1458. Max Dot Product of Two Subsequences

Hard Array Dynamic Programming Leetcode Link

Problem Description

In this problem, we are given two integer arrays `nums1` and `nums2`. We need to find the maximum dot product between non-empty subsequences of `nums1` and `nums2` that have the same length. A subsequence is derived from the original array by potentially removing some elements without changing the order of the remaining elements. The dot product of two sequences of the same length is the sum of the pairwise products of their corresponding elements.

To illustrate, if we have a subsequence `[a1, a2, a3, ..., ai]` from `nums1` and `[b1, b2, b3, ..., bi]` from `nums2`, the dot product is `a1*b1 + a2*b2 + a3*b3 + ... + ai*bi`.

Our goal is to find such subsequences from `nums1` and `nums2` that when we calculate their dot product, we get the maximum possible value.

Intuition

The intuition behind the solution comes from recognizing that this problem can be solved optimally by breaking it down into simpler subproblems. This is a hint that dynamic programming might be a useful approach. More specifically, we can use a 2D dynamic programming table `dp` where `dp[i][j]` represents the maximum dot product between the first `i` elements of `nums1` and the first `j` elements of `nums2`.

When we compute the entry `dp[i][j]`, we consider the following possibilities to maximize our result:

1. `dp[i - 1][j]` — the maximum dot product without including the current element from `nums1`.
2. `dp[i][j - 1]` — the maximum dot product without including the current element from `nums2`.
3. `dp[i - 1][j - 1] + nums1[i - 1] * nums2[j - 1]` — the maximum dot product including the current elements from both `nums1` and `nums2`.
4. If `dp[i - 1][j - 1]` is less than 0, we only consider the product `nums1[i - 1] * nums2[j - 1]` because we would not want to diminish our result by adding a negative dot product from previous elements.

By computing the maximum of these options at each entry of the `dp` table, we ensure that we have considered all possibilities and end up with the maximum dot product of subsequences of the same length from `nums1` and `nums2`.

Solution Approach

The solution to this problem involves using a 2D dynamic programming approach, which is a common pattern when dealing with problems of sequences and subproblems that depend on previous decisions.

We start by creating a 2D array `dp` with dimensions `(m + 1) x (n + 1)`, where `m` is the length of `nums1` and `n` is the length of `nums2`. We use `m + 1` and `n + 1` because we want to have an extra row and column to handle the base cases where the subsequence length is zero from either of the arrays. We initialize all values in the `dp` array to negative infinity (`-inf`) to represent the minimum possible dot product.

The dynamic programming algorithm iteratively fills the `dp` array as follows:

1. We loop over each possible subsequence length for `nums1` (denoted by `i`) and `nums2` (denoted by `j`) starting from 1 because index 0 is the base case representing an empty subsequence.
2. For each `i, j` pair, we calculate the dot product of the last elements `v` by multiplying `nums1[i - 1] * nums2[j - 1]`.
3. We then fill in `dp[i][j]` by taking the maximum of:
   - `dp[i - 1][j]`: The max dot product without including `nums1[i - 1]`;
   - `dp[i][j - 1]`: The max dot product without including `nums2[j - 1]`;
   - `max(dp[i - 1][j - 1], 0) + v`: The max dot product including the current elements of both arrays. We use `max(dp[i - 1][j - 1], 0)` because if the previous dot product is negative, we would get a better result by just taking the current product `v`.

The reason we initialize with negative infinity and consider the `max(dp[i-1][j-1], 0)` in our recurrence is to ensure that we do not forcefully include negative products that would decrease the total maximum dot product. However, we need to include at least one pair of elements from both subsequences, as the result cannot be an empty subsequence according to the problem statement.

After filling the `dp` array, the last cell `dp[m][n]` contains the maximum dot product of non-empty subsequences of `nums1` and `nums2`.

Here is the code snippet again that reflects this approach:

```python
class Solution:
    def maxDotProduct(self, nums1: List[int], nums2: List[int]) -> int:
        m, n = len(nums1), len(nums2)
        dp = [[-inf] * (n + 1) for _ in range(m + 1)]
        for i in range(1, m + 1):
            for j in range(1, n + 1):
                v = nums1[i - 1] * nums2[j - 1]
                dp[i][j] = max(dp[i - 1][j], dp[i][j - 1], max(dp[i - 1][j - 1], 0) + v)
        return dp[-1][-1]
```

This code correctly solves the problem by leveraging the power of dynamic programming to optimize the process of finding the maximum dot product.

Example Walkthrough

Let's consider two small arrays for simplicity:

- `nums1 = [2, 3]`
- `nums2 = [1, 2]`

Using the dynamic programming approach described in the problem solution, we will create a 2D `dp` array with dimensions 3×3 (since `nums1` and `nums2` both have lengths 2, and we include an extra row and column for the base case):

```
dp array initialization (values are -inf except dp[0][*] and dp[*][0] which could be set as 0 for base case):
[  0, (-inf), (-inf) ]
[ (-inf), (-inf), (-inf) ]
[ (-inf), (-inf), (-inf) ]
```

We start populating the `dp` array at `dp[1][1]`:

- At `i = 1`, `j = 1`:
  - `nums1[i - 1] * nums2[j - 1]` is `2 * 1` which equals `2`.
  - The entries `dp[i - 1][j]`, `dp[i][j - 1]`, and `dp[i - 1][j - 1]` are all `0` as they refer to base cases.
  - We select the maximum: `max(0, 0, 0 + 2)` which is `2`.
  - Now, `dp[1][1]` is updated to `2`.
- At `i = 1`, `j = 2`:
  - `nums1[i - 1] * nums2[j - 1]` is `2 * 2` which equals `4`.
  - We consider the maximum of `dp[1][1]`, `dp[1][0]`, and `dp[0][1] + 4` which are `2`, `0`, and `4`, respectively.
  - The maximum value is `4`, so we update `dp[1][2]` to `4`.
- At `i = 2`, `j = 1`:
  - `nums1[i - 1] * nums2[j - 1]` is `3 * 1` which equals `3`.
  - We take the maximum of `dp[2][0]`, `dp[1][1]`, and `0 + 3` (since `dp[i - 1][j - 1]` is `0`), which are `0`, `2`, and `3`.
  - The maximum value is `3`, so `dp[2][1]` is updated to `3`.
- At `i = 2`, `j = 2`:
  - `nums1[i - 1] * nums2[j - 1]` is `3 * 2` which equals `6`.
  - Now, we consider `max(dp[2][1], dp[1][2], dp[1][1] + 6)` which are `3`, `4`, and `8`.
  - The maximum value is `8`, so we update `dp[2][2]` to `8`.

After finishing the iteration, the final `dp` array looks like this:

```
[  0,    0,    0  ]
[  0,    2,    4  ]
[  0,    3,    8  ]
```

- The value `dp[2][2]` which is `8` represents the maximum dot product of non-empty subsequences of `nums1` and `nums2`.

Thus, the maximum dot product obtained from the subsequences `[2, 3]` from `nums1` and `[1, 2]` from `nums2` is `8`, which is the dot product of the two arrays. Since these arrays are both the full length of the originals and we are looking for any subsequences, this would indeed be the maximum dot product in this simple example.

Python Solution

```python
from typing import List

class Solution:
    def max_dot_product(self, nums1: List[int], nums2: List[int]) -> int:
        # Initialize the lengths of the two input lists
        len_nums1, len_nums2 = len(nums1), len(nums2)

        # Initialize a 2D DP array filled with negative infinity
        dp = [[float('-inf')] * (len_nums2 + 1) for _ in range(len_nums1 + 1)]

        # Build the DP table row by row, column by column
        for i in range(1, len_nums1 + 1):
            for j in range(1, len_nums2 + 1):
                # Calculate the dot product of the current elements
                dot_product = nums1[i - 1] * nums2[j - 1]

                # Update the DP table by considering:
                # 1. The previous row at the same column
                # 2. The same row at the previous column
                # 3. The previous row and column plus the current dot_product,
                #    ensuring that if the previous value is negative, zero is used instead
                dp[i][j] = max(dp[i - 1][j], dp[i][j - 1], max(dp[i - 1][j - 1], 0) + dot_product)

        # Return the last element of the DP array which contains the maximum dot product
        return dp[-1][-1]

```

Java Solution

```java
class Solution {

    // Method to calculate the maximum dot product between two arrays
    public int maxDotProduct(int[] nums1, int[] nums2) {
        // Lengths of the input arrays
        int length1 = nums1.length, length2 = nums2.length;

        // Create a DP table with an extra row and column for the base case
        int[][] dpTable = new int[length1 + 1][length2 + 1];

        // Initialize the DP table with the minimum integer values
        for (int[] row : dpTable) {
            Arrays.fill(row, Integer.MIN_VALUE);
        }

        // Iterate over the arrays to populate the DP table
        for (int i = 1; i <= length1; ++i) {
            for (int j = 1; j <= length2; ++j) {
                // Taking the maximum between the current value, ignoring current elements of nums1 or nums2
                dpTable[i][j] = Math.max(dpTable[i - 1][j], dpTable[i][j - 1]);

                // Determine the maximum value by considering the current elements and previous subsequence's result
                dpTable[i][j] = Math.max(dpTable[i][j], Math.max(0, dpTable[i - 1][j - 1]) + nums1[i - 1] * nums2[j - 1]);
            }
        }

        // Return the result from the DP table which contains the maximum dot product
        return dpTable[length1][length2];
    }
}

```

C++ Solution

```cpp
#include <vector>
#include <algorithm>
#include <climits>

class Solution {
public:
    // Function to calculate the maximum dot product between two sequences.
    int maxDotProduct(vector<int>& nums1, vector<int>& nums2) {
        int numRows = nums1.size(); // Size of the first sequence.
        int numCols = nums2.size(); // Size of the second sequence.

        // Create a 2D DP (Dynamic Programming) table with all elements initialized to INT_MIN.
        vector<vector<int>> dp(numRows + 1, vector<int>(numCols + 1, INT_MIN));

        // Building the DP table by considering each possible pair of elements from nums1 and nums2.
        for (int i = 1; i <= numRows; ++i) {
            for (int j = 1; j <= numCols; ++j) {
                // Current dot product value.
                int currentDotProduct = nums1[i - 1] * nums2[j - 1];

                // Choosing the maximum between not taking the current pair, or taking the current pair.
                // First, consider the maximum value from ignoring the current pair (up or left in DP table).
                dp[i][j] = max(dp[i - 1][j], dp[i][j - 1]);

                // Then, consider the maximum value from taking the current pair, which is the current dot product
                // plus the maximum dot product without both elements (up-left diagonally in DP table), unless negative,
                // in which case use zero (as dot products with negative results don't contribute to the maximum).
                dp[i][j] = max(dp[i][j], max(0, dp[i - 1][j - 1]) + currentDotProduct);
            }
        }

        // The maximum dot product for the sequences will be in the bottom-right corner of the DP table.
        return dp[numRows][numCols];
    }
};

```

Typescript Solution

```typescript
function maxDotProduct(nums1: number[], nums2: number[]): number {
    const numRows = nums1.length; // Size of the first sequence
    const numCols = nums2.length; // Size of the second sequence

    // Create a 2D DP (Dynamic Programming) table initialized to negative infinity
    const dp: number[][] = Array.from({ length: numRows + 1 }, () =>
                                      Array(numCols + 1).fill(-Infinity));

    // Building the DP table by considering each possible pair of elements from nums1 and nums2
    for (let i = 1; i <= numRows; i++) {
        for (let j = 1; j <= numCols; j++) {
            // Current dot product value
            const currentDotProduct = nums1[i - 1] * nums2[j - 1];

            // Choose the maximum between not taking the current pair, or taking the current pair
            // First, consider the maximum value from ignoring the current pair (above or to the left in DP table)
            dp[i][j] = Math.max(dp[i - 1][j], dp[i][j - 1]);

            // Then, consider the maximum value from taking the current pair, which includes the current dot product
            // plus the maximum dot product without both elements (diagonally above to the left in DP table); if this value
            // is negative, use zero instead, as dot products with negative results don't contribute to the maximum
            dp[i][j] = Math.max(dp[i][j], Math.max(0, dp[i - 1][j - 1]) + currentDotProduct);
        }
    }

    // The maximum dot product for the sequences will be in the bottom-right corner of the DP table
    return dp[numRows][numCols];
}

```

Time and Space Complexity

The time complexity of the given code is `O(m * n)`, where `m` is the length of `nums1` and `n` is the length of `nums2`. This is because there are two nested loops, each iterating through the elements of `nums1` and `nums2` respectively.

The space complexity of the code is also `O(m * n)`. This is due to the allocation of a 2D array `dp` of size `(m + 1) * (n + 1)` to store the intermediate results for each pair of indices `(i, j)`.