Chapter 12

More on Graph Theory

In this second chapter on graph theory, we study some of the fundamental numbers that are associated with a graph. The most famous of all these numbers is the *chromatic number* because of its association with the *four-color problem.* This problem, which for over 100 years was an unsolved problem,[1] asks the following: Consider a map that is drawn on the plane or on the surface of a sphere in which the countries are connected regions. We want to color each region with one color so that neighboring regions are colored differently. Will four colors always suffice to color any map in this way? The short answer is *yes.* The long answer is that the proof requires[2] an elaborate argument and depends substantially on calculations by computer. The four-color problem can be restated in terms of graphs. Choose a vertex-point in the interior of each country, and join two vertex-points by an edge-curve whenever the two countries share a border.[3] In this way, we obtain a plane-graph (and hence a planar graph) which is called the *dual graph* of the map. Coloring the regions of a map so that neighboring regions are colored differently is equivalent to coloring the vertices[4] of its dual graph in such a way that two vertices which are adjacent are colored differently. Thus, the four-color problem can be restated as follows: Every planar graph is four-colorable. In this chapter, we shall prove that every planar graph is five-colorable, and, more generally, we shall investigate colorings of graphs and other graphical parameters of interest.

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[1] A problem being unsolved for over 100 years is not automatically famous. What made the four-color problem so famous is that it is easily stated and understood by almost anyone. And it is very appealing!

[2] At least the currently known proof does. But a proof that four colors do suffice is beyond an attack by amateur means. The elementary approaches have been tried and have failed. For a very brief history of the four-color problem, see Section 1.4.

[3] Two countries which have only one, or, more generally, only finitely many points in common are not considered to have a common border.

[4] More precisely, we think of assigning colors to the vertices.

12.1 Chromatic Number

In this section we consider only graphs, since the presence of either more than one edge joining a pair of distinct vertices or loops has no essential effect on the types of questions treated here.

Let $G = (V, E)$ be a graph. A *vertex-coloring* of G is an assignment of a color to each of the vertices of G in such a way that adjacent vertices are assigned different colors. If the colors are chosen from a set of k colors, then the vertex-coloring is called a *k-vertex-coloring*, abbreviated *k-coloring*, whether or not all k colors are used. If G has a k-coloring, then G is said to be *k-colorable*. The smallest k, such that G is k-colorable, is called the *chromatic number* of G, denoted by $\chi(G)$. The actual nature[5] of the colors used is of no consequence. Thus, sometimes we describe the colors as red, blue, green, ... , while at other times we simply use the integers 1, 2, 3, ... to designate the colors. Isomorphic graphs have the same chromatic number.

A *null graph* is defined to be a graph without any edges.[6] A null graph of order n is denoted by N_n.

Theorem 12.1.1 *Let G be a graph of order $n \geq 1$. Then*

$$1 \leq \chi(G) \leq n.$$

Moreover, $\chi(G) = n$ if and only if G is a complete graph, and $\chi(G) = 1$ if and only if G is a null graph.

Proof. The inequalities in the theorem are obvious, since any graph with at least one vertex requires at least one color, and any assignment of n distinct colors to the vertices of G is a vertex-coloring. In any vertex-coloring of K_n, no two vertices can be assigned the same color; hence, $\chi(K_n) = n$. Suppose that G is not a complete graph. Then there are two vertices x and y that are not adjacent. Assigning x and y the same color and the remaining $n-2$ vertices different colors, we obtain an $(n-1)$-coloring of G, and hence $\chi(G) \leq n-1$. Assigning all vertices of N_n the same color is a vertex-coloring, and hence $\chi(N_n) = 1$. Suppose that G is not a null graph. Then there are vertices x and y that are adjacent and thus cannot be assigned the same color in any vertex-coloring of G. Hence, in this case $\chi(G) \geq 2$. □

Corollary 12.1.2 *Let G be a graph and let H be a subgraph of G. Then $\chi(G) \geq \chi(H)$. If G has a subgraph[7] equal to a complete graph K_p of order p, then*

$$\chi(G) \geq p.$$

[5] Should we say *color*?

[6] A null graph is not necessarily an empty graph, since it may have vertices. The *empty graph* is a graph without any vertices. Thus, a graph $G = (V, E)$ is a null graph if and only if $E = \emptyset$, while G is the empty graph if and only if $V = \emptyset$ (and hence $E = \emptyset$). The empty graph is a very special null graph, namely, the null graph of order 0. Confusing? Not to worry. Just remember that a null graph has no edges.

[7] This subgraph will necessarily be an induced subgraph.

Proof. It follows from the definition of chromatic number that, if H is any subgraph of G, then $\chi(G) \geq \chi(H)$. Hence, by Theorem 12.1.1, $\chi(G) \geq \chi(K_p) = p$. □

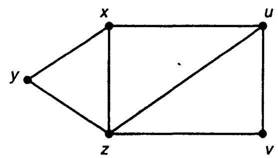


Figure 12.1

Example. Let G be the graph shown in Figure 12.1. Since G has a subgraph equal to K_3, the chromatic number of G is at least 3. Coloring the vertices x and v red, the vertices u and y blue, and the vertex z green, we obtain a 3-coloring of G. Hence, $\chi(G) = 3$. □

Let $G = (V, E)$ be a graph that is k-colored, using the colors $1, 2, \ldots, k$. Let V_i denote the subset of vertices that are assigned the color i, $(i = 1, 2, \ldots, k)$. Then $V_1, V_2, \ldots, V_k$ is a partition of V, called a *color partition* for G. Moreover, the induced subgraphs $G_{V_1}, G_{V_2}, \ldots, G_{V_k}$ are null graphs. Conversely, if we can partition the vertices into k parts, with each part inducing a null graph, then the chromatic number is at most k. Hence, another way to describe the chromatic number of G is that $\chi(G)$ *is the smallest integer k such that the vertices of G can be partitioned into k sets with each set inducing a null graph.* In the coloring of the graph in Figure 12.1 described in the preceding example, the partition is $\{x, v\}$ (the red vertices), $\{u, y\}$ (the blue vertices), and $\{z\}$ (the green vertices). Using these ideas, we can now obtain another lower bound on the chromatic number of a graph.

Corollary 12.1.3 *Let $G = (V, E)$ be a graph of order n and let q be the largest order of an induced subgraph of G equal to a null graph N_q. Then*

$$\chi(G) \geq \left\lceil \frac{n}{q} \right\rceil.$$

Proof. Let $\chi(G) = k$ and let $V_1, V_2, \ldots, V_k$ be a color partition for G. Then $|V_i| \leq q$ for each i, and we obtain

$$n = |V| = \sum_{i=1}^{k} |V_i| \leq \sum_{i=1}^{k} q = k \times q.$$

Hence,

$$\chi(G) = k \geq \frac{n}{q}.$$

Since $\chi(G)$ is an integer, the corollary follows. □

Example. Continuing with the graph in Figure 12.1, an examination of the graph reveals that the largest order of an induced null subgraph is $q = 2$ (that is, of every three vertices at least two are adjacent). Thus, by Corollary 12.1.3, we again obtain

$$\chi(G) \geq \left\lceil \frac{5}{2} \right\rceil = 3.$$

□

According to Theorem 12.1.1, the graphs with chromatic number 1 are the null graphs. It is then natural to ask for a characterization of graphs with chromatic number 2. Graphs with chromatic number 2 have a color partition with two sets. This should bring to mind bipartite graphs.

Theorem 12.1.4 *Let G be a graph with at least one edge. Then $\chi(G) = 2$ if and only if G is bipartite.*

Proof. The chromatic number of a graph with at least one edge is at least 2. If G is a bipartite graph, then, coloring the left vertices red and the right vertices blue,[8] we obtain a 2-coloring of G. Conversely, the color partition arising from a 2-coloring is a bipartition for G, establishing the bipartiteness of G. □

It follows from Theorems 11.4.1 and 12.1.4 that the chromatic number of a graph that is not a null graph equals 2 if and only if each cycle has even length. Graphs with chromatic number 3 can have a very complicated structure and do not admit a simple characterization.

Example. *A scheduling problem.* Many scheduling problems can be formulated as problems that ask for the chromatic number (but often will settle for a number not much larger than the chromatic number) of a graph. The basic idea is that we associate a graph with a scheduling problem whose vertices are the "tasks" to be scheduled, putting an edge between two tasks whenever they conflict, and hence cannot be scheduled at the same time. A color partition for G furnishes a schedule without any conflicts. The chromatic number of the graph thus equals the smallest number of time slots in a schedule with no conflicts.

For instance, suppose we want to schedule nine tasks $a, b, c, d, e, f, g, h, i$, where each task conflicts with the task that immediately follows it in the list and i conflicts with a. The "conflict" graph G in this case is a graph of order 9 whose edges are arranged in a cycle of length 9. Of any five vertices of this graph, at least two are adjacent. Hence, the q in Corollary 12.1.3 is at most 4, and it follows that $\chi(G) \geq 3$.

[8] Of course we could have said "coloring the left vertices left and the right vertices right," using left and right as our two colors.

It is easy to find a 3-coloring so that $\chi(G) = 3$. Thus, this scheduling problem requires three time slots. □

The determination of the chromatic number of a graph is a difficult problem, and there is no known good algorithm[9] for it. Therefore, it is of importance to have estimates for the chromatic number of a graph and some means for finding a vertex-coloring in which the number of colors used is "not too large." In Corollaries 12.1.2 and 12.1.3, we have given two lower bounds for the chromatic number. Theorem 12.1.1 contains an upper bound, namely, $n-1$ for a graph of order n, which is not a complete graph, but this bound is rather poor. One would hope to be able to do better. Indeed, we show that a better bound can be obtained from the degrees of the vertices, and there is a simple algorithm for obtaining a vertex-coloring that does not exceed this bound. This algorithm is another example of a greedy algorithm,[10] which proceeds sequentially by "choosing the first available color," ignoring the consequences this may have for later choices. We use the positive integers to color the vertices, and thus we can speak about one color being smaller than another.

Greedy algorithm for vertex-coloring

Let G be a graph in which the vertices have been listed in some order $x_1, x_2, \ldots, x_n$.

(1) Assign the color 1 to vertex x_1.

(2) For each $i = 2, 3, \ldots, n$, let p be the smallest color such that none of the vertices $x_1, \ldots, x_{i-1}$ which are adjacent to x_i is colored p, and assign the color p to x_i.

Theorem 12.1.5 *Let G be a graph for which the maximum degree of a vertex is Δ. Then the greedy algorithm produces a $(\Delta+1)$-coloring[11] of the vertices of G, and hence*

$$\chi(G) \leq \Delta + 1.$$

Proof. In words, the greedy algorithm considers each vertex in turn and assigns to it the smallest color which has not already been assigned to a vertex to which it is adjacent. In particular, two adjacent vertices are never assigned the same color, and hence the greedy algorithm does produce a vertex-coloring. There are at most Δ vertices adjacent to vertex x_i, and hence, at most, Δ of the vertices $x_1, \ldots, x_{i-1}$ are adjacent to x_i. Therefore, when we consider vertex x_i in step (2) of the algorithm,

[9] One for which the number of steps required grows like a polynomial function of the order of the graph. Most experts believe that no good algorithm is possible.

[10] A greedy algorithm for a minimum weight spanning tree is given in Section 11.7. Unlike that greedy algorithm, which actually constructed a minimum weight spanning tree, the current algorithm gives only an upper bound for the chromatic number.

[11] Remember that a $(\Delta + 1)$-coloring does not mean that all $\Delta + 1$ colors are actually used.

at least one of the colors $1, 2, \ldots, \Delta + 1$ has not already been assigned to a vertex adjacent to x_i, and the algorithm assigns the smallest of these to x_i. It follows that the greedy algorithm produces a $(\Delta + 1)$-coloring of the vertices of G. □

The greedy algorithm just *might* color the vertices of G in the fewest possible number, namely, $\chi(G)$, of colors. How well or how badly it does depends on the order in which the vertices are listed before the algorithm is applied. Let $V_1, V_2, \ldots, V_{\chi(G)}$ be a color partition arising from a vertex coloring using $\chi(G)$ colors.[12] Suppose we list the vertices of V_1 first, followed by the vertices of V_2, . . . , followed by the vertices of $V_{\chi(G)}$.[13] It is easy to see that the greedy algorithm colors the vertices in V_1 with the color 1, the vertices in V_2 with one of the colors 1 or 2, ... , the vertices in $V_{\chi(G)}$ with one of the colors $1, 2, \ldots, \chi(G)$. Thus, with this listing of the vertices, the greedy algorithm colors the vertices with the fewest possible number of colors.

Example. Consider a complete bipartite graph $K_{1,n}$. The largest degree of a vertex is $\Delta = n$. Thus, by Theorem 12.1.5, the greedy algorithm produces an $(n + 1)$-coloring. In fact, it does a lot better. No matter how the vertices are listed, the greedy algorithm colors the vertices with only two colors, the minimum possible number of colors. Thus, the greedy algorithm sometimes can give a much better coloring than is suggested by Theorem 12.1.5.

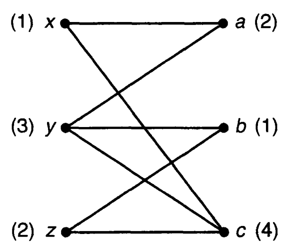


Figure 12.2

Now consider the bipartite graph drawn in Figure 12.2, and list the vertices as x, a, b, y, z, c. Then the colors assigned to these vertices by the greedy algorithm are, respectively, $1, 2, 1, 3, 2, 4$. Hence, the greedy algorithm produces a 4-coloring, yet the chromatic number is 2. □

The upper bound for the chromatic number given in Theorem 12.1.5 can be improved, except for two classes of graphs. These are the complete graphs K_n, for which $\Delta = n - 1$ and $\chi(G) = n$, and the graphs C_n of odd order n whose edges are arranged

[12]Of course, knowing this implies that we already know $\chi(G)$. Our point is that, if we were very lucky in the way we listed the vertices of the graph, then the greedy algorithm could produce a coloring using the smallest number of colors.

[13]All we want to do is to keep the vertices of the same color together.

in a cycle (of odd length), for which $\Delta = 2$ and $\chi(G) = 3$. The proof of the next theorem of Brooks[14] is omitted.

Theorem 12.1.6 *Let G be a connected graph for which the maximum degree of a vertex is Δ. If G is neither a complete graph K_n nor an odd cycle graph C_n, then $\chi(G) \leq \Delta$.*

A conclusion from our discussion of chromatic number is that coloring the vertices of a graph (so that adjacent vertices are colored differently) is hard if we want to use the fewest number of colors. We now remove the restriction that the number of colors is minimum, but consider a more difficult question: Given a graph G and a set $\{1, 2, \ldots, k\}$ of k colors, how many k-colorings of G are there? If we know that $\chi(G) > k$, then the question is easy and the answer is 0.[15]

For each nonnegative integer k, the number of k-colorings of the vertices of a graph G is denoted by

$$p_G(k).$$

If $\chi(G) > k$, then $p_G(k) = 0$. For example, for a complete graph, we have

$$p_{K_n}(k) = k(k-1)\ldots(k-(n-1)) = [k]_n,$$

since each vertex must be a different color.[16] For a null graph, we have

$$p_{N_n}(k) = k^n,$$

since we can arbitrarily assign colors to each of the vertices.[17]

Example. We determine $p_G(k)$ for the graph G in Figure 12.1. First we color the vertices x, y, z. These vertices can be colored in

$$k(k-1)(k-2)$$

ways, since each has to receive a different color. Next, we color u and observe that it must receive a color different from that of x and z. There are $k-2$ ways to color u.

[14]R. L. Brooks, On Coloring the Nodes of a Network, *Proc. Cambridge Philos. Soc.*, 37 (1941), 194–197.

[15]If $\chi(G) > k$, but we do not have that information, then the question is much more difficult. This is because, in answering it, we are implicity determining whether or not $\chi(G) \leq k$: $\chi(G) \leq k$ if and only if the the number of ways to color G with k colors is not 0.

[16]$[k]_n$ is the function that was introduced in Section 8.2 and counts the number of n-permutations of a set of k distinct objects. In the situation here, the k objects are the k colors and the n-permutations are the assignments of a color to each of the n vertices of K_n. Since each pair of vertices is adjacent in K_n, all vertices have to be colored differently.

[17]We recall from Chapter 2 that k^n counts the number of n-permutations of a set of k objects (the k colors here) in which unlimited repetition is allowed. Since no vertices of N_n are adjacent, we can freely repeat colors.

Finally, v can receive any of the colors other than the (distinct) colors of u and z, and hence there are $k-2$ ways to color v. Thus,

$$p_G(k) = k(k-1)(k-2) \times (k-2) \times (k-2) = k(k-1)(k-2)^3. \qquad \Box$$

It is not hard to count the number of ways to color the vertices of a tree. What is surprising is that, for each k, the number of k-colorings of a tree depends only on the number of vertices of the tree, and not on which tree is being considered!

Theorem 12.1.7 *Let T be a tree of order n. Then*

$$p_T(k) = k(k-1)^{n-1}.$$

Proof. We grow T as described in Section 11.5 and color the vertices as we do. The starting vertex can be colored with any one of the k colors. Each new vertex y we add is adjacent to only one of the previous vertices x. Hence, y can be colored with any one of the $k-1$ colors different from the color of x. Thus, each of the $n-1$ vertices, other than the first, can be colored in $k-1$ ways, and the formula follows. $\Box$

The observant reader will have noticed that, thus far, each of the formulas obtained for the number of ways to color the vertices of a graph has turned out to be a polynomial function of the number k of colors. Indeed, this is no accident and is a general phenomenon: $p_G(k)$ is always a polynomial function of k. We now turn to proving this fact. As a result of this property, $p_G(k)$ is called the *chromatic polynomial* of the graph G. The chromatic polynomial of G evaluated at k gives the number of k-colorings of G. The chromatic number of G is the smallest nonnegative integer that is not a root of the chromatic polynomial.

The fact that $p_G(k)$ is a polynomial rests on a simple observation. Let x and y be two vertices of G that are adjacent. Let G_1 be the graph obtained from G by removing the edge $\{x, y\}$ joining x and y. The k-colorings of G_1 can be partitioned into two parts, $C(k)$ and $D(k)$. In the first part, $C(k)$, we put those k-colorings of G_1 in which x and y are assigned the same color. In the second part, $D(k)$, we put those k-colorings in which x and y are assigned different colors. Thus,

$$p_{G_1}(k) = |C(k)| + |D(k)|.$$

Since x and y are adjacent in G, there is a one-to-one correspondence between the k-colorings of G_1, in which x and y are assigned different colors, and the k-colorings of G. Hence,

$$p_G(k) = |D(k)|.$$

Let G_2 be the graph obtained from G by *identifying* the vertices x and y. This means that we delete the edge $\{x, y\}$, replace x and y by one new vertex, denoted $\overline{xy}$, and join $\overline{xy}$ to any vertex that is joined either to x or y in G.[18] There is a one-to-one correspondence between the k-colorings of G_1, in which x and y are assigned the same color, and the k-colorings of G_2. Therefore,

$$p_{G_2}(k) = |C(k)|.$$

Combining the previous three equations, we get

$$p_{G_1}(k) = p_G(k) + p_{G_2}(k),$$

from which it follows that

$$p_G(k) = p_{G_1}(k) - p_{G_2}(k). \tag{12.1}$$

In words, the number of k-colorings of G can be obtained by finding the number of k-colorings of G_1 (in which the edge $\{x, y\}$ has been removed, making it possible for x and y to be assigned the same color) and subtracting the number of k-colorings of G_2 (in which the vertices x and y have been identified so that they must be assigned the same color). Why is this a useful observation?

The order of G_1 is the same as the order of G, and G_1 has one fewer edge than G. The order of G_2 is one less than the order of G, and G_2 has at least one fewer edge than G. Put another way, G_1 and G_2 are closer (in terms of the number of edges) to a null graph than G is. Thus, our observation suggests an algorithm to determine the number of k-colorings of G: Continue to remove edges and identify vertices until all graphs so obtained are null graphs. By (12.1), the number of k-colorings of G can be expressed in terms of the number of k-colorings of each of these null graphs. But we know what the number of k-colorings of a null graph is; the number of k-colorings of a null graph of order p is k^p. Hence, we can obtain the number of k-colorings of G by subtracting and adding the number of k-colorings of null graphs.[19] In addition, since k^p is a polynomial in k (a monomial, actually), the number of k-colorings of G, being a sum of such monomials or their negatives, is a polynomial in k; that is, the chromatic polynomial of G is indeed a polynomial. Before formalizing the previous discussions, we consider an example.

Example. Let G be a cycle graph C_5 of order 5 whose edges are arranged in a cycle. Choosing any edge of G and applying (12.1), we see that

$$p_G(k) = p_{G_1}(k) - p_{G_2}(k),$$

[18] We can think of moving x and y together until they coincide. This may create a multiple edge, in which case we delete one copy.

[19] Null graphs may be very uninteresting, but as we have just seen they have an important role to play in graph colorings.

where G_1 is a tree of order 5 whose edges are arranged in a path and G_2 is a cycle graph C_4 of order 4. By Theorem 12.1.7, $p_{G_1}(k) = k(k-1)^4$.[20] We do to G_2 what we did to G and obtain

$$p_{G_2}(k) = k(k-1)^3 - p_{G_3}(k),$$

where G_3 is a cycle graph C_3 of order 3. Since G_3 is a complete graph K_3 with $p_{G_3}(k) = k(k-1)(k-2)$, we obtain

$$p_G(k) = k(k-1)^4 - (k(k-1)^3 - k(k-1)(k-2)).$$

This simplifies to

$$p_G(k) = k(k-1)(k-2)(k^2 - 2k + 2).$$

Note that $p_G(0) = 0$, $p_G(1) = 0$, $p_G(2) = 0$ and $p_G(3) > 0$. Hence, $\chi(G) = 3$, a fact that is easy to establish directly. □

Let G be a graph and let $\alpha = \{x, y\}$ be an edge of G. We now denote the graph obtained from G by deleting the edge α by $G_{\ominus\alpha}$. We also denote the graph obtained from G by identifying x and y (as previously defined) by $G_{\otimes\alpha}$. We say that $G_{\otimes\alpha}$ is obtained from G by *contracting* the edge α. Thus, (12.1) can be rewritten as

$$p_G(k) = p_{G_{\ominus\alpha}}(k) - p_{G_{\otimes\alpha}}(k). \tag{12.2}$$

As already implied, repeated use of deletion and contraction gives an algorithm for determining $p_G(k)$. In the next algorithm, we consider objects $(\pm, H)$, where H is a graph. For the purposes of the algorithm, we call such an object a *signed graph*, a graph with either a plus sign $+$ or minus sign $-$ associated with it.

Algorithm for computing the chromatic polynomial of a graph

Let $G = (V, E)$ be a graph.

(1) Put $\mathcal{G} = \{(+, G)\}$.

(2) While there exists a signed graph in $\mathcal{G}$ that is not a null graph, do the following:

  (i) Choose a nonnull signed graph (ϵ, H) in $\mathcal{G}$ and an edge α of H.

  (ii) Remove (ϵ, H) from $\mathcal{G}$ and put in the two signed graphs $(\epsilon, H_{\ominus\alpha})$ and $(-\epsilon, H_{\otimes\alpha})$.

(3) Put $p_G(k) = \sum \epsilon k^p$, where the summation extends over all signed graphs (ϵ, H) in $\mathcal{G}$ and p is the order of H.

[20]This illustrates an important point in this process, namely, if one obtains a graph whose chromatic polynomial is known, then make use of that information. One doesn't necessarily have to reduce all graphs to null graphs.

In words, we start with G with a $+$ attached to it. Using the deletion/contraction process, we reduce G and all resulting graphs to null graphs, keeping track of the associated sign as determined by multiple applications of (12.2). When there are no remaining graphs with an edge, we compute the order p of each null graph so obtained and then form the monomial $\pm k^p$, which is its chromatic polynomial, adjusted for sign. By repeated use of (12.2), adding all these polynomials, we obtain the chromatic polynomial of G. In particular, since the sum of monomials is a polynomial, we obtain a polynomial. In the deletion/contraction process, exactly one graph is a null graph of the same order as G. This graph results by successive deletion of all edges of G, without any contraction, and contributes the monomial k^n with a $+$ sign. All other graphs have fewer than n vertices and contribute monomials of degree strictly less than n. We have thus proved the next theorem.

Theorem 12.1.8 *Let G be a graph of order $n \geq 1$. Then the number of k-colorings of G is a polynomial in k of degree equal to n (with leading coefficient equal to 1) and this polynomial—the chromatic polynomial of G—is computed correctly by the preceding algorithm.* □

It is straightforward to see that, if a graph G is disconnected, then its chromatic polynomial is the product of the chromatic polynomials of its connected components. In particular, the chromatic number is the largest of the chromatic numbers of its connected components. In the next theorem, we generalize this observation. The resulting formula can sometimes be used to shorten the computation of the chromatic polynomial of a graph.

Let $G = (V, E)$ be a connected graph and let U be a subset of the vertices of G. Then U is called an *articulation set* of G, provided that the subgraph G_{V-U} induced[21] by the vertices not in U is disconnected. If G is not complete, then G contains two nonadjacent vertices a and b, and hence $U = V - \{a, b\}$ is an articulation set with $V - U = \{a, b\}$. A complete graph does not have an articulation set. Therefore, a connected graph has an articulation set if and only if it is not complete.

Lemma 12.1.9 *Let G be a graph and assume that G contains a subgraph H equal to a complete graph K_r. Then the chromatic polynomial of G is divisible by the chromatic polynomial $[k]_r$ of K_r.*

Proof. In any k-coloring of G, the vertices of H are all colored differently. Moreover, each choice of colors for the vertices of H can be extended to the same number $q(k)$ of colorings for the remaining vertices of G. Hence, $p_G(k) = [k]_r q(k)$. □

[21] Recall that the vertices of this subgraph are those in $V - U$, and two vertices are adjacent in G_{V-U} if and only if they are adjacent in G.

Theorem 12.1.10 *Let U be an articulation set of G and suppose that the induced subgraph G_U is a complete graph K_r. Let the connected components of G_{V-U} be the induced subgraphs $G_{U_1}, \dots, G_{U_t}$. For $i = 1, \dots, t$, let $H_i = G_{U \cup U_i}$ be the subgraph of G induced by $U \cup U_i$. Then*

$$p_G(k) = \frac{p_{H_1}(k) \times \cdots \times p_{H_t}(k)}{([k]_r)^{t-1}}.$$

In particular, the chromatic number of G is the largest of the chromatic numbers of $H_1, \dots, H_t$.

Proof. The graphs $H_1, \dots, H_t$ all have the vertices of U in common but are otherwise pairwise disjoint. Each k-coloring of G can be obtained by first choosing a k-coloring of H_1 (there are $p_{H_1}(k)$ such colorings and now all the vertices of U are colored) and then completing the colorings of each $H_i, (i = 2, \dots, t)$ (each in $p_{H_i}(k)/[k]_r$ ways, by Lemma 12.1.9). □

Example. Let G be the graph drawn in Figure 12.3. Let $U = \{a, b, c\}$. Applying Theorem 12.1.10, we see that

$$p_G(k) = \frac{(q(k))^3}{(k(k-1)(k-2))^2},$$

where $q(k)$ is the chromatic polynomial of a complete graph G' of order 4 with one missing edge. It is simple to calculate (in fact, use Theorem 12.1.10 again) that $q(k) = k(k-1)(k-2)^2$. Hence,

$$p_G(k) = k(k-1)(k-2)^4.$$

□

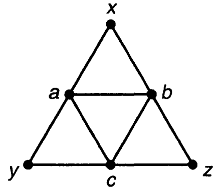


Figure 12.3

12.2 Plane and Planar Graphs

Let $G = (V, E)$ be a planar general graph and let G' be a planar representation of G. Thus, G' is a plane-graph and G' consists of a collection of points in the plane, called vertex-points because they correspond to the vertices of G, and a collection of curves, called edge-curves because they correspond to the edges of G. Also, an edge-curve α is a simple curve that passes through a vertex-point x if and only if the vertex x of G is incident with the edge α of G.[22] Only endpoints can be common points of edge-curves.

The plane graph G' divides the plane into a number of regions that are bounded by one or more of the edge-curves.[23] Exactly one of these regions extends infinitely far.

Example. The plane-graph shown in Figure 12.4 has 10 vertex-points, 14 edge-curves, and 6 regions. Each of the regions is bordered by some of the edge-curves, but we must be be very careful how we count the edge-curves. The regions R_2, R_3, R_5, and R_6 are bordered by one, two, six, and two edge-curves, respectively. The region R_4 is bordered by 10 edge-curves (and not 4 or 7). This is because, as we traverse R_4 by walking around its border, three of the edge-curves are traversed twice (see the dashed line in Figure 12.4). The region R_1 is bordered[24] by 7 edge-curves. In sum, we count the number of edge-curves bordering regions in such a way that each edge-curve is counted twice, either because it borders two different regions or because it borders the same region twice. □

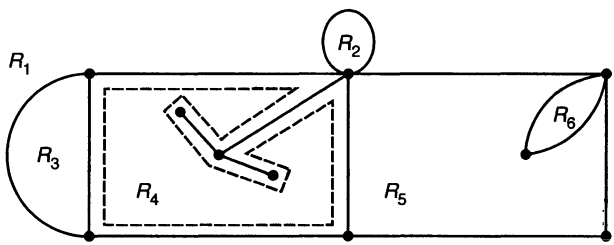


Figure 12.4

Let G' be a plane-graph with n vertex-points, e edge-curves, and r regions. Let the number of edge-curves bordering the regions be, respectively,

$$f_1, f_2, \ldots, f_r.$$

[22]Recall that we give the same label to a vertex and its corresponding vertex-point and the same label to an edge and its corresponding edge-curve.

[23]Thus, a plane-graph has points, curves, and now regions.

[24]R_1 might appear to be bordered by none of the edge-curves, since it extends infinitely far in all directions. However, a geometrical figure drawn in the plane can also be thought of as drawn on a sphere. Loosely speaking, we put a large sphere on top of the figure and then "wrap" the sphere with the plane. The infinite region is now some finite region on the sphere surrounding the north pole. Note also that a region may have "interior" border curves as, for example, R_4 does.

Then, using the convention established in the preceding example, we have

$$f_1 + f_2 + \cdots + f_r = 2e. \tag{12.3}$$

We now derive a relationship among n, e, and r which implies in particular that any two of them determine the third. This relationship is known as *Euler's formula.*

Theorem 12.2.1 *Let G be a plane-graph of order n with e edge-curves and assume that G is connected. Then the number r of regions into which G divides the plane satisfies*

$$r = e - n + 2. \tag{12.4}$$

Proof. First, assume that G is a tree. Then $e = n-1$ and $r = 1$ (the only region is the infinite region that is bordered twice by each edge-curve). Hence, (12.4) holds in this case. Now assume that G is not a tree. Since G is connected, it has a spanning tree T with $n' = n$ vertices, $e' = n - 1$ edges, and $r' = 1$ regions, where $r' = e' - n' + 2$. We can think of starting with the edge-curves of T and then inserting one new edge-curve at a time until we have G. Each time we insert an edge-curve, we divide an existing region into two regions. Hence, each time we insert another edge-curve, e' increases by 1, r' increases by 1, and n' stays the same (n' is always n). Therefore, starting with $r' = e' - n' + 2$ for a spanning tree, this relationship persists as we include the remaining edge-curves, and the theorem is proved. □

Euler's formula has an important consequence for planar graphs (with no loops and multiple edges).

Theorem 12.2.2 *Let G be a connected planar graph. Then there is a vertex of G whose degree is at most* 5.

Proof. Let G' be a planar representation of G. Since a graph has no loops, no region of G' is bordered by only one edge-curve. Similarly, since a graph has no multiple edges, no region is bordered by only two edge-curves (unless G has exactly one edge). Thus, in (12.3), each f_i satisfies $f_i \geq 3$, and hence we have

$$3r \leq 2e, \text{ or equivalently, } \frac{2e}{3} \geq r.$$

Using this inequality in Euler's formula, we get

$$\frac{2e}{3} \geq r = e - n + 2, \text{ or, equivalently, } e \leq 3n - 6. \tag{12.5}$$

Let $d_1, d_2, \ldots, d_n$ be the degrees of the vertices of G. By Theorem 11.1.1, we have

$$d_1 + d_2 + \cdots + d_n = 2e.$$

Hence, the average of the degrees of G satisfies

$$\frac{d_1 + d_2 + \cdots + d_n}{n} = \frac{2e}{n} \leq \frac{6n-12}{n} < 6.$$

Since the average of the degrees is less than 6, some vertex must have degree 5 or less. $\square$

If a graph G has a subgraph that is not planar, then G is not planar. Thus, in attempting to describe planar graphs, it is of interest to find nonplanar graphs G, each of whose subgraphs, other than G itself, is planar.

Example. A complete graph K_n is planar if and only if $n \leq 4$.

If $n \leq 4$, then K_n is planar. Now consider K_5. As shown in the proof of Theorem 12.2.2 (see (12.5)), the number n of vertices and the number e of edges of a planar graph satisfies $e \leq 3n-6$. Since K_5 has $n = 5$ vertices and $e = 10$ edges, this inequality is not satisfied and hence K_5 is not planar. Since K_5 is not planar, K_n is not planar for all $n \geq 5$. $\square$

Example. A complete bipartite graph $K_{p,q}$ is planar if and only if $p \leq 2$ or $q \leq 2$.

It is easy to draw a planar representations of $K_{p,q}$ if $p \leq 2$ or $q \leq 2$. Now consider $K_{3,3}$. A bipartite graph does not have any cycles of length 3; hence, in a planar representation of a planar bipartite graph, each region is bordered by at least four edge-curves. Arguing as in the proof of Theorem 12.2.2, we obtain $r \leq e/2$. Applying Euler's formula, we get

$$\frac{e}{2} \geq e - n + 2; \text{ equivalently, } 2n - 4 \geq e.$$

Since $K_{3,3}$ has $n = 6$ vertices and $e = 9$ edges, this inequality is not satisfied and hence $K_{3,3}$ is not planar. Since $K_{3,3}$ is not planar, $K_{p,q}$ is not planar if both $p \geq 3$ and $q \geq 3$. $\square$

Let $G = (V, E)$ be a nonplanar graph and let $\{x, y\}$ be any edge of G. Let G' be obtained from G by choosing a new vertex z not in V and replacing the edge $\{x, y\}$ with the two edges $\{x, z\}$ and $\{z, y\}$. We say that G' is obtained from G by *subdividing the edge* $\{x, y\}$. If G is not planar, then clearly G' is also not planar.[25] A graph H is called a *subdivison* of a graph G, provided that H can be obtained from G by successively subdividing edges. If H is a subdivision of G, then we can think of H as obtained from G by inserting several new vertices (possibly none) on each of its edges. For example, the graphs in Figure 12.5 are subdivisions of $K_{3,3}$ and K_5, respectively. It follows that each of these graphs is not planar.

[25]If there were a planar representation of G', then by "erasing" the vertex-point z we obtain a planar representation of G.

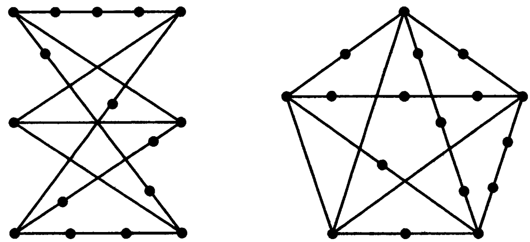


Figure 12.5

A nonplanar graph cannot contain a subdivision of a K_5 or a $K_{3,3}$. It is a remarkable theorem of Kuratowski[26] that the converse holds as well. We state this theorem without proof.

Theorem 12.2.3 *A graph G is planar if and only if it does not have a subgraph that is a subdivision of a K_5 or of a $K_{3,3}$.*

Loosely speaking, Theorem 12.2.3 says that a graph that is not planar has to contain a subgraph that either looks like a K_5 or looks like a $K_{3,3}$. Thus, the two graphs K_5 and $K_{3,3}$ are the only two "obstructions" to planarity. As noted by Wagner[27] and Harary and Tutte,[28] planar graphs can also be characterized by using the notion of contraction of an edge in place of subdivision of an edge. A graph H is a *contraction* of a graph G, provided that H can be obtained from G by successively contracting edges.

Theorem 12.2.4 *A graph G is planar if and only if it does not contain a subgraph that contracts to a K_5 or a $K_{3,3}$.*

12.3 A Five-Color Theorem

In this section we show that the chromatic mumber of a planar graph is at most 5. This was first proved by P. J. Heawood in 1890 after he discovered an error in a "proof" published in 1879 by A. Kempe, in which Kempe claimed that the chromatic number of a planar graph is at most 4. Although Kempe's proof was wrong, it contained good ideas, which Heawood used to prove his five-color theorem. As described in the introduction to this Chapter, and also in Section 1.4, a proof that the chromatic

[26] K. Kuratowski, Sur le problème des courbes gauches en topologie, *Fund. Math.*, 15 (1930), 271–283.

[27] K. Wagner, Über eine Eigenschaft der ebenen Komplexe, *Math. Ann.*, 114 (1937), 570–590.

[28] F. Harary and W. T. Tutte, A Dual Form of Kuratowski's Theorem, *Canadian Math. Bull.*, 8 (1965), 17–20.

number of every planar graph is at most 4 has now been obtained, and it relies heavily on computer checking.

There is an easy proof, which uses Theorem 12.2.2, of the fact that the chromatic number of a planar graph G is at most 6. Indeed, suppose there is a planar graph whose chromatic number is 7 or more, and let G be such a graph with the minimum number of vertices. By Theorem 12.2.2, G has a vertex x of degree at most 5. Removing x (and all incident edges) from G leaves a planar graph G' with one fewer vertex. The minimality assumption on G implies that G' has a 6-coloring. Since x is adjacent in G to at most five vertices, we can take a 6-coloring of G' and assign a color to x in such a way as to produce a 6-coloring of G, a contradiction. It follows that the chromatic number of every planar graph is 6 or less. It is harder, but not terribly so, to prove that a planar graph has a 5-coloring, but the jump from five colors to four colors is a giant one.

Before proving that five colors suffice to color the vertices of any planar graph, we make one observation. In the previous section, we showed that a complete graph K_5 of order 5 is not planar, and hence a planar graph cannot contain five vertices, the members of every pair of which are adjacent. It is erroneous to conclude from this that every planar graph has a 5-coloring. For instance, with 3 replacing 5, a cycle graph C_5 of order 5 does not have a K_3 as a subgraph, yet its chromatic number is 3 and it does not have a 2-coloring. So it does *not* simply suffice to say that there do not exist five vertices such that each must be assigned different colors and hence a 4-coloring is possible.

The next theorem is an important step in the proof of the five-color theorem. It applies to nonplanar graphs as well as planar graphs.

Theorem 12.3.1 *Let there be given a k-coloring of the vertices of a graph $H = (U, F)$. Let two of the colors be red and blue, and let W be the subset of vertices in U that are assigned either the color red or the color blue. Let $H_{r,b}$ be the subgraph of H induced by the vertices in W and let $C_{r,b}$ be a connected component of $H_{r,b}$. Interchanging the colors red and blue assigned to the vertices of $C_{r,b}$, we obtain another k-coloring of H.*

Proof. Suppose that after the colors red and blue have been interchanged in $C_{r,b}$, there are two adjacent vertices which are now colored the same. This color must be either red or blue (say, red). If x and y are both vertices of $C_{r,b}$, then before we switched colors, x and y were colored blue which is impossible. If neither x nor y is a vertex in $C_{r,b}$, then their colors weren't switched and so they both started out with color red, again impossible. Thus, one of x and y is a vertex in $C_{r,b}$ and the other isn't (say, x is in $C_{r,b}$ and y is not). Therefore, x started out with the color blue and y started out with the color red. Since x and y are adjacent and each is assigned the color red or blue, they must be in the same connected component of $H_{r,b}$, contradicting the fact that x is in the connected component $C_{r,b}$ of $H_{r,b}$ and y isn't. □

Theorem 12.3.2 *The chromatic number of a planar graph is at most* 5.

Proof. Let G be a planar graph of order n. If $n \leq 5$, then surely $\chi(G) \leq 5$. We now let $n > 5$ and prove the theorem by induction on n. We assume that G is drawn in the plane as a plane-graph. By Theorem 12.2.2, there is a vertex x whose degree is at most 5. Let H be the subgraph of order $n-1$ of G induced by the vertices different from x. By the induction hypothesis, there is a 5-coloring of H. If the degree of x is 4 or less, then we can assign to x one of the colors not equal to the colors of the vertices adjacent to x and obtain a 5-coloring of G.[29] Now suppose that the degree of x is 5. There are 5 vertices adjacent to x. If two of these vertices are assigned the same color, then, as before, there is a color we can assign x in order to obtain a 5-coloring of G. So we now further suppose that each of the vertices y_1, y_2, y_3, y_4, y_5 adjacent to x is assigned a different color. As in Figure 12.6, the vertices $y_1, \ldots, y_5$ are labeled consecutively around vertex x; the colors are the numbers $1, 2, 3, 4$, and 5 with y_j colored j, $(j = 1, 2, 3, 4, 5)$.

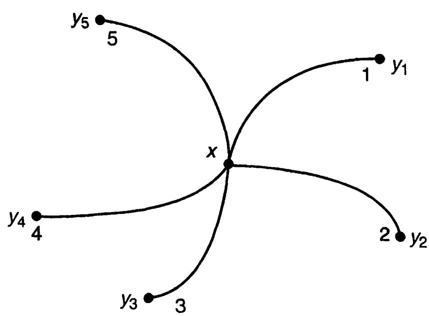


Figure 12.6

We consider the subgraph $H_{1,3}$ of H induced by the vertices of colors 1 and 3. If y_1 and y_3 are in different connected components of $H_{1,3}$, then we apply Lemma 12.1.1 to H and obtain a 5-coloring in which y_1 and y_3 are colored the same. This frees up a color for x, and we obtain a 5-coloring of G. Now assume that y_1 and y_3 are in the same connected component of $H_{1,3}$. Then there is a path in H joining y_1 and y_3 such that the colors of the vertices on the path alternate between 1 and 3. This path, along with the edge-curve joining x and y_1 and the edge-curve joining x and y_3, determine a closed curve γ. Of the remaining three vertices y_2, y_4, and y_5 adjacent to x, one of them is inside γ and two are outside γ, or the other way around. See Figure 12.7, in which y_2 is inside γ and y_4 and y_5 are outside. We now consider the subgraph $H_{2,4}$ of H induced by the vertices of colors 2 and 4. But (see Figure 12.7) vertices y_2 and y_4 cannot be in the same connected component of $H_{2,4}$ since y_2 is in the interior of a simple closed curve and y_4 is in the exterior of that curve. Switching the colors 2 and

[29]This is just like our proof that six colors suffice to color the vertices of a planar graph. But for a 5-coloring, we are not yet done, since we now have to deal with the case that x has degree 5.

4 of the vertices in the connected component of $H_{2,4}$ that contains x_2, we obtain by Lemma 12.1.1 a 5-coloring of H in which none of the vertices adjacent to x is assigned color 2. We now assign the color 2 to x and obtain a 5-coloring of G. □

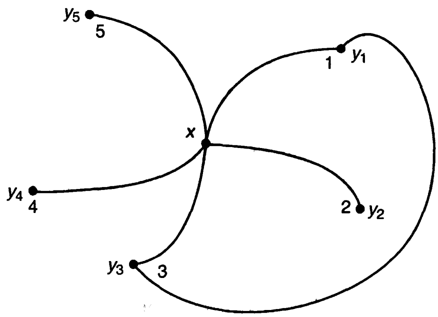


Figure 12.7

In 1943, Hadwiger[30] made a conjecture about the chromatic number of graphs, which, except in a few cases, is still unsolved. This is perhaps not too surprising since the truth of one instance of this conjecture is equivalent to the existence of a 4-coloring of any planar graph. This conjecture asserts: *A connected graph G whose chromatic number satisfies $\chi(G) \geq p$ can be contracted to a K_p. Equivalently, if G cannot be contracted to a K_p, then $\chi(G) < p$.* The converse of the conjecture is false; that is, it is possible for a graph to be contractable to a K_p and yet have chromatic number less than p. For instance, a graph of order 4 whose edges are arranged in a cycle has chromatic number 2, yet the graph itself can be contracted to a K_3 by contraction of one edge.

Theorem 12.3.3 *Hadwiger's conjecture holds for $p = 5$ if and only if every planar graph has a 4-coloring.*

Partial Proof. We prove only that if Hadwiger's conjecture holds for $p = 5$, then every planar graph G has a 4-coloring. Let G be a planar graph and suppose that G is contractable to a K_5. A contraction of a planar graph is also planar, and this implies that K_5 is planar, a statement we know to be false. Hence, G is not contractable to a K_5, and hence the truth of Hadwiger's conjecture for $p = 5$ implies that $\chi(G) \leq 4$. □

Hadwiger's conjecture is also known to be true for $p \leq 4$ and for $p = 6$. We verify Hadwiger's conjecture for $p = 2$ and 3 in the next theorem and leave its validity for $p = 4$ as a challenging exercise.

[30] H. Hadwiger, Über eine Klassifikation der Streckenkomplexe, *Vierteljschr. Naturforsch. Ges., Zurich*, 88 (1943), 133–142.

Theorem 12.3.4 *Let $p \leq 3$. If G is a connected graph with chromatic number $\chi(G) \geq p$, then G can be contracted to a K_p.*

Proof. If $p = 1$, then by contracting each edge, we arrive at a K_1. If $p = 2$, then G has at least one edge α, and by contracting all edges except for α, we arrive at a K_2. Now, suppose $p = 3$ and $\chi(G) \geq 3$. Since $\chi(G) \geq 3$, G is not bipartite, and by Theorem 11.4.1, G has a cycle of odd length. Let γ be an odd cycle of smallest length in G. Then the only edges joining vertices of γ are the edges of γ, for otherwise we could find an odd cycle of length shorter than γ. By contracting all the edges of G except for the edges of γ, we arrive at γ. We may further contract edges to obtain a K_3. □

12.4 Independence Number and Clique Number

Let $G = (V, E)$ be a graph of order n. A set of vertices U of G is called *independent*,[31] provided that no two of its vertices are adjacent. Equivalently, U is independent provided the subgraph G_U of G induced by the vertices in U is a null graph. Thus, the chromatic number $\chi(G)$ equals the smallest integer k such that the vertices of G can be partitioned into k independent sets. Each subset of an independent set is also an independent set. Consequently, we seek large independent sets. The largest number of vertices in an independent set is called the *independence number* of the graph G and is denoted by $\alpha(G)$. The independence number is the largest number of vertices that can be colored the same in a vertex-coloring of G. Corollary 12.1.3 can be rephrased as

$$\chi(G) \geq \left\lceil \frac{n}{\alpha(G)} \right\rceil.$$

For a null graph N_n, a complete graph K_n, and a complete bipartite graph $K_{m,n}$, we have

$$\alpha(N_n) = n, \quad \alpha(K_n) = 1, \quad \text{and} \quad \alpha(K_{m,n}) = \max\{m, n\}.$$

The determination of the independence number of a graph is, in general, a difficult computational problem.

Example. Let G be the graph in Figure 12.8. Then $\{a, e\}$ is an independent set that is not a subset of any larger independent set. Also, $\{b, c, d\}$ is an independent set with the same property. Of any four vertices, two are adjacent, and hence we have $\alpha(G) = 3$. □

[31] Sometimes also called *stable*.

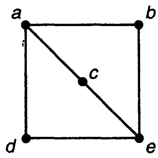


Figure 12.8

Example. A zoo wishes to place various species of animals in the same enclosure. Obviously, if one species preys on another, then both should not be put in the same enclosure. What is the largest number of species that can be placed in one enclosure?

We form the *zoo graph* G whose vertices are the different animal species in the zoo, and we put an edge between two species if and only if one of them preys on the other. The largest number of species that can be placed in the same enclosure equals the independence number $\alpha(G)$ of G. How many enclosures are required in order to accommodate all the species? The answer is the chromatic number $\chi(G)$ of G. $\square$

Example. (*The problem of the eight queens*). Consider an 8-by-8 chessboard and the chess piece known as a *queen.* In chess, a queen can attack any piece that lies in its row or column or in one of the two diagonals containing it. If nine queens are placed on the board, then necessarily, two lie in the same row and thus can attack one another. Is it possible to place eight queens on the board so that no queen can attack another?

Let G be the *queens graph* of the chessboard. The vertices of G are the squares of the board, with two squares adjacent if and only if a queen placed on one of the squares can attack a queen placed on the other. Our question thus asks whether the independence number of the queens graph equals 8. In fact, $\alpha(G) = 8$ and there are 92 different ways to place eight nonattacking queens on the board. One of these is shown in Figure 12.9. $\square$

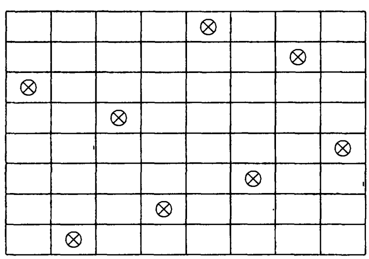


Figure 12.9

Let $G = (V, E)$ be a graph and let U be an independent set of vertices that is not a subset of any larger independent set. Thus, no two vertices in U are adjacent, and each vertex not in U is adjacent to at least one vertex in U.[32] A set of vertices with the latter property is called a dominating set. More precisely, a set W of vertices of G is a *dominating set*, provided that each vertex not in W is adjacent to at least one vertex in W. Vertices in W may or may not be adjacent. Clearly, if W is a dominating set, then any set of vertices containing W is also a dominating set. The problem is to find the smallest number of vertices in a dominating set. The smallest number of vertices in a dominating set is the *domination number* of G and is denoted by $\text{dom}(G)$.

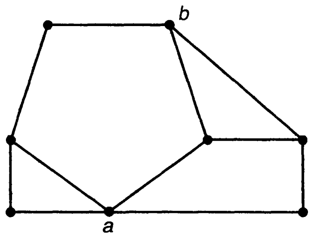


Figure 12.10

Example. Consider a building, perhaps housing an art gallery, consisting of a complicated array of corridors. It is desired to place guards throughout the building so that each part of the building is visible, and therefore protected, by at least one guard. How many guards must be employed to safeguard the building?

We construct a graph G whose vertices are the places where two or more corridors come together or where one corridor ends and whose edges correspond to the corridors. For example, we might have the corridor graph shown in Figure 12.10. The least number of guards that can protect the building equals the domination number $\text{dom}(G)$ of G. For the graph G in Figure 12.10, it is not difficult to check that $\text{dom}(G) = 2$ and that $\{a, b\}$ is a dominating set of two vertices. □

For a null graph, complete graph, and complete bipartite graph, we have

$$\text{dom}(N_n) = n, \ \text{dom}(K_n) = 1, \ \text{and } \text{dom}(K_{m,n}) = 2 \text{ if } m, n \geq 2.$$

In general, it is very difficult to compute the domination number of a graph. The domination number of a disconnected graph is clearly the sum of the domination numbers of its connected components. For a connected graph, we have a simple inequality.

Theorem 12.4.1 *Let G be a connected graph of order $n \geq 2$. Then*

$$\text{dom}(G) \leq \left\lfloor \frac{n}{2} \right\rfloor .$$

[32]If not, then U could be enlarged and so wouldn't be largest.

Proof. Let T be a spanning tree of G. Then surely

$$\mathrm{dom}(G) \leq \mathrm{dom}(T),$$

and hence it suffices to prove the inequality for trees of order $n \geq 2$. We use induction on n. If $n = 2$, then either vertex of T is a dominating set and hence $\mathrm{dom}(T) = 1 = \lfloor 2/2 \rfloor$. Now suppose that $n \geq 3$. Let y be a vertex that is adjacent to a pendent vertex x of T. Let T^* be the graph obtained from T by removing the vertex y and all edges incident with y. The connected components of T^* are trees, at least one of which is a tree of order 1. Let $T_1, \ldots, T_k$ be the trees of order at least 2. Let their orders be $n_1 \geq 2, \ldots, n_k \geq 2$, respectively. Then $n_1 + \cdots + n_k \leq n - 2$. By the induction hypothesis, each T_i has a dominating set of size at most $\lfloor n_i/2 \rfloor$. The union of these dominating sets along with y gives a dominating set of T of size at most

$$\begin{aligned} 1 + \left\lfloor \frac{n_1}{2} \right\rfloor + \cdots + \left\lfloor \frac{n_k}{2} \right\rfloor &\leq 1 + \left\lfloor \frac{n_1 + \cdots + n_k}{2} \right\rfloor \\ &\leq 1 + \left\lfloor \frac{n-2}{2} \right\rfloor = \left\lfloor \frac{n}{2} \right\rfloor . \end{aligned}$$

□

A *clique* in a graph G is a subset U of vertices, each pair of which is adjacent, equivalently, the subgraph induced by U is a complete graph. The largest number of vertices in a clique is called the *clique number* of G and is denoted by $\omega(G)$. For a null graph, complete graph, and complete bipartite graph, we have

$$\omega(N_n) = 1, \quad \omega(K_n) = n \quad \text{and } \omega(K_{m,n}) = 2.$$

The notion of a clique of a graph is "complementary" to that of independence in the following sense. Let $\overline{G} = (V, \overline{E})$ be the *complementary graph* of G. Recall that the complementary graph of G has the same set of vertices as G, and two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G. It follows from definitions that, for a subset U of V, U is an independent set of G if and only if U is a clique of $\overline{G}$, and U is a clique of G if and only if U is an independent set of $\overline{G}$. In particular, we have

$$\alpha(G) = \omega(\overline{G}) \quad \text{and } \omega(G) = \alpha(\overline{G}).$$

The chromatic number and clique number are related by the inequality (cf. Theorem 12.1.2)

$$\chi(G) \geq \omega(G). \tag{12.6}$$

Every bipartite graph G with at least one edge satisfies $\chi(G) = \omega(G) = 2$. A cycle graph C_n of odd order $n > 3$ with n edges arranged in a cycle satisfies $\chi(C_n) = 3 > 2 = \omega(C_n)$.

Since independence and clique are complementary notions, and since a vertex-coloring is a partition of the vertices of a graph into independent sets, it is natural to consider the notion complementary to that of vertex-coloring. Replacing independent set with clique in the definition of vertex-coloring, we obtain the following definitions. A *clique-partition* of a graph G is a partition of its vertices into cliques. The smallest number of cliques in a clique-partition of G is the *clique-partition number* of G, denoted by $\theta(G)$. We have

$$\chi(G) = \theta(\overline{G}) \quad \text{and } \theta(G) = \chi(\overline{G}).$$

The inequality "complementary" to that in (12.6) is

$$\theta(G) \geq \alpha(G). \tag{12.7}$$

This holds because two nonadjacent vertices cannot be in the same clique.

It is natural to investigate graphs for which equality holds in (12.6) (graphs whose chromatic number equals their clique number), and graphs for which equality holds in (12.7) (graphs whose clique-partition number equals its independence number). Graphs for which equality holds in either of these inequalities need not be too special. For instance, let H be any graph with chromatic number equal to p (thus its clique number satisfies $\omega(H) \leq p$). Let G be a graph with two connected components, one of which is H and the other of which is a K_p. Then we have $\chi(G) = p$ and $\omega(G) = p$, and hence equality holds in (12.6), no matter what the structure of H. Some structure can be imposed by requiring that (12.6) hold not only for G but for *every* induced subgraph of G.

A graph G is called *χ-perfect*, provided that $\chi(H) = \omega(H)$ for every induced subgraph H of G. The graph G is *θ-perfect*, provided that $\theta(H) = \alpha(H)$ for every induced subgraph H of G. It was conjectured by Berge[33] in 1961 and proved by Lovász[34] in 1972 that there is only one kind of perfection. We state this theorem without proof.

Theorem 12.4.2 *A graph G is χ-perfect if and only if it is θ-perfect. Equivalently, G is χ-perfect if and only if $\overline{G}$ is χ-perfect.*

As a result of this theorem we now refer to *perfect graphs*, and we show the existence of a large class of perfect graphs.

Let $G = (V, E)$ be a graph. A *chord* of a cycle of G is an edge that joins two nonconsecutive vertices of the cycle. A chord is thus an edge that joins two vertices of the cycle but that is not itself an edge of the cycle. A cycle of length 3 cannot have any chords. A graph is *chordal*, provided that each cycle of length greater than 3 has a chord. A chordal graph has no chordless cycles. An induced subgraph of a chordal graph is also a chordal graph.

[33] C. Berge, Färbung von Graphen, deren sämtliche bzw. deren ungerade Kreise starr sind, *Wiss Z. Martin-Luther-Univ., Halle-Wittenberg Math.-Natur, Reihe*, (1961), 114–115.

[34] L. Lovász, Normal Hypergraphs and the Perfect Graph Conjecture, *Discrete Math.*, 2 (1972), 253–267.

Example. Since induced subgraphs of complete graphs are complete graphs, and induced subgraphs of bipartite graphs are bipartite graphs, complete graphs and all bipartite graphs are perfect. A complete graph K_n is a chordal graph as is every tree.[35] A complete bipartite graph $K_{m,n}$ with $m \geq 2$ and $n \geq 2$ is not a chordal graph, since such a graph has a chordless cycle of length 4. The graph obtained from a complete graph K_n by removing one edge is a chordal graph, since every cycle of K_n of length greater than 3 has at least two chords.

□

A special class of chordal graphs arises by considering intervals on a line. A closed interval on the real line is denoted by

$$[a, b] = \{x : a \leq x \leq b\}.$$

Let

$$I_1 = [a_1, b_1],\ I_2 = [a_2, b_2],\ \ldots,\ I_n = [a_n, b_n] \tag{12.8}$$

be a family of closed intervals. Let G be the graph whose set of vertices is $\{I_1, I_2, \ldots, I_n\}$ where two intervals I_i and I_j are adjacent if and only if $I_i \cap I_j \neq \emptyset$. Such a graph G is called a *graph of intervals*, and any graph isomorphic to a graph of intervals is called an *interval graph*. Thus, the vertices of an interval graph can be thought of as intervals with two vertices adjacent if and only if the intervals have at least one point in common.

Example. A complete graph K_n is an interval graph. We choose the intervals (12.8) with

$$a_1 < a_2 < \cdots < a_n < b_n < \cdots < b_2 < b_1.$$

If $i \neq j$ and $i < j$, then $I_j \subset I_i$, and thus $I_i \cap I_j \neq \emptyset$. Hence, the graph of intervals is a complete graph.

Now let G be the graph of order 4 obtained from K_4 by removing one edge. We choose the intervals (12.8) $(n = 4)$ with

$$a_4 < a_1 < a_3 < b_4 < a_2 < b_1 < b_2 < b_3.$$

Except for the two intervals I_2 and I_4, every pair of intervals has a nonempty intersection. □

Theorem 12.4.3 *Every interval graph is a chordal graph.*

Proof. Let G be an interval graph with intervals $I_1, I_2, \ldots, I_n$ as given in (12.8). Suppose that $k > 3$ and that

$$I_{j_1} - I_{j_2} - \cdots - I_{j_k} - I_{j_1}$$

[35] If a graph doesn't have any cycles, it surely cannot have a chordless cycle.

is a cycle of length k. We show that at least one of the intervals of the cycle has a nonempty intersection with the interval two away from it on the cycle. We assume the contrary and obtain a contradiction. Suppose that I_m, I_p, I_q, I_r are four consecutive intervals on the cycle for which $I_m \cap I_q = \emptyset$ and $I_p \cap I_r = \emptyset$, so that there is no chord joining I_m and I_q and no chord joining I_p and I_r. Then

$$I_m \cap I_p \neq \emptyset,\ I_p \cap I_q \neq \emptyset,\ I_q \cap I_r \neq \emptyset,\ I_m \cap I_q = \emptyset, \text{ and } I_p \cap I_r = \emptyset.$$

If $a_q < a_p$ and $b_p < b_q$, then $I_p \subset I_q$, and hence $\emptyset \neq I_m \cap I_p \subset I_m \cap I_q$, a contradiction. Therefore, either $a_p \leq a_q$ or $b_q \leq b_p$. If $a_p \leq a_q$, then $a_q \leq a_r$. If $b_q \leq b_p$, then $b_r \leq b_q$. Thus, for three consecutive intervals I_p, I_q, I_r of the cycle, we have one of

$$a_p \leq a_q \leq a_r \text{ or } b_r \leq b_q \leq b_p. \tag{12.9}$$

Now, let $p = j_1$ and first suppose that $a_{j_1} \leq a_{j_2}$. Then, iteratively using (12.9), we obtain

$$a_{j_1} \leq a_{j_2} \leq \cdots \leq a_{j_k} \leq a_{j_1},$$

and we conclude that all of the intervals have the same left endpoint. If $b_{j_2} \leq b_{j_1}$, then, in a similar way, we conclude that all of the intervals have the same right endpoint. In either case, all of the intervals of the cycle have a point in common, contradicting our assumption that intervals two apart on the cycle have no point in common. This contradiction establishes the validity of the theorem. □

To conclude this section we show that chordal graphs, and hence interval graphs, are perfect. We require another lemma for the proof. Recall that a subset U of the vertices of a graph $G = (V, E)$ is an articulation set, provided that the subgraph G_{V-U} induced by the vertices not in U is disconnected. The lemma demonstrates that the chromatic number of a graph equals its clique number if certain smaller induced graphs have this property.

Lemma 12.4.4 *Let $G = (V, E)$ be a connected graph and let U be an articulation set of G such that the subgraph G_U induced by U is a complete graph. Let the connected components of the induced subgraph G_{V-U} be $G_1 = (U_1, E_1), \ldots, G_t = (U_t, E_t)$. Assume that the induced graphs $G_{U_i \cup U}$ satisfy*

$$\chi(G_{U_i \cup U}) = \omega(G_{U_i \cup U}) \quad (i = 1, 2, \ldots, t).$$

Then

$$\chi(G) = \omega(G).$$

Proof. Let $k = \omega(G)$. Because each clique of $G_{U_i \cup U}$ is a clique of G we have

$$\omega(G_{U_i \cup U}) \leq k \quad (i = 1, 2, \ldots, t).$$

Since vertices in different U_i's are not adjacent, each clique of G is a clique of $G_{U_j \cup U}$ for some j. Hence, for at least one j,

$$\omega(G_{U_j} \cup U) = k.$$

We now use the hypotheses and Theorem 12.1.10 to obtain

$$\begin{aligned} \chi(G) &= \max\{\chi(G_{U_1 \cup U}), \ldots, \chi(G_{U_t \cup U})\} \\ &= \max\{\omega(G_{U_1 \cup U}), \ldots, \omega(G_{U_t \cup U})\} \\ &= k = \omega(G). \end{aligned}$$

□

An articulation set U is a *minimal articulation set*, provided that, for all subsets $W \subseteq U$ with $W \neq U$, W is not an articulation set. In the next theorem we show that minimal articulation sets in chordal graphs induce a complete subgraph.

Theorem 12.4.5 *Let $G = (V, E)$ be a connected chordal graph and let U be a minimal articulation set of G. Then the subgraph G_U induced by U is a complete graph.*

Proof. We assume to the contrary that G_U is not a complete graph and obtain a contradiction. Let a and b be vertices in U that are not adjacent. Since U is an articulation set, the graph G_{V-U} has at least two connected components, $G_1 = (U_1, E_1)$ and $G_2 = (U_2, E_2)$. If a were not adjacent to any vertex of G_1, then it would follow that $U - \{a\}$ is also an articulation set. Since U is a minimal articulation set, we conclude that a is adjacent to at least one vertex in U_1. In a similar way we conclude that a is adjacent to a vertex in U_2 and that b is adjacent to at least one vertex in U_1 and at least one vertex in U_2. Since G_1 and G_2 are connected, there is a path γ_1 joining a to b, all of whose vertices different from a and b belong to U_1, and there is a path γ_2 joining b to a, all of whose vertices different from a and b belong to U_2 . We may choose γ_1 and γ_2 so that they have the shortest possible length. It follows that γ_1 followed by γ_2,

$$\gamma = \gamma_1, \gamma_2,$$

is a cycle in G of length at least 4. Moreover, since we have chosen γ_1 and γ_2 to have the shortest length, the only possible chord of γ is an edge joining a and b. Since a and b were chosen to be nonadjacent, we conclude that γ does not have a chord, contradicting the hypothesis that G is a chordal graph. □

We now prove that chordal graphs are perfect.

Theorem 12.4.6 *Every chordal graph is perfect.*

Proof. Since an induced subgraph of a chordal graph is also a chordal graph, it suffices to prove only that for a chordal graph G we have $\chi(G) = \omega(G)$.

Let G be a chordal graph of order n. We prove by induction on n that

$$\chi(G) = \omega(G).$$

Since complete graphs are known to be perfect, we assume that G is not complete. Then G has an articulation set and hence a minimal articulation set U. By Theorem 12.4.5, G_U is a complete graph. Let $G_1 = (U_1, E_1), \ldots, G_t = (U_t, E_t)$ be the connected components of G_{V-U}. By the induction hypothesis, each of the graphs $G_{U_i \cup U}$ satisfies

$$\chi(G_{U_i \cup U}) = \omega(G_{U_i \cup U}) \quad (i = 1, 2, \ldots, t).$$

Now, applying Lemma 12.4.4, we conclude that $\chi(G) = \omega(G)$. □

From Theorems 12.4.3 and 12.4.6, we immediately obtain the next corollary.

Corollary 12.4.7 *Every interval graph is a perfect graph.* □

A considerable amount of effort has been expended in attempts to characterize perfect graphs. These efforts have been largely directed toward resolving the following *conjecture* of Berge:[36]

> *A graph G is perfect if and only if neither G nor its complementary graph $\overline{G}$ has an induced subgraph equal to a cycle of odd length greater than three without any chords.*

This conjecture was resolved recently in the affirmative.[37] We leave to the Exercises the verification that, if either G or its complementary graph $\overline{G}$ has an induced subgraph equal to a chordless cycle of odd length greater than 3, then G is not perfect.

12.5 Matching Number

For our discussion in this section, we need only consider graphs.

Let $G = (V, E)$ be a graph. We consider the analogue of the notion of independence of vertices for edges. Recall that a set U of vertices in V is independent provided that no two of the vertices in U are joined by an edge. A set M of edges in E is a *matching*

[36] C. Berge, Färbung von Graphen, deren sämtliche bzw. deren ungerade Kreise starr sind, *Wiss. Z. Martin-Luther-Univ., Halle-Wittenberg Math.-Natur, Reihe*, (1961), 114–115.

[37] M. Chudnovsky, N. Robertson, P. Seymour, and R. Thomas, The Strong Perfect Graph Theorem, *Ann. of Math.* (2) 164 (2006), 51—229.

provided that no two of the edges in M have a vertex in common.[38] Since edges contain two vertices, if G has n vertices, then a matching M can have at most $n/2$ edges. The matching M *meets a vertex* x provided one of its edges (and thus only one of its edges) contains the vertex x. The matching M is called a *perfect matching* of G provided that it meets every vertex of G. If G has a perfect matching, then necessarily its number n of vertices is even. A perfect matching is also called a *1-factor* of G. The *matching number* of a graph G is the largest number of edges in a matching in G and is denoted by $\rho(G)$.

Example. As is easily verified, the complete graph K_n has a perfect matching if and only if n is even. In fact, if n is even, we can obtain a perfect matching by iteratively choosing an edge that does not have a common vertex with any of the edges previously chosen. In general, we have $\rho(K_n) = \lfloor n/2 \rfloor$. A cycle C_n of n vertices also has a perfect matching if and only if n is even; in fact, it has exactly two perfect matchings when n is even. We also have $\rho(C_n) = \lfloor n/2 \rfloor$. A path P_n of n vertices also satisfies $\rho(P_n) = \lfloor n/2 \rfloor$. The complete bipartite graph $K_{m,n}$ has a perfect matching if and only if $m = n$; this is because a perfect matching must pair up the left vertices with the right vertices. In general, we have $\rho(K_{m,n}) = \min\{m, n\}$. □

We first consider matchings in bipartite graphs. In fact, we already have done so in a disguised form in Chapter 9. Let $G = (V, E)$ be a bipartite graph with bipartition X, Y. Thus each edge of G has one of its vertices in X and one in Y. Let's list the vertices of X and Y as

$$X : x_1, x_2, \ldots, x_n \text{ and } Y : y_1, y_2, \ldots, y_m.$$

The graph G is a subgraph of the complete bipartite graph $K_{m,n}$ with bipartition X, Y. With the bipartite graph we associate a family $\mathcal{A}_G = (A_1, A_2, \ldots, A_n)$ of subsets of Y as follows:

$$A_i = \{y_j : \{x_i, y_j\} \text{ is an edge of } G\}, \quad (i = 1, 2, \ldots, n).$$

Thus A_i consists of all the vertices of Y to which x_i is joined by an edge. This construction is clearly reversible in that given a family $\mathcal{A}$ of subsets of Y we can construct a bipartite graph G such that $\mathcal{A} = \mathcal{A}_G$. So, speaking informally, a family of sets and a bipartite graph are different ways of representing the same mathematical idea.

[38]Why does this constitute "independence" of edges? Take the graph $G = (V, E)$ and form a new graph $L(G) = (E, S)$, with the edges of G as the new vertices, whose new edges are pairs of edges of G that have a vertex in common. Then a set of vertices of $L(G)$ (that is, edges of G), is independent provided no two are joined by an edge in $L(G)$ (that is, do not have a common vertex in G and so form a matching in G). The graph $L(G)$ is called the *line graph* of G. A good way to picture the line graph of G is to take a picture of G and insert a new vertex on each edge and join two new vertices if the edges on which they lie have a common vertex (then erase all the original vertices and edges, or use a different color to distinguish between old and new so that you don't forget which graph you started with). Try it with your favorite graph G; for instance, what is the line graph of K_3? of $K_{1,3}$? of K_4?

Suppose that $(e_1, e_2, \ldots, e_n)$ is a system of distinct representatives (SDR) of the family $\mathcal{A}_G$. Then e_i is an element of A_i for each i, and the elements $e_1, e_2, \ldots, e_n$ are distinct. This implies that

$$M = \{\{x_1, e_1\}, \{x_2, e_2\}, \ldots, \{x_n, e_n\}\}$$

is a set of n edges of G, and no two of the edges of M have a vertex in common. Thus M is a matching of n edges of G. Conversely, from a matching of n edges of G, we obtain an SDR of $\mathcal{A}_G$. The same type of reasoning gives the following result.

Theorem 12.5.1 *Let $G = (V, E)$ be a bipartite graph with bipartition X, Y with associated family $\mathcal{A}_G$ of subsets of Y. Let t be a positive integer. Then from a subfamily*

$$(A_{i_1}, A_{i_2}, \ldots, A_{i_t}) \text{ of } t \text{ sets of } \mathcal{A}_G \text{ with an SDR } (e_{i_1}, e_{i_2}, \ldots, e_{i_t}), \tag{12.10}$$

we obtain a matching

$$\{x_{i_1}, e_{i_1}\}, \{x_{i_2}, e_{i_2}\}, \ldots, \{x_{i_t}, e_{i_t}\} \text{ of } G \text{ of } t \text{ edges.} \tag{12.11}$$

Conversely, from a matching (12.11) *of G of t edges, we get a subfamily* (12.10) *of $\mathcal{A}_G$ of t sets with $(e_{i_1}, e_{i_2}, \ldots, e_{i_t})$ as SDR.*

Thus the largest number of sets in a subfamily of $\mathcal{A}_G$ with an SDR equals the matching number $\rho(G)$ of G. □

According to Corollary 9.2.3, the largest number of sets in a subfamily of $\mathcal{A}_G$ with an SDR is equal to

$$\min\{|A_{i_1} \cup A_{i_2} \cup \cdots \cup A_{i_k}| + n - k\} \tag{12.12}$$

where the minimum is taken over all choices of $k = 1, 2, \ldots, n$ and all choices of k indices $i_1, i_2, \ldots, i_k$ with $1 \le i_1 < i_2 < \cdots < i_k$. Thus this gives us an expression (as a minimum) for the matching number of a bipartite graph G. We now rework this expression to one which refers to the graph G in a more compact form.

A subset W of the set V of vertices of a graph is a *cover of the edges* of G, abbreviated to a *cover* of G, provided every edge has at least one of its vertices in W. A cover of a complete graph K_n can omit at most one vertex since every two vertices are joined by an edge. Two natural covers of a bipartite graph G with bipartition X, Y are X and Y. The smallest number of vertices in a cover of G is denoted by $c(G)$.

Lemma 12.5.2 *Let $G = (V, E)$ be a graph. Then a subset W of the set V of vertices is a cover if and only if the complementary set of vertices $V \setminus W$ is an independent set.*

Proof. First assume that W is a cover. Then every edge has at least one of its vertices in W, and so no edge has both of its vertices in $V \setminus W$. Thus $V \setminus W$ is an independent set. Conversely, assume U is an independent set of vertices of V. Then no edge has both of its vertices in U and so must have at least one of its vertices in $V \setminus U$. □

The following theorem is known as the König-Egerváry theorem.[39]

Theorem 12.5.3 *Let $G = (V, E)$ be a bipartite graph. Then*

$$\rho(G) = c(G), \tag{12.13}$$

that is, the largest number of edges in a matching equals the smallest number of vertices in a cover.

Proof. Let X, Y be a bipartition of G, and let $\mathcal{A}_G$ be the associated family of subsets of Y. First let M be a matching with $|M| = \rho(G)$. Since no two edges in M have a vertex in common, just to cover the edges in M requires $|M|$ vertices. Hence we need at least this many vertices to cover all the edges of G, and so $c(G) \geq |M| = \rho(G)$.

We now show that $c(G) \leq \rho(G)$. By Theorem 12.5.1 and equation (12.12),

$$\rho(G) = \min\{|A_{i_1} \cup A_{i_2} \cup \cdots \cup A_{i_k}| + n - k\}. \tag{12.14}$$

Choose an l from $1, 2, \ldots, n$, and indices $i_1, i_2, \ldots, i_l$ with $1 \leq i_1 < i_2 < \cdots < i_l \leq n$ giving the minimum value in (12.14):

$$\rho(G) = |A_{i_1} \cup A_{i_2} \cup \cdots \cup A_{i_l}| + n - l.$$

Let $\{j_1, j_2, \ldots, j_{n-l}\} = \{1, 2, \ldots, n\} \setminus \{i_1, i_2, \ldots, i_l\}$, the set of indices different from $i_1, i_2, \ldots, i_l$. Let $X' = \{x_{j_1}, x_{j_2}, \ldots, x_{j_{n-l}}\}$ be the subset of vertices of X corresponding to the indices $\{j_1, j_2, \ldots, j_{n-l}\}$, and let $Y' = Y \setminus (A_{i_1} \cup A_{i_2} \cup \cdots \cup A_{i_l})$ be the subset of those vertices of Y which are not in any of the sets $A_{i_1}, A_{i_2}, \ldots, A_{i_l}$. Then $W = X' \cup Y'$ is a cover of G. This is because there cannot exist an edge from any x_{i_t} to any vertex in $Y \setminus Y'$, for if there were we would contradict the definition of Y'. Hence $X' \cup Y'$ is a cover of size

$$|X'| + |Y'| = n - l + |A_{i_1} \cup A_{i_2} \cup \cdots \cup A_{i_l})| = \rho(G).$$

Since we have a cover W of G with $|W| = \rho(G)$, we conclude that $c(G) \leq \rho(G)$. Putting together the two inequalities $c(G) \leq \rho(G)$ and $\rho(G) \leq c(G)$, we conclude that $\rho(G) = c(G)$. □

Example. Consider the complete graph K_n with n vertices. Then $c(K_n) = n - 1$, since every pair of vertices is joined by an edge. But $\rho(K_n) = \lfloor n/2 \rfloor$, as already

[39] D. König: Graphen und Matrizen, *Mat. Lapok*, 38 (1931), 116–119; E. Egerváry: On Combinatorial Properties of Matrices (Hungarian with German summary), *Mat. Lapok*, 38 (1931), 16–28.

remarked. So if $n \geq 3$, then $c(G) > \rho(G)$; indeed the difference between $c(K_n)$ and $\rho(K_n)$ is $\lfloor (n-1)/2 \rfloor$, which grows without bound as n grows larger. Thus Theorem 12.5.3 does not hold for all graphs. On the other hand, the nonbipartite graph G with six vertices obtained from K_3 by attaching three new edges, one from each of the vertices of K_3 to three new vertices, satisfies $\rho(G) = 3$ (the three new edges form a matching) and $c(G) = 3$ (the three original vertices form a cover). □

As the preceding example shows, a graph G may or may not satisfy $\rho(G) = c(G)$. There is, however, a formula for $\rho(G)$ in the same spirit as Theorem 12.5.3 in the sense that $\rho(G)$ (the *largest number of edges in a matching*) equals the smallest value of another expression (for bipartite graphs it is the smallest number of vertices in a cover). We now describe without proof a theorem which, for any graph G, expresses $\rho(G)$ as the smallest value of a certain expression. We first need some new notions.

Let $G = (V, E)$ be a graph. Let U be a subset of the vertices and let $G_{V \setminus U} = (V \setminus U, F)$ be the subgraph induced on the vertices of G not in U. Thus $G_{V \setminus U}$ is obtained from G be removing all the vertices in U and every edge with at least one of its vertices in U. Even though the graph G may be connected, the graph $G_{V \setminus U}$ may not be, and so it will have a number of connected components. Some of these connected components may have an odd number of vertices and some may have an even number of vertices. It turns out that we need to consider the connected components of $G_{V \setminus U}$ with an odd number of vertices. We call a connected component with an odd number of vertices an *odd component*. Let $oc(G_{V \setminus U})$ be the number of odd components of $G_{V \setminus U}$. The following theorem characterizes graphs with a perfect matching.[40]

Theorem 12.5.4 *Let $G = (V, E)$ be a graph. Then G has a perfect matching if and only if*

$$oc(G_{V \setminus U}) \leq |U| \text{ for every } U \subseteq V, \tag{12.15}$$

that is, removing a set of vertices does not create more odd components than the number of vertices removed.

Note that by taking $U = \emptyset$ in (12.15) we get that $oc(G) \leq 0$, that is, G has no odd components, which means that every connected component of G has an even number of vertices, and so G itself has an even number of vertices.

We only verify here that condition (12.15) is a necessary condition for G to have a perfect matching. Now assume that $U \neq \emptyset$, and let the odd components of $G_{V \setminus U}$ be $G_{U_1}, G_{U_2}, \ldots, G_{U_k}$. Since $|U_i|$ is odd, in a perfect matching M of G, there must be at least one edge from some vertex in U_i to some vertex z_i in U. This is true for each $i = 1, 2, \ldots, k$ and, since M is a perfect matching, the vertices $z_1, z_2, \ldots, z_k$ are distinct. Hence $|U| \geq k = oc(G_{V \setminus U})$.

[40]This theorem was first proved by W. T. Tutte in 1947; The Factorization of Linear Graphs, *J. London Math. Soc.*, 22 (1947), 107–111; other more elementary proofs are now available in the mathematical literature, for example, D. B. West: *Introduction to Graph Theory*, 2nd edition, Prentice Hall, 2001, 136–138.

In analogy to Theorem 12.5.3, there is a formula for the matching number $\rho(G)$ of a graph, called the Berge-Tutte formula.

Theorem 12.5.5 *Let $G(V, E)$ be a graph with n vertices. Then*

$$\rho(G) = \min\{n - (oc(G_{V\setminus U}) - |U|)\},$$

where the minimum is taken over all $U \subseteq V$.

It is not too difficult to derive Theorem 12.5.5 from Theorem 12.5.4. First we show that $\rho(G) \le n - (oc(G_{V\setminus U}) - |U|)$ for each subset U of the vertices. Then we show that the upper bound is attained by introducing a complete graph K_d with $d = \max\{oc(G_{V\setminus U}) - |U|\}$ new vertices and joining each of the new vertices to all vertices of G.

12.6 Connectivity

Graphs are either connected or disconnected. But it is evident that some connected graphs are "more connected" than others.

Example. We could measure how connected a graph is by measuring how difficult it is to disconnect the graph. But how shall we measure the difficulty required to disconnect a graph? There are two natural ways for doing this. Consider, for instance, a tree of order $n \ge 3$ that forms a path. If we take a vertex other than one of the two end vertices of the path and remove it (and, of course, the two incident edges), the result is a disconnected graph. Indeed, a path is not special among trees in this regard. If we take any tree and remove a vertex other than a pendent vertex, the result is a disconnected graph. Thus, a tree is not very connected. It is necessary to remove only one vertex in order to disconnect it. If, instead of removing vertices (and their incident edges), we remove only edges (and none of the vertices), a tree still comes out as "almost disconnected": Removing any edge leaves a disconnected graph. In contrast, a complete graph K_n of order n can never be disconnected by removing vertices because removing vertices always leaves us with a smaller complete graph. If, instead of removing vertices, we remove edges, we can disconnect K_n: If we remove all of the $n-1$ edges incident with a particular vertex, then we are left with a disconnected graph.[41] A simple calculation reveals that K_n cannot be disconnected by removing fewer than $n-1$ edges. Thus, by either manner of reckoning,[42] a complete graph K_n is very connected and a tree is not very connected. The main purpose of this section is to define formally these two notions of connectivity and to discuss some of their implications. □

[41] Indeed, a K_{n-1} and a vertex separate from it.

[42] And, as we would expect, for any reasonable way to measure how connected a graph is.

In order to simplify our exposition, we assume throughout this section that all graphs have order $n \geq 2$. Thus we don't deal with the trivial graph with only one vertex.

Let $G = (V, E)$ be a graph of order n. If G is a complete graph K_n, then we define its *vertex-connectivity* to be

$$\kappa(K_n) = n - 1.$$

Otherwise, we define the *vertex-connectivity* of G to be

$$\kappa(G) = \min\{|U| : G_{V \setminus U} \text{ is disconnected }\},$$

the smallest number of vertices whose removal leaves a disconnected graph. Equivalently, the connectivity of a noncomplete graph equals the smallest size of an articulation set (as defined in Section 12.1). A noncomplete graph has a pair of nonadjacent vertices a and b. Removing all vertices different from a and b leaves a disconnected graph, and hence $\kappa(G) \leq n - 2$ if G is a noncomplete graph of order n. The connectivity of a disconnected graph is clearly 0. Thus, we have the next elementary result.

Theorem 12.6.1 *Let G be a graph of order n. Then*

$$0 \leq \kappa(G) \leq n - 1,$$

with equality on the left if and only if G is disconnected and with equality on the right if and only if G is a complete graph. □

The *edge-connectivity* of a graph G is defined to be the minimum number of edges whose removal disconnects G and is denoted by $\lambda(G)$. The edge-connectivity of a disconnected graph G satisfies $\lambda(G) = 0$. A connected graph G has edge-connectivity equal to 1 if and only if it has a bridge. The edge-connectivity of a complete graph K_n satisfies $\lambda(K_n) = n - 1$. If we remove all the edges of a graph that are incident with a specified vertex x, then we obviously obtain a disconnected graph. Thus, the edge-connectivity of a graph G satisfies $\lambda(G) \leq \delta(G)$, where $\delta(G)$ denotes the *smallest degree of a vertex* of G. The basic relation between vertex-connectivity and edge-connectivity is contained in the next theorem.[43]

Theorem 12.6.2 *For each graph G, we have*

$$\kappa(G) \leq \lambda(G) \leq \delta(G).$$

[43]This theorem was first proved by H. Whitney, Congruent Graphs and the Connectivity of Graphs, *American J. Math.*, 54 (1932), 150–168. The proof given here is from R. A. Brualdi and J. Csima, A note on Vertex- and Edge-Connectivity, *Bulletin of the Institute of Combinatorics and Its Applications*, 2 (1991), 67–70.

Proof. We have verified the second inequality in the preceding paragraph. We now verify the first inequality. Let G have order n. If G is a complete graph K_n, then $\kappa(G) = \lambda(G) = n - 1$. We henceforth assume that G is not complete. If G is disconnected, the inequality holds since $\kappa(G) = \lambda(G) = 0$. So we assume that G is connected. Let F be a set of $\lambda(G)$ edges whose removal leaves a disconnected graph H. Then H has two connected components,[44] with vertex sets V_1 and V_2, respectively, where $|V_1| + |V_2| = n$. If F consists of all possible edges joining vertices in V_1 to vertices in V_2, then we must have $|F| \geq n - 1$; hence, $\lambda(G) \geq n - 1$, implying that $\lambda(G) = n - 1$ and, contrary to assumption, that G is complete. Thus, there exist vertices a in V_1 and b in V_2 such that a and b are not adjacent in G. For each edge α in F, we choose one vertex as follows: If a is a vertex of α, we choose the other vertex of α (the one in V_2); otherwise, we choose the vertex of α that is in V_1. The resulting set U of vertices satisfies $|U| \leq |F|$. Moreover, removing the vertices U from G results in a disconnected graph, since there can be no path from a to b. Thus,

$$\kappa(G) \leq |U| \leq |F| = \lambda(G),$$

completing the proof of the theorem. □

Example. Suppose that, in a communication system, there are n stations,[45] some of which are linked by a direct communication line. We assume that the system is connected in the sense that each station can communicate with every other station through intermediary communication links. Thus, we have a natural connected graph G of order n in which the vertices correspond to the stations and the edges to the direct links. Now, links may fail and stations may get shut down, and this affects communication. The vertex-connectivity and edge-connectivity of G are intimately related to the reliability of the system. Indeed, as many as $\kappa(G) - 1$ of the stations may be shut down and the others will still be able to communicate among themselves. As many as $\lambda(G) - 1$ of the links may fail and all of the stations will still be able to communicate with each other. □

Let G be a graph. Then G is connected if and only if its vertex-connectivity satisfies $\kappa(G) \geq 1$. If k is an integer and $\kappa(G) \geq k$, then G is called *k-connected.* Thus, the 1-connected graphs are the connected graphs. Notice that, if a graph is k-connected, then it is also $(k - 1)$-connected. The vertex-connectivity of a graph equals the largest integer k such that the graph is k-connected. In the remainder of this section we investigate the structure of 2-connected graphs and show, in particular, that the edges (but not the vertices in general) of a graph are naturally partitioned into its "2-connected parts."[46] We define an *articulation vertex* of a graph G to be a

[44] If there were more than two components, we could disconnect G by removing fewer edges.

[45] Or, we might have n chips in a computer.

[46] Since 1-connected means "connected," we know that the vertices of a graph, and hence the edges, are naturally partitioned into its 1-connected parts, that is, its connected components. When we consider the 2-connected parts, we get only a natural partition of the edges.

vertex a whose removal disconnects G, that is, a vertex such that $\{a\}$ is an articulation set.

Theorem 12.6.3 *Let G be a graph of order $n \geq 3$. Then the following three assertions are equivalent:*

(1) *G is 2-connected.*

(2) *G is connected and does not have an articulation vertex.*

(3) *For each triple of vertices a, b, c, there is a path joining a and b that does not contain c.*

Proof. If $\kappa(G) \geq 2$, then G is connected and does not have an articulation vertex. Conversely, since $n \geq 3$, if G is connected and without articulation vertices, then $\kappa(G) \geq 2$. Thus, assertions (1) and (2) are equivalent.

Now assume that (2) holds. Let a, b, c be a triple of vertices. Since G has no articulation vertices, removing c does not disconnect G. Hence, there is a path joining a and b that does not contain c, and assertion (3) holds. Conversely, assume that (3) holds. Then G is surely connected. Suppose that c is an articulation vertex of G. Removing c disconnects G; choosing a and b in different connected components of the resulting graph, we contradict (3). Hence, G has no articulation vertex and (2) holds. Therefore, (2) and (3) are also equivalent. □

The reason for the assumption $n \geq 3$ in Theorem 12.6.3 is that a complete graph K_2 is connected and does not have an articulation vertex; that is, satisfies (2) but does not satisfy (1), since we have $\kappa(K_2) = 1$.

Let $G = (V, E)$ be a connected graph of order $n \geq 2$. A *block* of G is a maximal induced subgraph of G that is connected and has no articulation vertex. More precisely, let U be a subset of the vertices of G. Then the induced subgraph G_U is a block of G, provided that G_U is connected and has no articulation vertex, and for all subsets W of the vertices of G with $U \subseteq W$ and $U \neq W$, either the induced subgraph G_W is not connected or it has an articulation vertex. It follows from Theorem 12.6.3 that the blocks of G are either the complete graph K_2 or are 2-connected.

Example. Let G be the graph in Figure 12.11. Then the blocks are the induced subgraphs G_U with U equal to

$$\{a, b\}, \{b, c, d, e\}, \{c, f, g, h\}, \{h, i\}, \{i, j\}, \{i, k\}.$$

Four of the blocks are K_2's, and two of the blocks are 2-connected. Notice that, while some of the blocks may have a vertex in common, each edge of G belongs to exactly one block. □

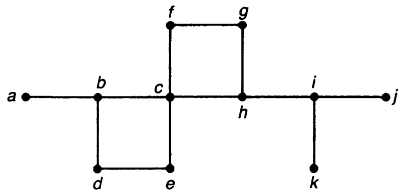


Figure 12.11

Theorem 12.6.4 *Let $G = (V, E)$ be a connected graph of order $n \geq 2$, and let*

$$G_{U_1} = (U_1, E_1), G_{U_2} = (U_2, E_2), \ldots, G_{U_r} = (U_r, E_r)$$

be the blocks of G. Then $E_1, E_2, \ldots, E_r$ is a partition of the set E of edges of G,[47] *and each pair of blocks has at most one vertex in common.*

Proof. Each edge of G belongs to some block, since a block can be a K_2. A block that is a K_2 cannot have an edge in common with any other block, and hence has at most one vertex in common with any other block. Thus, we need consider only blocks G_{U_i} and G_{U_j} $(i \neq j)$ of order at least 3 and hence blocks that are 2-connected. If we show that these blocks can have at most one vertex in common, then it will follow that an edge cannot be in two different blocks.

Suppose that $U_i \cap U_j$ contains at least two vertices. Then, since U_i and U_j have a nonempty intersection, the induced graph $G_{U_i \cup U_j}$ is connected. Let x be any vertex in $U_i \cup U_j$. Since G_{U_i} and G_{U_j} are 2-connected, $G_{U_i - \{x\}}$ and $G_{U_j - \{x\}}$ are connected. Moreover, since U_i and U_j have two vertices in common, $G_{U_i \cup U_j - \{x\}}$ is connected. It follows that the induced graph $G_{U_i \cup U_j}$ is 2-connected. This gives us a larger 2-connected induced subgraph and contradicts the assumption that G_{U_i} and G_{U_j} are blocks (and hence maximal 2-connected induced subgraphs). Therefore, two distinct blocks can have at most one common vertex. $\square$

We conclude this section with another characterization of graphs that are 2-connected.

Theorem 12.6.5 *Let $G = (V, E)$ be a graph of order $n \geq 3$. Then G is 2-connected if and only if, for each pair a, b of distinct vertices, there is a cycle containing both a and b.*

Proof. If each pair of distinct vertices of G is on a cycle, then surely G is connected and has no articulation vertex. Hence, by Theorem 12.6.3, G is 2-connected.

Now assume that G is 2-connected. Let a and b be distinct vertices of G. Let U be the set of all vertices x different from a for which there exists a cycle containing

[47]Thus, each edge of G belongs to exactly one block.

both a and x. We first show that $U \neq \emptyset$; that is, there is at least one cycle containing a. Let $\{a, y\}$ be any edge containing a. By Theorem 12.6.1, $\lambda(G) \geq \kappa(G) \geq 2$, and hence the deletion of the edge $\{a, y\}$ does not disconnect G. Consequently, there is a path joining a and y that does not use the edge $\{a, y\}$, and thus a cycle containing both a and y. Therefore, $U \neq \emptyset$.

Suppose, contrary to what we wish to prove, that b is not in U. Let z be a vertex in U whose distance p to b is as small as possible, and let γ be a path from z to b of length p. Since z is in U, there is a cycle γ_1 containing both a and z. The cycle γ_1 contains two paths, γ_1' and γ_1'', joining a to z. Since G is 2-connected, it follows from Theorem 12.6.2 that there is a path γ_2 joining a and b that does not contain the vertex z. Let u be the first vertex of γ that is also a vertex of γ_2.[48] Let v be the last vertex of γ_2 which is also a vertex of γ_1.[49] The vertex v belongs either to γ_1' or to γ_1'', let us say to γ_1'. Then, following a to v along γ_1', v to u along γ_2, u to z along γ, and z back to a along γ_1'', we construct a cycle containing both a and u. Thus, u is in U. But since u is closer to b than z, we contradict our choice of z. We conclude that b is in U, and hence there is a cycle containing both a and b. □

An alternative formulation of the characterization of 2-connected graphs in Theorem 12.6.5 is given in the next corollary.

Corollary 12.6.6 *Let G be a graph with at least three vertices. Then G is 2-connected if and only if, for each pair a, b of distinct vertices, there are two paths joining a and b whose only common vertices are a and b.*

The corollary is a special case of a theorem of Menger[50] that characterizes k-connected graphs for any k. We state this theorem without proof; it is the "undirected version" of Menger's theorem for digraphs proved in Section 13.2.

Theorem 12.6.7 *Let k be a positive integer and let G be a graph of order $n \geq k + 1$. Then G is k-connected if and only if, for each pair a, b of distinct vertices, there are k paths joining a and b such that each pair of paths has only the vertices a and b in common.*

If $k = 1$, then the theorem asserts that a graph is 1-connected (i.e., is connected) if and only if each pair of vertices is joined by a path.

12.7 Exercises

1. Prove that isomorphic graphs have the same chromatic number and the same chromatic polynomial.

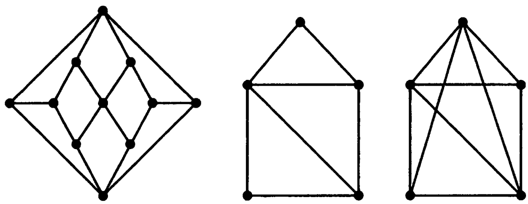
[48] Such a vertex exists, since b is a vertex of γ, which is also a vertex of γ_2.
[49] Such a vertex exists, since a is a vertex of γ_2, which is also a vertex of γ_1.
[50] K. Menger, Zur allgemeinen Kurventheorie, *Fund. Math.*, 10 (1927), 95–115.

2. Prove that the chromatic number of a disconnected graph is the largest of the chromatic numbers of its connected components.

3. Prove that the chromatic polynomial of a disconnected graph equals the product of the chromatic polynomials of its connected components.

4. Prove that the chromatic number of a cycle graph C_n of odd length equals 3.

5. Determine the chromatic numbers of the following graphs:



6. Prove that a graph with chromatic number equal to k has at least $\binom{k}{2}$ edges.

7. Prove that the greedy algorithm always produces a coloring of the vertices of $K_{m,n}$ in two colors $(m, n \geq 1)$.

8. Let G be a graph of order $n \geq 1$ with chromatic polynomial $p_G(k)$.

   (a) Prove that the constant term of $p_G(k)$ equals 0.

   (b) Prove that the coefficient of k in $p_G(k)$ is nonzero if and only if G is connected.

   (c) Prove that the coefficient of k^{n-1} in $p_G(k)$ equals $-m$, where m is the number of edges of G.

9. Let G be a graph of order n whose chromatic polynomial is $p_G(k) = k(k-1)^{n-1}$ (i.e., the chromatic polynomial of G is the same as that of a tree of order n). Prove that G is a tree.

10. What is the chromatic number of the graph obtained from K_n by removing one edge?

11. Prove that the chromatic polynomial of the graph obtained from K_n by removing an edge equals
$$[k]_n + [k]_{n-1}.$$

12. What is the chromatic number of the graph obtained from K_n by removing two edges with a common vertex?

13. What is the chromatic number of the graph obtained from K_n by removing two edges without a common vertex?

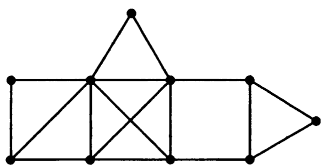
14. Prove that the chromatic polynomial of a cycle graph C_n equals

$$(k-1)^n + (-1)^n(k-1).$$

15. Prove that the chromatic number of a graph that has exactly one cycle of odd length is 3.

16. Prove that the polynomial $k^4 - 4k^3 + 3k^2$ is not the chromatic polynomial of any graph.

17. Use Theorem 12.1.10 to determine the chromatic number of the following graph:



18. Use the algorithm for computing the chromatic polynomial of a graph to determine the chromatic polynomial of the graph Q_3 of vertices and edges of a three-dimensional cube.

19. Find a planar graph that has two different planar representations such that, for some integer f, one has a region bounded by f edge-curves and the other has no such region.

20. Give an example of a planar graph with chromatic number 4 that does not contain a K_4 as an induced subgraph.

21. A plane is divided into regions by a finite number of straight lines. Prove that the regions can be colored with two colors in such a way that regions which share a boundary are colored differently.

22. Repeat Exercise 21, with circles replacing straight lines.

23. Let G be a connected planar graph of order n having $e = 3n - 6$ edges. Prove that, in any planar representation of G, each region is bounded by exactly 3 edge-curves.

24. Prove that a connected graph can always be contracted to a single vertex.

25. Verify that a contraction of a planar graph is planar.

26. Let G be a planar graph of order n in which every vertex has the same degree k. Prove that $k \leq 5$.

27. Let G be a planar graph of order $n \geq 2$. Prove that G has at least two vertices whose degrees are at most 5.

28. A graph is called *color-critical* provided each subgraph obtained by removing a vertex has a smaller chromatic number. Let $G = (V, E)$ be a color-critical graph. Prove the following:

    (a) $\chi(G_{V-\{x\}}) = \chi(G) - 1$ for every vertex x.

    (b) G is connected.

    (c) Each vertex of G has degree at least equal to $\chi(G) - 1$.

    (d) G does not have an articulation set U such that G_U is a complete graph.

    (e) Every graph H has an induced subgraph G such that $\chi(G) = \chi(H)$ and G is color-critical.

29. Let $p \geq 3$ be an integer. Prove that a graph, each of whose vertices has degree at least $p - 1$, contains a cycle of length greater than or equal to p. Then use Exercise 28 to show that a graph with chromatic number equal to p contains a cycle of length at least p.

30. $*$ Let G be a graph without any articulation vertices such that each vertex has degree at least 3. Prove that G contains a subgraph that can be contracted to a K_4. (*Hint*: Begin with a cycle of largest length p. By Exercise 29, we have $p \geq 4$. Now use Exercise 28 to obtain a proof of Hadwiger's conjecture for $p = 4$.)

31. Let G be a connected graph. Let T be a spanning tree of G. Prove that T contains a spanning subgraph T' such that, for each vertex v, the degree of v in G and the degree of v in T' are equal modulo 2.

32. Find a solution to the problem of the 8 queens that is different from that given in Figure 12.9.

33. Prove that the independence number of a tree of order n is at least $\lceil n/2 \rceil$.

34. Prove that the complement of a disconnected graph is connected.

35. Let H be a spanning subgraph of a graph G. Prove that $\text{dom}(G) \leq \text{dom}(H)$.

36. For each integer $n \geq 2$, determine a tree of order n whose domination number equals $\lfloor n/2 \rfloor$.

37. Determine the domination number of the graph Q_3 of vertices and edges of a three-dimensional cube.

38. Determine the domination number of a cycle graph C_n.

39. For $n = 5$ and 6, show that the domination number of the queens graph of an n-by-n chessboard is at most 3 by finding three squares on which to place queens so that every other square is attacked by at least one of the queens.

40. Show that the domination number of the queens graph of a 7-by-7 chessboard is at most 4.

41. * Show that the domination number of the queens graph of an 8-by-8 chessboard is at most 5.

42. Prove that an induced subgraph of an interval graph is an interval graph.

43. Prove that an induced subgraph of a chordal graph is chordal.

44. Prove that the only connected bipartite graphs that are chordal are trees.

45. Prove that all bipartite graphs are perfect.

46. Let G be a graph such that either G or its complement $\overline{G}$ has an induced subgraph equal to a chordless cycle of odd length greater than 3. Prove that G is not perfect.

47. Let k be a positive integer, and let G be a bipartite graph in which every vertex has degree k.

    (a) Prove that G has a perfect matching.

    (b) Prove that the edges of G can be partitioned into k perfect matchings.

48. Consider the graph Q_n of vertices and edges of the n-dimensional cube. Using induction,

    (a) Prove that Q_n has a perfect matching for each $n \geq 1$.

    (b) Prove that Q_n has at least $2^{2^{n-2}}$ perfect matchings.

49. Prove that if a tree has a perfect matching, then it has exactly one perfect matching.

50. Use Theorem 12.5.4 to prove the following theorem of Petersen (1891): A graph with every vertex of degree 3 and edge-connectivity at least 2 has a perfect matching.

51. The *Petersen graph* $\mathcal{P}$ is the graph whose vertices are the ten 2-subsets of $\{1, 2, 3, 4, 5\}$ in which two vertices are joined by an edge if and only if their 2-subsetss are disjoint.

(a) Draw a picture of the Petersen graph. (It can be drawn as a pentagon with a disjoint pentagram inside it—so 10 vertices and 10 edges—where there are an additional five edges joining each vertex of the pentagon to the corresponding vertex of the pentagram.)

(b) Verify that for each pair of vertices of $\mathcal{P}$ that are not joined by an edge, there is exactly one vertex joined by an edge to both.

(c) Verify that the smallest length of a cycle of $\mathcal{P}$ is 5.

52. Prove that the edge-connectivity of K_n equals $n-1$.

53. Give an example of a graph G different from a complete graph for which $\kappa(G) = \lambda(G)$.

54. Give an example of a graph G for which $\kappa(G) < \lambda(G)$.

55. Give an example of a graph G for which $\kappa(G) < \lambda(G) < \delta(G)$.

56. Determine the edge-connectivity of the complete bipartite graphs $K_{m,n}$.

57. Let G be a graph of order n with vertex degrees $d_1, d_2, \ldots, d_n$. Assume that the degrees have been arranged so that $d_1 \leq d_2 \leq \cdots \leq d_n$. Prove that, if $d_k \geq k$ for all $k \leq n - d_n - 1$, then G is a connected graph.

58. Let G be a graph of order n in which every vertex has degree equal to d.

    (a) How large must d be in order to *guarantee* that G is connected?

    (b) How large must d be in order to *guarantee* that G is 2-connected?

59. Determine the blocks of the graph given in Figure 12.12.

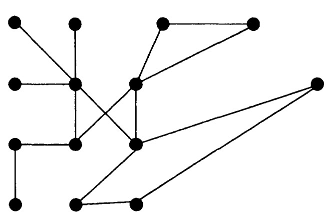


Figure 12.12

60. Prove that the blocks of a tree are all K_2's.

61. Let G be a connected graph. Prove that an edge of G is a bridge if and only if it is the edge of a block equal to a K_2.

62. Let G be a graph. Prove that G is 2-connected if and only if, for each vertex x and each edge α, there is a cycle that contains both the vertex x and the edge α.

63. Let G be a graph each of whose vertices has positive degree. Prove that G is 2-connected if and only if, for each pair of edges α_1, α_2, there is a cycle containing both α_1 and α_2.

64. Prove that a connected graph of order $n \geq 2$ has at least two vertices that are not articulation vertices. (*Hint*: Take the two end vertices of a longest path.