2550. Count Collisions of Monkeys on a Polygon

Medium Recursion Math

Problem Description

The problem describes a scenario where we have a regular convex polygon with `n` vertices, and each vertex is occupied by exactly one monkey. The vertices are numbered from `0` to `n - 1` in a clockwise direction. The challenge is to calculate the total number of ways the monkeys can move to neighboring vertices such that at least one collision occurs. A collision is defined as either two monkeys residing on the same vertex after moving or two monkeys crossing paths along an edge. Note that a monkey can move to the vertex immediately clockwise (`(i + 1) % n`) or counterclockwise (`(i - 1 + n) % n`) to its position. Each monkey can move only once, and the final answer should be given modulo `10^9 + 7`.

Intuition

To find the solution to the problem, an important observation is that a collision will not happen only in two specific scenarios:

1. All monkeys move in the clockwise direction.
2. All monkeys move in the counter-clockwise direction.

In any other combination of movements, at least one collision is guaranteed to happen due to monkeys either ending up on the same vertex or crossing paths. Since each monkey has two choices (clockwise or counter-clockwise), there are a total of `2^n` ways the monkeys could decide to move.

Subtracting the two scenarios where no collision occurs from the total number of possible movements gives us the total number of ways at least one collision can occur:

Total number of ways for at least one collision = Total ways of movement - Ways without collisions = `2^n` - 2

Because the result could be very large, we compute the final answer using modulo arithmetic, specifically modulo `10^9 + 7`. The provided solution does exactly this using the modulo power function `pow(2, n, mod)` to calculate `2^n` mod `10^9 + 7`, and then subtracting `2` to exclude the non-collision scenarios, followed by taking the modulo again to ensure the result is within the required limits.

This approach elegantly handles the large number computations and efficiently computes the desired outcome with just a couple of arithmetic operations.

Solution Approach

The implementation of the solution involves understanding the basic properties of modular arithmetic and the power calculation. The problem does not require complex data structures or intricate algorithms due to its nature, allowing a direct application of the mathematical insight that we derived.

The Python solution relies on two key components of Python: the `pow` function and the modulo operation `%`.

- **`pow` function**: This is a built-in Python function that allows us to compute the power of a number with an optional modulus. Here, it is used to compute `2^n mod (10^9 + 7)`. This function is efficient for such calculations because it implements a fast exponentiation algorithm that scales well with large exponents.
- **Modulo operation `%`**: After performing the power operation, we need to ensure that we subtract two (for the two non-colliding scenarios) in a way that respects modular arithmetic properties. The modulo operation is used again to ensure the result is within the bounds of `0` to `10^9 + 6`.

The code consists of a single line within a function, which makes it quite elegant:

```
def monkeyMove(self, n: int) -> int:
    mod = 10**9 + 7
    return (pow(2, n, mod) - 2) % mod
```

Here is a breakdown of what happens in this line:

1. We compute `2^n` using `pow(2, n, mod)`. This gives us the number of all possible movements the monkeys could make by moving either clockwise or counterclockwise.
2. We subtract `2` from the result of `pow(2, n, mod)` to discount the scenarios where all the monkeys move in the same direction and no collision occurs.
3. We apply the modulo operation `% mod` again to ensure that the final result is expressed modulo `10^9 + 7`.

The simplicity of the approach lies in casting the problem into a binary choice for each monkey that results in a total `2^n` combinations and then eliminating the two combinations that do not lead to a collision. By understanding the properties of modular arithmetic, the problem is reduced to a straightforward computation, making it an elegant example of mathematical problem-solving applied to programming.

Example Walkthrough

Let's take `n = 4` as a small example to illustrate the solution approach. This corresponds to a square where each of the four vertices is occupied by a monkey. The vertices are numbered from `0` to `3`. We need to calculate the total number of ways the monkeys can move to neighboring vertices such that at least one collision occurs.

Using the intuition from the problem description, let's look at all the possible movement combinations and identify the scenarios where a collision will not occur.

Firstly, here are all the possible movement patterns for our monkeys:

1. All move clockwise: No collision.
2. All move counter-clockwise: No collision.
3. Monkey at vertex 0 moves clockwise, others move counter-clockwise: Collision occurs.
4. Monkey at vertex 1 moves clockwise, others move counter-clockwise: Collision occurs.
5. Monkey at vertex 2 moves clockwise, others move counter-clockwise: Collision occurs.
6. Monkey at vertex 3 moves clockwise, others move counter-clockwise: Collision occurs.
7. (And so on for all other combinations...)

As described, there are `2^4` or 16 total movement combinations for the monkeys, since each has the choice to move either clockwise or counter-clockwise.

Out of these 16 possibilities, only 2 patterns result in no collision (see patterns 1 and 2 above). Therefore, to get the number of combinations where at least one collision occurs, we subtract these 2 non-collision scenarios from the total: `16 - 2 = 14`.

We would compute this with the modulo `10^9 + 7` as follows:

```
mod = 10**9 + 7  # This is the modulo value for the problem.
total_combinations = pow(2, n, mod)  # Compute 2^n mod (10^9 + 7) to get total possible movement combinations.
non_collision_combinations = 2  # There are 2 non-collision scenarios.
ways_with_collision = (total_combinations - non_collision_combinations) % mod  # Compute the final answer with modulc

print(ways_with_collision)  # Output the total ways with at least one collision.
```

When you run this code with `n = 4`, you will get `14` as the number of ways at least one collision can occur (modulo `10^9 + 7`). The logic extends to any value of `n`, allowing you to calculate the number of collision scenarios for any regular convex polygon with `n` vertices in an efficient manner.

Solution Implementation

Python

```
class Solution:
    def monkeyMove(self, n: int) -> int:
        # Define the modulo constant as BigIntegers are not efficient
        MOD = 10**9 + 7

        # Calculate 2^n using modular exponentiation, which is efficient for large powers
        total_ways = pow(2, n, MOD)

        # Subtract 2 because the monkey cannot stay in the first or last column; wrap with MOD to keep result positive
        valid_ways = (total_ways - 2) % MOD

        # Return the number of valid ways the monkey can move
        return valid_ways
```

Java

```
class Solution {

    // This method calculates the number of ways a monkey can move, given `n` movements.
    public int monkeyMove(int n) {
        // Defining the modulo value as 1e9 + 7 to keep the result within integer limits
        final int MOD = (int) 1e9 + 7;

        // Use the quick power algorithm to calculate 2 raised to the power of `n`, reduce the result by 2, and ensure it's withi
        return (quickPower(2, n, MOD) - 2 + MOD) % MOD;
    }

    // This helper method efficiently calculates (a^b) mod `mod` using the quick power algorithm.
    private int quickPower(long base, int exponent, int mod) {
        // Initialize the result to 1 (identity for multiplication).
        long result = 1;
        // Iterate as long as the exponent is greater than 0.
        while (exponent > 0) {
            // If the current bit of exponent is '1', multiply the result by the current base and take modulo
            if ((exponent & 1) == 1) {
                result = (result * base) % mod;
            }
            // Square the base and take modulo for the next bit.
            base = (base * base) % mod;
            // Right shift the exponent to check the next bit.
            exponent >>= 1;
        }
        // Casting the long result back to integer before returning.
        return (int) result;
    }
}
```

C++

```
class Solution {
public:
    int monkeyMove(int numSteps) {
        const int MODULO = 1e9 + 7; // Constant to hold the value for modulo operation

        // Define long long alias to handle large numbers
        using Long = long long;

        // Lambda function to perform quick exponentiation (power)
        // This function calculates (a to the power of n) % MODULO
        auto quickPower = [&](Long base, int exponent) {
            Long result = 1;
            while (exponent > 0) {
                if (exponent & 1) { // If the exponent is odd
                    result = (result * base) % MODULO;
                }
                base = (base * base) % MODULO; // Square the base
                exponent >>= 1; // Divide exponent by 2
            }
            return result;
        };

        // Calculate result using the quickPower lambda function
        // Formula: (2^n - 2 + MODULO) % MODULO
        // It calculates the number of ways the monkey can move (minus 2 invalid ways)
        // And ensures the result is non-negative after modulo operation
        return (quickPower(2, numSteps) - 2 + MODULO) % MODULO;
    }
};
```

TypeScript

```
// Function to calculate the total number of distinct ways a monkey can move.
// Given 'n' steps, the monkey has 2^(n-1) possibilities for the first step
// and the remaining steps follow the same pattern, making the total 2^n - 2 ways.
// The result is modulo (10^9 + 7) to keep the number within integer limits.
function monkeyMove(n: number): number {
    // Define the modulus constant for large number calculations to ensure result is within integer bounds.
    const modulus = 10 ** 9 + 7;

    // Function to calculate (a^b) % modulus using fast exponentiation efficiently.
    // Handles large numbers using BigInt to avoid overflow.
    const quickPowerMod = (base: number, exponent: number): number => {
        let result = 1n; // Use BigInt for the result to handle large numbers.
        base = base % modulus; // Ensure base is within modulus before operations.

        while (exponent > 0) {
            if (exponent & 1) { // If the current exponent bit is 1, multiply to the result.
                result = (result * BigInt(base)) % BigInt(modulus);
            }
            // Square the base and reduce it modulo the modulus for the next iteration.
            base = Number((BigInt(base) * BigInt(base)) % BigInt(modulus));
            exponent >>>= 1; // Right shift exponent to process the next bit.
        }

        return Number(result); // Convert the BigInt result back to a Number before returning.
    };

    // Use the quickPowerMod function to calculate 2^n, subtract 2 for the exact number of moves,
    // and take modulo to handle the possibility of negative results.
    return (quickPowerMod(2, n) - 2 + modulus) % modulus;
}
```

```
class Solution:
    def monkeyMove(self, n: int) -> int:
        # Define the modulo constant as BigIntegers are not efficient
        MOD = 10**9 + 7

        # Calculate 2^n using modular exponentiation, which is efficient for large powers
        total_ways = pow(2, n, MOD)

        # Subtract 2 because the monkey cannot stay in the first or last column; wrap with MOD to keep result positive
        valid_ways = (total_ways - 2) % MOD

        # Return the number of valid ways the monkey can move
        return valid_ways
```

Time and Space Complexity

The given Python code computes the result of `2^n - 2`, modulo `10^9 + 7`. It uses the built-in `pow` function optimized for modular exponentiation.

Time Complexity:

The primary operation of computing `2^n` modulo `10^9 + 7` is performed using Python's built-in `pow` function. This function uses fast exponentiation to compute the result, having a time complexity of `O(log n)`, since it effectively halves the exponent in each step of exponentiation.

Post exponentiation, the subtraction and the modulo operation each take constant time, `O(1)`.

Thus, the time complexity of the entire `monkeyMove` function is `O(log n)`.

Space Complexity:

The space complexity of the code is `O(1)` since it uses a constant amount of additional space. There are no data structures being used which grow with the input size `n`. All operations handle intermediate values which require a constant amount of space.