862. Shortest Subarray with Sum at Least K

Hard | Queue | Array | Binary Search | Prefix Sum | Sliding Window | Monotonic Queue | Heap (Priority Queue)

Problem Description

The problem asks us to find the length of the shortest contiguous subarray within an integer array `nums` such that the sum of its elements is at least a given integer `k`. If such a subarray does not exist, we are to return `-1`. The attention is on finding the minimum-length subarray that meets or exceeds the sum condition.

Intuition

To solve this efficiently, we utilize a monotonic queue and prefix sums technique. The intuition behind using prefix sums is that they allow us to quickly calculate the sum of any subarray in constant time. This makes the task of finding subarrays with a sum of at least `k` much faster, as opposed to calculating the sum over and over again for different subarrays.

A prefix sum array `s` is an array that holds the sum of all elements up to the current index. So for any index `i`, `s[i]` is the sum of `nums[0] + nums[1] + ... + nums[i-1]`.

To understand the monotonic queue, which is a Double-Ended Queue (deque) in this case, let's look at its properties:

1. It is used to maintain a list of indices of prefix sums in increasing order.
2. When we add a new prefix sum, we remove all the larger sums at the end of the queue because the new sum and any future sums would always be better choices (smaller subarray) for finding a subarray of sum `k`.
3. We also continuously check if the current prefix sum minus the prefix sum at the start of the queue is at least `k`. If it is, we found a candidate subarray, and update `ans` with the subarray's length. Then we can pop that element off the queue since we've already considered this subarray and it won't be needed for future calculations.

In summary, the prefix sums help us quickly compute subarray sums, and the monotonic queue lets us store and traverse candidate subarray ranges efficiently, ensuring we always have the smallest length subarray that meets the sum condition, thus arriving at the optimal solution.

Solution Approach

The solution makes use of prefix sums and a monotonic queue, specifically a deque, to achieve an efficient algorithm to find the shortest subarray summing to at least `k`. Let's explore the steps involved:

1. **Prefix Sum Calculation**: We initiate the computation by creating a list called `s` that contains the cumulative sum of the `nums` list, by using the `accumulate` function with an `initial` parameter set to `0`. This denotes that the first element of `s` is `0` and is a requirement to consider subarrays starting at index `0`.

2. **Initialization**: A deque `q` is initialized to maintain the monotonic queue of indices. A variable `ans` is initialized to `inf` (infinity) which will later store the smallest length of a valid subarray.

3. **Iterate Over Prefix Sums**: We iterate over each value `v` in the prefix sums array `s` and its index `i`.

4. **Deque Front Comparison**: While there are elements in `q` and the current prefix sum `v` minus the prefix sum at `q[0]` (the front of the deque) is greater than or equal to `k`, we've found a subarray that meets the requirement. We then compute its length `i - q.popleft()` and update `ans` with the minimum of `ans` and this length. The `popleft()` operation removes this index from the deque as it is no longer needed.

5. **Deque Back Optimization**: Before appending the current index `i` to `q`, we pop indices from the back of the deque if their corresponding prefix sums are greater than or equal to `v` because these are not conducive to the smallest length requirement and will not be optimal candidates for future comparisons.

6. **Index Appending**: Append the current index `i` to `q`. This index represents the right boundary for the potential subarray sums evaluated in future iterations.

After the loop, two cases may arise:

- If `ans` remains `inf`, it means no valid subarray summing to at least `k` was found, so we return `-1`.
- Otherwise, we return the value stored in `ans` as it represents the length of the shortest subarray that fulfills the condition.

Overall, the algorithm smartly maintains a set of candidate indices for the start of the subarray in a deque, ensuring that only those that can potentially give a smaller length subarray are considered. The key here is to understand how the prefix sum helps us quickly calculate the sum of a subarray and how the monotonically increasing property of the queue ensures we always get the shortest length possible. The use of these data structures makes the solution capable of working in O(n) time complexity.

Example Walkthrough

Let's illustrate the solution approach with an example. Suppose we have the following array and target sum `k`:

```
nums = [2, 1, 5, 2, 3, 2]
k = 7
```

We want to find the length of the smallest contiguous subarray with a sum greater than or equal to `7`. Here's how we would apply the solution approach:

1. **Prefix Sum Calculation**: We compute the prefix sum array `s`:

   ```
   s = [0, 2, 3, 8, 10, 13, 15]
   ```

   Notice that `s[0]` is `0` because we've added it artificially to account for subarrays starting at index 0.

2. **Initialization**: We create a deque `q` to maintain indices of prefix sums:

   ```
   q = []
   ```

   And initialize `ans` to `inf`:

   ```
   ans = inf
   ```

3. **Iterate Over Prefix Sums**: We iterate over `s`, looking for subarrays that sum up to at least `k`.

4. **Deque Front Comparison**: As we proceed, when we reach `s[3] = 8` (consider `nums[0..2]`), we find it is greater than or equal to `k`. The `q` is empty, so we just move on.

5. **Deque Back Optimization**: Before appending index `3` to `q`, we don't remove anything from `q` because it's still empty. So we append `3` into `q`.

6. **Index Appending**: Our `q` is now `[3]`.

   Continuing the iteration, we eventually come to `s[5] = 13`, which is the sum up to `nums[0..4]`.

   We have a non-empty `q`, and `s[5] - s[q[0]] = 13 - 8 = 5`, which is not greater than or equal to `k`. So we can't pop anything from the queue yet.

7. Once we reach `s[6] = 15`, we notice that `15 - 8 = 7` is exactly our `k`. We then calculate the subarray length `6 - q.popleft() = 6 - 3 = 3`. Now the `ans` becomes `3`, the smallest subarray `[5, 2, 3]` found so far.

8. **Deque Back Optimization** is also done each time before we append a new index, which keeps the indices in the deque monotonically increasing in sums.

After considering all elements in `s`, our `ans` is `3`, as no smaller subarray summing to at least `7` is found. So we return `3` as the length of the shortest subarray. If `ans` was still `infinity`, we would return `-1` indicating no such subarray was found.

Python Solution

```python
from collections import deque
from itertools import accumulate
from math import inf

class Solution:
    def shortest_subarray(self, nums: List[int], k: int) -> int:
        # Calculate the prefix sums of nums with an initial value of 0
        prefix_sums = list(accumulate(nums, initial=0))
        # Initialize a double-ended queue to store indices
        indices_deque = deque()
        # Set the initial answer to infinity, as we are looking for the minimum
        min_length = inf

        # Enumerate over the prefix sums to find the shortest subarray
        for current_index, current_sum in enumerate(prefix_sums):
            # If the current_sum minus the sum at the front of the deque is at least k,
            # update the min_length and pop from the deque
            while indices_deque and current_sum - prefix_sums[indices_deque[0]] >= k:
                min_length = min(min_length, current_index - indices_deque.popleft())
            # Remove indices from the back of the deque if their prefix sums are greater
            # than or equal to current_sum, as they are not useful anymore
            while indices_deque and prefix_sums[indices_deque[-1]] >= current_sum:
                indices_deque.pop()
            # Add the current index to the back of the deque
            indices_deque.append(current_index)

        # Return -1 if no such subarray exists, otherwise the length of the shortest subarray
        return -1 if min_length == inf else min_length

```

Java Solution

```java
class Solution {

    // Function to find the length of the shortest subarray with a sum at least 'k'
    public int shortestSubarray(int[] nums, int k) {
        int n = nums.length; // Get the length of the input array
        long[] prefixSums = new long[n + 1]; // Create an array to store prefix sums

        // Calculate prefix sums
        for (int i = 0; i < n; ++i) {
            prefixSums[i + 1] = prefixSums[i] + nums[i];
        }

        // Initialize a deque to keep track of indices
        Deque<Integer> indexDeque = new ArrayDeque<>();
        int minLength = n + 1; // Initialize the minimum length to an impossible value (larger than the array itself)

        // Iterate over the prefix sums
        for (int i = 0; i <= n; ++i) {

            // While the deque is not empty, check if the current sum minus the front value is >= k
            while (!indexDeque.isEmpty() && prefixSums[i] - prefixSums[indexDeque.peek()] >= k) {
                minLength = Math.min(minLength, i - indexDeque.poll()); // If true, update minLength
            }

            // While the deque is not empty, remove all indices from the back that have a prefix sum greater than or equal to the cur
            while (!indexDeque.isEmpty() && prefixSums[indexDeque.peekLast()] >= prefixSums[i]) {
                indexDeque.pollLast();
            }

            // Add the current index to the deque
            indexDeque.offer(i);
        }

        // If minLength is still greater than the length of the array, there is no valid subarray, return -1
        return minLength > n ? -1 : minLength;
    }
}

```

C++ Solution

```cpp
#include <vector>
#include <deque>
#include <algorithm> // For std::min

class Solution {
public:
    // Function to find the length of shortest subarray with sum at least K
    int shortestSubarray(vector<int>& nums, int k) {
        int n = nums.size();
        // Prefix sum array with an extra slot for ease of calculations
        vector<long> prefixSum(n + 1, 0);
        // Calculate the prefix sums
        for (int i = 0; i < n; ++i) {
            prefixSum[i + 1] = prefixSum[i] + nums[i];
        }

        // Double ended queue to store indices of the prefix sums
        deque<int> indices;
        // Initialize the answer with maximum possible length + 1
        int minLength = n + 1;

        // Loop through all prefix sum entries
        for (int i = 0; i <= n; ++i) {
            // If the current subarray (from front of deque to i) has sum >= k
            while (!indices.empty() && prefixSum[i] - prefixSum[indices.front()] >= k) {
                // Update the minimum length
                minLength = min(minLength, i - indices.front());
                // Pop the front index since we found a shorter subarray ending at index i
                indices.pop_front();
            }
            // While the last index in the deque has a prefix sum larger than or equal to current
            // we can discard it, since better candidates for subarray start are available
            while (!indices.empty() && prefixSum[indices.back()] >= prefixSum[i]) {
                indices.pop_back();
            }
            // Add current index to the deque
            indices.push_back(i);
        }

        // If no valid subarray is found, minLength remains > n. Return -1 in that case.
        return minLength > n ? -1 : minLength;
    }
};

```

Typescript Solution

```typescript
// Importing required modules
import { Deque } from 'collections/deque'; // Assume a Deque implementation like "collections.js" or similar

// Function to find the length of shortest subarray with sum at least K
function shortestSubarray(nums: number[], k: number): number {
    let n = nums.length;
    // Prefix sum array with an extra slot for ease of calculations
    let prefixSum: number[] = new Array(n + 1).fill(0);
    // Calculate the prefix sums
    for (let i = 0; i < n; ++i) {
        prefixSum[i + 1] = prefixSum[i] + nums[i];
    }

    // Double-ended queue to store indices of the prefix sums
    let indices: Deque<number> = new Deque<number>();
    // Initialize the answer with maximum possible length + 1
    let minLength = n + 1;

    // Loop through all prefix sum entries
    for (let i = 0; i <= n; ++i) {
        // If the current subarray (from front of deque to i) has a sum >= k
        while (!indices.isEmpty() && prefixSum[i] - prefixSum[indices.peekFront()!] >= k) {
            // Update the minimum length
            minLength = Math.min(minLength, i - indices.peekFront()!);
            // Pop the front index since we found a shorter subarray ending at index i
            indices.shift();
        }
        // While the last index in the deque has a prefix sum larger than or equal to the current
        // we can discard it, since better candidates for subarray start are available
        while (!indices.isEmpty() && prefixSum[indices.peekBack()!] >= prefixSum[i]) {
            indices.pop();
        }
        // Add current index to the deque
        indices.push(i);
    }

    // If no valid subarray is found, minLength remains > n. Return -1 in that case.
    return minLength > n ? -1 : minLength;
}

// Example usage:
// const nums = [2, -1, 2];
// const k = 3;
// console.log(shortestSubarray(nums, k)); // Output should be 3

```

Time and Space Complexity

The given Python code is for finding the length of the shortest contiguous subarray whose sum is at least `k`. The code uses the concept of prefix sums and a monotonic queue to keep track of potential candidates for the shortest subarray.

Time Complexity

The time complexity of the code is `O(N)`, where `N` is the length of the input array `nums`. This is because:

- The prefix sum array `s` is computed using `itertools.accumulate`, which is a single pass through the array, thus taking `O(N)` time.
- The deque `q` is maintained by iterating through each element of the array once. Each element is added and removed from the deque at most once. Since the operations of adding to and popping from a deque are `O(1)`, the loop operations are also `O(N)` in total.
- Inside the loop, `q.popleft()` and `q.pop()` are each called at most once per iteration, and as a result, each element in `nums` contributes at most `O(1)` to the time complexity.

Space Complexity

The space complexity of the code is `O(N)` as well:

- The prefix sum array `s` requires `O(N)` space.
- The deque `q` potentially stores indices from the entire array in the worst-case scenario. In the worst case, it could hold all indices in `nums`, also requiring `O(N)` space.
- Apart from these two data structures, the code uses a constant amount of space for variables such as `ans` and `v`, which does not depend on the input size.