593. Valid Square

Medium | Geometry | Math

Problem Description

The problem requires determining whether four given points in a 2D plane form a square. The input includes the coordinates of each point, and the points are not necessarily given in any specific order. A valid square must have all four sides of equal length, and all four angles must be right angles (90 degrees). The task is to return `true` if the points form a valid square and `false` otherwise.

Intuition

To solve this problem, the intuition is to check every possible combination of three points out of the four to form two sides of a square and then calculate the lengths of these sides using the distance formula. The key idea is that for any three points which form a right angle, two sides squaring and adding them up should give the square of the third side (following the Pythagorean theorem). This condition should be true for all combinations of three points. This ensures that four right angles are formed. Moreover, the lengths of adjacent sides should be equal (and greater than zero to avoid degenerate cases).

The solution consists of a `check` function which performs the calculations and comparisons. It calculates the squared distances between pairs of points (`d1`, `d2`, `d3`) and checks for the two conditions: equality of any two distances to make sure the sides are equal, and the sum of these two equal distances should be equal to the third distance, ensuring a 90-degree angle between them. It also checks that the distance is non-zero to avoid considering points that overlap as valid sides.

The solution verifies these conditions for all combinations of three points out of the four: (p1, p2, p3), (p2, p3, p4), (p1, p3, p4), and (p1, p2, p4). If all combinations satisfy the condition, we conclude that the points can form a valid square.

Solution Approach

The implementation of the solution employs the concept of distance calculation between points and the comparison of these distances to validate the square's sides and angles. Here's how the approach unfolds step by step:

1. **Distance Formula:** To calculate the squared distance between two points (x1, y1) and (x2, y2) in 2D space, we use the distance formula $(x1 - x2)^2 + (y1 - y2)^2$. We avoid the square root computation for efficiency and because it can introduce floating-point precision issues. Squared distances are sufficient for comparison purposes.
2. **Check Function:** This helper function `check` takes three points `a`, `b`, and `c` as arguments and calculates the squared distances between them: `d1` is the distance between `a` and `b`, `d2` is the distance between `a` and `c`, and `d3` is the distance between `b` and `c`.
3. **Right Angle Validation:** Inside the `check` function, it checks if any pair of these distances (`d1`, `d2`, `d3`) are equal – indicating that two sides of a triangle formed by the points are equal – and if the sum of their squares equals the square of the third distance. This is essentially checking for a right triangle using the Pythagorean theorem.
4. **Non-zero Sides:** The `check` function also ensures that the distances are non-zero to prevent considering points that are the same (overlap) as valid sides of a square.
5. **Combining Checks:** The `validSquare` method calls the `check` function for all combinations of three points out of the four points given. This step is crucial as it verifies the square property from all angles and sides.

The input is in the form of four points `p1`, `p2`, `p3`, and `p4`, each represented as a list of two integers.

The `check` is written in Python as:

```python
def check(a, b, c):
    (x1, y1), (x2, y2), (x3, y3) = a, b, c
    d1 = (x1 - x2) * (x1 - x2) + (y1 - y2) * (y1 - y2)
    d2 = (x1 - x3) * (x1 - x3) + (y1 - y3) * (y1 - y3)
    d3 = (x2 - x3) * (x2 - x3) + (y2 - y3) * (y2 - y3)
    return any([
        d1 == d2 and d1 + d2 == d3 and d1,
        d2 == d3 and d2 + d3 == d1 and d2,
        d1 == d3 and d1 + d3 == d2 and d1,
    ])
```

The final return statement calls the `check` function multiple times to verify each three-point combination forms part of a square:

```python
return (
    check(p1, p2, p3) and check(p2, p3, p4) and check(p1, p3, p4) and check(p1, p2, p4)
)
```

Overall, the provided Python solution leverages simple geometry and efficient calculations without the need for complex data structures or algorithms. It is an elegant example of applying mathematical principles to solve a computational problem.

Example Walkthrough

Let's walk through the solution approach with a small example. Suppose we have the following four points that are potential vertices of a square:

- `p1 = (1, 1)`,
- `p2 = (1, 3)`,
- `p3 = (3, 1)`,
- `p4 = (3, 3)`.

These points are given in no specific order and might form a square in a 2D plane. We will use the functions and approach described in the solution to determine if these points make up a valid square.

First, the distance formula is used to check if the sides formed by these points are equal, and if the diagonals intersect at a right angle. Squared distances are calculated as follows:

- `d1` (distance between `p1` and `p2`): $(1 - 1)^2 + (1 - 3)^2 = 0 + 4 = 4$
- `d2` (distance between `p1` and `p3`): $(1 - 3)^2 + (1 - 1)^2 = 4 + 0 = 4$
- `d3` (distance between `p2` and `p3`): $(1 - 3)^2 + (3 - 1)^2 = 4 + 4 = 8$

The helper function `check` now verifies if two distances are equal and if the sum of their squares equals the third distance. For our distances, we have:

- `d1` equals `d2`, and `d1 + d2` equals `d3`. We also ensure `d1` is non-zero which it is. This combination reflects two equal sides of a triangle and a right angle between them.

We will repeat this process for the other combinations of points:

- When checking `p2`, `p3`, and `p4`, we obtain `d1 = 4`, `d2 = 4`, and `d3 = 8`, satisfying the conditions.
- For `p1`, `p3`, and `p4`, we compute `d1 = 4`, `d2 = 4`, and `d3 = 8` again, meeting the conditions.
- Lastly, for `p1`, `p2`, and `p4`, we find `d1 = 4`, `d2 = 4`, and `d3 = 8`, which also passes the check.

Each combination gives us two equal sides and a diagonal that conforms to the Pythagorean theorem, indicating four right angles in total. Since the points p1, p2, p3, and p4 satisfy the `check` for every combination, they form a valid square, and the function would return `true`.

To summarize, the solution involves using the distance formula to calculate squared distances between different points, followed by an application of the Pythagorean theorem to ensure right angles and equal sides. This simple yet effective solution illustrates how mathematical principles can be used to solve computational geometry problems.

Solution Implementation

Python

```python
from typing import List

class Solution:
    def validSquare(self, p1: List[int], p2: List[int], p3: List[int], p4: List[int]) -> bool:
        # Helper function to check whether three points form a right angle at the first point
        def is_right_angle(point_a, point_b, point_c):
            x1, y1 = point_a
            x2, y2 = point_b
            x3, y3 = point_c
            # Calculate the squared distances between the points
            dist_ab_2 = (x1 - x2) ** 2 + (y1 - y2) ** 2
            dist_ac_2 = (x1 - x3) ** 2 + (y1 - y3) ** 2
            dist_bc_2 = (x2 - x3) ** 2 + (y2 - y3) ** 2
            # Check if the triangle formed by the three points is a right triangle
            return any([
                dist_ab_2 == dist_ac_2 and dist_ab_2 + dist_ac_2 == dist_bc_2 and dist_ab_2,
                dist_ac_2 == dist_bc_2 and dist_ac_2 + dist_bc_2 == dist_ab_2 and dist_ac_2,
                dist_ab_2 == dist_bc_2 and dist_ab_2 + dist_bc_2 == dist_ac_2 and dist_ab_2
            ])

        # Check all combinations of points to ensure all the required conditions for a valid square are met
        return (
            is_right_angle(p1, p2, p3) and
            is_right_angle(p2, p3, p4) and
            is_right_angle(p3, p1, p4) and
            is_right_angle(p1, p2, p4)
        )

# Example usage:
# solution = Solution()
# print(solution.validSquare([0,0], [1,1], [1,0], [0,1]))  # The output would be True
```

Java

```java
class Solution {
    // Method to determine if four points form a valid square.
    public boolean validSquare(int[] p1, int[] p2, int[] p3, int[] p4) {
        // Check all combinations of three points to see if they form right triangles.
        return checkRightTriangle(p1, p2, p3) &&
               checkRightTriangle(p1, p3, p4) &&
               checkRightTriangle(p1, p2, p4) &&
               checkRightTriangle(p2, p3, p4);
    }

    // Helper method to check if three points form a right-angled triangle.
    private boolean checkRightTriangle(int[] pointA, int[] pointB, int[] pointC) {
        // Extract x and y coordinates of points.
        int xA = pointA[0], yA = pointA[1];
        int xB = pointB[0], yB = pointB[1];
        int xC = pointC[0], yC = pointC[1];

        // Compute squared distances between the points.
        int distanceAB = squaredDistance(xA, yA, xB, yB);
        int distanceAC = squaredDistance(xA, yA, xC, yC);
        int distanceBC = squaredDistance(xB, yB, xC, yC);

        // Check the conditions for forming a right-angled triangle (Pythagorean theorem):
        // two distances must be equal (the sides of the square) and the sum of their squares
        // must equal the square of the third distance (the diagonal of the square).
        // Ensure that no distance is zero to prevent degenerate triangles.
        return (distanceAB == distanceAC && distanceAB + distanceAC == distanceBC && distanceAB > 0) ||
               (distanceAB == distanceBC && distanceAB + distanceBC == distanceAC && distanceAB > 0) ||
               (distanceAC == distanceBC && distanceAC + distanceBC == distanceAB && distanceAC > 0);
    }

    // Helper method to calculate the squared distance between two points to avoid floating point errors.
    private int squaredDistance(int x1, int y1, int x2, int y2) {
        return (x1 - x2) * (x1 - x2) + (y1 - y2) * (y1 - y2);
    }
}
```

C++

```cpp
#include <vector>

class Solution {
public:
    // This function checks if four points form a valid square.
    bool validSquare(std::vector<int>& p1, std::vector<int>& p2,
                     std::vector<int>& p3, std::vector<int>& p4) {
        // A square should have 2 pairs of equal sides & 1 pair of equal diagonals

        return areSidesValid(p1, p2, p3) && areSidesValid(p1, p3, p4) &&
               areSidesValid(p1, p2, p4) && areSidesValid(p2, p3, p4);
    }

    // Helper function to check if 3 points form two equal sides and one diagonal of a square
    bool areSidesValid(std::vector<int>& a, std::vector<int>& b, std::vector<int>& c) {
        int ax = a[0], ay = a[1]; // Coordinates of point a
        int bx = b[0], by = b[1]; // Coordinates of point b
        int cx = c[0], cy = c[1]; // Coordinates of point c

        // Calculate squared distances between points
        int abSquaredDist = squaredDistance(ax, ay, bx, by);
        int acSquaredDist = squaredDistance(ax, ay, cx, cy);
        int bcSquaredDist = squaredDistance(bx, by, cx, cy);

        // Check for two equal sides and a diagonal; also, check that sides are not zero-length
        return (abSquaredDist == acSquaredDist && abSquaredDist + acSquaredDist == bcSquaredDist && abSquaredDist > 0) ||
               (abSquaredDist == bcSquaredDist && abSquaredDist + bcSquaredDist == acSquaredDist && abSquaredDist > 0) ||
               (acSquaredDist == bcSquaredDist && acSquaredDist + bcSquaredDist == abSquaredDist && acSquaredDist > 0);
    }

private:
    // Helper function to calculate squared distance between two points
    int squaredDistance(int x1, int y1, int x2, int y2) {
        return (x1 - x2) * (x1 - x2) + (y1 - y2) * (y1 - y2);
    }
};
```

TypeScript

```typescript
// This function checks if four points form a valid square.
function validSquare(p1: number[], p2: number[], p3: number[], p4: number[]): boolean {
    // A square should have 2 pairs of equal sides & 1 pair of equal diagonals

    return areSidesValid(p1, p2, p3) && areSidesValid(p1, p3, p4) &&
           areSidesValid(p1, p2, p4) && areSidesValid(p2, p3, p4);
}

// Helper function to check if 3 points form two equal sides and one diagonal of a square
function areSidesValid(a: number[], b: number[], c: number[]): boolean {
    let ax = a[0], ay = a[1]; // Coordinates of point a
    let bx = b[0], by = b[1]; // Coordinates of point b
    let cx = c[0], cy = c[1]; // Coordinates of point c

    // Calculate squared distances between points
    let abSquaredDist = squaredDistance(ax, ay, bx, by);
    let acSquaredDist = squaredDistance(ax, ay, cx, cy);
    let bcSquaredDist = squaredDistance(bx, by, cx, cy);

    // Check for two equal sides and a diagonal; also, check that sides are not zero-length
    return (abSquaredDist === acSquaredDist && abSquaredDist + acSquaredDist === bcSquaredDist && abSquaredDist > 0) ||
           (abSquaredDist === bcSquaredDist && abSquaredDist + bcSquaredDist === acSquaredDist && abSquaredDist > 0) ||
           (acSquaredDist === bcSquaredDist && acSquaredDist + bcSquaredDist === abSquaredDist && acSquaredDist > 0);
}

// Helper function to calculate squared distance between two points
function squaredDistance(x1: number, y1: number, x2: number, y2: number): number {
    return (x1 - x2) * (x1 - x2) + (y1 - y2) * (y1 - y2);
}
```


```python
from typing import List

class Solution:
    def validSquare(self, p1: List[int], p2: List[int], p3: List[int], p4: List[int]) -> bool:
        # Helper function to check whether three points form a right angle at the first point
        def is_right_angle(point_a, point_b, point_c):
            x1, y1 = point_a
            x2, y2 = point_b
            x3, y3 = point_c
            # Calculate the squared distances between the points
            dist_ab_2 = (x1 - x2) ** 2 + (y1 - y2) ** 2
            dist_ac_2 = (x1 - x3) ** 2 + (y1 - y3) ** 2
            dist_bc_2 = (x2 - x3) ** 2 + (y2 - y3) ** 2
            # Check if the triangle formed by the three points is a right triangle
            return any([
                dist_ab_2 == dist_ac_2 and dist_ab_2 + dist_ac_2 == dist_bc_2 and dist_ab_2,
                dist_ac_2 == dist_bc_2 and dist_ac_2 + dist_bc_2 == dist_ab_2 and dist_ac_2,
                dist_ab_2 == dist_bc_2 and dist_ab_2 + dist_bc_2 == dist_ac_2 and dist_ab_2
            ])

        # Check all combinations of points to ensure all the required conditions for a valid square are met
        return (
            is_right_angle(p1, p2, p3) and
            is_right_angle(p2, p3, p4) and
            is_right_angle(p3, p1, p4) and
            is_right_angle(p1, p2, p4)
        )

# Example usage:
# solution = Solution()
# print(solution.validSquare([0,0], [1,1], [1,0], [0,1]))  # The output would be True
```


Time and Space Complexity

The given Python code defines a method `validSquare` to check whether four points form a valid square.

Time Complexity:

To analyze the time complexity, we'll consider the significant operations in the given code.

1. The `check` function calculates the distances squared (to avoid floating point arithmetic) for three pairs of points. This requires a constant amount of operations, specifically, six subtractions and six multiplications for the distance calculations, plus comparisons and additions. So the time complexity for this part is `O(1)`.
2. The `validSquare` method calls the `check` function four times, once for each combination of three points out of four. Since the number of calls to `check` does not depend on the input size (it's always four), this also contributes a constant factor.

There are no loops or recursive calls that depend on the size of the input, hence the overall time complexity is `O(1)`.

Space Complexity:

Now let's examine the space complexity.

1. The `check` function uses a fixed amount of space for variables to store the distances and the points. No additional data structures that grow with input size are used, so its space complexity is `O(1)`.
2. The main `validSquare` method does not use any additional space apart from the space required for the `check` function and the input points.

Therefore, the space complexity is `O(1)`.

In summary:

- The **time complexity** of the code is `O(1)`.
- The **space complexity** of the code is `O(1)`.