2} \cos \frac{x}{2}\right).$$

Example 4. *Obtain the general solution of the differential equation*

$$y'' - 2y' + 2y = x + e^x \cos x.$$ (U.P.T.U. 2

Sol. The given equation is

$$(D^2 - 2D + 2)y = x + e^x \cos x \; ; \; D \equiv \frac{d}{dx}$$

Auxiliary equation is,

$$m^2 - 2m + 2 = 0$$

$$m = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$$

$$\therefore \quad \text{C.F.} = e^x (c_1 \cos x + c_2 \sin x)$$

$$\text{P.I.} = \frac{1}{D^2 - 2D + 2} (x + e^x \cos x)$$

$$= \frac{1}{D^2 - 2D + 2} (x) + \frac{1}{D^2 - 2D + 2} (e^x \cos x)$$

$$= \frac{1}{2 - 2D + D^2} (x) + e^x \cdot \frac{1}{(D+1)^2 - 2(D+1) + 2} (\cos x)$$

$$= \frac{1}{2} \left[1 - \left(\frac{2D - D^2}{2}\right)\right]^{-1} (x) + e^x \cdot \frac{1}{D^2 + 1} \cos x$$

$$= \frac{1}{2} \left[1 + \left(\frac{2D - D^2}{2}\right)\right] (x) + e^x \cdot x \cdot \frac{1}{2D} \cos x$$ | Case of fai

$$= \frac{1}{2} [1 + D] (x) + e^x \cdot \frac{x}{2} \sin x$$ | Leaving higher pow

$$= \frac{1}{2} (x + 1) + \frac{xe^x}{2} \sin x$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = e^x (c_1 \cos x + c_2 \sin x) + \frac{1}{2} (x + 1) + \frac{xe^x}{2} \sin x$$

where c_1 and c_2 are arbitrary constants of integration.

Example 5. *Solve* : $\frac{d^2y}{dx^2} - 4y = x \sinh x$.

Sol. Given equation is $(D^2 - 4)y = x \sinh x \; ; \; D \equiv \frac{d}{dx}$

Auxiliary equation is $m^2 - 4 = 0$ so that $m = \pm 2$

$\therefore$ C.F. $= c_1 e^{2x} + c_2 e^{-2x}$

$$\text{P.I.} = \frac{1}{D^2-4} x \sinh x = \frac{1}{D^2-4} x \left(\frac{e^x - e^{-x}}{2}\right)$$

$$= \frac{1}{2}\left[\frac{1}{D^2-4} e^x . x - \frac{1}{D^2-4} e^{-x} . x\right] = \frac{1}{2}\left[e^x \frac{1}{(D+1)^2-4} x - e^{-x} \frac{1}{(D-1)^2-4} x\right]$$

$$= \frac{1}{2}\left[e^x \frac{1}{D^2+2D-3} x - e^{-x} \frac{1}{D^2-2D-3} x\right]$$

$$= \frac{1}{2}\left[e^x \frac{1}{-3\left(1-\frac{2D}{3}-\frac{D^2}{3}\right)} x - e^{-x} \frac{1}{-3\left(1+\frac{2D}{3}-\frac{D^2}{3}\right)} x\right]$$

$$= -\frac{1}{6}\left[e^x \left\{1-\left(\frac{2D}{3}+\frac{D^2}{3}\right)\right\}^{-1} x - e^{-x}\left\{1+\left(\frac{2D}{3}-\frac{D^2}{3}\right)\right\}^{-1} x\right]$$

$$= -\frac{1}{6}\left[e^x \left(1+\frac{2D}{3}\ldots\ldots\right) x - e^{-x}\left(1-\frac{2D}{3}\ldots\ldots\right) x\right] = -\frac{1}{6}\left[e^x\left(x+\frac{2}{3}\right) - e^{-x}\left(x-\frac{2}{3}\right)\right]$$

$$= -\frac{x}{3}\left(\frac{e^x-e^{-x}}{2}\right) - \frac{2}{9}\left(\frac{e^x+e^{-x}}{2}\right) = -\frac{x}{3}\sinh x - \frac{2}{9}\cosh x$$

Hence the complete solution is $y = \text{C.F.} + \text{P.I.} = c_1 e^{2x} + c_2 e^{-2x} - \frac{x}{3}\sinh x - \frac{2}{9}\cosh x$ where c_1 and c_2 are arbitrary constants of integration.

Example 6. *Solve* : $\frac{d^4 y}{dx^4} - y = \cos x \cosh x.$ (Q. Bank U.P.T.U. 2001)

Sol. Given equation is $(D^4 - 1)y = \cos x \cosh x$, where $D \equiv \frac{d}{dx}$.

Auxiliary equation is

$m^4 - 1 = 0$ or $(m^2-1)(m^2+1) = 0$ so that $m = \pm 1, \pm i$

$$\therefore \quad \text{C.F.} = c_1 e^x + c_2 e^{-x} + e^{0x}(c_3 \cos x + c_4 \sin x)$$
$$= c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x$$

$$\text{P.I.} = \frac{1}{D^4-1}\cos x \cosh x = \frac{1}{D^4-1}\cos x\left(\frac{e^x+e^{-x}}{2}\right)$$

$$= \frac{1}{2}\left[\frac{1}{D^4-1} e^x \cos x + \frac{1}{D^4-1} e^{-x}\cos x\right]$$

$$= \frac{1}{2}\left[e^x \frac{1}{(D+1)^4-1}\cos x + e^{-x}\frac{1}{(D-1)^4-1}\cos x\right]$$

$$= \frac{1}{2}\left[e^x \frac{1}{D^4+4D^3+6D^2+4D}\cos x + e^{-x}\frac{1}{D^4-4D^3+6D^2-4D}\cos x\right]$$

$$= \frac{1}{2}\left[e^x \frac{1}{(-1^2)^2 + 4D(-1^2) + 6(-1^2) + 4D} \cos x \right.$$

$$\left. + e^{-x} \frac{1}{(-1^2)^2 - 4D(-1^2) + 6(-1^2) - 4D} \cos x\right]$$

$$= \frac{1}{2}\left[e^x \frac{1}{-5} \cos x + e^{-x} \frac{1}{-5} \cos x\right] = -\frac{1}{5}\left(\frac{e^x + e^{-x}}{2}\right) \cos x = -\frac{1}{5} \cosh x \cos x$$

Hence complete solution is

$$y = \text{C.F.} + \text{P.I.} = c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x - \frac{1}{5} \cos x \cosh x$$

where c_1, c_2, c_3 and c_4 are arbitrary constants of integration.

Example 7. (*i*) *Solve:* $(D^2 - 2D + 1)y = x\, e^x \sin x$ (U.P.T.U. 2005)

(*ii*) *Solve:* $\dfrac{d^2 y}{dx^2} - 2\dfrac{dy}{dx} + y = xe^x \cos x.$ (U.P.T.U. 2006)

Sol. (*i*) Auxiliary equation is

$$m^2 - 2m + 1 = 0$$

$$\Rightarrow \quad m = 1, 1$$

$$\therefore \quad \text{C.F.} = (c_1 + c_2 x)e^x$$

$$\text{P.I.} = \frac{1}{D^2 - 2D + 1}(x\, e^x \sin x) = \frac{1}{(D-1)^2}(x\, e^x \sin x)$$

$$= e^x \cdot \frac{1}{(D+1-1)^2}(x \sin x) = e^x \cdot \frac{1}{D^2}(x \sin x)$$

$$= e^x \cdot \frac{1}{D}(-x \cos x + \sin x) = -e^x (x \sin x + 2 \cos x)$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = (c_1 + c_2 x)\, e^x - e^x (x \sin x + 2 \cos x)$$

where c_1 and c_2 are arbitrary constants of integration.

(*ii*) Auxiliary equation is

$$m^2 - 2m + 1 = 0 \qquad \Rightarrow \quad m = 1, 1$$

$$\therefore \quad \text{C.F.} = (c_1 + c_2 x)e^x$$

$$\text{P.I.} = \frac{1}{D^2 - 2D + 1}(x\, e^x \cos x) = \frac{1}{(D-1)^2}(x e^x \cos x)$$

$$= e^x \cdot \frac{1}{(D+1-1)^2}(x \cos x) = e^x \cdot \frac{1}{D^2}(x \cos x)$$

$$= e^x \cdot \frac{1}{D}(x \sin x + \cos x) = e^x (-x \cos x + 2 \sin x)$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = (c_1 + c_2 x)\, e^x + e^x (-x \cos x + 2 \sin x)$$

where c_1 and c_2 are arbitrary constants of integration.

Example 8. *Solve* $(D^4 + 6D^3 + 11D^2 + 6D)y = 20\, e^{-2x} \sin x$ [U.P.T.U. (C.O.) 2005]

Sol. Auxiliary equation is

$$m^4 + 6m^3 + 11m^2 + 6m = 0$$

$$\Rightarrow \quad m(m^3 + 6m^2 + 11m + 6) = 0$$

$$\Rightarrow \quad m = 0, -1, -2, -3$$

$$\therefore \quad \text{C.F.} = c_1 + c_2 e^{-x} + c_3 e^{-2x} + c_4 e^{-3x}$$

$$\text{P.I.} = \frac{1}{D^4 + 6D^3 + 11D^2 + 6D} (20\, e^{-2x} \sin x)$$

$$= \frac{1}{D(D+1)(D+2)(D+3)} (20\, e^{-2x} \sin x)$$

$$= 20\, e^{-2x} \cdot \frac{1}{(D-2)(D-1)D(D+1)} (\sin x)$$

$$= 20\, e^{-2x} \cdot \frac{1}{D^4 - 2D^3 - D^2 + 2D} (\sin x)$$

$$= 20\, e^{-2x} \cdot \frac{1}{2 + 4D} \sin x$$

$$= 10\, e^{-2x} \frac{1 - 2D}{1 - 4D^2} \sin x$$

$$= 2\, e^{-2x} (\sin x - 2 \cos x)$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = c_1 + c_2 e^{-x} + c_3 e^{-2x} + c_4 e^{-3x} + 2e^{-2x}(\sin x - 2 \cos x)$$

where c_1, c_2, c_3 and c_4 are arbitrary constants of integration.

Example 9. *Solve the differential equation*

$$\frac{d^2 y}{dx^2} + 2 \frac{dy}{dx} + y = \frac{e^{-x}}{x + 2}.$$

Sol. Auxiliary equation is

$$m^2 + 2m + 1 = 0$$

$$\Rightarrow \quad (m + 1)^2 = 0 \quad \Rightarrow \quad m = -1, -1$$

$$\therefore \quad \text{C.F.} = (c_1 + c_2 x)\, e^{-x}$$

$$\text{P.I.} = \frac{1}{(D+1)^2} \left(\frac{e^{-x}}{x+2} \right) = e^{-x} \cdot \frac{1}{(D-1+1)^2} \left(\frac{1}{x+2} \right)$$

$$= e^{-x} \frac{1}{D^2} \left(\frac{1}{x+2} \right) = e^{-x} \cdot \frac{1}{D} \log (x+2)$$

$$= e^{-x} \left[\log (x+2) \, . \, x - \int \frac{1}{x+2} \, . \, x \, dx \right] = e^{-x} \left[x \log (x+2) - \int \left(1 - \frac{2}{x+2} \right) dx \right]$$

$$= e^{-x} [x \log (x+2) - x + 2 \log (x+2)] = e^{-x} [(x+2) \log (x+2) - x]$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = (c_1 + c_2 x) e^{-x} + e^{-x} [(x+2) \log (x+2) - x]$$

where c_1 and c_2 are arbitrary constants of integration.

Example 10. *Solve :* $(D^2 + 2D + 1)y = x \cos x$.

Sol. Auxiliary equation is

$$m^2 + 2m + 1 = 0$$

$$\Rightarrow \quad m = -1, -1$$

$$\text{C.F.} = (c_1 + c_2 x)\, e^{-x}$$

$$\text{P.I.} = \frac{1}{D^2 + 2D + 1}(x \cos x) = \text{Real part of } \frac{1}{D^2 + 2D + 1}(xe^{ix})$$

$$= \text{R.P. of } e^{ix} \cdot \frac{1}{(D+i)^2 + 2(D+i) + 1}(x) = \text{R.P. of } e^{ix} \cdot \frac{1}{D^2 + 2D(1+i) + 2i}(x)$$

$$= \text{R.P. of } \frac{e^{ix}}{2i}\left[1 + \frac{1+i}{i} D + \frac{D^2}{2i}\right]^{-1}(x)$$

$$= \text{R.P. of } \frac{e^{ix}}{2i}\left(1 - \frac{1+i}{i} D\right) x \qquad | \text{ Leaving higher powers}$$

$$= \text{R.P. of } \frac{e^{ix}}{2i}\left(x - \frac{1+i}{i}\right)$$

$$= \text{R.P. of } \frac{1}{2}(\cos x + i \sin x)(-ix + 1 + i) = \frac{1}{2}\cos x + \frac{1}{2}(x-1)\sin x$$

$\therefore$ The complete solution is given by

$$y = \text{C.F.} + \text{P.I.} = (c_1 + c_2 x)\, e^{-x} + \frac{1}{2}\cos x + \frac{1}{2}(x-1)\sin x$$

where c_1 and c_2 are the arbitrary constants of integration.

Example 11. *Solve the following differential equation :*

$$(D^2 - 4D + 4)\, y = 8x^2 e^{2x} \sin 2x.$$

[U.P.T.U. 2004 ; U.P.T.U. (B. Pharm.) 2005]

Sol. The auxiliary equation is

$$m^2 - 4m + 4 = 0 \Rightarrow m = 2, 2$$

$$\therefore \quad \text{C.F.} = (c_1 + c_2 x)\, e^{2x}$$

$$\text{P.I.} = \frac{1}{(D-2)^2}(8x^2 e^{2x} \sin 2x) = 8\, e^{2x} \cdot \frac{1}{(D+2-2)^2}(x^2 \sin 2x)$$

$$= 8\, e^{2x} \cdot \frac{1}{D^2}(x^2 \sin 2x) = 8\, e^{2x} \cdot \frac{1}{D}\int x^2 \sin 2x\, dx$$

$$= 8\, e^{2x} \cdot \frac{1}{D}\left[x^2 \cdot \left(-\frac{\cos 2x}{2}\right) - \int 2x \cdot \left(\frac{-\cos 2x}{2}\right) dx\right]$$

$$= 8\, e^{2x} \cdot \frac{1}{D}\left[\frac{-x^2}{2}\cos 2x + x \cdot \frac{\sin 2x}{2} - \int 1 \cdot \frac{\sin 2x}{2}\, dx\right]$$

$$= 8\, e^{2x} \cdot \frac{1}{D}\left[\frac{-x^2}{2}\cos 2x + \frac{x \sin 2x}{2} + \frac{\cos 2x}{4}\right]$$

$$= 8\, e^{2x} \cdot \left[\left(\frac{-x^2}{2}\right)\frac{\sin 2x}{2} - \int (-x)\frac{\sin 2x}{2}\, dx + \int x \frac{\sin 2x}{2}\, dx + \frac{\sin 2x}{8}\right]$$

$$= 8\, e^{2x}\left[\frac{-x^2}{4}\sin 2x + \frac{\sin 2x}{8} + \int x \sin 2x\, dx\right]$$

$$= 8\, e^{2x}\left[\left(\frac{1}{8} - \frac{x^2}{4}\right)\sin 2x + x \cdot \left(\frac{-\cos 2x}{2}\right) - \int 1 \cdot \left(\frac{-\cos 2x}{2}\right) dx\right]$$

$$= 8\, e^{2x}\left[\left(\frac{1}{8} - \frac{x^2}{4}\right)\sin 2x - \frac{x}{2}\cos 2x + \frac{\sin 2x}{4}\right]$$

$$= 8\, e^{2x}\left[\left(\frac{3}{8} - \frac{x^2}{4}\right)\sin 2x - \frac{x}{2}\cos 2x\right]$$

$$= e^{2x}\left[(3 - 2x^2)\sin 2x - 4x\cos 2x\right]$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = (c_1 + c_2 x)\, e^{2x} + e^{2x}[(3 - 2x^2)\sin 2x - 4x\cos 2x]$$

where c_1 and c_2 are arbitrary constants of integration.

Example 12. *A body executes damped forced vibrations given by the equation*

$$\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + b^2 x = e^{-kt}\sin \omega t.$$

Solve the equation for both the cases, when $\omega^2 \neq b^2 - k^2$ and $\omega^2 = b^2 - k^2$.

[U.P.T.U. 2001, 2003, 2004 ; U.P.T.U. (C.O.) 2005]

Sol. Auxiliary equation is

$$m^2 + 2km + b^2 = 0 \;\Rightarrow\; m = -k \pm \sqrt{k^2 - b^2}$$

For damped force vibrations

$$k^2 < b^2$$

$$\therefore \quad m = -k \pm i\sqrt{b^2 - k^2}$$

Case I. When $\omega^2 \neq b^2 - k^2$

$$\therefore \quad \text{C.F.} = e^{-kt}\{c_1 \cos\sqrt{b^2 - k^2}\, t + c_2 \sin\sqrt{b^2 - k^2}\, t\}$$

$$\text{P.I.} = \frac{1}{D^2 + 2kD + b^2}(e^{-kt}\sin \omega t)$$

$$= e^{-kt} \cdot \frac{1}{(D - k)^2 + 2k(D - k) + b^2}\sin \omega t$$

$$= e^{-kt} \cdot \frac{1}{D^2 + (b^2 - k^2)}\sin \omega t = e^{-kt} \cdot \frac{1}{-\omega^2 + b^2 - k^2}\sin \omega t$$

$$= \frac{e^{-kt}}{b^2 - k^2 - \omega^2} \sin \omega t$$

Hence complete solution is

$$x = e^{-kt} \{c_1 \cos \sqrt{b^2 - k^2}\, t + c_2 \sin \sqrt{b^2 - k^2}\, t\} + \frac{e^{-kt}}{b^2 - k^2 - \omega^2} \sin \omega t$$

where c_1 and c_2 are arbitrary constants of integration.

Case II. When $\omega^2 = b^2 - k^2$

$$\text{C.F.} = e^{-kt} (c_1 \cos \omega t + c_2 \sin \omega t)$$

$$\text{P.I.} = \frac{1}{D^2 + 2kD + b^2} (e^{-kt} \sin \omega t)$$

$$= e^{-kt} \cdot \frac{1}{(D - k)^2 + 2k(D - k) + b^2} \sin \omega t$$

$$= e^{-kt} \cdot \frac{1}{D^2 + b^2 - k^2} \sin \omega t$$

$$= e^{-kt} \cdot t \cdot \frac{1}{2D} \sin \omega t \qquad | \text{ Case of fai}$$

$$= \frac{te^{-kt}}{2} \left(\frac{-\cos \omega t}{\omega} \right) = -\frac{t}{2\omega} e^{-kt} \cos \omega t$$

Hence complete solution is

$$x = \text{C.F.} + \text{P.I.} = e^{-kt} (c_1 \cos \omega t + c_2 \sin \omega t) - \frac{t}{2\omega} e^{-kt} \cos \omega t$$

where c_1 and c_2 are arbitrary constants of integration.

TEST YOUR KNOWLEDGE

Solve the following differential equations :

1. $(D - a)^2 y = e^{ax} f''(x)$
2. $(D^2 - 2D)y = e^x \sin x$
3. $(D^2 - 2D + 4)\, y = e^x \cos x$
4. $(D^2 + 3D + 2)\, y = e^{2x} \sin x$
5. $(D^2 - 4D + 4)\, y = x^2 + e^x + \cos 2x$
6. $\frac{d^2y}{dx^2} + y = e^{-x} + \cos x + x^3 + e^x \sin x$
7. (*i*) $(D^2 - 3D + 2)\, y = xe^x + \sin 2x$ (*U.P.T.U. 2008*)
   (*ii*) $(D^2 - 1)\, y = xe^x + \cos^2 x$ [*U.P.T.U. (SUM) 20*
8. $(D - 1)^2 (D^2 + 1)^2 y = \sin^2 \frac{x}{2} + e^x + x$
9. $(D^2 - 6D + 13)y = 8e^{3x} \sin 4x + 2^x$
10. $(D^2 + 4)y = e^x \sin^2 x$
11. $(D^3 + 2D^2 + D)y = x^2 e^{2x} + \sin^2 x$
12. $(D^2 - 4D + 3)y = e^x \cos 2x + \cos 3x$
13. $(D^2 + 4D + 8)y = 12e^{-2x} \sin x \sin 3x$ (*Q. Bank U.P.T.U. 20*
14. $\frac{d^2y}{dx^2} + 2y = x^2 e^{3x} + e^x \cos 2x$
15. $(D - 1)^2 y = e^x \sec^2 x \tan x$
16. $(D^3 - 7D - 6)y = (x + 1)\, e^{2x}$
17. $(D^2 - 1)\, y = x \sin x + x^2 e^x$
18. $(D^2 - 1)\, y = x^2 \cos x$ [*U.P.T.U. (SUM) 200*

Answers

1. $y = e^{ax}[c_1 + c_2 x + f(x)]$

2. $y = c_1 + c_2 e^{2x} - \frac{1}{2} e^{x} \sin x$

3. $y = e^{x}(c_1 \cos\sqrt{3}\,x + c_2 \sin\sqrt{3}\,x) + \frac{1}{2} e^{x} \cos x$

4. $y = c_1 e^{-2x} + c_2 e^{-x} - \frac{e^{2x}}{170}(7\cos x - 11 \sin x)$

5. $y = (c_1 + c_2 x) e^{2x} + \frac{1}{8}(2x^2 + 4x + 3) - \frac{1}{8} \sin 2x + e^{x}$

6. $y = c_1 \cos x + c_2 \sin x + \frac{1}{2} e^{-x} + \frac{1}{2} x \sin x + x^3 - 6x - \frac{1}{5} e^{x}(2\cos x - \sin x)$

7. (i) $y = c_1 e^{x} + c_2 e^{2x} - e^{x}\left(\frac{x^2}{2} + x\right) + \frac{1}{20}(3\cos 2x - \sin 2x)$

   (ii) $y = c_1 e^{x} + c_2 e^{-x} + \frac{1}{4} e^{x}(x^2 - x) - \frac{1}{2} - \frac{1}{10} \cos 2x$

8. $y = (c_1 + c_2 x)e^{x} + (c_3 + c_4 x)\cos x + (c_5 + c_6 x)\sin x + \frac{1}{2} - \frac{x^2}{32} \sin x + \frac{x^2}{8} e^{x} + x + 2$

9. $y = e^{3x}(c_1 \cos 2x + c_2 \sin 2x) - \frac{2}{3} e^{3x} \sin 4x + \frac{2^x}{(\log 2)^2 - 6\log 2 + 13}$

10. $y = c_1 \cos 2x + c_2 \sin 2x + \frac{e^{x}}{10} - \frac{e^{x}}{34}(4 \sin 2x + \cos 2x)$

11. $y = c_1 + (c_2 + c_3 x)e^{-x} + \frac{e^{2x}}{18}\left(x^2 - \frac{7}{3}x + \frac{11}{6}\right) + \frac{1}{100}(3 \sin 2x + 4\cos 2x) + \frac{x}{2}$

12. $y = c_1 e^{x} + c_2 e^{3x} - \frac{1}{8} e^{x}(\sin 2x + \cos 2x) - \frac{1}{30}(2 \sin 3x + \cos 3x)$

13. $y = e^{-2x}(c_1 \cos 2x + c_2 \sin 2x) + \frac{1}{2} e^{-2x}(3x \sin 2x + \cos 4x)$

14. $y = c_1 \cos\sqrt{2}\,x + c_2 \sin\sqrt{2}\,x + \frac{e^{3x}}{11}\left(x^2 - \frac{12}{11}x + \frac{50}{121}\right) + \frac{e^{x}}{17}(4 \sin 2x - \cos 2x)$

15. $y = e^{x}\left(c_1 + c_2 x + \frac{1}{2}\tan x\right)$

16. $y = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{3x} - \frac{e^{2x}}{144}(12x + 17)$

17. $y = c_1 e^{x} + c_2 e^{-x} - \frac{1}{2}(x \sin x + \cos x) + \frac{x e^{x}}{12}(2x^2 - 3x + 3).$

18. $y = c_1 e^{x} + c_2 e^{-x} + x \sin x + \left(\frac{1 - x^2}{2}\right) \cos x.$

1.18.5. Case V. When Q is any other function of x

Resolve $f(D)$ into linear factors.

Let $\quad f(D) \equiv (D - m_1)(D - m_2) \ldots\ldots (D - m_n)$

Then $\quad \text{P.I.} = \frac{1}{f(D)} Q = \frac{1}{(D - m_1)(D - m_2)\ldots\ldots(D - m_n)} Q$

$$= \left(\frac{A_1}{D - m_1} + \frac{A_2}{D - m_2} + \ldots\ldots + \frac{A_n}{D - m_n}\right) Q \qquad \text{(Partial Fractions)}$$

$$= A_1 \frac{1}{D - m_1} Q + A_2 \frac{1}{D - m_2} Q + \ldots\ldots + A_n \frac{1}{D - m_n} Q$$

$$= A_1 e^{m_1 x} \int Q e^{-m_1 x} dx + A_2 e^{m_2 x} \int Q e^{-m_2 x} dx + \ldots\ldots + A_n e^{m_n x} \int Q e^{-m_n x} dx$$

Remark. Remember the following formulae :

(i) $\frac{1}{D - \alpha} Q = e^{\alpha x} \int e^{-\alpha x} Q\, dx$

(ii) $\frac{1}{D + \alpha} Q = e^{-\alpha x} \int e^{\alpha x} Q\, dx$

ILLUSTRATIVE EXAMPLES

Example 1. *Find the particular integral of* $(D^2 + a^2)y = \sec ax$.

(Q. Bank U.P.T.U. 2001)

Sol. $$\text{P.I.} = \frac{1}{D^2 + a^2} \sec ax$$

$$= \frac{1}{(D - ia)(D + ia)} \sec ax = \frac{1}{2ia}\left[\frac{1}{D - ia} - \frac{1}{D + ia}\right] \sec ax$$

$$= \frac{1}{2ia}\left[\frac{1}{D - ia}(\sec ax) - \frac{1}{D + ia}(\sec ax)\right] = \frac{1}{2ia}(P_1 - P_2)$$

where $$P_1 = \frac{1}{D - ia}(\sec ax)$$

$$= e^{iax} \int e^{-iax} \sec ax\, dx = e^{iax} \int (\cos ax - i \sin ax) \sec ax\, dx$$

$$= e^{iax} \int (1 - i \tan ax)\, dx = e^{iax} \left\{x + i\left(\frac{\log \cos ax}{a}\right)\right\}$$

$$P_2 = \frac{1}{D + ia}(\sec ax) = e^{-iax} \int e^{iax} \sec ax\, dx$$

$$= e^{-iax} \int (\cos ax + i \sin ax) \sec ax\, dx$$

$$= e^{-iax} \int (1 + i \tan ax)\, dx = e^{-iax} \left\{x - i\left(\frac{\log \cos ax}{a}\right)\right\}$$

$$\therefore \quad \text{P.I.} = \frac{1}{2ia}\left[e^{iax}\left\{x + i\left(\frac{\log \cos ax}{a}\right)\right\} - e^{-iax}\left\{x - i\left(\frac{\log \cos ax}{a}\right)\right\}\right]$$

$$= \frac{1}{2ia}\left[x(e^{iax} - e^{-iax}) + i\left(\frac{\log \cos ax}{a}\right)(e^{iax} + e^{-iax})\right]$$

$$= \frac{1}{2ia}\left[2ix \sin ax + \frac{i}{a}(\log \cos ax)(2 \cos ax)\right] = \frac{1}{a}\left[x \sin ax + \frac{1}{a} \cos ax \log \cos ax\right].$$

Example 2. *Find the general solution of the differential equation*

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^{e^x}.$$

Sol. Auxiliary equation is

$$m^2 + 3m + 2 = 0$$

$$\Rightarrow \quad (m+1)(m+2) = 0 \quad \Rightarrow \quad m = -1, -2$$

$$\therefore \quad \text{C.F.} = c_1 e^{-x} + c_2 e^{-2x}$$

$$\text{P.I.} = \frac{1}{(D+1)(D+2)} e^{e^x} = \left(\frac{1}{D+1} - \frac{1}{D+2}\right) e^{e^x}$$

$$= \frac{1}{D+1}(e^{e^x}) - \frac{1}{D+2}(e^{e^x}) = P_1 - P_2$$

where $P_1 = \frac{1}{D+1}(e^{e^x}) = e^{-x} \int e^x e^{e^x} dx = e^{-x} \int e^z dz$ | Put $e^x = z \quad \therefore \quad e^x dx = dz$

$$= e^{-x} e^{e^x}$$

$P_2 = \frac{1}{D+2}(e^{e^x}) = e^{-2x} \int e^{2x} e^{e^x} dx = e^{-2x} \int z e^z dz,$ where $e^x = z \ \therefore \ e^x dx = dz$

$$= e^{-2x}(z-1)e^z = e^{-2x}(e^x - 1) e^{e^x} = (e^{-x} - e^{-2x}) e^{e^x}$$

$$\therefore \quad \text{P.I.} = e^{-x} e^{e^x} - (e^{-x} - e^{-2x}) e^{e^x} = e^{-2x} e^{e^x}$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = c_1 e^{-x} + c_2 e^{-2x} + e^{-2x} e^{e^x}$$

where c_1 and c_2 are arbitrary constants of integration.

Example 3. *Solve* : $\frac{d^2y}{dx^2} + y = \operatorname{cosec} x.$

Sol. Given equation is $(D^2 + 1)y = \operatorname{cosec} x$

Auxiliary equation is

$$m^2 + 1 = 0 \Rightarrow m = \pm i$$

$$\therefore \quad \text{C.F.} = c_1 \cos x + c_2 \sin x$$

$$\text{P.I.} = \frac{1}{D^2+1} \operatorname{cosec} x = \frac{1}{(D+i)(D-i)} \operatorname{cosec} x$$

$$= \frac{1}{2i}\left(\frac{1}{D-i} - \frac{1}{D+i}\right) \operatorname{cosec} x \qquad \text{(Partial Fractions)}$$

$$= \frac{1}{2i}\left(\frac{1}{D-i} \operatorname{cosec} x - \frac{1}{D+i} \operatorname{cosec} x\right)$$

Now $\frac{1}{D-i} \operatorname{cosec} x = e^{ix} \int \operatorname{cosec} x \, e^{-ix} dx$

$$= e^{ix} \int \operatorname{cosec} x (\cos x - i \sin x) dx = e^{ix} \int (\cot x - i) dx$$

$$= e^{ix} (\log \sin x - ix)$$

Changing i to $-i$, we have $\frac{1}{D+i} \operatorname{cosec} x = e^{-ix} (\log \sin x + ix)$

$$\therefore \quad \text{P.I.} = \frac{1}{2i} [e^{ix} (\log \sin x - ix) - e^{-ix} (\log \sin x + ix)]$$

$$= \log \sin x \left(\frac{e^{ix} - e^{-ix}}{2i}\right) - x\left(\frac{e^{ix} + e^{-ix}}{2}\right)$$

$$= (\log \sin x) . \sin x - x \cos x$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = c_1 \cos x + c_2 \sin x + \sin x \log \sin x - x \cos x$$

where c_1 and c_2 are arbitrary constants of integration.

Example 4. *Solve* : $\frac{d^2y}{dx^2} + a^2y = \tan ax.$ (Q. Bank U.P.T.U. 2

Sol. Given equation is $(D^2 + a^2)y = \tan ax$

Auxiliary equation is $m^2 + a^2 = 0 \Rightarrow m = \pm ia$

$$\therefore \quad \text{C.F.} = c_1 \cos ax + c_2 \sin ax$$

$$\text{P.I.} = \frac{1}{D^2 + a^2} \tan ax = \frac{1}{(D + ia)(D - ia)} \tan ax$$

$$= \frac{1}{2ia}\left[\frac{1}{D - ia} - \frac{1}{D + ia}\right] \tan ax \qquad \text{(Partial Fracti}$$

$$= \frac{1}{2ia}\left[\frac{1}{D - ia} \tan ax - \frac{1}{D + ia} \tan ax\right]$$

Now $\frac{1}{D - ia} \tan ax = e^{iax} \int \tan ax . e^{-iax}\, dx$

$$= e^{iax} \int \tan ax (\cos ax - i \sin ax) = e^{iax} \int \left(\sin ax - i \frac{\sin^2 ax}{\cos ax}\right) dx$$

$$= e^{iax} \int \left(\sin ax - i \frac{1 - \cos^2 ax}{\cos ax}\right) dx = e^{iax} \int [\sin ax - i(\sec ax - \cos ax)]\, dx$$

$$= e^{iax}\left[-\frac{\cos ax}{a} - \frac{i}{a} \log (\sec ax + \tan ax) + i \frac{\sin ax}{a}\right]$$

$$= -\frac{1}{a} e^{iax} [(\cos ax - i \sin ax) + i \log (\sec ax + \tan ax)]$$

$$= -\frac{1}{a} e^{iax} [e^{-iax} + i \log (\sec ax + \tan ax)] = -\frac{1}{a} [1 + ie^{iax} \log (\sec ax + \tan ax)]$$

Changing i to $-i$, we have $\frac{1}{D + ia} \tan ax = -\frac{1}{a} [1 - ie^{-iax} \log (\sec ax + \tan ax)]$

$$\therefore \quad \text{P.I.} = \frac{1}{2ia}\left[-\frac{1}{a}\{1 + ie^{iax} \log (\sec ax + \tan ax)\} + \frac{1}{a}\{1 - ie^{-iax} \log (\sec ax + \tan ax)\}\right]$$

$$= -\frac{1}{a^2} \log (\sec ax + \tan ax)\left(\frac{e^{iax} + e^{-iax}}{2}\right) = -\frac{1}{a^2} \log (\sec ax + \tan ax) . \cos ax$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = c_1 \cos ax + c_2 \sin ax - \frac{1}{a^2} \cos ax \log (\sec ax + \tan ax)$$

where c_1 and c_2 are arbitrary constants of integration.

Example 5. *Solve the differential equation*

$$\frac{d^2y}{dx^2} + y = x - \cot x.$$

Sol. Auxiliary equation is

$$m^2 + 1 = 0 \Rightarrow m = \pm i$$

$$\therefore \quad \text{C.F.} = c_1 \cos x + c_2 \sin x$$

$$\text{P.I.} = \frac{1}{D^2+1}(x - \cot x) = \frac{1}{D^2+1}(x) - \frac{1}{(D-i)(D+i)} \cot x$$

$$= (1 + D^2)^{-1}(x) - \frac{1}{2i}\left[\frac{1}{D-i} - \frac{1}{D+i}\right] \cot x$$

$$= (1 - D^2)(x) - \frac{1}{2i}\left[\frac{1}{D-i}(\cot x) - \frac{1}{D+i}(\cot x)\right]$$

$$= x - \frac{1}{2i}(P_1 - P_2) \qquad ...(1)$$

Now,
$$P_1 = \frac{1}{D-i} \cot x = e^{ix} \int e^{-ix} \cot x \, dx$$

$$= e^{ix} \int (\cos x - i \sin x) \cot x \, dx = e^{ix} \int \left(\frac{\cos^2 x}{\sin x} - i \cos x\right) dx$$

$$= e^{ix} \int (\operatorname{cosec} x - \sin x - i \cos x) \, dx$$

$$= e^{ix} [\log(\operatorname{cosec} x - \cot x) + \cos x - i \sin x]$$

$$= e^{ix} [\log(\operatorname{cosec} x - \cot x) + e^{-ix}]$$

$$P_2 = \frac{1}{D+i} \cot x = e^{-ix} [\log(\operatorname{cosec} x - \cot x) + e^{ix}]$$

$\therefore$ From (1),

$$\text{P.I.} = x - \frac{1}{2i} [e^{ix} \{\log(\operatorname{cosec} x - \cot x) + e^{-ix}\} - e^{-ix} \{\log(\operatorname{cosec} x - \cot x) + e^{ix}\}]$$

$$= x - \frac{1}{2i} [(e^{ix} - e^{-ix}) \log(\operatorname{cosec} x - \cot x)]$$

$$= x - \sin x \log(\operatorname{cosec} x - \cot x)$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = c_1 \cos x + c_2 \sin x + x - \sin x \log(\operatorname{cosec} x - \cot x)$$

where c_1 and c_2 are arbitrary constants of integration.

TEST YOUR KNOWLEDGE

Solve the following differential equations :

1. $(D^2 + 4)y = \tan 2x$
2. $(D^2 + 1)y = \sec x$
3. $(D^2 + a^2)y = \operatorname{cosec} ax$
4. $(D^2 + 2D + 2)y = e^{-x} \sec^3 x$
5. $(D^2 - 2D + 2)y = e^x \tan x.$

Answers

1. $y = c_1 \cos 2x + c_2 \sin 2x - \frac{1}{4} \cos 2x \log(\sec 2x + \tan 2x)$
2. $y = c_1 \cos x + c_2 \sin x + x \sin x + \cos x \log \cos x$
3. $y = c_1 \cos ax + c_2 \sin ax + \frac{1}{a^2} \sin ax \log \sin ax - \frac{x}{a} \cos ax$
4. $y = e^{-x}\left(c_1 \cos x + c_2 \sin x + \frac{\sin x \tan x}{2}\right)$
5. $y = e^x (c_1 \cos x + c_2 \sin x) - e^x \cos x \log(\sec x + \tan x)$.

1.19. HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS (EULER-CAUC EQUATIONS)

An equation of the form

$$x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + a_2 x^{n-2} \frac{d^{n-2}y}{dx^{n-2}} + \ldots\ldots + a_{n-1} x \frac{dy}{dx} + a_n y = Q$$

where a_i's are constants and Q is a function of x, is called Cauchy's homogeneous linear equa

Such equations can be reduced to linear differential equations with constant coeffic by the substitution

$$x = e^z \quad \text{or} \quad z = \log x$$

so that $$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{dy}{dz} \cdot \frac{1}{x} \quad \text{or} \quad x\frac{dy}{dx} = \frac{dy}{dz} = \text{D}y, \text{ where } \text{D} \equiv \frac{d}{dz}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right) = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x} \cdot \frac{d^2y}{dz^2} \cdot \frac{dz}{dx}$$

$$= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2y}{dz^2} \qquad \left(\because \frac{dz}{dx}\right.$$

or $$x^2 \frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz} = \text{D}^2y - \text{D}y = \text{D}(\text{D}-1)y$$

Similarly $$x^3 \frac{d^3y}{dx^3} = \text{D}(\text{D}-1)(\text{D}-2)y \text{ and so on.}$$

Substituting these values is equation (1), we get a linear differential equation w constant coefficients, which can be solved by the methods already discussed.

1.19.1. Steps for Solution :

1. Put $x = e^z$ so that $z = \log x$ and Let $\text{D} \equiv \frac{d}{dz}$
2. Replace $x\frac{d}{dx}$ by D,

   $x^2 \frac{d^2}{dx^2}$ by D(D − 1)

   $x^3 \frac{d^3}{dx^3}$ by D(D − 1) (D − 2) and so on.

3. By doing so, this type of equation reduces to linear differential equation with constant coefficients which is then solved as before.

ILLUSTRATIVE EXAMPLES

Example 1. *Solve* : $x^3 \frac{d^3y}{dx^3} + 2x^2 \frac{d^2y}{dx^2} + 2y = 10\left(x + \frac{1}{x}\right)$. [U.P.T.U. (C.O.) 2009]

Sol. Put $x = e^z$ so that $z = \log x$ and let $D \equiv \frac{d}{dz}$ then the given differential equation reduces to

$$[D(D-1)(D-2) + 2D(D-1) + 2]y = 10(e^z + e^{-z})$$

or

$$(D^3 - D^2 + 2)y = 10\,(e^z + e^{-z})$$

which is a linear equation with constant coefficients.

Its Auxiliary equation is

$$m^3 - m^2 + 2 = 0 \quad \text{or} \quad (m+1)(m^2 - 2m + 2) = 0$$

$$\therefore \quad m = -1, \frac{2 \pm \sqrt{4-8}}{2} = -1, 1 \pm i$$

$$\therefore \quad \text{C.F.} = c_1 e^{-z} + e^z (c_1 \cos z + c_3 \sin z) = \frac{c_1}{x} + x\,[c_2 \cos(\log x) + c_3 \sin(\log x)]$$

$$\text{P.I.} = 10\frac{1}{D^3 - D^2 + 2}(e^z + e^{-z}) = 10\left(\frac{1}{D^3 - D^2 + 2}e^z + \frac{1}{D^3 - D^2 + 2}e^{-z}\right)$$

$$= 10\left(\frac{1}{1^3 - 1^2 + 2}e^z + z \cdot \frac{1}{3D^2 - 2D}e^{-z}\right) = 10\left(\frac{1}{2}e^z + z \cdot \frac{1}{3(-1)^2 - 2(-1)}e^{-z}\right)$$

$$= 5e^z + 2ze^{-z} = 5x + \frac{2}{x}\log x$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = \frac{c_1}{x} + x\,[c_2 \cos(\log x) + c_3 \sin(\log x)] + 5x + \frac{2}{x}\log x$$

where c_1, c_2 and c_3 are arbitrary constants of integration.

Example 2. *Solve* : $x^3 \frac{d^3y}{dx^3} + 3x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = x + \log x$. (U.P.T.U. 2001)

Sol. Put $x = e^z$ so that $z = \log x$ and let $D \equiv \frac{d}{dz}$ then the given differential equation reduces to

$$[D(D-1)(D-2) + 3D(D-1) + D + 1]\,y = e^z + z$$

$$\Rightarrow \quad (D^3 + 1)y = e^z + z$$

Auxiliary equation is

$$m^3 + 1 = 0$$

$$\Rightarrow \quad (m+1)(m^2 - m + 1) = 0 \qquad \Rightarrow \qquad m = -1, \frac{1 \pm \sqrt{3}i}{2}$$

$$\therefore \quad \text{C.F.} = c_1 e^{-z} + e^{z/2}\left(c_2 \cos\frac{\sqrt{3}}{2}z + c_3 \sin\frac{\sqrt{3}}{2}z\right)$$

$$\text{P.I.} = \frac{1}{D^3+1}(e^z + z) = \frac{1}{D^3+1}(e^z) + \frac{1}{1+D^3}(z)$$

$$= \frac{e^z}{2} + (1+D^3)^{-1}(z) = \frac{e^z}{2} + (1-D^3)(z) \qquad | \text{ Leaving higher terms}$$

$$= \frac{e^z}{2} + z$$

$\therefore$ The complete solution is

$$y = c_1 e^{-z} + e^{z/2}\left(c_2 \cos\frac{\sqrt{3}}{2}z + c_3 \sin\frac{\sqrt{3}}{2}z\right) + \frac{e^z}{2} + z$$

$$\therefore \quad y = \frac{c_1}{x} + \sqrt{x}\left[c_2 \cos\frac{\sqrt{3}}{2}(\log x) + c_3 \sin\frac{\sqrt{3}}{2}(\log x)\right] + \frac{x}{2} + \log x$$

where c_1, c_2 and c_3 are the arbitrary constants of integration.

Example 3. *Solve the following homogeneous differential equation*

$$x^2 \frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 5y = x^2 \sin(\log x). \qquad \text{(Q. Bank U.P.T.U. 200}$$

Sol. Put $x = e^z$ so that $z = \log x$ and let $D \equiv \frac{d}{dz}$ then the given differential equatio reduces to

$$[D(D-1) - 3D + 5]y = e^{2z}\sin z$$

$$\Rightarrow \quad (D^2 - 4D + 5)y = e^{2z}\sin z$$

Auxiliary equation is

$$m^2 - 4m + 5 = 0$$

$$\Rightarrow \quad m = \frac{4 \pm \sqrt{16-20}}{2} = \frac{4 \pm 2i}{2} = 2 \pm i$$

$$\therefore \quad \text{C.F.} = e^{2z}(c_1 \cos z + c_2 \sin z)$$

$$\text{P.I.} = \frac{1}{D^2 - 4D + 5}(e^{2z}\sin z) = e^{2z} \cdot \frac{1}{(D+2)^2 - 4(D+2) + 5}\sin z$$

$$= e^{2z} \cdot \frac{1}{D^2+1}(\sin z) = e^{2z} \cdot z \cdot \frac{1}{2D}\sin z = -\frac{z}{2}e^{2z}\cos z$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = e^{2z}(c_1 \cos z + c_2 \sin z) - \frac{z}{2}e^{2z}\cos z$$

$$= x^2\{c_1 \cos(\log x) + c_2 \sin(\log x)\} - \frac{x^2}{2}\log x \cos(\log x)$$

where c_1 and c_2 are arbitrary constants of integration.

Example 4. *Solve* : $x^2 \dfrac{d^2y}{dx^2} + 4x\dfrac{dy}{dx} + 2y = e^x$. [U.P.T.U. (C.O.) 2005]

Sol. Put $x = e^z$ so that $z = \log x$ and let $D \equiv \dfrac{d}{dz}$ then the given differential equation reduces to

$$\{D(D-1) + 4D + 2\}y = e^{e^z}$$

$$(D^2 + 3D + 2)y = e^{e^z}$$

Auxiliary equation is

$$m^2 + 3m + 2 = 0$$

$$\Rightarrow \quad (m+1)(m+2) = 0 \Rightarrow m = -1, -2$$

$$\therefore \quad \text{C.F.} = c_1e^{-z} + c_2e^{-2z}$$

$$\text{P.I.} = \frac{1}{D^2+3D+2}(e^{e^z}) = \left(\frac{1}{D+1} - \frac{1}{D+2}\right)e^{e^z}$$

$$= \frac{1}{D+1}(e^{e^z}) - \frac{1}{D+2}e^{e^z} = e^{-z}\int e^z \cdot e^{e^z}\,dz - e^{-2z}\int e^{2z}e^{e^z}\,dz$$

$$= e^{-z}e^{e^z} - e^{-2z}(e^z - 1)e^{e^z} = e^{-2z}e^{e^z}$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = c_1e^{-z} + c_2e^{-2z} + e^{-2z}\,e^{e^z} = \frac{c_1}{x} + \frac{c_2}{x^2} + \frac{1}{x^2}e^x$$

where c_1 and c_2 are arbitrary constants of integration.

Example 5. *Solve:* $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - \lambda^2 y = 0$ (U.P.T.U. 2007)

Sol. Put $x = e^z$ so that $z = \log x$ and let $D \equiv \dfrac{d}{dz}$ then the given differential equation reduces to

$$[D(D-1) + D - \lambda^2]y = 0$$

$$\Rightarrow \quad (D^2 - \lambda^2)y = 0$$

Auxiliary equation is

$$m^2 - \lambda^2 = 0$$

$$\Rightarrow \quad m = \pm\lambda$$

$$\therefore \quad \text{C.F.} = c_1e^{\lambda z} + c_2e^{-\lambda z}$$

$$\text{P.I.} = 0$$

Hence, the complete solution is

$$y = \text{C.F.} + \text{P.I.} = c_1e^{\lambda z} + c_2e^{-\lambda z}$$

$$\Rightarrow \quad y = c_1x^{\lambda} + c_2x^{-\lambda}$$

where c_1 and c_2 are arbitrary constants of integration.