Recurrence Relation.

deff -> An equation that express 'a_n' in terms of one or more of number of previous terms of the sequence (with some enetical cond^n) is called recurrence relation.

Exp Fibonacci sequence.

$$\langle \overset{a_0}{1}, \overset{a_1}{1}, 2, 3, 5, 8 \cdots \rangle$$

$$\boxed{a_n = a_{n-1} + a_{n-2} \quad , \quad n \geq 2}$$

$a_3 = ?$

$$a_3 = a_{3-1} + a_{3-2}$$
$$= a_2 + a_1$$

$a_2 = ?$, $a_2 = a_{2-1} + a_{2-2}$

$$= a_1 + a_0$$
$$= 1 + 1$$
$$= 2$$

as we know

$a_0 = 1$
$a_1 = 1$

$$a_3 = a_2 + a_1$$
$$= 2 + 1$$
$$= 3$$

Exp 2 $\langle 3, 9, 27, 81, \cdots \rangle$

So we can write.

$$\boxed{a_n = 3a_{n-1}}$$

initial cond^n -> $a_0 = 3, a_1 = 9$

If $a_3 = ?$

$a_3 = 3a_{3-1}$
$= 3a_2$ -> $a_2 = ?$

$a_2 = 3a_{2-1}$
$= 3a_1$
$= 3 \times 9 = 27$

=> $\boxed{a_3 = 3 \times 27 = 81}$

$a_3 = 81$

If asked : $a_{227} = ?$

$a_{227} = 3a_{227-1} = 3a_{226}$

=> $a_{226} = ?$, => $3a_{225}$

its very difficult to find the value of a_{227}?

so that we can write, the express^n in diff form.

$\boxed{a_n = 3(3^n)}$ ← closed form of a recursive function or

Lecture-3

Linear Recurrence relations with constant coefficients.

Its representation is in the form of

$$\boxed{C_0 a_n + C_1 a_{n-1} + C_2 a_{n-2} + \cdots\cdots K_n C_{n-k} = f(n).}$$

where C_i are all constant

If $f(n) \Rightarrow 0$

then the above equation is known as homogenous equation.

Exp → $\boxed{3a_n + 4a_{n-1} = 0}$

If $f(n) \neq 0$ then it is known as nonhomogenous equation

$\boxed{3a_n + 4a_{n-1} = n^2}$, here $\boxed{f(n) = n^2}$

Order of Recurrence relation.:

It is the difference between highest & lowest subscript. in numeric function.

Exp $C_0 a_n + C_1 a_{n-1} + \cdots\cdots + C_k a_{n-k} = f(n)$

order $= n - (n-k)$

$= n - n + k \Rightarrow \boxed{\text{order} = k}$

e.g. $a_n + 2a_{n-1} = 0$

degree = ?

$\Rightarrow n-(n-1) = 1$

$\rightarrow f(n)=0$, so it is a homogeneous equation.

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3 different methods are there to solve the recurrence relation.

- → Iteration
- → characteristic roots
- → generating function.

(a) Iteration method.

$T_p = T(p-1) + 3 \quad , p \geq 1$

given that $T(0) = 2$

find $T(3)$ by using I.o.M ?

Soln

$T_p = T(p-1) + 3$

$\boxed{T_3 = T_2 + 3}$ —— ①

$T_2 = T_1 + 3$ —— ②

$T_1 = T_0 + 3$ —— ③

$T_1 = 2+3 = 5$

$T_2 = 5+3 = 8$

$T_3 = 8+3 = 11$

but if it asked $T_{100} = ?$ ← difficult to find.

i-5

$$\boxed{T_p = T_{(p-1)} + 3} \quad — (1)$$

we have to solve it for getting certain pattern

case-1

$T_{p-1} = T_{(p-2)} + 3$

$\Rightarrow T_p - 3 = T_{(p-2)} + 3$

$\Rightarrow \boxed{T_p = T_{(p-2)} + 6} \quad — (2)$

case-2

$T_{p-2} = T_{p-3} + 3$

$\Rightarrow T_p - 6 = T_{p-3} + 3$

$\Rightarrow \boxed{T_p = T_{p-3} + 9} \quad — (3)$

using eqn - 1, 2, & 3

we can generalized it

$$\boxed{T_p = T_{(p-k)} + k \cdot 3}$$

Initial condn given: $T(0) = 2$

so we can start such a way that can use the initial condition.

let $k = p$

$\Rightarrow \boxed{T_p = T_0 + p3}$

$\Rightarrow \boxed{T_p = 2 + p3}$ ← closed form, which is free from any previous relation

$T_{100} = 2 + 100 \times 3$

$= 2 + 300$

$= 302$

check for T_3

$\Rightarrow T_3 = 2 \times 3 \times 3$

(5) characteristics root Method.

it specifically consist of 3 steps.

(1) step-I → write characteristics equation.

(2) step2 → find roots of

Let roots are $r_1, r_2, \cdots r_n$

(3) step-3 → If all the roots are different then the general solution is

$$a_n = c_1\lambda_1^n + c_2\lambda_2^n + \cdots + c_n\lambda_n^n$$

Exp $a_n - 6a_{n-1} + 8a_{n-2} = 0$, $a_0 = 4$, $a_1 = 10$.

here we can write

$a_n \Rightarrow r^n$, $a_{n-1} \Rightarrow r^{n-1}$, $a_{n-2} \Rightarrow r^{n-2}$

$$r^n - 6r^{n-1} + 8r^{n-2} = 0 \quad \text{— (1)}$$

divide both side by $\underline{\underline{r^{n-2}}}$ (lowest value)

$$\Rightarrow r^2 - 6r + 8 = 0$$

$$\Rightarrow r \Rightarrow \frac{6 \pm \sqrt{36 - 4.1.8}}{2.1} = \frac{6 \pm \sqrt{36-32}}{2} = \frac{6 \pm 2}{2}$$

$$= 4 \text{ or } 2$$

$$\boxed{r = 4 \text{ or } 2}$$

Follow the step-3

$$\boxed{a_n = c_1 2^n + c_2 4^n}$$ — (2) ← general solution.

For particular sol^n

Initial cond^n => $a_0 = 4$
$a_1 = 10$

$a_n = C_1 2^n + C_2 4^n$

$n=0 \Rightarrow a_0 = C_1 2^0 + C_2 4^0$

$\Rightarrow \boxed{4 = C_1 + C_2}$ — (3)

$n=1 \Rightarrow a_1 = C_1 2 + C_2 4.$

$\Rightarrow \boxed{10 = 2C_1 + 4C_2}$ — (4)

by solving (3) & (4) we can find out. C_1 & C_2

$\Rightarrow$ eq^n 3×2, − eq^n (4)

$\Rightarrow \boxed{8 = 2C_1 + 2C_2}$ — (5)

$\Rightarrow 2 = 2C_2 \Rightarrow \boxed{C_2 = 1}$

$\boxed{4 = C_1 + C_2}$

$\Rightarrow C_1 = 4 - C_2 = 4 - 1 = 3$

$\boxed{C_1 = 3}$

My particular sol^n (PS) => $a_n = C_1 2^n + C_2 4^n$

$\Rightarrow \boxed{a_n = 3 2^n + 4^n}$ — (5)

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Q5 $a_n - 2a_{n-1} + a_{n-2} = 0$, $a_0 = 2$, $a_1 = 6$

$r^n - 2r^{n-1} + r^{n-2} = 0$

$\Rightarrow$ devided by r^{n-2} on both side.

$\Rightarrow r^2 - 2r + 1 = 0$

$r \Rightarrow \frac{2 \pm \sqrt{4 - 4 \cdot 1}}{2} = \frac{2 \pm 0}{2} = 1$

$a_n = c_1 r_1^n + c_2 r_2^n$ (x) }

~~$a_n = c_1 + c_2$~~ ↳ it is a special case.

$a_n = c_1 \lambda_1^n + c_2 \lambda_2^n + \cdots$

if roots are equal.

$$\boxed{a_n = (c_1 + c_2 n)\lambda_1^n + c_3 \lambda_2^n + \cdots + c_k \lambda_{k-1}^n}$$

$\Rightarrow \boxed{a_n = (c_1 + c_2 n) 1^n}$ — ① (General solⁿ)

$a_0 = 2$, $a_1 = 6$

$\Rightarrow a_0 = [c_1 + c_2 (0)] \times 1$

$\Rightarrow \boxed{2 = c_1}$

$a_1 = c_1 + c_2 \Rightarrow 6 = 2 + c_2 = 4$

$\Rightarrow \boxed{c_2 = 4}$

PS $\rightarrow$ $\boxed{a_n = (2 + 4 \cdot n) 1^n}$ —— ②

(i) generating function (G.F)

Diff → {A} be any sequence having terms $a_0, a_1, a_2 \ldots a_n$.

& G.F., $G(A,z)$ of a sequence 'A' is infinite series.

which can represented as

$$\boxed{G(A,z) = \sum_{n=0}^{\infty} a_n z^n}$$

$$= a_0 + a_1 z + a_2 z^2 + \ldots \infty$$

generating function for some standard forms.

case-I

$a_n = c$, $n \geq 0$

$$G(A,z) = \sum_{n=0}^{\infty} z^n$$

$$\Rightarrow \sum_{n=0}^{\infty} c \cdot z^n = c \sum_{n=0}^{\infty} z^n$$

$$= c\,(\underbrace{1 + z + z^2 \ldots \infty}_{\text{geometric progression}})$$

$$= c\left(\frac{1}{1-z}\right), \quad |z| < 1 \qquad \text{sum} = \frac{\text{Initial}}{1 - \text{ratio}}$$

$$\Rightarrow \boxed{G(A,z) = \frac{c}{1-z}}$$

$= c\left(\frac{a \cdot (r^n - 1)}{r - 1}\right)$

$= c\!\!\!/$

(x) it should not be because it used for divergence.

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Case-2 when $a_n = b^n$, $n \geq 0$

$$G(A,z) = \sum_{n=0}^{\infty} b^n z^n$$

$$= \sum_{n=0}^{\infty} (bz)^n$$

$$= 1 + bz + (bz)^2 + \cdots$$

↳ it is in geometric progression

$$\boxed{G(A,z) = \frac{1}{1-bz}}$$

Case-3 when $a_n = cb^n$

$$G(A,z) = \sum_{n=0}^{\infty} cb^n z^n$$

$$= c \sum_{n=0}^{\infty} (bz)^n$$

$$\boxed{G(A,z) = \frac{c}{(1-bz)}}$$

Case-4 when $a_n = n$

$$G(A,z) = \sum_{n=0}^{\infty} n \cdot z^n$$

$$\Rightarrow 0 + z + 2z^2 + 3z^3 \cdots$$

$$\Rightarrow z(1 + 2z + 3z^2 + \cdots)$$

$$\Rightarrow z(1-z)^{-2}$$

↳ using binomial theorem

Prov $|z| < 1$

Steps to solve Recurrence Relation.

Let. $C_0 a_n + C_1 a_{n-1} + \cdots + C_n a_{n-k} = c$

for all $n \geq k$.

step-I

multiply both side by z^n.

& sum up from $n = k$, to ∞

step-II write each term in the form of $G(A,z)$.

step-III solve $G(A,z)$, then using standard generating function equation. as a_n can found as

① $a_n = c$	② $a_n = b^n$	③ $a_n = cb^n$	④ $a_n = n$
$G(A,z) = \frac{c}{1-z}$	$G(A,z) = \frac{1}{1-bz}$	$\Rightarrow G(A,z) = \frac{c}{1-bz}$	$G(A,z) = \frac{z}{(1-z)}$

Exp Find G.F for the sequence of Recurrence relation.

$a_n + 2a_{n-1} = 0$, with initial condition $\Rightarrow a_0 = 5$

step-I $\sum_{n=1}^{\infty} a_n z^n + 2a_{n-1} z^n = 0$

↳ obj is to make $n = 1 \to 0$

we know that $G(A,z) = \sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + \cdots$

$\Rightarrow \sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + \cdots$

$= a_0 + \sum^{\infty} a_n z^n$

$$\Rightarrow \left(\sum_{n=0}^{\infty} a_n z^n - a_0\right) + \sum_{n\geq 1}^{\infty} 2 \cdot a_{n-1} \cdot z^{n-1} \cdot z = 0$$

$$\Rightarrow \left(\sum_{n=0}^{\infty} a_n z^n - a_0\right) + 2z\left(\sum_{n=1}^{\infty} a_{n-1} \cdot z^{n-1}\right) = 0$$

$$\Rightarrow \left(\sum_{n=0}^{\infty} a_n z^n - a_0\right) + 2z\left(\sum_{n=0}^{\infty} a_n \cdot z^n\right) = 0$$

$$\Rightarrow G(A,z) - a_0 + 2z(G(A,z)) = 0$$

$$\Rightarrow G(A,z) - 5 + 2z(G(A,z)) = 0$$

$$\Rightarrow \cancel{G(A,z) =} \; t$$

$$\Rightarrow G(A,z)(1+2z) = 5 \quad \Rightarrow \boxed{G(A,z) = \frac{5}{1+2z}}$$

step-3

$$G(A,z) = \frac{5}{1-(-2z)} \Rightarrow \frac{c}{1-bz}$$

$$\boxed{\begin{aligned} a_n &= c\,b^n \\ &= 5(-2)^n \end{aligned}} \leftarrow \text{sequence.}$$

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Solve recurrence relation

as $a_r - 7a_{r-1} + 10a_{r-2} = 0.$

given that $a_0 = 0,\ a_1 = 3.$

$\Rightarrow r^n - 7r^{n-1} + 10r^{n-2} = 0$

$\Rightarrow r^2 - 7r + 10 = 0$

$$\frac{+7 \pm \sqrt{49 - 4.1.10}}{2.} = \frac{7 \pm \sqrt{49-40}}{2} = \frac{7 \pm 3}{2}$$

$\therefore r = 5$ or 2

general sol^n

$$\boxed{a_n = c_1 \lambda_1^n + c_2 \lambda_2^n}$$

$$\boxed{a_n = c_1 5^n + c_2 2^n}$$

particular sol^n $\rightarrow$ $a_0 = c_1 + c_2 \Rightarrow c_1 + c_2 = 0$

$\Rightarrow \boxed{c_1 = -c_2}$

$a_1 = c_1 5 + c_2 . 2$

$\Rightarrow 3 = 5c_1 + 2c_2$

$= -5c_2 + 2c_2 = -3c_2 = 3$

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Q) obtained particular sol^n for.

$a_r + 5a_{r-1} + 6a_{r-2} = 3r+2$

$r^2 + 5r + 6 = 0$

$r = 2, or, 3$

general sol^n $\Rightarrow$ $a_n = c_1 \alpha_1^n + c_2 \alpha_2^n$

$$\boxed{a_n = c_1 2^n + c_2 3^n}$$

P. sol^n $\rightarrow$ $3r+2, \Rightarrow c_1 + c_2 r^1$

$a_r + 5a_{r-1} + 6a_{r-2} = 3r+2$

$\Rightarrow (c_1 + c_2 r) + 5[c_1 + c_2(r-1)] + 6[c_1 + c_2(r-2)]$

$= 3r+2$

$\Rightarrow c_1 + c_2 r + 5c_1 + 5c_2 r - 5c_2 + 6c_1 + 6c_2 r - 12c_2 = 3r+2$

$\Rightarrow (c_1 + 5c_1 - 5c_2 + 6c_1 - 12c_2) + r(c_2 + 5c_2 + 6c_2) = 3r+2$

$\Rightarrow (\cancel{6c_1 + 6c_1} -$

$\Rightarrow (12c_1 - 17c_2) + r(12c_2) = 3r+2$

$\Rightarrow \cancel{r} 12c_2 = 3\cancel{r}$

$\Rightarrow \boxed{c_2 = 3/12 = \frac{1}{4}}$

$12c_1 - 17c_2 = 2$

$\Rightarrow 12 \times c_1 - 17 \times \frac{1}{4} = 2$

$\Rightarrow 12c_1 = 2 + \frac{17}{4} = \frac{8+17}{4} = \frac{25}{4}$

P.T.O