1277. Count Square Submatrices with All Ones

Given a `m * n` matrix of ones and zeros, return how many square submatrices have all ones.

Example 1:

Input:

```
matrix =
[
  [0,1,1,1],
  [1,1,1,1],
  [0,1,1,1]
]
```

Output: 15

Explanation:

There are 10 squares of side 1.

There are 4 squares of side 2.

There is 1 square of side 3.

Total number of squares $= 10 + 4 + 1 = 15$.

Example 2:

Input:

```
matrix =
[
  [1,0,1],
  [1,1,0],
  [1,1,0]
]
```

Output: 7

Explanation:

There are 6 squares of side 1.

There is 1 square of side 2.

Total number of squares $= 6 + 1 = 7$.

Constraints:

- $1 \le$ `arr.length` ≤ 300
- $1 \le$ `arr[0].length` ≤ 300
- $0 \le$ `arr[i][j]` ≤ 1

Solution

First, let's say that a square is located at a cell `(i,j)` if its bottom right corner is located at that cell.

We can observe that if there is a length x square with all ones located at `(i,j)`, then there exists squares with all ones that have lengths $x-1, x-2 \ldots, 3, 2, 1$. This is because if a square submatrix with length x located there exists, then square submatrices with lengths $x-1, x-2 \ldots 3, 2, 1$ must exist as well.

For each cell `(i,j)` we'll find the length of the largest square submatrix that has all ones with its bottom right corner located at `(i,j)`. Let's denote this value as $x_{i,j}$. We can observe that the number of square submatrices with all ones and its bottom right corner located at `(i,j)` is simply $x_{i,j}$. To find $x_{i,j}$, we can brute force through all possible lengths and check if a square submatrix with that respective length exists. Once we find $x_{i,j}$ for each cell `(i,j)`, our final answer is the sum of all $x_{i,j}$.

Our full solution will involve dynamic programming.

Let `dp[i][j]` represent $x_{i,j}$.

Since dynamic programming uses the answers to sub-problems to calculate answers to a larger problem, let's try to see how we can use other values from `dp` to calculate `dp[i][j]` for some cell `(i,j)`.

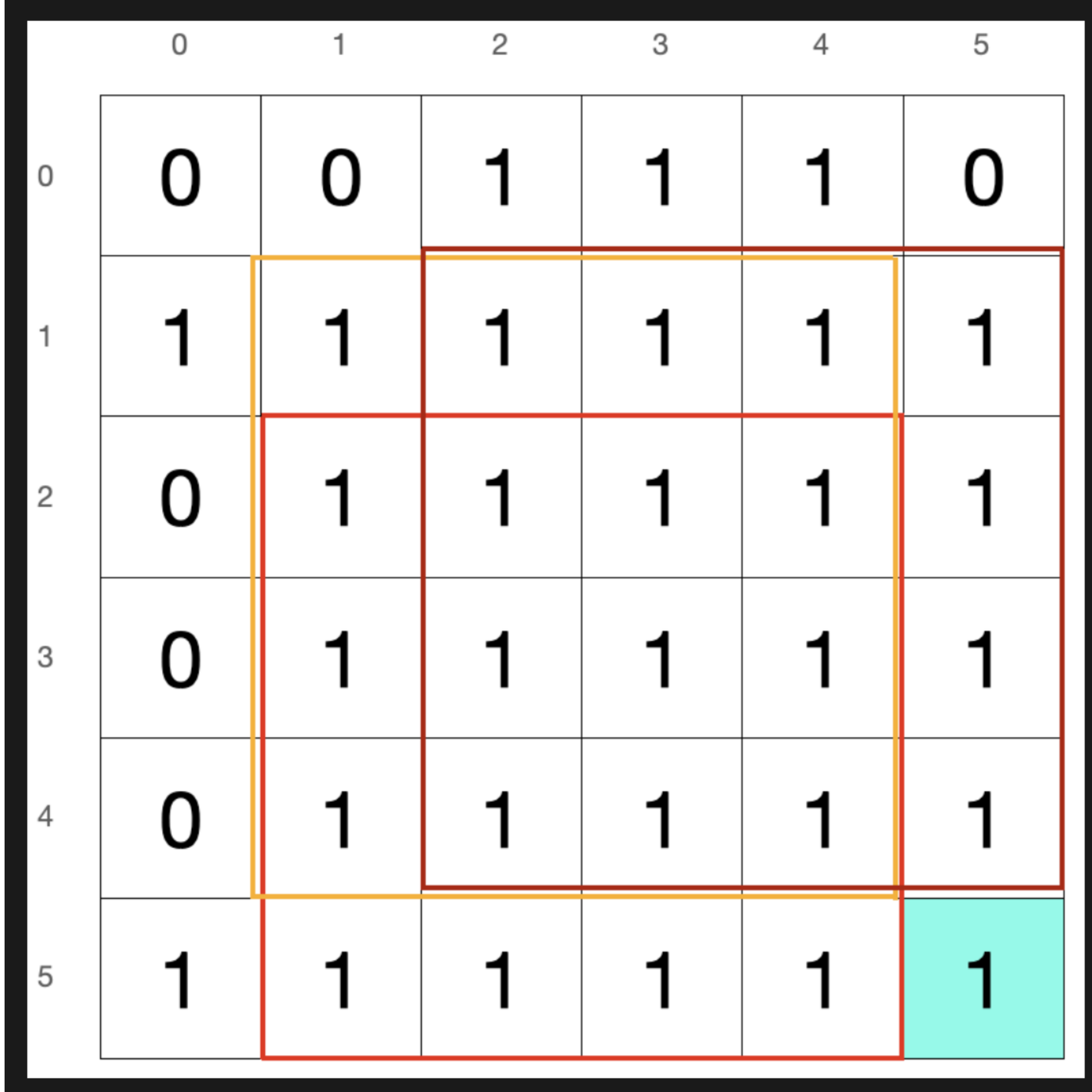
Let's say that a square submatrix is located at a cell `(i,j)` if its bottom right corner is located there.

First, let's assume `dp[i][j] = k` for some positive integer k. This means that the largest square submatrix with all ones located at `(i,j)` has length k. We can observe that this means there exists square submatrices with all ones at `(i,j-1)`, `(i-1,j)`, and `(i-1,j-1)` that have length $k-1$. In addition, the cell `(i,j)` is also 1.

If we look at all the cells that are covered by the square submatrices at `(i,j-1)`, `(i-1,j)`, and `(i-1,j-1)` with length $k-1$ and the cell at `(i,j)`, we obtain a square with length k located at `(i,j)`.

Example

Here is a diagram with $k = 5$ and the cell at `(5,5)` to help visualize this observation.



From the observation mentioned above, we can deduce that `dp[i-1][j], dp[i][j-1], dp[i-1][j-1]` $\ge k-1$.

For `dp[i][j]` to be greater or equal to k, then `(i,j)` must be 1 and `dp[i-1][j], dp[i][j-1], dp[i-1][j-1]` all have to be greater or equal to $k-1$.

This translates into `dp[i][j] = min(dp[i-1][j],dp[i][j-1],dp[i-1][j-1]) + 1` if the cell `(i,j)` is a 1. For most cases, this is how we can calculate `dp[i][j]`.

If the cell is in the first row or column however (i.e. $i = 0$ or $j = 0$), then `dp[i][j]` is the same value as the cell `(i,j)`.

Obviously, if the cell `(i,j)` is 0, then `dp[i][j] = 0`.

Our final answer is just the sum of all values stored in `dp`.

Time Complexity

We can calculate the value of a cell in `dp` in $\mathcal{O}(1)$ and since there are $\mathcal{O}(MN)$ cells, our time complexity is $\mathcal{O}(MN)$.

Time Complexity: $\mathcal{O}(MN)$

Space Complexity

We store $\mathcal{O}(MN)$ cells in `dp` so our space complexity is $\mathcal{O}(MN)$.

Space Complexity: $\mathcal{O}(MN)$

Bonus: We can use the space optimization mentioned in this article to optimize memory to $O(N)$. Since we only use rows $i-1$ and i in `dp` to calculate `dp` values for row i, we only need to maintain two rows of memory for `dp`, which is $O(N)$.

```cpp
class Solution {
   public:
    int countSquares(vector<vector<int>>& matrix) {
        int m = matrix.size();
        int n = matrix[0].size();  // dimensions for matrix
        int ans = 0;
        vector<vector<int>> dp(m, vector<int>(n));
        for (int i = 0; i < m; i++) {
            for (int j = 0; j < n; j++) {
                if (i == 0 || j == 0) {  // cell is in first row or column
                    dp[i][j] = matrix[i][j];
                } else if (matrix[i][j] == 1) {
                    dp[i][j] = min({dp[i - 1][j], dp[i][j - 1], dp[i - 1][j - 1]}) + 1;
                }
                ans += dp[i][j];
            }
        }
        return ans;
    }
};
```

```java
class Solution {
    public int countSquares(int[][] matrix) {
        int m = matrix.length;
        int n = matrix[0].length; // dimensions for matrix
        int[][] dp = new int[m][n];
        int ans = 0;
        for (int i = 0; i < m; i++) {
            for (int j = 0; j < n; j++) {
                if (i == 0 || j == 0) { // cell is in first row or column
                    dp[i][j] = matrix[i][j];
                } else if (matrix[i][j] == 1) {
                    dp[i][j] = Math.min(dp[i - 1][j], Math.min(dp[i][j - 1], dp[i - 1][j - 1])) + 1;
                }
                ans += dp[i][j];
            }
        }
        return ans;
    }
}
```

```python
class Solution:
    def countSquares(self, matrix: List[List[int]]) -> int:
        m = len(matrix)
        n = len(matrix[0])  # dimensions for matrix
        dp = [[0] * n for a in range(m)]
        ans = 0
        for i in range(m):
            for j in range(n):
                if i == 0 or j == 0:  # cell is in first row or column
                    dp[i][j] = matrix[i][j]
                elif matrix[i][j] == 1:
                    dp[i][j] = min(dp[i - 1][j], dp[i][j - 1], dp[i - 1][j - 1]) + 1
                ans += dp[i][j]
        return ans
```