General Rule for finding the Particular integral

Case-I $\quad F(x, y) = e^{ax+by}$

$$\text{P.I.} \quad \frac{e^{ax+by}}{f(D, D')} = \frac{e^{ax+by}}{f(a, b)} \qquad [\text{Put } D = a,\ D' = b]$$

$$f(a,b) \neq 0$$

$$x \cdot \frac{e^{ax+by}}{f(a, b)} \qquad \text{If } f(a,b) = 0$$

eg-1.

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = e^{x+2y}$$

$$(D^2 - D'^2)z = e^{x+2y}$$

A.E

$$m^2 - 1 = 0$$

$$m = \pm 1$$

$$\text{C.F.} = f_1(y+x) + f_2(y-x)$$

P.I.

$$\frac{e^{x+2y}}{(D^2 - D'^2)} = \frac{e^{x+2y}}{1-4} = -\frac{1}{3} e^{x+2y}$$

$$z = \text{C.F.} + \text{P.I.}$$

$$= f_1(y+x) + f_2(y-x) - \frac{1}{3} e^{x+2y}$$

$Z = e$

eg.2

$$(D^2 - 5DD' + 6D'^2)z = e^{x+y}$$

A.E.

$$m^2 - 5m + 6 = 0$$

$$m = 2, 3$$

$$\text{C.F.} = f_1(y+2x) + f_2(y+3x)$$

P.I.

$$\frac{e^{x+y}}{(D^2 - 5DD' + 6)} = \frac{1}{2} e^{x+y}$$

$$z = \text{C.F.} + \text{P.I.}$$

eg 1. $(D^3 - 3D^2D' + 4D'^3)z = e^{x+2y}$ $(-1, 2, 2)$

2. $(D^2 - 4DD' + 4D'^2)z = e^{2x+y}$

case II $F(x,y) = \sin(ax+by)$ or $\cos(ax+by)$

P.I. $$\frac{\sin(ax+by)}{f(D^2, DD', D'^2)} = \frac{1}{f(-a^2, -ab, -b^2)}\sin(ax+by)$$

$f(-a^2, -ab, -b^2) \neq 0$

eg. $(D^2 + 2DD' + D'^2)z = \sin(2x+3y)$

A.E $m^2 + 2m + 1 = 0$

$m = -1, -1$

C.f. $= f_1(y-x) + x f_2(y-x)$

P.I $$\frac{\sin(2x+3y)}{(D^2 + 2DD' + D'^2)}$$

$D^2 \to -4$

$DD' \to -6$

$D'^2 \to -9$

$$\frac{\sin(2x+3y)}{(-4-12-9)}$$

$$= -\frac{1}{25}\sin(2x+3y)$$

$z =$ C.f. + P.I.

$= f_1(y-x) + x f_2(y-x) - \frac{1}{25}\sin(2x+3y)$

eg. $(D^3 - 4D^2D' + 4D'^3)z = 2\sin(3x+2y)$

$z = f_1(y) + f_2(y+2x) + x f_3(y+2x) + \frac{2}{3}\cos(3x+2y)$

eg. $(D^2 + DD' - 6D'^2)z = \cos(2x+y)$

eg. $(D^2 + DD' - 6D'^2)z = \cos(2x+y)$

A.E. $m^2 + m - 6 = 0$

$m = 2, -3$

C.F. $f_1(y+2x) + f_2(y-3x)$

P.I. $\dfrac{\cos(2x+y)}{(D^2+DD'-6D'^2)}$

$= x.\dfrac{\cos(2x+y)}{(2D+D')}$

$= x\left[\dfrac{D}{2D^2+DD'}\cos(2x+y)\right]$

$= x\left[\dfrac{D}{-8-2}\cos(2x+y)\right]$

$= +\dfrac{x}{10}\left[2\sin(2x+y)\right]$

$= \dfrac{x}{5}\sin(2x+y)$

$z = $ C.F. $+$ P.I.

eg. $(D^2 - 2DD' + D'^2)z = \sin x$

Solⁿ. $f_1(x+y) + x f_2(x+y) - \sin x$

... the following equations:

1. $\frac{\partial^2 z}{\partial x^2}-\frac{\partial^2 z}{\partial y^2}=e^{x+2y}$ **Ans.** $z=f_1(y+x)+f_2(y-x)-\frac{e^{x+2y}}{3}$

2. $\frac{\partial^2 z}{\partial x^2}-5\frac{\partial^2 z}{\partial x\partial y}+6\frac{\partial^2 z}{\partial y^2}=e^{x+y}$ **Ans.** $z=f_1(y+2x)+f_2(y+3x)+\frac{1}{2}e^{x+y}$

3. $\frac{\partial^2 z}{\partial x^2}-4\frac{\partial^2 z}{\partial x\partial y}+4\frac{\partial^2 z}{\partial y^2}=e^{2x+y}$ **Ans.** $z=f_1(y+2x)+x\,f_2(y+2x)+\frac{x^2}{2}e^{2x+y}$

4. $\frac{\partial^2 z}{\partial x^2}-7\frac{\partial^2 z}{\partial x\partial y}+12\frac{\partial^2 z}{\partial y^2}=e^{x-y}$ **Ans.** $z=f_1(y+3x)+f_2(y+4x)+\frac{1}{20}e^{x-y}$

5. $\frac{\partial^3 z}{\partial x^3}-2\frac{\partial^3 z}{\partial x^2\partial y}=2e^{2x-y}$ **Ans.** $z=f_1(y)+x\,f_2(y)+f_3(y+2x)+\frac{1}{8}e^{2x-y}$

6. $(D^2-2DD'+D'^2)z=e^{x+2y}$ **Ans.** $z=f_1(y+x)+x\,f_2(y+x)+e^{x+2y}$

7. $\frac{\partial^2 z}{\partial x^2}-\frac{\partial^2 z}{\partial y^2}-2\frac{\partial z}{\partial x}+2\frac{\partial z}{\partial y}=e^{2x+3y}$ **Ans.** $z=f_1(y+x)+e^{2x}f_2(y-x)-\frac{1}{3}e^{2x+3y}$

8. $\frac{\partial^2 z}{\partial x^2}-5\frac{\partial^2 z}{\partial x\partial y}+6\frac{\partial^2 z}{\partial y^2}=\exp(3x-2y)$ **Ans.** $z=f_1(y+2x)+f_2(y+3x)+\frac{1}{63}e^{3x-2y}$

Case II. When R.H.S. = sin $(ax+by)$ or cos $(ax+by)$

Example 7. *Solve the linear partial differential equation*

$$\frac{\partial^2 z}{\partial x^2}+2\frac{\partial^2 z}{\partial x\,\partial y}+\frac{\partial^2 z}{\partial y^2}=\sin(2x+3y)$$ *(U.P. II Semester Summer 2006)*

Solution. We have,

$$\frac{\partial^2 z}{\partial x^2}+2\frac{\partial^2 z}{\partial x\,\partial y}+\frac{\partial^2 z}{\partial y^2}=\sin(2x+3y)$$

$$(D^2+2DD'+D'^2)z=\sin(2x+3y)\quad\text{where } D=\frac{\partial}{\partial x}\text{ and } D'=\frac{\partial}{\partial y}$$

Put $$D=m,\quad D'=1$$

The auxiliary equation is

$$m^2+2m+1=0$$

$$(m+1)^2=0\ \Rightarrow\ m=-1,-1$$

$$\text{C.F.} = f_1(y-x)+x f_2(y-x)$$

$$\Rightarrow \quad \text{P.I.} = \frac{1}{D^2+2DD'+D'^2}\sin(2x+3y)$$

$$= \frac{1}{-4+2(-6)-9}\sin(2x+3y)$$

$$= \frac{1}{-25}\sin(2x+3y) \qquad \begin{bmatrix} D^2=-2^2=-4 \\ D'^2=-3^2=-9 \\ DD'=-2\times 3=-6 \end{bmatrix}$$

Hence, the complete solution is

$$z = C.F. + P.I.$$

$$\Rightarrow \quad z = f_1(y-x)+x f_2(y-x)+\frac{1}{-25}\sin(2x+3y)$$

$$\Rightarrow \quad z = f_1(y-x)+x f_2(y-x)-\frac{1}{25}\sin(2x+3y)$$

Ans.

Example 8. *Solve*

$$\frac{\partial^3 z}{\partial x^3}-4\frac{\partial^3 z}{\partial x^2\partial y}+4\frac{\partial^3 z}{\partial x\partial y^2}=2\sin(3x+2y)$$

Solution. We have, $\frac{\partial^3 z}{\partial x^3}-4\frac{\partial^3 z}{\partial x^2\partial y}+4\frac{\partial^3 z}{\partial x\partial y^2}=2\sin(3x+2y)$

Putting $\frac{\partial}{\partial x}=D,\ \frac{\partial}{\partial y}=D'$

$$D^3z-4D^2D'z+4DD'^2z=2\sin(3x+2y)$$

A.E. is $D^3-4D^2D'+4DD'^2=0 \Rightarrow D(D^2-4DD'+4D'^2)=0$

Put $D=m,\ D'=1$

$$m(m^2-4m+4)=0 \Rightarrow m(m-2)^2=0 \Rightarrow m=0,2,2$$

$$\text{C.F.} = f_1(y)+f_2(y+2x)+x f_3(y+2x)$$

$$\text{P.I.} = \frac{1}{D^3-4D^2D'+4DD'^2}2\sin(3x+2y) = 2\frac{1}{D(D^2-4DD'+4D'^2)}\sin(3x+2y)$$

$$= 2.\frac{1}{D[-9-4(-6)+4(-4)]}\sin(3x+2y) = -\frac{2}{D}\sin(3x+2y)$$

$$= -\frac{2}{3}[-\cos(3x+2y)] = \frac{2}{3}\cos(3x+2y) \qquad \left[\frac{1}{D}=\int\right]$$

General solution is

$$z = f_1(y)+f_2(y+2x)+x f_3(y+2x)+\frac{2}{3}\cos(3x+2y)$$

Ans.

Example 9. *Solve* $\frac{\partial^2 z}{\partial x^2}-\frac{\partial^2 z}{\partial x\partial y}=\sin x\cos 2y$ (U.P., II Semester, June, 2010, 2008)

Solution. We have, $\frac{\partial^2 z}{\partial x^2}-\frac{\partial^2 z}{\partial x\partial y}=\sin x\cos 2y$

The given equation can be written in the form

$$(D^2 - DD')z = \sin x \cos 2y \text{ where } D = \frac{\partial}{\partial x},\ D' = \frac{\partial}{\partial y}$$

Writing $D = m$ and $D' = 1$, the auxiliary equation is

$$m^2 - m = 0 \quad \Rightarrow \quad m(m-1) = 0, \Rightarrow m = 0, 1$$

$$\text{C.F.} = f_1(y) + f_2(y+x)$$

$$\text{P.I.} = \frac{1}{D^2 - DD'} \sin x \cos 2y$$

$$= \frac{1}{D^2 - DD'} \frac{1}{2}[\sin(x+2y) + \sin(x-2y)]$$

$$= \frac{1}{2}\frac{1}{D^2 - DD'}\sin(x+2y) + \frac{1}{2}\frac{1}{D^2 - DD'}\sin(x-2y)$$

Put $D^2 = -1$, $DD' = -2$ in the first integral and $D^2 = -1$, $DD' = 2$ in the second integral.

$$= \frac{1}{2}\left[\frac{\sin(x+2y)}{-1-(-2)}\right] + \frac{1}{2}\left[\frac{\sin(x-2y)}{-1-(2)}\right] = \frac{1}{2}\sin(x+2y) - \frac{1}{6}\sin(x-2y)$$

Hence the complete solution is $z = \text{C.F.} + \text{P.I.}$

$$z = f_1(y) + f_2(y+x) + \frac{1}{2}\sin(x+2y) - \frac{1}{6}\sin(x-2y)$$ **Ans.**

Example 10. *Solve* $(D^2 + DD' - 6D'^2)z = \cos(2x+y)$

Solution. $(D^2 + DD' - 6D'^2)z = \cos(2x+y)$

A.E. is $m^2 + m - 6 = 0$ or $m = 2, -3$

$$\text{C.F.} = f_1(y+2x) + f_2(y-3x)$$

$$\text{P.I.} = \frac{1}{D^2 + DD' - 6D'^2}\cos(2x+y)$$

$$D^2 + DD' - 6D'^2 = -4 - 2 - 6(-1) = 0$$

It is a case of failure.

ow

$$\text{P.I.} = \frac{1}{D^2 + DD' - 6D'^2}\cos(2x+y)$$

$$= x\frac{1}{2D + D'}\cos(2x+y) = x\frac{D}{2D^2 + DD'}\cos(2x+y)$$

$$= x\frac{D}{2\times(-4) - 2}\cos(2x+y) = -\frac{x}{10}D\cos(2x+y)$$

$$= 2\frac{x}{10}\sin(2x+y) = \frac{x}{5}\sin(2x+y)$$

ce, the complete solution is

$$z = f_1(y+2x) + f_2(y-3x) + \frac{x}{5}\sin(2x+y)$$ **Ans.**

Example 11. *Solve the partial differential equation :*

$$\frac{\partial^2 z}{\partial x^2} - 3\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = e^{2x-y} + e^{x+y} + \cos(x+2y) \quad \text{(U.P. II Semester Summer 2006)}$$

Solution. Given equation is

$$\frac{\partial^2 z}{\partial x^2} - 3\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = e^{2x-y} + e^{x+y} + \cos(x+2y)$$

Given equation can be written as :

$$(D^2 - 3DD' + 2D'^2)\, z = e^{2x-y} + e^{x+y} + \cos(x+2y)$$

The auxiliary equation is

$$m^2 - 3m + 2 = 0$$
$$m^2 - 2m - m + 2 = 0$$
$$m(m-2) - 1(m-2) = 0$$
$$(m-1)(m-2) = 0$$
$$m = 1, 2$$

Hence, $\quad$ C.F. $= \phi_1(y+x) + \phi_2(y+2x)$

Now $\quad$ P.I. $= \dfrac{1}{(D-D')(D-2D')}\{e^{2x-y} + e^{x+y} + \cos(x+2y)\}$

$$= \frac{1}{(D-D')(D-2D')}e^{2x-y} + \frac{1}{(D-D')(D-2D')}e^{x+y} + \frac{1}{(D-D')(D-2D')}\cos(x+2y)$$

$$= I_1 + I_2 + I_3$$

Let $\quad I_1 = \dfrac{1}{(D-D')(D-2D')}e^{2x-y}$ $\quad$ (Replacing D by 2 and D' by – 1.)

$$= \frac{1}{(2+1)(2+2)}e^{2x-y} = \frac{1}{12}e^{2x-y}$$

Now, $\quad I_2 = \dfrac{1}{(D-D')(D-2D')}e^{x+y},$ $\quad$ (Replacing D by 1 and D' by 1)

$$= \frac{1}{(D-D')(-1)}e^{x+y} = -\frac{1}{(D-D')}e^{x+y}$$

$$= -x\frac{1}{1}e^{x+y} = -x\,e^{x+y}$$

Now, $\quad I_3 = \dfrac{1}{(D-D')(D-2D')}\cos(x+2y)$

$$= \frac{1}{D^2 - 3DD' + 2D'^2}\cos(x+2y)$$

$= \dfrac{1}{-1-3(-2)+2(-4)}\cos(x+2y)$ (Replacing D^2 by -1; DD' by -2; D'^2 by –

$$= \frac{1}{-1+6-8}\cos(x+2y) = -\frac{1}{3}\cos(x+2y)$$

P.I. $= I_1 + I_2 + I_3$

Thus required $P.I. = \frac{1}{12} e^{2x-y} - xe^{x+y} - \frac{1}{3}\cos(x+2y)$

Hence, the complete solution is

$$z = C.F. + P.I.$$

$$= \phi_1(y+x) + \phi_2(y+2x) + \frac{1}{12}e^{2x-y} - xe^{x+y} - \frac{1}{3}\cos(x+2y)$$ **Ans.**

Example 12. *Solve the equation*

$$(D^3 - 7DD'^2 - 6D'^3)z = \sin(x+2y) + e^{2x+y}$$

Solution. $(D^3 - 7DD'^2 - 6D'^3)z = \sin(x+2y) + e^{2x+y}$ ... (1)

Its auxiliary equation is

$m^3 - 7m - 6 = 0$ or $(m+1)(m+2)(m-3) = 0 \Rightarrow m = -1, -2, 3$

$\therefore$ C.F. $= f_1(y-x) + f_2(y-2x) + f_3(y+3x)$

$$\text{P.I.} = \frac{1}{D^3 - 7DD'^2 - 6D'^3}[\sin(x+2y) + e^{2x+y}]$$

$$= \frac{1}{D^3 - 7DD'^2 - 6D'^3}\sin(x+2y) + \frac{1}{D^3 - 7DD'^2 - 6D'^3}e^{2x+y}$$

$$= \frac{1}{D^2D - 7DD'^2 - 6D'^3}\sin(x+2y) + \frac{e^{2x+y}}{(2)^3 - 7(2)(1)^2 - 6(1)^3}$$

Put $D^2 = -1,\ D'^2 = -2^2$

$$= \frac{1}{-D - 7D(-4) - 6(-4)D'}\sin(x+2y) + \frac{e^{2x+y}}{8-14-6}$$

$$= \frac{1}{27D + 24D'}\sin(x+2y) - \frac{1}{12}e^{2x+y} = \frac{1}{3}\frac{1}{9D+8D'}\sin(x+2y) - \frac{1}{12}e^{2x+y}$$

$$= \frac{1}{3}\frac{D}{9D^2 + 8DD'}\sin(x+2y) - \frac{1}{12}e^{2x+y} = \frac{1}{3}\frac{D}{9(-1)+8(-2)}\sin(x+2y) - \frac{1}{12}e^{2x+y}$$

$$= -\frac{1}{75}D\sin(x+2y) - \frac{1}{12}e^{2x+y} = -\frac{1}{75}\cos(x+2y) - \frac{1}{12}e^{2x+y}$$

Hence, the complete solution is

$$z = f_1(y-x) + f_2(y-2x) + f_3(y+3x) - \frac{1}{75}\cos(x+2y) - \frac{1}{12}e^{2x+y}$$ **Ans.**

Example 13. *Solve the P.D.E.* $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = \sin x.$ *(U.P II Semester 2009, 2004)*

Solution. Here, we have

$$\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = \sin x$$

$$\Rightarrow (D^2 - 2DD' + D'^2)z = \sin x$$

Its auxiliary equation is

$$(m^2 - 2m + 1) = 0 \Rightarrow (m-1)^2 = 0 \Rightarrow m = 1, 1$$

$$\text{C.F} = f_1(x+y) + x f_2(x+y)$$

$$\text{P.I.} = \frac{1}{D^2 - 2DD' + D'^2} \sin x$$

$$= \left[\frac{1}{-1-0+0}\right] \sin x \qquad \begin{bmatrix} D=1 \\ D'=0 \end{bmatrix}$$

$$= -\sin x$$

Hence, the complete solution is

$$z = \text{C.F.} + \text{P.I.}$$

$$= f_1(x+y) + x f_2(x+y) - \sin x \qquad \textbf{Ans.}$$

EXERCISE 14.3

1. Find the P.I. of $(D^2 + DD')z = \sin(x+y)$ *(GBTU II Sem., Jan. 2012)* **Ans.** $-\frac{1}{2}\sin(x+y)$

Solve the following equations :

2. $[2D^2 - 5DD' + 2D'^2]z = 5\sin(2x+y)$ **Ans.** $z = f_1(y+2x) + f_2(2y+x) - \frac{5}{3}x\cos(2x+y)$

3. $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} = \cos(x+2y)$ **Ans.** $z = f_1(y) + f_2(y+x) + \cos(x+2y)$

4. $(D^2 - DD')z = \cos x \cos 2y$ **Ans.** $z = f_1(y) + f_2(y+x) + \frac{1}{2}\cos(x+2y) - \frac{1}{6}\cos(x-2y)$

5. $(D^2 + 2D'D + D'^2)z = \sin(x+2y)$ **Ans.** $z = f_1(y-x) + x f_2(y-x) - \frac{1}{9}\sin(x+2y)$

6. $\frac{\partial^2 z}{\partial x^2} - 3\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = e^{2x+3y} + \sin(x+2y)$ **Ans.** $z = f\left(y + \frac{x}{2}\right) + f_2(y+2x) + \frac{1}{4}e^{2x+3y} - \frac{1}{15}\sin(x-2y)$

7. $r - 2s = \sin x \cos 2y$ **Ans.** $z = f_1(y) + f_2(y+2x) + \frac{1}{15}(\sin x \cos 2y) + 4\sin 2y \cos x$

8. $(D^2 + D'^2)z = \cos mx. \cos ny$ **Ans.** $z = f_1(y+ix) + f_2(y-ix) - \frac{\cos mx. \cos ny}{(m^2+n^2)}$

9. $(D^3 - 7DD'^2 - 6D'^3)z = \sin(x+2y) + e^{3x+y}$

Ans. $z = f_1(y-x) + f_2(y-2x) + f_3(y+3x) - \frac{1}{75}\cos(x+2y) + \frac{x}{20}e^{3x+y}$

10. $(D^2 - DD')z = \cos 2y(\sin x + \cos x)$ *(U.P; II Semester, 2003)*

Ans. $z = f_1(y) + f_2(y+x) + \frac{1}{2}[\sin(x+2y) + \cos(x+2y)] - \frac{1}{6}[\sin(x-2y) + \cos(x-2y)]$

Case III. When R.H.S. $= \phi(ax+by)$ polynomial

Example 14. *Find the general integral of the equation*

$$\frac{\partial^2 z}{\partial x^2} + 3\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = x + y$$

Solution. $\frac{\partial^2 z}{\partial x^2} + 3\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = x + y$