Chapter 1

Permutations and Combinations

1.1. Two Basic Counting Principles

In our everyday lives, we often need to enumerate "events" such as, the arrangement of objects in a certain way, the partition of things under a certain condition, the distribution of items according to a certain specification, and so on. For instance, we may come across counting problems of the following types:

"How many ways are there to arrange 5 boys and 3 girls in a row so that no two girls are adjacent?"

"How many ways are there to divide a group of 10 people into three groups consisting of 4, 3 and 2 people respectively, with 1 person rejected?"

These are two very simple examples of counting problems related to what we call "permutations" and "combinations". Before we introduce in the next three sections what permutations and combinations are, we state in this section two principles that are fundamental in all kinds of counting problems.

The Addition Principle (AP) Assume that there are

$$\begin{array}{llll} n_1 & \text{ways for the event} & E_1 & \text{to occur,} \\ n_2 & \text{ways for the event} & E_2 & \text{to occur,} \\ & & \vdots & \\ n_k & \text{ways for the event} & E_k & \text{to occur,} \end{array}$$

where $k \geq 1$. If these ways for the different events to occur are pairwise disjoint, then the number of ways for at least one of the events $E_1, E_2, \ldots,$ or E_k to occur is $n_1 + n_2 + \cdots + n_k = \sum_{i=1}^{k} n_i$.

Example 1.1.1. One can reach city Q from city P by sea, air and road. Suppose that there are 2 ways by sea, 3 ways by air and 2 ways by road (see Figure 1.1.1). Then by (AP), the total number of ways from P to Q by sea, air or road is $2 + 3 + 2 = 7$. ∎

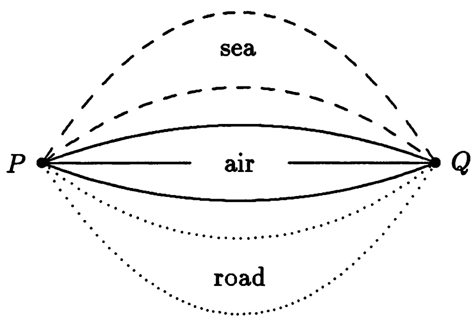


Figure 1.1.1.

An equivalent form of (AP), using set-theoretic terminology, is given below.

> Let $A_1, A_2, \ldots, A_k$ be any k finite sets, where $k \geq 1$. If the given sets are pairwise disjoint, i.e., $A_i \cap A_j = \emptyset$ for $i, j = 1, 2, ..., k$, $i \neq j$, then
>
> $$\left|\bigcup_{i=1}^{k} A_i\right| = |A_1 \cup A_2 \cup \cdots \cup A_k| = \sum_{i=1}^{k} |A_i|.$$

Example 1.1.2. Find the number of ordered pairs (x, y) of integers such that $x^2 + y^2 \leq 5$.

Solution. We may divide the problem into 6 disjoint cases: $x^2+y^2=0,1,\ldots,5$. Thus for $i=0,1,\ldots,5$, let

$$S_i=\{(x,y)\mid x,y\in\mathbf{Z},\ \ x^2+y^2=i\}.$$

It can be checked that

$S_0=\{(0,0)\}$,

$S_1=\{(1,0),(-1,0),(0,1),(0,-1)\}$,

$S_2=\{(1,1),(1,-1),(-1,1),(-1,-1)\}$,

$S_3=\emptyset$,

$S_4=\{(0,2),(0,-2),(2,0),(-2,0)\}$, and

$S_5=\{(1,2),(1,-2),(2,1),(2,-1),(-1,2),(-1,-2),(-2,1),(-2,-1)\}$.

Thus by (AP), the desired number of ordered pairs is

$$\sum_{i=0}^{5}|S_i|=1+4+4+0+4+8=21.\quad\blacksquare$$

Remarks. **1)** In the above example, one can find out the answer "21" simply by listing all the required ordered pairs (x,y). The above method, however, provides us with a systematical way to obtain the answer.

2) One may also divide the above problem into disjoint cases: $x^2=0,1,\ldots,5$, find out the number of required ordered pairs in each case, and obtain the desired answer by applying (AP).

The Multiplication Principle (MP) Assume that an event E can be decomposed into r ordered events $E_1,E_2,\ldots,E_r$, and that there are

n_1 ways for the event E_1 to occur,
n_2 ways for the event E_2 to occur,
$\vdots$
n_r ways for the event E_r to occur.

Then the total number of ways for the event E to occur is given by:

$$n_1\times n_2\times\cdots\times n_r=\prod_{i=1}^{r}n_i.$$

Example 1.1.3. To reach city D from city A, one has to pass through city B and then city C as shown in Figure 1.1.2.

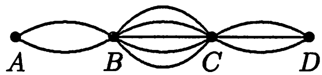


Figure 1.1.2.

If there are 2 ways to travel from A to B, 5 ways from B to C, and 3 ways from C to D, then by (MP), the number of ways from A to D via B and C is given by $2 \times 5 \times 3 = 30$. ∎

An equivalent form of (MP) using set-theoretic terminology, is stated below.

> Let
>
> $$\prod_{i=1}^{r} A_i = A_1 \times A_2 \times \cdots \times A_r = \{(a_1, a_2, ..., a_r) \mid a_i \in A_i, i = 1, 2, ..., r\}$$
>
> denote the cartesian product of the finite sets $A_1, A_2, ..., A_r$. Then
>
> $$\left|\prod_{i=1}^{r} A_i\right| = |A_1| \times |A_2| \times \cdots \times |A_r| = \prod_{i=1}^{r} |A_i|.$$

A sequence of numbers $a_1a_2 \ldots a_n$ is called a *k-ary sequence*, where $n, k \in \mathbf{N}$, if $a_i \in \{0, 1, ..., k-1\}$ for each $i = 1, 2, ..., n$. The *length* of the sequence $a_1a_2 \ldots a_n$ is defined to be n, which is the number of terms contained in the sequence. At times, such a sequence may be denoted by $(a_1, a_2, ..., a_n)$. A k-ary sequence is also called a *binary*, *ternary*, or *quaternary* sequence when $k = 2, 3$ or 4, respectively. Thus, $\{000, 001, 010, 100, 011, 101, 110, 111\}$ is the set of all $8 (= 2^3)$ binary sequences of length 3. For given $k, n \in \mathbf{N}$, how many different k-ary sequences of length n can we form? This will be discussed in the following example. You will find the result useful later on.

Example 1.1.4. To form a k-ary sequence $a_1a_2...a_n$ of length n, we first select an a_1 from the set $B = \{0, 1, ..., k-1\}$; then an a_2 from the same set B; and so on until finally an a_n again from B. Since there are k choices in each step, the number of distinct k-ary sequences of length n is, by (MP), $\underbrace{k \times k \times \cdots \times k}_{n} = k^n$. ∎

Example 1.1.5. Find the number of positive divisors of 600, inclusive of 1 and 600 itself.

Solution. We first note that the number '600' has a unique prime factorization, namely, $600 = 2^3 \times 3^1 \times 5^2$. It thus follows that a positive integer m is a divisor of 600 if and only if m is of the form $m = 2^a \times 3^b \times 5^c$, where $a, b, c \in \mathbf{Z}$ such that $0 \le a \le 3$, $0 \le b \le 1$ and $0 \le c \le 2$. Accordingly, the number of positive divisors of '600' is the number of ways to form the triples (a, b, c) where $a \in \{0, 1, 2, 3\}$, $b \in \{0, 1\}$ and $c \in \{0, 1, 2\}$, which by (MP), is equal to $4 \times 2 \times 3 = 24$. ∎

Remark. By applying (MP) in a similar way, one obtains the following general result.

If a natural number n has as its prime factorization,

$$n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$$

where the p_i's are distinct primes and the k_i's are positive integers, then the number of positive divisors of n is given by $\prod_{i=1}^{r}(k_i + 1)$.

In the above examples, we have seen how (AP) and (MP) were separately used to solve some counting problems. Very often, solving a more complicated problem may require a 'joint' application of both (AP) and (MP). To illustrate this, we give the following example.

Example 1.1.6. Let $X = \{1, 2, ..., 100\}$ and let

$$S = \{(a, b, c) \mid a, b, c \in X, a < b \text{ and } a < c\}.$$

Find $|S|$.

Solution. The problem may be divided into disjoint cases by considering $a = 1, 2, \ldots, 99$.

For $a = k \in \{1, 2, \ldots, 99\}$, the number of choices for b is $100 - k$ and that for c is also $100 - k$. Thus the number of required ordered triples (k, b, c) is $(100 - k)^2$, by (MP). Since k takes on the values $1, 2, \ldots, 99$, by applying (AP), we have

$$|S| = 99^2 + 98^2 + \cdots + 1^2.$$

Using the formula $\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1)$, we finally obtain

$$|S| = \frac{1}{6} \times 99 \times 100 \times 199 = 328350. \quad \blacksquare$$

As mathematical statements, both (AP) and (MP) are really 'trivial'. This could be a reason why they are very often neglected by students. Actually, they are very fundamental in solving counting problems. As we shall witness in this book, a given counting problem, no matter how complicated it is, can always be 'decomposed' into some simpler 'sub-problems' that in turn can be counted by using (AP) and/or (MP).

1.2. Permutations

At the beginning of Section 1.1, we mentioned the following problem: "How many ways are there to arrange 5 boys and 3 girls in a row so that no two girls are adjacent?" This is a typical example of a more general problem of arranging some distinct objects subject to certain additional conditions.

Let $A = \{a_1, a_2, \ldots, a_n\}$ be a given set of n distinct objects. For $0 \leq r \leq n$, an *r-permutation* of A is a way of arranging any r of the objects of A in a row. When $r = n$, an n-permutation of A is simply called a *permutation* of A.

Example 1.2.1. Let $A = \{a, b, c, d\}$. All the 3-permutations of A are shown below:

$$\begin{array}{cccccc} abc, & acb, & bac, & bca, & cab, & cba, \\ abd, & adb, & bad, & bda, & dab, & dba, \\ acd, & adc, & cad, & cda, & dac, & dca, \\ bcd, & bdc, & cbd, & cdb, & dbc, & dcb. \end{array}$$

There are altogether 24 in number. $\blacksquare$

Let P_r^n denote the number of r-permutations of A. Thus $P_3^4 = 24$ as shown in Example 1.2.1. In what follows, we shall derive a formula for P_r^n by applying (MP).

An r-permutation of A can be formed in r steps, as described below: First, we choose an object from A and put it in the first position (see Figure 1.2.1). Next we choose an object from the remaining ones in A and put it in the second position. We proceed on until the rth-step in which we choose an object from the remaining $(n-r+1)$ elements in A and put it in the rth-position.

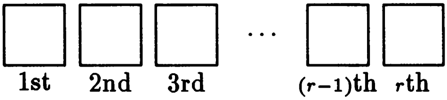


Figure 1.2.1

There are n choices in step 1, $(n-1)$ choices in step 2, ..., $n-(r-1)$ choices in step r. Thus by (MP),

$$P_r^n = n(n-1)(n-2)\cdots(n-r+1). \tag{1.2.1}$$

If we use the factorial notation: $n! = n(n-1)\cdots 2\cdot 1$, then

$$P_r^n = \frac{n!}{(n-r)!}. \tag{1.2.2}$$

Remark. By convention, $0! = 1$. Note that $P_0^n = 1$ and $P_n^n = n!$.

Example 1.2.2. Let $E = \{a, b, c, ..., x, y, z\}$ be the set of the 26 English alphabets. Find the number of 5-letter words that can be formed from E such that the first and last letters are distinct vowels and the remaining three are distinct consonants.

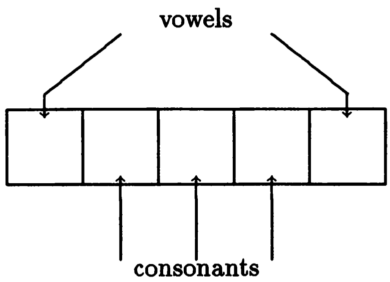


Figure 1.2.2.

Solution. There are 5 vowels and 21 consonants in E. A required 5-letter word can be formed in the following way.

Step 1. Choose a 2-permutation of $\{a, e, i, o, u\}$ and then put the first vowel in the 1^{st} position and the second vowel in the 5^{th} position (see Figure 1.2.2).

Step 2. Choose a 3-permutation of $E \backslash \{a, e, i, o, u\}$ and put the 1st, 2nd and 3rd consonants of the permutation in the 2nd, 3rd and 4th positions respectively (see Figure 1.2.2).

There are P_2^5 choices in Step 1 and P_3^{21} choices in Step 2. Thus by (MP), the number of such 5-letter words is given by

$$P_2^5 \times P_3^{21} = (5 \times 4) \times (21 \times 20 \times 19) = 159600. \quad \blacksquare$$

Example 1.2.3. There are 7 boys and 3 girls in a gathering. In how many ways can they be arranged in a row so that

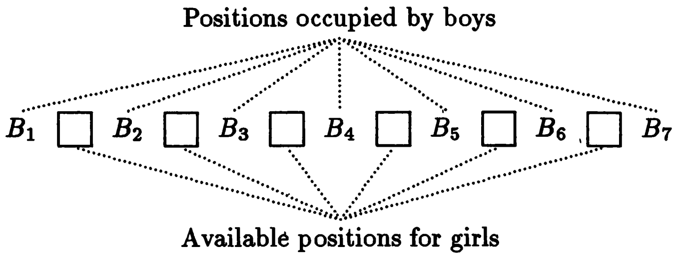
(i) the 3 girls form a single block (i.e. there is no boy between any two of the girls)?

(ii) the two end-positions are occupied by boys and no girls are adjacent?

Solution. (i) Since the 3 girls must be together, we can treat them as a single entity. The number of ways to arrange 7 boys together with this entity is $(7 + 1)!$. As the girls can permute among themselves within the entity in 3! ways, the desired number of ways is, by (MP),

$$8! \times 3!.$$

(ii) We first consider the arrangements of boys and then those of girls. There are 7! ways to arrange the boys. Fix an arbitrary one of the arrangements. Since the end-positions are occupied by boys, there are only 6 spaces available for the 3 girls G_1, G_2 and G_3.



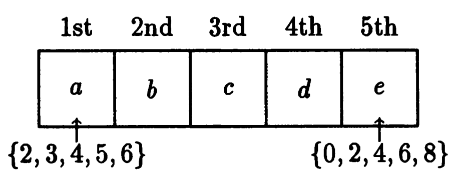
G_1 has 6 choices. Since no two girls are adjacent, G_2 has 5 choices and G_3 has 4. Thus by (MP), the number of such arrangements is

$$7! \times 6 \times 5 \times 4. \quad \blacksquare$$

Remark. Example 1.2.3 can also be solved by considering the arrangements for the girls first. This will be discussed in Example 1.7.2.

Example 1.2.4. Between 20000 and 70000, find the number of even integers in which no digit is repeated.

Solution. Let *abcde* be a required even integer. As shown in the following diagram, the 1st digit a can be chosen from $\{2,3,4,5,6\}$ and the 5th digit e can be chosen from $\{0,2,4,6,8\}$.



Since $\{2,3,4,5,6\} \cap \{0,2,4,6,8\} = \{2,4,6\}$, we divide the problem into 2 disjoint cases:

Case 1. $a \in \{2,4,6\}$.

In this case, a has 3 choices, e then has $4(= 5-1)$ choices, and bcd has $P_3^{(10-2)} = P_3^8$ choices. By (MP), there are

$$3 \times 4 \times P_3^8 = 4032$$

such even numbers.

Case 2. $a \in \{3,5\}$.

In this case, a has 2 choices, e has 5 choices and again bcd has P_3^8 choices. By (MP), there are

$$2 \times 5 \times P_3^8 = 3360$$

such even numbers.

Now, by (AP), the total number of required even numbers is $4032 + 3360 = 7392$. ∎

Example 1.2.5. Let S be the set of natural numbers whose digits are chosen from $\{1,3,5,7\}$ such that no digits are repeated. Find

(i) $|S|$;

(ii) $\sum_{n \in S} n$.

Solution (i) We divide S into 4 disjoint subsets consisting of:

(1) 1-digit numbers: $1,3,5,7$;

(2) 2-digit numbers: $13, 15, \ldots$;

(3) 3-digit numbers: $135, 137, \ldots$;

(4) 4-digit numbers: $1357, 1375, \ldots$;

and find $|S|$ by applying (AP). Thus for $i = 1,2,3,4$, let S_i denote the set of i-digit natural numbers formed by 1,3,5,7 with no repetition. Then $S = S_1 \cup S_2 \cup S_3 \cup S_4$ and by (AP),

$$\begin{aligned}|S| = \sum_{i=1}^{4} |S_i| &= P_1^4 + P_2^4 + P_3^4 + P_4^4 \\ &= 4 + 12 + 24 + 24 = 64.\end{aligned}$$

(ii) Let $\alpha = \sum_{n \in S} n$. It is tedious to determine α by summing up all the 64 numbers in S. Instead, we use the following method.

Let α_1 denote the sum of unit-digits of the numbers in S; α_2 that of ten-digits of the numbers in S; α_3 that of hundred-digits of the numbers in S; and α_4 that of thousand-digits of the numbers in S. Then

$$\alpha = \alpha_1 + 10\alpha_2 + 100\alpha_3 + 1000\alpha_4.$$

We first count α_1. Clearly, the sum of unit-digits of the numbers in S_1 is

$$1 + 3 + 5 + 7 = 16.$$

In S_2, there are P_1^3 numbers whose unit-digits are, respectively, 1, 3, 5 and 7. Thus the sum of the unit-digits of the number is S_2 is

$$P_1^3 \times (1 + 3 + 5 + 7) = 3 \times 16 = 48.$$

In S_3, there are P_2^3 numbers whose unit-digits are, respectively, 1, 3, 5 and 7. Thus the sum of unit-digits of the numbers in S_3 is

$$P_2^3 \times (1 + 3 + 5 + 7) = 6 \times 16 = 96.$$

In S_4, there are P_3^3 numbers whose unit-digits are, respectively, 1, 3, 5 and 7. Thus the sum of unit-digits of the numbers in S_4 is

$$P_3^3 \times (1 + 3 + 5 + 7) = 6 \times 16 = 96.$$

Hence by (AP),

$$\alpha_1 = 16 + 48 + 96 + 96 = 256.$$

Similarly, we have:

$$\begin{aligned} \alpha_2 =& P_1^3 \times (1 + 3 + 5 + 7) + P_2^3 \times (1 + 3 + 5 + 7) \\ &+ P_3^3 \times (1 + 3 + 5 + 7) = 240; \\ \alpha_3 =& (P_2^3 + P_3^3) \times (1 + 3 + 5 + 7) = 192; \\ \text{and} \quad \alpha_4 =& P_3^3 \times (1 + 3 + 5 + 7) = 96. \end{aligned}$$

Thus,

$$\begin{aligned} \alpha &= \alpha_1 + 10\alpha_2 + 100\alpha_3 + 1000\alpha_4 \\ &= 256 + 2400 + 19200 + 96000 \\ &= 117856. \quad \blacksquare \end{aligned}$$

Remark. There is a shortcut to compute the sum $\alpha = \sum(n \mid n \in S)$ in part (ii). Observe that the 4 numbers in S_1 can be paired off as $\{1,7\}$ and $\{3,5\}$ so that the sum of the two numbers in each pair is equal to 8 and the 12 numbers in S_2 can be paired off as $\{13,75\}$, $\{15,73\}$, $\{17,71\}$, $\{35,53\}$, ... so that the sum of the two numbers in each pair is 88. Likewise, the 24 numbers in S_3 and the 24 numbers in S_4 can be paired off so that the sum of the two numbers in each pair is equal to 888 and 8888 respectively. Thus

$$\begin{aligned}\alpha &= 8 \times \frac{4}{2} + 88 \times \frac{12}{2} + 888 \times \frac{24}{2} + 8888 \times \frac{24}{2} \\ &= 117856.\end{aligned}$$

1.3. Circular Permutations

The permutations discussed in Section 1.2 involved arrangements of objects *in a row*. There are permutations which require arranging objects in a circular closed curve. These are called circular permutations.

Consider the problem of arranging 3 distinct objects a, b, c in 3 positions around a circle. Suppose the 3 positions are *numbered* (1), (2) and (3) as shown in Figure 1.3.1. Then the three arrangements of a, b, c shown in the figure can be viewed as the permutations:

$$abc, \quad cab, \quad bca$$

respectively.

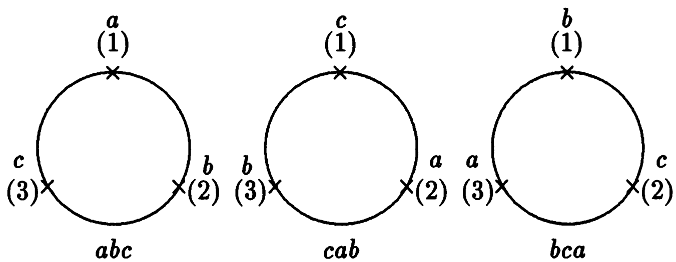


Figure 1.3.1.

In this case, such "circular permutations" are identical with the usual permutations, and thus there is nothing new worth discussing. To get something interesting, let us now neglect the numbering of the positions (and thus only "relative positions" of objects are concerned). As shown in Figure 1.3.2, any of the 3 arrangements is a rotation of every other; i.e., the relative positions of the objects are invariant under rotation. In this case, we shall agree to say that the 3 arrangements of Figure 1.3.2 are identical. In general, two circular permutations of the same objects are *identical* if any one of them can be obtained by a rotation of the other.

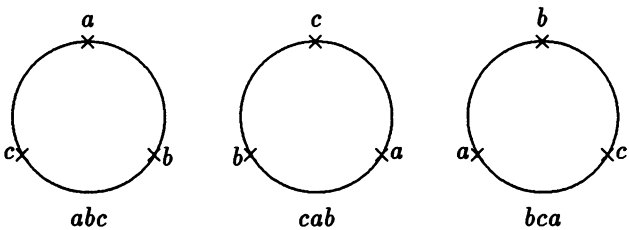


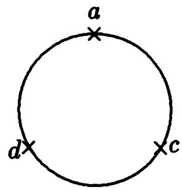
Figure 1.3.2.

Let A be a set of n distinct objects. For $0 \leq r \leq n$, an ***r-circular permutation*** of A is a circular permutation of any r distinct objects taken from A. Let Q_r^n denote the number of r-circular permutations of A. We shall derive a formula for Q_r^n.

Example 1.3.1. Let $A = \{a, b, c, d\}$. There are altogether $P_3^4(= 24)$ 3-permutations of A and they are shown in Example 1.2.1. These 24 3-permutations are re-grouped into 8 subsets as shown below:

abc *cab* *bca*	*acb* *bac* *cba*
abd *dab* *bda*	*adb* *bad* *dba*
acd *dac* *cda*	*adc* *cad* *dca*
bcd *dbc* *cdb*	*bdc* *cbd* *dcb*

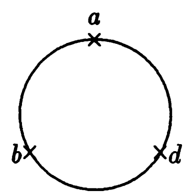
It is noted that every 3-circular permutation of A gives rise to a unique such subset. For instance,



$\Longrightarrow \quad \{acd, dac, cda\}$

Conversely, every such subset corresponds to a unique 3-circular permutation of A. For instance,

$\{adb, bad, dba\} \quad \Longrightarrow$

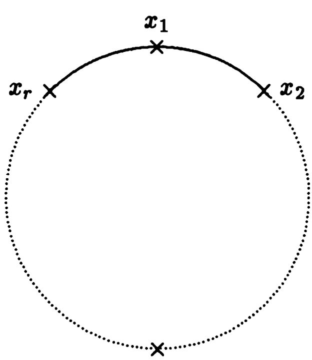


Thus we see that

$$Q_3^4 = \frac{24}{3} = 8. \quad \blacksquare$$

Example 1.3.1 tells us that $Q_3^4 = \frac{1}{3}P_3^4$. What is the relation between Q_r^n and P_r^n in general?

A circular permutation of r distinct objects $x_1, x_2, ..., x_r$ shown below:



gives rise to a unique subset of r r-permutations:

$$x_1x_2\cdots x_r,\ x_rx_1x_2\cdots x_{r-1},\ \ldots,\ x_2x_3\cdots x_rx_1$$

obtained through a rotation of the circular permutation. Conversely, every such subset of r r-permutations of A corresponds to a unique r-circular permutation of A. Since all the r-permutations of A can be equally divided into such subsets, we have

$$Q_r^n = \frac{P_r^n}{r} . \tag{1.3.1}$$

In particular,

$$Q_n^n = \frac{P_n^n}{n} = (n-1)! . \tag{1.3.2}$$

Example 1.3.2. In how many ways can 5 boys and 3 girls be seated around a table if

(i) there is no restriction?

(ii) boy B_1 and girl G_1 are not adjacent?

(iii) no girls are adjacent?

Solution (i) The number of ways is $Q_8^8 = 7!$.

(ii) The 5 boys and 2 girls not including G_1 can be seated in $(7-1)!$ ways. Given such an arrangement as shown in Figure 1.3.3, G_1 has $5(= 7-2)$ choices for a seat not adjacent to B_1. Thus the desired number of ways is

$$6! \times 5 = 3600.$$

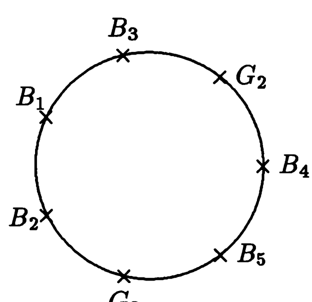


Figure 1.3.3.

We may obtain another solution by using what we call the Principle of Complementation as given below:

Principle of Complementation (CP) If A is a subset of a finite universal set $\mathcal{U}$, then

$$|\mathcal{U} \setminus A| = |\mathcal{U}| - |A|.$$

Now, the number of ways to arrange the 5 boys and 3 girls around a table so that boy B_1 and girl G_1 are adjacent (treating $\{B_1, G_1\}$ as an entity) is

$$(7-1)! \times 2 = 1440.$$

Thus the desired number of ways is by (CP),

$$7! - 1440 = 3600.$$

(iii) We first seat the 5 boys around the table in $(5-1)! = 4!$ ways. Given such an arrangement as shown in Figure 1.3.4, there are 5 ways to seat girl G_1. As no girls are adjacent, G_2 and G_3 have 4 and 3 choices respectively. Thus the desired number of ways is

$$4! \times 5 \times 4 \times 3 = 1440. \quad \blacksquare$$

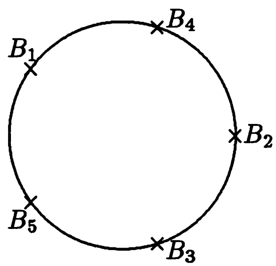


Figure 1.3.4.

Example 1.3.3. Find the number of ways to seat n married couples around a table in each of the following cases:

(i) Men and women alternate;

(ii) Every woman is next to her husband.

Solution. **(i)** The n men can first be seated in $(n-1)!$ ways. The n women can then be seated in the n spaces between two men in $n!$ ways. Thus the number of such arrangements is $(n-1)! \times n!$.

(ii) Each couple is first treated as an entity. The number of ways to arrange the n entities around the table is $(n-1)!$. Since the two people in each entity can be permuted in 2! ways, the desired number of ways is

$$(n-1)! \times 2^n.$$

Remark. A famous and much more difficult problem related to the above problem is the following: How many ways are there to seat n married couples $(n \geq 3)$ around a table such that men and women alternate and each woman is not adjacent to her husband? This problem, known as the problem of *ménages*, was first introduced by the French mathematician Francis Edward Anatole Lucas (1842 – 1891). A solution to this problem will be given in Chapter 4.

1.4. Combinations

Let A be a set of n distinct objects. A *combination* of A is simply a subset of A. More precisely, for $0 \leq r \leq n$, an *r-combination* of A is an r-element subset of A. Thus, for instance, if $A = \{a, b, c, d\}$, then the following consists of all the 3-combinations of A:

$$\{a, b, c\},\ \{a, b, d\},\ \{a, c, d\},\ \{b, c, d\}.$$

There are 4 in number. Let C_r^n or $\binom{n}{r}$ (which is read 'n choose r') denote the number of r-combinations of an n-element set A. Then the above example says that $C_3^4 = \binom{4}{3} = 4$. We shall soon derive a formula for C_r^n.

What is the difference between a permutation and a combination of a set of objects? A permutation is an *arrangement* of certain objects and thus the ordering of objects is important, whereas a combination is just a *set* of objects and thus the ordering of objects is immaterial. As a matter of fact, every r-permutation of A can be obtained in the following way:

Step 1. Form an r-combination B of A.

Step 2. Arrange the r objects of B in a row.

This provides us with a means to relate the numbers P_r^n and C_r^n. Indeed, we have by (MP):

$$P_r^n = C_r^n \times r!$$

and thus

$$C_r^n = \binom{n}{r} = \frac{P_r^n}{r!} = \frac{n!}{r!(n-r)!}. \tag{1.4.1}$$

In particular,

$$C_0^n = \binom{n}{0} = 1 \quad \text{and} \quad C_n^n = \binom{n}{n} = 1.$$

Note that

$$C_r^n = \frac{n!}{r!(n-r)!} = \frac{n!}{(n-r)!(n-(n-r))!} = C_{n-r}^n,$$

i.e.

$$\binom{n}{r} = \binom{n}{n-r}. \tag{1.4.2}$$

For convenience, we show in Table 1.4.1 the values of $\binom{n}{r}$, where $0 \le r \le n \le 9$. For instance, we have $\binom{6}{3} = 20$ and $\binom{9}{4} = 126$.

$n \setminus r$	0	1	2	3	4	5	6	7	8	9
0	1									
1	1	1								
2	1	2	1							
3	1	3	3	1						
4	1	4	6	4	1					
5	1	5	10	10	5	1				
6	1	6	15	20	15	6	1			
7	1	7	21	35	35	21	7	1		
8	1	8	28	56	70	56	28	8	1	
9	1	9	36	84	126	126	84	36	9	1

Table 1.4.1. The values of $\binom{n}{r}$, $0 \le r \le n \le 9$

One can see from Table 1.4.1 that

$$\binom{8}{3} + \binom{8}{4} = 56 + 70 = 126 = \binom{9}{4}.$$

In general, we have:

Example 1.4.1. Prove that

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}, \tag{1.4.3}$$

where $n, r \in \mathbf{N}$ with $r \leq n$.

Proof. *Algebraic Proof.* By (1.4.1),

$$\begin{aligned}
\binom{n-1}{r-1} + \binom{n-1}{r} &= \frac{(n-1)!}{(r-1)!(n-r)!} + \frac{(n-1)!}{r!(n-1-r)!} \\
&= \frac{(n-1)!r + (n-1)!(n-r)}{r!(n-r)!} \\
&= \frac{(n-1)!(r+n-r)}{r!(n-r)!} \\
&= \frac{n!}{r!(n-r)!} = \binom{n}{r}. \quad \blacksquare
\end{aligned}$$

Combinatorial Proof. Let $A = \{1, 2, ..., n\}$. By definition, there are $\binom{n}{r}$ ways to form r-combinations S of A. We shall count the number of such S in a different way.

Every r-combination S of A either contains "1" or not. If $1 \in S$, the number of ways to form S is $\binom{n-1}{r-1}$. If $1 \notin S$, the number of ways to form S is $\binom{n-1}{r}$. Thus by (AP), we have

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}. \quad \blacksquare$$

Remark. In the second proof, we fix an enumeration problem and count it in two different ways, giving rise to an equality relating two different expressions. This is a useful way to derive combinatorial identities.

Example 1.4.2. By Example 1.1.4, there are 2^7 binary sequences of length 7. How many such sequences are there which contain 3 0's and 4 1's?

Solution. To form such a sequence of length 7:

$$\begin{array}{ccccccc}
\square & \square & \square & \square & \square & \square & \square \\
(1) & (2) & (3) & (4) & (5) & (6) & (7)
\end{array}$$

we first select 3 of the 7 spaces for '0' and then leave the remaining spaces for '1'. There are $\binom{7}{3}$ ways in the first step and $\binom{4}{4} = 1$ way in the next. Thus the number of such binary sequences is given by $\binom{7}{3}$. $\blacksquare$

Remarks. **(1)** In the above example, you may first select 4 of the 7 spaces for '1' and obtain the answer $\binom{7}{4}$, which is equal to $\binom{7}{3}$ by identity (1.4.2).

(2) In general, the number of binary sequences of length n with m 0's and $(n-m)$ 1's, where $0 \le m \le n$, is given by $\binom{n}{m}$.

Example 1.4.3. In how many ways can a committee of 5 be formed from a group of 11 people consisting of 4 teachers and 7 students if

(i) there is no restriction in the selection?

(ii) the committee must include exactly 2 teachers?

(iii) the committee must include at least 3 teachers?

(iv) a particular teacher and a particular student cannot be both in the committee?

Solution. **(i)** The number of ways is $\binom{11}{5} = 11!/(5!6!) = 462$.

(ii) We first select 2 teachers from 4 and then $(5-2)$ students from 7. The number of ways is

$$\binom{4}{2}\binom{7}{3} = 6 \times 35 = 210.$$

(iii) There are two cases: either 3 teachers or 4 teachers are in the committee. In the former case, the number of ways is

$$\binom{4}{3}\binom{7}{2} = 4 \times 21 = 84,$$

while in the latter, the number of ways is

$$\binom{4}{4}\binom{7}{1} = 7.$$

Thus by (AP), the desired number of ways is $84 + 7 = 91$.

(iv) Let T be the particular teacher and S the particular student. We first find the number of ways to form a committee of 5 which includes both T and S. Evidently, such a committee of 5 can be formed by taking the union of $\{T, S\}$ and a subset of 3 from the remaining 9 people. Thus the number of ways to form a committee of 5 including T and S is $\binom{9}{3} = 84$.

Hence the number of ways to form a committee of 5 which does not include both T and S is by (CP):

$$\binom{11}{5} - \binom{9}{3} = 462 - 84 = 378, \quad \text{by (i).} \quad \blacksquare$$

Suppose that there are 8 players a, b, c, d, e, f, g, h taking part in the singles event of a tennis championship. In the first round of the competition, they are divided into 4 pairs so that the two players in each pair play against each other. There are several ways to do so. For instance,

$$\begin{array}{lllll} (1) & a \text{ vs } b, & c \text{ vs } f, & d \text{ vs } h, & e \text{ vs } g, \end{array}$$

or

$$\begin{array}{lllll} (2) & a \text{ vs } h, & b \text{ vs } g, & c \text{ vs } f, & d \text{ vs } e. \end{array}$$

What is the number of ways that this arrangement can be made?

To phrase this sort of questions mathematically, let A be a set of $2n$ distinct objects. A *pairing* of A is a partition of A into 2-element subsets; i.e., a collection of pairwise disjoint 2-element subsets whose union is A. For instance, if A is the set of 8 players $\{a, b, c, d, e, f, g, h\}$ as given above, then

$$\begin{aligned} &\{\{a, b\}, \{c, f\}, \{d, h\}, \{e, g\}\} \\ \text{and} \quad &\{\{a, h\}, \{b, g\}, \{c, f\}, \{d, e\}\} \end{aligned}$$

are different pairings of A. We note that the order of the subsets and the order of the 2 elements in each subset are immaterial.

Example 1.4.4. Let A be a $2n$-element set where $n \geq 1$. Find the number of different pairings of A.

Solution. We shall give 3 different methods for solving the problem.

Method 1. Pick an arbitrary element, say x, of A. The number of ways to select x's partner, say y, is $2n - 1$ (and $\{x, y\}$ forms a 2-element subset). Pick an arbitrary element, say z, from the $2n-2$ elements of $A \backslash \{x, y\}$. The number of ways to select z's partner is $2n - 3$. Continuing in this manner and applying (MP), the desired number of ways is given by

$$(2n - 1)(2n - 3) \cdots 5 \cdot 3 \cdot 1.$$

Method 2. First, form a 2-element subset of A and put it in position (1) as shown below. There are $\binom{2n}{2}$ ways to do so.

$$\underset{(1)}{\underline{\{\ ,\ \}}}\quad \underset{(2)}{\underline{\{\ ,\ \}}}\quad \underset{(3)}{\underline{\{\ ,\ \}}}\quad \underline{\{\ ,\ \}}\quad \cdots\quad \underset{(n)}{\underline{\{\ ,\ \}}}$$

Next, form a 2-element subset from the remainder of A and put it in the position (2). There are $\binom{2n-2}{2}$ ways to do so. Continuing in this manner and applying (MP), we see that the number of ways of arranging the n 2-element subsets *in a row* is:

$$\binom{2n}{2}\binom{2n-2}{2}\cdots\binom{4}{2}\binom{2}{2}.$$

Since the order of the n subsets is immaterial, the desired number of ways is:

$$\frac{\binom{2n}{2}\binom{2n-2}{2}\cdots\binom{4}{2}\binom{2}{2}}{n!}.$$

Method 3. We first arrange the $2n$ elements of A in a row by putting them in the $2n$ spaces as shown below:

$$\{\underset{(1)}{\quad},\underset{(2)}{\quad}\},\ \{\underset{(3)}{\quad},\underset{(4)}{\quad}\},\ \cdots,\ \{\underset{(2n-1)}{\quad},\underset{(2n)}{\quad}\}$$

There are $(2n)!$ ways to do so. Since the order of the elements in each 2-element subset and the order of the n subsets are immaterial, the desired number of ways is given by

$$\frac{(2n)!}{\underbrace{2!\times 2!\times\cdots\times 2!}_{n}\times n!}=\frac{(2n)!}{n!\times 2^n}.\quad \blacksquare$$

It can be checked that the above 3 answers are all the same.

The above problem can be generalized in the following way. Let A be a set of kn distinct elements, where $k, n \in \mathbf{N}$. A *k-grouping* of A is a partition of A into k-element subsets; i.e., a collection of pairwise disjoint k-element subsets whose union is A. Thus if $A = \{a_1, a_2, \ldots, a_{12}\}$, then

$$\{\{a_1, a_4, a_9, a_{12}\}, \{a_2, a_5, a_8, a_{10}\}, \{a_3, a_6, a_7, a_{11}\}\}$$

is a 4-grouping of A. Clearly, a pairing of a $2n$-element set A is a 2-grouping of A. What is the number of different k-groupings of a set with kn elements? (See Problem 1.43.)

Example 1.4.5. (IMO, 1989/3) Let n and k be positive integers and let S be a set of n points in the plane such that

(i) no three points of S are collinear, and

(ii) for any point P of S, there are at least k points of S equidistant from P.

Prove that $k < \frac{1}{2} + \sqrt{2n}$.

Proof. For convenience, we call a line segment in the plane an *edge* if it joins up any two points in S. Let ℓ be the number of edges in the plane. We shall consider the quantity ℓ.

First, since there are n distinct points in S and any two of them determine an edge, we have,

$$\ell = \binom{n}{2}. \tag{1}$$

Next, for each point P of S, by condition (ii), one can draw a circle with centre P whose circumference $C(P)$ contains at least k points of S. Clearly, the points of S on $C(P)$ determine at least $\binom{k}{2}$ edges. As there are n points P in S, the total number of these edges, counted with repetition, is at least $n\binom{k}{2}$.

Now, let us look at those edges which are counted more than once. An edge is counted more than once when and only when it is a common chord of at least 2 circles. Since two circles can have at most one common chord and there are n such circles, the number of common chords, counted with repetition, is at most $\binom{n}{2}$. Thus

$$\ell \geq n\binom{k}{2} - \binom{n}{2}. \tag{2}$$

Combining (1) with (2), we have

$$n\binom{k}{2} - \binom{n}{2} \leq \binom{n}{2}$$

or

$$n\binom{k}{2} \leq 2\binom{n}{2},$$

which implies that

$$k^2 - k - 2(n-1) \leq 0.$$

Hence

$$\begin{aligned} k &\le \frac{1+\sqrt{1+8(n-1)}}{2} \\ &< \frac{1}{2} + \frac{1}{2}\sqrt{8n} = \frac{1}{2} + \sqrt{2n}, \end{aligned}$$

as required. ∎

Comments. (1) In the above proof, the quantity "ℓ" is first introduced, and it is then counted as well as estimated from two different perpectives, thereby leading to the inequality: $n\binom{k}{2} - \binom{n}{2} \le \ell = \binom{n}{2}$. This is a common and useful technique, in combinatorics, in establishing inequalities linking up some parameters.

(2) From the proof above, we see that condition (i) is not necessary since, even if A, B, C are three collinear points, AB, BC, CA are regarded as three distinct edges in the above argument.

In Section 1.3, we studied circular permutations, which are arrangements of objects around a circle. We shall extend such arrangements to more than one circle.

Example 1.4.6. If there must be at least one person in each table, in how many ways can 6 people be seated

(i) around two tables?

(ii) around three tables?

(We assume that the tables are indistinguishable.)

Solution. (i) For 2 tables, there are 3 cases to consider according to the numbers of people to be seated around the 2 respective tables, namely,

$$(1)\quad 5+1 \qquad (2)\quad 4+2 \qquad (3)\quad 3+3.$$

Case (1). There $\binom{6}{5}$ ways to divide the 6 people into 2 groups of sizes 5 and 1 each. By formula (1.3.2), the 5 people chosen can be seated around a table in $(5-1)!$ ways and the 1 chosen in $0!$ way around the other. Thus by (MP), the number of ways in this case is

$$\binom{6}{5} \times 4! \times 0! = 144.$$

Case (2). There are $\binom{6}{4}$ ways to divide the 6 people into 2 groups of size 4 and 2 each. Thus, again, the number of ways in this case is

$$\binom{6}{4} \times 3! \times 1! = 90.$$

Case (3). We have to be careful in this case. The number of ways to divide the 6 people into 2 groups of size 3 each is $\frac{1}{2}\binom{6}{3}$ (why?). Thus the number of arrangements is

$$\frac{1}{2}\binom{6}{3} \times 2! \times 2! = 40.$$

Hence by (AP), the desired number of arrangements is $144+90+40 = 274$.

(ii) For 3 tables, there are also 3 cases to consider depending on the number of people distributed to the 3 respective tables, namely,

$$(1)\quad 4+1+1 \qquad (2)\quad 3+2+1 \qquad (3)\quad 2+2+2.$$

The number of arrangements in these cases are given below:

$$\begin{aligned}
&(1)\quad \frac{1}{2}\binom{6}{4}\binom{2}{1} \times 3! \times 0! \times 0! = 90;\\
&(2)\quad \binom{6}{3}\binom{3}{2} \times 2! \times 1! = 120;\\
&(3)\quad \frac{1}{3!}\binom{6}{2}\binom{4}{2} \times 1! \times 1! \times 1! = 15.
\end{aligned}$$

Hence by (AP), the desired number of arrangements is $90 + 120 + 15 = 225$. ∎

Given $r, n \in \mathbf{Z}$ with $0 \leq n \leq r$, let $s(r, n)$ denote the number of ways to arrange r distinct objects around n (indistinguishable) circles such that each circle has at least one object. These numbers $s(r, n)$ are called the *Stirling numbers of the first kind*, named after James Stirling (1692–1770). From Example 1.4.6, we see that $s(6, 2) = 274$ and $s(6, 3) = 225$. Other obvious results are:

$$\begin{aligned}
s(r, 0) &= 0 && \text{if } r \geq 1,\\
s(r, r) &= 1 && \text{if } r \geq 0,\\
s(r, 1) &= (r-1)! && \text{for } r \geq 2,\\
s(r, r-1) &= \tbinom{r}{2} && \text{for } r \geq 2.
\end{aligned}$$

The following result, which resembles (1.4.3) in Example 1.4.1, tells us how to compute $s(r, n)$ for larger r and n from smaller r and n.

Example 1.4.7. Show that

$$s(r,n) = s(r-1,n-1) + (r-1)s(r-1,n) \tag{1.4.4}$$

where $r, n \in \mathbf{N}$ with $n \leq r$.

Proof. For simplicity, we denote the r distinct objects by $1, 2, ..., r$. Consider the object "1". In any arrangement of the objects, either (i) "1" is the only object in a circle or (ii) "1" is mixed with others in a circle. In case (i), there are $s(r-1, n-1)$ ways to form such arrangements. In case (ii), first of all, the $r-1$ objects $2, 3, ..., r$ are put in n circles in $s(r-1, n)$ ways; then "1" can be placed in one of the $r-1$ distinct spaces to the "immediate right" of the corresponding $r-1$ distinct objects. By (MP), there are $(r-1)s(r-1, n)$ ways to form such arrangements in case (ii). The identity now follows from the definition of $s(r, n)$ and (AP). ∎

Using the initial values $s(0,0) = 1, s(r,0) = 0$ for $r \geq 1$ and $s(r,1) = (r-1)!$ for $r \geq 1$, and applying the identity (1.4.4), one can easily find out the values of $s(r, n)$ for small r and n. For r, n with $0 \leq n \leq r \leq 9$, the values of $s(r, n)$ are recorded in Table 1.4.2.

$r \backslash n$	0	1	2	3	4	5	6	7	8	9
0	1									
1	0	1								
2	0	1	1							
3	0	2	3	1						
4	0	6	11	6	1					
5	0	24	50	35	10	1				
6	0	120	274	225	85	15	1			
7	0	720	1764	1624	735	175	21	1		
8	0	5040	13068	13132	6769	1960	322	28	1	
9	0	40320	109584	118124	67284	22449	4536	546	36	1

Table 1.4.2. The values of $s(r, n)$, $0 \leq n \leq r \leq 9$

1.5. The Injection and Bijection Principles

Suppose that a group of n students attend a lecture in a lecture theatre with 200 seats. Assume that no student occupies more than one seat and no two students share a seat. If it is known that every student has a seat, then we must have $n \leq 200$. If it is known, furthermore, that no seat is vacant, then we are sure that $n = 200$ without actually counting the number of students. This is an example which illustrates two simple principles that we are going to state. Before doing so, we first give some definitions. Let A, B be finite sets. A mapping $f : A \to B$ from A to B is *injective* (or *one-one*) if $f(a_1) \neq f(a_2)$ in B whenever $a_1 \neq a_2$ in A. f is *surjective* (or *onto*) if for any $b \in B$, there exists $a \in A$ such that $f(a) = b$. f is *bijective* if f is both injective and surjective. Every injective (resp., surjective and bijective) mapping is also called an *injection* (resp., a *surjection* and a *bijection*).

The Injection Principle (IP) Let A and B be two finite sets. If there is an injection from A to B, then $|A| \leq |B|$.

The Bijection Principle (BP) Let A and B be two finite sets. If there is an bijection from A to B, then $|A| = |B|$.

Just like (AP), (MP) and (CP), the two principles (IP) and (BP) as given above are also trivially true. However, as we will see below, they are also useful and powerful as tools for solving counting problems.

Example 1.5.1. A student wishes to walk from the corner X to the corner Y through streets as given in the street map shown in Figure 1.5.1. How many shortest routes are there from X to Y available to the student?

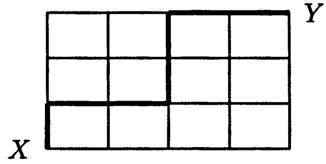


Figure 1.5.1.

Solution. Let A be the set of all shortest routes from X to Y. We shall find $|A|$.

We first note that every route in A consists of 7 continuous segments (a segment is part of a street connecting two adjacent junctions) of which 4 are horizontal and 3 vertical. Thus if we use a '0' to denote a horizontal segment and a '1' to denote a vertical segment, then every route in A can be uniquely represented by a binary sequence of length 7 with 4 0's and 3 1's (for instance, the shortest route shown by bold line segments in Figure 1.5.1 is represented by 1001100). This way of representing a route clearly defines a mapping f from A to the set B of all binary sequences of length 7 with 4 0's and 3 1's. It is easy to see that f is both one-one and onto, and hence it is a bijection from A to B. Thus by (BP) and Example 1.4.2, we have $|A| = |B| = \binom{7}{3}$. ∎

Remark. The street map of Figure 1.5.1 is a 5×4 rectangular grid. In general, if it is an $(m+1) \times (n+1)$ rectangular grid consisting of $m+1$ vertical streets and $n+1$ horizontal streets, then the number of shortest routes form the southwest corner X to the northeast corner Y is equal to the number of binary sequences of length $m+n$ with m 0's and n 1's, which by Remark (2) of Example 1.4.2, is given by

$$\binom{m+n}{m} \text{ or } \binom{m+n}{n}.$$

Given a set X, the *power set* of X, denoted by $\mathcal{P}(X)$, is the set of all subsets of X, inclusive of X and the empty set $\emptyset$. Thus, for instance, if $X = \{1,2,3\}$, then

$$\mathcal{P}(X) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, X\}.$$

We observe that $|\mathcal{P}(X)| = 8$. In general, what can be said about $|\mathcal{P}(X)|$ if X consists of n distinct elements?

Example 1.5.2. Show that if $|X| = n$, then $|\mathcal{P}(X)| = 2^n$ for all $n \in \mathbf{N}$.

Proof. We may assume that $X = \{1, 2, ..., n\}$. Now, let

$$B = \{a_1 a_2 \dots a_n \mid a_i = 0 \text{ or } 1, \;\; i = 1, 2, ..., n\}$$

be the set of all binary sequences of length n.

Define a mapping $f : \mathcal{P}(X) \to B$ as follows: For each $S \in \mathcal{P}(X)$ (i.e., $S \subseteq X$), we put

$$f(S) = b_1 b_2 \dots b_n$$

where

$$b_i = \begin{cases} 1 & \text{if } i \in S, \\ 0 & \text{if } i \notin S. \end{cases}$$

(For instance, if $X = \{1, 2, 3, 4, 5\}$, $S_1 = \{4\}$ and $S_2 = \{2, 3, 5\}$, then $f(S_1) = 00010$ and $f(S_2) = 01101$.) It is easy to see that f is a bijection from $\mathcal{P}(X)$ to B. Thus by (BP), $|\mathcal{P}(X)| = |B|$. Since $|B| = 2^n$ by Example 1.1.4, we have $|\mathcal{P}(X)| = 2^n$, as required. ∎

Example 1.5.3. Let $X = \{1, 2, ..., n\}$, where $n \in \mathbf{N}$. Show that the number of r-combinations of X which contain no consecutive integers is given by

$$\binom{n - r + 1}{r},$$

where $0 \leq r \leq n - r + 1$.

As an illustration, consider $X = \{1, 2, ..., 7\}$. All the 3-combinations of X containing no consecutive integers are listed below:

$$\begin{array}{lllll} \{1,3,5\}, & \{1,3,6\}, & \{1,3,7\}, & \{1,4,6\}, & \{1,4,7\}, \\ \{1,5,7\}, & \{2,4,6\}, & \{2,4,7\}, & \{2,5,7\}, & \{3,5,7\}. \end{array}$$

There are 10 in number and $\binom{7-3+1}{3} = 10$.

Proof. Let A be the set of r-combinations of X containing no consecutive integers, and B be the set of r-combinations of Y, where

$$Y = \{1, 2, ..., n - (r - 1)\}.$$

We shall establish a bijection from A to B.

Let $S = \{s_1, s_2, s_3, ..., s_r\}$ be a member in A. We may assume that $s_1 < s_2 < s_3 < \cdots < s_r$. Define

$$f(S) = \{s_1, s_2 - 1, s_3 - 2, ..., s_r - (r - 1)\}.$$

Observe that as s_i and s_{i+1} are non-consecutive, all the numbers in $f(S)$ are distinct. Thus $f(S) \in B$, and so f is a mapping from A to B. It is easy to see that f is injective. To see that f is surjective, let $T = \{t_1, t_2, t_3, ..., t_r\}$ be a member in B. Consider

$$S = \{t_1, t_2 + 1, t_3 + 2, ..., t_r + (r - 1)\}.$$

It can be checked that S is a member in A. Also $f(S) = T$ by definition.

This shows that $f : A \to B$ is a bijection. Hence by (BP), we have

$$|A| = |B| = \binom{n - r + 1}{r}. \quad \blacksquare$$

Remark. The above problem can be extended in the following way. Given $m \in \mathbf{N}$, a set $S = \{a_1, a_2, ..., a_r\}$ of positive integers, where $a_1 < a_2 < ... < a_r$, is said to be *m-separated* if $a_i - a_{i-1} \geq m$ for each $i = 2, 3, ..., r$. Thus S is 2-separated if and only if S contains no consecutive integers. Let $X = \{1, 2, ..., n\}$, where $n \in \mathbf{N}$. Using (BP), we can also find a formula for the number of r-element subsets of X which are m-separated. Readers who are interested may see Problem 1.91.

In the above three examples, three sets A (namely, the set of shortest routes, the power set $\mathcal{P}(X)$ and the set of r-combinations containing no consecutive integers) and their counterparts B (namely, the set of binary sequences of length 7 with 4 0's, the set of binary sequences of length n and the set of r-combinations, respectively) are considered. By establishing a bijection from A to B, we have by (BP), $|A| = |B|$. Note that in each case, the enumeration of $|A|$ by itself is not straightforward, while that of $|B|$ is fairly standard and so much easier. Thus with (BP), we can sometimes transform a hard problem into an easier one. This is always a crucial step in the process of solving a problem.

The above three examples involve only the applications of (BP). In what follows, we shall give an interesting example which makes use of both (BP) and (IP).

Example 1.5.4. (IMO, 1989/6) A permutation $x_1x_2...x_{2n}$ of the set $\{1,2,...,2n\}$, where $n \in \mathbf{N}$, is said to have property P if $|x_i - x_{i+1}| = n$ for at least one i in $\{1,2,...,2n-1\}$. Show that, for each n, there are more permutations with property P than without.

This problem, proposed by Poland, was placed last in the set of the six problems for the 1989 IMO, and was considered a more difficult problem among the six. The original proof given by the proposer makes use of recurrence relations which is rather long and looks hard. However, it was a pleasant surprise that a contestant from the China team was able to produce a shorter and more elegant proof of the result. Before we see this proof, let us first try to understand the problem better.

Let $S = \{1,2,...,2n\}$. Clearly, a permutation of S does not have property P if and only if for each $k = 1,2,...,n$, the pair of numbers k and $n+k$ are not adjacent in the permutation. For $n = 2$, the set A of permutations without P and the set B of permutations with P are given below:

$$\begin{aligned} A =& \{1234,\ 1432,\ 2143,\ 2341,\ 3214,\ 3412,\ 4123,\ 4321\}, \\ B =& \{1\underline{24}3,\ \underline{13}\,\underline{24},\ \underline{13}\,\underline{42},\ 1\underline{42}3,\ 2\underline{13}4,\ 2\underline{31}4,\ \underline{24}\,\underline{13},\ \underline{24}\,\underline{31}, \\ & \underline{31}\,\underline{24},\ \underline{31}\,\underline{42},\ 3\underline{24}1,\ 3\underline{42}1,\ 4\underline{13}2,\ \underline{42}\,\underline{13},\ \underline{42}\,\underline{31},\ 4\underline{31}2\}. \end{aligned}$$

Clearly, $|B| = 16 > 8 = |A|$.

Proof. The case when $n = 1$ is trivial. Assume that $n \geq 2$. Let A (resp., B) be the set of permutations of $S = \{1,2,...,2n\}$ without property P (resp., with P). To show that $|B| > |A|$, by (IP) and (BP), it suffices to establish a mapping $f : A \rightarrow B$ which is injective but not surjective.

For convenience, any number in the pair $\{k, n+k\}$ $(k = 1,2,...,n)$ is called the *partner* of the other. If k and $n+k$ are adjacent in a permutation, the pair $\{k, n+k\}$ is called an *adjacent pair of partners*.

Let $\alpha = x_1x_2...x_{2n}$ be an element in A. Since α does not have property P, the partner of x_1 is x_r where $3 \leq r \leq 2n$. Now we put

$$f(\alpha) = x_2x_3...x_{r-1}\underline{x_1x_r}x_{r+1}...x_{2n}$$

by taking x_1 away and placing it just in front of its partner x_r. In $f(\alpha)$, it is clear that $\{x_1, x_r\}$ is the only adjacent pair of partners. (Thus, for instance, $f(\underline{1}2\underline{3}4) = 2\underline{13}4$ and $f(\underline{2}1\underline{4}3) = 1\underline{24}3$.) Obviously, $f(\alpha) \in B$ and f defines a mapping from A to B.

We now claim that f is injective. Let

$$\alpha = x_1 x_2 \ldots x_{2n}$$

$$\beta = y_1 y_2 \cdots y_{2n}$$

be elements of A in which x_1's partner is x_r and y_1's partner is y_s, where $3 \le r, s \le 2n$. Suppose $f(\alpha) = f(\beta)$; i.e.,

$$x_2 x_3 \ldots x_{r-1} \underline{x_1 x_r} \ldots x_{2n} = y_2 y_3 \cdots y_{s-1} \underline{y_1 y_s} \cdots y_{2n}.$$

Since $\{x_1, x_r\}$ (resp., $\{y_1, y_s\}$) is the only adjacent pair of partners in $f(\alpha)$ (resp., $f(\beta)$), we must have $r = s$, $x_1 = y_1$ and $x_r = y_s$. These, in turn, imply that $x_i = y_i$ for all $i = 1, 2, \ldots, 2n$ and so $\alpha = \beta$, showing that f is injective.

Finally, we note that $f(A)$ consists of all permutations of S having exactly one adjacent pair of partners while there are permutations of S in B which contain more than one adjacent pair of partners. Thus we have $f(A) \subset B$, showing that f is not surjective. The proof is thus complete. ∎

1.6. Arrangements and Selections with Repetitions

In the previous sections we studied arrangements and selections of elements from a set in which no repetitions are allowed. In this section we shall consider arrangements and selections in which elements are allowed to be repeated.

Example 1.6.1. Let $A = \{a, b, c\}$. All the 2-permutations of A with repetitions allowed are given below:

$$aa,\ ab,\ ac,\ ba,\ bb,\ bc,\ ca,\ cb,\ cc.$$

There are 9 in number. ∎

In general, we have:

(I) The number of r-permutations of the set

$$A = \{a_1, a_2, \ldots, a_n\},$$

where $r, n \in \mathbf{N}$, with repetitions allowed, is given by n^r.

Proof of (I). There are n choices for the first object of an r-permutation. Since repetitions are allowed, there are again n choices for each of the remaining $r-1$ objects of an r-permutation. Thus the number of such permutations is, by (MP), $\underbrace{n \cdot n \cdots n}_{r} = n^r$. ∎

Example 1.6.2. A 4-storey house is to be painted by some 6 different colours such that each storey is painted in one colour. How many ways are there to paint the house?

Solution. This is the number of 4-permutations of the set $\{1, 2, ..., 6\}$ of 6 colours with repetitions allowed. By (I), the desired number is 6^4. ∎

For those permutations considered in (I), an element of the set A can be repeated any number of times. We now consider another type of permutations in which the number of times an element can be repeated is limited.

Example 1.6.3. Find the number of permutations of the 5 letters: a, a, a, b, c.

Solution. Let α be the desired number of such permutations. Fix one of them, say $abaac$. Imagine now that the 3 a's are distinct, say a_1, a_2, a_3. We then observe from the following exhibition

$$\boxed{abaac} \quad \longleftrightarrow \quad \boxed{\begin{array}{l} a_1ba_2a_3c \\ a_1ba_3a_2c \\ a_2ba_1a_3c \\ a_2ba_3a_1c \\ a_3ba_1a_2c \\ a_3ba_2a_1c \end{array}}$$

that "$abaac$" corresponds to a set of $3! = 6$ permutations of the set $\{a_1, a_2, a_3, b, c\}$ keeping the pattern of $abaac$, and vice versa.

Since there are $5!$ permutations of $\{a_1, a_2, a_3, b, c\}$, we have

$$\alpha \cdot 3! = 5!,$$

$$\text{i.e.,} \quad \alpha = \frac{5!}{3!} = 20. \quad ∎$$

In general, we have:

> (II) Consider a collection of r objects, in which r_1 are of type 1, r_2 are of type 2, ..., and r_n are of type n, where $r_1 + r_2 + \cdots + r_n = r$. The number of different permutations of the collection of objects, denoted by $P(r; r_1, r_2, ..., r_n)$, is given by
> $$P(r; r_1, r_2, ..., r_n) = \frac{r!}{r_1! r_2! \cdots r_n!}.$$

One may extend the idea shown in Example 1.6.3 to prove (II). We give a different approach here.

Proof of (II). In any permutation of the collection of r objects, there are r_i positions to place the r_i objects of type i, for each $i = 1, 2, ..., n$. Different choices of positions give rise to different permutations.

We first choose r_1 positions from the r distinct positions and place the r_1 identical objects of type 1 at the r_1 positions chosen. There are $\binom{r}{r_1}$ ways to do so. Next, we choose r_2 positions from the remaining $r - r_1$ positions and place the r_2 identical objects of type 2 at the r_2 positions chosen. There are $\binom{r-r_1}{r_2}$ ways to do so. We proceed in this manner till the final step when we choose r_n positions from the remaining $r - (r_1 + r_2 + \cdots + r_{n-1})(= r_n)$ positions and place the r_n identical objects of type n at the r_n positions left. There are $\binom{r-(r_1+r_2+\cdots+r_{n-1})}{r_n}$ ways to do so. By (MP) and formula (1.4.1), we then have

$$P(r; r_1, r_2, ..., r_n) = \binom{r}{r_1}\binom{r - r_1}{r_2} \cdots \binom{r - (r_1 + r_2 + \cdots r_{n-1})}{r_n}$$
$$= \frac{r!}{r_1!(r - r_1)!} \cdot \frac{(r - r_1)!}{r_2!(r - r_1 - r_2)!} \cdots \frac{(r - r_1 - r_2 - \cdots - r_{n-1})!}{r_n!(r - r_1 - r_2 - \cdots - r_n)!}$$
$$= \frac{r!}{r_1! r_2! \cdots r_n!}. \quad \blacksquare$$

The results (I) and (II) may be rephrased in a more convenient way by using a notion, called a multi-set. Just like a set, a *multi-set* is a collection of objects, but its members need not be distinct. Thus, for instance, $M = \{a, b, a, c, b, a\}$ is a multi-set consisting of 3 a's, 2 b's and 1 c. This multi-set can be written in a neater way as $M = \{3 \cdot a, 2 \cdot b, c\}$. In general, the multi-set

$$M = \{r_1 \cdot a_1, r_2 \cdot a_2, ..., r_n \cdot a_n\},$$

where $n, r_1, r_2, ..., r_n$ are nonnegative integers and $a_1, a_2, ..., a_n$ are distinct objects, consists of r_1 a_1's, r_2 a_2's, ... and r_n a_n's. For each $i = 1, 2, ..., n$, the number r_i is called the ***repetition number*** of the object a_i. For convenience, given an object a, we may write $\infty \cdot a$ to indicate that a can be repeated an infinite number of times. Thus a multi-set in which b and e occur an infinite number of times, and a, c, d have, respectively, the repetition numbers 2, 7, 4, is denoted by $\{2 \cdot a, \infty \cdot b, 7 \cdot c, 4 \cdot d, \infty \cdot e\}$.

An ***r-permutation*** of $M = \{r_1 \cdot a_1, r_2 \cdot a_2, \ldots, r_n \cdot a_n\}$ is an arrangement of r objects taken from M with at most r_i of a_i $(i = 1, 2, ..., n)$ in a row. A ***permutation*** of M is an arrangement of all the objects of M in a row. An ***r-permutation*** of the multi-set $\{\infty \cdot a_1, \infty \cdot a_2, \ldots, \infty \cdot a_n\}$ is similarly defined, except that the number of a_i's chosen is not limited for all i.

Using the above terminology, we may re-state the results (I) and (II) as follows:

(I) The number of r-permutations of the multi-set

$$\{\infty \cdot a_1, \infty \cdot a_2, \ldots, \infty \cdot a_n\}$$

is given by n^r.

(II) Let $M = \{r_1 \cdot a_1, r_2 \cdot a_2, ..., r_n \cdot a_n\}$ and $r = r_1 + r_2 + \cdots + r_n$. Then the number $P(r; r_1, r_2, ..., r_n)$ of permutations of M is given by

$$P(r; r_1, r_2, ..., r_n) = \frac{r!}{r_1! r_2! \cdots r_n!}.$$

Example 1.6.4. Find the number of ternary sequences of length 10 having two 0's, three 1's and five 2's.

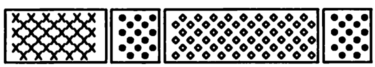
Solution. The number of such ternary sequences is the number of permutations of the multi-set $\{2 \cdot 0, 3 \cdot 1, 5 \cdot 2\}$, which is equal to

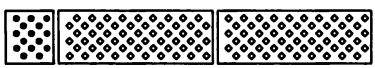
$$\frac{10!}{2!3!5!} = 2520$$

by (II). ∎

Example 1.6.5. Find the number of ways to pave a 1×7 rectangle by 1×1, 1×2 and 1×3 blocks, assuming that blocks of the same size are indistinguishable.

As an illustration, two ways of paving are shown below:





For $i = 1, 2, 3$, we let b_i denote an $1 \times i$ block. Thus the first way shown above may be represented by $b_2b_1b_3b_1$, which is a permutation of $\{2 \cdot b_1, b_2, b_3\}$, while the second way by $b_1b_3b_3$, which is a permutation of $\{b_1, 2 \cdot b_3\}$. Note that in each case, the sum of the sub-indices of b_i's is "7".

Solution. From the above illustration, we see that the desired number of ways is equal to the number of permutations of some b_i's such that the sum of the sub-indices of such b_i's is 7. The following 8 cases cover all the possibilities:

(i)	$\{7 \cdot b_1\}$	(ii)	$\{5 \cdot b_1, b_2\}$
(iii)	$\{4 \cdot b_1, b_3\}$	(iv)	$\{3 \cdot b_1, 2 \cdot b_2\}$
(v)	$\{2 \cdot b_1, b_2, b_3\}$	(vi)	$\{b_1, 3 \cdot b_2\}$
(vii)	$\{b_1, 2 \cdot b_3\}$	(viii)	$\{2 \cdot b_2, b_3\}$.

For each case, the number of permutations of the multi-set is shown below:

(i)	1	(ii)	$\frac{6!}{5!} = 6$
(iii)	$\frac{5!}{4!} = 5$	(iv)	$\frac{5!}{3!2!} = 10$
(v)	$\frac{4!}{2!} = 12$	(vi)	$\frac{4!}{3!} = 4$
(vii)	$\frac{3!}{2!} = 3$	(viii)	$\frac{3!}{2!} = 3.$

Thus the desired number of ways is

$$1 + 6 + 5 + 10 + 12 + 4 + 3 + 3 = 44. \quad \blacksquare$$

Example 1.6.6. Show that $(4n)!$ is a multiple of $2^{3n} \cdot 3^n$, for each natural number n.

Proof. Consider the multi-set

$$M = \{4 \cdot a_1, 4 \cdot a_2, ..., 4 \cdot a_n\}.$$

By (II),

$$P(4n; \underbrace{4, 4, ..., 4}_{n}) = \frac{(4n)!}{(4!)^n} = \frac{(4n)!}{(2^3 \cdot 3)^n} = \frac{(4n)!}{2^{3n} \cdot 3^n}.$$

The result now follows as $P(4n; 4, 4, ..., 4)$ is a whole number. $\blacksquare$

We now turn our attention to the problem of counting the number of combinations with repetitions.

Let $A = \{1, 2, 3, 4\}$. Then there are $\binom{4}{3} = 4$ ways to form 3-combinations of A in which no elements are repeated. Suppose now *elements are allowed to be repeated.* How many 3-combinations can be formed? One can find out the answer simply by listing all such 3-combinations as shown below. There are altogether 20 in number.

$$\begin{array}{lllll}
\{1,1,1\}, & \{1,2,2\}, & \{1,3,4\}, & \{2,2,4\}, & \{3,3,3\}, \\
\{1,1,2\}, & \{1,2,3\}, & \{1,4,4\}, & \{2,3,3\}, & \{3,3,4\}, \\
\{1,1,3\}, & \{1,2,4\}, & \{2,2,2\}, & \{2,3,4\}, & \{3,4,4\}, \\
\{1,1,4\}, & \{1,3,3\}, & \{2,2,3\}, & \{2,4,4\}, & \{4,4,4\}.
\end{array}$$

Let

$$M = \{\infty \cdot a_1, \infty \cdot a_2, ..., \infty \cdot a_n\}$$

be a given multi-set where $n \in \mathbf{N}$. A multi-set of the form

$$\{m_1 \cdot a_1, m_2 \cdot a_2, ..., m_n \cdot a_n\},$$

where m_i's are nonnegative integers, is called a $(m_1 + m_2 + \cdots + m_n)$-element *multi-subset* of M. Thus, as shown above, there are 20 3-element multi-subsets of the multi-set $\{\infty \cdot 1, \infty \cdot 2, \infty \cdot 3, \infty \cdot 4\}$. For a nonnegative integer r, let H_r^n denote the number of r-element multi-subsets of M. The above example shows that $H_3^4 = 20$. We shall find a formula for H_r^n. To get to this, let us consider the following example.

Example 1.6.7. There are 3 types of sandwiches, namely chicken (C), fish (F) and ham (H), available in a restaurant. A boy wishes to place an order of 6 sandwiches. Assuming that there is no limit in the supply of sandwiches of each type, how many such orders can the boy place?

Solution. This problem amounts to computing H_6^3. It is tedious to find H_6^3 by listing all 6-element multi-subsets of $\{\infty \cdot C, \infty \cdot F, \infty \cdot H\}$ as how we did before. We introduce an indirect way here.

The table below shows 4 different orders:

	C	F	H
(1)	o o	o	o o o
(2)	o	o o o o	o
(3)		o o	o o o o
(4)	o o o		o o o

(1) 2 chicken, 1 fish and 3 ham sandwiches,
(2) 1 chicken, 4 fish and 1 ham sandwiches,
(3) 2 fish and 4 ham sandwiches,
(4) 3 chicken and 3 ham sandwiches.

It is now interesting to note from the table that if we treat a "vertical stroke" as a '1', then order (1) can be uniquely represented by the binary sequence

$$00101000,$$

while (2), (3) and (4) respectively by

$$01000010,$$

$$10010000,$$

and

$$00011000.$$

In this way, we find that every order of 6 sandwiches corresponds to a binary sequence of length 8 with 6 0's and 2 1's, and different orders correspond to different binary sequences. On the other hand, every such binary sequence represents an order of 6 sandwiches. For instance, 01001000 represents the order of 1 chicken, 2 fish and 3 ham sandwiches. Thus, we see that there is a bijection between the set of such orders and the set of binary sequences with 6 0's and 2 1's. Hence by (BP) and Remark (2) of Example 1.4.2, the desired number of ways is $H_6^3 = \binom{8}{2}$. ∎

So, can you now generalize the above idea to obtain a formula for H_r^n?

Look at the following table. The first row of the table shows the n types of objects of the multi-set $M = \{\infty \cdot a_1, \infty \cdot a_2, ..., \infty \cdot a_n\}$ which are separated by $n-1$ vertical strokes.

$$\begin{array}{c c|c|c|c|c} & a_1 & a_2 & a_3 & & a_n \\ \hline S: & \underbrace{\circ\,\circ \ldots \circ}_{r_1} & \underbrace{\circ\,\circ \ldots \circ}_{r_2} & \underbrace{\circ\,\circ \ldots \circ}_{r_3} & \cdots & \underbrace{\circ\,\circ \ldots \circ}_{r_n} \end{array}$$

Using this framework, every multi-subset $S = \{r_1 \cdot a_1, r_2 \cdot a_2, ..., r_n \cdot a_n\}$ of M, where $r_i \geq 0$ for all i, can be represented by a row having r_i 0's within the interval under a_i. If we treat each vertical stroke as an '1', then every r-element multi-subset of M corresponds to a unique binary sequence of length $r + n - 1$ with r 0's and $(n-1)$ 1's. This correspondence is indeed a bijection between the family of all r-element multi-subsets of M and the family of all such binary sequences. Thus by (BP) and Remark (2) of Example 1.4.2, we obtain the following result.

(III) Let $M = \{\infty \cdot a_1, \infty \cdot a_2, ..., \infty \cdot a_n\}$. The number H_r^n of r-element multi-subsets of M is given by

$$H_r^n = \binom{r+n-1}{r}.$$

Result (III) can be proved in various ways. We give another proof below.

Another Proof of (III). For convenience, we represent a_i by i, $i = 1, 2, ..., n$, and so $M = \{\infty \cdot 1, \infty \cdot 2, ..., \infty \cdot n\}$. Let A be the family of all r-element multi-subsets of M and B be the family of all r-combinations of the set $\{1, 2, ..., r+n-1\}$. Define a mapping $f : A \to B$ as follows: For each r-element multi-subset $S = \{b_1, b_2, ..., b_r\}$ of M, where $1 \leq b_1 \leq b_2 \leq ... \leq b_r \leq n$, let

$$f(S) = \{b_1, b_2 + 1, b_3 + 2, ..., b_r + (r-1)\}.$$

It should be noted that members in $f(S)$ are distinct and so $f(S) \in B$. It is easy to see that f is injective. To show that f is surjective, let $T = \{c_1, c_2, ..., c_r\}$ be an r-combination of $\{1, 2, ..., r+n-1\}$ with $c_1 < c_2 < \cdots < c_r$. Consider

$$S = \{c_1, c_2 - 1, c_3 - 2, ..., c_r - (r-1)\}.$$

Observe that S is an r-element multi-subset of M and by definition, $f(S) = T$. This shows that f is surjective.

We thus conclude that f is a bijection from A to B. Hence by (BP),

$$H_r^n = |A| = |B| = \binom{r+n-1}{r}. \quad \blacksquare$$

Remarks. (1) Let $M' = \{p_1 \cdot a_1, p_2 \cdot a_2, ..., p_n \cdot a_n\}$ be a multi-set. From the above discussion, we see that the number of r-element multi-subsets

$$\{r_1 \cdot a_1, r_2 \cdot a_2, ..., r_n \cdot a_n\},$$

($0 \leq r_i \leq p_i$, for all i) of M' is also given by $\binom{r+n-1}{r}$ if $r \leq p_i$ for all i.

On the other hand, if $r > p_i$ for some i, then the above statement is invalid. This case will be studied in Chapter 5.

(2) The reader might have noticed that there is some similarity between the above proof and that given in Example 1.5.3. Indeed, the rule defining f here is the same as that defining f^{-1} there.

1.7. Distribution Problems

We consider in this section the following problem: Count the number of ways of distributing r objects into n *distinct* boxes satisfying certain conditions. We split our consideration into two cases: (1) objects are distinct, (2) objects are identical (or indistinguishable).

Case (1) Distributing *r distinct* objects into *n distinct* boxes.

(i) If each box can hold *at most one* object, then the number of ways to distribute the objects is given by

$$n(n-1)(n-2)\cdots(n-r+1) = P_r^n,$$

since object 1 can be put into any of the n boxes, object 2 into any of the $n-1$ boxes left, and so on.

(ii) If each box can hold any number of objects, then the number of ways to distribute the objects is given by

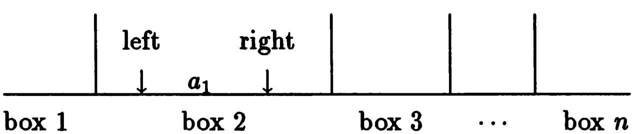
$$\underbrace{n \cdot n \cdots n}_{r} = n^r,$$

as each object can be put into any of the n boxes.

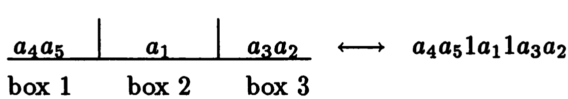
(iii) Assume that each box can hold any number of objects and the orderings of objects in each box count.

In this case, the 1st object, say a_1, can be put in any of the n places (namely, the n boxes); and the 2nd object, say a_2, can be put in any of the $n+1$ places (the $n-1$ boxes not containing a_1 plus the left and right positions of a_1 in the box containing a_1). Similarly, the 3rd object can be put in any of the $n+2$ places due to the presents of a_1 and a_2, and so on. Thus the number of ways that an arrangement can be made in this case is given by

$$n(n+1)(n+2)\cdots(n+(r-1)).$$



There is another way to solve the problem. As shown below,



$n=3$, $r=5$

one can establish a bijection between the set of such distributions of r distinct objects $a_1, a_2, \ldots, a_r$ into n distinct boxes and the set of arrangements of the multi-set $\{a_1, a_2, \ldots, a_r, (n-1)\cdot 1\}$ (we treate each vertical stroke separating adjacent boxes as a '1'). Thus by (BP) and result (II) in Section 1.6, the desired number of ways is given by

$$\frac{(n-1+r)!}{(n-1)!},$$

which agrees with the above result.

Case (2) Distributing r *identical* objects into n *distinct* boxes.

(i) Assume that each box can hold at most one object (and thus $r \leq n$).

In this case, there is a 1–1 correspondence between the ways of distribution and the ways of selecting r boxes from the given n distinct boxes. Thus the number of ways this can be done is given by $\binom{n}{r}$.

(ii) Assume that each box can hold any number of objects.

In this case, a way of distribution can be represented by

$$\{r_1 \cdot a_1, r_2 \cdot a_2, ..., r_n \cdot a_n\},$$

where r_i's are nonnegative integers with $r_1 + r_2 + \cdots + r_n = r$, which means that r_i objects are put in box i, $i = 1, 2, ..., n$. Thus a way of distribution can be considered as an r-element multi-subset of $M = \{\infty \cdot a_1, \infty \cdot a_2, ..., \infty \cdot a_n\}$, and conversely, every r-element multi-subset of M represents a way of distribution. Hence, the number of ways this can be done is given by

$$H_r^n = \binom{r+n-1}{r},$$

by result (III) in Section 1.6.

(iii) Assume that each box holds at least one object (and thus $r \geq n$); i.e., no box is empty.

In this case, we first put one object in each box to fulfill the requirement (this can be done in one way), and then distribute the remaining $r - n$ objects in the boxes in an arbitrary way. By (MP) and the result in (ii), the desired number of ways is given by

$$\binom{(r-n)+n-1}{r-n} = \binom{r-1}{r-n}.$$

By identity (1.4.2), this can also be written as

$$\binom{r-1}{n-1}.$$

Example 1.7.1. How many ways are there to arrange the letters of the word 'VISITING' if no two I's are adjacent?

Solution. *Method 1.* The letters used are V, S, T, N, G and 3 I's. We first arrange V, S, T, N, G in a row. There are 5! ways. Take one of these arrangements as shown below.

___	V	___	S	___	T	___	N	___	G	___
1st		2nd		3rd		4th		5th		6th

There are 6 spaces separated by the 5 letters. The problem is now reduced to that of distributing the 3 identical I's in the 6 places such that each place can hold at most one I (no 2 I's are adjacent). By Case (2)(i), the number of ways to do so is given by $\binom{6}{3}$. Thus by (MP), the desired number of ways is

$$5!\binom{6}{3}. \quad \blacksquare$$

In the above method, we first consider "V, S, T, N, G" and then 3 I's. In the next method, we reverse the order.

Method 2. We first arrange the 3 I's in a row in one way:

___	I	___	I	___	I	___
1st		2nd		3rd		4th

Then treat the 5 letters "V, S, T, N, G" as 5 identical "x". Since no 2 I's are adjacent, one 'x' must be put in the 2nd and 3rd places (this can be done in one way):

___	I	x	I	x	I	___
1st		2nd		3rd		4th

Now, the remaining 3 x's can be put in the 4 places arbitrarily in $\binom{3+4-1}{3} = \binom{6}{3}$ ways by Case (2)(ii). Finally, by restoring the original letters that each "x" represents (so 5! ways to arrange them) and by applying (MP), the desired number of ways is given by

$$\binom{6}{3} \cdot 5!. \quad \blacksquare$$

Example 1.7.2. (Example 1.2.3(ii) revisited) In how many ways can 7 boys and 3 girls be arranged in a row so that the 2 end-positions are occupied by boys and no girls are adjacent?

In Example 1.2.3, this problem was solved by considering the arrangement of boys followed by that of girls. In the following solution, we reverse the order.

Solution. The 3 girls can be arranged in 3! ways. Fix one of them:

$$\underset{\text{1st}}{\_\_} \quad G_1 \quad \underset{\text{2nd}}{\_\_} \quad G_2 \quad \underset{\text{3rd}}{\_\_} \quad G_3 \quad \underset{\text{4th}}{\_\_}$$

Then treat the 7 boys as 7 identical "x". To meet the requirements, one "x" must be placed in each of the 4 places separated by the 3 girls as shown below:

$$x \quad G_1 \quad x \quad G_2 \quad x \quad G_3 \quad x$$

Now the remaining 3 x's can be put in the 4 places arbitrarily in $\binom{3+4-1}{3} = \binom{6}{3}$ ways by Case (2)(ii). Finally, by restoring the meaning of 'x' and applying (MP), we obtain the desired number of ways:

$$3! \cdot \binom{6}{3} \cdot 7! = 7! \cdot 6 \cdot 5 \cdot 4. \quad \blacksquare$$

Example 1.7.3. (Example 1.5.3 revisited) Let $X = \{1, 2, ..., n\}$, where $n \in N$. Show that the number of r-combinations of X which contain no consecutive integers is given by $\binom{n-r+1}{r}$, where $0 \le r \le n - r + 1$.

Proof. We first establish a bijection between the set A of all such r-combinations of X and the set B of all binary sequences of length n with r 1's such that there is at least a '0' between any two 1's.

Define a mapping $f : A \to B$ as follows: given such an r-combination $S = \{k_1, k_2, ..., k_r\}$ of X, where $1 \le k_1 < k_2 < \ldots < k_r \le n$, let $f(S) = b_1 b_2 \cdots b_n$, where

$$b_i = \begin{cases} 1 & \text{if } i = k_1, k_2, ..., k_r, \\ 0 & \text{otherwise.} \end{cases}$$

For instance, if $n = 8$ and $r = 3$, then

$$f(\{2,4,7\}) = 01010010$$

$$\text{and} \qquad f(\{1,5,8\}) = 10001001.$$

It is easy to check that f is a bijection between A and B. Thus $|A| = |B|$.

Our next task is to count $|B|$. Observe that a binary sequence in B can be regarded as a way of distributing $n - r$ identical objects into $r + 1$ distinct boxes such that the 2nd, 3rd, ... and rth boxes are all nonempty as shown below:

$$\begin{array}{ccc|c|c|c}
01010010 & \longleftrightarrow & 0 & 0 & 00 & 0 \\ \hline
 & & \text{1st} & \text{2nd} & \text{3rd} & \text{4th}
\end{array}$$

$$\begin{array}{ccc|c|c|c}
10001001 & \longleftrightarrow & & 000 & 00 & \\ \hline
 & & \text{1st} & \text{2nd} & \text{3rd} & \text{4th}
\end{array}$$

To get one such distribution, we first put one object each in the 2nd, 3rd, ... and rth boxes. We then distribute the remaining $(n-r)-(r-1) = n-2r+1$ objects in an arbitrary way to the $r + 1$ boxes including the 1st and last boxes. The first step can be done in one way while the second step, by the result of Case (2)(ii), in

$$\binom{(n-2r+1)+(r+1)-1}{n-2r+1}$$

ways. Thus we have

$$|A| = |B| = \binom{n-r+1}{n-2r+1} = \binom{n-r+1}{r},$$

by identity (1.4.2). ∎

We now turn our attention to consider the following important and typical problem in combinatorics, namely, finding the number of integer solutions to the linear equation:

$$x_1 + x_2 + \cdots + x_n = r \tag{1.7.1}$$

in n unknowns $x_1, x_2, ..., x_n$, where r and n are integers with $r \geq 0$ and $n \geq 1$.

An *integer solution* to the equation (1.7.1) is an n-tuple $(e_1, e_2, ..., e_n)$ of integers satisfying (1.7.1) when x_i is substituted by e_i, for each $i = 1, 2, ..., n$. Thus, for instance, $(-1, -4, 7), (2, 0, 0), (0, 1, 1), (0, 2, 0)$ and $(0, 0, 2)$ are some integer solutions to the equation

$$x_1 + x_2 + x_3 = 2.$$

There are infinitely many integer solutions to (1.7.1). In this section, we shall confine ourselves to "nonnegative" integer solutions (i.e., $e_i \geq 0$, for all i).

Example 1.7.4. Show that the number of nonnegative integer solutions to equation (1.7.1) is given by

$$\binom{r+n-1}{r}.$$

Proof. Every nonnegative integer solution $(e_1, e_2, ..., e_n)$ to (1.7.1) corresponds to a way of distributing r identical objects to n distinct boxes as shown below:

$$\underset{\text{box } 1}{\overset{e_1}{\boxed{\circ\cdots\circ}}} + \underset{\text{box } 2}{\overset{e_2}{\boxed{\circ\cdots\circ}}} + \cdots + \underset{\text{box } n}{\overset{e_n}{\boxed{\circ\cdots\circ}}} = r.$$

Clearly, different solutions to (1.7.1) correspond to different ways of distribution. On the other hand, every such way of distribution corresponds to a nonnegative integer solution to (1.7.1). Thus by (BP) and the result in Case (2)(ii), the desired number is given by

$$\binom{r+n-1}{r}. \blacksquare$$

We now review all the problems in this chapter which give rise to the important number $\binom{r+n-1}{r}$.

The number of ways of selecting r objects from n different types of objects with repetitions allowed

$=$ the number of r-element multi-subsets of the multi-set

$$\{\infty \cdot a_1, \infty \cdot a_2, \ldots, \infty \cdot a_n\}$$

$=$ the number of ways of distributing r identical objects into n distinct boxes

$=$ the number of nonnegative integer solutions to the equation

$$x_2 + x_2 + \cdots + x_n = r$$

$= \binom{r+n-1}{r}$

$= H_r^n$.

Some problems of distributing objects (identical or distinct) into distinct boxes have just been studied. In what follows, we shall study a problem of distributing distinct objects into identical boxes. Problems of distributing identical objects into identical boxes will be discussed in Chapter 5.

Given nonnegative integers r and n, the *Stirling number of the second kind*, denoted by $S(r,n)$, is defined as the number of ways of distributing r distinct objects into n identical boxes such that no box is empty.

The following results are obvious.

(i) $S(0,0) = 1$,

(ii) $S(r,0) = S(0,n) = 0$ for all $r, n \in \mathbf{N}$,

(iii) $S(r,n) > 0$ if $r \geq n \geq 1$,

(iv) $S(r,n) = 0$ if $n > r \geq 1$,

(v) $S(r,1) = 1$ for $r \geq 1$,

(vi) $S(r,r) = 1$ for $r \geq 1$.

We also have (see Problem 1.84):

$$\begin{aligned}
&\text{(vii)} && S(r,2) = 2^{r-1} - 1,\\
&\text{(viii)} && S(r,3) = \tfrac{1}{2}(3^{r-1}+1) - 2^{r-1},\\
&\text{(ix)} && S(r,r-1) = \tbinom{r}{2},\\
&\text{(x)} && S(r,r-2) = \tbinom{r}{3} + 3\tbinom{r}{4}.
\end{aligned}$$

The following result bears some analogy to those given in Example 1.4.1 and Example 1.4.7.

Example 1.7.5. Show that

$$S(r,n) = S(r-1,n-1) + nS(r-1,n) \tag{1.7.2}$$

where $r, n \in \mathbf{N}$ with $r \geq n$.

Proof. Let a_1 be a particular object of the r distinct objects. In any way of distributing the r objects into n identical boxes such that no box is empty, either (i) a_1 is the only object in a box or (ii) a_1 is mixed with others in a box. In case (i), the number of ways to do so is $S(r-1, n-1)$. In case (ii), the $r-1$ objects (excluding a_1) are first put in the n boxes in $S(r-1, n)$ ways; then a_1 can be put in any of the boxes in n ways (why?). Thus the number of ways this can be done in case (ii) is $nS(r-1,n)$. The result now follows by (AP). ∎

Using some initial values of $S(r,n)$ and applying the identity (1.7.2), one can easily construct the following table:

Let $A = \{1, 2, ..., r\}$. For $n \in \mathbf{N}$, an *n-partition* of A is a collection $\{S_1, S_2, \ldots, S_n\}$ of n nonempty subsets of A such that

$$\begin{aligned}
&\text{(i) } S_i \cap S_j = \emptyset \text{ for } i \neq j\\
\text{and}\quad &\text{(ii) } \bigcup_{i=1}^{n} S_i = A.
\end{aligned}$$

A *partition* of A is an n-partition of A for some $n = 1, 2, ..., r$.

A binary relation R on A is an *equivalence relation* on A if

(i) R is reflexive; i.e., aRa for all $a \in A$,

(ii) R is symmetric; i.e., if $a, b \in A$ and aRb, then bRa, and

(iii) R is transitive; i.e., if $a, b, c \in A$, aRb and bRc, then aRc.

r \ n	0	1	2	3	4	5	6	7	8	9
0	1									
1	0	1								
2	0	1	1							
3	0	1	3	1						
4	0	1	7	6	1					
5	0	1	15	25	10	1				
6	0	1	31	90	65	15	1			
7	0	1	63	301	350	140	21	1		
8	0	1	127	966	1701	1050	266	28	1	
9	0	1	255	3025	7770	6951	2646	462	36	1

Table 1.7.1. The values of $S(r,n)$, $0 \le n \le r \le 9$

Let $\mathcal{S} = \{S_1, S_2, ..., S_n\}$ be a partition of A. Define a binary relation R on A by putting

$$xRy \Leftrightarrow x, y \in S_i \quad \text{for some } i = 1, 2, ..., n.$$

It can be checked that R is an equivalence relation on A called the *equivalence relation induced by* $\mathcal{S}$; and in this way, different partitions of A induce different equivalence relations on A.

Conversely, given an equivalence relation R on A and $a \in A$, let

$$[a] = \{x \in A \mid xRa\}$$

be the *equivalence class determined by a*. Then it can be checked that the set

$$\mathcal{S} = \{[a] \mid a \in A\}$$

of subsets of A is a partition of A such that the equivalence relation induced by $\mathcal{S}$ is R.

The above discussion shows that there is a bijection between the family of partitions of A and the family of equivalence relations on A.

It is obvious that a way of distributing r distinct objects $1, 2, ..., r$ to n identical boxes such that no box is empty can be regarded as an n-partition of the set $A = \{1, 2, ..., r\}$. Thus, by definition, $S(r, n)$ counts the number of n-partitions of A, and therefore

$$\begin{aligned}\sum_{n=1}^{r} S(r, n) &= \text{ the number of partitions of } \{1, 2, ..., r\} \\ &= \text{ the number of equivalence relations on } \{1, 2, ..., r\}.\end{aligned}$$

The sum $\sum_{n=1}^{r} S(r, n)$, usually denoted by B_r, is called a *Bell number* after E.T. Bell (1883 – 1960). The first few Bell numbers are:

$$B_1 = 1, \; B_2 = 2, \; B_3 = 5, \; B_4 = 15, \; B_5 = 52, \; B_6 = 203, \ldots .$$

Exercise 1

1. Find the number of ways to choose a pair $\{a, b\}$ of distinct numbers from the set $\{1, 2, ..., 50\}$ such that

   (i) $|a - b| = 5$; (ii) $|a - b| \le 5$.

2. There are 12 students in a party. Five of them are girls. In how many ways can these 12 students be arranged in a row if

   (i) there are no restrictions?

   (ii) the 5 girls must be together (forming a block)?

   (iii) no 2 girls are adjacent?

   (iv) between two particular boys A and B, there are no boys but exactly 3 girls?

3. m boys and n girls are to be arranged in a row, where $m, n \in \mathbf{N}$. Find the number of ways this can be done in each of the following cases:

   (i) There are no restrictions;

   (ii) No boys are adjacent ($m \le n + 1$);

   (iii) The n girls form a single block;

   (iv) A particular boy and a particular girl must be adjacent.

4. How many 5-letter words can be formed using A, B, C, D, E, F, G, H, I, J,

   (i) if the letters in each word must be distinct?

   (ii) if, in addition, A, B, C, D, E, F can only occur as the first, third or fifth letters while the rest as the second or fourth letters?

5. Find the number of ways of arranging the 26 letters in the English alphabet in a row such that there are exactly 5 letters between x and y.

6. Find the number of *odd* integers between 3000 and 8000 in which no digit is repeated.

7. Evaluate
$$1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \cdots + n \cdot n!,$$
where $n \in \mathbf{N}$.

8. Evaluate
$$\frac{1}{(1+1)!} + \frac{2}{(2+1)!} + \cdots + \frac{n}{(n+1)!},$$
where $n \in \mathbf{N}$.

9. Prove that for each $n \in \mathbf{N}$,
$$(n+1)(n+2)\cdots(2n)$$
is divisible by 2^n. (Spanish Olympiad, 1985)

10. Find the number of common positive divisors of 10^{40} and 20^{30}.

11. In each of the following, find the number of positive divisors of n (inclusive of n) which are multiples of 3:
(i) $n = 210$; (ii) $n = 630$; (iii) $n = 151200$.

12. Show that for any $n \in \mathbf{N}$, the number of positive divisors of n^2 is always odd.

13. Show that the number of positive divisors of "$\underbrace{111\ldots1}_{1992}$" is even.

14. Let $n, r \in \mathbf{N}$ with $r \leq n$. Prove each of the following identities:
(i) $P_r^n = nP_{r-1}^{n-1}$,
(ii) $P_r^n = (n-r+1)P_{r-1}^n$,
(iii) $P_r^n = \frac{n}{n-r}P_r^{n-1}$, where $r < n$,
(iv) $P_r^{n+1} = P_r^n + rP_{r-1}^n$,
(v) $P_r^{n+1} = r! + r(P_{r-1}^n + P_{r-1}^{n-1} + \cdots + P_{r-1}^r)$.

15. In a group of 15 students, 5 of them are female. If exactly 3 female students are to be selected, in how many ways can 9 students be chosen from the group
(i) to form a committee?
(ii) to take up 9 different posts in a committee?

16. Ten chairs have been arranged in a row. Seven students are to be seated in seven of them so that no two students share a common chair. Find the number of ways this can be done if no two empty chairs are adjacent.

17. Eight boxes are arranged in a row. In how many ways can five distinct balls be put into the boxes if each box can hold at most one ball and no two boxes without balls are adjacent?

18. A group of 20 students, including 3 particular girls and 4 particular boys, are to be lined up in two rows with 10 students each. In how many ways can this be done if the 3 particular girls must be in the front row while the 4 particular boys be in the back?

19. In how many ways can 7 boys and 2 girls be lined up in a row such that the girls must be separated by exactly 3 boys?

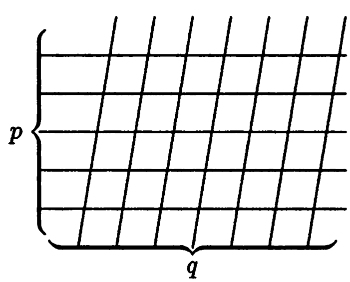
20. In a group of 15 students, 3 of them are female. If at least one female student is to be selected, in how many ways can 7 students be chosen from the group

    (i) to form a committee?

    (ii) to take up 7 different posts in a committee?

21. Find the number of $(m + n)$-digit binary sequences with m 0's and n 1's such that no two 1's are adjacent, where $n \leq m + 1$.

22. Two sets of parallel lines with p and q lines each are shown in the following diagram:

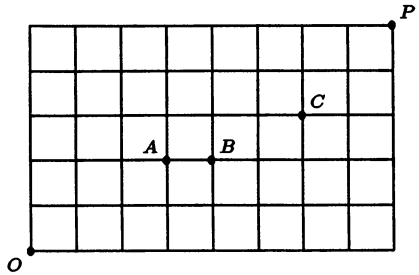


    Find the number of parallelograms formed by the lines?

23. There are 10 girls and 15 boys in a junior class, and 4 girls and 10 boys in a senior class. A committee of 7 members is to be formed from these 2 classes. Find the number of ways this can be done if the committee must have exactly 4 senior students and exactly 5 boys.

24. A box contains 7 identical white balls and 5 identical black balls. They are to be drawn randomly, one at a time without replacement, until the box is empty. Find the probability that the 6th ball drawn is white, while before that exactly 3 black balls are drawn.

25. In each of the following cases, find the number of shortest routes from O to P in the street network shown below:



(i) The routes must pass through the junction A;

(ii) The routes must pass through the street AB;

(iii) The routes must pass through junctions A and C;

(iv) The street AB is closed.

26. Find the number of ways of forming a group of $2k$ people from n couples, where $k, n \in \mathbf{N}$ with $2k \leq n$, in each of the following cases:

(i) There are k couples in such a group;

(ii) No couples are included in such a group;

(iii) At least one couple is included in such a group;

(iv) Exactly two couples are included in such a group.

27. Let $S = \{1, 2, \ldots, n+1\}$ where $n \geq 2$, and let

$$T = \{(x, y, z) \in S^3 \mid x < z \quad \text{and} \quad y < z\}.$$

Show by counting $|T|$ in two different ways that

$$\sum_{k=1}^{n} k^2 = |T| = \binom{n+1}{2} + 2\binom{n+1}{3}.$$

28. Consider the following set of points in the $x-y$ plane:

$$A = \{(a, b) \mid a, b \in \mathbf{Z},\ 0 \leq a \leq 9 \quad \text{and} \quad 0 \leq b \leq 5\}.$$

Find

(i) the number of rectangles whose vertices are points in A;

(ii) the number of squares whose vertices are points in A.

29. Fifteen points $P_1, P_2, \ldots, P_{15}$ are drawn in the plane in such a way that besides P_1, P_2, P_3, P_4, P_5 which are collinear, no other 3 points are collinear. Find

    (i) the number of straight lines which pass through at least 2 of the 15 points;

    (ii) the number of triangles whose vertices are 3 of the 15 points.

30. In each of the following 6-digit natural numbers:

$$333333, \ 225522, \ 118818, \ 707099,$$

every digit in the number appears at least twice. Find the number of such 6-digit natural numbers.

31. In each of the following 7-digit natural numbers:

$$1001011, \ 5550000, \ 3838383, \ 7777777,$$

every digit in the number appears at least 3 times. Find the number of such 7-digit natural numbers.

32. Let $X = \{1, 2, 3, ..., 1000\}$. Find the number of 2-element subsets $\{a, b\}$ of X such that the product $a \cdot b$ is divisible by 5.

33. Consider the following set of points in the $x - y$ plane:

$$A = \{(a, b) \mid a, b \in \mathbf{Z} \quad \text{and} \quad |a| + |b| \leq 2\}.$$

Find

(i) $|A|$;

(ii) the number of straight lines which pass through at least 2 points in A; and

(iii) the number of triangles whose vertices are points in A.

34. Let P be a convex n-gon, where $n \geq 6$. Find the number of triangles formed by any 3 vertices of P that are pairwise nonadjacent in P.

35. 6 boys and 5 girls are to be seated around a table. Find the number of ways that this can be done in each of the following cases:
   (i) There are no restrictions;
   (ii) No 2 girls are adjacent;
   (iii) All girls form a single block;
   (iv) A particular girl G is adjacent to two particular boys B_1 and B_2.

36. Show that the number of r-circular permutations of n distinct objects, where $1 \le r \le n$, is given by $\frac{n!}{(n-r)! \cdot r}$.

37. Let $k, n \in \mathbf{N}$. Show that the number of ways to seat kn people around k distinct tables such that there are n people in each table is given by $\frac{(kn)!}{n^k}$.

38. Let $r \in \mathbf{N}$ such that
$$\frac{1}{\binom{9}{r}} - \frac{1}{\binom{10}{r}} = \frac{11}{6\binom{11}{r}}.$$
Find the value of r.

39. Prove each of the following identities:
   (a) $\displaystyle\binom{n}{r} = \frac{n}{r}\binom{n-1}{r-1}$, where $n \ge r \ge 1$;
   (b) $\displaystyle\binom{n}{r} = \frac{n-r+1}{r}\binom{n}{r-1}$, where $n \ge r \ge 1$;
   (c) $\displaystyle\binom{n}{r} = \frac{n}{n-r}\binom{n-1}{r}$, where $n > r \ge 0$;
   (d) $\displaystyle\binom{n}{m}\binom{m}{r} = \binom{n}{r}\binom{n-r}{m-r}$, where $n \ge m \ge r \ge 0$.

40. Prove the identity $\binom{n}{r} = \binom{n}{n-r}$ by (BP).

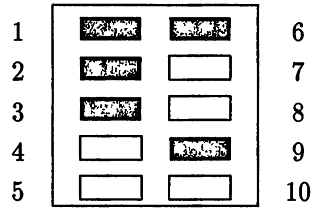
41. Let $X = \{1, 2, ..., n\}$, $\mathcal{A} = \{A \subseteq X \mid n \notin A\}$, and $\mathcal{B} = \{A \subseteq X \mid n \in A\}$. Show that $|\mathcal{A}| = |\mathcal{B}|$ by (BP).

42. Let $r, n \in \mathbf{N}$. Show that the product
$$(n+1)(n+2)\cdots(n+r)$$
of r consecutive positive integers is divisible by $r!$.

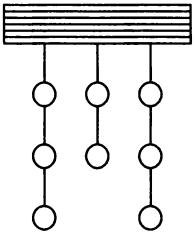
43. Let A be a set of kn elements, where $k, n \in \mathbf{N}$. A k-grouping of A is a partition of A into k-element subsets. Find the number of different k-groupings of A.

44. Twenty five of King Arthur's knights are seated at their customary round table. Three of them are chosen – all choices of three being equally likely – and are sent off to slay a troublesome dragon. Let P be the probability that at least two of the three had been sitting next to each other. If P is written as a fraction in lowest terms, what is the sum of the numerator and denominator? (AIME, 1983/7) (Readers who wish to get more information about the AIME may write to Professor Walter E. Mientka, AMC Executive Director, Department of Mathematics & Statistics, University of Nebraska, Lincohn, NE 68588-0322, USA.)

45. One commercially available ten-button lock may be opened by depressing – in any order – the correct five buttons. The sample shown below has $\{1, 2, 3, 6, 9\}$ as its combination. Suppose that these locks are redesigned so that sets of as many as nine buttons or as few as one button could serve as combinations. How many additional combinations would this allow? (AIME, 1988/1)



46. In a shooting match, eight clay targets are arranged in two hanging columns of three each and one column of two, as pictured. A marksman is to break all eight targets according to the following rules: (1) The marksman first chooses a column from which a target is to be broken. (2) The marksman must then break the lowest remaining unbroken target in the chosen column. If these rules are followed, in how many different orders can the eight targets be broken? (AIME, 1990/8)



47. Using the numbers 1, 2, 3, 4, 5, we can form $5!(= 120)$ 5-digit numbers in which the 5 digits are all distinct. If these numbers are listed in increasing order:

$$\underset{\text{1st}}{12345}, \quad \underset{\text{2nd}}{12354}, \quad \underset{\text{3rd}}{12435}, \quad \ldots, \quad \underset{\text{120th}}{54321},$$

find (i) the position of the number 35421; (ii) the 100th number in the list.

48. The $P_3^4(= 24)$ 3-permutations of the set $\{1, 2, 3, 4\}$ can be arranged in the following way, called the lexicographic ordering:

$$123,\ 124,\ 132,\ 134,\ 142,\ 143,\ 213,\ 214,\ 231,\ 234,$$

$$241,\ 243,\ 312,\ \ldots,\ 431,\ 432.$$

Thus the 3-permutations "132" and "214" appear at the 3rd and 8th positions of the ordering respectively. There are $P_4^9(= 3024)$ 4-permutations of the set $\{1, 2, ..., 9\}$. What are the positions of the 4-permutations "4567" and "5182" in the corresponding lexicographic ordering of the 4 permutations of $\{1, 2, ..., 9\}$?

49. The $\binom{5}{3}(= 10)$ 3-element subsets of the set $\{1, 2, 3, 4, 5\}$ can be arranged in the following way, called the lexicographic ordering:

$$\{1,2,3\},\ \{1,2,4\},\ \{1,2,5\},\ \{1,3,4\},\ \{1,3,5\},\ \{1,4,5\},$$

$$\{2,3,4\},\ \{2,3,5\},\ \{2,4,5\},\ \{3,4,5\}.$$

Thus the subset $\{1, 3, 5\}$ appears at the 5th position of the ordering. There are $\binom{10}{4}$ 4-element subsets of the set $\{1, 2, ..., 10\}$. What are the positions of the subsets $\{3, 4, 5, 6\}$ and $\{3, 5, 7, 9\}$ in the corresponding lexicographic ordering of the 4-element subsets of $\{1, 2, ..., 10\}$?

50. Six scientists are working of a secret project. They wish to lock up the documents in a cabinet so that the cabinet can be opened when and only when three or more of the scientists are present. What is the smallest number of locks needed? What is the smallest number of keys each scientist must carry?

51. A 10-storey building is to be painted with some 4 different colours such that each storey is painted with one colour. It is not necessary that all 4 colours must be used. How many ways are there to paint the building if

(i) there are no other restrictions?

(ii) any 2 adjacent stories must be painted with different colours?

52. Find the number of all multi-subsets of $M = \{r_1 \cdot a_1, r_2 \cdot a_2, \cdots, r_n \cdot a_n\}$.

53. Let $r, b \in \mathbf{N}$ with $r \leq n$. A permutation $x_1 x_2 \cdots x_{2n}$ of the set $\{1, 2, ..., 2n\}$ is said to have property $P(r)$ if $|x_i - x_{i+1}| = r$ for at least one i in $\{1, 2, ..., 2n-1\}$. Show that, for each n and r, there are more permutations with property $P(r)$ than without.

54. Prove by a combinatorial argument that each of the following numbers is always an integer for each $n \in \mathbf{N}$:

(i) $\dfrac{(3n)!}{2^n \cdot 3^n}$,

(ii) $\dfrac{(6n)!}{5^n \cdot 3^{2n} \cdot 2^{4n}}$,

(iii) $\dfrac{(n^2)!}{(n!)^n}$,

(iv) $\dfrac{(n!)!}{(n!)^{(n-1)!}}$.

55. Find the number of r-element multi-subsets of the multi-set

$$M = \{1 \cdot a_1, \infty \cdot a_2, \infty \cdot a_3, \ldots, \infty \cdot a_n\}.$$

56. Six distinct symbols are transmitted through a communication channel. A total of 18 blanks are to be inserted between the symbols with at least 2 blanks between every pair of symbols. In how many ways can the symbols and blanks be arranged?

57. In how many ways can the following 11 letters: A, B, C, D, E, F, X, X, X, Y, Y be arranged in a row so that every Y lies between two X's (not necessarily adjacent)?

58. Two n-digit integers (leading zero allowed) are said to be *equivalent* if one is a permutation of the other. For instance, 10075, 01057 and 00751 are equivalent 5-digit integers.

(i) Find the number of 5-digit integers such that no two are equivalent.

(ii) If the digits 5,7,9 can appear at most once, how many nonequivalent 5-digit integers are there?

59. How many 10-letter words are there using the letters a, b, c, d, e, f if

(i) there are no restrictions?

(ii) each vowel (a and e) appears 3 times and each consonant appears once?

(iii) the letters in the word appear in alphabetical order?

(iv) each letter occurs at least once and the letters in the word appear in alphabetical order?

60. Let $r, n, k \in \mathbf{N}$ such that $r \geq nk$. Find the number of ways of distributing r identical objects into n distinct boxes so that each box holds at least k objects.

61. Find the number of ways of arranging the 9 letters $r, s, t, u, v, w, x, y, z$ in a row so that y always lies between x and z (x and y, or y and z need not be adjacent in the row).

62. Three girls A, B and C, and nine boys are to be lined up in a row. In how many ways can this be done if B must lie between A and C, and A, B must be separated by exactly 4 boys?

63. Five girls and eleven boys are to be lined up in a row such that from left to right, the girls are in the order: G_1, G_2, G_3, G_4, G_5. In how many ways can this be done if G_1 and G_2 must be separated by at least 3 boys, and there is at most one boy between G_4 and G_5?

64. Given $r, n \in \mathbf{N}$ with $r \geq n$, let $L(r, n)$ denote the number of ways of distributing r distinct objects into n identical boxes so that no box is empty and the objects in each box are arranged in a row. Find $L(r, n)$ in terms of r and n.

65. Find the number of integer solutions to the equation:

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 60$$

in each of the following cases:

(i) $x_i \geq i - 1$ for each $i = 1, 2, ..., 6$;

(ii) $x_1 \geq 2$, $x_2 \geq 5$, $2 \leq x_3 \leq 7$, $x_4 \geq 1$, $x_5 \geq 3$ and $x_6 \geq 2$.

66. Find the number of integer solutions to the equation:

$$x_1 + x_2 + x_3 + x_4 = 30$$

in each of the following cases:

(i) $x_i \geq 0$ for each $i = 1, 2, 3, 4$;

(ii) $2 \leq x_1 \leq 7$ and $x_i \geq 0$ for each $i = 2, 3, 4$;

(iii) $x_1 \geq -5$, $x_2 \geq -1$, $x_3 \geq 1$ and $x_4 \geq 2$.

67. Find the number of quadruples (w, x, y, z) of nonnegative integers which satisfy the inequality

$$w + x + y + z \leq 1992.$$

68. Find the number of nonnegative integer solutions to the equation:

$$5x_1 + x_2 + x_3 + x_4 = 14.$$

69. Find the number of nonnegative integer solutions to the equation:

$$rx_1 + x_2 + \cdots + x_n = kr,$$

where $k, r, n \in \mathbf{N}$.

70. Find the number of nonnegative integer solutions to the equation:

$$3x_1 + 5x_2 + x_3 + x_4 = 10.$$

71. Find the number of positive integer solutions to the equation:

$$(x_1 + x_2 + x_3)(y_1 + y_2 + y_3 + y_4) = 77.$$

72. Find the number of nonnegative integer solutions to the equation:

$$(x_1 + x_2 + \cdots + x_n)(y_1 + y_2 + \cdots + y_n) = p,$$

where $n \in \mathbf{N}$ and p is a prime.

73. There are 5 ways to express "4" as a sum of 2 nonnegative integers in which the order counts:

$$4 = 4 + 0 = 3 + 1 = 2 + 2 = 1 + 3 = 0 + 4.$$

Given $r, n \in \mathbf{N}$, what is the number of ways to express r as a sum of n nonnegative integers in which the order counts?

74. There are 6 ways to express "5" as a sum of 3 positive integers in which the order counts:

$$5 = 3+1+1 = 2+2+1 = 2+1+2 = 1+3+1 = 1+2+2 = 1+1+3.$$

Given $r, n \in \mathbf{N}$ with $r \geq n$, what is the number of ways to express r as a sum of n positive integers in which the order counts?

75. A positive integer d is said to be *ascending* if in its decimal representation: $d = d_m d_{m-1} \ldots d_2 d_1$ we have

$$0 < d_m \leq d_{m-1} \leq \cdots \leq d_2 \leq d_1.$$

For instance, 1337 and 2455566799 are ascending integers. Find the number of ascending integers which are less than 10^9.

76. A positive integer d is said to be *strictly ascending* if in its decimal representation: $d = d_m d_{m-1} \ldots d_2 d_1$ we have

$$0 < d_m < d_{m-1} < \cdots < d_2 < d_1.$$

For instance, 145 and 23689 are strictly ascending integers. Find the number of strictly ascending integers which are less than (i) 10^9, (ii) 10^5.

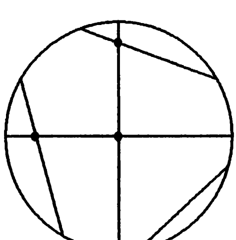
77. Let $A = \{1, 2, \ldots, n\}$, where $n \in \mathbf{N}$.

(i) Given $k \in A$, show that the number of subsets of A in which k is the maximum number is given by 2^{k-1}.

(ii) Apply (i) to show that

$$\sum_{i=0}^{n-1} 2^i = 2^n - 1.$$

78. In a given circle, $n \geq 2$ arbitrary chords are drawn such that no three are concurrent within the interior of the circle. Suppose m is the number of points of intersection of the chords within the interior. Find, in terms of n and m, the number r of line segments obtained through dividing the chords by their points of intersection. (In the following example, $n = 5, m = 3$ and $r = 11$.)



79. There are $p \geq 6$ points given on the circumference of a circle, and every two of the points are joined by a chord.

    (i) Find the number of such chords.

    Assume that no 3 chords are concurrent within the interior of the circle.

    (ii) Find the number of points of intersection of these chords within the interior of the circle.

    (iii) Find the number of line segments obtained through dividing the chords by their points of intersection.

    (iv) Find the number of triangles whose vertices are the points of intersection of the chords within the interior of the circle.

80. In how many ways can $n+1$ different prizes be awarded to n students in such a way that each student has at least one prize?

81. (a) Let $n, m, k \in \mathbf{N}$, and let $\mathbf{N}_k = \{1, 2, ..., k\}$. Find

    (i) the number of mappings from $\mathbf{N}_n$ to $\mathbf{N}_m$.

    (ii) the number of 1-1 mappings from $\mathbf{N}_n$ to $\mathbf{N}_m$, where $n \leq m$.

    (b) A mapping $f : \mathbf{N}_n \to \mathbf{N}_m$ is *strictly increasing* if $f(a) < f(b)$ whenever $a < b$ in $\mathbf{N}_n$. Find the number of strictly increasing mappings from $\mathbf{N}_n$ to $\mathbf{N}_m$, where $n \leq m$.

    (c) Express the number of mappings from $\mathbf{N}_n$ *onto* $\mathbf{N}_m$ in terms of $S(n, m)$ (the Stirling number of the second kind).

82. Given $r, n \in \mathbf{Z}$ with $0 \leq n \leq r$, the Stirling number $s(r, n)$ of the first kind is defined as the number of ways to arrange r distinct objects around n identical circles such that each circle has at least one object. Show that

    (i) $s(r, 1) = (r-1)!$ for $r \geq 1$;

    (ii) $s(r, 2) = (r-1)!(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{r-1})$ for $r \geq 2$;

    (iii) $s(r, r-1) = \binom{r}{2}$ for $r \geq 2$;

    (iv) $s(r, r-2) = \frac{1}{24}r(r-1)(r-2)(3r-1)$ for $r \geq 2$;

    (v) $\sum_{n=0}^{r} s(r, n) = r!$.

83. The Stirling numbers of the first kind occur as the coefficients of x^n in the expansion of

$$x(x+1)(x+2)\cdots(x+r-1).$$

For instance, when $r = 3$,

$$\begin{aligned} x(x+1)(x+2) &= 2x + 3x^2 + x^3 \\ &= s(3,1)x + s(3,2)x^2 + s(3,3)x^3; \end{aligned}$$

and when $r = 5$,

$$\begin{aligned} &x(x+1)(x+2)(x+3)(x+4) \\ &= 24x + 50x^2 + 35x^3 + 10x^4 + x^5 \\ &= s(5,1)x + s(5,2)x^2 + s(5,3)x^3 + s(5,4)x^4 + s(5,5)x^5. \end{aligned}$$

Show that

$$x(x+1)(x+2)\cdots(x+r-1) = \sum_{n=0}^{r} s(r,n)x^n,$$

where $r \in \mathbf{N}$.

84. Given $r, n \in \mathbf{Z}$ with $0 \le n \le r$, the Stirling number $S(r,n)$ of the second kind is defined as the number of ways of distributing r distinct objects into n identical boxes such that no box is empty. Show that

(i) $S(r,2) = 2^{r-1} - 1$;

(ii) $S(r,3) = \frac{1}{2}(3^{r-1}+1) - 2^{r-1}$;

(iii) $S(r,r-1) = \binom{r}{2}$;

(iv) $S(r,r-2) = \binom{r}{3} + 3\binom{r}{4}$.

85. Let $(x)_0 = 1$ and for $n \in \mathbf{N}$, let

$$(x)_n = x(x-1)(x-2)\cdots(x-n+1).$$

The Stirling numbers of the second kind occur as the coefficients of $(x)_n$ when x^r is expressed in terms of $(x)_n$'s. For instance, when $r = 2, 3$ and 4, we have, respectively,

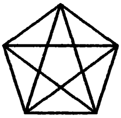
$$\begin{aligned} x^2 &= x + x(x-1) = (x)_1 + (x)_2 \\ &= S(2,1)(x)_1 + S(2,2)(x)_2, \\ x^3 &= x + 3x(x-1) + x(x-1)(x-2) \\ &= S(3,1)(x)_1 + S(3,2)(x)_2 + S(3,3)(x)_3, \\ x^4 &= x + 7x(x-1) + 6x(x-1)(x-2) + x(x-1)(x-2)(x-3) \\ &= S(4,1)(x)_1 + S(4,2)(x)_2 + S(4,3)(x)_3 + S(4,4)(x)_4. \end{aligned}$$

Show that for $r = 0, 1, 2, \ldots,$

$$x^r = \sum_{n=0}^{r} S(r, n)(x)_n.$$

86. Suppose that m chords of a given circle are drawn in such a way that no three are concurrent in the interior of the circle. If n denotes the number of points of intersection of the chords within the circle, show that the number of regions divided by the chords in the circle is $m + n + 1$.

87. For $n \geq 4$, let $r(n)$ denote the number of interior regions of a convex n-gon divided by all its diagonals if no three diagonals are concurrent within the n-gon. For instance, as shown in the following diagrams, $r(4) = 4$ and $r(5) = 11$. Prove that $r(n) = \binom{n}{4} + \binom{n-1}{2}$.



88. Let $n \in \mathbf{N}$. How many solutions are there in ordered positive integer pairs (x, y) to the equation

$$\frac{xy}{x+y} = n?$$

(Putnam, 1960)

89. Let $S = \{1, 2, 3, \ldots, 1992\}$. In each of the following cases, find the number of 3-element subsets $\{a, b, c\}$ of S satisfying the given condition:

(i) $3|(a + b + c)$;

(ii) $4|(a + b + c)$.

90. A sequence of 15 random draws, one at a time with replacement, is made from the set

$$\{A, B, C, \ldots, X, Y, Z\}$$

of the English alphabet. What is the probability that the string:

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occurs as a block in the sequence?

91. A set $S = \{a_1, a_2, \ldots, a_r\}$ of positive integers, where $r \in \mathbf{N}$ and $a_1 < a_2 < \ldots < a_r$, is said to be *m-separated* ($m \in \mathbf{N}$) if $a_i - a_{i-1} \geq m$, for each $i = 2, 3, \ldots, r$. Let $X = \{1, 2, \ldots, n\}$. Find the number of r-element subsets of X which are m-separated, where $0 \leq r \leq n - (m-1)(r-1)$.

92. Let $a_1, a_2, \ldots, a_n$ be positive real numbers, and let S_k be the sum of products of $a_1, a_2, \ldots, a_n$ taken k at a time. Show that

$$S_k S_{n-k} \geq \binom{n}{k}^2 a_1 a_2 \cdots a_n,$$

for $k = 1, 2, \ldots, n-1$. (APMO, 1990)

93. For $\{1, 2, 3, \ldots, n\}$ and each of its nonempty subsets, a unique *alternating sum* is defined as follows: Arrange the numbers in the subset in decreasing order and then, beginning with the largest, alternately add and subtract successive numbers. (For example, the alternating sum for $\{1, 2, 4, 6, 9\}$ is $9 - 6 + 4 - 2 + 1 = 6$ and for $\{5\}$ it is simply 5.) Find the sum of all such alternating sums for $n = 7$. (AIME, 1983/13)

94. A gardener plants three maple trees, four oak trees and five birch trees in a row. He plants them in random order, each arrangement being equally likely. Let $\frac{m}{n}$ in lowest terms be the probability that no two birch trees are next to one another. Find $m + n$. (AIME, 1984/11)

95. In a tournament each player played exactly one game against each of the other players. In each game the winner was awarded 1 point, the loser got 0 points, and each of the two players earned 1/2 point if the game was a tie. After the completion of the tournament, it was found that exactly half of the points earned by each player were earned in games against the ten players with the least number of points. (In particular, each of the ten lowest scoring players earned half of her/his points against the other nine of the ten). What was the total number of players in the tournament? (AIME, 1985/14)

96. Let S be the sum of the base 10 logarithms of all of the proper divisors of 1,000,000. (By a proper divisor of a natural number we mean a positive integral divisor other than 1 and the number itself.) What is the integer nearest to S? (AIME, 1986/8)

97. In a sequence of coin tosses one can keep a record of the number of instances when a tail is immediately followed by a head, a head is immediately followed by a head, etc.. We denote these by TH, HH, etc.. For example, in the sequence $HHTTHHHHTHHTTTT$ of 15 coin tosses we observe that there are five HH, three HT, two TH and four TT subsequences. How many different sequences of 15 coin tosses will contain exactly two HH, three HT, four TH and five TT subsequences? (AIME, 1986/13)

98. An ordered pair (m, n) of non-negative integers is called "simple" if the addition $m + n$ in base 10 requires no carrying. Find the number of simple ordered pairs of non-negative integers that sum to

    (i) 1492; (AIME, 1987/1) (ii) 1992.

99. Let m/n, in lowest terms, be the probability that a randomly chosen positive divisor of 10^{99} is an integer multiple of 10^{88}. Find $m + n$. (AIME, 1988/5)

100. A convex polyhedron has for its faces 12 squares, 8 regular hexagons, and 6 regular octagons. At each vertex of the polyhedron one square, one hexagon, and one octagon meet. How many segments joining vertices of the polyhedron lie in the interior of the polyhedron rather than along an edge or a face? (AIME, 1988/10)

101. Someone observed that $6! = 8 \cdot 9 \cdot 10$. Find the largest positive integer n for which $n!$ can be expressed as the product of $n - 3$ consecutive positive integers. (AIME, 1990/11)

102. Let $S = \{1, 2, ..., n\}$. Find the number of subsets A of S satisfying the following conditions:

    $A = \{a, a+d, \ldots, a+kd\}$ for some positive integers a, d and k, and

    $A \cup \{x\}$ is no longer an A.P. with common difference d for each $x \in S \setminus A$.

    (Note that $|A| \geq 2$ and any sequence of two terms is considered as an A.P.) (Chinese Math. Competition, 1991)

103. Find all natural numbers $n > 1$ and $m > 1$ such that

$$1!3!5! \cdots (2n-1)! = m!\,.$$

    (Proposed by I. Cucurezeanu, see *Amer. Math. Monthly*, **94** (1987), 190.)

104. Show that for $n \in \mathbf{N}$,

$$\sum_{r=0}^{n} P_r^n = \lfloor n!e \rfloor,$$

where $\lfloor x \rfloor$ denotes the greatest integer $\leq x$ and $e = 2.718 \cdots$. (Proposed by D. Ohlsen, see *The College Math. J.* **20** (1989), 260.)

105. Let $S = \{1, 2, ..., 1990\}$. A 31-element subset A of S is said to be *good* if the sum $\sum_{a \in A} a$ is divisible by 5. Find the number of 31-element subsets of S which are good. (Proposed by the Indian Team at the 31st IMO.)

106. Let S be a 1990-element set and let $\mathcal{P}$ be a set of 100-ary sequences $(a_1, a_2, ..., a_{100})$, where a_i's are distinct elements of S. An ordered pair (x, y) of elements of S is said to *appear* in $(a_1, a_2, ..., a_{100})$ if $x = a_i$ and $y = a_j$ for some i, j with $1 \leq i < j \leq 100$. Assume that every ordered pair (x, y) of elements of S appears in at most one member in $\mathcal{P}$. Show that

$$|\mathcal{P}| \leq 800.$$

(Proposed by the Iranian Team at the 31st IMO.)

107. Let $M = \{r_1 \cdot a_1, r_2 \cdot a_2, \ldots, r_n \cdot a_n\}$ be a multi-set with $r_1 + r_2 + \cdots + r_n = r$. Show that the number of r-permutations of M is equal to the number of $(r-1)$-permutations of M.

108. Prove that it is impossible for seven distinct straight lines to be situated in the Euclidean plane so as to have at least six points where exactly three of these lines intersect and at least four points where exactly two of these lines intersect. (Putnam, 1973)

109. For what $n \in \mathbf{N}$ does there exist a permutation $(x_1, x_2, \ldots, x_n)$ of $(1, 2, ..., n)$ such that the differences $|x_k - k|$, $1 \leq k \leq n$, are all distinct? (Prospoosed by M.J. Pelling, see *Amer. Math. Monthly*, **96** (1989), 843-844.)

110. Numbers $d(n, m)$, where n, m are integers and $0 \leq m \leq n$, are defined by

$$d(n, 0) = d(n, n) = 1 \quad \text{for all } n \geq 0$$

and

$$m \cdot d(n, m) = m \cdot d(n-1, m) + (2n - m) \cdot d(n-1, m-1)$$

for $0 < m < n$. Prove that all the $d(n, m)$ are integers. (Great Britian, 1987)

111. A difficult mathematical competition consisted of a Part I and a Part II with a combined total of 28 problems. Each contestant solved 7 problems altogether. For each pair of problems, there were exactly two contestants who solved both of them. Prove that there was a contestant who, in Part I, solved either no problems or at least four problems. (USA MO, 1984/4)

112. Suppose that five points in a plane are situated so that no two of the straight lines joining them are parallel, perpendicular, or coincident. From each point perpendiculars are drawn to all the lines joining the other four points. Determine the maximum number of intersections that these perpendiculars can have. (IMO, 1964/5)

113. Let n distinct points in the plane be given. Prove that fewer than $2n^{\frac{3}{2}}$ pairs of them are at unit distance apart. (Putnam, 1978)

114. If c and m are positive integers each greater than 1, find the number $n(c,m)$ of ordered c-tuples $(n_1, n_2, ..., n_c)$ with entries from the initial segment $\{1, 2, ..., m\}$ of the positive integers such that $n_2 < n_1$ and $n_2 \leq n_3 \leq \cdots \leq n_c$. (Proposed by D. Spellman, see *Amer. Math. Monthly*, **94** (1987), 383-384.)

115. Let $X = \{x_1, x_2, \ldots, x_m\}$, $Y = \{y_1, y_2, \ldots, y_n\}$ $(m, n \in \mathbf{N})$ and $A \subseteq X \times Y$. For $x_i \in X$, let

$$A(x_i, \cdot) = (\{x_i\} \times Y) \cap A$$

and for $y_j \in Y$, let

$$A(\cdot, y_j) = (X \times \{y_j\}) \cap A.$$

(i) Prove the following *Fubini Principle*:

$$\sum_{i=1}^{m} |A(x_i, \cdot)| = |A| = \sum_{j=1}^{n} |A(\cdot, y_j)|.$$

(ii) Using (i), or otherwise, solve the following problem: There are $n \geq 3$ given points in the plane such that any three of them form a right-angled triangle. Find the largest possible value of n.

(23rd Moscow MO)