2400. Number of Ways to Reach a Position After Exactly k Steps

Medium Math Dynamic Programming Combinatorics Leetcode Link

Problem Description

You start at a `startPos` on an infinite number line and want to reach an `endPos`. You can take steps to the left or right, and you have to take exactly `k` steps to do this. The goal is to find how many different ways you can do this. A way is different from another if the sequence of steps is different. Since you are moving on an infinite number line, negative positions are possible. The position reached after `k` steps must be exactly `endPos`. Since the number of ways could be very large, you need to return the result modulo `10^9 + 7` to keep the number manageable.

Intuition

Given the need to move a specific number of steps and end at a specific position, this is a classic problem that can be addressed using Dynamic Programming or memoization to avoid redundant calculations.

First, consider the absolute difference between the `startPos` and `endPos`: this is the minimum number of moves required to get from start to end without any extra steps. From there, any additional steps must eventually be counteracted by an equal number of steps in the opposite direction.

To calculate the number of ways, you can use recursion. With each recursive call, you decrement `k` because you're making a step. If `k` reaches 0, you check if you've arrived at `endPos`: if so, that's one valid way; if not, you return 0 as it's not a valid sequence of steps.

The solution uses a depth-first search (DFS) approach where each 'node' is a position after a certain number of steps. The DFS explores all possible sequences of left and right steps until `k` steps are performed. Memoization (`@cache` decorator) is key here, as it stores the result of previous computations. This is crucial since there are many overlapping subproblems, especially as `k` grows larger.

The `dfs(i, j)` function checks whether you can reach the relative position `i` (`startPos - endPos`) in `j` steps. It evaluates two scenarios at each step: moving right and moving left. The modulus operation is used on the sum of possibilities to keep the numbers within the constraints of the modulo.

Since movement on the number line is symmetrical, taking an absolute of the `i` when moving left is a significant optimization. Movements `X` steps to the right and `Y` steps to the left result in the same position as `Y` steps to the right and `X` steps to the left if `X` and `Y` are flipped. So you can always consider moves to the right and moves toward zero, enabling the `dfs` function to memoize more effectively as it encounters fewer unique `(i, j)` pairs.

Solution Approach

The solution employs depth-first search (DFS) with memoization to systematically explore all possible step sequences that amount to exactly `k` steps and arrive at `endPos`. The following patterns and algorithms are used:

1. **Recursion**: The `dfs(i, j)` function calls itself to explore each possibility. It increments or decrements the current position, respectively, for a right or left step, and decreases the remaining steps `j` by one.
2. **Base Cases**: The function has two critical base cases:
   - If `j == 0` (no more steps left), check if the current position `i` is 0 (which means `startPos` == `endPos`). If so, return 1, representing one valid way to reach the end position; otherwise return 0, as it's not possible to reach `endPos` without steps left.
   - If `i > j` (the current position is farther from 0 than the number of steps remaining) or `j < 0` (somehow the steps went negative, which should not happen in this approach), return 0 because reaching `endPos` is impossible.
3. **Memoization**: The `@cache` decorator is used in Python to store the results of `dfs` calls, which prevents redundant calculations of the same `(i, j)` state. Since many positions will be visited multiple times with the same number of remaining steps, storing these results significantly speeds up the computation.
4. **Modulo Arithmetic**: The modulo operation `% mod` ensures that intermediate sums do not exceed the problem's specified limit (`10**9 + 7`). This is done to avoid integer overflow and is a common practice in problems dealing with large numbers.
5. **Symmetry and Absolute Value**: When a step is taken to the left, the function calls itself with `abs(i - 1)` instead of `-(i - 1)`. This is because taking a step to the left from position `i` is equivalent to taking a step to the right from position `-i` due to the line's symmetry. Using the absolute value maximizes the effectiveness of memoization because it reduces the number of unique states.

The code does not explicitly create a data structure for memoization; it relies on the `cache` mechanism provided by the Python `functools` module, which internally creates a mapping of arguments to the `dfs` function to their results.

Finally, the result of the `dfs` function call for `abs(startPos - endPos), k` gives us the number of ways to reach from `startPos` to `endPos` in `k` steps. It respects the symmetry of the problem and allows for an efficient computation of the result.

Example Walkthrough

Let's go through a small example to illustrate the solution approach.

Suppose you start at `startPos = 1` and want to reach `endPos = 3`. You have `k = 3` steps to do it. How many different ways can you do this?

1. First, check the absolute difference between `startPos` and `endPos`, which is `|3 - 1| = 2`. This is the minimum number of steps to reach from start to end without any extra steps.
2. To account for the exact `k` steps, you can take 1 additional step to the left and then move back to the right, resulting in 3 total steps (right → left → right).

Let's use the `dfs(i, j)` recursion with memoization to explore the possibilities:

- The initial call is `dfs(abs(1-3), 3)` which simplifies to `dfs(2, 3)`.
- The `dfs` function will now calculate this in two ways: consider a step to the left and consider a step to the right.
- When considering a step to the right (`dfs(2+1, 2)`), the state becomes `dfs(3, 2)`.
  - Here, since `i = 3` is greater than `j = 2`, we return `0` because you can't reach position `0` in just `2` steps.
- When considering a step to the left (`dfs(abs(2-1), 2)`), the state becomes `dfs(1, 2)`.
  - Now, for `dfs(1, 2)`, again we consider steps left and right:
    - Step to the right: `dfs(1+1, 1)` simplifies to `dfs(2, 1)`, which returns 0 because `i > j`.
    - Step to the left: `dfs(abs(1-1), 1)` simplifies to `dfs(0, 1)`.
      - At this point we have `dfs(0, 1)`. Considering further steps:
        - Step to the right: `dfs(0+1, 0)` simplifies to `dfs(1, 0)`, which returns 0 because `i` is not equal to `0`.
        - Step to the left: `dfs(abs(0-1), 0)` simplifies to `dfs(1, 0)`, which also returns 0.

However, from the initial `dfs(1, 2)`, if we make one more DFS call with one step to the left (`dfs(abs(1-1), 1)`), the result is `dfs(0, 1)`. This state has already been evaluated when it was encountered via the `dfs(1, 2)` from the first step to the right. Hence, it will not compute again but fetch the result from the memoization cache, which should be zero since we learned no way exists to end at `0` starting from `1` with `1` step.

To summarize, from `startPos = 1` to `endPos = 3` with `k = 3` steps, there are no valid ways to achieve the end position.

The use of memoization is crucial in this example because it avoids the re-computation of states that have already been visited, such as `dfs(1, 2)` being re-computed after `dfs(1, 0)`. By caching the results, we speed up the computation and also maintain the count of possible ways within the required modulo.

Python Solution

```python
from functools import lru_cache  # Python 3 caching decorator for optimization

class Solution:
    def numberOfWays(self, start_pos: int, end_pos: int, num_steps: int) -> int:
        # Define the modulo constant to avoid large numbers
        MOD = 10 ** 9 + 7

        @lru_cache(maxsize=None)  # Cache the results of the function calls
        def dfs(current_pos: int, steps_remaining: int) -> int:
            # We can't reach the target if we are too far or steps are negative
            if current_pos > steps_remaining or steps_remaining < 0:
                return 0
            # If no steps remaining, check if we are at the start
            if steps_remaining == 0:
                return 1 if current_pos == 0 else 0
            # Calculate the number of ways by going one step forward or backward
            # Use modulo operation to keep numbers in a reasonable range
            return (dfs(current_pos + 1, steps_remaining - 1) +
                    dfs(abs(current_pos - 1), steps_remaining - 1)) % MOD

        # The absolute difference gives the minimum number of steps required
        # Call the dfs function with the absolute difference and the number of steps
        return dfs(abs(start_pos - end_pos), num_steps)

solution_instance = Solution()
print(solution_instance.numberOfWays(1, 2, 3))  # Example call to the method

```

Java Solution

```java
class Solution {
    // A memoization table to avoid redundant calculations
    private Integer[][] memo;
    // Define a constant for the mod value as required by the problem
    private static final int MOD = 1000000007;

    public int numberOfWays(int startPos, int endPos, int k) {
        // Initialize the memoization table with null values
        memo = new Integer[k + 1][2 * k + 1];
        // Calculate the difference in positions as the first dimension for memo
        int offset = k; // Offset is used to handle negative indices
        // Call the helper method to compute the result
        return dfs(Math.abs(startPos - endPos), k, offset);
    }

    // A helper method to find the number of ways using depth-first search
    private int dfs(int posDiff, int stepsRemain, int offset) {
        // If position difference is greater than the remaining steps, or steps are negative, return 0 as it's not possible
        if (posDiff > stepsRemain || stepsRemain < 0) {
            return 0;
        }
        // If there are no remaining steps, check if the current position difference is zero
        if (stepsRemain == 0) {
            return posDiff == 0 ? 1 : 0;
        }
        // Check the memo table to avoid redundant calculations
        if (memo[posDiff][stepsRemain + offset] != null) {
            return memo[posDiff][stepsRemain + offset];
        }
        // Calculate the result by considering a step in either direction
        long ans = dfs(posDiff + 1, stepsRemain - 1, offset) + dfs(Math.abs(posDiff - 1), stepsRemain - 1, offset);
        ans %= MOD;
        // Save the result to the memoization table before returning
        memo[posDiff][stepsRemain + offset] = (int)ans;
        return memo[posDiff][stepsRemain + offset];
    }
}

```

C++ Solution

```cpp
#include <vector>
#include <functional>
#include <cstring>

class Solution {
public:
    int numberOfWays(int startPos, int endPos, int k) {
        const int MOD = 1e9 + 7; // Define the modulo constant.
        // Create a memoization table initialized to -1 for DP.
        std::vector<std::vector<int>> dp(k + 1, std::vector<int>(k + 1, -1));

        // Define the function 'dfs' for Depth First Search with memoization.
        std::function<int(int, int)> dfs = [&](int distance, int stepsRemaining) -> int {
            if (distance > stepsRemaining || stepsRemaining < 0) {
                return 0; // Base case: If distance is more than steps or steps are negative, return 0.
            }
            if (stepsRemaining == 0) {
                return distance == 0 ? 1 : 0; // Base case: If no steps left, return 1 if distance is 0, else 0.
            }
            if (dp[distance][stepsRemaining] != -1) {
                return dp[distance][stepsRemaining]; // If value is memoized, return it.
            }
            // Calculate the number of ways by moving to the right or to the left and apply modulo.
            dp[distance][stepsRemaining] = (dfs(distance + 1, stepsRemaining - 1) +
                                            dfs(std::abs(distance - 1), stepsRemaining - 1)) % MOD;
            return dp[distance][stepsRemaining]; // Return the number of ways for current state.
        };

        // Call 'dfs' with the absolute distance from startPos to endPos and k steps available.
        return dfs(std::abs(startPos - endPos), k);
    }
};

```

Typescript Solution

```typescript
// The function calculates the number of ways to reach the end position from the start position
// in exactly k steps.
function numberOfWays(startPos: number, endPos: number, k: number): number {
    const mod = 10 ** 9 + 7; // Define the modulo for large numbers to avoid overflow

    // Initialize a memoization table to store intermediate results with default value -1
    const memoTable = new Array(k + 1).fill(0).map(() => new Array(k + 1).fill(-1));

    // Depth-first search function to calculate the number of paths recursively
    const dfs = (currentPos: number, stepsRemaining: number): number => {
        // Base case: If the current position is greater than the remaining steps
        // or remaining steps are less than 0 then return 0 since it's not possible
        if (currentPos > stepsRemaining || stepsRemaining < 0) {
            return 0;
        }
        // Base case: If no steps are remaining, check if we are at starting position
        if (stepsRemaining === 0) {
            return currentPos === 0 ? 1 : 0;
        }

        // If result is already calculated for the given state, return the cached result
        if (memoTable[currentPos][stepsRemaining] !== -1) {
            return memoTable[currentPos][stepsRemaining];
        }

        // Recursive case: sum the number of ways from moving forward and backward
        // Note: When moving backward, if currentPos is already 0, it will stay at 0 since
        // we can't move to a negative position on the number line
        memoTable[currentPos][stepsRemaining] =
            dfs(currentPos + 1, stepsRemaining - 1) + // Move forward
            dfs(Math.abs(currentPos - 1), stepsRemaining - 1); // Move backward

        memoTable[currentPos][stepsRemaining] %= mod; // Apply modulo operation

        return memoTable[currentPos][stepsRemaining];
    };

    // Kick off the dfs with the absolute distance to cover and the total steps.
    return dfs(Math.abs(startPos - endPos), k);
}

```

Time and Space Complexity

The given Python code defines a `Solution` class with a method `numberOfWays` that calculates the number of ways to reach the `endPos` from the `startPos` in exactly `k` steps. The solution uses a depth-first search (DFS) approach with memoization.

Time Complexity

The time complexity of this algorithm largely depends on the parameters `startPos`, `endPos`, and `k`, and how many unique states the DFS function generates.

There are at most `k + 1` levels to explore because each level corresponds to a step, and we do not go beyond `k` steps. At each step, the number of positions that we could be at increases, however due to the absolute value used in the second call to `dfs`, two calls may result in the same next state (e.g., `dfs(1, j-1)` and `dfs(-1, j-1)` both lead to `dfs(1, j-1)`), effectively reducing the number of unique states. Therefore, for each step, there can be up to `2 * (j - 1) + 1` unique positions to consider because we could either move left or right from each position or stay at the same position.

Considering memoization, which ensures that each unique state is only computed once, the time complexity can be estimated by the number of unique `(i, j)` pairs for which the `dfs` function is called. Hence, the total number of unique calls to `dfs` is `O(k^2)`, leading to a time complexity of `T(n) = O(k^2)`.

Space Complexity

The space complexity is determined by the size of the cache (to memoize the DFS calls) and the maximum depth of the recursion stack (which occurs when the DFS explores from `j = k` down to `j = 0`).

The cache potentially stores all unique states that the DFS visits, which we've established above is `O(k^2)`.

The maximum depth of the recursive call stack occurs when the function continually recurses without finding any cached results, which would mean a maximum depth of `k`. This is a very rare case, though, because memoization ensures that even deep recursive chains would quickly find cached results and not recurse to their maximum possible depth.

However, for the sake of computation, we consider the worst-case scenario without memoization's benefits, which is `O(k)`.

Therefore, the space complexity is determined by the larger of the cache's size or the recursion stack's depth. In this case, the cache's size is larger; hence the space complexity is `S(n) = O(k^2)`.