5

Diagonalization

This chapter is concerned with the so-called *diagonalization problem.* For a given linear operator T on a finite-dimensional vector space V, we seek answers to the following questions.

1. Does there exist an ordered basis β for V such that $[\mathsf{T}]_\beta$ is a diagonal matrix?
2. If such a basis exists, how can it be found?

Since computations involving diagonal matrices are simple, an affirmative answer to question 1 leads us to a clearer understanding of how the operator T acts on V, and an answer to question 2 enables us to obtain easy solutions to many practical problems that can be formulated in a linear algebra context. We consider some of these problems and their solutions in this chapter; see, for example, Section 5.3.

A solution to the diagonalization problem leads naturally to the concepts of *eigenvalue* and *eigenvector.* Aside from the important role that these concepts play in the diagonalization problem, they also prove to be useful tools in the study of many nondiagonalizable operators, as we will see in Chapter 7.

5.1 EIGENVALUES AND EIGENVECTORS

In Example 3 of Section 2.5, we were able to obtain a formula for the reflection of R^2 about the line $y = 2x$. The key to our success was to find a basis β' for which $[\mathsf{T}]_{\beta'}$ is a diagonal matrix. We now introduce the name for an operator or matrix that has such a basis.

Definitions. *A linear operator* T *on a finite-dimensional vector space* V *is called* **diagonalizable** *if there is an ordered basis* β *for* V *such that* $[\mathsf{T}]_\beta$

is a diagonal matrix. A square matrix A is called **diagonalizable** *if L_A is diagonalizable.*

We want to determine when a linear operator T on a finite-dimensional vector space V is diagonalizable and, if so, how to obtain an ordered basis $\beta = \{v_1, v_2, \ldots, v_n\}$ for V such that $[\mathsf{T}]_\beta$ is a diagonal matrix. Note that, if $D = [\mathsf{T}]_\beta$ is a diagonal matrix, then for each vector $v_j \in \beta$, we have

$$\mathsf{T}(v_j) = \sum_{i=1}^{n} D_{ij} v_i = D_{jj} v_j = \lambda_j v_j,$$

where $\lambda_j = D_{jj}$.

Conversely, if $\beta = \{v_1, v_2, \ldots, v_n\}$ is an ordered basis for V such that $\mathsf{T}(v_j) = \lambda_j v_j$ for some scalars $\lambda_1, \lambda_2, \ldots, \lambda_n$, then clearly

$$[\mathsf{T}]_\beta = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}.$$

In the preceding paragraph, each vector v in the basis β satisfies the condition that $\mathsf{T}(v) = \lambda v$ for some scalar λ. Moreover, because v lies in a basis, v is nonzero. These computations motivate the following definitions.

Definitions. *Let T be a linear operator on a vector space V. A nonzero vector $v \in \mathsf{V}$ is called an* **eigenvector** *of T if there exists a scalar λ such that $\mathsf{T}(v) = \lambda v$. The scalar λ is called the* **eigenvalue** *corresponding to the eigenvector v.*

Let A be in $\mathsf{M}_{n\times n}(F)$. A nonzero vector $v \in \mathsf{F}^n$ is called an **eigenvector** *of A if v is an eigenvector of L_A; that is, if $Av = \lambda v$ for some scalar λ. The scalar λ is called the* **eigenvalue** *of A corresponding to the eigenvector v.*

The words *characteristic vector* and *proper vector* are also used in place of *eigenvector*. The corresponding terms for *eigenvalue* are *characteristic value* and *proper value.*

Note that a vector is an eigenvector of a matrix A if and only if it is an eigenvector of L_A. Likewise, a scalar λ is an eigenvalue of A if and only if it is an eigenvalue of L_A. Using the terminology of eigenvectors and eigenvalues, we can summarize the preceding discussion as follows.

Theorem 5.1. *A linear operator T on a finite-dimensional vector space V is diagonalizable if and only if there exists an ordered basis β for V consisting of eigenvectors of T. Furthermore, if T is diagonalizable, $\beta = \{v_1, v_2, \ldots, v_n\}$ is an ordered basis of eigenvectors of T, and $D = [\mathsf{T}]_\beta$, then D is a diagonal matrix and D_{jj} is the eigenvalue corresponding to v_j for $1 \le j \le n$.*

To *diagonalize* a matrix or a linear operator is to find a basis of eigenvectors and the corresponding eigenvalues.

Before continuing our study of the diagonalization problem, we consider three examples of eigenvalues and eigenvectors.

Example 1

Let

$$A = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}, \quad v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \text{and} \quad v_2 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}.$$

Since

$$\mathsf{L}_A(v_1) = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \end{pmatrix} = -2\begin{pmatrix} 1 \\ -1 \end{pmatrix} = -2v_1,$$

v_1 is an eigenvector of L_A, and hence of A. Here $\lambda_1 = -2$ is the eigenvalue corresponding to v_1. Furthermore,

$$\mathsf{L}_A(v_2) = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 15 \\ 20 \end{pmatrix} = 5\begin{pmatrix} 3 \\ 4 \end{pmatrix} = 5v_2,$$

and so v_2 is an eigenvector of L_A, and hence of A, with the corresponding eigenvalue $\lambda_2 = 5$. Note that $\beta = \{v_1, v_2\}$ is an ordered basis for R^2 consisting of eigenvectors of both A and L_A, and therefore A and L_A are diagonalizable. Moreover, by Theorem 5.1,

$$[\mathsf{L}_A]_\beta = \begin{pmatrix} -2 & 0 \\ 0 & 5 \end{pmatrix}. \quad \blacklozenge$$

Example 2

Let T be the linear operator on R^2 that rotates each vector in the plane through an angle of $\pi/2$. It is clear geometrically that for any nonzero vector v, the vectors v and $\mathsf{T}(v)$ are not collinear; hence $\mathsf{T}(v)$ is not a multiple of v. Therefore T has no eigenvectors and, consequently, no eigenvalues. Thus there exist operators (and matrices) with no eigenvalues or eigenvectors. Of course, such operators and matrices are not diagonalizable. ♦

Example 3

Let $\mathsf{C}^\infty(R)$ denote the set of all functions $f\colon R \to R$ having derivatives of all orders. (Thus $\mathsf{C}^\infty(R)$ includes the polynomial functions, the sine and cosine functions, the exponential functions, etc.) Clearly, $\mathsf{C}^\infty(R)$ is a subspace of the vector space $\mathcal{F}(R, R)$ of all functions from R to R as defined in Section 1.2. Let $\mathsf{T}\colon \mathsf{C}^\infty(R) \to \mathsf{C}^\infty(R)$ be the function defined by $\mathsf{T}(f) = f'$, the derivative of f. It is easily verified that T is a linear operator on $\mathsf{C}^\infty(R)$. We determine the eigenvalues and eigenvectors of T.

Suppose that f is an eigenvector of T with corresponding eigenvalue λ. Then $f' = \mathsf{T}(f) = \lambda f$. This is a first-order differential equation whose solutions are of the form $f(t) = ce^{\lambda t}$ for some constant c. Consequently, every real number λ is an eigenvalue of T, and λ corresponds to eigenvectors of the form $ce^{\lambda t}$ for $c \neq 0$. Note that for $\lambda = 0$, the eigenvectors are the nonzero constant functions. ◆

In order to obtain a basis of eigenvectors for a matrix (or a linear operator), we need to be able to determine its eigenvalues and eigenvectors. The following theorem gives us a method for computing eigenvalues.

Theorem 5.2. *Let $A \in \mathsf{M}_{n\times n}(F)$. Then a scalar λ is an eigenvalue of A if and only if* $\det(A - \lambda I_n) = 0$.

Proof. A scalar λ is an eigenvalue of A if and only if there exists a nonzero vector $v \in \mathsf{F}^n$ such that $Av = \lambda v$, that is, $(A - \lambda I_n)(v) = \mathit{0}$. By Theorem 2.5 (p. 71), this is true if and only if $A - \lambda I_n$ is not invertible. However, this result is equivalent to the statement that $\det(A - \lambda I_n) = 0$. ∎

Definition. *Let $A \in \mathsf{M}_{n\times n}(F)$. The polynomial $f(t) = \det(A - tI_n)$ is called the* ***characteristic polynomial***[1] *of A.*

Theorem 5.2 states that the eigenvalues of a matrix are the zeros of its characteristic polynomial. When determining the eigenvalues of a matrix or a linear operator, we normally compute its characteristic polynomial, as in the next example.

Example 4

To find the eigenvalues of

$$A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \in \mathsf{M}_{2\times 2}(R),$$

we compute its characteristic polynomial:

$$\det(A - tI_2) = \det\begin{pmatrix} 1-t & 1 \\ 4 & 1-t \end{pmatrix} = t^2 - 2t - 3 = (t-3)(t+1).$$

It follows from Theorem 5.2 that the only eigenvalues of A are 3 and -1. ◆

[1]The observant reader may have noticed that the entries of the matrix $A - tI_n$ are not scalars in the field F. They are, however, scalars in another field $F(t)$, the field of quotients of polynomials in t with coefficients from F. Consequently, any results proved about determinants in Chapter 4 remain valid in this context.

It is easily shown that similar matrices have the same characteristic polynomial (see Exercise 12). This fact enables us to define the characteristic polynomial of a linear operator as follows.

Definition. *Let T be a linear operator on an n-dimensional vector space V with ordered basis β. We define the* **characteristic polynomial** $f(t)$ *of T to be the characteristic polynomial of $A = [\mathsf{T}]_\beta$. That is,*

$$f(t) = \det(A - tI_n).$$

The remark preceding this definition shows that the definition is independent of the choice of ordered basis β. Thus if T is a linear operator on a finite-dimensional vector space V and β is an ordered basis for V, then λ is an eigenvalue of T if and only if λ is an eigenvalue of $[\mathsf{T}]_\beta$. We often denote the characteristic polynomial of an operator T by $\det(\mathsf{T} - tI)$.

Example 5

Let T be the linear operator on $\mathsf{P}_2(R)$ defined by $\mathsf{T}(f(x)) = f(x)+(x+1)f'(x)$, let β be the standard ordered basis for $\mathsf{P}_2(R)$, and let $A = [\mathsf{T}]_\beta$. Then

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}.$$

The characteristic polynomial of T is

$$\begin{aligned} \det(A - tI_3) &= \det \begin{pmatrix} 1-t & 1 & 0 \\ 0 & 2-t & 2 \\ 0 & 0 & 3-t \end{pmatrix} \\ &= (1-t)(2-t)(3-t) \\ &= -(t-1)(t-2)(t-3). \end{aligned}$$

Hence λ is an eigenvalue of T (or A) if and only if $\lambda = 1, 2$, or 3. ♦

Examples 4 and 5 suggest that the characteristic polynomial of an $n \times n$ matrix A is a polynomial of degree n. The next theorem tells us even more. It can be proved by a straightforward induction argument.

Theorem 5.3. *Let $A \in \mathsf{M}_{n\times n}(F)$.*

(a) *The characteristic polynomial of A is a polynomial of degree n with leading coefficient $(-1)^n$.*

(b) *A has at most n distinct eigenvalues.*

Proof. Exercise. ∎

Theorem 5.2 enables us to determine all the eigenvalues of a matrix or a linear operator on a finite-dimensional vector space provided that we can compute the zeros of the characteristic polynomial. Our next result gives us a procedure for determining the eigenvectors corresponding to a given eigenvalue.

Theorem 5.4. *Let* T *be a linear operator on a vector space* V*, and let* λ *be an eigenvalue of* T*. A vector* $v \in \mathsf{V}$ *is an eigenvector of* T *corresponding to* λ *if and only if* $v \neq 0$ *and* $v \in \mathsf{N}(\mathsf{T} - \lambda\mathsf{I})$*.*

Proof. Exercise. ∎

Example 6

To find all the eigenvectors of the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$$

in Example 4, recall that A has two eigenvalues, $\lambda_1 = 3$ and $\lambda_2 = -1$. We begin by finding all the eigenvectors corresponding to $\lambda_1 = 3$. Let

$$B_1 = A - \lambda_1 I = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} - \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix}.$$

Then

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathsf{R}^2$$

is an eigenvector corresponding to $\lambda_1 = 3$ if and only if $x \neq 0$ and $x \in \mathsf{N}(\mathsf{L}_{B_1})$; that is, $x \neq 0$ and

$$\begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2x_1 + x_2 \\ 4x_1 - 2x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Clearly the set of all solutions to this equation is

$$\left\{ t \begin{pmatrix} 1 \\ 2 \end{pmatrix} : t \in R \right\}.$$

Hence x is an eigenvector corresponding to $\lambda_1 = 3$ if and only if

$$x = t \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad \text{for some } t \neq 0.$$

Now suppose that x is an eigenvector of A corresponding to $\lambda_2 = -1$. Let

$$B_2 = A - \lambda_2 I = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} - \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}.$$

Then

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathsf{N}(\mathsf{L}_{B_2})$$

if and only if x is a solution to the system

$$\begin{aligned} 2x_1 + x_2 &= 0 \\ 4x_1 + 2x_2 &= 0. \end{aligned}$$

Hence

$$\mathsf{N}(\mathsf{L}_{B_2}) = \left\{ t \begin{pmatrix} 1 \\ -2 \end{pmatrix} : t \in R \right\}.$$

Thus x is an eigenvector corresponding to $\lambda_2 = -1$ if and only if

$$x = t \begin{pmatrix} 1 \\ -2 \end{pmatrix} \qquad \text{for some } t \neq 0.$$

Observe that

$$\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right\}$$

is a basis for R^2 consisting of eigenvectors of A. Thus L_A, and hence A, is diagonalizable. ◆

Suppose that β is a basis for F^n consisting of eigenvectors of A. The corollary to Theorem 2.23 assures us that if Q is the $n \times n$ matrix whose columns are the vectors in β, then $Q^{-1}AQ$ is a diagonal matrix. In Example 6, for instance, if

$$Q = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix},$$

then

$$Q^{-1}AQ = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}.$$

Of course, the diagonal entries of this matrix are the eigenvalues of A that correspond to the respective columns of Q.

To find the eigenvectors of a linear operator T on an n-dimensional vector space, select an ordered basis β for V and let $A = [\mathsf{T}]_\beta$. Figure 5.1 is the special case of Figure 2.2 in Section 2.4 in which $\mathsf{V} = \mathsf{W}$ and $\beta = \gamma$. Recall that for $v \in \mathsf{V}$, $\phi_\beta(v) = [v]_\beta$, the coordinate vector of v relative to β. We show that $v \in \mathsf{V}$ is an eigenvector of T corresponding to λ if and only if $\phi_\beta(v)$

$$
\begin{tikzcd}
\mathsf{V} \arrow[r, "\mathsf{T}"] \arrow[d, "\phi_\beta"'] & \mathsf{V} \arrow[d, "\phi_\beta"] \\
\mathsf{F}^n \arrow[r, "\mathsf{L}_A"] & \mathsf{F}^n
\end{tikzcd}
.
$$

Figure 5.1

is an eigenvector of A corresponding to λ. Suppose that v is an eigenvector of T corresponding to λ. Then $\mathsf{T}(v) = \lambda v$. Hence

$$A\phi_\beta(v) = \mathsf{L}_A\phi_\beta(v) = \phi_\beta\mathsf{T}(v) = \phi_\beta(\lambda v) = \lambda\phi_\beta(v).$$

Now $\phi_\beta(v) \neq \mathit{0}$, since ϕ_β is an isomorphism; hence $\phi_\beta(v)$ is an eigenvector of A. This argument is reversible, and so we can establish that if $\phi_\beta(v)$ is an eigenvector of A corresponding to λ, then v is an eigenvector of T corresponding to λ. (See Exercise 13.)

An equivalent formulation of the result discussed in the preceding paragraph is that for an eigenvalue λ of A (and hence of T), a vector $y \in \mathsf{F}^n$ is an eigenvector of A corresponding to λ if and only if $\phi_\beta^{-1}(y)$ is an eigenvector of T corresponding to λ.

Thus we have reduced the problem of finding the eigenvectors of a linear operator on a finite-dimensional vector space to the problem of finding the eigenvectors of a matrix. The next example illustrates this procedure.

Example 7

Let T be the linear operator on $\mathsf{P}_2(R)$ defined in Example 5, and let β be the standard ordered basis for $\mathsf{P}_2(R)$. Recall that T has eigenvalues 1, 2, and 3 and that

$$A = [\mathsf{T}]_\beta = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}.$$

We consider each eigenvalue separately.

Let $\lambda_1 = 1$, and define

$$B_1 = A - \lambda_1 I = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix}.$$

Then

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathsf{R}^3$$

is an eigenvector corresponding to $\lambda_1 = 1$ if and only if $x \neq 0$ and $x \in \mathsf{N}(\mathsf{L}_{B_1})$; that is, x is a nonzero solution to the system

$$\begin{aligned} x_2 \qquad &= 0 \\ x_2 + 2x_3 &= 0 \\ 2x_3 &= 0. \end{aligned}$$

Notice that this system has three unknowns, x_1, x_2, and x_3, but one of these, x_1, does not actually appear in the system. Since the values of x_1 do not affect the system, we assign x_1 a parametric value, say $x_1 = a$, and solve the system for x_2 and x_3. Clearly, $x_2 = x_3 = 0$, and so the eigenvectors of A corresponding to $\lambda_1 = 1$ are of the form

$$a\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = ae_1$$

for $a \neq 0$. Consequently, the eigenvectors of T corresponding to $\lambda_1 = 1$ are of the form

$$\phi_\beta^{-1}(ae_1) = a\phi_\beta^{-1}(e_1) = a \cdot 1 = a$$

for any $a \neq 0$. Hence the nonzero constant polynomials are the eigenvectors of T corresponding to $\lambda_1 = 1$.

Next let $\lambda_2 = 2$, and define

$$B_2 = A - \lambda_2 I = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$

It is easily verified that

$$\mathsf{N}(\mathsf{L}_{B_2}) = \left\{ a\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} : a \in R \right\},$$

and hence the eigenvectors of T corresponding to $\lambda_2 = 2$ are of the form

$$\phi_\beta^{-1}\left(a\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right) = a\phi_\beta^{-1}(e_1 + e_2) = a(1 + x)$$

for $a \neq 0$.

Finally, consider $\lambda_3 = 3$ and

$$B_3 = A - \lambda_3 I = \begin{pmatrix} -2 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since

$$\mathsf{N}(\mathsf{L}_{B_3}) = \left\{ a\begin{pmatrix}1\\2\\1\end{pmatrix} : a \in R \right\},$$

the eigenvectors of T corresponding to $\lambda_3 = 3$ are of the form

$$\phi_\beta^{-1}\left(a\begin{pmatrix}1\\2\\1\end{pmatrix}\right) = a\phi_\beta^{-1}(e_1 + 2e_2 + e_3) = a(1 + 2x + x^2)$$

for $a \neq 0$.

For each eigenvalue, select the corresponding eigenvector with $a = 1$ in the preceding descriptions to obtain $\gamma = \{1, 1+x, 1+2x+x^2\}$, which is an ordered basis for $\mathsf{P}_2(R)$ consisting of eigenvectors of T. Thus T is diagonalizable, and

$$[\mathsf{T}]_\gamma = \begin{pmatrix}1 & 0 & 0\\0 & 2 & 0\\0 & 0 & 3\end{pmatrix}. \quad \blacklozenge$$

We close this section with a geometric description of how a linear operator T acts on an eigenvector in the context of a vector space V over R. Let v be an eigenvector of T and λ be the corresponding eigenvalue. We can think of $\mathsf{W} = \text{span}(\{v\})$, the one-dimensional subspace of V spanned by v, as a line in V that passes through $\mathit{0}$ and v. For any $w \in \mathsf{W}$, $w = cv$ for some scalar c, and hence

$$\mathsf{T}(w) = \mathsf{T}(cv) = c\mathsf{T}(v) = c\lambda v = \lambda w;$$

so T acts on the vectors in W by multiplying each such vector by λ. There are several possible ways for T to act on the vectors in W, depending on the value of λ. We consider several cases. (See Figure 5.2.)

CASE 1. If $\lambda > 1$, then T moves vectors in W farther from $\mathit{0}$ by a factor of λ.

CASE 2. If $\lambda = 1$, then T acts as the identity operator on W.

CASE 3. If $0 < \lambda < 1$, then T moves vectors in W closer to $\mathit{0}$ by a factor of λ.

CASE 4. If $\lambda = 0$, then T acts as the zero transformation on W.

CASE 5. If $\lambda < 0$, then T reverses the orientation of W; that is, T moves vectors in W from one side of $\mathit{0}$ to the other.

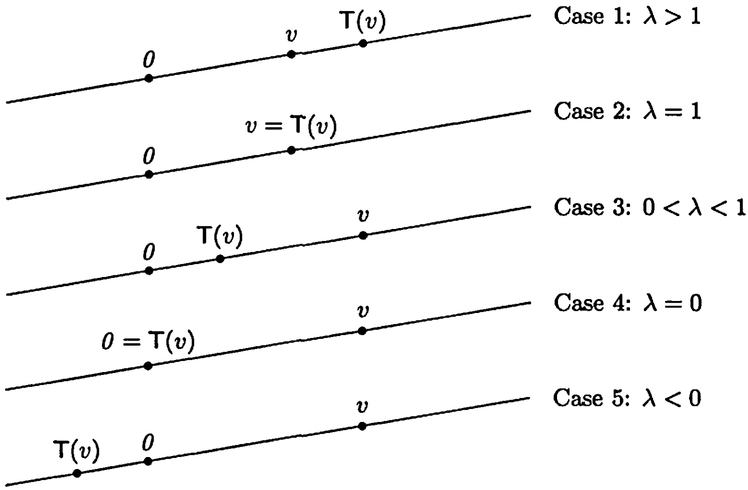

Figure 5.2: The action of T on $\mathsf{W} = \mathrm{span}(\{v\})$ when v is an eigenvector of T.

To illustrate these ideas, we consider the linear operators in Examples 3, 4, and 2 of Section 2.1.

For the operator T on R^2 defined by $\mathsf{T}(a_1, a_2) = (a_1, -a_2)$, the reflection about the x-axis, e_1 and e_2 are eigenvectors of T with corresponding eigenvalues 1 and -1, respectively. Since e_1 and e_2 span the x-axis and the y-axis, respectively, T acts as the identity on the x-axis and reverses the orientation of the y-axis.

For the operator T on R^2 defined by $\mathsf{T}(a_1, a_2) = (a_1, 0)$, the projection on the x-axis, e_1 and e_2 are eigenvectors of T with corresponding eigenvalues 1 and 0, respectively. Thus, T acts as the identity on the x-axis and as the zero operator on the y-axis.

Finally, we generalize Example 2 of this section by considering the operator that rotates the plane through the angle θ, which is defined by

$$\mathsf{T}_\theta(a_1, a_2) = (a_1 \cos\theta - a_2 \sin\theta, a_1 \sin\theta + a_2 \cos\theta).$$

Suppose that $0 < \theta < \pi$. Then for any nonzero vector v, the vectors v and $\mathsf{T}_\theta(v)$ are not collinear, and hence T_θ maps no one-dimensional subspace of R^2 into itself. But this implies that T_θ has no eigenvectors and therefore no eigenvalues. To confirm this conclusion, let β be the standard ordered basis for R^2, and note that the characteristic polynomial of T_θ is

$$\det([\mathsf{T}_\theta]_\beta - tI_2) = \det\begin{pmatrix} \cos\theta - t & -\sin\theta \\ \sin\theta & \cos\theta - t \end{pmatrix} = t^2 - (2\cos\theta)t + 1,$$

which has no real zeros because, for $0 < \theta < \pi$, the discriminant $4\cos^2\theta - 4$ is negative.

EXERCISES

1. Label the following statements as true or false.
   (a) Every linear operator on an n-dimensional vector space has n distinct eigenvalues.
   (b) If a real matrix has one eigenvector, then it has an infinite number of eigenvectors.
   (c) There exists a square matrix with no eigenvectors.
   (d) Eigenvalues must be nonzero scalars.
   (e) Any two eigenvectors are linearly independent.
   (f) The sum of two eigenvalues of a linear operator T is also an eigenvalue of T.
   (g) Linear operators on infinite-dimensional vector spaces never have eigenvalues.
   (h) An $n \times n$ matrix A with entries from a field F is similar to a diagonal matrix if and only if there is a basis for F^n consisting of eigenvectors of A.
   (i) Similar matrices always have the same eigenvalues.
   (j) Similar matrices always have the same eigenvectors.
   (k) The sum of two eigenvectors of an operator T is always an eigenvector of T.

2. For each of the following linear operators T on a vector space V and ordered bases β, compute $[\mathsf{T}]_\beta$, and determine whether β is a basis consisting of eigenvectors of T.
   (a) $\mathsf{V} = \mathsf{R}^2$, $\mathsf{T}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 10a - 6b \\ 17a - 10b \end{pmatrix}$, and $\beta = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix} \right\}$
   (b) $\mathsf{V} = \mathsf{P}_1(R)$, $\mathsf{T}(a + bx) = (6a - 6b) + (12a - 11b)x$, and $\beta = \{3 + 4x, 2 + 3x\}$
   (c) $\mathsf{V} = \mathsf{R}^3$, $\mathsf{T}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 3a + 2b - 2c \\ -4a - 3b + 2c \\ -c \end{pmatrix}$, and

   $$\beta = \left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \right\}$$

   (d) $\mathsf{V} = \mathsf{P}_2(R)$, $\mathsf{T}(a + bx + cx^2) =$

   $$(-4a + 2b - 2c) - (7a + 3b + 7c)x + (7a + b + 5c)x^2,$$

   and $\beta = \{x - x^2, -1 + x^2, -1 - x + x^2\}$

(e) $\mathsf{V} = \mathsf{P}_3(R)$, $\mathsf{T}(a + bx + cx^2 + dx^3) =$

$$-d + (-c + d)x + (a + b - 2c)x^2 + (-b + c - 2d)x^3,$$

and $\beta = \{1 - x + x^3, 1 + x^2, 1, x + x^2\}$

(f) $\mathsf{V} = \mathsf{M}_{2\times 2}(R)$, $\mathsf{T}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -7a - 4b + 4c - 4d & b \\ -8a - 4b + 5c - 4d & d \end{pmatrix}$, and

$$\beta = \left\{\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}\right\}$$

3. For each of the following matrices $A \in \mathsf{M}_{n\times n}(F)$,

(i) Determine all the eigenvalues of A.

(ii) For each eigenvalue λ of A, find the set of eigenvectors corresponding to λ.

(iii) If possible, find a basis for F^n consisting of eigenvectors of A.

(iv) If successful in finding such a basis, determine an invertible matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$.

(a) $A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$ for $F = R$

(b) $A = \begin{pmatrix} 0 & -2 & -3 \\ -1 & 1 & -1 \\ 2 & 2 & 5 \end{pmatrix}$ for $F = R$

(c) $A = \begin{pmatrix} i & 1 \\ 2 & -i \end{pmatrix}$ for $F = C$

(d) $A = \begin{pmatrix} 2 & 0 & -1 \\ 4 & 1 & -4 \\ 2 & 0 & -1 \end{pmatrix}$ for $F = R$

4. For each linear operator T on V, find the eigenvalues of T and an ordered basis β for V such that $[\mathsf{T}]_\beta$ is a diagonal matrix.

(a) $\mathsf{V} = \mathsf{R}^2$ and $\mathsf{T}(a, b) = (-2a + 3b, -10a + 9b)$
(b) $\mathsf{V} = \mathsf{R}^3$ and $\mathsf{T}(a, b, c) = (7a - 4b + 10c, 4a - 3b + 8c, -2a + b - 2c)$
(c) $\mathsf{V} = \mathsf{R}^3$ and $\mathsf{T}(a, b, c) = (-4a + 3b - 6c, 6a - 7b + 12c, 6a - 6b + 11c)$
(d) $\mathsf{V} = \mathsf{P}_1(R)$ and $\mathsf{T}(ax + b) = (-6a + 2b)x + (-6a + b)$
(e) $\mathsf{V} = \mathsf{P}_2(R)$ and $\mathsf{T}(f(x)) = xf'(x) + f(2)x + f(3)$
(f) $\mathsf{V} = \mathsf{P}_3(R)$ and $\mathsf{T}(f(x)) = f(x) + f(2)x$
(g) $\mathsf{V} = \mathsf{P}_3(R)$ and $\mathsf{T}(f(x)) = xf'(x) + f''(x) - f(2)$
(h) $\mathsf{V} = \mathsf{M}_{2\times 2}(R)$ and $\mathsf{T}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & b \\ c & a \end{pmatrix}$

(i) $\mathsf{V} = \mathsf{M}_{2\times 2}(R)$ and $\mathsf{T}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d \\ a & b \end{pmatrix}$

(j) $\mathsf{V} = \mathsf{M}_{2\times 2}(R)$ and $\mathsf{T}(A) = A^t + 2 \cdot \mathrm{tr}(A) \cdot I_2$

5. Prove Theorem 5.4.

6. Let T be a linear operator on a finite-dimensional vector space V, and let β be an ordered basis for V. Prove that λ is an eigenvalue of T if and only if λ is an eigenvalue of $[\mathsf{T}]_\beta$.

7. Let T be a linear operator on a finite-dimensional vector space V. We define the **determinant** of T, denoted $\det(\mathsf{T})$, as follows: Choose any ordered basis β for V, and define $\det(\mathsf{T}) = \det([\mathsf{T}]_\beta)$.

   (a) Prove that the preceding definition is independent of the choice of an ordered basis for V. That is, prove that if β and γ are two ordered bases for V, then $\det([\mathsf{T}]_\beta) = \det([\mathsf{T}]_\gamma)$.
   (b) Prove that T is invertible if and only if $\det(\mathsf{T}) \neq 0$.
   (c) Prove that if T is invertible, then $\det(\mathsf{T}^{-1}) = [\det(\mathsf{T})]^{-1}$.
   (d) Prove that if U is also a linear operator on V, then $\det(\mathsf{TU}) = \det(\mathsf{T}) \cdot \det(\mathsf{U})$.
   (e) Prove that $\det(\mathsf{T} - \lambda\mathsf{I}_\mathsf{V}) = \det([\mathsf{T}]_\beta - \lambda I)$ for any scalar λ and any ordered basis β for V.

8. (a) Prove that a linear operator T on a finite-dimensional vector space is invertible if and only if zero is not an eigenvalue of T.
   (b) Let T be an invertible linear operator. Prove that a scalar λ is an eigenvalue of T if and only if λ^{-1} is an eigenvalue of T^{-1}.
   (c) State and prove results analogous to (a) and (b) for matrices.

9. Prove that the eigenvalues of an upper triangular matrix M are the diagonal entries of M.

10. Let V be a finite-dimensional vector space, and let λ be any scalar.

   (a) For any ordered basis β for V, prove that $[\lambda\mathsf{I}_\mathsf{V}]_\beta = \lambda I$.
   (b) Compute the characteristic polynomial of $\lambda\mathsf{I}_\mathsf{V}$.
   (c) Show that $\lambda\mathsf{I}_\mathsf{V}$ is diagonalizable and has only one eigenvalue.

11. A **scalar matrix** is a square matrix of the form λI for some scalar λ; that is, a scalar matrix is a diagonal matrix in which all the diagonal entries are equal.

   (a) Prove that if a square matrix A is similar to a scalar matrix λI, then $A = \lambda I$.
   (b) Show that a diagonalizable matrix having only one eigenvalue is a scalar matrix.

(c) Prove that $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is not diagonalizable.

12. (a) Prove that similar matrices have the same characteristic polynomial.
(b) Show that the definition of the characteristic polynomial of a linear operator on a finite-dimensional vector space V is independent of the choice of basis for V.

13. Let T be a linear operator on a finite-dimensional vector space V over a field F, let β be an ordered basis for V, and let $A = [\mathsf{T}]_\beta$. In reference to Figure 5.1, prove the following.
(a) If $v \in \mathsf{V}$ and $\phi_\beta(v)$ is an eigenvector of A corresponding to the eigenvalue λ, then v is an eigenvector of T corresponding to λ.
(b) If λ is an eigenvalue of A (and hence of T), then a vector $y \in \mathsf{F}^n$ is an eigenvector of A corresponding to λ if and only if $\phi_\beta^{-1}(y)$ is an eigenvector of T corresponding to λ.

14.† For any square matrix A, prove that A and A^t have the same characteristic polynomial (and hence the same eigenvalues).

15.† (a) Let T be a linear operator on a vector space V, and let x be an eigenvector of T corresponding to the eigenvalue λ. For any positive integer m, prove that x is an eigenvector of T^m corresponding to the eigenvalue λ^m.
(b) State and prove the analogous result for matrices.

16. (a) Prove that similar matrices have the same trace. *Hint:* Use Exercise 13 of Section 2.3.
(b) How would you define the trace of a linear operator on a finite-dimensional vector space? Justify that your definition is well-defined.

17. Let T be the linear operator on $\mathsf{M}_{n\times n}(R)$ defined by $\mathsf{T}(A) = A^t$.
(a) Show that ± 1 are the only eigenvalues of T.
(b) Describe the eigenvectors corresponding to each eigenvalue of T.
(c) Find an ordered basis β for $\mathsf{M}_{2\times 2}(R)$ such that $[\mathsf{T}]_\beta$ is a diagonal matrix.
(d) Find an ordered basis β for $\mathsf{M}_{n\times n}(R)$ such that $[\mathsf{T}]_\beta$ is a diagonal matrix for $n > 2$.

18. Let $A, B \in \mathsf{M}_{n\times n}(C)$.
(a) Prove that if B is invertible, then there exists a scalar $c \in C$ such that $A + cB$ is not invertible. *Hint:* Examine $\det(A + cB)$.

(b) Find nonzero 2×2 matrices A and B such that both A and $A+cB$ are invertible for all $c \in C$.

19.† Let A and B be similar $n \times n$ matrices. Prove that there exists an n-dimensional vector space V, a linear operator T on V, and ordered bases β and γ for V such that $A = [\mathsf{T}]_\beta$ and $B = [\mathsf{T}]_\gamma$. *Hint:* Use Exercise 14 of Section 2.5.

20. Let A be an $n \times n$ matrix with characteristic polynomial

$$f(t) = (-1)^n t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0.$$

Prove that $f(0) = a_0 = \det(A)$. Deduce that A is invertible if and only if $a_0 \neq 0$.

21. Let A and $f(t)$ be as in Exercise 20.

(a) Prove that $f(t) = (A_{11} - t)(A_{22} - t) \cdots (A_{nn} - t) + q(t)$, where $q(t)$ is a polynomial of degree at most $n-2$. *Hint:* Apply mathematical induction to n.

(b) Show that $\operatorname{tr}(A) = (-1)^{n-1} a_{n-1}$.

22.† (a) Let T be a linear operator on a vector space V over the field F, and let $g(t)$ be a polynomial with coefficients from F. Prove that if x is an eigenvector of T with corresponding eigenvalue λ, then $g(\mathsf{T})(x) = g(\lambda)x$. That is, x is an eigenvector of $g(\mathsf{T})$ with corresponding eigenvalue $g(\lambda)$.

(b) State and prove a comparable result for matrices.

(c) Verify (b) for the matrix A in Exercise 3(a) with polynomial $g(t) = 2t^2 - t + 1$, eigenvector $x = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, and corresponding eigenvalue $\lambda = 4$.

23. Use Exercise 22 to prove that if $f(t)$ is the characteristic polynomial of a diagonalizable linear operator T, then $f(\mathsf{T}) = \mathsf{T}_0$, the zero operator. (In Section 5.4 we prove that this result does not depend on the diagonalizability of T.)

24. Use Exercise 21(a) to prove Theorem 5.3.

25. Prove Corollaries 1 and 2 of Theorem 5.3.

26. Determine the number of distinct characteristic polynomials of matrices in $\mathsf{M}_{2\times 2}(Z_2)$.

5.2 DIAGONALIZABILITY

In Section 5.1, we presented the diagonalization problem and observed that not all linear operators or matrices are diagonalizable. Although we are able to diagonalize operators and matrices and even obtain a necessary and sufficient condition for diagonalizability (Theorem 5.1 p. 246), we have not yet solved the diagonalization problem. What is still needed is a simple test to determine whether an operator or a matrix can be diagonalized, as well as a method for actually finding a basis of eigenvectors. In this section, we develop such a test and method.

In Example 6 of Section 5.1, we obtained a basis of eigenvectors by choosing one eigenvector corresponding to each eigenvalue. In general, such a procedure does not yield a basis, but the following theorem shows that any set constructed in this manner is linearly independent.

Theorem 5.5. *Let* T *be a linear operator on a vector space* V, *and let* $\lambda_1, \lambda_2, \ldots, \lambda_k$ *be distinct eigenvalues of* T. *If* $v_1, v_2, \ldots, v_k$ *are eigenvectors of* T *such that* λ_i *corresponds to* v_i $(1 \leq i \leq k)$, *then* $\{v_1, v_2, \ldots, v_k\}$ *is linearly independent.*

Proof. The proof is by mathematical induction on k. Suppose that $k = 1$. Then $v_1 \neq 0$ since v_1 is an eigenvector, and hence $\{v_1\}$ is linearly independent. Now assume that the theorem holds for $k-1$ distinct eigenvalues, where $k-1 \geq 1$, and that we have k eigenvectors $v_1, v_2, \ldots, v_k$ corresponding to the distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$. We wish to show that $\{v_1, v_2, \ldots, v_k\}$ is linearly independent. Suppose that $a_1, a_2, \ldots, a_k$ are scalars such that

$$a_1v_1 + a_2v_2 + \cdots + a_kv_k = 0. \tag{1}$$

Applying $\mathsf{T} - \lambda_k\mathsf{I}$ to both sides of (1), we obtain

$$a_1(\lambda_1 - \lambda_k)v_1 + a_2(\lambda_2 - \lambda_k)v_2 + \cdots + a_{k-1}(\lambda_{k-1} - \lambda_k)v_{k-1} = 0.$$

By the induction hypothesis $\{v_1, v_2, \ldots, v_{k-1}\}$ is linearly independent, and hence

$$a_1(\lambda_1 - \lambda_k) = a_2(\lambda_2 - \lambda_k) = \cdots = a_{k-1}(\lambda_{k-1} - \lambda_k) = 0.$$

Since $\lambda_1, \lambda_2, \ldots, \lambda_k$ are distinct, it follows that $\lambda_i - \lambda_k \neq 0$ for $1 \leq i \leq k-1$. So $a_1 = a_2 = \cdots = a_{k-1} = 0$, and (1) therefore reduces to $a_kv_k = 0$. But $v_k \neq 0$ and therefore $a_k = 0$. Consequently $a_1 = a_2 = \cdots = a_k = 0$, and it follows that $\{v_1, v_2, \ldots, v_k\}$ is linearly independent. ∎

Corollary. *Let* T *be a linear operator on an* n*-dimensional vector space* V. *If* T *has* n *distinct eigenvalues, then* T *is diagonalizable.*

Proof. Suppose that T has n distinct eigenvalues $\lambda_1, \ldots, \lambda_n$. For each i choose an eigenvector v_i corresponding to λ_i. By Theorem 5.5, $\{v_1, \ldots, v_n\}$ is linearly independent, and since $\dim(\mathsf{V}) = n$, this set is a basis for V. Thus, by Theorem 5.1 (p. 246), T is diagonalizable. ∎

Example 1

Let

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \in \mathsf{M}_{2\times 2}(R).$$

The characteristic polynomial of A (and hence of L_A) is

$$\det(A - tI) = \det \begin{pmatrix} 1-t & 1 \\ 1 & 1-t \end{pmatrix} = t(t-2),$$

and thus the eigenvalues of L_A are 0 and 2. Since L_A is a linear operator on the two-dimensional vector space R^2, we conclude from the preceding corollary that L_A (and hence A) is diagonalizable. ◆

The converse of Theorem 5.5 is false. That is, it is not true that if T is diagonalizable, then it has n distinct eigenvalues. For example, the identity operator is diagonalizable even though it has only one eigenvalue, namely, $\lambda = 1$.

We have seen that diagonalizability requires the existence of eigenvalues. Actually, diagonalizability imposes a stronger condition on the characteristic polynomial.

Definition. *A polynomial $f(t)$ in $\mathsf{P}(F)$* **splits over** F *if there are scalars $c, a_1, \ldots, a_n$ (not necessarily distinct) in F such that*

$$f(t) = c(t-a_1)(t-a_2)\cdots(t-a_n).$$

For example, $t^2 - 1 = (t+1)(t-1)$ splits over R, but $(t^2+1)(t-2)$ does not split over R because t^2+1 cannot be factored into a product of linear factors. However, $(t^2+1)(t-2)$ does split over C because it factors into the product $(t+i)(t-i)(t-2)$. If $f(t)$ is the characteristic polynomial of a linear operator or a matrix over a field F, then the statement that $f(t)$ splits is understood to mean that it splits over F.

Theorem 5.6. *The characteristic polynomial of any diagonalizable linear operator splits.*

Proof. Let T be a diagonalizable linear operator on the n-dimensional vector space V, and let β be an ordered basis for V such that $[\mathsf{T}]_\beta = D$ is a diagonal matrix. Suppose that

$$D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix},$$

and let $f(t)$ be the characteristic polynomial of T. Then

$$f(t) = \det(D - tI) = \det \begin{pmatrix} \lambda_1 - t & 0 & \cdots & 0 \\ 0 & \lambda_2 - t & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n - t \end{pmatrix}$$

$$= (\lambda_1 - t)(\lambda_2 - t)\cdots(\lambda_n - t) = (-1)^n(t - \lambda_1)(t - \lambda_2)\cdots(t - \lambda_n). \quad \blacksquare$$

From this theorem, it is clear that if T is a diagonalizable linear operator on an n-dimensional vector space that fails to have distinct eigenvalues, then the characteristic polynomial of T must have repeated zeros.

The converse of Theorem 5.6 is false; that is, the characteristic polynomial of T may split, but T need not be diagonalizable. (See Example 3, which follows.) The following concept helps us determine when an operator whose characteristic polynomial splits is diagonalizable.

Definition. *Let λ be an eigenvalue of a linear operator or matrix with characteristic polynomial $f(t)$. The* **(algebraic) multiplicity** *of λ is the largest positive integer k for which $(t - \lambda)^k$ is a factor of $f(t)$.*

Example 2

Let

$$A = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 4 \\ 0 & 0 & 4 \end{pmatrix},$$

which has characteristic polynomial $f(t) = -(t-3)^2(t-4)$. Hence $\lambda = 3$ is an eigenvalue of A with multiplicity 2, and $\lambda = 4$ is an eigenvalue of A with multiplicity 1. ♦

If T is a diagonalizable linear operator on a finite-dimensional vector space V, then there is an ordered basis β for V consisting of eigenvectors of T. We know from Theorem 5.1 (p. 246) that $[\mathsf{T}]_\beta$ is a diagonal matrix in which the diagonal entries are the eigenvalues of T. Since the characteristic polynomial of T is $\det([\mathsf{T}]_\beta - tI)$, it is easily seen that each eigenvalue of T must occur as a diagonal entry of $[\mathsf{T}]_\beta$ exactly as many times as its multiplicity. Hence

β contains as many (linearly independent) eigenvectors corresponding to an eigenvalue as the multiplicity of that eigenvalue. So the number of linearly independent eigenvectors corresponding to a given eigenvalue is of interest in determining whether an operator can be diagonalized. Recalling from Theorem 5.4 (p. 250) that the eigenvectors of T corresponding to the eigenvalue λ are the nonzero vectors in the null space of $\mathsf{T} - \lambda\mathsf{I}$, we are led naturally to the study of this set.

Definition. *Let T be a linear operator on a vector space V, and let λ be an eigenvalue of T. Define $\mathsf{E}_\lambda = \{x \in \mathsf{V} \colon \mathsf{T}(x) = \lambda x\} = \mathsf{N}(\mathsf{T} - \lambda\mathsf{I}_\mathsf{V})$. The set E_λ is called the* ***eigenspace*** *of T corresponding to the eigenvalue λ. Analogously, we define the* ***eigenspace*** *of a square matrix A to be the eigenspace of L_A.*

Clearly, E_λ is a subspace of V consisting of the zero vector and the eigenvectors of T corresponding to the eigenvalue λ. The maximum number of linearly independent eigenvectors of T corresponding to the eigenvalue λ is therefore the dimension of E_λ. Our next result relates this dimension to the multiplicity of λ.

Theorem 5.7. *Let T be a linear operator on a finite-dimensional vector space V, and let λ be an eigenvalue of T having multiplicity m. Then $1 \le \dim(\mathsf{E}_\lambda) \le m$.*

Proof. Choose an ordered basis $\{v_1, v_2, \ldots, v_p\}$ for E_λ, extend it to an ordered basis $\beta = \{v_1, v_2, \ldots, v_p, v_{p+1}, \ldots, v_n\}$ for V, and let $A = [\mathsf{T}]_\beta$. Observe that v_i $(1 \le i \le p)$ is an eigenvector of T corresponding to λ, and therefore

$$A = \begin{pmatrix} \lambda I_p & B \\ O & C \end{pmatrix}.$$

By Exercise 21 of Section 4.3, the characteristic polynomial of T is

$$\begin{aligned} f(t) = \det(A - tI_n) &= \det\begin{pmatrix} (\lambda - t)I_p & B \\ O & C - tI_{n-p} \end{pmatrix} \\ &= \det((\lambda - t)I_p)\det(C - tI_{n-p}) \\ &= (\lambda - t)^p g(t), \end{aligned}$$

where $g(t)$ is a polynomial. Thus $(\lambda - t)^p$ is a factor of $f(t)$, and hence the multiplicity of λ is at least p. But $\dim(\mathsf{E}_\lambda) = p$, and so $\dim(\mathsf{E}_\lambda) \le m$. ∎

Example 3

Let T be the linear operator on $\mathsf{P}_2(R)$ defined by $\mathsf{T}(f(x)) = f'(x)$. The matrix representation of T with respect to the standard ordered basis β for

$\mathsf{P}_2(R)$ is

$$[\mathsf{T}]_\beta = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Consequently, the characteristic polynomial of T is

$$\det([\mathsf{T}]_\beta - tI) = \det \begin{pmatrix} -t & 1 & 0 \\ 0 & -t & 2 \\ 0 & 0 & -t \end{pmatrix} = -t^3.$$

Thus T has only one eigenvalue ($\lambda = 0$) with multiplicity 3. Solving $\mathsf{T}(f(x)) = f'(x) = \mathit{0}$ shows that $\mathsf{E}_\lambda = \mathsf{N}(\mathsf{T} - \lambda\mathsf{I}) = \mathsf{N}(\mathsf{T})$ is the subspace of $\mathsf{P}_2(R)$ consisting of the constant polynomials. So $\{1\}$ is a basis for E_λ, and therefore $\dim(\mathsf{E}_\lambda) = 1$. Consequently, there is no basis for $\mathsf{P}_2(R)$ consisting of eigenvectors of T, and therefore T is not diagonalizable. ◆

Example 4

Let T be the linear operator on R^3 defined by

$$\mathsf{T}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 4a_1 & & + \ a_3 \\ 2a_1 & + 3a_2 & + 2a_3 \\ a_1 & & + 4a_3 \end{pmatrix}.$$

We determine the eigenspace of T corresponding to each eigenvalue. Let β be the standard ordered basis for R^3. Then

$$[\mathsf{T}]_\beta = \begin{pmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{pmatrix},$$

and hence the characteristic polynomial of T is

$$\det([\mathsf{T}]_\beta - tI) = \det \begin{pmatrix} 4-t & 0 & 1 \\ 2 & 3-t & 2 \\ 1 & 0 & 4-t \end{pmatrix} = -(t-5)(t-3)^2.$$

So the eigenvalues of T are $\lambda_1 = 5$ and $\lambda_2 = 3$ with multiplicities 1 and 2, respectively.

Since

$$\mathsf{E}_{\lambda_1} = \mathsf{N}(\mathsf{T} - \lambda_1\mathsf{I}) = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathsf{R}^3 : \begin{pmatrix} -1 & 0 & 1 \\ 2 & -2 & 2 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\},$$

E_{λ_1} is the solution space of the system of linear equations

$$\begin{aligned} -x_1 \quad\quad\; + \; x_3 &= 0 \\ 2x_1 - 2x_2 + 2x_3 &= 0 \\ x_1 \quad\quad\; - \; x_3 &= 0. \end{aligned}$$

It is easily seen (using the techniques of Chapter 3) that

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right\}$$

is a basis for E_{λ_1}. Hence $\dim(\mathsf{E}_{\lambda_1}) = 1$.

Similarly, $\mathsf{E}_{\lambda_2} = \mathsf{N}(\mathsf{T} - \lambda_2 \mathsf{I})$ is the solution space of the system

$$\begin{aligned} x_1 + \; x_3 &= 0 \\ 2x_1 + 2x_3 &= 0 \\ x_1 + \; x_3 &= 0. \end{aligned}$$

Since the unknown x_2 does not appear in this system, we assign it a parametric value, say, $x_2 = s$, and solve the system for x_1 and x_3, introducing another parameter t. The result is the general solution to the system

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = s \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \quad \text{for } s, t \in R.$$

It follows that

$$\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

is a basis for E_{λ_2}, and $\dim(\mathsf{E}_{\lambda_2}) = 2$.

In this case, the multiplicity of each eigenvalue λ_i is equal to the dimension of the corresponding eigenspace E_{λ_i}. Observe that the union of the two bases just derived, namely,

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\},$$

is linearly independent and hence is a basis for R^3 consisting of eigenvectors of T. Consequently, T is diagonalizable. ♦

Examples 3 and 4 suggest that an operator whose characteristic polynomial splits is diagonalizable if and only if the dimension of each eigenspace is equal to the multiplicity of the corresponding eigenvalue. This is indeed true, as we now show. We begin with the following lemma, which is a slight variation of Theorem 5.5.

Lemma. *Let* T *be a linear operator, and let* $\lambda_1, \lambda_2, \ldots, \lambda_k$ *be distinct eigenvalues of* T. *For each* $i = 1, 2, \ldots, k$, *let* $v_i \in \mathsf{E}_{\lambda_i}$, *the eigenspace corresponding to* λ_i. *If*

$$v_1 + v_2 + \cdots + v_k = \mathit{0},$$

then $v_i = \mathit{0}$ *for all* i.

Proof. Suppose otherwise. By renumbering if necessary, suppose that, for $1 \leq m \leq k$, we have $v_i \neq \mathit{0}$ for $1 \leq i \leq m$, and $v_i = \mathit{0}$ for $i > m$. Then, for each $i \leq m$, v_i is an eigenvector of T corresponding to λ_i and

$$v_1 + v_2 + \cdots + v_m = \mathit{0}.$$

But this contradicts Theorem 5.5, which states that these v_i's are linearly independent. We conclude, therefore, that $v_i = \mathit{0}$ for all i. ∎

Theorem 5.8. *Let* T *be a linear operator on a vector space* V, *and let* $\lambda_1, \lambda_2, \ldots, \lambda_k$ *be distinct eigenvalues of* T. *For each* $i = 1, 2, \ldots, k$, *let* S_i *be a finite linearly independent subset of the eigenspace* E_{λ_i}. *Then* $S = S_1 \cup S_2 \cup \cdots \cup S_k$ *is a linearly independent subset of* V.

Proof. Suppose that for each i

$$S_i = \{v_{i1}, v_{i2}, \ldots, v_{in_i}\}.$$

Then $S = \{v_{ij} \colon 1 \leq j \leq n_i, \text{ and } 1 \leq i \leq k\}$. Consider any scalars $\{a_{ij}\}$ such that

$$\sum_{i=1}^{k} \sum_{j=1}^{n_i} a_{ij} v_{ij} = \mathit{0}.$$

For each i, let

$$w_i = \sum_{j=1}^{n_i} a_{ij} v_{ij}.$$

Then $w_i \in \mathsf{E}_{\lambda_i}$ for each i, and $w_1 + \cdots + w_k = \mathit{0}$. Therefore, by the lemma, $w_i = \mathit{0}$ for all i. But each S_i is linearly independent, and hence $a_{ij} = 0$ for all j. We conclude that S is linearly independent. ∎

Theorem 5.8 tells us how to construct a linearly independent subset of eigenvectors, namely, by collecting bases for the individual eigenspaces. The next theorem tells us when the resulting set is a basis for the entire space.

Theorem 5.9. *Let* T *be a linear operator on a finite-dimensional vector space* V *such that the characteristic polynomial of* T *splits. Let* $\lambda_1, \lambda_2, \ldots, \lambda_k$ *be the distinct eigenvalues of* T. *Then*

(a) T *is diagonalizable if and only if the multiplicity of* λ_i *is equal to* $\dim(\mathsf{E}_{\lambda_i})$ *for all* i.

(b) *If* T *is diagonalizable and* β_i *is an ordered basis for* E_{λ_i} *for each* i, *then* $\beta = \beta_1 \cup \beta_2 \cup \cdots \cup \beta_k$ *is an ordered basis*[2] *for* V *consisting of eigenvectors of* T.

Proof. For each i, let m_i denote the multiplicity of λ_i, $d_i = \dim(\mathsf{E}_{\lambda_i})$, and $n = \dim(\mathsf{V})$.

First, suppose that T is diagonalizable. Let β be a basis for V consisting of eigenvectors of T. For each i, let $\beta_i = \beta \cap \mathsf{E}_{\lambda_i}$, the set of vectors in β that are eigenvectors corresponding to λ_i, and let n_i denote the number of vectors in β_i. Then $n_i \leq d_i$ for each i because β_i is a linearly independent subset of a subspace of dimension d_i, and $d_i \leq m_i$ by Theorem 5.7. The n_i's sum to n because β contains n vectors. The m_i's also sum to n because the degree of the characteristic polynomial of T is equal to the sum of the multiplicities of the eigenvalues. Thus

$$n = \sum_{i=1}^{k} n_i \leq \sum_{i=1}^{k} d_i \leq \sum_{i=1}^{k} m_i = n.$$

It follows that

$$\sum_{i=1}^{k} (m_i - d_i) = 0.$$

Since $(m_i - d_i) \geq 0$ for all i, we conclude that $m_i = d_i$ for all i.

Conversely, suppose that $m_i = d_i$ for all i. We simultaneously show that T is diagonalizable and prove (b). For each i, let β_i be an ordered basis for E_{λ_i}, and let $\beta = \beta_1 \cup \beta_2 \cup \cdots \cup \beta_k$. By Theorem 5.8, β is linearly independent. Furthermore, since $d_i = m_i$ for all i, β contains

$$\sum_{i=1}^{k} d_i = \sum_{i=1}^{k} m_i = n$$

[2]We regard $\beta_1 \cup \beta_2 \cup \cdots \cup \beta_k$ as an ordered basis in the natural way—the vectors in β_1 are listed first (in the same order as in β_1), then the vectors in β_2 (in the same order as in β_2), etc.

vectors. Therefore β is an ordered basis for V consisting of eigenvectors of V, and we conclude that T is diagonalizable. ∎

This theorem completes our study of the diagonalization problem. We summarize our results.

Test for Diagonalization

Let T be a linear operator on an n-dimensional vector space V. Then T is diagonalizable if and only if both of the following conditions hold.

1. The characteristic polynomial of T splits.
2. For each eigenvalue λ of T, the multiplicity of λ equals $n - \operatorname{rank}(\mathsf{T} - \lambda\mathsf{I})$.

These same conditions can be used to test if a square matrix A is diagonalizable because diagonalizability of A is equivalent to diagonalizability of the operator L_A.

If T is a diagonalizable operator and $\beta_1, \beta_2, \ldots, \beta_k$ are ordered bases for the eigenspaces of T, then the union $\beta = \beta_1 \cup \beta_2 \cup \cdots \cup \beta_k$ is an ordered basis for V consisting of eigenvectors of T, and hence $[\mathsf{T}]_\beta$ is a diagonal matrix.

When testing T for diagonalizability, it is usually easiest to choose a convenient basis α for V and work with $B = [\mathsf{T}]_\alpha$. If the characteristic polynomial of B splits, then use condition 2 above to check if the multiplicity of each *repeated* eigenvalue of B equals $n - \operatorname{rank}(B - \lambda I)$. (By Theorem 5.7, condition 2 is automatically satisfied for eigenvalues of multiplicity 1.) If so, then B, and hence T, is diagonalizable.

If T is diagonalizable and a basis β for V consisting of eigenvectors of T is desired, then we first find a basis for each eigenspace of B. The union of these bases is a basis γ for F^n consisting of eigenvectors of B. Each vector in γ is the coordinate vector relative to α of an eigenvector of T. The set consisting of these n eigenvectors of T is the desired basis β.

Furthermore, if A is an $n \times n$ diagonalizable matrix, we can use the corollary to Theorem 2.23 (p. 115) to find an invertible $n \times n$ matrix Q and a diagonal $n \times n$ matrix D such that $Q^{-1}AQ = D$. The matrix Q has as its columns the vectors in a basis of eigenvectors of A, and D has as its jth diagonal entry the eigenvalue of A corresponding to the jth column of Q.

We now consider some examples illustrating the preceding ideas.

Example 5

We test the matrix

$$A = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix} \in \mathsf{M}_{3\times 3}(R)$$

for diagonalizability.

The characteristic polynomial of A is $\det(A-tI) = -(t-4)(t-3)^2$, which splits, and so condition 1 of the test for diagonalization is satisfied. Also A has eigenvalues $\lambda_1 = 4$ and $\lambda_2 = 3$ with multiplicities 1 and 2, respectively. Since λ_1 has multiplicity 1, condition 2 is satisfied for λ_1. Thus we need only test condition 2 for λ_2. Because

$$A - \lambda_2 I = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

has rank 2, we see that $3 - \operatorname{rank}(A - \lambda_2 I) = 1$, which is not the multiplicity of λ_2. Thus condition 2 fails for λ_2, and A is therefore not diagonalizable. ◆

Example 6

Let T be the linear operator on $\mathsf{P}_2(R)$ defined by

$$\mathsf{T}(f(x)) = f(1) + f'(0)x + (f'(0) + f''(0))x^2.$$

We first test T for diagonalizability. Let α denote the standard ordered basis for $\mathsf{P}_2(R)$ and $B = [\mathsf{T}]_\alpha$. Then

$$B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}.$$

The characteristic polynomial of B, and hence of T, is $-(t-1)^2(t-2)$, which splits. Hence condition 1 of the test for diagonalization is satisfied. Also B has the eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 2$ with multiplicities 2 and 1, respectively. Condition 2 is satisfied for λ_2 because it has multiplicity 1. So we need only verify condition 2 for $\lambda_1 = 1$. For this case,

$$3 - \operatorname{rank}(B - \lambda_1 \mathsf{I}) = 3 - \operatorname{rank}\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} = 3 - 1 = 2,$$

which is equal to the multiplicity of λ_1. Therefore T is diagonalizable.

We now find an ordered basis γ for R^3 of eigenvectors of B. We consider each eigenvalue separately.

The eigenspace corresponding to $\lambda_1 = 1$ is

$$\mathsf{E}_{\lambda_1} = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathsf{R}^3 \colon \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \mathit{0} \right\},$$

which is the solution space for the system

$$x_2 + x_3 = 0,$$

and has

$$\gamma_1 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \right\}$$

as a basis.

The eigenspace corresponding to $\lambda_2 = 2$ is

$$\mathsf{E}_{\lambda_2} = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathsf{R}^3 \colon \begin{pmatrix} -1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \mathit{0} \right\},$$

which is the solution space for the system

$$\begin{aligned} -x_1 + x_2 + x_3 &= 0 \\ x_2 \qquad &= 0, \end{aligned}$$

and has

$$\gamma_2 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

as a basis.

Let

$$\gamma = \gamma_1 \cup \gamma_2 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Then γ is an ordered basis for R^3 consisting of eigenvectors of B.

Finally, observe that the vectors in γ are the coordinate vectors relative to α of the vectors in the set

$$\beta = \{1, -x + x^2, 1 + x^2\},$$

which is an ordered basis for $\mathsf{P}_2(R)$ consisting of eigenvectors of T. Thus

$$[\mathsf{T}]_\beta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}. \quad \blacklozenge$$

Our next example is an application of diagonalization that is of interest in Section 5.3.

Example 7

Let

$$A = \begin{pmatrix} 0 & -2 \\ 1 & 3 \end{pmatrix}.$$

We show that A is diagonalizable and find a 2×2 matrix Q such that $Q^{-1}AQ$ is a diagonal matix. We then show how to use this result to compute A^n for any positive integer n.

First observe that the characteristic polynomial of A is $(t-1)(t-2)$, and hence A has two distinct eigenvalues, $\lambda_1 = 1$ and $\lambda_2 = 2$. By applying the corollary to Theorem 5.5 to the operator L_A, we see that A is diagonalizable. Moreover,

$$\gamma_1 = \left\{ \begin{pmatrix} -2 \\ 1 \end{pmatrix} \right\} \quad \text{and} \quad \gamma_2 = \left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$$

are bases for the eigenspaces E_{λ_1} and E_{λ_2}, respectively. Therefore

$$\gamma = \gamma_1 \cup \gamma_2 = \left\{ \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$$

is an ordered basis for R^2 consisting of eigenvectors of A. Let

$$Q = \begin{pmatrix} -2 & -1 \\ 1 & 1 \end{pmatrix},$$

the matrix whose columns are the vectors in γ. Then, by the corollary to Theorem 2.23 (p. 115),

$$D = Q^{-1}AQ = [\mathsf{L}_A]_\beta = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

To find A^n for any positive integer n, observe that $A = QDQ^{-1}$. Therefore

$$\begin{aligned} A^n &= (QDQ^{-1})^n \\ &= (QDQ^{-1})(QDQ^{-1}) \cdots (QDQ^{-1}) \\ &= QD^nQ^{-1} \\ &= Q \begin{pmatrix} 1^n & 0 \\ 0 & 2^n \end{pmatrix} Q^{-1} \\ &= \begin{pmatrix} -2 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2^n \end{pmatrix} \begin{pmatrix} -1 & -1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 2-2^n & 2-2^{n+1} \\ -1+2^n & -1+2^{n+1} \end{pmatrix}. \end{aligned}$$ ◆

We now consider an application that uses diagonalization to solve a system of differential equations.

Systems of Differential Equations

Consider the system of differential equations

$$\begin{aligned} x_1' &= 3x_1 + x_2 + x_3 \\ x_2' &= 2x_1 + 4x_2 + 2x_3 \\ x_3' &= -x_1 - x_2 + x_3, \end{aligned}$$

where, for each i, $x_i = x_i(t)$ is a differentiable real-valued function of the real variable t. Clearly, this system has a solution, namely, the solution in which each $x_i(t)$ is the zero function. We determine all of the solutions to this system.

Let $x\colon R \to \mathsf{R}^3$ be the function defined by

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}.$$

The **derivative** of x, denoted x', is defined by

$$x'(t) = \begin{pmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{pmatrix}.$$

Let

$$A = \begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{pmatrix}$$

be the coefficient matrix of the given system, so that we can rewrite the system as the matrix equation $x' = Ax$.

It can be verified that for

$$Q = \begin{pmatrix} -1 & 0 & -1 \\ 0 & -1 & -2 \\ 1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix},$$

we have $Q^{-1}AQ = D$. Substitute $A = QDQ^{-1}$ into $x' = Ax$ to obtain $x' = QDQ^{-1}x$ or, equivalently, $Q^{-1}x' = DQ^{-1}x$. The function $y\colon R \to \mathsf{R}^3$ defined by $y(t) = Q^{-1}x(t)$ can be shown to be differentiable, and $y' = Q^{-1}x'$ (see Exercise 16). Hence the original system can be written as $y' = Dy$.

Since D is a diagonal matrix, the system $y' = Dy$ is easy to solve. Setting

$$y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{pmatrix},$$

we can rewrite $y' = Dy$ as

$$\begin{pmatrix} y_1'(t) \\ y_2'(t) \\ y_3'(t) \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{pmatrix} = \begin{pmatrix} 2y_1(t) \\ 2y_2(t) \\ 4y_3(t) \end{pmatrix}.$$

The three equations

$$\begin{aligned} y_1' &= 2y_1 \\ y_2' &= 2y_2 \\ y_3' &= 4y_3 \end{aligned}$$

are independent of each other, and thus can be solved individually. It is easily seen (as in Example 3 of Section 5.1) that the general solution to these equations is $y_1(t) = c_1e^{2t}$, $y_2(t) = c_2e^{2t}$, and $y_3(t) = c_3e^{4t}$, where c_1, c_2, and c_3 are arbitrary constants. Finally,

$$\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = x(t) = Qy(t) = \begin{pmatrix} -1 & 0 & -1 \\ 0 & -1 & -2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} c_1e^{2t} \\ c_2e^{2t} \\ c_3e^{4t} \end{pmatrix}$$

$$= \begin{pmatrix} -c_1e^{2t} & & - & c_3e^{4t} \\ & - c_2e^{2t} & - & 2c_3e^{4t} \\ c_1e^{2t} & + c_2e^{2t} & + & c_3e^{4t} \end{pmatrix}$$

yields the general solution of the original system. Note that this solution can be written as

$$x(t) = e^{2t}\left[c_1\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}\right] + e^{4t}\left[c_3\begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}\right].$$

The expressions in brackets are arbitrary vectors in E_{λ_1} and E_{λ_2}, respectively, where $\lambda_1 = 2$ and $\lambda_2 = 4$. Thus the general solution of the original system is $x(t) = e^{2t}z_1 + e^{4t}z_2$, where $z_1 \in \mathsf{E}_{\lambda_1}$ and $z_2 \in \mathsf{E}_{\lambda_2}$. This result is generalized in Exercise 15.

Direct Sums*

Let T be a linear operator on a finite-dimensional vector space V. There is a way of decomposing V into simpler subspaces that offers insight into the behavior of T. This approach is especially useful in Chapter 7, where we study nondiagonalizable linear operators. In the case of diagonalizable operators, the simpler subspaces are the eigenspaces of the operator.

Definition. *Let* $\mathsf{W}_1, \mathsf{W}_2, \ldots, \mathsf{W}_k$ *be subspaces of a vector space* V. *We define the* **sum** *of these subspaces to be the set*

$$\{v_1 + v_2 + \cdots + v_k \colon v_i \in \mathsf{W}_i \text{ for } 1 \leq i \leq k\},$$

which we denote by $\mathsf{W}_1 + \mathsf{W}_2 + \cdots + \mathsf{W}_k$ or $\sum_{i=1}^{k} \mathsf{W}_i$.

It is a simple exercise to show that the sum of subspaces of a vector space is also a subspace.

Example 8

Let $\mathsf{V} = \mathsf{R}^3$, let W_1 denote the xy-plane, and let W_2 denote the yz-plane. Then $\mathsf{R}^3 = \mathsf{W}_1 + \mathsf{W}_2$ because, for any vector $(a, b, c) \in \mathsf{R}^3$, we have

$$(a, b, c) = (a, 0, 0) + (0, b, c),$$

where $(a, 0, 0) \in \mathsf{W}_1$ and $(0, b, c) \in \mathsf{W}_2$. ♦

Notice that in Example 8 the representation of (a, b, c) as a sum of vectors in W_1 and W_2 is not unique. For example, $(a, b, c) = (a, b, 0) + (0, 0, c)$ is another representation. Because we are often interested in sums for which representations are unique, we introduce a condition that assures this outcome. The definition of *direct sum* that follows is a generalization of the definition given in the exercises of Section 1.3.

Definition. *Let* $\mathsf{W}_1, \mathsf{W}_2, \ldots, \mathsf{W}_k$ *be subspaces of a vector space* V. *We call* V *the* **direct sum** *of the subspaces* $\mathsf{W}_1, \mathsf{W}_2, \ldots, \mathsf{W}_k$ *and write* $\mathsf{V} = \mathsf{W}_1 \oplus \mathsf{W}_2 \oplus \cdots \oplus \mathsf{W}_k$, *if*

$$\mathsf{V} = \sum_{i=1}^{k} \mathsf{W}_i$$

and

$$\mathsf{W}_j \cap \sum_{i \neq j} \mathsf{W}_i = \{0\} \quad \text{for each } j \ (1 \leq j \leq k).$$

Example 9

Let $\mathsf{V} = \mathsf{R}^4$, $\mathsf{W}_1 = \{(a, b, 0, 0) \colon a, b, \in R\}$, $\mathsf{W}_2 = \{(0, 0, c, 0) \colon c \in R\}$, and $\mathsf{W}_3 = \{(0, 0, 0, d) \colon d \in R\}$. For any $(a, b, c, d) \in \mathsf{V}$,

$$(a, b, c, d) = (a, b, 0, 0) + (0, 0, c, 0) + (0, 0, 0, d) \in \mathsf{W}_1 + \mathsf{W}_2 + \mathsf{W}_3.$$

Thus

$$\mathsf{V} = \sum_{i=1}^{3} \mathsf{W}_i.$$

To show that V is the direct sum of W_1, W_2, and W_3, we must prove that $\mathsf{W}_1 \cap (\mathsf{W}_2 + \mathsf{W}_3) = \mathsf{W}_2 \cap (\mathsf{W}_1 + \mathsf{W}_3) = \mathsf{W}_3 \cap (\mathsf{W}_1 + \mathsf{W}_2) = \{0\}$. But these equalities are obvious, and so $\mathsf{V} = \mathsf{W}_1 \oplus \mathsf{W}_2 \oplus \mathsf{W}_3$. ◆

Our next result contains several conditions that are equivalent to the definition of a direct sum.

Theorem 5.10. *Let $\mathsf{W}_1, \mathsf{W}_2, \ldots, \mathsf{W}_k$ be subspaces of a finite-dimensional vector space V. The following conditions are equivalent.*

(a) $\mathsf{V} = \mathsf{W}_1 \oplus \mathsf{W}_2 \oplus \cdots \oplus \mathsf{W}_k$.

(b) $\mathsf{V} = \sum_{i=1}^{k} \mathsf{W}_i$ *and, for any vectors $v_1, v_2, \ldots, v_k$ such that $v_i \in \mathsf{W}_i$ $(1 \le i \le k)$, if $v_1 + v_2 + \cdots + v_k = 0$, then $v_i = 0$ for all i.*

(c) *Each vector $v \in \mathsf{V}$ can be uniquely written as $v = v_1 + v_2 + \cdots + v_k$, where $v_i \in \mathsf{W}_i$.*

(d) *If γ_i is an ordered basis for W_i $(1 \le i \le k)$, then $\gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_k$ is an ordered basis for V.*

(e) *For each $i = 1, 2, \ldots, k$, there exists an ordered basis γ_i for W_i such that $\gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_k$ is an ordered basis for V.*

Proof. Assume (a). We prove (b). Clearly

$$\mathsf{V} = \sum_{i=1}^{k} \mathsf{W}_i.$$

Now suppose that $v_1, v_2, \ldots, v_k$ are vectors such that $v_i \in \mathsf{W}_i$ for all i and $v_1 + v_2 + \cdots + v_k = 0$. Then for any j

$$-v_j = \sum_{i \neq j} v_i \in \sum_{i \neq j} \mathsf{W}_i.$$

But $-v_j \in \mathsf{W}_j$ and hence

$$-v_j \in \mathsf{W}_j \cap \sum_{i \neq j} \mathsf{W}_i = \{0\}.$$

So $v_j = 0$, proving (b).

Now assume (b). We prove (c). Let $v \in \mathsf{V}$. By (b), there exist vectors $v_1, v_2, \ldots, v_k$ such that $v_i \in \mathsf{W}_i$ and $v = v_1 + v_2 + \cdots + v_k$. We must show that this representation is unique. Suppose also that $v = w_1 + w_2 + \cdots + w_k$, where $w_i \in \mathsf{W}_i$ for all i. Then

$$(v_1 - w_1) + (v_2 - w_2) + \cdots + (v_k - w_k) = 0.$$

But $v_i - w_i \in \mathsf{W}_i$ for all i, and therefore $v_i - w_i = 0$ for all i by (b). Thus $v_i = w_i$ for all i, proving the uniqueness of the representation.

Now assume (c). We prove (d). For each i, let γ_i be an ordered basis for W_i. Since

$$\mathsf{V} = \sum_{i=1}^{k} \mathsf{W}_i$$

by (c), it follows that $\gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_k$ generates V. To show that this set is linearly independent, consider vectors $v_{ij} \in \gamma_i$ ($j = 1, 2, \ldots, m_i$ and $i = 1, 2, \ldots, k$) and scalars a_{ij} such that

$$\sum_{i,j} a_{ij} v_{ij} = 0.$$

For each i, set

$$w_i = \sum_{j=1}^{m_i} a_{ij} v_{ij}.$$

Then for each i, $w_i \in \operatorname{span}(\gamma_i) = \mathsf{W}_i$ and

$$w_1 + w_2 + \cdots + w_k = \sum_{i,j} a_{ij} v_{ij} = 0.$$

Since $0 \in \mathsf{W}_i$ for each i and $0 + 0 + \cdots + 0 = w_1 + w_2 + \cdots + w_k$, (c) implies that $w_i = 0$ for all i. Thus

$$0 = w_i = \sum_{j=1}^{m_i} a_{ij} v_{ij}$$

for each i. But each γ_i is linearly independent, and hence $a_{ij} = 0$ for all i and j. Consequently $\gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_k$ is linearly independent and therefore is a basis for V.

Clearly (e) follows immediately from (d).

Finally, we assume (e) and prove (a). For each i, let γ_i be an ordered basis for W_i such that $\gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_k$ is an ordered basis for V. Then

$$\mathsf{V} = \operatorname{span}(\gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_k)$$

$$= \text{span}(\gamma_1) + \text{span}(\gamma_2) + \cdots + \text{span}(\gamma_k) = \sum_{i=1}^{k} \mathsf{W}_i$$

by repeated applications of Exercise 14 of Section 1.4. Fix j $(1 \leq j \leq k)$, and suppose that, for some nonzero vector $v \in \mathsf{V}$,

$$v \in \mathsf{W}_j \cap \sum_{i \neq j} \mathsf{W}_i.$$

Then

$$v \in \mathsf{W}_j = \text{span}(\gamma_j) \quad \text{and} \quad v \in \sum_{i \neq j} \mathsf{W}_i = \text{span}\left(\bigcup_{i \neq j} \gamma_i\right).$$

Hence v is a nontrivial linear combination of both γ_j and $\left(\bigcup_{i \neq j} \gamma_i\right)$, so that v can be expressed as a linear combination of $\gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_k$ in more than one way. But these representations contradict Theorem 1.8 (p. 43), and so we conclude that

$$\mathsf{W}_j \cap \sum_{i \neq j} \mathsf{W}_i = \{0\},$$

proving (a). ∎

With the aid of Theorem 5.10, we are able to characterize diagonalizability in terms of direct sums.

Theorem 5.11. *A linear operator* T *on a finite-dimensional vector space* V *is diagonalizable if and only if* V *is the direct sum of the eigenspaces of* T.

Proof. Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be the distinct eigenvalues of T.

First suppose that T is diagonalizable, and for each i choose an ordered basis γ_i for the eigenspace E_{λ_i}. By Theorem 5.9, $\gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_k$ is a basis for V, and hence V is a direct sum of the E_{λ_i}'s by Theorem 5.10.

Conversely, suppose that V is a direct sum of the eigenspaces of T. For each i, choose an ordered basis γ_i of E_{λ_i}. By Theorem 5.10, the union $\gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_k$ is a basis for V. Since this basis consists of eigenvectors of T, we conclude that T is diagonalizable. ∎

Example 10

Let T be the linear operator on R^4 defined by

$$\mathsf{T}(a, b, c, d) = (a, b, 2c, 3d).$$

It is easily seen that T is diagonalizable with eigenvalues $\lambda_1 = 1$, $\lambda_2 = 2$, and $\lambda_3 = 3$. Furthermore, the corresponding eigenspaces coincide with the subspaces W_1, W_2, and W_3 of Example 9. Thus Theorem 5.11 provides us with another proof that $\mathsf{R}^4 = \mathsf{W}_1 \oplus \mathsf{W}_2 \oplus \mathsf{W}_3$. ♦

EXERCISES

1. Label the following statements as true or false.
   (a) Any linear operator on an n-dimensional vector space that has fewer than n distinct eigenvalues is not diagonalizable.
   (b) Two distinct eigenvectors corresponding to the same eigenvalue are always linearly dependent.
   (c) If λ is an eigenvalue of a linear operator T, then each vector in E_λ is an eigenvector of T.
   (d) If λ_1 and λ_2 are distinct eigenvalues of a linear operator T, then $\mathsf{E}_{\lambda_1} \cap \mathsf{E}_{\lambda_2} = \{0\}$.
   (e) Let $A \in \mathsf{M}_{n\times n}(F)$ and $\beta = \{v_1, v_2, \ldots, v_n\}$ be an ordered basis for F^n consisting of eigenvectors of A. If Q is the $n \times n$ matrix whose jth column is v_j $(1 \le j \le n)$, then $Q^{-1}AQ$ is a diagonal matrix.
   (f) A linear operator T on a finite-dimensional vector space is diagonalizable if and only if the multiplicity of each eigenvalue λ equals the dimension of E_λ.
   (g) Every diagonalizable linear operator on a nonzero vector space has at least one eigenvalue.

   The following two items relate to the optional subsection on direct sums.

   (h) If a vector space is the direct sum of subspaces $\mathsf{W}_1, \mathsf{W}_2, \ldots, \mathsf{W}_k$, then $\mathsf{W}_i \cap \mathsf{W}_j = \{0\}$ for $i \neq j$.
   (i) If
   $$\mathsf{V} = \sum_{i=1}^{k} \mathsf{W}_i \quad \text{and} \quad \mathsf{W}_i \cap \mathsf{W}_j = \{0\} \quad \text{for } i \neq j,$$
   then $\mathsf{V} = \mathsf{W}_1 \oplus \mathsf{W}_2 \oplus \cdots \oplus \mathsf{W}_k$.

2. For each of the following matrices $A \in \mathsf{M}_{n\times n}(R)$, test A for diagonalizability, and if A is diagonalizable, find an invertible matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$.

   (a) $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}$

   (d) $\begin{pmatrix} 7 & -4 & 0 \\ 8 & -5 & 0 \\ 6 & -6 & 3 \end{pmatrix}$ (e) $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}$ (f) $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix}$

(g) $\begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{pmatrix}$

3. For each of the following linear operators T on a vector space V, test T for diagonalizability, and if T is diagonalizable, find a basis β for V such that $[\mathsf{T}]_\beta$ is a diagonal matrix.
   (a) $\mathsf{V} = \mathsf{P}_3(R)$ and T is defined by $\mathsf{T}(f(x)) = f'(x) + f''(x)$, respectively.
   (b) $\mathsf{V} = \mathsf{P}_2(R)$ and T is defined by $\mathsf{T}(ax^2 + bx + c) = cx^2 + bx + a$.
   (c) $\mathsf{V} = \mathsf{R}^3$ and T is defined by

$$\mathsf{T}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} a_2 \\ -a_1 \\ 2a_3 \end{pmatrix}.$$

   (d) $\mathsf{V} = \mathsf{P}_2(R)$ and T is defined by $\mathsf{T}(f(x)) = f(0) + f(1)(x + x^2)$.
   (e) $\mathsf{V} = \mathsf{C}^2$ and T is defined by $\mathsf{T}(z, w) = (z + iw, iz + w)$.
   (f) $\mathsf{V} = \mathsf{M}_{2\times 2}(R)$ and T is defined by $\mathsf{T}(A) = A^t$.

4. Prove the matrix version of the corollary to Theorem 5.5: If $A \in \mathsf{M}_{n\times n}(F)$ has n distinct eigenvalues, then A is diagonalizable.

5. State and prove the matrix version of Theorem 5.6.

6. (a) Justify the test for diagonalizability and the method for diagonalization stated in this section.
   (b) Formulate the results in (a) for matrices.

7. For

$$A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} \in \mathsf{M}_{2\times 2}(R),$$

find an expression for A^n, where n is an arbitrary positive integer.

8. Suppose that $A \in \mathsf{M}_{n\times n}(F)$ has two distinct eigenvalues, λ_1 and λ_2, and that $\dim(\mathsf{E}_{\lambda_1}) = n - 1$. Prove that A is diagonalizable.

9. Let T be a linear operator on a finite-dimensional vector space V, and suppose there exists an ordered basis β for V such that $[\mathsf{T}]_\beta$ is an upper triangular matrix.
   (a) Prove that the characteristic polynomial for T splits.
   (b) State and prove an analogous result for matrices.

   The converse of (a) is treated in Exercise 32 of Section 5.4.

10. Let T be a linear operator on a finite-dimensional vector space V with the distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ and corresponding multiplicities $m_1, m_2, \ldots, m_k$. Suppose that β is a basis for V such that $[\mathsf{T}]_\beta$ is an upper triangular matrix. Prove that the diagonal entries of $[\mathsf{T}]_\beta$ are $\lambda_1, \lambda_2, \ldots, \lambda_k$ and that each λ_i occurs m_i times $(1 \leq i \leq k)$.

11. Let A be an $n \times n$ matrix that is similar to an upper triangular matrix and has the distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ with corresponding multiplicities $m_1, m_2, \ldots, m_k$. Prove the following statements.

(a) $\operatorname{tr}(A) = \displaystyle\sum_{i=1}^{k} m_i \lambda_i$

(b) $\det(A) = (\lambda_1)^{m_1}(\lambda_2)^{m_2} \cdots (\lambda_k)^{m_k}$.

12. Let T be an invertible linear operator on a finite-dimensional vector space V.

(a) Recall that for any eigenvalue λ of T, λ^{-1} is an eigenvalue of T^{-1} (Exercise 8 of Section 5.1). Prove that the eigenspace of T corresponding to λ is the same as the eigenspace of T^{-1} corresponding to λ^{-1}.

(b) Prove that if T is diagonalizable, then T^{-1} is diagonalizable.

13. Let $A \in \mathsf{M}_{n\times n}(F)$. Recall from Exercise 14 of Section 5.1 that A and A^t have the same characteristic polynomial and hence share the same eigenvalues with the same multiplicities. For any eigenvalue λ of A and A^t, let E_λ and E'_λ denote the corresponding eigenspaces for A and A^t, respectively.

(a) Show by way of example that for a given common eigenvalue, these two eigenspaces need not be the same.

(b) Prove that for any eigenvalue λ, $\dim(\mathsf{E}_\lambda) = \dim(\mathsf{E}'_\lambda)$.

(c) Prove that if A is diagonalizable, then A^t is also diagonalizable.

14. Find the general solution to each system of differential equations.

(a) $\begin{aligned} x' &= x + y \\ y' &= 3x - y \end{aligned}$ (b) $\begin{aligned} x_1' &= 8x_1 + 10x_2 \\ x_2' &= -5x_1 - 7x_2 \end{aligned}$

(c) $\begin{aligned} x_1' &= x_1 \phantom{{}+x_2} + x_3 \\ x_2' &= \phantom{x_1+{}} x_2 + x_3 \\ x_3' &= \phantom{x_1+x_2+{}} 2x_3 \end{aligned}$

15. Let

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

be the coefficient matrix of the system of differential equations

$$\begin{aligned} x_1' &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ x_2' &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ &\vdots \\ x_n' &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n. \end{aligned}$$

Suppose that A is diagonalizable and that the distinct eigenvalues of A are $\lambda_1, \lambda_2, \ldots, \lambda_k$. Prove that a differentiable function $x\colon R \to \mathsf{R}^n$ is a solution to the system if and only if x is of the form

$$x(t) = e^{\lambda_1 t}z_1 + e^{\lambda_2 t}z_2 + \cdots + e^{\lambda_k t}z_k,$$

where $z_i \in \mathsf{E}_{\lambda_i}$ for $i = 1, 2, \ldots, k$. Use this result to prove that the set of solutions to the system is an n-dimensional real vector space.

16. Let $C \in \mathsf{M}_{m\times n}(R)$, and let Y be an $n \times p$ matrix of differentiable functions. Prove $(CY)' = CY'$, where $(Y')_{ij} = Y'_{ij}$ for all i, j.

Exercises 17 through 19 are concerned with *simultaneous diagonalization.*

Definitions. *Two linear operators T and U on a finite-dimensional vector space V are called* **simultaneously diagonalizable** *if there exists an ordered basis β for V such that both $[\mathsf{T}]_\beta$ and $[\mathsf{U}]_\beta$ are diagonal matrices. Similarly, $A, B \in \mathsf{M}_{n\times n}(F)$ are called* **simultaneously diagonalizable** *if there exists an invertible matrix $Q \in \mathsf{M}_{n\times n}(F)$ such that both $Q^{-1}AQ$ and $Q^{-1}BQ$ are diagonal matrices.*

17. (a) Prove that if T and U are simultaneously diagonalizable linear operators on a finite-dimensional vector space V, then the matrices $[\mathsf{T}]_\beta$ and $[\mathsf{U}]_\beta$ are simultaneously diagonalizable for any ordered basis β.
(b) Prove that if A and B are simultaneously diagonalizable matrices, then L_A and L_B are simultaneously diagonalizable linear operators.

18. (a) Prove that if T and U are simultaneously diagonalizable operators, then T and U commute (i.e., $\mathsf{TU} = \mathsf{UT}$).
(b) Show that if A and B are simultaneously diagonalizable matrices, then A and B commute.

The converses of (a) and (b) are established in Exercise 25 of Section 5.4.

19. Let T be a diagonalizable linear operator on a finite-dimensional vector space, and let m be any positive integer. Prove that T and T^m are simultaneously diagonalizable.

Exercises 20 through 23 are concerned with direct sums.

20. Let $\mathsf{W}_1, \mathsf{W}_2, \ldots, \mathsf{W}_k$ be subspaces of a finite-dimensional vector space V such that

$$\sum_{i=1}^{k} \mathsf{W}_i = \mathsf{V}.$$

Prove that V is the direct sum of $\mathsf{W}_1, \mathsf{W}_2, \ldots, \mathsf{W}_k$ if and only if

$$\dim(\mathsf{V}) = \sum_{i=1}^{k} \dim(\mathsf{W}_i).$$

21. Let V be a finite-dimensional vector space with a basis β, and let $\beta_1, \beta_2, \ldots, \beta_k$ be a partition of β (i.e., $\beta_1, \beta_2, \ldots, \beta_k$ are subsets of β such that $\beta = \beta_1 \cup \beta_2 \cup \cdots \cup \beta_k$ and $\beta_i \cap \beta_j = \varnothing$ if $i \neq j$). Prove that $\mathsf{V} = \operatorname{span}(\beta_1) \oplus \operatorname{span}(\beta_2) \oplus \cdots \oplus \operatorname{span}(\beta_k)$.

22. Let T be a linear operator on a finite-dimensional vector space V, and suppose that the distinct eigenvalues of T are $\lambda_1, \lambda_2, \ldots, \lambda_k$. Prove that

$$\operatorname{span}(\{x \in \mathsf{V} \colon x \text{ is an eigenvector of } \mathsf{T}\}) = \mathsf{E}_{\lambda_1} \oplus \mathsf{E}_{\lambda_2} \oplus \cdots \oplus \mathsf{E}_{\lambda_k}.$$

23. Let $\mathsf{W}_1, \mathsf{W}_2, \mathsf{K}_1, \mathsf{K}_2, \ldots, \mathsf{K}_p, \mathsf{M}_1, \mathsf{M}_2, \ldots, \mathsf{M}_q$ be subspaces of a vector space V such that $\mathsf{W}_1 = \mathsf{K}_1 \oplus \mathsf{K}_2 \oplus \cdots \oplus \mathsf{K}_p$ and $\mathsf{W}_2 = \mathsf{M}_1 \oplus \mathsf{M}_2 \oplus \cdots \oplus \mathsf{M}_q$. Prove that if $\mathsf{W}_1 \cap \mathsf{W}_2 = \{0\}$, then

$$\mathsf{W}_1 + \mathsf{W}_2 = \mathsf{W}_1 \oplus \mathsf{W}_2 = \mathsf{K}_1 \oplus \mathsf{K}_2 \oplus \cdots \oplus \mathsf{K}_p \oplus \mathsf{M}_1 \oplus \mathsf{M}_2 \oplus \cdots \oplus \mathsf{M}_q.$$

5.3* MATRIX LIMITS AND MARKOV CHAINS

In this section, we apply what we have learned thus far in Chapter 5 to study the *limit* of a sequence of powers $A, A^2, \ldots, A^n, \ldots$, where A is a square matrix with complex entries. Such sequences and their limits have practical applications in the natural and social sciences.

We assume familiarity with limits of sequences of real numbers. The limit of a sequence of complex numbers $\{z_m \colon m = 1, 2, \ldots\}$ can be defined in terms of the limits of the sequences of the real and imaginary parts: If $z_m = r_m + is_m$, where r_m and s_m are real numbers, and i is the imaginary number such that $i^2 = -1$, then

$$\lim_{m\to\infty} z_m = \lim_{m\to\infty} r_m + i \lim_{m\to\infty} s_m,$$

provided that $\lim\limits_{m\to\infty} r_m$ and $\lim\limits_{m\to\infty} s_m$ exist.

Definition. *Let $L, A_1, A_2, \ldots$ be $n \times p$ matrices having complex entries. The sequence $A_1, A_2, \ldots$ is said to* **converge** *to the $n \times p$ matrix L, called the* **limit** *of the sequence, if*

$$\lim_{m\to\infty} (A_m)_{ij} = L_{ij}$$

for all $1 \le i \le n$ and $1 \le j \le p$. To designate that L is the limit of the sequence, we write

$$\lim_{m\to\infty} A_m = L.$$

Example 1

If

$$A_m = \begin{pmatrix} 1 - \frac{1}{m} & \left(-\frac{3}{4}\right)^m & \frac{3m^2}{m^2+1} + i\left(\frac{2m+1}{m-1}\right) \\ \left(\frac{i}{2}\right)^m & 2 & \left(1 + \frac{1}{m}\right)^m \end{pmatrix},$$

then

$$\lim_{m\to\infty} A_m = \begin{pmatrix} 1 & 0 & 3+2i \\ 0 & 2 & e \end{pmatrix},$$

where e is the base of the natural logarithm. ◆

A simple, but important, property of matrix limits is contained in the next theorem. Note the analogy with the familiar property of limits of sequences of real numbers that asserts that if $\lim_{m\to\infty} a_m$ exists, then

$$\lim_{m\to\infty} ca_m = c\left(\lim_{m\to\infty} a_m\right).$$

Theorem 5.12. *Let $A_1, A_2, \ldots$ be a sequence of $n \times p$ matrices with complex entries that converges to the matrix L. Then for any $P \in \mathsf{M}_{r\times n}(C)$ and $Q \in \mathsf{M}_{p\times s}(C)$,*

$$\lim_{m\to\infty} PA_m = PL \quad \text{and} \quad \lim_{m\to\infty} A_mQ = LQ.$$

Proof. For any i $(1 \le i \le r)$ and j $(1 \le j \le p)$,

$$\lim_{m\to\infty} (PA_m)_{ij} = \lim_{m\to\infty} \sum_{k=1}^{n} P_{ik}(A_m)_{kj}$$

$$= \sum_{k=1}^{n} P_{ik} \cdot \lim_{m\to\infty} (A_m)_{kj} = \sum_{k=1}^{n} P_{ik} L_{kj} = (PL)_{ij}.$$

Hence $\lim_{m\to\infty} PA_m = PL$. The proof that $\lim_{m\to\infty} A_m Q = LQ$ is similar. ∎

Corollary. *Let $A \in \mathsf{M}_{n\times n}(C)$ be such that $\lim_{m\to\infty} A^m = L$. Then for any invertible matrix $Q \in \mathsf{M}_{n\times n}(C)$,*

$$\lim_{m\to\infty} (QAQ^{-1})^m = QLQ^{-1}.$$

Proof. Since

$$(QAQ^{-1})^m = (QAQ^{-1})(QAQ^{-1}) \cdots (QAQ^{-1}) = QA^mQ^{-1},$$

we have

$$\lim_{m\to\infty} (QAQ^{-1})^m = \lim_{m\to\infty} QA^mQ^{-1} = Q\left(\lim_{m\to\infty} A^m\right) Q^{-1} = QLQ^{-1}$$

by applying Theorem 5.12 twice. ∎

In the discussion that follows, we frequently encounter the set

$$S = \{\lambda \in C \colon |\lambda| < 1 \text{ or } \lambda = 1\}.$$

Geometrically, this set consists of the complex number 1 and the interior of the unit disk (the disk of radius 1 centered at the origin). This set is of interest because if λ is a complex number, then $\lim_{m\to\infty} \lambda^m$ exists if and only $\lambda \in S$. This fact, which is obviously true if λ is real, can be shown to be true for complex numbers also.

The following important result gives necessary and sufficient conditions for the existence of the type of limit under consideration.

Theorem 5.13. *Let A be a square matrix with complex entries. Then $\lim_{m\to\infty} A^m$ exists if and only if both of the following conditions hold.*

(a) *Every eigenvalue of A is contained in S.*
(b) *If 1 is an eigenvalue of A, then the dimension of the eigenspace corresponding to 1 equals the multiplicity of 1 as an eigenvalue of A.*

One proof of this theorem, which relies on the theory of Jordan canonical forms (Section 7.2), can be found in Exercise 19 of Section 7.2. A second proof, which makes use of Schur's theorem (Theorem 6.14 of Section 6.4), can be found in the article by S. H. Friedberg and A. J. Insel, "Convergence of matrix powers," *Int. J. Math. Educ. Sci. Technol.*, 1992, Vol. 23, no. 5, pp. 765-769.

The necessity of condition (a) is easily justified. For suppose that λ is an eigenvalue of A such that $\lambda \notin S$. Let v be an eigenvector of A corresponding to λ. Regarding v as an $n \times 1$ matrix, we see that

$$\lim_{m\to\infty} (A^m v) = \left(\lim_{m\to\infty} A^m\right) v = Lv$$

by Theorem 5.12, where $L = \lim_{m\to\infty} A^m$. But $\lim_{m\to\infty} (A^m v) = \lim_{m\to\infty} (\lambda^m v)$ diverges because $\lim_{m\to\infty} \lambda^m$ does not exist. Hence if $\lim_{m\to\infty} A^m$ exists, then condition (a) of Theorem 5.13 must hold.

Although we are unable to prove the necessity of condition (b) here, we consider an example for which this condition fails. Observe that the characteristic polynomial for the matrix

$$B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

is $(t-1)^2$, and hence B has eigenvalue $\lambda = 1$ with multiplicity 2. It can easily be verified that $\dim(\mathsf{E}_\lambda) = 1$, so that condition (b) of Theorem 5.13 is violated. A simple mathematical induction argument can be used to show that

$$B^m = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix},$$

and therefore that $\lim_{m\to\infty} B^m$ does not exist. We see in Chapter 7 that if A is a matrix for which condition (b) fails, then A is similar to a matrix whose upper left 2×2 submatrix is precisely this matrix B.

In most of the applications involving matrix limits, the matrix is diagonalizable, and so condition (b) of Theorem 5.13 is automatically satisfied. In this case, Theorem 5.13 reduces to the following theorem, which can be proved using our previous results.

Theorem 5.14. *Let $A \in \mathsf{M}_{n\times n}(C)$ satisfy the following two conditions.*

(i) *Every eigenvalue of A is contained in S.*

(ii) *A is diagonalizable.*

Then $\lim_{m\to\infty} A^m$ exists.

Proof. Since A is diagonalizable, there exists an invertible matrix Q such that $Q^{-1}AQ = D$ is a diagonal matrix. Suppose that

$$D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}.$$

Because $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A, condition (i) requires that for each i, either $\lambda_i = 1$ or $|\lambda_i| < 1$. Thus

$$\lim_{m\to\infty} \lambda_i{}^m = \begin{cases} 1 & \text{if } \lambda_i = 1 \\ 0 & \text{otherwise.} \end{cases}$$

But since

$$D^m = \begin{pmatrix} \lambda_1{}^m & 0 & \cdots & 0 \\ 0 & \lambda_2{}^m & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n{}^m \end{pmatrix},$$

the sequence $D, D^2, \ldots$ converges to a limit L. Hence

$$\lim_{m\to\infty} A^m = \lim_{m\to\infty} (QDQ^{-1})^m = QLQ^{-1}$$

by the corollary to Theorem 5.12. ∎

The technique for computing $\lim\limits_{m\to\infty} A^m$ used in the proof of Theorem 5.14 can be employed in actual computations, as we now illustrate. Let

$$A = \begin{pmatrix} \frac{7}{4} & -\frac{9}{4} & -\frac{15}{4} \\ \frac{3}{4} & \frac{7}{4} & \frac{3}{4} \\ \frac{3}{4} & -\frac{9}{4} & -\frac{11}{4} \end{pmatrix}.$$

Using the methods in Sections 5.1 and 5.2, we obtain

$$Q = \begin{pmatrix} 1 & 3 & -1 \\ -3 & -2 & 1 \\ 2 & 3 & -1 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix}$$

such that $Q^{-1}AQ = D$. Hence

$$\lim_{m\to\infty} A^m = \lim_{m\to\infty} (QDQ^{-1})^m = \lim_{m\to\infty} QD^mQ^{-1} = Q\left(\lim_{m\to\infty} D^m\right)Q^{-1}$$

$$= \begin{pmatrix} 1 & 3 & -1 \\ -3 & -2 & 1 \\ 2 & 3 & -1 \end{pmatrix} \left[\lim_{m\to\infty} \begin{pmatrix} 1 & 0 & 0 \\ 0 & (-\frac{1}{2})^m & 0 \\ 0 & 0 & (\frac{1}{4})^m \end{pmatrix}\right] \begin{pmatrix} -1 & 0 & 1 \\ -1 & 1 & 2 \\ -5 & 3 & 7 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 3 & -1 \\ -3 & -2 & 1 \\ 2 & 3 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 1 \\ -1 & 1 & 2 \\ -5 & 3 & 7 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ -2 & 0 & 2 \end{pmatrix}.$$

Next, we consider an application that uses the limit of powers of a matrix. Suppose that the population of a certain metropolitan area remains constant but there is a continual movement of people between the city and the suburbs. Specifically, let the entries of the following matrix A represent the probabilities that someone living in the city or in the suburbs on January 1 will be living in each region on January 1 of the next year.

$$\begin{array}{lcc} & \text{Currently living in the city} & \text{Currently living in the suburbs} \\ \text{Living next year in the city} & \multicolumn{2}{c}{\multirow{2}{*}{$\begin{pmatrix} 0.90 & 0.02 \\ 0.10 & 0.98 \end{pmatrix} = A$}} \\ \text{Living next year in the suburbs} & & \end{array}$$

For instance, the probability that someone living in the city (on January 1) will be living in the suburbs next year (on January 1) is 0.10. Notice that since the entries of A are probabilities, they are nonnegative. Moreover, the assumption of a constant population in the metropolitan area requires that the sum of the entries of each column of A be 1.

Any square matrix having these two properties (nonnegative entries and columns that sum to 1) is called a **transition matrix** or a **stochastic matrix.** For an arbitrary $n \times n$ transition matrix M, the rows and columns correspond to n **states,** and the entry M_{ij} represents the probability of moving from state j to state i in one **stage.**

In our example, there are two states (residing in the city and residing in the suburbs). So, for example, A_{21} is the probability of moving from the city to the suburbs in one stage, that is, in one year. We now determine the

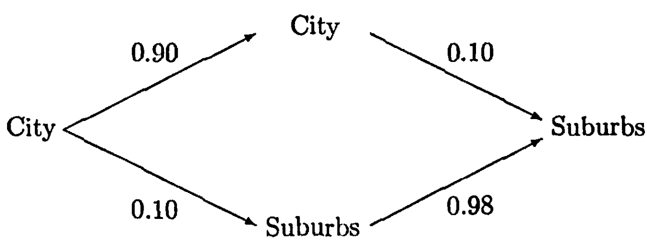

Figure 5.3

probability that a city resident will be living in the suburbs after 2 years. There are two different ways in which such a move can be made: remaining in the city for 1 year and then moving to the suburbs, or moving to the suburbs during the first year and remaining there the second year. (See

Figure 5.3.) The probability that a city dweller remains in the city for the first year is 0.90, whereas the probability that the city dweller moves to the suburbs during the first year is 0.10. Hence the probability that a city dweller stays in the city for the first year and then moves to the suburbs during the second year is the product (0.90)(0.10). Likewise, the probability that a city dweller moves to the suburbs in the first year and remains in the suburbs during the second year is the product (0.10)(0.98). Thus the probability that a city dweller will be living in the suburbs after 2 years is the sum of these products, $(0.90)(0.10) + (0.10)(0.98) = 0.188$. Observe that this number is obtained by the same calculation as that which produces $(A^2)_{21}$, and hence $(A^2)_{21}$ represents the probability that a city dweller will be living in the suburbs after 2 years. In general, for any transition matrix M, the entry $(M^m)_{ij}$ represents the probability of moving from state j to state i in m stages.

Suppose additionally that 70% of the 2000 population of the metropolitan area lived in the city and 30% lived in the suburbs. We record these data as a column vector:

$$\begin{array}{l} \text{Proportion of city dwellers} \\ \text{Proportion of suburb residents} \end{array} \quad \begin{pmatrix} 0.70 \\ 0.30 \end{pmatrix} = P.$$

Notice that the rows of P correspond to the states of residing in the city and residing in the suburbs, respectively, and that these states are listed in the same order as the listing in the transition matrix A. Observe also that the column vector P contains nonnegative entries that sum to 1; such a vector is called a **probability vector**. In this terminology, each column of a transition matrix is a probability vector. It is often convenient to regard the entries of a transition matrix or a probability vector as proportions or percentages instead of probabilities, as we have already done with the probability vector P.

In the vector AP, the first coordinate is the sum $(0.90)(0.70)+(0.02)(0.30)$. The first term of this sum, $(0.90)(0.70)$, represents the proportion of the 2000 metropolitan population that remained in the city during the next year, and the second term, $(0.02)(0.30)$, represents the proportion of the 2000 metropolitan population that moved into the city during the next year. Hence the first coordinate of AP represents the proportion of the metropolitan population that was living in the city in 2001. Similarly, the second coordinate of

$$AP = \begin{pmatrix} 0.636 \\ 0.364 \end{pmatrix}$$

represents the proportion of the metropolitan population that was living in the suburbs in 2001. This argument can be easily extended to show that the coordinates of

$$A^2P = A(AP) = \begin{pmatrix} 0.57968 \\ 0.42032 \end{pmatrix}$$

represent the proportions of the metropolitan population that were living in each location in 2002. In general, the coordinates of A^mP represent the *proportion of the metropolitan population that will be living in the city and* suburbs, respectively, after m stages (m years after 2000).

Will the city eventually be depleted if this trend continues? In view of the preceding discussion, it is natural to define the eventual proportion of the city dwellers and suburbanites to be the first and second coordinates, respectively, of $\lim\limits_{m\to\infty} A^mP$. We now compute this limit. It is easily shown that A is diagonalizable, and so there is an invertible matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$. In fact,

$$Q = \begin{pmatrix} \frac{1}{6} & -\frac{1}{6} \\ \frac{5}{6} & \frac{1}{6} \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 1 & 0 \\ 0 & 0.88 \end{pmatrix}.$$

Therefore

$$L = \lim_{m\to\infty} A^m = \lim_{m\to\infty} QD^mQ^{-1} = Q\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}Q^{-1} = \begin{pmatrix} \frac{1}{6} & \frac{1}{6} \\ \frac{5}{6} & \frac{5}{6} \end{pmatrix}.$$

Consequently

$$\lim_{m\to\infty} A^mP = LP = \begin{pmatrix} \frac{1}{6} \\ \frac{5}{6} \end{pmatrix}.$$

Thus, eventually, $\frac{1}{6}$ of the population will live in the city and $\frac{5}{6}$ will live in the suburbs each year. Note that the vector LP satisfies $A(LP) = LP$. Hence LP is both a probability vector and an eigenvector of A corresponding to the eigenvalue 1. Since the eigenspace of A corresponding to the eigenvalue 1 is one-dimensional, there is only one such vector, and LP is independent of the initial choice of probability vector P. (See Exercise 15.) For example, had the 2000 metropolitan population consisted entirely of city dwellers, the limiting outcome would be the same.

In analyzing the city–suburb problem, we gave probabilistic interpretations of A^2 and AP, showing that A^2 is a transition matrix and AP is a probability vector. In fact, the product of any two transition matrices is a transition matrix, and the product of any transition matrix and probability vector is a probability vector. A proof of these facts is a simple corollary of the next theorem, which characterizes transition matrices and probability vectors.

Theorem 5.15. *Let M be an $n\times n$ matrix having real nonnegative entries, let v be a column vector in R^n having nonnegative coordinates, and let $u \in \mathsf{R}^n$ be the column vector in which each coordinate equals 1. Then*

(a) *M is a transition matrix if and only if $M^t u = u$;*
(b) *v is a probability vector if and only if $u^t v = (1)$.*

Proof. Exercise. ∎

Corollary.
(a) *The product of two $n \times n$ transition matrices is an $n \times n$ transition matrix. In particular, any power of a transition matrix is a transition matrix.*
(b) *The product of a transition matrix and a probability vector is a probability vector.*

Proof. Exercise. ∎

The city–suburb problem is an example of a process in which elements of a set are each classified as being in one of several fixed states that can switch over time. In general, such a process is called a **stochastic process**. The switching to a particular state is described by a probability, and in general this probability depends on such factors as the state in question, the time in question, some or all of the previous states in which the object has been (including the current state), and the states that other objects are in or have been in.

For instance, the object could be an American voter, and the state of the object could be his or her preference of political party; or the object could be a molecule of $\mathbf{H_2O}$, and the states could be the three physical states in which $\mathbf{H_2O}$ can exist (solid, liquid, and gas). In these examples, all four of the factors mentioned above influence the probability that an object is in a particular state at a particular time.

If, however, the probability that an object in one state changes to a different state in a fixed interval of time depends only on the two states (and not on the time, earlier states, or other factors), then the stochastic process is called a **Markov process**. If, in addition, the number of possible states is finite, then the Markov process is called a **Markov chain**. We treated the city–suburb example as a two-state Markov chain. Of course, a Markov process is usually only an idealization of reality because the probabilities involved are almost never constant over time.

With this in mind, we consider another Markov chain. A certain community college would like to obtain information about the likelihood that students in various categories will graduate. The school classifies a student as a sophomore or a freshman depending on the number of credits that the student has earned. Data from the school indicate that, from one fall semester to the next, 40% of the sophomores will graduate, 30% will remain sophomores, and 30% will quit permanently. For freshmen, the data show that 10% will graduate by next fall, 50% will become sophomores, 20% will remain freshmen, and 20% will quit permanently. During the present year, 50% of the students at the school are sophomores and 50% are freshmen. Assuming that the trend indicated by the data continues indefinitely, the school would like to know

1. the percentage of the present students who will graduate, the percentage who will be sophomores, the percentage who will be freshmen, and the percentage who will quit school permanently by next fall;
2. the same percentages as in item 1 for the fall semester two years hence; and
3. the probability that one of its present students will eventually graduate.

The preceding paragraph describes a four-state Markov chain with the following states:

1. having graduated
2. being a sophomore
3. being a freshman
4. having quit permanently.

The given data provide us with the transition matrix

$$A = \begin{pmatrix} 1 & 0.4 & 0.1 & 0 \\ 0 & 0.3 & 0.5 & 0 \\ 0 & 0 & 0.2 & 0 \\ 0 & 0.3 & 0.2 & 1 \end{pmatrix}$$

of the Markov chain. (Notice that students who have graduated or have quit permanently are assumed to remain indefinitely in those respective states. Thus a freshman who quits the school and returns during a later semester is not regarded as having changed states—the student is assumed to have remained in the state of being a freshman during the time he or she was not enrolled.) Moreover, we are told that the present distribution of students is half in each of states 2 and 3 and none in states 1 and 4. The vector

$$P = \begin{pmatrix} 0 \\ 0.5 \\ 0.5 \\ 0 \end{pmatrix}$$

that describes the initial probability of being in each state is called the **initial probability vector** for the Markov chain.

To answer question 1, we must determine the probabilities that a present student will be in each state by next fall. As we have seen, these probabilities are the coordinates of the vector

$$AP = \begin{pmatrix} 1 & 0.4 & 0.1 & 0 \\ 0 & 0.3 & 0.5 & 0 \\ 0 & 0 & 0.2 & 0 \\ 0 & 0.3 & 0.2 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0.5 \\ 0.5 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.25 \\ 0.40 \\ 0.10 \\ 0.25 \end{pmatrix}.$$

Hence by next fall, 25% of the present students will graduate, 40% will be sophomores, 10% will be freshmen, and 25% will quit the school permanently. Similarly,

$$A^2P = A(AP) = \begin{pmatrix} 1 & 0.4 & 0.1 & 0 \\ 0 & 0.3 & 0.5 & 0 \\ 0 & 0 & 0.2 & 0 \\ 0 & 0.3 & 0.2 & 1 \end{pmatrix} \begin{pmatrix} 0.25 \\ 0.40 \\ 0.10 \\ 0.25 \end{pmatrix} = \begin{pmatrix} 0.42 \\ 0.17 \\ 0.02 \\ 0.39 \end{pmatrix}$$

provides the information needed to answer question 2: within two years 42% of the present students will graduate, 17% will be sophomores, 2% will be freshmen, and 39% will quit school.

Finally, the answer to question 3 is provided by the vector LP, where $L = \lim_{m\to\infty} A^m$. For the matrices

$$Q = \begin{pmatrix} 1 & 4 & 19 & 0 \\ 0 & -7 & -40 & 0 \\ 0 & 0 & 8 & 0 \\ 0 & 3 & 13 & 1 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.3 & 0 & 0 \\ 0 & 0 & 0.2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

we have $Q^{-1}AQ = D$. Thus

$$L = \lim_{m\to\infty} A^m = Q\left(\lim_{m\to\infty} D^m\right)Q^{-1}$$

$$= \begin{pmatrix} 1 & 4 & 19 & 0 \\ 0 & -7 & -40 & 0 \\ 0 & 0 & 8 & 0 \\ 0 & 3 & 13 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{4}{7} & \frac{27}{56} & 0 \\ 0 & -\frac{1}{7} & -\frac{5}{7} & 0 \\ 0 & 0 & \frac{1}{8} & 0 \\ 0 & \frac{3}{7} & \frac{29}{56} & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \frac{4}{7} & \frac{27}{56} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{3}{7} & \frac{29}{56} & 1 \end{pmatrix}.$$

So

$$LP = \begin{pmatrix} 1 & \frac{4}{7} & \frac{27}{56} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{3}{7} & \frac{29}{56} & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0.5 \\ 0.5 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{59}{112} \\ 0 \\ 0 \\ \frac{53}{112} \end{pmatrix},$$

and hence the probability that one of the present students will graduate is $\frac{59}{112}$.

In the preceding two examples, we saw that $\lim_{m\to\infty} A^m P$, where A is the transition matrix and P is the initial probability vector of the Markov chain, gives the eventual proportions in each state. In general, however, the limit of powers of a transition matrix need not exist. For example, if

$$M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

then $\lim_{m\to\infty} M^m$ does not exist because odd powers of M equal M and even powers of M equal I. The reason that the limit fails to exist is that condition (a) of Theorem 5.13 does not hold for M (-1 is an eigenvalue). In fact, it can be shown (see Exercise 20 of Section 7.2) that the only transition matrices A such that $\lim_{m\to\infty} A^m$ does not exist are precisely those matrices for which condition (a) of Theorem 5.13 fails to hold.

But even if the limit of powers of the transition matrix exists, the computation of the limit may be quite difficult. (The reader is encouraged to work Exercise 6 to appreciate the truth of the last sentence.) Fortunately, there is a large and important class of transition matrices for which this limit exists and is easily computed—this is the class of *regular* transition matrices.

Definition. *A transition matrix is called* **regular** *if some power of the matrix contains only positive entries.*

Example 2

The transition matrix

$$\begin{pmatrix} 0.90 & 0.02 \\ 0.10 & 0.98 \end{pmatrix}$$

of the Markov chain used in the city–suburb problem is clearly regular because each entry is positive. On the other hand, the transition matrix

$$A = \begin{pmatrix} 1 & 0.4 & 0.1 & 0 \\ 0 & 0.3 & 0.5 & 0 \\ 0 & 0 & 0.2 & 0 \\ 0 & 0.3 & 0.2 & 1 \end{pmatrix}$$

of the Markov chain describing community college enrollments is not regular because the first column of A^m is

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

for any power m.

Observe that a regular transition matrix may contain zero entries. For example,

$$M = \begin{pmatrix} 0.9 & 0.5 & 0 \\ 0 & 0.5 & 0.4 \\ 0.1 & 0 & 0.6 \end{pmatrix}$$

is regular because every entry of M^2 is positive. ♦

The remainder of this section is devoted to proving that, for a regular transition matrix A, the limit of the sequence of powers of A exists and has identical columns. From this fact, it is easy to compute this limit. In the course of proving this result, we obtain some interesting bounds for the magnitudes of eigenvalues of any square matrix. These bounds are given in terms of the sum of the absolute values of the rows and columns of the matrix. The necessary terminology is introduced in the definitions that follow.

Definitions. *Let $A \in \mathsf{M}_{n\times n}(C)$. For $1 \le i, j \le n$, define $\rho_i(A)$ to be the sum of the absolute values of the entries of row i of A, and define $\nu_j(A)$ to be equal to the sum of the absolute values of the entries of column j of A. Thus*

$$\rho_i(A) = \sum_{j=1}^{n} |A_{ij}| \quad \text{for } i = 1, 2, \ldots n$$

and

$$\nu_j(A) = \sum_{i=1}^{n} |A_{ij}| \quad \text{for } j = 1, 2, \ldots n.$$

The **row sum** *of A, denoted $\rho(A)$, and the* **column sum** *of A, denoted $\nu(A)$, are defined as*

$$\rho(A) = \max\{\rho_i(A)\colon 1 \le i \le n\} \quad \text{and} \quad \nu(A) = \max\{\nu_j(A)\colon 1 \le j \le n\}.$$

Example 3

For the matrix

$$A = \begin{pmatrix} 1 & -i & 3-4i \\ -2+i & 0 & 6 \\ 3 & 2 & i \end{pmatrix},$$

$\rho_1(A) = 7$, $\rho_2(A) = 6 + \sqrt{5}$, $\rho_3(A) = 6$, $\nu_1(A) = 4 + \sqrt{5}$, $\nu_2(A) = 3$, and $\nu_3(A) = 12$. Hence $\rho(A) = 6 + \sqrt{5}$ and $\nu(A) = 12$. ♦

Our next results show that the smaller of $\rho(A)$ and $\nu(A)$ is an upper bound for the absolute values of eigenvalues of A. In the preceding example, for instance, A has no eigenvalue with absolute value greater than $6+\sqrt{5}$.

To obtain a geometric view of the following theorem, we introduce some terminology. For an $n\times n$ matrix A, we define the ith **Gerschgorin disk** C_i to be the disk in the complex plane with center A_{ii} and radius $r_i = \rho_i(A) - |A_{ii}|$; that is,

$$C_i = \{z \in C\colon |z - A_{ii}| < r_i\}.$$

For example, consider the matrix

$$A = \begin{pmatrix} 1+2i & 1 \\ 2i & -3 \end{pmatrix}.$$

For this matrix, C_1 is the disk with center $1+2i$ and radius 1, and C_2 is the disk with center -3 and radius 2. (See Figure 5.4.)

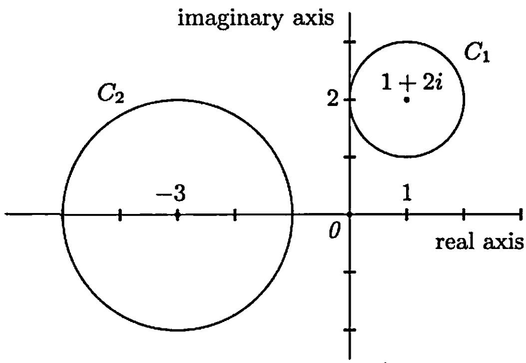

Figure 5.4

Gershgorin's disk theorem, stated below, tells us that all the eigenvalues of A are located within these two disks. In particular, we see that 0 is *not* an eigenvalue, and hence by Exercise 8(c) of section 5.1, A is invertible.

Theorem 5.16 (Gerschgorin's Disk Theorem). *Let $A \in \mathsf{M}_{n\times n}(C)$. Then every eigenvalue of A is contained in a Gerschgorin disk.*

Proof. Let λ be an eigenvalue of A with the corresponding eigenvector

$$v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}.$$

Then v satisfies the matrix equation $Av = \lambda v$, which can be written

$$\sum_{j=1}^{n} A_{ij} v_j = \lambda v_i \quad (i = 1, 2, \ldots, n). \tag{2}$$

Suppose that v_k is the coordinate of v having the largest absolute value; note that $v_k \neq 0$ because v is an eigenvector of A.

We show that λ lies in C_k, that is, $|\lambda - A_{kk}| \leq r_k$. For $i = k$, it follows from (2) that

$$\begin{aligned}
|\lambda v_k - A_{kk} v_k| &= \left| \sum_{j=1}^{n} A_{kj} v_j - A_{kk} v_k \right| = \left| \sum_{j \neq k} A_{kj} v_j \right| \\
&\leq \sum_{j \neq k} |A_{kj}| |v_j| \leq \sum_{j \neq k} |A_{kj}| |v_k| \\
&= |v_k| \sum_{j \neq k} |A_{kj}| = |v_k| r_k.
\end{aligned}$$

Thus

$$|v_k| |\lambda - A_{kk}| \leq |v_k| r_k;$$

so

$$|\lambda - A_{kk}| \leq r_k$$

because $|v_k| > 0$. ∎

Corollary 1. *Let λ be any eigenvalue of $A \in \mathsf{M}_{n \times n}(C)$. Then $|\lambda| \leq \rho(A)$.*

Proof. By Gerschgorin's disk theorem, $|\lambda - A_{kk}| \leq r_k$ for some k. Hence

$$\begin{aligned}
|\lambda| &= |(\lambda - A_{kk}) + A_{kk}| \leq |\lambda - A_{kk}| + |A_{kk}| \\
&\leq r_k + |A_{kk}| = \rho_k(A) \leq \rho(A).
\end{aligned}$$ ∎

Corollary 2. *Let λ be any eigenvalue of $A \in \mathsf{M}_{n \times n}(C)$. Then*

$$|\lambda| \leq \min\{\rho(A), \nu(A)\}.$$

Proof. Since $|\lambda| \leq \rho(A)$ by Corollary 1, it suffices to show that $|\lambda| \leq \nu(A)$. By Exercise 14 of Section 5.1, λ is an eigenvalue of A^t, and so $|\lambda| \leq \rho(A^t)$ by Corollary 1. But the rows of A^t are the columns of A; consequently $\rho(A^t) = \nu(A)$. Therefore $|\lambda| \leq \nu(A)$. ∎

The next corollary is immediate from Corollary 2.

Corollary 3. *If λ is an eigenvalue of a transition matrix, then $|\lambda| \leq 1$.*

The next result asserts that the upper bound in Corollary 3 is attained.

Theorem 5.17. *Every transition matrix has 1 as an eigenvalue.*

Proof. Let A be an $n \times n$ transition matrix, and let $u \in \mathsf{R}^n$ be the column vector in which each coordinate is 1. Then $A^t u = u$ by Theorem 5.15, and hence u is an eigenvector of A^t corresponding to the eigenvalue 1. But since A and A^t have the same eigenvalues, it follows that 1 is also an eigenvalue of A. ∎

Suppose that A is a transition matrix for which some eigenvector corresponding to the eigenvalue 1 has only nonnegative coordinates. Then some multiple of this vector is a probability vector P as well as an eigenvector of A corresponding to eigenvalue 1. It is interesting to observe that if P is the initial probability vector of a Markov chain having A as its transition matrix, then the Markov chain is completely static. For in this situation, $A^m P = P$ for every positive integer m; hence the probability of being in each state never changes. Consider, for instance, the city–suburb problem with

$$P = \begin{pmatrix} \frac{1}{6} \\ \frac{5}{6} \end{pmatrix}.$$

Theorem 5.18. *Let $A \in \mathsf{M}_{n\times n}(C)$ be a matrix in which each entry is positive, and let λ be an eigenvalue of A such that $|\lambda| = \rho(A)$. Then $\lambda = \rho(A)$ and $\{u\}$ is a basis for E_λ, where $u \in \mathsf{C}^n$ is the column vector in which each coordinate equals 1.*

Proof. Let v be an eigenvector of A corresponding to λ, with coordinates $v_1, v_2, \ldots, v_n$. Suppose that v_k is the coordinate of v having the largest absolute value, and let $b = |v_k|$. Then

$$\begin{aligned} |\lambda| b = |\lambda||v_k| = |\lambda v_k| &= \left| \sum_{j=1}^{n} A_{kj} v_j \right| \leq \sum_{j=1}^{n} |A_{kj} v_j| \\ &= \sum_{j=1}^{n} |A_{kj}||v_j| \leq \sum_{j=1}^{n} |A_{kj}| b = \rho_k(A) b \leq \rho(A) b. \end{aligned} \tag{3}$$

Since $|\lambda| = \rho(A)$, the three inequalities in (3) are actually equalities; that is,

(a) $\left| \sum_{j=1}^{n} A_{kj} v_j \right| = \sum_{j=1}^{n} |A_{kj} v_j|$,

(b) $\sum_{j=1}^{n} |A_{kj}||v_j| = \sum_{j=1}^{n} |A_{kj}|b$, and

(c) $\rho_k(A) = \rho(A)$.

We see in Exercise 15(b) of Section 6.1 that (a) holds if and only if all the terms $A_{kj}v_j$ $(j = 1, 2, \ldots, n)$ are nonnegative multiples of some nonzero complex number z. Without loss of generality, we assume that $|z| = 1$. Thus there exist nonnegative real numbers $c_1, c_2, \ldots, c_n$ such that

$$A_{kj}v_j = c_j z. \tag{4}$$

By (b) and the assumption that $A_{kj} \neq 0$ for all k and j, we have

$$|v_j| = b \quad \text{for } j = 1, 2, \ldots, n. \tag{5}$$

Combining (4) and (5), we obtain

$$b = |v_j| = \left| \frac{c_j}{A_{kj}} z \right| = \frac{c_j}{A_{kj}} \quad \text{for } j = 1, 2, \ldots, n,$$

and therefore by (4), we have $v_j = bz$ for all j. So

$$v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} bz \\ bz \\ \vdots \\ bz \end{pmatrix} = bzu,$$

and hence $\{u\}$ is a basis for E_λ.

Finally, observe that all of the entries of Au are positive because the same is true for the entries of both A and u. But $Au = \lambda u$, and hence $\lambda > 0$. Therefore, $\lambda = |\lambda| = \rho(A)$. ∎

Corollary 1. *Let $A \in \mathsf{M}_{n\times n}(C)$ be a matrix in which each entry is positive, and let λ be an eigenvalue of A such that $|\lambda| = \nu(A)$. Then $\lambda = \nu(A)$, and the dimension of $\mathsf{E}_\lambda = 1$.*

Proof. Exercise. ∎

Corollary 2. *Let $A \in \mathsf{M}_{n\times n}(C)$ be a transition matrix in which each entry is positive, and let λ be any eigenvalue of A other than 1. Then $|\lambda| < 1$. Moreover, the eigenspace corresponding to the eigenvalue 1 has dimension 1.*

Proof. Exercise. ∎

Our next result extends Corollary 2 to regular transition matrices and thus shows that regular transition matrices satisfy condition (a) of Theorems 5.13 and 5.14.

Theorem 5.19. *Let A be a regular transition matrix, and let λ be an eigenvalue of A. Then*

(a) $|\lambda| \leq 1$.

(b) *If* $|\lambda| = 1$, *then* $\lambda = 1$, *and* $\dim(\mathsf{E}_\lambda) = 1$.

Proof. Statement (a) was proved as Corollary 3 to Theorem 5.16.

(b) Since A is regular, there exists a positive integer s such that A^s has only positive entries. Because A is a transition matrix and the entries of A^s are positive, the entries of $A^{s+1} = A^s(A)$ are positive. Suppose that $|\lambda| = 1$. Then λ^s and λ^{s+1} are eigenvalues of A^s and A^{s+1}, respectively, having absolute value 1. So by Corollary 2 to Theorem 5.18, $\lambda^s = \lambda^{s+1} = 1$. Thus $\lambda = 1$. Let E_λ and E'_λ denote the eigenspaces of A and A^s, respectively, corresponding to $\lambda = 1$. Then $\mathsf{E}_\lambda \subseteq \mathsf{E}'_\lambda$ and, by Corollary 2 to Theorem 5.18, $\dim(\mathsf{E}'_\lambda) = 1$. Hence $\mathsf{E}_\lambda = \mathsf{E}'_\lambda$, and $\dim(\mathsf{E}_\lambda) = 1$. ∎

Corollary. *Let A be a regular transition matrix that is diagonalizable. Then* $\lim_{m\to\infty} A^m$ *exists.*

The preceding corollary, which follows immediately from Theorems 5.19 and 5.14, is not the best possible result. In fact, it can be shown that if A is a regular transition matrix, then the multiplicity of 1 as an eigenvalue of A is 1. Thus, by Theorem 5.7 (p. 264), condition (b) of Theorem 5.13 is satisfied. So if A is a regular transition matrix, $\lim_{m\to\infty} A^m$ exists regardless of whether A is or is not diagonalizable. As with Theorem 5.13, however, the fact that the multiplicity of 1 as an eigenvalue of A is 1 cannot be proved at this time. Nevertheless, we state this result here (leaving the proof until Exercise 21 of Section 7.2) and deduce further facts about $\lim_{m\to\infty} A^m$ when A is a regular transition matrix.

Theorem 5.20. *Let A be an $n \times n$ regular transition matrix. Then*

(a) *The multiplicity of* 1 *as an eigenvalue of A is* 1.

(b) $\lim_{m\to\infty} A^m$ *exists.*

(c) $L = \lim_{m\to\infty} A^m$ *is a transition matrix.*

(d) $AL = LA = L$.

(e) *The columns of L are identical. In fact, each column of L is equal to the unique probability vector v that is also an eigenvector of A corresponding to the eigenvalue* 1.

(f) *For any probability vector w,* $\lim_{m\to\infty}(A^m w) = v$.

Proof. (a) See Exercise 21 of Section 7.2.

(b) This follows from (a) and Theorems 5.19 and 5.13.

(c) By Theorem 5.15, we must show that $u^t L = u^t$. Now A^m is a transition matrix by the corollary to Theorem 5.15, so

$$u^t L = u^t \lim_{m\to\infty} A^m = \lim_{m\to\infty} u^t A^m = \lim_{m\to\infty} u^t = u^t,$$

and it follows that L is a transition matrix.

(d) By Theorem 5.12,

$$AL = A \lim_{m\to\infty} A^m = \lim_{m\to\infty} AA^m = \lim_{m\to\infty} A^{m+1} = L.$$

Similarly, $LA = L$.

(e) Since $AL = L$ by (d), each column of L is an eigenvector of A corresponding to the eigenvalue 1. Moreover, by (c), each column of L is a probability vector. Thus, by (a), each column of L is equal to the unique probability vector v corresponding to the eigenvalue 1 of A.

(f) Let w be any probability vector, and set $y = \lim_{m\to\infty} A^m w = Lw$. Then y is a probability vector by the corollary to Theorem 5.15, and also $Ay = ALw = Lw = y$ by (d). Hence y is also an eigenvector corresponding to the eigenvalue 1 of A. So $y = v$ by (e). ∎

Definition. *The vector v in Theorem 5.20(e) is called the* ***fixed probability vector*** *or* ***stationary vector*** *of the regular transition matrix A.*

Theorem 5.20 can be used to deduce information about the eventual distribution in each state of a Markov chain having a regular transition matrix.

Example 4

A survey in Persia showed that on a particular day 50% of the Persians preferred a loaf of bread, 30% preferred a jug of wine, and 20% preferred "thou beside me in the wilderness." A subsequent survey 1 month later yielded the following data: Of those who preferred a loaf of bread on the first survey, 40% continued to prefer a loaf of bread, 10% now preferred a jug of wine, and 50% preferred "thou"; of those who preferred a jug of wine on the first survey, 20% now preferred a loaf of bread, 70% continued to prefer a jug of wine, and 10% now preferred "thou"; of those who preferred "thou" on the first survey, 20% now preferred a loaf of bread, 20% now preferred a jug of wine, and 60% continued to prefer "thou."

Assuming that this trend continues, the situation described in the preceding paragraph is a three-state Markov chain in which the states are the three possible preferences. We can predict the percentage of Persians in each state for each month following the original survey. Letting the first, second, and third states be preferences for bread, wine, and "thou", respectively, we see that the probability vector that gives the initial probability of being in each state is

$$P = \begin{pmatrix} 0.50 \\ 0.30 \\ 0.20 \end{pmatrix},$$

and the transition matrix is

$$A = \begin{pmatrix} 0.40 & 0.20 & 0.20 \\ 0.10 & 0.70 & 0.20 \\ 0.50 & 0.10 & 0.60 \end{pmatrix}.$$

The probabilities of being in each state m months after the original survey are the coordinates of the vector $A^m P$. The reader may check that

$$AP = \begin{pmatrix} 0.30 \\ 0.30 \\ 0.40 \end{pmatrix}, \quad A^2P = \begin{pmatrix} 0.26 \\ 0.32 \\ 0.42 \end{pmatrix}, \quad A^3P = \begin{pmatrix} 0.252 \\ 0.334 \\ 0.414 \end{pmatrix}, \quad \text{and} \quad A^4P = \begin{pmatrix} 0.2504 \\ 0.3418 \\ 0.4078 \end{pmatrix}.$$

Note the apparent convergence of $A^m P$.

Since A is regular, the long-range prediction concerning the Persians' preferences can be found by computing the fixed probability vector for A. This vector is the unique probability vector v such that $(A - I)v = \mathit{0}$. Letting

$$v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix},$$

we see that the matrix equation $(A - I)v = \mathit{0}$ yields the following system of linear equations:

$$\begin{aligned} -0.60v_1 + 0.20v_2 + 0.20v_3 &= 0 \\ 0.10v_1 - 0.30v_2 + 0.20v_3 &= 0 \\ 0.50v_1 + 0.10v_2 - 0.40v_3 &= 0\,. \end{aligned}$$

It is easily shown that

$$\begin{pmatrix} 5 \\ 7 \\ 8 \end{pmatrix}$$

is a basis for the solution space of this system. Hence the unique fixed probability vector for A is

$$\begin{pmatrix} \frac{5}{5+7+8} \\ \frac{7}{5+7+8} \\ \frac{8}{5+7+8} \end{pmatrix} = \begin{pmatrix} 0.25 \\ 0.35 \\ 0.40 \end{pmatrix}.$$

Thus, in the long run, 25% of the Persians prefer a loaf of bread, 35% prefer a jug of wine, and 40% prefer "thou beside me in the wilderness."

Note that if

$$Q = \begin{pmatrix} 5 & 0 & -3 \\ 7 & -1 & -1 \\ 8 & 1 & 4 \end{pmatrix},$$

then

$$Q^{-1}AQ = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.2 \end{pmatrix}.$$

So

$$\lim_{m\to\infty} A^m = Q\left[\lim_{m\to\infty} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.2 \end{pmatrix}^m\right] Q^{-1} = Q \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} Q^{-1}$$

$$= \begin{pmatrix} 0.25 & 0.25 & 0.25 \\ 0.35 & 0.35 & 0.35 \\ 0.40 & 0.40 & 0.40 \end{pmatrix}. \quad \blacklozenge$$

Example 5

Farmers in Lamron plant one crop per year—either corn, soybeans, or wheat. Because they believe in the necessity of rotating their crops, these farmers do not plant the same crop in successive years. In fact, of the total acreage on which a particular crop is planted, exactly half is planted with each of the other two crops during the succeeding year. This year, 300 acres of corn, 200 acres of soybeans, and 100 acres of wheat were planted.

The situation just described is another three-state Markov chain in which the three states correspond to the planting of corn, soybeans, and wheat, respectively. In this problem, however, the amount of land devoted to each crop, rather than the percentage of the total acreage (600 acres), is given. By converting these amounts into fractions of the total acreage, we see that the transition matrix A and the initial probability vector P of the Markov chain are

$$A = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \quad \text{and} \quad P = \begin{pmatrix} \frac{300}{600} \\ \frac{200}{600} \\ \frac{100}{600} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{3} \\ \frac{1}{6} \end{pmatrix}.$$

The fraction of the total acreage devoted to each crop in m years is given by the coordinates of A^mP, and the eventual proportions of the total acreage used for each crop are the coordinates of $\lim_{m\to\infty} A^mP$. Thus the eventual amounts of land devoted to each crop are found by multiplying this limit by the total acreage; that is, the eventual amounts of land used for each crop are the coordinates of $600 \cdot \lim_{m\to\infty} A^m P$.

Since A is a regular transition matrix, Theorem 5.20 shows that $\lim_{m\to\infty} A^m$ is a matrix L in which each column equals the unique fixed probability vector for A. It is easily seen that the fixed probability vector for A is

$$\begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}.$$

Hence

$$L = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix};$$

so

$$600 \cdot \lim_{m\to\infty} A^m P = 600LP = \begin{pmatrix} 200 \\ 200 \\ 200 \end{pmatrix}.$$

Thus, in the long run, we expect 200 acres of each crop to be planted each year. (For a direct computation of $600 \cdot \lim_{m\to\infty} A^m P$, see Exercise 14.) ◆

In this section, we have concentrated primarily on the theory of regular transition matrices. There is another interesting class of transition matrices that can be represented in the form

$$\begin{pmatrix} I & B \\ O & C \end{pmatrix},$$

where I is an identity matrix and O is a zero matrix. (Such transition matrices are not regular since the lower left block remains O in any power of the matrix.) The states corresponding to the identity submatrix are called **absorbing states** because such a state is never left once it is entered. A Markov chain is called an **absorbing Markov chain** *if it is possible to go from an arbitrary state into an absorbing state in a finite number of stages.* Observe that the Markov chain that describes the enrollment pattern in a community college is an absorbing Markov chain with states 1 and 4 as its absorbing states. Readers interested in learning more about absorbing Markov chains are referred to *Introduction to Finite Mathematics* (third edition) by

J. Kemeny, J. Snell, and G. Thompson (Prentice-Hall, Inc., Englewood Cliffs, N. J., 1974) or *Discrete Mathematical Models* by Fred S. Roberts (Prentice-Hall, Inc., Englewood Cliffs, N. J., 1976).

An Application

In species that reproduce sexually, the characteristics of an offspring with respect to a particular genetic trait are determined by a pair of genes, one inherited from each parent. The genes for a particular trait are of two types, which are denoted by G and g. The gene G represents the dominant characteristic, and g represents the recessive characteristic. Offspring with genotypes GG or Gg exhibit the dominant characteristic, whereas offspring with genotype gg exhibit the recessive characteristic. For example, in humans, brown eyes are a dominant characteristic and blue eyes are the corresponding recessive characteristic; thus the offspring with genotypes GG or Gg are brown-eyed, whereas those of type gg are blue-eyed.

Let us consider the probability of offspring of each genotype for a male parent of genotype Gg. (We assume that the population under consideration is large, that mating is random with respect to genotype, and that the distribution of each genotype within the population is independent of sex and life expectancy.) Let

$$P = \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

denote the proportion of the adult population with genotypes GG, Gg, and gg, respectively, at the start of the experiment. This experiment describes a three-state Markov chain with the following transition matrix:

$$\begin{array}{lll} & & \text{Genotype of female parent} \\ & & \begin{array}{ccc} \text{GG} & \text{Gg} & \text{gg} \end{array} \\ \begin{array}{l} \text{Genotype} \\ \text{of} \\ \text{offspring} \end{array} & \begin{array}{c} \text{GG} \\ \text{Gg} \\ \text{gg} \end{array} & \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{4} & \frac{1}{2} \end{pmatrix} = B. \end{array}$$

It is easily checked that B^2 contains only positive entries; so B is regular. Thus, by permitting only males of genotype Gg to reproduce, the proportion of offspring in the population having a certain genotype will stabilize at the fixed probability vector for B, which is

$$\begin{pmatrix} \frac{1}{4} \\ \frac{1}{2} \\ \frac{1}{4} \end{pmatrix}.$$

Now suppose that similar experiments are to be performed with males of genotypes GG and gg. As already mentioned, these experiments are three-state Markov chains with transition matrices

$$A = \begin{pmatrix} 1 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 0 & 0 & 0 \\ 1 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 1 \end{pmatrix},$$

respectively. In order to consider the case where all male genotypes are permitted to reproduce, we must form the transition matrix $M = pA + qB + rC$, which is the linear combination of A, B, and C weighted by the proportion of males of each genotype. Thus

$$M = \begin{pmatrix} p + \frac{1}{2}q & \frac{1}{2}p + \frac{1}{4}q & 0 \\ \frac{1}{2}q + r & \frac{1}{2}p + \frac{1}{2}q + \frac{1}{2}r & p + \frac{1}{2}q \\ 0 & \frac{1}{4}q + \frac{1}{2}r & \frac{1}{2}q + r \end{pmatrix}.$$

To simplify the notation, let $a = p + \frac{1}{2}q$ and $b = \frac{1}{2}q + r$. (The numbers a and b represent the proportions of G and g genes, respectively, in the population.) Then

$$M = \begin{pmatrix} a & \frac{1}{2}a & 0 \\ b & \frac{1}{2} & a \\ 0 & \frac{1}{2}b & b \end{pmatrix},$$

where $a + b = p + q + r = 1$.

Let p', q', and r' denote the proportions of the first-generation offspring having genotypes GG, Gg, and gg, respectively. Then

$$\begin{pmatrix} p' \\ q' \\ r' \end{pmatrix} = MP = \begin{pmatrix} ap + \frac{1}{2}aq \\ bp + \frac{1}{2}q + ar \\ \frac{1}{2}bq + br \end{pmatrix} = \begin{pmatrix} a^2 \\ 2ab \\ b^2 \end{pmatrix}.$$

In order to consider the effects of unrestricted matings among the first-generation offspring, a new transition matrix $\widetilde{M}$ must be determined based upon the distribution of first-generation genotypes. As before, we find that

$$\widetilde{M} = \begin{pmatrix} p' + \frac{1}{2}q' & \frac{1}{2}p' + \frac{1}{4}q' & 0 \\ \frac{1}{2}q' + r' & \frac{1}{2}p' + \frac{1}{2}q' + \frac{1}{2}r' & p' + \frac{1}{2}q' \\ 0 & \frac{1}{4}q' + \frac{1}{2}r' & \frac{1}{2}q' + r' \end{pmatrix} = \begin{pmatrix} a' & \frac{1}{2}a' & 0 \\ b' & \frac{1}{2} & a' \\ 0 & \frac{1}{2}b' & b' \end{pmatrix},$$

where $a' = p' + \frac{1}{2}q'$ and $b' = \frac{1}{2}q' + r'$. However

$$a' = a^2 + \frac{1}{2}(2ab) = a(a+b) = a \quad \text{and} \quad b' = \frac{1}{2}(2ab) + b^2 = b(a+b) = b.$$

Thus $\widetilde{M} = M$; so the distribution of second-generation offspring among the three genotypes is

$$\begin{aligned}\widetilde{M}(MP) = M^2P &= \begin{pmatrix} a^3 + a^2b \\ a^2b + ab + ab^2 \\ ab^2 + b^3 \end{pmatrix} = \begin{pmatrix} a^2(a+b) \\ ab(a+1+b) \\ b^2(a+b) \end{pmatrix} = \begin{pmatrix} a^2 \\ 2ab \\ b^2 \end{pmatrix} \\ &= MP,\end{aligned}$$

the same as the first-generation offspring. In other words, MP is the fixed probability vector for M, and genetic equilibrium is achieved in the population after only one generation. (This result is called the *Hardy Weinberg law.*) Notice that in the important special case that $a = b$ (or equivalently, that $p = r$), the distribution at equilibrium is

$$MP = \begin{pmatrix} a^2 \\ 2ab \\ b^2 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{2} \\ \frac{1}{4} \end{pmatrix}.$$

EXERCISES

1. Label the following statements as true or false.
   (a) If $A \in \mathsf{M}_{n\times n}(C)$ and $\lim_{m\to\infty} A^m = L$, then, for any invertible matrix $Q \in \mathsf{M}_{n\times n}(C)$, we have $\lim_{m\to\infty} QA^mQ^{-1} = QLQ^{-1}$.
   (b) If 2 is an eigenvalue of $A \in \mathsf{M}_{n\times n}(C)$, then $\lim_{m\to\infty} A^m$ does not exist.
   (c) Any vector

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathsf{R}^n$$

   such that $x_1 + x_2 + \cdots + x_n = 1$ is a probability vector.
   (d) The sum of the entries of each row of a transition matrix equals 1.
   (e) The product of a transition matrix and a probability vector is a probability vector.

(f) Let z be any complex number such that $|z| < 1$. Then the matrix

$$\begin{pmatrix} 1 & z & -1 \\ z & 1 & 1 \\ -1 & 1 & z \end{pmatrix}$$

does not have 3 as an eigenvalue.

(g) Every transition matrix has 1 as an eigenvalue.

(h) No transition matrix can have -1 as an eigenvalue.

(i) If A is a transition matrix, then $\lim_{m\to\infty} A^m$ exists.

(j) If A is a regular transition matrix, then $\lim_{m\to\infty} A^m$ exists and has rank 1.

2. Determine whether $\lim_{m\to\infty} A^m$ exists for each of the following matrices A, and compute the limit if it exists.

(a) $\begin{pmatrix} 0.1 & 0.7 \\ 0.7 & 0.1 \end{pmatrix}$ (b) $\begin{pmatrix} -1.4 & 0.8 \\ -2.4 & 1.8 \end{pmatrix}$ (c) $\begin{pmatrix} 0.4 & 0.7 \\ 0.6 & 0.3 \end{pmatrix}$

(d) $\begin{pmatrix} -1.8 & 4.8 \\ -0.8 & 2.2 \end{pmatrix}$ (e) $\begin{pmatrix} -2 & -1 \\ 4 & 3 \end{pmatrix}$ (f) $\begin{pmatrix} 2.0 & -0.5 \\ 3.0 & -0.5 \end{pmatrix}$

(g) $\begin{pmatrix} -1.8 & 0 & -1.4 \\ -5.6 & 1 & -2.8 \\ 2.8 & 0 & 2.4 \end{pmatrix}$ (h) $\begin{pmatrix} 3.4 & -0.2 & 0.8 \\ 3.9 & 1.8 & 1.3 \\ -16.5 & -2.0 & -4.5 \end{pmatrix}$

(i) $\begin{pmatrix} -\frac{1}{2}-2i & 4i & \frac{1}{2}+5i \\ 1+2i & -3i & -1-4i \\ -1-2i & 4i & 1+5i \end{pmatrix}$

(j) $\begin{pmatrix} \dfrac{-26+i}{3} & \dfrac{-28-4i}{3} & 28 \\ \dfrac{-7+2i}{3} & \dfrac{-5+i}{3} & 7-2i \\ \dfrac{-13+6i}{6} & \dfrac{-5+6i}{6} & \dfrac{35-20i}{6} \end{pmatrix}$

3. Prove that if $A_1, A_2, \ldots$ is a sequence of $n \times p$ matrices with complex entries such that $\lim_{m\to\infty} A_m = L$, then $\lim_{m\to\infty} (A_m)^t = L^t$.

4. Prove that if $A \in \mathsf{M}_{n\times n}(C)$ is diagonalizable and $L = \lim_{m\to\infty} A^m$ exists, then either $L = I_n$ or $\operatorname{rank}(L) < n$.

5. Find 2×2 matrices A and B having real entries such that $\lim_{m\to\infty} A^m$, $\lim_{m\to\infty} B^m$, and $\lim_{m\to\infty} (AB)^m$ all exist, but

$$\lim_{m\to\infty} (AB)^m \neq (\lim_{m\to\infty} A^m)(\lim_{m\to\infty} B^m).$$

6. A hospital trauma unit has determined that 30% of its patients are ambulatory and 70% are bedridden at the time of arrival at the hospital. A month after arrival, 60% of the ambulatory patients have recovered, 20% remain ambulatory, and 20% have become bedridden. After the same amount of time, 10% of the bedridden patients have recovered, 20% have become ambulatory, 50% remain bedridden, and 20% have died. Determine the percentages of patients who have recovered, are ambulatory, are bedridden, and have died 1 month after arrival. Also determine the eventual percentages of patients of each type.

7. A player begins a game of chance by placing a marker in box 2, marked *Start*. (See Figure 5.5.) A die is rolled, and the marker is moved one square to the left if a 1 or a 2 is rolled and one square to the right if a 3, 4, 5, or 6 is rolled. This process continues until the marker lands in square 1, in which case the player wins the game, or in square 4, in which case the player loses the game. What is the probability of winning this game? *Hint:* Instead of diagonalizing the appropriate transition matrix

Win	Start		Lose
1	2	3	4

Figure 5.5

A, it is easier to represent e_2 as a linear combination of eigenvectors of A and then apply A^n to the result.

8. Which of the following transition matrices are regular?

(a) $\begin{pmatrix} 0.2 & 0.3 & 0.5 \\ 0.3 & 0.2 & 0.5 \\ 0.5 & 0.5 & 0 \end{pmatrix}$ (b) $\begin{pmatrix} 0.5 & 0 & 1 \\ 0.5 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ (c) $\begin{pmatrix} 0.5 & 0 & 0 \\ 0.5 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

(d) $\begin{pmatrix} 0.5 & 0 & 1 \\ 0.5 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ (e) $\begin{pmatrix} \frac{1}{3} & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ \frac{1}{3} & 0 & 1 \end{pmatrix}$ (f) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.7 & 0.2 \\ 0 & 0.3 & 0.8 \end{pmatrix}$

(g) $\begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & 1 & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 & 1 \end{pmatrix}$ (h) $\begin{pmatrix} \frac{1}{4} & \frac{1}{4} & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & 1 & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 & 1 \end{pmatrix}$

9. Compute $\lim_{m\to\infty} A^m$ if it exists, for each matrix A in Exercise 8.

10. Each of the matrices that follow is a regular transition matrix for a three-state Markov chain. In all cases, the initial probability vector is

$$P = \begin{pmatrix} 0.3 \\ 0.3 \\ 0.4 \end{pmatrix}.$$

For each transition matrix, compute the proportions of objects in each state after two stages and the eventual proportions of objects in each state by determining the fixed probability vector.

(a) $\begin{pmatrix} 0.6 & 0.1 & 0.1 \\ 0.1 & 0.9 & 0.2 \\ 0.3 & 0 & 0.7 \end{pmatrix}$ (b) $\begin{pmatrix} 0.8 & 0.1 & 0.2 \\ 0.1 & 0.8 & 0.2 \\ 0.1 & 0.1 & 0.6 \end{pmatrix}$ (c) $\begin{pmatrix} 0.9 & 0.1 & 0.1 \\ 0.1 & 0.6 & 0.1 \\ 0 & 0.3 & 0.8 \end{pmatrix}$

(d) $\begin{pmatrix} 0.4 & 0.2 & 0.2 \\ 0.1 & 0.7 & 0.2 \\ 0.5 & 0.1 & 0.6 \end{pmatrix}$ (e) $\begin{pmatrix} 0.5 & 0.3 & 0.2 \\ 0.2 & 0.5 & 0.3 \\ 0.3 & 0.2 & 0.5 \end{pmatrix}$ (f) $\begin{pmatrix} 0.6 & 0 & 0.4 \\ 0.2 & 0.8 & 0.2 \\ 0.2 & 0.2 & 0.4 \end{pmatrix}$

11. In 1940, a county land-use survey showed that 10% of the county land was urban, 50% was unused, and 40% was agricultural. Five years later, a follow-up survey revealed that 70% of the urban land had remained urban, 10% had become unused, and 20% had become agricultural. Likewise, 20% of the unused land had become urban, 60% had remained unused, and 20% had become agricultural. Finally, the 1945 survey showed that 20% of the agricultural land had become unused while 80% remained agricultural. Assuming that the trends indicated by the 1945 survey continue, compute the percentages of urban, unused, and agricultural land in the county in 1950 and the corresponding eventual percentages.

12. A diaper liner is placed in each diaper worn by a baby. If, after a diaper change, the liner is soiled, then it is discarded and replaced by a new liner. Otherwise, the liner is washed with the diapers and reused, except that each liner is discarded and replaced after its third use (even if it has never been soiled). The probability that the baby will soil any diaper liner is one-third. If there are only new diaper liners at first, eventually what proportions of the diaper liners being used will be new, once used, and twice used? *Hint:* Assume that a diaper liner ready for use is in one of three states: new, once used, and twice used. After its use, it then transforms into one of the three states described.

13. In 1975, the automobile industry determined that 40% of American car owners drove large cars, 20% drove intermediate-sized cars, and 40% drove small cars. A second survey in 1985 showed that 70% of the large-car owners in 1975 still owned large cars in 1985, but 30% had changed to an intermediate-sized car. Of those who owned intermediate-sized cars in 1975, 10% had switched to large cars, 70% continued to drive intermediate-sized cars, and 20% had changed to small cars in 1985. Finally, of the small-car owners in 1975, 10% owned intermediate-sized cars and 90% owned small cars in 1985. Assuming that these trends continue, determine the percentages of Americans who own cars of each size in 1995 and the corresponding eventual percentages.

14. Show that if A and P are as in Example 5, then

$$A^m = \begin{pmatrix} r_m & r_{m+1} & r_{m+1} \\ r_{m+1} & r_m & r_{m+1} \\ r_{m+1} & r_{m+1} & r_m \end{pmatrix},$$

where

$$r_m = \frac{1}{3}\left[1 + \frac{(-1)^m}{2^{m-1}}\right].$$

Deduce that

$$600(A^m P) = A^m \begin{pmatrix} 300 \\ 200 \\ 100 \end{pmatrix} = \begin{pmatrix} 200 + \dfrac{(-1)^m}{2^m}(100) \\ 200 \\ 200 + \dfrac{(-1)^{m+1}}{2^m}(100) \end{pmatrix}.$$

15. Prove that if a 1-dimensional subspace W of R^n contains a nonzero vector with all nonnegative entries, then W contains a unique probability vector.

16. Prove Theorem 5.15 and its corollary.

17. Prove the two corollaries of Theorem 5.18.

18. Prove the corollary of Theorem 5.19.

19. Suppose that M and M' are $n \times n$ transition matrices.

(a) Prove that if M is regular, N is any $n \times n$ transition matrix, and c is a real number such that $0 < c \leq 1$, then $cM + (1-c)N$ is a regular transition matrix.

(b) Suppose that for all i, j, we have that $M'_{ij} > 0$ whenever $M_{ij} > 0$. Prove that there exists a transition matrix N and a real number c with $0 < c \leq 1$ such that $M' = cM + (1-c)N$.

(c) Deduce that if the nonzero entries of M and M' occur in the same positions, then M is regular if and only if M' is regular.

The following definition is used in Exercises 20–24.

Definition. *For $A \in \mathsf{M}_{n\times n}(C)$, define $e^A = \lim_{m\to\infty} B_m$, where*

$$B_m = I + A + \frac{A^2}{2!} + \cdots + \frac{A^m}{m!}$$

(see Exercise 22). Thus e^A is the sum of the infinite series

$$I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots,$$

and B_m is the mth partial sum of this series. (Note the analogy with the power series

$$e^a = 1 + a + \frac{a^2}{2!} + \frac{a^3}{3!} + \cdots,$$

which is valid for all complex numbers a.)

20. Compute e^O and e^I, where O and I denote the $n \times n$ zero and identity matrices, respectively.

21. Let $P^{-1}AP = D$ be a diagonal matrix. Prove that $e^A = Pe^DP^{-1}$.

22. Let $A \in \mathsf{M}_{n\times n}(C)$ be diagonalizable. Use the result of Exercise 21 to show that e^A exists. (Exercise 21 of Section 7.2 shows that e^A exists for every $A \in \mathsf{M}_{n\times n}(C)$.)

23. Find $A, B \in \mathsf{M}_{2\times 2}(R)$ such that $e^Ae^B \neq e^{A+B}$.

24. Prove that a differentiable function $x\colon R \to \mathsf{R}^n$ is a solution to the system of differential equations defined in Exercise 15 of Section 5.2 if and only if $x(t) = e^{tA}v$ for some $v \in \mathsf{R}^n$, where A is defined in that exercise.

5.4 INVARIANT SUBSPACES AND THE CAYLEY–HAMILTON THEOREM

In Section 5.1, we observed that if v is an eigenvector of a linear operator T, then T maps the span of $\{v\}$ into itself. Subspaces that are mapped into themselves are of great importance in the study of linear operators (see, e.g., Exercises 28–32 of Section 2.1).

Definition. *Let* T *be a linear operator on a vector space* V. *A subspace* W *of* V *is called a* **T-invariant subspace** *of* V *if* $\mathsf{T}(\mathsf{W}) \subseteq \mathsf{W}$, *that is, if* $\mathsf{T}(v) \in \mathsf{W}$ *for all* $v \in \mathsf{W}$.

Example 1

Suppose that T is a linear operator on a vector space V. Then the following subspaces of V are T-invariant:

1. $\{0\}$
2. V
3. R(T)
4. N(T)
5. E_λ, for any eigenvalue λ of T.

The proofs that these subspaces are T-invariant are left as exercises. (See Exercise 3.) ♦

Example 2

Let T be the linear operator on R^3 defined by

$$\mathsf{T}(a, b, c) = (a + b, b + c, 0).$$

Then the xy-plane $= \{(x, y, 0) : x, y \in R\}$ and the x-axis $= \{(x, 0, 0) : x \in R\}$ are T-invariant subspaces of R^3. ♦

Let T be a linear operator on a vector space V, and let x be a nonzero vector in V. The subspace

$$\mathsf{W} = \mathrm{span}(\{x, \mathsf{T}(x), \mathsf{T}^2(x), \ldots\})$$

is called the **T-cyclic subspace** of V **generated by** x. It is a simple matter to show that W is T-invariant. In fact, W is the "smallest" T-invariant subspace of V containing x. That is, any T-invariant subspace of V containing x must also contain W (see Exercise 11). Cyclic subspaces have various uses. We apply them in this section to establish the Cayley–Hamilton theorem. In Exercise 31, we outline a method for using cyclic subspaces to compute the characteristic polynomial of a linear operator without resorting to determinants. Cyclic subspaces also play an important role in Chapter 7, where we study matrix representations of nondiagonalizable linear operators.

Example 3

Let T be the linear operator on R^3 defined by

$$\mathsf{T}(a,b,c) = (-b+c, a+c, 3c).$$

We determine the T-cyclic subspace generated by $e_1 = (1,0,0)$. Since

$$\mathsf{T}(e_1) = \mathsf{T}(1,0,0) = (0,1,0) = e_2$$

and

$$\mathsf{T}^2(e_1) = \mathsf{T}(\mathsf{T}(e_1)) = \mathsf{T}(e_2) = (-1,0,0) = -e_1,$$

it follows that

$$\operatorname{span}(\{e_1, \mathsf{T}(e_1), \mathsf{T}^2(e_1), \ldots\}) = \operatorname{span}(\{e_1, e_2\}) = \{(s,t,0) \colon s,t \in R\}. \qquad \blacklozenge$$

Example 4

Let T be the linear operator on $\mathsf{P}(R)$ defined by $\mathsf{T}(f(x)) = f'(x)$. Then the T-cyclic subspace generated by x^2 is $\operatorname{span}(\{x^2, 2x, 2\}) = \mathsf{P}_2(R)$. ◆

The existence of a T-invariant subspace provides the opportunity to define a new linear operator whose domain is this subspace. If T is a linear operator on V and W is a T-invariant subspace of V, then the restriction T_W of T to W (see Appendix B) is a mapping from W to W, and it follows that T_W is a linear operator on W (see Exercise 7). As a linear operator, T_W inherits certain properties from its parent operator T. The following result illustrates one way in which the two operators are linked.

Theorem 5.21. *Let T be a linear operator on a finite-dimensional vector space V, and let W be a T-invariant subspace of V. Then the characteristic polynomial of T_W divides the characteristic polynomial of T.*

Proof. Choose an ordered basis $\gamma = \{v_1, v_2, \ldots, v_k\}$ for W, and extend it to an ordered basis $\beta = \{v_1, v_2, \ldots, v_k, v_{k+1}, \ldots, v_n\}$ for V. Let $A = [\mathsf{T}]_\beta$ and $B_1 = [\mathsf{T}_\mathsf{W}]_\gamma$. Then, by Exercise 12, A can be written in the form

$$A = \begin{pmatrix} B_1 & B_2 \\ O & B_3 \end{pmatrix}.$$

Let $f(t)$ be the characteristic polynomial of T and $g(t)$ the characteristic polynomial of T_W. Then

$$f(t) = \det(A - tI_n) = \det\begin{pmatrix} B_1 - tI_k & B_2 \\ O & B_3 - tI_{n-k} \end{pmatrix} = g(t) \cdot \det(B_3 - tI_{n-k})$$

by Exercise 21 of Section 4.3. Thus $g(t)$ divides $f(t)$. ∎

Example 5

Let T be the linear operator on R^4 defined by

$$\mathsf{T}(a,b,c,d) = (a+b+2c-d,\, b+d,\, 2c-d,\, c+d),$$

and let $\mathsf{W} = \{(t,s,0,0)\colon t,s \in R\}$. Observe that W is a T-invariant subspace of R^4 because, for any vector $(a,b,0,0) \in \mathsf{R}^4$,

$$\mathsf{T}(a,b,0,0) = (a+b,b,0,0) \in \mathsf{W}.$$

Let $\gamma = \{e_1, e_2\}$, which is an ordered basis for W. Extend γ to the standard ordered basis β for R^4. Then

$$B_1 = [\mathsf{T_W}]_\gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad A = [\mathsf{T}]_\beta = \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

in the notation of Theorem 5.21. Let $f(t)$ be the characteristic polynomial of T and $g(t)$ be the characteristic polynomial of $\mathsf{T_W}$. Then

$$f(t) = \det(A - tI_4) = \det \begin{pmatrix} 1-t & 1 & 2 & -1 \\ 0 & 1-t & 0 & 1 \\ 0 & 0 & 2-t & -1 \\ 0 & 0 & 1 & 1-t \end{pmatrix}$$

$$= \det \begin{pmatrix} 1-t & 1 \\ 0 & 1-t \end{pmatrix} \cdot \det \begin{pmatrix} 2-t & -1 \\ 1 & 1-t \end{pmatrix}$$

$$= g(t) \cdot \det \begin{pmatrix} 2-t & -1 \\ 1 & 1-t \end{pmatrix}. \quad \blacklozenge$$

In view of Theorem 5.21, we may use the characteristic polynomial of $\mathsf{T_W}$ to gain information about the characteristic polynomial of T itself. In this regard, cyclic subspaces are useful because the characteristic polynomial of the restriction of a linear operator T to a cyclic subspace is readily computable.

Theorem 5.22. *Let T be a linear operator on a finite-dimensional vector space V, and let W denote the T-cyclic subspace of V generated by a nonzero vector $v \in \mathsf{V}$. Let $k = \dim(\mathsf{W})$. Then*

(a) $\{v, \mathsf{T}(v), \mathsf{T}^2(v), \ldots, \mathsf{T}^{k-1}(v)\}$ *is a basis for* W.

(b) *If* $a_0 v + a_1\mathsf{T}(v) + \cdots + a_{k-1}\mathsf{T}^{k-1}(v) + \mathsf{T}^k(v) = 0$, *then the characteristic polynomial of* $\mathsf{T_W}$ *is* $f(t) = (-1)^k(a_0 + a_1 t + \cdots + a_{k-1}t^{k-1} + t^k)$.

Proof. (a) Since $v \neq 0$, the set $\{v\}$ is linearly independent. Let j be the largest positive integer for which

$$\beta = \{v, \mathsf{T}(v), \ldots, \mathsf{T}^{j-1}(v)\}$$

is linearly independent. Such a j must exist because V is finite-dimensional. Let $\mathsf{Z} = \operatorname{span}(\beta)$. Then β is a basis for Z. Furthermore, $\mathsf{T}^j(v) \in \mathsf{Z}$ by Theorem 1.7 (p. 39). We use this information to show that Z is a T-invariant subspace of V. Let $w \in \mathsf{Z}$. Since w is a linear combination of the vectors of β, there exist scalars $b_0, b_1, \ldots, b_{j-1}$ such that

$$w = b_0 v + b_1 \mathsf{T}(v) + \cdots + b_{j-1}\mathsf{T}^{j-1}(v),$$

and hence

$$\mathsf{T}(w) = b_0 \mathsf{T}(v) + b_1 \mathsf{T}^2(v) + \cdots + b_{j-1}\mathsf{T}^{j}(v).$$

Thus $\mathsf{T}(w)$ is a linear combination of vectors in Z, and hence belongs to Z. So Z is T-invariant. Furthermore, $v \in \mathsf{Z}$. By Exercise 11, W is the smallest T-invariant subspace of V that contains v, so that $\mathsf{W} \subseteq \mathsf{Z}$. Clearly, $\mathsf{Z} \subseteq \mathsf{W}$, and so we conclude that $\mathsf{Z} = \mathsf{W}$. It follows that β is a basis for W, and therefore $\dim(\mathsf{W}) = j$. Thus $j = k$. This proves (a).

(b) Now view β (from (a)) as an ordered basis for W. Let $a_0, a_1, \ldots, a_{k-1}$ be the scalars such that

$$a_0 v + a_1 \mathsf{T}(v) + \cdots + a_{k-1}\mathsf{T}^{k-1}(v) + \mathsf{T}^k(v) = 0.$$

Observe that

$$[\mathsf{T_W}]_\beta = \begin{pmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{k-1} \end{pmatrix},$$

which has the characteristic polynomial

$$f(t) = (-1)^k(a_0 + a_1 t + \cdots + a_{k-1}t^{k-1} + t^k)$$

by Exercise 19. Thus $f(t)$ is the characteristic polynomial of $\mathsf{T_W}$, proving (b). ∎

Example 6

Let T be the linear operator of Example 3, and let $\mathsf{W} = \operatorname{span}(\{e_1, e_2\})$, the T-cyclic subspace generated by e_1. We compute the characteristic polynomial $f(t)$ of $\mathsf{T_W}$ in two ways: by means of Theorem 5.22 and by means of determinants.

(a) *By means of Theorem 5.22.* From Example 3, we have that $\{e_1, e_2\}$ is a cycle that generates W, and that $\mathsf{T}^2(e_1) = -e_1$. Hence

$$1e_1 + 0\mathsf{T}(e_1) + \mathsf{T}^2(e_1) = 0.$$

Therefore, by Theorem 5.22(b),

$$f(t) = (-1)^2(1 + 0t + t^2) = t^2 + 1.$$

(b) *By means of determinants.* Let $\beta = \{e_1, e_2\}$, which is an ordered basis for W. Since $\mathsf{T}(e_1) = e_2$ and $\mathsf{T}(e_2) = -e_1$, we have

$$[\mathsf{T_W}]_\beta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and therefore,

$$f(t) = \det \begin{pmatrix} -t & -1 \\ 1 & -t \end{pmatrix} = t^2 + 1. \quad \blacklozenge$$

The Cayley–Hamilton Theorem

As an illustration of the importance of Theorem 5.22, we prove a well-known result that is used in Chapter 7. The reader should refer to Appendix E for the definition of $f(\mathsf{T})$, where T is a linear operator and $f(x)$ is a polynomial.

Theorem 5.23 (Cayley–Hamilton). *Let T be a linear operator on a finite-dimensional vector space V, and let $f(t)$ be the characteristic polynomial of T. Then $f(\mathsf{T}) = \mathsf{T}_0$, the zero transformation. That is, T "satisfies" its characteristic equation.*

Proof. We show that $f(\mathsf{T})(v) = 0$ for all $v \in \mathsf{V}$. This is obvious if $v = 0$ because $f(\mathsf{T})$ is linear; so suppose that $v \neq 0$. Let W be the T-cyclic subspace generated by v, and suppose that $\dim(\mathsf{W}) = k$. By Theorem 5.22(a), there exist scalars $a_0, a_1, \ldots, a_{k-1}$ such that

$$a_0v + a_1\mathsf{T}(v) + \cdots + a_{k-1}\mathsf{T}^{k-1}(v) + \mathsf{T}^k(v) = 0.$$

Hence Theorem 5.22(b) implies that

$$g(t) = (-1)^k(a_0 + a_1t + \cdots + a_{k-1}t^{k-1} + t^k)$$

is the characteristic polynomial of $\mathsf{T_W}$. Combining these two equations yields

$$g(\mathsf{T})(v) = (-1)^k(a_0\mathsf{I} + a_1\mathsf{T} + \cdots + a_{k-1}\mathsf{T}^{k-1} + \mathsf{T}^k)(v) = 0.$$

By Theorem 5.21, $g(t)$ divides $f(t)$; hence there exists a polynomial $q(t)$ such that $f(t) = q(t)g(t)$. So

$$f(\mathsf{T})(v) = q(\mathsf{T})g(\mathsf{T})(v) = q(\mathsf{T})(g(\mathsf{T})(v)) = q(\mathsf{T})(0) = 0. \quad \blacksquare$$

Example 7

Let T be the linear operator on R^2 defined by $\mathsf{T}(a,b) = (a+2b, -2a+b)$, and let $\beta = \{e_1, e_2\}$. Then

$$A = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix},$$

where $A = [\mathsf{T}]_\beta$. The characteristic polynomial of T is, therefore,

$$f(t) = \det(A - tI) = \det\begin{pmatrix} 1-t & 2 \\ -2 & 1-t \end{pmatrix} = t^2 - 2t + 5.$$

It is easily verified that $\mathsf{T}_0 = f(\mathsf{T}) = \mathsf{T}^2 - 2\mathsf{T} + 5\mathsf{I}$. Similarly,

$$f(A) = A^2 - 2A + 5I = \begin{pmatrix} -3 & 4 \\ -4 & -3 \end{pmatrix} + \begin{pmatrix} -2 & -4 \\ 4 & -2 \end{pmatrix} + \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \quad \blacklozenge$$

Example 7 suggests the following result.

Corollary (Cayley–Hamilton Theorem for Matrices). *Let A be an $n \times n$ matrix, and let $f(t)$ be the characteristic polynomial of A. Then $f(A) = O$, the $n \times n$ zero matrix.*

Proof. See Exercise 15. ∎

Invariant Subspaces and Direct Sums*[3]

It is useful to decompose a finite-dimensional vector space V into a direct sum of as many T-invariant subspaces as possible because the behavior of T on V can be inferred from its behavior on the direct summands. For example, T is diagonalizable if and only if V can be decomposed into a direct sum of one-dimensional T-invariant subspaces (see Exercise 36). In Chapter 7, we consider alternate ways of decomposing V into direct sums of T-invariant subspaces if T is not diagonalizable. We proceed to gather a few facts about direct sums of T-invariant subspaces that are used in Section 7.4. The first of these facts is about characteristic polynomials.

Theorem 5.24. *Let T be a linear operator on a finite-dimensional vector space V, and suppose that $\mathsf{V} = \mathsf{W}_1 \oplus \mathsf{W}_2 \oplus \cdots \oplus \mathsf{W}_k$, where W_i is a T-invariant subspace of V for each i $(1 \le i \le k)$. Suppose that $f_i(t)$ is the characteristic polynomial of $\mathsf{T}_{\mathsf{W}_i}$ $(1 \le i \le k)$. Then $f_1(t) \cdot f_2(t) \cdot \cdots \cdot f_k(t)$ is the characteristic polynomial of T.*

[3]This subsection uses optional material on direct sums from Section 5.2.

Proof. The proof is by mathematical induction on k. In what follows, $f(t)$ denotes the characteristic polynomial of T. Suppose first that $k = 2$. Let β_1 be an ordered basis for W_1, β_2 an ordered basis for W_2, and $\beta = \beta_1 \cup \beta_2$. Then β is an ordered basis for V by Theorem 5.10(d) (p. 276). Let $A = [\mathsf{T}]_\beta$, $B_1 = [\mathsf{T}_{\mathsf{W}_1}]_{\beta_1}$, and $B_2 = [\mathsf{T}_{\mathsf{W}_2}]_{\beta_2}$. By Exercise 34, it follows that

$$A = \begin{pmatrix} B_1 & O \\ O' & B_2 \end{pmatrix},$$

where O and O' are zero matrices of the appropriate sizes. Then

$$f(t) = \det(A - tI) = \det(B_1 - tI) \cdot \det(B_2 - tI) = f_1(t) \cdot f_2(t)$$

as in the proof of Theorem 5.21, proving the result for $k = 2$.

Now assume that the theorem is valid for $k-1$ summands, where $k-1 \geq 2$, and suppose that V is a direct sum of k subspaces, say,

$$\mathsf{V} = \mathsf{W}_1 \oplus \mathsf{W}_2 \oplus \cdots \oplus \mathsf{W}_k.$$

Let $\mathsf{W} = \mathsf{W}_1 + \mathsf{W}_2 + \cdots + \mathsf{W}_{k-1}$. It is easily verified that W is T-invariant and that $\mathsf{V} = \mathsf{W} \oplus \mathsf{W}_k$. So by the case for $k = 2$, $f(t) = g(t) \cdot f_k(t)$, where $g(t)$ is the characteristic polynomial of T_W. Clearly $\mathsf{W} = \mathsf{W}_1 \oplus \mathsf{W}_2 \oplus \cdots \oplus \mathsf{W}_{k-1}$, and therefore $g(t) = f_1(t) \cdot f_2(t) \cdot \cdots \cdot f_{k-1}(t)$ by the induction hypothesis. We conclude that $f(t) = g(t) \cdot f_k(t) = f_1(t) \cdot f_2(t) \cdot \cdots \cdot f_k(t)$. ∎

As an illustration of this result, suppose that T is a diagonalizable linear operator on a finite-dimensional vector space V with distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$. By Theorem 5.11 (p. 278), V is a direct sum of the eigenspaces of T. Since each eigenspace is T-invariant, we may view this situation in the context of Theorem 5.24. For each eigenvalue λ_i, the restriction of T to E_{λ_i} has characteristic polynomial $(\lambda_i - t)^{m_i}$, where m_i is the dimension of E_{λ_i}. By Theorem 5.24, the characteristic polynomial $f(t)$ of T is the product

$$f(t) = (\lambda_1 - t)^{m_1} (\lambda_2 - t)^{m_2} \cdots (\lambda_k - t)^{m_k}.$$

It follows that the multiplicity of each eigenvalue is equal to the dimension of the corresponding eigenspace, as expected.

Example 8

Let T be the linear operator on R^4 defined by

$$\mathsf{T}(a, b, c, d) = (2a - b, a + b, c - d, c + d),$$

and let $\mathsf{W}_1 = \{(s, t, 0, 0) \colon s, t \in R\}$ and $\mathsf{W}_2 = \{(0, 0, s, t) \colon s, t \in R\}$. Notice that W_1 and W_2 are each T-invariant and that $\mathsf{R}^4 = \mathsf{W}_1 \oplus \mathsf{W}_2$. Let $\beta_1 = \{e_1, e_2\}$, $\beta_2 = \{e_3, e_4\}$, and $\beta = \beta_1 \cup \beta_2 = \{e_1, e_2, e_3, e_4\}$. Then β_1 is an ordered basis for W_1, β_2 is an ordered basis for W_2, and β is an ordered basis for R^4. Let $A = [\mathsf{T}]_\beta$, $B_1 = [\mathsf{T}_{\mathsf{W}_1}]_{\beta_1}$, and $B_2 = [\mathsf{T}_{\mathsf{W}_2}]_{\beta_2}$. Then

$$B_1 = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix},$$

and

$$A = \begin{pmatrix} B_1 & O \\ O & B_2 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Let $f(t)$, $f_1(t)$, and $f_2(t)$ denote the characteristic polynomials of T, $\mathsf{T}_{\mathsf{W}_1}$, and $\mathsf{T}_{\mathsf{W}_2}$, respectively. Then

$$f(t) = \det(A - tI) = \det(B_1 - tI) \cdot \det(B_2 - tI) = f_1(t) \cdot f_2(t). \qquad \blacklozenge$$

The matrix A in Example 8 can be obtained by joining the matrices B_1 and B_2 in the manner explained in the next definition.

Definition. *Let $B_1 \in \mathsf{M}_{m\times m}(F)$, and let $B_2 \in \mathsf{M}_{n\times n}(F)$. We define the* ***direct sum*** *of B_1 and B_2, denoted $B_1 \oplus B_2$, as the $(m+n)\times(m+n)$ matrix A such that*

$$A_{ij} = \begin{cases} (B_1)_{ij} & \text{for } 1 \le i,j \le m \\ (B_2)_{(i-m),(j-m)} & \text{for } m+1 \le i,j \le n+m \\ 0 & \text{otherwise.} \end{cases}$$

If $B_1, B_2, \ldots, B_k$ are square matrices with entries from F, then we define the ***direct sum*** *of $B_1, B_2, \ldots, B_k$ recursively by*

$$B_1 \oplus B_2 \oplus \cdots \oplus B_k = (B_1 \oplus B_2 \oplus \cdots \oplus B_{k-1}) \oplus B_k.$$

If $A = B_1 \oplus B_2 \oplus \cdots \oplus B_k$, then we often write

$$A = \begin{pmatrix} B_1 & O & \cdots & O \\ O & B_2 & \cdots & O \\ \vdots & \vdots & & \vdots \\ O & O & \cdots & B_k \end{pmatrix}.$$

Example 9

Let

$$B_1 = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, \quad B_2 = (3), \quad \text{and} \quad B_3 = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}.$$

Then

$$B_1 \oplus B_2 \oplus B_3 = \left(\begin{array}{cc|c|ccc} 1 & 2 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 3 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right). \quad \blacklozenge$$

The final result of this section relates direct sums of matrices to direct sums of invariant subspaces. It is an extension of Exercise 34 to the case $k \geq 2$.

Theorem 5.25. *Let* T *be a linear operator on a finite-dimensional vector space* V, *and let* $\mathsf{W}_1, \mathsf{W}_2, \ldots, \mathsf{W}_k$ *be* T*-invariant subspaces of* V *such that* $\mathsf{V} = \mathsf{W}_1 \oplus \mathsf{W}_2 \oplus \cdots \oplus \mathsf{W}_k$. *For each* i, *let* β_i *be an ordered basis for* W_i, *and let* $\beta = \beta_1 \cup \beta_2 \cup \cdots \cup \beta_k$. *Let* $A = [\mathsf{T}]_\beta$ *and* $B_i = [\mathsf{T}_{\mathsf{W}_i}]_{\beta_i}$ *for* $i = 1, 2, \ldots, k$. *Then* $A = B_1 \oplus B_2 \oplus \cdots \oplus B_k$.

Proof. See Exercise 35. ∎

EXERCISES

1. Label the following statements as true or false.
   (a) There exists a linear operator T with no T-invariant subspace.
   (b) If T is a linear operator on a finite-dimensional vector space V and W is a T-invariant subspace of V, then the characteristic polynomial of T_W divides the characteristic polynomial of T.
   (c) Let T be a linear operator on a finite-dimensional vector space V, and let v and w be in V. If W is the T-cyclic subspace generated by v, W' is the T-cyclic subspace generated by w, and $\mathsf{W} = \mathsf{W}'$, then $v = w$.
   (d) If T is a linear operator on a finite-dimensional vector space V, then for any $v \in \mathsf{V}$ the T-cyclic subspace generated by v is the same as the T-cyclic subspace generated by $\mathsf{T}(v)$.
   (e) Let T be a linear operator on an n-dimensional vector space. Then there exists a polynomial $g(t)$ of degree n such that $g(\mathsf{T}) = \mathsf{T}_0$.
   (f) Any polynomial of degree n with leading coefficient $(-1)^n$ is the characteristic polynomial of some linear operator.
   (g) If T is a linear operator on a finite-dimensional vector space V, and if V is the direct sum of k T-invariant subspaces, then there is an ordered basis β for V such that $[\mathsf{T}]_\beta$ is a direct sum of k matrices.

2. For each of the following linear operators T on the vector space V, determine whether the given subspace W is a T-invariant subspace of V.

   (a) $\mathsf{V} = \mathsf{P}_3(R)$, $\mathsf{T}(f(x)) = f'(x)$, and $\mathsf{W} = \mathsf{P}_2(R)$
   (b) $\mathsf{V} = \mathsf{P}(R)$, $\mathsf{T}(f(x)) = xf(x)$, and $\mathsf{W} = \mathsf{P}_2(R)$
   (c) $\mathsf{V} = \mathsf{R}^3$, $\mathsf{T}(a,b,c) = (a+b+c, a+b+c, a+b+c)$, and $\mathsf{W} = \{(t,t,t) \colon t \in R\}$
   (d) $\mathsf{V} = \mathsf{C}([0,1])$, $\mathsf{T}(f(t)) = \left[\int_0^1 f(x)\,dx\right] t$, and $\mathsf{W} = \{f \in \mathsf{V} \colon f(t) = at + b \text{ for some } a \text{ and } b\}$
   (e) $\mathsf{V} = \mathsf{M}_{2\times 2}(R)$, $\mathsf{T}(A) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A$, and $\mathsf{W} = \{A \in \mathsf{V} \colon A^t = A\}$

3. Let T be a linear operator on a finite-dimensional vector space V. Prove that the following subspaces are T-invariant.

   (a) $\{0\}$ and V
   (b) $\mathsf{N}(\mathsf{T})$ and $\mathsf{R}(\mathsf{T})$
   (c) E_λ, for any eigenvalue λ of T

4. Let T be a linear operator on a vector space V, and let W be a T-invariant subspace of V. Prove that W is $g(\mathsf{T})$-invariant for any polynomial $g(t)$.

5. Let T be a linear operator on a vector space V. Prove that the intersection of any collection of T-invariant subspaces of V is a T-invariant subspace of V.

6. For each linear operator T on the vector space V, find an ordered basis for the T-cyclic subspace generated by the vector z.

   (a) $\mathsf{V} = \mathsf{R}^4$, $\mathsf{T}(a,b,c,d) = (a+b, b-c, a+c, a+d)$, and $z = e_1$.
   (b) $\mathsf{V} = \mathsf{P}_3(R)$, $\mathsf{T}(f(x)) = f''(x)$, and $z = x^3$.
   (c) $\mathsf{V} = \mathsf{M}_{2\times 2}(R)$, $\mathsf{T}(A) = A^t$, and $z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.
   (d) $\mathsf{V} = \mathsf{M}_{2\times 2}(R)$, $\mathsf{T}(A) = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} A$, and $z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

7. Prove that the restriction of a linear operator T to a T-invariant subspace is a linear operator on that subspace.

8. Let T be a linear operator on a vector space with a T-invariant subspace W. Prove that if v is an eigenvector of T_W with corresponding eigenvalue λ, then the same is true for T.

9. For each linear operator T and cyclic subspace W in Exercise 6, compute the characteristic polynomial of T_W in two ways, as in Example 6.

10. For each linear operator in Exercise 6, find the characteristic polynomial $f(t)$ of T, and verify that the characteristic polynomial of T_W (computed in Exercise 9) divides $f(t)$.

11. Let T be a linear operator on a vector space V, let v be a nonzero vector in V, and let W be the T-cyclic subspace of V generated by v. Prove that

(a) W is T-invariant.
(b) Any T-invariant subspace of V containing v also contains W.

12. Prove that $A = \begin{pmatrix} B_1 & B_2 \\ O & B_3 \end{pmatrix}$ in the proof of Theorem 5.21.

13. Let T be a linear operator on a vector space V, let v be a nonzero vector in V, and let W be the T-cyclic subspace of V generated by v. For any $w \in \mathsf{V}$, prove that $w \in \mathsf{W}$ if and only if there exists a polynomial $g(t)$ such that $w = g(\mathsf{T})(v)$.

14. Prove that the polynomial $g(t)$ of Exercise 13 can always be chosen so that its degree is less than $\dim(\mathsf{W})$.

15. Use the Cayley–Hamilton theorem (Theorem 5.23) to prove its corollary for matrices. *Warning:* If $f(t) = \det(A - tI)$ is the characteristic polynomial of A, it is tempting to "prove" that $f(A) = O$ by saying "$f(A) = \det(A - AI) = \det(O) = 0$." But this argument is nonsense. Why?

16. Let T be a linear operator on a finite-dimensional vector space V.

(a) Prove that if the characteristic polynomial of T splits, then so does the characteristic polynomial of the restriction of T to any T-invariant subspace of V.
(b) Deduce that if the characteristic polynomial of T splits, then any nontrivial T-invariant subspace of V contains an eigenvector of T.

17. Let A be an $n \times n$ matrix. Prove that

$$\dim(\operatorname{span}(\{I_n, A, A^2, \ldots\})) \leq n.$$

18. Let A be an $n \times n$ matrix with characteristic polynomial

$$f(t) = (-1)^n t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0.$$

(a) Prove that A is invertible if and only if $a_0 \neq 0$.
(b) Prove that if A is invertible, then

$$A^{-1} = (-1/a_0)[(-1)^n A^{n-1} + a_{n-1}A^{n-2} + \cdots + a_1 I_n].$$

(c) Use (b) to compute A^{-1} for

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & -1 \end{pmatrix}.$$

19. Let A denote the $k \times k$ matrix

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & -a_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -a_{k-2} \\ 0 & 0 & \cdots & 1 & -a_{k-1} \end{pmatrix},$$

where $a_0, a_1, \ldots, a_{k-1}$ are arbitrary scalars. Prove that the characteristic polynomial of A is

$$(-1)^k(a_0 + a_1t + \cdots + a_{k-1}t^{k-1} + t^k).$$

Hint: Use mathematical induction on k, expanding the determinant along the first row.

20. Let T be a linear operator on a vector space V, and suppose that V is a T-cyclic subspace of itself. Prove that if U is a linear operator on V, then $\mathsf{UT} = \mathsf{TU}$ if and only if $\mathsf{U} = g(\mathsf{T})$ for some polynomial $g(t)$. *Hint:* Suppose that V is generated by v. Choose $g(t)$ according to Exercise 13 so that $g(\mathsf{T})(v) = \mathsf{U}(v)$.

21. Let T be a linear operator on a two-dimensional vector space V. Prove that either V is a T-cyclic subspace of itself or $\mathsf{T} = c\mathsf{I}$ for some scalar c.

22. Let T be a linear operator on a two-dimensional vector space V and suppose that $\mathsf{T} \neq c\mathsf{I}$ for any scalar c. Show that if U is any linear operator on V such that $\mathsf{UT} = \mathsf{TU}$, then $\mathsf{U} = g(\mathsf{T})$ for some polynomial $g(t)$.

23. Let T be a linear operator on a finite-dimensional vector space V, and let W be a T-invariant subspace of V. Suppose that $v_1, v_2, \ldots, v_k$ are eigenvectors of T corresponding to distinct eigenvalues. Prove that if $v_1 + v_2 + \cdots + v_k$ is in W, then $v_i \in \mathsf{W}$ for all i. *Hint:* Use mathematical induction on k.

24. Prove that the restriction of a diagonalizable linear operator T to any nontrivial T-invariant subspace is also diagonalizable. *Hint:* Use the result of Exercise 23.

25. (a) Prove the converse to Exercise 18(a) of Section 5.2: If T and U are diagonalizable linear operators on a finite-dimensional vector space V such that $UT = TU$, then T and U are simultaneously diagonalizable. (See the definitions in the exercises of Section 5.2.) *Hint:* For any eigenvalue λ of T, show that E_λ is U-invariant, and apply Exercise 24 to obtain a basis for E_λ of eigenvectors of U.

(b) State and prove a matrix version of (a).

26. Let T be a linear operator on an n-dimensional vector space V such that T has n distinct eigenvalues. Prove that V is a T-cyclic subspace of itself. *Hint:* Use Exercise 23 to find a vector v such that $\{v, T(v), \ldots, T^{n-1}(v)\}$ is linearly independent.

Exercises 27 through 32 require familiarity with quotient spaces as defined in Exercise 31 of Section 1.3. Before attempting these exercises, the reader should first review the other exercises treating quotient spaces: Exercise 35 of Section 1.6, Exercise 40 of Section 2.1, and Exercise 24 of Section 2.4.

For the purposes of Exercises 27 through 32, T is a fixed linear operator on a finite-dimensional vector space V, and W is a nonzero T-invariant subspace of V. We require the following definition.

Definition. *Let T be a linear operator on a vector space V, and let W be a T-invariant subspace of V. Define $\overline{T}\colon V/W \to V/W$ by*

$$\overline{T}(v+W) = T(v) + W \quad \text{for any } v+W \in V/W.$$

27. (a) Prove that $\overline{T}$ is well defined. That is, show that $\overline{T}(v+W) = \overline{T}(v'+W)$ whenever $v+W = v'+W$.

(b) Prove that $\overline{T}$ is a linear operator on V/W.

(c) Let $\eta\colon V \to V/W$ be the linear transformation defined in Exercise 40 of Section 2.1 by $\eta(v) = v+W$. Show that the diagram of Figure 5.6 commutes; that is, prove that $\eta T = \overline{T}\eta$. (This exercise does not require the assumption that V is finite-dimensional.)

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\eta"'] & V \arrow[d, "\eta"] \\ V/W \arrow[r, "\overline{T}"] & V/W \end{tikzcd}$$

Figure 5.6

28. Let $f(t)$, $g(t)$, and $h(t)$ be the characteristic polynomials of T, T_W, and $\overline{T}$, respectively. Prove that $f(t) = g(t)h(t)$. *Hint:* Extend an ordered basis $\gamma = \{v_1, v_2, \ldots, v_k\}$ for W to an ordered basis $\beta = \{v_1, v_2, \ldots, v_k, v_{k+1}, \ldots, v_n\}$ for V. Then show that the collection of cosets $\alpha = \{v_{k+1} + \mathsf{W}, v_{k+2} + \mathsf{W}, \ldots, v_n + \mathsf{W}\}$ is an ordered basis for V/W, and prove that

$$[\mathsf{T}]_\beta = \begin{pmatrix} B_1 & B_2 \\ O & B_3 \end{pmatrix},$$

where $B_1 = [\mathsf{T}]_\gamma$ and $B_3 = [\overline{\mathsf{T}}]_\alpha$.

29. Use the hint in Exercise 28 to prove that if T is diagonalizable, then so is $\overline{\mathsf{T}}$.

30. Prove that if both T_W and $\overline{\mathsf{T}}$ are diagonalizable and have no common eigenvalues, then T is diagonalizable.

The results of Theorem 5.22 and Exercise 28 are useful in devising methods for computing characteristic polynomials without the use of determinants. This is illustrated in the next exercise.

31. Let $A = \begin{pmatrix} 1 & 1 & -3 \\ 2 & 3 & 4 \\ 1 & 2 & 1 \end{pmatrix}$, let $\mathsf{T} = \mathsf{L}_A$, and let W be the cyclic subspace of R^3 generated by e_1.

(a) Use Theorem 5.22 to compute the characteristic polynomial of T_W.
(b) Show that $\{e_2 + \mathsf{W}\}$ is a basis for R^3/W, and use this fact to compute the characteristic polynomial of $\overline{\mathsf{T}}$.
(c) Use the results of (a) and (b) to find the characteristic polynomial of A.

32. Prove the converse to Exercise 9(a) of Section 5.2: If the characteristic polynomial of T splits, then there is an ordered basis β for V such that $[\mathsf{T}]_\beta$ is an upper triangular matrix. *Hints:* Apply mathematical induction to $\dim(\mathsf{V})$. First prove that T has an eigenvector v, let $\mathsf{W} = \operatorname{span}(\{v\})$, and apply the induction hypothesis to $\overline{\mathsf{T}}\colon \mathsf{V}/\mathsf{W} \to \mathsf{V}/\mathsf{W}$. Exercise 35(b) of Section 1.6 is helpful here.

Exercises 33 through 40 are concerned with direct sums.

33. Let T be a linear operator on a vector space V, and let $\mathsf{W}_1, \mathsf{W}_2, \ldots, \mathsf{W}_k$ be T-invariant subspaces of V. Prove that $\mathsf{W}_1 + \mathsf{W}_2 + \cdots + \mathsf{W}_k$ is also a T-invariant subspace of V.

34. Give a direct proof of Theorem 5.25 for the case $k = 2$. (This result is used in the proof of Theorem 5.24.)

35. Prove Theorem 5.25. *Hint:* Begin with Exercise 34 and extend it using mathematical induction on k, the number of subspaces.

36. Let T be a linear operator on a finite-dimensional vector space V. Prove that T is diagonalizable if and only if V is the direct sum of one-dimensional T-invariant subspaces.

37. Let T be a linear operator on a finite-dimensional vector space V, and let $\mathsf{W}_1, \mathsf{W}_2, \ldots, \mathsf{W}_k$ be T-invariant subspaces of V such that $\mathsf{V} = \mathsf{W}_1 \oplus \mathsf{W}_2 \oplus \cdots \oplus \mathsf{W}_k$. Prove that

$$\det(\mathsf{T}) = \det(\mathsf{T}_{\mathsf{W}_1}) \det(\mathsf{T}_{\mathsf{W}_2}) \cdots \det(\mathsf{T}_{\mathsf{W}_k}).$$

38. Let T be a linear operator on a finite-dimensional vector space V, and let $\mathsf{W}_1, \mathsf{W}_2, \ldots, \mathsf{W}_k$ be T-invariant subspaces of V such that $\mathsf{V} = \mathsf{W}_1 \oplus \mathsf{W}_2 \oplus \cdots \oplus \mathsf{W}_k$. Prove that T is diagonalizable if and only if $\mathsf{T}_{\mathsf{W}_i}$ is diagonalizable for all i.

39. Let $\mathcal{C}$ be a collection of diagonalizable linear operators on a finite-dimensional vector space V. Prove that there is an ordered basis β such that $[\mathsf{T}]_\beta$ is a diagonal matrix for all $\mathsf{T} \in \mathcal{C}$ if and only if the operators of $\mathcal{C}$ commute under composition. (This is an extension of Exercise 25.) *Hints for the case that the operators commute:* The result is trivial if each operator has only one eigenvalue. Otherwise, establish the general result by mathematical induction on $\dim(\mathsf{V})$, using the fact that V is the direct sum of the eigenspaces of some operator in $\mathcal{C}$ that has more than one eigenvalue.

40. Let $B_1, B_2, \ldots, B_k$ be square matrices with entries in the same field, and let $A = B_1 \oplus B_2 \oplus \cdots \oplus B_k$. Prove that the characteristic polynomial of A is the product of the characteristic polynomials of the B_i's.

41. Let

$$A = \begin{pmatrix} 1 & 2 & \cdots & n \\ n+1 & n+2 & \cdots & 2n \\ \vdots & \vdots & & \vdots \\ n^2-n+1 & n^2-n+2 & \cdots & n^2 \end{pmatrix}.$$

Find the characteristic polynomial of A. *Hint:* First prove that A has rank 2 and that $\mathrm{span}(\{(1, 1, \ldots, 1), (1, 2, \ldots, n)\})$ is L_A-invariant.

42. Let $A \in \mathsf{M}_{n \times n}(R)$ be the matrix defined by $A_{ij} = 1$ for all i and j. Find the characteristic polynomial of A.

INDEX OF DEFINITIONS FOR CHAPTER 5