483. Smallest Good Base

Hard Math Binary Search Leetcode Link

Problem Description

The LeetCode problem is asking us to find the *smallest good base* for a given integer `n` that is represented as a string. A *good base* `k` for `n` means that if we were to write `n` in base `k`, all its digits would be `1's`. For example, the number `7` in base `2` is written as `111`, so `2` is a good base for `7`. The challenge is to find the *smallest* such base, which is always greater than or equal to `2`.

Intuition

To approach this problem, the key observation is that `n` can be expressed as a sum of geometric series when using a base `k` in which all digits are 1's.

Mathematically, for a base `k` and `m` digits in base `k` representation, `n` can be expressed as:

n = 1 + k^1 + k^2 + ... + k^(m-1)

This is a geometric series, and we need to find the smallest `k` (good base) for which there is some `m` that satisfies the equation.

Since `n` consists entirely of `1's` for a good base `k`, the larger the base `k`, the fewer digits `m` we'll need, and vice versa. This means that for larger `m`, `k` will be smaller.

The maximum `m` is bounded due to the size of `n` — `n`'s binary representation is the longest `m` could be since binary (base `2`) has the most `1's` for any given number. Therefore, the loop counter `m` starts from `63` (since a 64-bit integer has a maximum of 63 `1's` plus 1 sign bit).

The solution works by iterating over possible values of `m` from this maximum `m` down to `2`. For each `m`, it performs a binary search to find the smallest `k` such that the sum of the geometric series equals `n`. Once we find such `k`, we return it as the result. If we fail to find such `k` for all `m`, then the smallest good base is `n-1` (because the only representation of `n` with all `1's` using base `n-1` is `11`).

To speed up the process, the `cal(k, m)` function is defined to calculate the sum of the geometric series efficiently. This avoids recalculating powers of `k` multiple times.

The binary search is conducted within the range of `[2, num - 1]` for each `m`. Whenever the `cal(mid, m)` function, which represents the sum of the series for base `mid` and `m` digits, yields a value less than `num` (our original number), we know that the base is too small and needs to be larger; thus, we adjust our search range accordingly.

The solution, therefore, combines the understanding of geometric series with binary search to find the smallest good base within an optimized time complexity.

Solution Approach

The solution involves the implementation of an iterative approach combined with binary search. Let's go through the steps and the algorithm utilized:

1. Convert the input string `n` to an integer `num` to perform numerical operations.
2. Loop through `m` starting from `63` down to `2`. This represents the possible lengths of the number `n` when written in base `k`, all in 1's. The number `63` is used because for a 64-bit integer, the maximum length of pure `1's` (excluding the sign bit) is `63`.
3. For each value of `m`, perform a binary search to find the smallest base `k` that satisfies the condition that all digits of `n` base `k` are `1's`. Set the initial search range with `l` (left) as `2` and `r` (right) as `num - 1`, indicating the minimum and maximum potential bases, respectively.
4. Conduct the binary search:
   - Compute the middle point `mid` between `l` and `r` as `(l + r) >> 1`, where `>> 1` is a bitwise right shift equivalent to division by 2.
   - Calculate the sum of the geometric series using `cal(mid, m)`.
     - The `cal` function takes a base `k` and a length `m`, iteratively multiplies the base (geometric progression), and accumulates the result in `s`, initializing with `1` (for the first digit, which is always `1`).
   - Check if the computed series sum is greater than or equal to `num`. If so, update `r` to `mid`, indicating that the current base may be too large, or the right range from mid to `r` does not contain the smallest base.
   - Otherwise, update the left range boundary `l` to `mid + 1`, as the current base `mid` is too small to represent `num` with all `1's`.
   - This process narrows down the search space until the left and right boundaries converge.
5. After the binary search loop concludes, the function checks if the value discovered at `l` produces a sum equal to `num` using `cal(l, m)`. If it does, this base `l` is the smallest good base for the given `m`, and it is returned as a string.
6. If no suitable base `k` is found across all `m` values in the loop, which means `n` cannot be written as all `1's` in any other base than itself minus 1, the function returns `num - 1` as a string. This corresponds to the base `n-1` since any number `n` in base `n-1` is `11`.

The solution makes efficient use of binary search within an iterative loop to significantly narrow down the possible candidates for a good base and arrive at the smallest possible one. It leverages the mathematical properties of geometric series for verification within the binary search.

Example Walkthrough

Let's assume `n` is given as the string "13". First, we convert this string to an integer `num = 13` to perform arithmetic operations.

Starting with `m` equal to `63` and decreasing, we're looking for the smallest base `k` such that 13 can be written as a series of `1's` – this would take too long computationally, though, so for the sake of the example, let's consider smaller `m` values. We will start with `m = 3`, as larger values of `m` would result in smaller values of `k`, and we are searching for the smallest `k`.

When `m = 3`, our equation `n = 1 + k^1 + k^2` should equal `13`. So in this step, we'll perform a binary search between `2` and `12` (num - 1) to find the smallest `k`.

During each iteration of the binary search, we:

1. Find `mid`. For the first iteration, `l` is `2`, `r` is `12`, so `mid` will be `(2 + 12) >> 1` which equals `7`.
2. Calculate `cal(mid, m)`. Using `mid = 7`, we find that `cal(7, 3) = 1 + 7 + 49 = 57`.
3. Since `57` is greater than `13`, `mid` is too large. We adjust our range and set `r` to `mid - 1`, which is now `6`.
4. Find the new `mid`, which is now `(2 + 6) >> 1` which equals `4`.
5. Calculating `cal(4, 3)` gives us `1 + 4 + 16 = 21`, which is again greater than `13`, so we adjust `r` again to `mid - 1`, now `3`.
6. New `mid` is `(2 + 3) >> 1` which equals `2`.
7. Calculate `cal(2, 3)` giving us `1 + 2 + 4 = 7`, which is less than `13`, so we adjust `l` to `mid + 1`, now `3`.
8. Since `l` now equals `r`, the search concludes.

We find that using `k = 3`, `cal(3, 3)` equals `1 + 3 + 9 = 13`, which matches our `num`. Therefore, base `k = 3` can represent the number `13` as `111` in base `3`, and given this is the smallest `k` we've found, we return `k = 3` as the smallest good base for the number `13`.

```

For the given integer `n` of value "13" (which we convert to the integer `num = 13` for processing), we aim to find the smallest base

We start with `m = 3`, a possible length of the all `1's` representation (i.e., `111`):

1. Conduct a binary search for `k` between `2` and `12`.
   - First iteration: `mid = 7`. Calculate `cal(7, 3) = 57`. This is greater than `13`, so set `r` to `6`.
   - Second iteration: `mid = 4`. Calculate `cal(4, 3) = 21`. Still greater than `13`, set `r` to `3`.
   - Third iteration: `mid = 2`. Calculate `cal(2, 3) = 7`. Less than `13`, set `l` to `3`.
2. When `l` and `r` converge, we find that `cal(3, 3)` equals `13`.
3. Return base `k = 3` as the smallest good base, which represents `13` as `111` in this base.
```

Python Solution

```python
class Solution:
    def smallestGoodBase(self, n: str) -> str:
        # Helper function to calculate the sum of a geometric series
        def calculate_sum(base, term_count):
            power_product = sum_product = 1
            for i in range(term_count):
                power_product *= base
                sum_product += power_product
            return sum_product

        # Convert input string to integer
        num = int(n)
        # Try to find the smallest base by iterating from the largest term count down to 2
        for term_count in range(63, 1, -1):
            # Binary search for the good base
            left, right = 2, num - 1
            while left < right:
                mid = (left + right) // 2
                if calculate_sum(mid, term_count) >= num:
                    right = mid
                else:
                    left = mid + 1
            # Check if we found the exact sum that matches the given number
            if calculate_sum(left, term_count) == num:
                return str(left)
        # If no good base is found, return num - 1,
        # which is always a good base (n = k^1 + 1)
        return str(num - 1)

```

Java Solution

```java
class Solution {
    // Finds the smallest base for a number with the properties of a good base
    public String smallestGoodBase(String n) {
        long num = Long.parseLong(n); // Convert string to long integer

        // loop to check all possible lengths starting from the highest possible
        for (int length = 63; length >= 2; --length) {
            long base = getBaseForGivenLength(length, num);
            if (base != -1) {
                return String.valueOf(base); // if a valid base is found, return it
            }
        }
        // If no good base is found, return n-1 as base as per the mathematical property
        return String.valueOf(num - 1);
    }

    // Helper method to get a base for a given range and target number
    private long getBaseForGivenLength(int length, long targetNumber) {
        long left = 2, right = targetNumber - 1;
        while (left < right) { // Binary search to find the good base
            long mid = (left + right) >>> 1; // Use unsigned right shift for division by 2
            long result = calculatePowerSum(mid, length);

            if (result >= targetNumber) {
                right = mid; // Adjust right boundary
            } else {
                left = mid + 1; // Adjust left boundary
            }
        }
        return calculatePowerSum(right, length) == targetNumber ? right : -1;
    }

    // Helper method to calculate the sum of powers for a given base and length
    private long calculatePowerSum(long base, int length) {
        long power = 1; // Start with k^0
        long sum = 0;

        for (int i = 0; i < length; ++i) {
            if (Long.MAX_VALUE - sum < power) {
                return Long.MAX_VALUE;
            }
            sum += power; // Add current power of base to sum

            // Check if next multiplication would cause overflow
            if (Long.MAX_VALUE / power < base) {
                power = Long.MAX_VALUE;
            } else {
                power *= base; // Otherwise, multiply power by base
            }
        }
        return sum;
    }
}

```

C++ Solution

```cpp
class Solution {
public:
    // Function to find the smallest good base of a number as a string
    string smallestGoodBase(string n) {
        // Convert the input number n to a long integer
        long value = stol(n);

        // Calculate the maximum possible value of m, assuming base 2 (binary)
        int maxM = floor(log(value) / log(2));

        // Start iterating from the largest possible m to 1
        for (int m = maxM; m > 1; --m) {
            // Calculate the base k for the current m using nth root
            int base = pow(value, 1.0 / m);

            // Initialize multiplier (mul) and sum (s) for geometric progression
            long mul = 1, sum = 1;

            // Calculate the sum of the sequence with m terms
            for (int i = 0; i < m; ++i) {
                mul *= base; // multiply by base each time
                sum += mul;  // add the term to the sum
            }

            // If sum equals to the value, we've found the smallest good base
            if (sum == value) {
                return to_string(base);
            }
        }

        // If no other base found, the smallest good base is value - 1
        // since a K-base number system of K+1 (here v) would always be written as 10...0 (which equals K+1).
        return to_string(value - 1);
    }
};

```

Typescript Solution

```typescript
// Import the required function from JavaScript Math object
import { log10, pow, floor } from 'math';

// Function to find the smallest good base for a number given as a string
function smallestGoodBase(n: string): string {
    // Convert the input string to a number
    let value: number = parseInt(n);

    // Calculate the maximum possible value of m assuming the base is 2 (binary) system
    let maxM: number = floor(log10(value) / log10(2));

    // Iterate from the largest possible value of m to 1
    for (let m = maxM; m > 1; m--) {
        // Calculate the base (k) for the current value of m using the nth root
        let base: number = pow(value, 1.0 / m);

        // Initialize variables for the geometric progression
        let mul: number = 1; // Multiplier
        let sum: number = 1; // Sum of the geometric sequence

        // Calculate the sum of the sequence with m terms
        for (let i = 0; i < m; i++) {
            mul *= base; // Multiply by the base for each term
            sum += mul;  // Add the computed term to the sum
        }

        // If the sum is equal to the original number, we've found the smallest good base
        if (sum === value) {
            return base.toString();
        }
    }

    // If no base was found, return value - 1, which is always a good base for any number
    return (value - 1).toString();
}

// Please note the usage of 'log10' instead of 'log' in TypeScript.
// TypeScript uses the built-in JavaScript Math object's log10 method for base 10 logarithms.

```

Time and Space Complexity

Time Complexity

The time complexity of the algorithm is determined by the nested loop:

1. The outer loop runs for each possible value of `m`, which ranges from 63 to 2, resulting in a maximum of 62 iterations. This is because `m` represents the maximum length of digits in base `k` representation for the number `n`, and since the largest number in this context is `2^64 - 1`, the maximum length of `m` is 63.
2. The inner loop is a binary search, which runs in `O(log(n))` time, where `n` is the given number. In each iteration of this binary search, the function `cal` is called which performs, at maximum, `m` multiplications.

Given that `m` is at most 63, and for each `m` we perform a binary search which takes `O(log(n))` time, the overall time complexity of the inner loop is `O(m * log(num))`.

3. The `cal` function itself runs in `O(m)` time, since it contains a loop that iterates `m` times.

Combining these aspects together, the total time complexity of the code is `O(m * m * log(num))` or, more concretely, `O(63 * log(num))` because `m` is a constant at most 63.

Space Complexity

The space complexity of the algorithm is `O(1)`:

1. The space used by the algorithm is constant, as there are only a few integer variables being used and no additional space (like data structures) that grow with the size of the input.
2. The `cal` function uses a constant amount of space as well, as the variables `p` and `s` are just integers and do not require space that scales with the input size.

Hence, there is no significant space usage that scales with the size of the input.