2

Combinatorial Identities and Recursions

This chapter begins with a discussion of the generalized distributive law and its consequences, which include the multinomial and binomial theorems. We then study algebraic and combinatorial proofs of identities involving binomial coefficients, factorials, summations, etc. We also introduce *recursions*, which provide ways to enumerate classes of combinatorial objects whose cardinalities are not given by closed formulas. We use recursions to obtain information about more intricate combinatorial objects including set partitions, integer partitions, equivalence relations, surjections, and lattice paths.

2.1 Generalized Distributive Law

Suppose we have a product of several factors, where each factor consists of a sum of some terms. How can we simplify such a product of sums? The following example suggests the answer.

2.1. Example. Suppose A, B, C, T, U, V are $n \times n$ matrices. Let us simplify the matrix product

$$(C+V)(A+U)(B+C+T).$$

Using the distributive laws for matrices several times, we first compute

$$(C+V)(A+U) = C(A+U) + V(A+U) = CA + CU + VA + VU.$$

Now we multiply this on the right by the matrix $B+C+T$. Using the distributive laws again, we obtain

$$CA(B+C+T) + CU(B+C+T) + VA(B+C+T) + VU(B+C+T)$$

$$= CAB+CAC+CAT+CUB+CUC+CUT+VAB+VAC+VAT+VUB+VUC+VUT.$$

Observe that the final answer is a sum of many terms, where each term can be viewed as a *word* drawn from the set of words $\{C, V\} \times \{A, U\} \times \{B, C, T\}$. We obtain such a word by choosing a first matrix from the first factor $C+V$, then choosing a second matrix from the second factor $A+U$, and then choosing a third matrix from the third factor $B+C+T$. This sequence of choices can be done in 12 ways, and accordingly there are 12 terms in the final sum.

The pattern in the previous example holds in general. Intuitively, to multiply together some factors, each of which is a sum of some terms, we choose one term from each factor and multiply these terms together. Then we add together all possible products obtained in this way. We will now give a rigorous proof of this result, which ultimately follows from the distributive laws for a (possibly non-commutative) ring. For convenience, we now state the relevant definitions from abstract algebra. Readers unfamiliar with abstract algebra may replace the abstract rings used below by the set of $n \times n$ matrices with real entries (which is a particular example of a ring).

2.2. Definition: Rings. A *ring* consists of a set R and two binary operations $+$ (addition) and $\cdot$ (multiplication) with domain $R \times R$, subject to the following axioms.

$$\begin{array}{ll}
\forall x, y \in R,\ x + y \in R & \text{(closure under addition)} \\
\forall x, y, z \in R,\ x + (y + z) = (x + y) + z & \text{(associativity of addition)} \\
\forall x, y \in R,\ x + y = y + x & \text{(commutativity of addition)} \\
\exists 0_R \in R, \forall x \in R, x + 0_R = x = 0_R + x & \text{(existence of additive identity)} \\
\forall x \in R, \exists -x \in R, x + (-x) = 0_R = (-x) + x & \text{(existence of additive inverses)} \\
\forall x, y \in R,\ x \cdot y \in R & \text{(closure under multiplication)} \\
\forall x, y, z \in R,\ x \cdot (y \cdot z) = (x \cdot y) \cdot z & \text{(associativity of multiplication)} \\
\exists 1_R \in R, \forall x \in R, x \cdot 1_R = x = 1_R \cdot x & \text{(existence of multiplicative identity)} \\
\forall x, y, z \in R,\ x \cdot (y + z) = x \cdot y + x \cdot z & \text{(left distributive law)} \\
\forall x, y, z \in R,\ (x + y) \cdot z = x \cdot z + y \cdot z & \text{(right distributive law)}
\end{array}$$

We often write xy instead of $x \cdot y$. R is a *commutative ring* iff R satisfies the additional axiom

$$\forall x, y \in R,\ xy = yx \qquad \text{(commutativity of multiplication).}$$

2.3. Definition: Fields. A *field* is a commutative ring F with $1_F \neq 0_F$ such that every nonzero element of F has a multiplicative inverse:

$$\forall x \in F, x \neq 0_F \Rightarrow \exists y \in F, xy = 1_F = yx.$$

Let R be a ring, and suppose $x_1, x_2, \ldots, x_n \in R$. Because addition is associative, we can unambiguously write a sum like $x_1 + x_2 + x_3 + \cdots + x_n$ without parentheses (see 2.148). Similarly, associativity of multiplication implies that we can write the product $x_1 x_2 \cdots x_n$ without parentheses. Because addition in the ring is commutative, we can permute the summands in a sum like $x_1 + x_2 + \cdots + x_n$ without changing the answer. More formally, for any bijection $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$, we have

$$x_{f(1)} + x_{f(2)} + \cdots + x_{f(n)} = x_1 + x_2 + \cdots + x_n \in R$$

(see 2.149). It follows that if $\{x_i : i \in I\}$ is a finite indexed family of ring elements, then the sum of all these elements (denoted $\sum_{i \in I} x_i$) is well defined. Similarly, if A is a finite subset of R, then $\sum_{x \in A} x$ is well defined. On the other hand, the products $\prod_{i \in I} x_i$ and $\prod_{x \in A} x$ are *not* well defined (for R non-commutative) unless we specify in advance a total ordering on I and A.

Now we are ready to derive the general distributive law for non-commutative rings. The idea is to keep iterating the left and right distributive laws to obtain successively more general formulas. We divide the proof into a sequence of lemmas.

2.4. Lemma. Let R be a ring, n a positive integer, and $x, y_1, y_2, \ldots, y_n \in R$. Then

$$x(y_1 + y_2 + \cdots + y_n) = xy_1 + xy_2 + \cdots + xy_n; \qquad (y_1 + y_2 + \cdots + y_n)x = y_1 x + y_2 x + \cdots + y_n x.$$

Proof. We prove the first equation by induction on n. The case $n = 1$ is immediate, while the case $n = 2$ is the left distributive law for R. Now assume that $x(y_1 + y_2 + \cdots + y_n) = xy_1 + xy_2 + \cdots + xy_n$ is known for some $n \geq 2$; let us prove the corresponding formula for $n + 1$. In the left distributive law for R, let $y = y_1 + \cdots + y_n$ and $z = y_{n+1}$. Using this and the induction hypothesis, we calculate

$$x(y_1 + \cdots + y_n + y_{n+1}) = x(y + z) = xy + xz = x(y_1 + \cdots + y_n) + xy_{n+1} = xy_1 + \cdots + xy_n + xy_{n+1}.$$

This proves the first equation. The second is proved similarly from the right distributive law. $\square$

Now suppose $x \in R$ and $\{y_i : i \in I\}$ is a finite indexed family of elements of R. The results in the previous lemma can be written as follows:

$$x \cdot \left(\sum_{i \in I} y_i\right) = \sum_{i \in I} (x \cdot y_i); \tag{2.1}$$

$$\left(\sum_{i \in I} y_i\right) \cdot x = \sum_{i \in I} (y_i \cdot x). \tag{2.2}$$

2.5. Lemma. Let R be a ring, and suppose $\{u_i : i \in I\}$ and $\{v_j : j \in J\}$ are two finite indexed families of ring elements. Then

$$\left(\sum_{i \in I} u_i\right) \cdot \left(\sum_{j \in J} v_j\right) = \sum_{(i,j) \in I \times J} (u_i \cdot v_j). \tag{2.3}$$

Proof. Applying (2.2) with $x = \sum_{j \in J} v_j \in R$ and $y_i = u_i$ for all i, we obtain first

$$\left(\sum_{i \in I} u_i\right) \cdot \left(\sum_{j \in J} v_j\right) = \sum_{i \in I} \left(u_i \cdot \sum_{j \in J} v_j\right).$$

Now, for each $i \in I$, apply (2.1) with $x = u_i$ and $\sum_{i \in I} y_i$ replaced by $\sum_{j \in J} v_j$ to obtain

$$\sum_{i \in I} \left(u_i \cdot \sum_{j \in J} v_j\right) = \sum_{i \in I} \left(\sum_{j \in J} (u_i \cdot v_j)\right).$$

Finally, since addition in R is commutative, the iterated sum in the last formula is equal to the single sum

$$\sum_{(i,j) \in I \times J} (u_i \cdot v_j)$$

over the new index set $I \times J$. The lemma follows. □

2.6. Theorem (Generalized Distributive Law). Suppose R is a ring, $I_1, \ldots, I_n$ are finite index sets, and $\{x_{k,i_k} : i_k \in I_k\}$ are indexed families of ring elements for $1 \le k \le n$. Then

$$\left(\sum_{i_1 \in I_1} x_{1,i_1}\right) \cdot \left(\sum_{i_2 \in I_2} x_{2,i_2}\right) \cdot \ldots \cdot \left(\sum_{i_n \in I_n} x_{n,i_n}\right) = \sum_{(i_1,\ldots,i_n) \in I_1 \times \cdots \times I_n} (x_{1,i_1} \cdot x_{2,i_2} \cdot \ldots \cdot x_{n,i_n}). \tag{2.4}$$

We can also write this as

$$\prod_{k=1}^{n} \left(\sum_{i_k \in I_k} x_{k,i_k}\right) = \sum_{(i_1,\ldots,i_n) \in I_1 \times \cdots \times I_n} \left(\prod_{k=1}^{n} x_{k,i_k}\right), \tag{2.5}$$

provided we remember that the *order* of the factors in $\prod_{k=1}^{n} u_k = u_1 u_2 \cdots u_n$ is crucial.

Proof. We use induction on n. There is nothing to prove if $n = 1$, and the case $n = 2$ was proved in the previous lemma. Now assume the result holds for some $n \ge 2$; we prove the result for $n+1$ factors. Using $\prod_{k=1}^{n+1} v_k = (\prod_{k=1}^{n} v_k) \cdot v_{n+1}$, then the induction hypothesis for n factors, then the result for 2 factors, we compute

$$\begin{aligned}
\prod_{k=1}^{n+1} \sum_{i_k \in I_k} x_{k,i_k} &= \left(\prod_{k=1}^{n} \sum_{i_k \in I_k} x_{k,i_k}\right) \cdot \left(\sum_{i_{n+1} \in I_{n+1}} x_{n+1,i_{n+1}}\right) \\
&= \left(\sum_{(i_1,\ldots,i_n) \in I_1 \times \cdots \times I_n} \prod_{k=1}^{n} x_{k,i_k}\right) \cdot \left(\sum_{i_{n+1} \in I_{n+1}} x_{n+1,i_{n+1}}\right) \\
&= \sum_{((i_1,\ldots,i_n),i_{n+1}) \in (I_1 \times \cdots \times I_n) \times I_{n+1}} \left(\left(\prod_{k=1}^{n} x_{k,i_k}\right) \cdot x_{n+1,i_{n+1}}\right).
\end{aligned}$$

By commutativity of addition, the final expression is equal to

$$\sum_{(i_1,\ldots,i_{n+1}) \in I_1 \times \cdots \times I_{n+1}} \prod_{k=1}^{n+1} x_{k,i_k}.$$

This completes the induction. ☐

Here is a formula that follows from the generalized distributive law.

2.7. Theorem. If $y_1, \ldots, y_n, z_1, \ldots, z_n$ are elements of a *commutative* ring R, then

$$\prod_{k=1}^{n} (y_k + z_k) = \sum_{S \subseteq \{1,2,\ldots,n\}} \prod_{k \in S} z_k \prod_{k \notin S} y_k.$$

Proof. Write $x_{k,0} = y_k$ and $x_{k,1} = z_k$ for $1 \leq k \leq n$, and let $I_1 = I_2 = \cdots = I_n = \{0,1\}$. Using 2.6 gives

$$\prod_{k=1}^{n} (y_k + z_k) = \prod_{k=1}^{n} (x_{k,0} + x_{k,1}) = \sum_{(i_1,\ldots,i_n) \in \{0,1\}^n} \prod_{k=1}^{n} x_{k,i_k}.$$

Now, we can use the bijection in 1.38 to convert the sum over binary words in $\{0,1\}^n$ to a sum over subsets $S \subseteq \{1,2,\ldots,n\}$. Suppose $(i_1,\ldots,i_n)$ corresponds to S under the bijection. Then $k \in S$ iff $i_k = 1$ iff $x_{k,i_k} = z_k$, while $k \notin S$ iff $i_k = 0$ iff $x_{k,i_k} = y_k$. It follows that the summand indexed by S is

$$\prod_{k=1}^{n} x_{k,i_k} = \prod_{k: i_k = 1} x_{k,i_k} \prod_{k: i_k = 0} x_{k,i_k} = \prod_{k \in S} z_k \prod_{k \notin S} y_k.$$

Note that the first equality here used the commutativity of R. ☐

2.2 Multinomial and Binomial Theorems

We now deduce some consequences of the generalized distributive law. In particular, we derive the non-commutative and commutative versions of the multinomial theorem and the binomial theorem.

2.8. Theorem. Suppose R is a ring and $A_1, \ldots, A_n$ are finite subsets of R. Then

$$\left(\sum_{w_1 \in A_1} w_1\right) \cdot \left(\sum_{w_2 \in A_2} w_2\right) \cdot \ldots \cdot \left(\sum_{w_n \in A_n} w_n\right) = \sum_{(w_1, w_2, \cdots, w_n) \in A_1 \times A_2 \times \cdots \times A_n} w_1 w_2 \cdots w_n.$$

Proof. In 2.6, choose the index sets $I_k = A_k$, and define $x_{k,i_k} = i_k \in A_k \subseteq R$ for each $k \leq n$ and each $i_k \in I_k$. Then $\sum_{w_k \in I_k} x_{k,w_k} = \sum_{w_k \in A_k} w_k$ for each k, and $\prod_{k=1}^{n} x_{k,w_k} = w_1 w_2 \cdots w_n$. Thus the formula in the theorem is a special case of (2.4). $\square$

To emphasize the combinatorial nature of the previous result, we can write it as follows:

$$\left(\sum_{w_1 \in A_1} w_1\right) \cdot \left(\sum_{w_2 \in A_2} w_2\right) \cdot \ldots \cdot \left(\sum_{w_n \in A_n} w_n\right) = \sum_{\text{words } w \in A_1 \times A_2 \times \cdots \times A_n} w.$$

Intuitively, we simplify a given product of sums by choosing one letter w_i from each factor, concatenating (multiplying) these letters to get a word, and adding all the words obtainable in this way.

2.9. Non-Commutative Multinomial Theorem. Suppose R is a ring, $n \in \mathbb{N}$, and $A = \{z_1, \ldots, z_s\} \subseteq R$. Then

$$(z_1 + z_2 + \cdots + z_s)^n = \sum_{w \in A^n} w_1 w_2 \cdots w_n.$$

Proof. Take $A_1 = A_2 = \cdots = A_n = A$ in 2.8. $\square$

2.10. Example. If A and B are two $n \times n$ matrices, then

$$(A+B)^3 = AAA + AAB + ABA + ABB + BAA + BAB + BBA + BBB.$$

If $A, B, \ldots, Z$ are 26 matrices, then

$$\begin{aligned} (A + B + \cdots + Z)^4 &= AAAA + AAAB + AABA + \cdots + ZZZY + ZZZZ \\ &= \text{the sum of all 4-letter words.} \end{aligned}$$

2.11. Remark. Our statement of the non-commutative multinomial theorem tacitly assumed that $z_1, \ldots, z_s$ were *distinct* ring elements. This assumption can be dropped at the expense of a slight notation change. More precisely, if $\{z_i : i \in I\}$ is a finite indexed family of ring elements, then it follows from 2.6 that

$$\left(\sum_{i \in I} z_i\right)^n = \sum_{w \in I^n} z_{w_1} z_{w_2} \cdots z_{w_n}.$$

Similar comments apply to the theorems below.

2.12. Commutative Multinomial Theorem. Suppose R is a ring, $n \in \mathbb{N}$, and $z_1, \ldots, z_s \in R$ are elements of R that *commute* (meaning $z_i z_j = z_j z_i$ for all i, j). Then

$$(z_1 + z_2 + \cdots + z_s)^n = \sum_{n_1 + n_2 + \cdots + n_s = n} \binom{n}{n_1, n_2, \ldots, n_s} z_1^{n_1} z_2^{n_2} \cdots z_s^{n_s}.$$

The summation here extends over all ordered sequences $(n_1, n_2, \ldots, n_s)$ of nonnegative integers that sum to n.

Proof. Let $A = \{z_1, z_2, \ldots, z_s\} \subseteq R$. Let X be the set of all ordered sequences $(n_1, n_2, \ldots, n_s)$ of nonnegative integers that sum to n. The non-commutative multinomial theorem gives

$$(z_1 + z_2 + \cdots + z_s)^n = \sum_{w \in A^n} w_1 w_2 \cdots w_n.$$

The set A^n of n-letter words over the alphabet A is the disjoint union of the sets $\mathcal{R}(z_1^{n_1} z_2^{n_2} \cdots z_s^{n_s})$, as $(n_1, \ldots, n_s)$ ranges over X. By commutativity of addition, we therefore have

$$(z_1 + z_2 + \cdots + z_s)^n = \sum_{(n_1,\ldots,n_s) \in X} \sum_{w \in \mathcal{R}(z_1^{n_1} \cdots z_s^{n_s})} w_1 w_2 \cdots w_n.$$

Now we use the assumption that all the z_i's commute with one another. This assumption allows us to reorder any product of z_i's so that all z_1's come first, followed by the z_2's, etc. Given $w \in \mathcal{R}(z_1^{n_1} \cdots z_s^{n_s})$, reordering the letters of w gives

$$w_1 w_2 \cdots w_n = z_1^{n_1} z_2^{n_2} \cdots z_s^{n_s}.$$

Thus,

$$\begin{aligned}(z_1 + z_2 + \cdots + z_s)^n &= \sum_{(n_1,\ldots,n_s) \in X} \sum_{w \in \mathcal{R}(z_1^{n_1} \cdots z_s^{n_s})} z_1^{n_1} \cdots z_s^{n_s} \\ &= \sum_{(n_1,\ldots,n_s) \in X} z_1^{n_1} \cdots z_s^{n_s} \cdot \left(\sum_{w \in \mathcal{R}(z_1^{n_1} \cdots z_s^{n_s})} 1_R \right).\end{aligned}$$

The inner sum is $|\mathcal{R}(z_1^{n_1} \cdots z_s^{n_s})| = \binom{n}{n_1, n_2, \ldots, n_s}$ (or more precisely, the sum of this many copies of 1_R). Thus, we obtain the formula in the statement of the theorem. □

2.13. Example. If $xy = yx$ and $xz = zx$ and $yz = zy$, then

$$(x + y + z)^3 = x^3 + y^3 + z^3 + 3x^2y + 3x^2z + 3y^2z + 3xy^2 + 3xz^2 + 3yz^2 + 6xyz.$$

2.14. Commutative Binomial Theorem. Suppose R is a ring, $n \in \mathbb{N}$, and $x, y \in R$ are ring elements such that $xy = yx$. Then

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Proof. Apply the commutative multinomial theorem with $s = 2$, $z_1 = x$, and $z_2 = y$ to get

$$(x + y)^n = \sum_{n_1 + n_2 = n} \binom{n}{n_1, n_2} x^{n_1} y^{n_2}.$$

Let $n_1 = k$; note that the possible values of k are $0, 1, \ldots, n$. Once n_1 has been chosen, n_2 is uniquely determined as $n_2 = n - n_1 = n - k$. Also, $\binom{n}{n_1, n_2} = \binom{n}{k}$, so the formula becomes

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$ □

2.15. Example. What is the coefficient of z^7 in $(2z-5)^9$? We apply the binomial theorem taking $x = 2z$ and $y = -5$ and $n = 9$. We have

$$(2z-5)^9 = \sum_{k=0}^{9} \binom{9}{k} (2z)^k (-5)^{9-k}.$$

The only summand involving z^7 is the $k = 7$ summand. The corresponding coefficient is

$$\binom{9}{7} 2^7 (-5)^2 = 115,200.$$

2.16. Remark. If r is any real number and x is a real number such that $|x| < 1$, there exists a power series expansion for $(1+x)^r$ that is often called the *generalized binomial formula*. This power series is discussed in 7.68.

2.3 Combinatorial Proofs

Consider the problem of proving an identity of the form $A = B$, where A and B are formulas that may involve factorials, binomial coefficients, powers, etc. One way to prove such an identity is to give an *algebraic proof* using tools like the binomial theorem or other algebraic techniques. Another way to prove such an identity is to find a *combinatorial proof.* A combinatorial proof establishes the equality of two formulas by exhibiting a set of objects whose cardinality is given by both formulas. Thus, the main steps in a combinatorial proof of $A = B$ are as follows.

- Define a set S of objects.
- Give a counting argument (using the sum rule, product rule, bijections, etc.) to prove that $|S| = A$.
- Give a different counting argument to prove that $|S| = B$.
- Conclude that $A = B$.

We now give some examples illustrating this technique and its variations.

2.17. Theorem. For all $n \in \mathbb{N}$,

$$2^n = \sum_{k=0}^{n} \binom{n}{k}.$$

Proof. We give an algebraic proof and a combinatorial proof.
Algebraic Proof. By the binomial theorem, we know that

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \qquad (x, y \in \mathbb{R}).$$

Setting $x = y = 1$ yields the desired formula.
Combinatorial Proof. Fix $n \in \mathbb{N}$. Let S be the set of all subsets of $\{1, 2, \ldots, n\}$. As shown

earlier, $|S| = 2^n$ since we can build a typical subset by either including or excluding each of the n available elements. On the other hand, note that S is the disjoint union

$$S = S_0 \cup S_1 \cup \cdots \cup S_n,$$

where S_k consists of all k-element subsets of $\{1, 2, \ldots, n\}$. As shown earlier, $|S_k| = \binom{n}{k}$. By the sum rule, we therefore have

$$|S| = \sum_{k=0}^{n} |S_k| = \sum_{k=0}^{n} \binom{n}{k}.$$

Thus,

$$2^n = |S| = \sum_{k=0}^{n} \binom{n}{k}.$$

$\square$

2.18. Theorem. For all integers n, k with $0 \leq k \leq n$, we have

$$\binom{n}{k} = \binom{n}{n-k}.$$

Proof. Again we give both an algebraic proof and a combinatorial proof.
Algebraic Proof. Using the explicit formula for binomial coefficients involving factorials, we calculate

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)!(n-(n-k))!} = \binom{n}{n-k}.$$

Combinatorial Proof. For this proof, we will define two different sets of objects. Fix n and k. Let S be the set of all k-element subsets of $\{1, 2, \ldots, n\}$, and let T be the set of all $(n-k)$-element subsets of $\{1, 2, \ldots, n\}$. We have already shown that $|S| = \binom{n}{k}$ and $|T| = \binom{n}{n-k}$. We complete the proof by exhibiting a bijection $\phi : S \to T$, which shows that $|S| = |T|$. Given $A \in S$, define $\phi(A) = \{1, 2, \ldots, n\} \sim A$. Since A has k elements, $\phi(A)$ has $n-k$ elements and is thus an element of T. The inverse of this map is the map $\phi' : T \to S$ given by $\phi'(B) = \{1, 2, \ldots, n\} \sim B$ for $B \in T$. Note that ϕ and ϕ' are both restrictions of the "set complement" map $I : \mathcal{P}(\{1, 2, \ldots, n\}) \to \mathcal{P}(\{1, 2, \ldots, n\})$ given by $I(A) = \{1, 2, \ldots, n\} \sim A$. Since $I \circ I$ is the identity map on $\mathcal{P}(\{1, 2, \ldots, n\})$, it follows that ϕ' is the two-sided inverse of ϕ. $\square$

2.19. Theorem. For $0 \leq k \leq n$,

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$

Proof. In terms of factorials, we are trying to prove that

$$\sum_{k=0}^{n} \frac{(n!)^2}{(k!)^2((n-k)!)^2} = \frac{(2n)!}{n!n!}.$$

An algebraic proof of this formula is not evident. So we proceed to look for a combinatorial proof.

Define S to be the set of all n-element subsets of $X = \{1, 2, \ldots, 2n\}$. This choice of S was motivated by our knowledge that $|S| = \binom{2n}{n}$, which is the right side of the desired

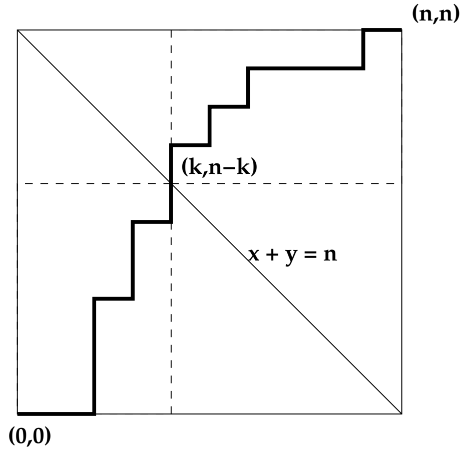


FIGURE 2.1
A combinatorial proof using lattice paths.

identity. To complete the proof, we count S in a new way. Let $X_1 = \{1, 2, \ldots, n\}$ and $X_2 = \{n+1, \ldots, 2n\}$. For $0 \le k \le n$, define

$$S_k = \{A \in S : |A \cap X_1| = k \text{ and } |A \cap X_2| = n - k\}.$$

Evidently, S is the disjoint union of the S_k's, so that $|S| = \sum_{k=0}^{n} |S_k|$ by the sum rule. To compute $|S_k|$, we build a typical object $A \in S_k$ by making two choices. First, choose the k-element subset $A \cap X_1$ in any of $\binom{n}{k}$ ways. Second, choose the $(n-k)$-element subset $A \cap X_2$ in any of $\binom{n}{n-k} = \binom{n}{k}$ ways. We see that $|S_k| = \binom{n}{k}^2$ by the product rule. Thus, $|S| = \sum_{k=0}^{n} \binom{n}{k}^2$, completing the proof. □

One can often find different combinatorial proofs of a given identity. For example, here is an alternate proof of the previous identity using lattice paths. Let S be the set of all lattice paths from the origin to (n,n); we know that $|S| = \binom{2n}{n,n} = \binom{2n}{n}$. For $0 \le k \le n$, let S_k be the set of all paths in S passing through the point $(k, n-k)$ on the line $x + y = n$. Every path in S must go through exactly one such point for some k between 0 and n, so S is the disjoint union of $S_0, S_1, \ldots, S_n$. See Figure 2.1. To build a path in S_k, first choose a path from $(0,0)$ to $(k, n-k)$ in any of $\binom{n}{k,n-k} = \binom{n}{k}$ ways. Second, choose a path from $(k, n-k)$ to (n,n). This is a path in a rectangle of width $n-k$ and height $n-(n-k) = k$, so there are $\binom{n}{n-k,k} = \binom{n}{k}$ ways to make this second choice. By the sum and product rules, we conclude that

$$|S| = \sum_{k=0}^{n} |S_k| = \sum_{k=0}^{n} \binom{n}{k}^2.$$

Lattice paths can often be used to give elegant, visually appealing combinatorial proofs of identities involving binomial coefficients. We conclude this section with two more examples of this kind.

2.20. Theorem. For all integers $a \ge 0$ and $b \ge 1$,

$$\binom{a+b}{a,b} = \sum_{k=0}^{a} \binom{k+b-1}{k, b-1}.$$

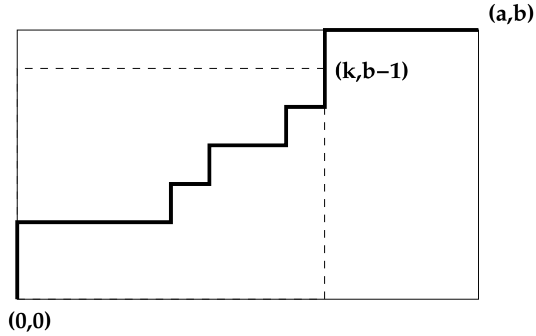


FIGURE 2.2
Another combinatorial proof using lattice paths.

Proof. Let S be the set of all lattice paths from the origin to (a,b). We already know that $|S| = \binom{a+b}{a,b}$. For $0 \le k \le a$, let S_k be the set of paths $\pi \in S$ such that the last north step of π lies on the line $x = k$. See Figure 2.2. We can build a path in S_k by choosing any lattice path from the origin to $(k, b-1)$ in $\binom{k+b-1}{k,b-1}$ ways, and then appending one north step and $a-k$ east steps. Thus, the required identity follows from the sum rule. If we classify the paths by the final east step instead, we obtain the dual identity (for $a \ge 1$, $b \ge 0$)

$$\binom{a+b}{a,b} = \sum_{j=0}^{b} \binom{a-1+j}{a-1,j}.$$

This identity also follows from the previous one by the known symmetry $\binom{a+b}{a,b} = \binom{b+a}{b,a}$. □

2.21. Theorem (Chu-Vandermonde Identity). For all integers $a, b, c \ge 0$,

$$\binom{a+b+c+1}{a,b+c+1} = \sum_{k=0}^{a} \binom{k+b}{k,b}\binom{a-k+c}{a-k,c}.$$

Proof. Let S be the set of all lattice paths from the origin to $(a, b+c+1)$. We know that $|S| = \binom{a+b+c+1}{a,b+c+1}$. For $0 \le k \le a$, let S_k be the set of paths $\pi \in S$ that contain the north step from (k,b) to $(k,b+1)$. Since every path in π must cross the line $y = b+1/2$ by taking a north step between the lines $x = 0$ and $x = a$, we see that S is the disjoint union of $S_0, S_1, \ldots, S_a$. See Figure 2.3. Now, we can build a path in S_k as follows. First, choose a lattice path from the origin to (k,b) in $\binom{k+b}{k,b}$ ways. Second, append a north step to this path. Third, choose a lattice path from $(k,b+1)$ to $(a,b+c+1)$. This is a path in a rectangle of width $a-k$ and height c, so there are $\binom{a-k+c}{a-k,c}$ ways to make this choice. Thus, $|S_k| = \binom{k+b}{k,b} \cdot 1 \cdot \binom{a-k+c}{a-k,c}$ by the product rule. The desired identity now follows from the sum rule. □

We remark that 2.20 is the special case of 2.21 obtained by setting $c = 0$ and replacing b by $b-1$.

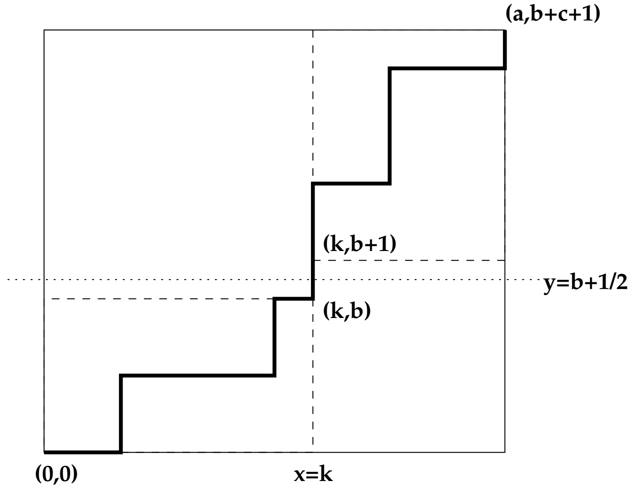


FIGURE 2.3
A third combinatorial proof using lattice paths.

2.4 Recursions

Suppose we are given some unknown quantities $a_0, a_1, \ldots, a_n, \ldots$. A *closed formula* for these quantities is an expression of the form $a_n = f(n)$, where the right side is some explicit formula involving the integer n but not involving any of the unknown quantities a_i. In contrast, a *recursive formula* for a_n is an expression of the form $a_n = f(n, a_0, a_1, \ldots, a_{n-1})$, where the right side is a formula that does involve one or more of the unknown quantities a_i. A recursive formula is usually accompanied by one or more *initial conditions*, which are non-recursive expressions for a_0 and possibly other a_i's. Similar definitions apply to doubly indexed sequences $a_{n,k}$.

Now consider the problem of counting sets of combinatorial objects. Suppose we have several related families of objects, say $T_0, T_1, \ldots, T_n, \ldots$. We think of the index n as somehow measuring the size of the objects in T_n. Sometimes we can give an explicit description of the objects in T_n (using the sum and product rules) leading to a closed formula for $|T_n|$. In many cases, however, it is more natural to give a *recursive* description of T_n, which tells us how to construct a typical object in T_n by assembling smaller objects of the same kind from the sets $T_0, \ldots, T_{n-1}$. Such an argument leads to a recursive formula for $|T_n|$ in terms of one or more of the quantities $|T_0|, \ldots, |T_{n-1}|$. If we suspect that $|T_n|$ is also given by a certain closed formula, we can then prove this fact using induction. We use the following example to illustrate these ideas.

2.22. Theorem: Recursion for Subsets. For each integer $n \geq 0$, let T_n be the set of all subsets of $\{1, 2, \ldots, n\}$, and let $a_n = |T_n|$. We derive a recursive formula for a_n as follows. Suppose $n \geq 1$ and we are trying to build a typical subset $A \in T_n$. We can do this recursively by first choosing a subset $A' \subseteq \{1, 2, \ldots, n-1\}$ in any of $|T_{n-1}| = a_{n-1}$ ways, and then either adding or not adding the element n to this subset (2 possibilities). By the product rule, we conclude that

$$a_n = a_{n-1} \cdot 2 \qquad (n \geq 1).$$

The initial condition is $a_0 = 1$, since $T_0 = \{\emptyset\}$.

Using the recursion and initial condition, we calculate:

$$(a_0, a_1, a_2, a_3, a_4, a_5, \ldots) = (1, 2, 4, 8, 16, 32, \ldots).$$

The pattern suggests that $a_n = 2^n$ for all $n \geq 0$. (We have already proved this earlier, but we wish to reprove this fact using our recursion.) We will prove that $a_n = 2^n$ by induction on n. In the base case ($n = 0$), we have $a_0 = 1 = 2^0$ by the initial condition. Assume that $n > 0$ and that $a_{n-1} = 2^{n-1}$ (this is the induction hypothesis). Using the recursion and the induction hypothesis, we see that

$$a_n = 2a_{n-1} = 2(2^{n-1}) = 2^n.$$

This completes the proof by induction.

2.23. Example: Fibonacci Words. Let W_n be the set of all words in $\{0, 1\}^n$ that do not have two consecutive zeroes, and let $f_n = |W_n|$. We now derive a recursion and initial condition for the sequence of f_n's. First, direct enumeration shows that $f_0 = 1$ and $f_1 = 2$. Suppose $n \geq 2$. We use the sum rule to find a formula for $|W_n| = f_n$. Given $w \in W_n$, w starts with either 0 or 1. If $w_1 = 0$, then we are forced to have $w_2 = 1$, and then $w' = w_3w_4 \cdots w_n$ can be an arbitrary word in W_{n-2}. Therefore, there are f_{n-2} words in W_n starting with 0. On the other hand, if $w_1 = 1$, then $w' = w_2 \cdots w_n$ can be an arbitrary word in W_{n-1}. Therefore, there are f_{n-1} words in W_n starting with 1. By the sum rule,

$$f_n = f_{n-1} + f_{n-2} \qquad (n \geq 2).$$

Using this recursion and the initial conditions, we compute

$$(f_0, f_1, f_2, f_3, f_4, f_5, \ldots) = (1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, \ldots).$$

This sequence is called the *Fibonacci sequence*. We will find an explicit closed formula for f_n later (see 2.134(a) or §7.14).

Now we consider some examples involving doubly indexed families of combinatorial objects. We begin by revisiting the enumeration of k-permutations, subsets, multisets, and anagrams. We will reprove some of our earlier counting results by recursive methods.

2.24. Recursion for k-Permutations. For all integers $n, k \geq 0$, let $P(n, k)$ be the number of k-permutations of an n-element set. (One can show bijectively that every n-element set has the same number of k-permutations, so that $P(n, k)$ is a well-defined integer. Alternatively, we could define $P(n, k)$ to be the number of k-permutations of a particular n-element set like $\{1, 2, \ldots, n\}$. However, in the latter case, the argument in the text must be modified accordingly. Similar comments apply to later examples.) Recall that a k-permutation is an ordered sequence $x_1x_2 \cdots x_k$ of *distinct* elements from the given n-element set. Observe that $P(n, k) = 0$ whenever $k > n$. On the other hand, $P(n, 0) = 1$ for all $n \geq 0$ since the empty sequence is the unique 0-permutation of any n-element set. Now assume that $0 < k \leq n$. We can build a typical k-permutation $x = x_1x_2 \cdots x_k$ of a given n-element set X as follows. First, choose x_1 in any of n ways. For the second choice, note that $x' = x_2x_3 \cdots x_k$ can be any $(k-1)$-permutation of the $(n-1)$-element set $X \sim \{x_1\}$. There are $P(n-1, k-1)$ choices for x', by definition. The product rule thus gives us the recursion

$$P(n, k) = nP(n-1, k-1) \qquad (0 < k \leq n).$$

The initial conditions are $P(n, 0) = 1$ for all n and $P(n, k) = 0$ for all $k > n$.

In §1.4, we used the product rule to prove that $P(n, k) = n(n-1) \cdots (n-k+1) =$

$n!/(n-k)!$ for $0 \leq k \leq n$. Let us now reprove this result using our recursion. We proceed by induction on n. In the base case, $n = 0$ and hence $k = 0$. The initial condition gives

$$P(n, k) = P(0, 0) = 1 = 0!/(0-0)! = n!/(n-k)!.$$

For the induction step, assume that $n > 0$ and that

$$P(n-1, j) = (n-1)!/(n-1-j)! \qquad (0 \leq j \leq n-1).$$

Fix k with $0 \leq k \leq n$. If $k = 0$, the initial condition gives

$$P(n, k) = P(n, 0) = 1 = n!/(n-0)! = n!/(n-k)!.$$

If $k > 0$, we use the recursion and induction hypothesis (applied to $j = k-1$) to compute

$$P(n, k) = nP(n-1, k-1) = n\frac{(n-1)!}{((n-1)-(k-1))!} = \frac{n!}{(n-k)!}.$$

This completes the proof by induction.

2.25. Recursion for k-element Subsets. For all integers $n, k \geq 0$, let $C(n, k)$ be the number of k-element subsets of $\{1, 2, \ldots, n\}$. Observe that $C(n, k) = 0$ whenever $k > n$. On the other hand, the initial condition $C(n, 0) = 1$ follows since the empty set is the unique zero-element subset of any set. Similarly, $C(n, n) = 1$ since $\{1, 2, \ldots, n\}$ is the only n-element subset of itself. Let us now derive a recursion for $C(n, k)$ assuming that $0 < k < n$. A typical k-element subset A of $\{1, 2, \ldots, n\}$ either *does* or *does not* contain n as a member. In the former case, we can construct A by choosing any $(k-1)$-element subset of $\{1, 2, \ldots, n-1\}$ in $C(n-1, k-1)$ ways, and then appending n as the final member of A. In the latter case, we can construct A by choosing any k-element subset of $\{1, 2, \ldots, n-1\}$ in $C(n-1, k)$ ways. By the sum rule, we deduce *Pascal's recursion*

$$C(n, k) = C(n-1, k-1) + C(n-1, k) \qquad (0 < k < n).$$

For $n > 0$, this recursion even holds for $k = 0$ and $k = n$, provided we use the conventions that $C(a, b) = 0$ whenever $b < 0$ or $b > a$.

In §1.8, we proved that $C(n, k) = \frac{n!}{k!(n-k)!} = \binom{n}{k}$. Let us reprove this result using the recursion and initial conditions. We proceed by induction on n. The base case $n = k = 0$ follows since $C(0, 0) = 1 = \frac{0!}{0!(0-0)!}$. Assume $n > 0$ and that

$$C(n-1, j) = \frac{(n-1)!}{j!(n-1-j)!} \qquad (0 \leq j \leq n-1).$$

Fix k with $0 \leq k \leq n$. If $k = 0$, the initial condition gives

$$C(n, k) = 1 = \frac{n!}{0!(n-0)!}$$

as desired. Similarly, the result holds when $k = n$. If $0 < k < n$, we use the recursion and induction hypothesis (applied to $j = k-1$ and to $j = k$, which are integers between 0 and

$$
\begin{array}{lccccccccccccccccc}
n=0: & & & & & & & & & 1 & & & & & & & & \\
n=1: & & & & & & & & 1 & & 1 & & & & & & & \\
n=2: & & & & & & & 1 & & 2 & & 1 & & & & & & \\
n=3: & & & & & & 1 & & 3 & & 3 & & 1 & & & & & \\
n=4: & & & & & 1 & & 4 & & 6 & & 4 & & 1 & & & & \\
n=5: & & & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & & \\
n=6: & & & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & & \\
n=7: & & 1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1 & \\
n=8: & 1 & & 8 & & 28 & & 56 & & 70 & & 56 & & 28 & & 8 & & 1
\end{array}
$$

FIGURE 2.4
Pascal's Triangle.

$n-1$) to compute

$$
\begin{aligned}
C(n,k) &= C(n-1,k-1)+C(n-1,k) \\
&= \frac{(n-1)!}{(k-1)!((n-1)-(k-1))!}+\frac{(n-1)!}{k!((n-1)-k)!} \\
&= \frac{(n-1)!k}{k!(n-k)!}+\frac{(n-1)!(n-k)}{k!(n-k)!} \\
&= \frac{(n-1)!}{k!(n-k)!}\cdot[k+(n-k)] \\
&= \frac{n!}{k!(n-k)!}.
\end{aligned}
$$

This completes the proof by induction.

The reader may wonder what good it is to have a recursion for $C(n,k)$, since we already proved by other methods the explicit formula $C(n,k)=\frac{n!}{k!(n-k)!}$. There are several answers to this question. One answer is that the recursion for the $C(n,k)$'s gives us a fast method for calculating these quantities that is more efficient than computing with factorials. One popular way of displaying this calculation is called *Pascal's Triangle*. We build this triangle by writing the $n+1$ numbers $C(n,0),C(n,1),\ldots,C(n,n)$ in the nth row from the top. If we position the entries as shown in Figure 2.4, then each entry is the sum of the two entries directly above it. We compute $C(n,k)$ by calculating rows 0 through n of this triangle.

Note that computing $C(n,k)$ via Pascal's recursion requires only addition operations. In contrast, calculation using the closed formula $\frac{n!}{k!(n-k)!}$ requires us to divide one large factorial by the product of two other factorials. For example, Pascal's Triangle quickly gives $C(8,4)=70$, while the closed formula gives $\binom{8}{4}=\frac{8!}{4!4!}=\frac{40,320}{24^2}=70$.

In bijective combinatorics, it turns out that the arithmetic operation of *division* is much harder to understand (from a combinatorial standpoint) than the operations of *addition* and *multiplication*. In particular, our original derivation of the formula $\binom{n}{k}=\frac{n!}{k!(n-k)!}$ was an indirect argument using the product rule, in which we divided by $k!$ at the end (§1.8). For later applications (e.g., listing all k-element subsets of a given n-element set, or randomly selecting a k-element subset), it is convenient to have a counting argument that does not rely on division. See Chapter 5 for more details.

A final reason for studying recursions for $C(n,k)$ is to emphasize that recursions are helpful and ubiquitous tools for studying combinatorial objects. Indeed, we will soon be studying combinatorial collections whose cardinalities may not be given by explicit closed formulas. Nevertheless, these cardinalities satisfy recursions that allow them to be computed quickly and efficiently.

2.5 Recursions for Multisets and Anagrams

This section continues to give examples of combinatorial recursions for objects we have studied before, namely multisets and anagrams.

2.26. Recursion for Multisets. In §1.11, we counted k-element multisets on an n-letter alphabet using bijective techniques. Now, we give a recursive analysis to reprove the enumeration results for multisets. For all integers $n, k \geq 0$, let $M(n, k)$ be the number of k-element multisets using letters from $\{1, 2, \ldots, n\}$. The initial conditions are $M(n, 0) = 1$ for all $n \geq 0$ and $M(0, k) = 0$ for all $k > 0$. We now derive a recursion for $M(n, k)$ assuming $n > 0$ and $k > 0$. A typical multiset counted by $M(n, k)$ either does not contain n at all or contains one or more copies of n. In the former case, the multiset is a k-element multiset using letters from $\{1, 2, \ldots, n-1\}$, and there are $M(n-1, k)$ such multisets. In the latter case, if we remove one copy of n from the multiset, we obtain an arbitrary $(k-1)$-element multiset using letters from $\{1, 2, \ldots, n\}$. There are $M(n, k-1)$ such multisets. By the sum rule, we obtain the recursion

$$M(n, k) = M(n-1, k) + M(n, k-1) \qquad (n > 0, k > 0).$$

One can now prove that for all $n \geq 0$ and all $k \geq 0$,

$$M(n, k) = \binom{k+n-1}{k, n-1} = \frac{(k+n-1)!}{k!(n-1)!}.$$

The proof is by induction on n, and is similar to the corresponding proof for $C(n, k)$. We leave this proof as an exercise.

If desired, we can use the recursion to compute values of $M(n, k)$. Here we use a left-justified table of entries in which the nth row contains the numbers $M(n, 0), M(n, 1), \ldots$. The values in the top row (where $n = 0$) and in the left column (where $k = 0$) are given by the initial conditions. Each remaining entry in the table is the sum of the number directly above it and the number directly to its left. See Figure 2.5. The reader will perceive that this is merely a shifted version of Pascal's Triangle.

2.27. Recursion for Multinomial Coefficients. Let $n_1, \ldots, n_s$ be nonnegative integers that add to n. Let $\{a_1, \ldots, a_s\}$ be a given s-letter alphabet, and let $C(n; n_1, \ldots, n_s) = |\mathcal{R}(a_1^{n_1} \cdots a_s^{n_s})|$ be the number of n-letter words that are rearrangements of n_i copies of a_i for $1 \leq i \leq s$. We proved in §1.9 that

$$C(n; n_1, \ldots, n_s) = \binom{n}{n_1, \ldots, n_s} = \frac{n!}{n_1! n_2! \cdots n_s!}.$$

We now give a new proof of this result using recursions.

Assume first that every n_i is positive. For $1 \leq i \leq s$, let T_i be the set of words in $T = \mathcal{R}(a_1^{n_1} \cdots a_s^{n_2})$ that begin with the letter a_i. T is the disjoint union of the sets T_i. To build a typical word $w \in T_i$, we start with the letter a_i and then append any element of

	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
$n=0:$	1	0	0	0	0	0	0
$n=1:$	1	1	1	1	1	1	1
$n=2:$	1	2	3	4	5	6	7
$n=3:$	1	3	6	10	15	21	28
$n=4:$	1	4	10	20	35	56	84
$n=5:$	1	5	15	35	70	126	210
$n=6:$	1	6	21	56	126	252	462

FIGURE 2.5
Table for computing $M(n,k)$.

$\mathcal{R}(a_1^{n_1}\cdots a_i^{n_i-1}\cdots a_s^{n_s})$. There are $C(n-1;n_1,\ldots,n_i-1,\ldots,n_s)$ ways to do this. Hence, by the sum rule,

$$C(n;n_1,\ldots,n_s)=\sum_{i=1}^{s}C(n-1;n_1,\ldots,n_i-1,\ldots,n_s).$$

If we adopt the convention that $C(n;n_1,\ldots,n_s)=0$ whenever any n_i is negative, then this recursion holds (with the same proof) for all choices of $n_i\geq 0$ and $n>0$. The initial condition is

$$C(0;0,0,\ldots,0)=1,$$

since the empty word is the unique rearrangement of zero copies of the given letters.

Now let us prove that

$$C(n;n_1,\ldots,n_s)=\frac{n!}{\prod_{k=1}^{s}n_k!}$$

by induction on n. In the base case, $n=n_1=\cdots=n_s=0$, and the desired formula follows from the initial condition. For the induction step, assume that $n>0$ and that

$$C(n-1;m_1,\ldots,m_s)=\frac{(n-1)!}{\prod_{k=1}^{s}m_k!}$$

whenever $m_1+\cdots+m_s=n-1$. Assume that we are given integers $n_k\geq 0$ that sum to n. Now, using the recursion and induction hypothesis, we compute as follows:

$$\begin{aligned}
C(n;n_1,\ldots,n_k) &= \sum_{k=1}^{s}C(n-1;n_1,\ldots,n_k-1,\ldots,n_s)\\
&= \sum_{k=1}^{s}\chi(n_k>0)\frac{(n-1)!}{(n_k-1)!\prod_{j\neq k}n_j!}\\
&= \sum_{k=1}^{s}\chi(n_k>0)\frac{(n-1)!n_k}{\prod_{j=1}^{s}n_j!}=\sum_{k=1}^{s}\frac{(n-1)!n_k}{\prod_{j=1}^{s}n_j!}\\
&= \frac{(n-1)!}{\prod_{j=1}^{s}n_j!}\left[\sum_{k=1}^{s}n_k\right]=\frac{n!}{\prod_{j=1}^{s}n_j!}.
\end{aligned}$$

2.6 Recursions for Lattice Paths

Recursive techniques allow us to count many collections of lattice paths. We first consider the situation of lattice paths in a rectangle.

2.28. Recursion for Paths in a Rectangle. For $a, b \geq 0$, let $L(a,b)$ be the number of lattice paths from the origin to (a,b). We have $L(a,0) = L(0,b) = 1$ for all $a, b \geq 0$. If $a > 0$ and $b > 0$, note that any lattice path ending at (a,b) arrives there via an east step or a north step. We obtain lattice paths of the first kind by taking any lattice path ending at $(a-1,b)$ and appending an east step. We obtain lattice paths of the second kind by taking any lattice path ending at $(a,b-1)$ and appending a north step. Hence, by the sum rule,

$$L(a,b) = L(a-1,b) + L(a,b-1) \qquad (a,b>0).$$

One can now show (by induction on $a+b$) that

$$L(a,b) = \binom{a+b}{a,b} = \frac{(a+b)!}{a!b!} \qquad (a,b \geq 0).$$

We can visually display and calculate the numbers $L(a,b)$ by labeling each lattice point (a,b) with the number $L(a,b)$. The initial conditions say that the lattice points on the axes are labeled 1. The recursion says that the label of some point (a,b) is the sum of the labels of the point $(a-1,b)$ to its immediate left and the point $(a,b-1)$ immediately below it. See Figure 2.6.

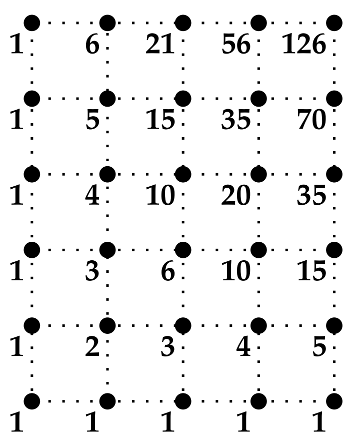


FIGURE 2.6
Recursive enumeration of lattice paths.

By modifying the boundary conditions, we can adapt the recursion in the previous example to count more complicated collections of lattice paths.

2.29. Recursion for Paths in a Triangle. For $b \geq a \geq 0$, let $T(a,b)$ be the number of lattice paths from the origin to (a,b) that always stay weakly above the line $y = x$. (In particular, $T(n,n)$ is the number of Dyck paths of order n.) By the same argument used above, we have

$$T(a,b) = T(a-1,b) + T(a,b-1) \qquad (b > a > 0).$$

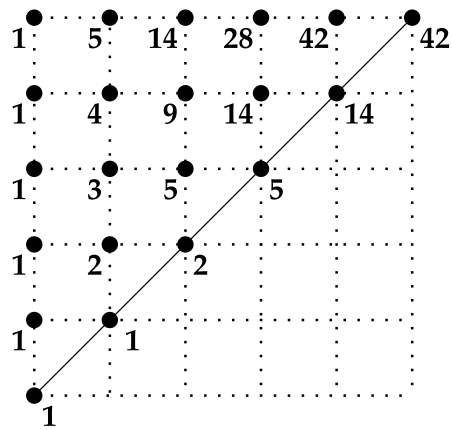


FIGURE 2.7
Recursive enumeration of lattice paths in a triangle.

On the other hand, when $a = b > 0$, a lattice path can only reach $(a, b) = (a, a)$ by taking an east step, since the point $(a, b-1)$ lies below $y = x$. Thus,

$$T(a, a) = T(a-1, a) \qquad (a > 0).$$

The initial conditions are $T(0, b) = 1$ for all $b \geq 0$. Figure 2.7 shows how to compute the numbers $T(a, b)$ by drawing a picture.

It turns out that there is an explicit closed formula for the numbers $T(a, b)$.

2.30. Theorem: Ballot Numbers. For $b \geq a \geq 0$, the number of lattice paths from the origin to (a, b) that always stay weakly above the line $y = x$ is

$$\frac{b-a+1}{b+a+1}\binom{a+b+1}{a}.$$

In particular, the number of Dyck paths of order n is

$$\frac{1}{2n+1}\binom{2n+1}{n} = C_n.$$

Proof. We show that

$$T(a, b) = \frac{b-a+1}{b+a+1}\binom{a+b+1}{a}$$

by induction on $a+b$. If $a+b = 0$, so that $a = b = 0$, then $T(0, 0) = 1 = \frac{0-0+1}{0+0+1}\binom{0+0+1}{0}$. Now assume that $a+b > 0$ and that $T(c, d) = \frac{d-c+1}{d+c+1}\binom{c+d+1}{c}$ whenever $d \geq c \geq 0$ and $c+d < a+b$. To prove the desired formula for $T(a, b)$, we consider cases based on the recursions and initial conditions. First, if $a = 0$ and $b \geq 0$, we have $T(a, b) = 1 = \frac{b-0+1}{b+0+1}\binom{0+b+1}{0}$. Second, if $a = b > 0$, we have

$$\begin{aligned}
T(a, b) &= T(a, a) = T(a-1, a) = \frac{2}{2a}\binom{2a}{a-1} \\
&= \frac{(2a)!}{a!(a+1)!} = \frac{1}{2a+1}\binom{2a+1}{a} \\
&= \frac{b-a+1}{b+a+1}\binom{a+b+1}{a}.
\end{aligned}$$

Third, if $b > a > 0$, we have

$$\begin{aligned} T(a,b) &= T(a-1,b)+T(a,b-1) = \frac{b-a+2}{a+b}\binom{a+b}{a-1} + \frac{b-a}{a+b}\binom{a+b}{a} \\ &= \frac{(b-a+2)(a+b-1)!}{(a-1)!(b+1)!} + \frac{(b-a)(a+b-1)!}{a!b!} \\ &= \left[\frac{a(b-a+2)}{a+b} + \frac{(b-a)(b+1)}{a+b}\right]\frac{(a+b)!}{a!(b+1)!} \\ &= \left[\frac{ab-a^2+2a+b^2-ab+b-a}{a+b}\right]\frac{(a+b+1)!}{(b+a+1)a!(b+1)!} \\ &= \left[\frac{(b-a+1)(a+b)}{a+b}\right]\frac{1}{b+a+1}\binom{a+b+1}{a} \\ &= \frac{b-a+1}{b+a+1}\binom{a+b+1}{a}. \quad \square \end{aligned}$$

The numbers $T(a,b)$ in the previous theorem are called *ballot numbers*, for the following reason. Let $\pi \in \{N,E\}^{a+b}$ be a lattice path counted by $T(a,b)$. Imagine that $a+b$ people are voting for two candidates ("candidate N" and "candidate E") by casting an ordered sequence of $a+b$ ballots. The path π records this sequence of ballots as follows: $\pi_j = N$ if the jth person votes for candidate N, and $\pi_j = E$ if the jth person votes for candidate E. The condition that π stays weakly above $y = x$ means that candidate N always has at least as many votes as candidate E at each stage in the election process. The condition that π ends at (a,b) means that candidate N has b votes and candidate E has a votes at the end of the election.

Returning to lattice paths, suppose we replace the boundary line $y = x$ by the line $y = mx$ (where m is any positive integer). We can then derive the following more general result.

2.31. Theorem: m-Ballot Numbers. Let m be a fixed positive integer. For $b \geq ma \geq 0$, the number of lattice paths from the origin to (a,b) that always stay weakly above the line $y = mx$ is

$$\frac{b-ma+1}{b+a+1}\binom{a+b+1}{a}.$$

In particular, the number of such paths ending at (n,mn) is

$$\frac{1}{(m+1)n+1}\binom{(m+1)n+1}{n}.$$

Proof. Let $T_m(a,b)$ be the number of paths ending at (a,b) that never go below $y = mx$. Arguing as before, we have $T_m(0,b) = 1$ for all $b \geq 0$; $T_m(a,b) = T_m(a-1,b) + T_m(a,b-1)$ whenever $b > ma > 0$; and $T_m(a,ma) = T_m(a-1,ma)$ since the point $(a,ma-1)$ lies below the line $y = mx$. One now proves that

$$T_m(a,b) = \frac{b-ma+1}{b+a+1}\binom{a+b+1}{a}$$

by induction on $a+b$. The proof is similar to the one given above, so we leave it as an exercise. For a bijective proof of this theorem, see 12.92. $\square$

When the slope m of the boundary line $y = mx$ is not an integer, we cannot use the formula in the preceding theorem. Nevertheless, the recursion (with appropriate initial

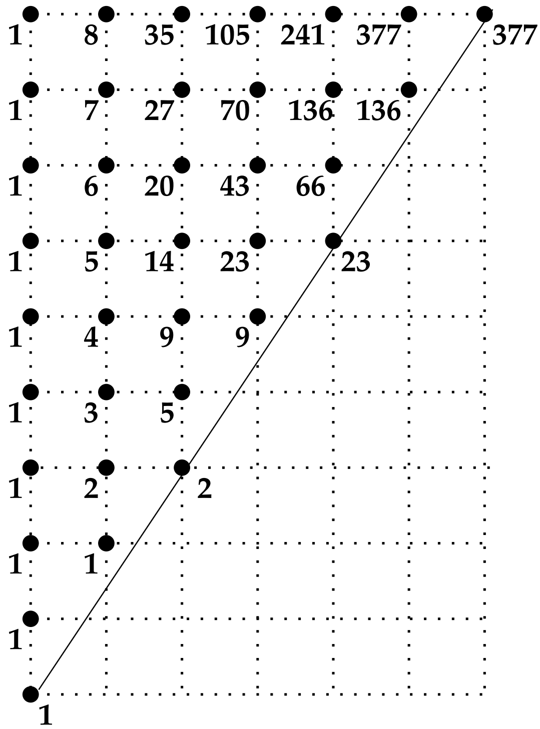


FIGURE 2.8
Recursive enumeration of lattice paths above $y = (3/2)x$.

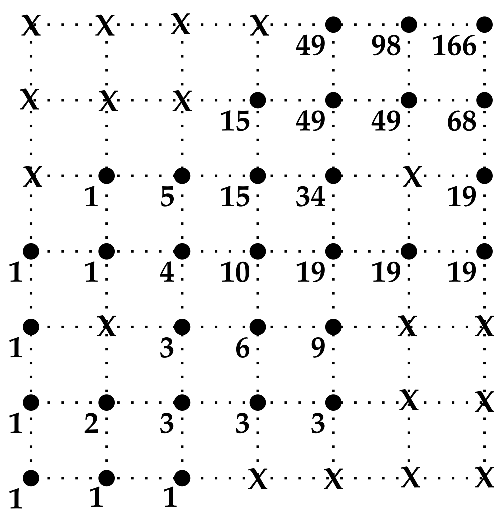


FIGURE 2.9
Recursive enumeration of lattice paths in an irregular region.

conditions) can still be used to count lattice paths bounded below by this line. For example, Figure 2.8 illustrates the enumeration of lattice paths from $(0,0)$ to $(6,9)$ that always stay weakly above $y = (3/2)x$.

We end this section with a general recursion for counting lattice paths in a given region.

2.32. Theorem: General Lattice Path Recursion. Suppose V is a given set of lattice points in $\mathbb{N} \times \mathbb{N}$ containing the origin. For $(a,b) \in V$, let $T_V(a,b)$ be the number of lattice paths from the origin to (a,b) that visit only lattice points in V. Then $T_V(0,0) = 1$ and

$$T_V(a,b) = T_V(a-1,b)\chi((a-1,b) \in V) + T_V(a,b-1)\chi((a,b-1) \in V) \qquad \text{for } (a,b) \neq (0,0).$$

The proof is immediate from the sum rule. Figure 2.9 illustrates the use of this recursion to count lattice paths contained in an irregular region. In the figure, lattice points in V are drawn as closed circles, while X's indicate certain forbidden lattice points that the path is not allowed to use.

2.7 Catalan Recursions

The recursions from the previous section provide one way of computing Catalan numbers, which are a special case of ballot numbers. This section discusses another recursion that involves only the Catalan numbers. This "convolution recursion" comes up in many settings, thus leading to many different combinatorial interpretations for the Catalan numbers.

2.33. Theorem: Catalan Recursion. The Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$ satisfy the recursion

$$C_n = \sum_{k=1}^{n} C_{k-1}C_{n-k} \qquad (n > 0)$$

and initial condition $C_0 = 1$.

Proof. Recall from 1.56 that C_n is the number of Dyck paths of order n. There is one Dyck path of order 0, so $C_0 = 1$. Fix $n > 0$, and let A be the set of Dyck paths ending at (n,n). For $1 \leq k \leq n$, let

$$A_k = \{\pi \in A : (k,k) \in \pi \text{ and } (j,j) \notin \pi \text{ for } 0 < j < k\}.$$

In other words, A_k consists of the Dyck paths of order n that return to the diagonal line $y = x$ for the first time at the point (k,k). See Figure 2.10. Suppose w is the word in $\{N,E\}^{2n}$ that encodes a path $\pi \in A_k$. Inspection of Figure 2.10 shows that we have the factorization $w = Nw_1Ew_2$, where w_1 encodes a Dyck path of order $k-1$ (starting at $(0,1)$ in the figure) and w_2 encodes a Dyck path of order $n-k$ (starting at (k,k) in the figure). We can uniquely construct all paths in A_k by choosing w_1 and w_2 and then setting $w = Nw_1Ew_2$. There are C_{k-1} choices for w_1 and C_{n-k} choices for w_2. By the product rule and sum rule,

$$C_n = |A| = \sum_{k=1}^{n} |A_k| = \sum_{k=1}^{n} C_{k-1}C_{n-k}.$$

□

The next result shows that the Catalan recursion uniquely determines the Catalan numbers.

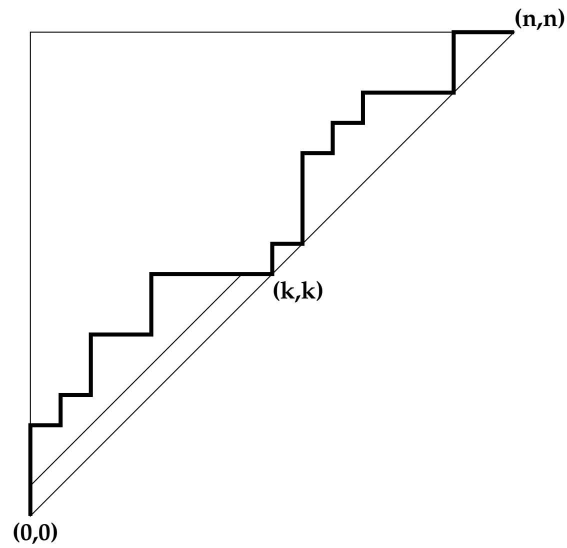


FIGURE 2.10
Proving the Catalan recursion by analyzing the first return to $y = x$.

2.34. Theorem. Suppose $(d_n : n \geq 0)$ is a sequence such that $d_0 = 1$ and

$$d_n = \sum_{k=1}^{n} d_{k-1} d_{n-k} \qquad (n > 0).$$

Then $d_n = C_n = \frac{1}{n+1}\binom{2n}{n}$ for all $n \geq 0$.

Proof. We argue by strong induction. For $n = 0$, we have $d_0 = 1 = C_0$. Assume that $n > 0$ and that $d_m = C_m$ for all $m < n$. Then

$$d_n = \sum_{k=1}^{n} d_{k-1} d_{n-k} = \sum_{k=1}^{n} C_{k-1} C_{n-k} = C_n. \qquad \square$$

We can now prove that various collections of objects are counted by the Catalan numbers. One proof method sets up a bijection between such objects and other objects (like Dyck paths) that are already known to be counted by Catalan numbers. A second proof method shows that the new collections of objects satisfy the Catalan recursion. We illustrate both methods in the examples below.

2.35. Example: Balanced Parentheses. For $n \geq 0$, let BP_n be the set of all words consisting of n left parentheses and n right parentheses, such that every left parenthesis can be matched with a right parenthesis later in the word. For example, BP_3 consists of the following five words:

`((())), (())(), ()(()), (()()), ()()().`

We show that $|BP_n| = C_n$ for all n by exhibiting a bijection between BP_n and the set of Dyck paths of order n. Given $w \in BP_3$, replace each left parenthesis by N (which encodes a north step) and each right parenthesis by E (which encodes an east step). One can check that a string w of n left and n right parentheses is balanced iff for every $i \leq 2n$, the number

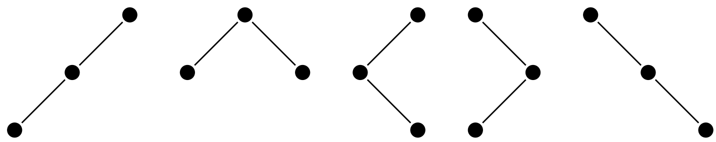


FIGURE 2.11
The five binary trees with three nodes.

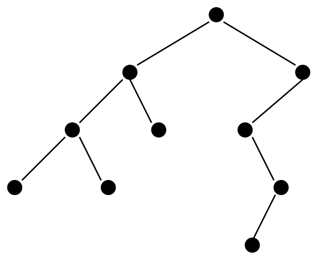


FIGURE 2.12
A binary tree with ten nodes.

of left parentheses in the prefix $w_1 w_2 \cdots w_i$ weakly exceeds the number of right parentheses in this prefix. Converting to north and east steps, this condition means that no lattice point on the path lies strictly below the line $y = x$. Thus we have mapped each $w \in BP_n$ to a Dyck path. This map is a bijection, so $|BP_n| = C_n$.

2.36. Example: Binary Trees. We recursively define the set of *binary trees with n nodes* as follows. The empty set is the unique binary tree with 0 nodes. If T_1 is a binary tree with h nodes and T_2 is a binary tree with k nodes, then the ordered triple $T = (\bullet, T_1, T_2)$ is a binary tree with $h + k + 1$ nodes. By definition, all binary trees arise by a finite number of applications of these rules. If $T = (\bullet, T_1, T_2)$ is a binary tree, we call T_1 the *left subtree* of T and T_2 the *right subtree* of T. Note that T_1 or T_2 (or both) may be empty. We can draw a picture of a nonempty binary tree T as follows. First, draw a *root node* of the binary tree at the top of the picture. If T_1 is nonempty, draw an edge leading down and left from the root node, and then draw the picture of T_1. If T_2 is nonempty, draw an edge leading down and right from the root node, and then draw the picture of T_2. For example, Figure 2.11 displays the five binary trees with three nodes. Figure 2.12 depicts a larger binary tree that is formally represented by the sequence

$$T = (\bullet, (\bullet, (\bullet, (\bullet, \emptyset, \emptyset), (\bullet, \emptyset, \emptyset)), (\bullet, \emptyset, \emptyset)), (\bullet, (\bullet, \emptyset, (\bullet, (\bullet, \emptyset, \emptyset), \emptyset)), \emptyset)).$$

Let BT_n denote the set of binary trees with n nodes. We show that $|BT_n| = C_n$ for all n by verifying that the sequence $(|BT_n| : n \geq 0)$ satisfies the Catalan recursion. First, $|BT_0| = 1$ by definition. Second, suppose $n \geq 1$. By the recursive definition of binary trees, we can uniquely construct a typical element of BT_n as follows. Fix k with $1 \leq k \leq n$. Choose a tree $T_1 \in BT_{k-1}$ with $k - 1$ nodes. Then choose a tree $T_2 \in BT_{n-k}$ with $n - k$ nodes. We assemble these trees (together with a new root node) to get a binary tree $T = (\bullet, T_1, T_2)$ with $(k - 1) + 1 + (n - k) = n$ nodes. By the sum and product rules, we have

$$|BT_n| = \sum_{k=1}^{n} |BT_{k-1}||BT_{n-k}|.$$

It follows from 2.34 that $|BT_n| = C_n$ for all $n \geq 0$.

2.37. Example: 231-avoiding permutations. Suppose $w = w_1 w_2 \cdots w_n$ is a permutation of n distinct integers. We say that w is *231-avoiding* iff there do not exist indices $i < k < p$ such that $w_p < w_i < w_k$. This means that no three-element subsequence $w_i \ldots w_k \ldots w_p$ in w has the property that w_p is the smallest number in $\{w_i, w_k, w_p\}$ and w_k is the largest number in $\{w_i, w_k, w_p\}$. For example, when $n = 4$, there are fourteen 231-avoiding permutations of $\{1, 2, 3, 4\}$:

$$1234,\ 1243,\ 1324,\ 1423,\ 1432,\ 2134,\ 2143,$$

$$3124,\ 3214,\ 4123,\ 4132,\ 4213,\ 4312,\ 4321.$$

The following ten permutations do contain occurrences of the pattern 231:

$$2314,\ 2341,\ 2431,\ 4231,\ 3421,\ 3412,\ 3142,\ 1342,\ 3241,\ 2413.$$

Let S_n^{231} be the set of 231-avoiding permutations of $\{1, 2, \ldots, n\}$. We prove that $|S_n^{231}| = C_n$ for all $n \geq 0$ by verifying the Catalan recursion. First, $|S_0^{231}| = 1 = C_0$ since the empty permutation is certainly 231-avoiding. Next, suppose $n > 0$. We construct a typical object $w \in S_n^{231}$ as follows. Consider cases based on the position of the letter n in w. Say $w_k = n$. For all $i < k$ and all $p > k$, we must have $w_i < w_p$; otherwise, the subsequence $w_i, w_k = n, w_p$ would be an occurrence of the forbidden 231 pattern. Assuming that $w_i < w_p$ whenever $i < k < p$, one checks that $w = w_1 w_2 \cdots w_n$ is 231-avoiding iff $w_1 w_2 \cdots w_{k-1}$ is 231-avoiding and $w_{k+1} \cdots w_n$ is 231-avoiding. Thus, for a fixed k, we can construct w by choosing an arbitrary 231-avoiding permutation w' of the $k-1$ letters $\{1, 2, \ldots, k-1\}$ in $|S_{k-1}^{231}|$ ways, then choosing an arbitrary 231-avoiding permutation w'' of the $n-k$ letters $\{k, \ldots, n-1\}$ in $|S_{n-k}^{231}|$ ways, and finally letting w be the concatenation of w', the letter n, and w''. By the sum and product rules, we have

$$|S_n^{231}| = \sum_{k=1}^{n} |S_{k-1}^{231}||S_{n-k}^{231}|.$$

By 2.34, $|S_n^{231}| = C_n$ for all $n \geq 0$.

2.38. Example: τ-avoiding permutations. Let $\tau : \{1, 2, \ldots, k\} \to \{1, 2, \ldots, k\}$ be a fixed permutation of k letters. A permutation w of $\{1, 2, \ldots, n\}$ is called *τ-avoiding* iff there do not exist indices $1 \leq i(1) < i(2) < \cdots < i(k) \leq n$ such that

$$w_{i(\tau^{-1}(1))} < w_{i(\tau^{-1}(2))} < \cdots < w_{i(\tau^{-1}(k))}.$$

This means that no subsequence of k entries of w consists of numbers in the same relative order as the numbers $\tau_1, \tau_2, \ldots, \tau_k$. For instance, $w = 15362784$ is not 2341-avoiding, since the subsequence 5684 matches the pattern 2341 (as does the subsequence 5674). On the other hand, w is 4321-avoiding, since there is no descending subsequence of w of length 4. Let S_n^τ denote the set of τ-avoiding permutations of $\{1, 2, \ldots, n\}$.

For general τ, the enumeration of τ-avoiding permutations is an extremely difficult problem that has stimulated much research in recent years. On the other hand, if τ is a permutation of $k = 3$ letters, then the number of τ-avoiding permutations of length n is always the Catalan number C_n, for all six possible choices of τ. We have already proved this in the last example for $\tau = 231$. The arguments in that example readily adapt to prove the Catalan recursion for $\tau = 132$, $\tau = 213$, and $\tau = 312$. However, more subtle arguments are needed to prove this result for $\tau = 123$ and $\tau = 321$ (see 12.65).

2.39. Remark. Let $(A_n : n \geq 0)$ and $(B_n : n \geq 0)$ be two families of combinatorial objects such that $|A_n| = C_n = |B_n|$ for all n. Suppose that we have an explicit bijective proof that the numbers $|A_n|$ satisfy the Catalan recursion. This means that we can describe a bijection g_n between the set A_n and the disjoint union of the sets $A_{k-1} \times A_{n-k}$ for $k = 1, 2, \ldots, n$. (Such a bijection is usually implicit in an argument involving the sum and product rules.) Suppose we have similar bijections h_n for the sets B_n. We can combine these bijections to obtain recursively defined bijections $f_n : A_n \to B_n$. First, there is a unique bijection $f_0 : A_0 \to B_0$, since $|A_0| = 1 = |B_0|$. Second, assume that $f_m : A_m \to B_m$ has already been defined for all $m < n$. Define $f_n : A_n \to B_n$ as follows. Given $x \in A_n$, suppose $g_n(x) = (k, y, z)$ where $1 \leq k \leq n$, $y \in A_{k-1}$, and $z \in A_{n-k}$. Set

$$f_n(x) = h_n^{-1}((k, f_{k-1}(y), f_{n-k}(z))).$$

The inverse map is defined analogously.

For example, let us recursively define a bijection ϕ from the set of binary trees to the set of Dyck paths such that trees with n nodes map to paths of order n. Linking together the first-return recursion for Dyck paths with the left/right-subtree recursion for binary trees as discussed in the previous paragraph, we obtain the rule

$$\phi(\emptyset) = \text{the empty word (denoted } \epsilon); \qquad \phi((\bullet, T_1, T_2)) = N\phi(T_1)E\phi(T_2).$$

For example, the one-node tree $(\bullet, \emptyset, \emptyset)$ maps to the Dyck path $N\epsilon E\epsilon = NE$. It then follows that

$$\phi(\bullet, (\bullet, \emptyset, \emptyset), \emptyset) = N(NE)E\epsilon = NNEE;$$

$$\phi(\bullet, \emptyset, (\bullet, \emptyset, \emptyset)) = N\epsilon E(NE) = NENE;$$

$$\phi(\bullet, (\bullet, \emptyset, \emptyset), (\bullet, \emptyset, \emptyset)) = N(NE)E(NE) = NNEENE;$$

and so on. Figure 2.13 illustrates the recursive computation of $\phi(T)$ for the binary tree T shown in Figure 2.12.

As another example, let us recursively define a bijection ψ from the set of binary trees to the set of 231-avoiding permutations such that trees with n nodes map to permutations of n letters. Linking together the two proofs of the Catalan recursion for binary trees and 231-avoiding permutations, we obtain the rule

$$\psi(\emptyset) = \epsilon, \qquad \psi((\bullet, T_1, T_2)) = \psi(T_1)\, n\, \psi'(T_2),$$

where $\psi'(T_2)$ is the permutation obtained by increasing each entry of $\psi(T_2)$ by $k - 1 = |T_1|$. Figure 2.14 illustrates the recursive computation of $\psi(T)$ for the binary tree T shown in Figure 2.12.

2.8 Integer Partitions

2.40. Definition: Integer Partitions. Let n be a nonnegative integer. An *integer partition* of n is a sequence $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$ of positive integers such that $\mu_1 + \mu_2 + \cdots + \mu_k = n$ and $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k$. Each μ_i is called a *part* of the partition. Let $p(n)$ be the number of integer partitions of n, and let $p(n, k)$ be the number of integer partitions of n into exactly k parts. If μ is a partition of n into k parts, we write $|\mu| = n$ and $\ell(\mu) = k$ and say that μ has *area* n and *length* k. Let Par denote the set of all integer partitions.

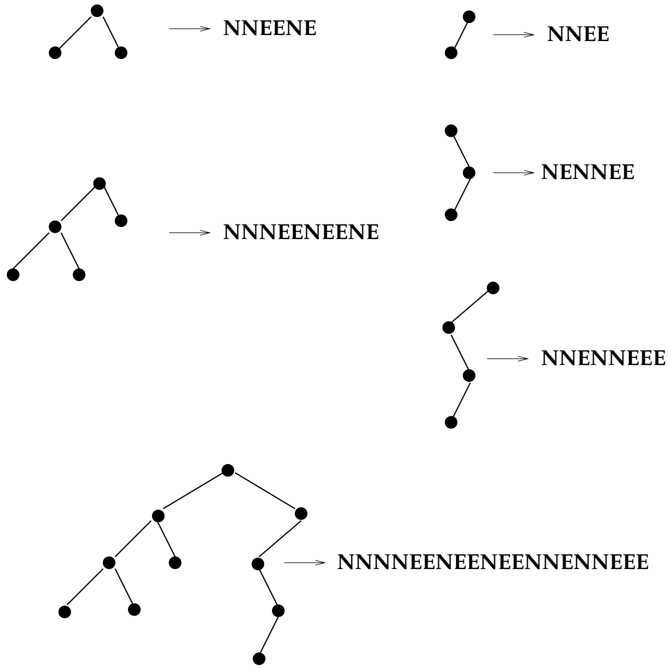


FIGURE 2.13
Mapping binary trees to Dyck paths.

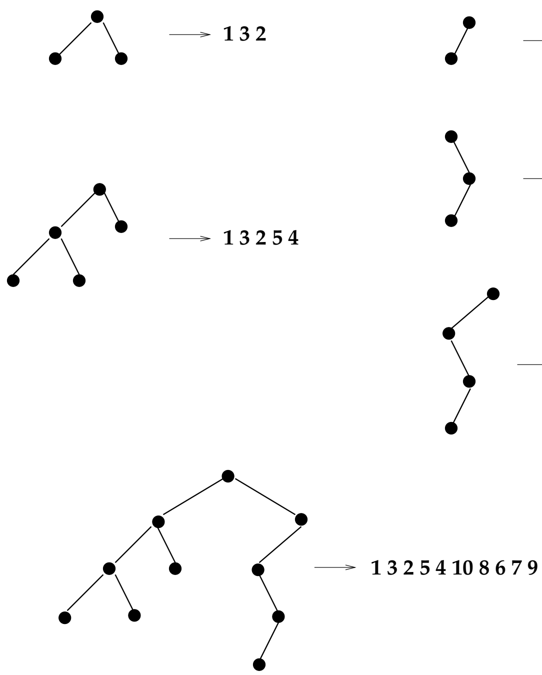


FIGURE 2.14
Mapping binary trees to 231-avoiding permutations.

2.41. Example. The integer partitions of 5 are

$$(5),\ (4,1),\ (3,2),\ (3,1,1),\ (2,2,1),\ (2,1,1,1),\ (1,1,1,1,1).$$

Thus, $p(5)=7$, $p(5,1)=1$, $p(5,2)=2$, $p(5,3)=2$, $p(5,4)=1$, and $p(5,5)=1$. As another example, the empty sequence is the unique integer partition of 0, so $p(0)=1=p(0,0)$.

An integer partition of n is a composition of n in which the parts appear in weakly decreasing order. Informally, we can think of an integer partition of n as a way of writing n as a sum of positive integers where the order of the summands does not matter.

We know from 1.41 that there are 2^{n-1} compositions of n. One might hope for a similar explicit formula for $p(n)$. Such a formula does exist (see 2.49 below), but it is extraordinarily complicated. Fortunately, the numbers $p(n,k)$ do satisfy a nice recursion.

2.42. Theorem: Recursion for Integer Partitions. Let $p(n,k)$ be the number of integer partitions of n into k parts. Then

$$p(n,k)=p(n-1,k-1)+p(n-k,k) \qquad (n,k>0).$$

The initial conditions are $p(n,k)=0$ for $k>n$ or $k<0$, $p(n,0)=0$ for $n>0$, and $p(0,0)=1$.

Proof. For all i,j, let $P(i,j)$ be the set of integer partitions of i into j parts. We have $|P(i,j)|=p(i,j)$. Fix $n,k>0$. The set $P(n,k)$ is the disjoint union of the two sets

$$\begin{aligned} Q &= \{(\mu_1,\ldots,\mu_k)\in P(n,k):\mu_k=1\},\\ R &= \{(\mu_1,\ldots,\mu_k)\in P(n,k):\mu_k>1\}. \end{aligned}$$

On one hand, the map $(\mu_1,\ldots,\mu_k)\mapsto(\mu_1,\ldots,\mu_{k-1})$ is a bijection from Q onto $P(n-1,k-1)$ with inverse $(\nu_1,\ldots,\nu_{k-1})\mapsto(\nu_1,\ldots,\nu_{k-1},1)$. So $|Q|=|P(n-1,k-1)|=p(n-1,k-1)$. On the other hand, the map $(\mu_1,\ldots,\mu_k)\mapsto(\mu_1-1,\mu_2-1,\ldots,\mu_k-1)$ is a bijection from R onto $P(n-k,k)$ with inverse $(\rho_1,\ldots,\rho_k)\mapsto(\rho_1+1,\ldots,\rho_k+1)$. So $|R|=|P(n-k,k)|=p(n-k,k)$. The sum rule now gives

$$p(n,k)=|P(n,k)|=|Q|+|R|=p(n-1,k-1)+p(n-k,k). \qquad \square$$

2.43. Theorem: Dual Recursion for Partitions. Let $p'(n,k)$ be the number of integer partitions of n with first part k. Then

$$p'(n,k)=p'(n-1,k-1)+p'(n-k,k) \qquad (n,k>0).$$

The initial conditions are $p'(n,k)=0$ for $k>n$ or $k<0$, $p'(n,0)=0$ for $n>0$, and (by convention) $p'(0,0)=1$.

Proof. For all i,j, let $P'(i,j)$ be the set of integer partitions of i with first part j. We have $|P'(i,j)|=p'(i,j)$. Fix $n,k>0$. The set $P'(n,k)$ is the disjoint union of the two sets

$$\begin{aligned} Q &= \{(\mu_1=k,\mu_2,\ldots,\mu_s)\in P'(n,k):\mu_1>\mu_2\}\\ R &= \{(\mu_1=k,\mu_2,\ldots,\mu_s)\in P'(n,k):\mu_1=\mu_2\}. \end{aligned}$$

(If μ has only one part, we take $\mu_2=0$ by convention.) On one hand, the map $(k,\mu_2,\ldots,\mu_s)\mapsto(k-1,\mu_2,\ldots,\mu_s)$ is a bijection from Q onto $P'(n-1,k-1)$ with inverse $(k-1,\nu_2,\ldots,\nu_s)\mapsto(k,\nu_2,\ldots,\nu_s)$. So $|Q|=|P'(n-1,k-1)|=p'(n-1,k-1)$. On the other hand, the map $(k,\mu_2,\ldots,\mu_s)\mapsto(\mu_2,\mu_3,\ldots,\mu_s)$ is a bijection from R onto $P'(n-k,k)$ with inverse $(\rho_1,\ldots,\rho_s)\mapsto(k,\rho_1,\ldots,\rho_s)$. So $|R|=|P'(n-k,k)|=p'(n-k,k)$. The sum rule now gives

$$p'(n,k)=|P'(n,k)|=|Q|+|R|=p'(n-1,k-1)+p'(n-k,k). \qquad \square$$

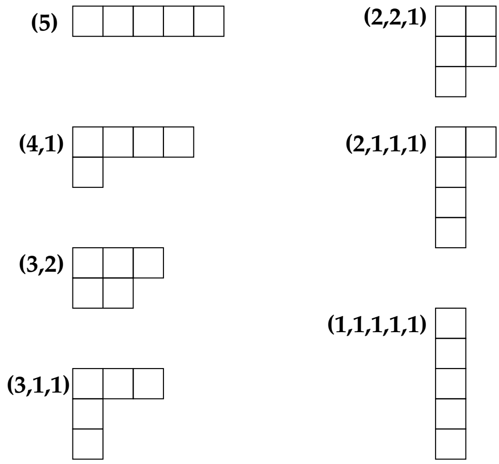


FIGURE 2.15
Partition diagrams.

2.44. Theorem: First Part vs. Number of Parts. The number of integer partitions of n into k parts equals the number of integer partitions of n with first part k.

Proof. We prove $p(n,k) = p'(n,k)$ for all n and all k by induction on n. The case $n = 0$ is true since the initial conditions show that $p(0,k) = p'(0,k)$ for all k. Now assume that $n > 0$ and that $p(m,k) = p'(m,k)$ for all $m < n$ and all k. If $k = 0$, then $p(n,0) = p'(n,0)$ follows from the initial conditions. If $k > 0$, compute

$$p(n,k) = p(n-1,k-1) + p(n-k,k) = p'(n-1,k-1) + p'(n-k,k) = p'(n,k). \qquad \square$$

We now describe a convenient way of visualizing integer partitions.

2.45. Definition: Diagram of a Partition. Let $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$ be an integer partition of n. The *diagram* of μ is the set

$$\mathrm{dg}(\mu) = \{(i,j) \in \mathbb{N} \times \mathbb{N} : 1 \leq i \leq k,\ 1 \leq j \leq \mu_i\}.$$

We can make a picture of $\mathrm{dg}(\mu)$ by drawing an array of n boxes, with μ_i left-justified boxes in row i. For example, Figure 2.15 illustrates the diagrams for the seven integer partitions of 5. Note that $|\mu| = \mu_1 + \cdots + \mu_k = |\,\mathrm{dg}(\mu)|$ is the total number of boxes in the diagram of μ.

2.46. Definition: Conjugate Partitions. Suppose μ is an integer partition of n. The *conjugate partition* of μ is the unique integer partition μ' of n satisfying

$$\mathrm{dg}(\mu') = \{(j,i) : (i,j) \in \mathrm{dg}(\mu)\}.$$

In other words, we obtain the diagram for μ' by interchanging the rows and columns in the diagram for μ. For example, Figure 2.16 shows that the conjugate of $\mu = (7,4,3,1,1)$ is $\mu' = (5,3,3,2,1,1,1)$.

We can now give pictorial proofs of some of the preceding results concerning $p(n,k)$ and $p'(n,k)$. Note that the length of a partition μ is the number of rows in $\mathrm{dg}(\mu)$, while the first part of μ is the number of columns of $\mathrm{dg}(\mu)$. Hence, conjugation gives a bijection

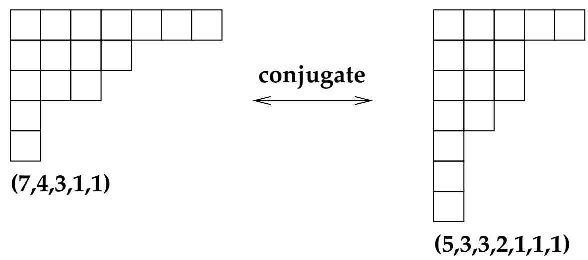


FIGURE 2.16
Conjugate of a partition.

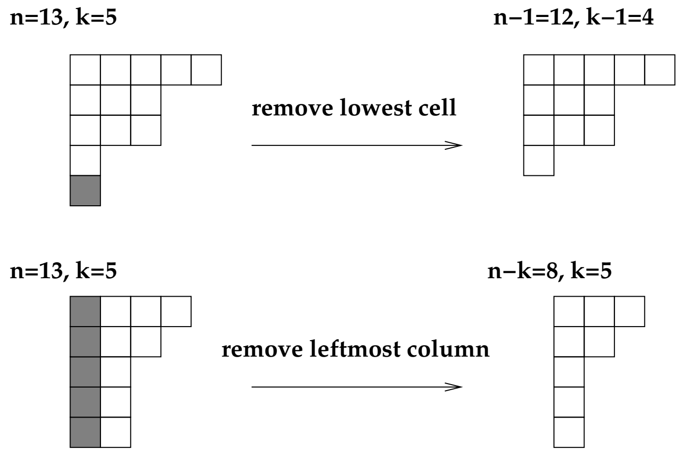


FIGURE 2.17
Pictorial proof of the recursion for $p(n,k)$.

between the partitions counted by $p(n,k)$ and the partitions counted by $p'(n,k)$, so that $p(n,k) = p'(n,k)$. Similarly, consider the proof of the recursion

$$p(n,k) = p(n-1,k-1) + p(n-k,k).$$

Suppose μ is a partition of n into k parts. If $\mathrm{dg}(\mu)$ has one box in the lowest row, we remove this box to get a typical partition counted by $p(n-1,k-1)$. If $\mathrm{dg}(\mu)$ has more than one box in the lowest row, we remove the entire first column of the diagram to get a typical partition counted by $p(n-k,k)$. See Figure 2.17.

Our next result counts integer partitions whose diagrams fit in a box with b rows and a columns.

2.47. Theorem: Enumeration of Partitions in a Box. The number of integer partitions μ such that $\mathrm{dg}(\mu) \subseteq \{1,2,\ldots,b\} \times \{1,2,\ldots,a\}$ is

$$\binom{a+b}{a,b} = \frac{(a+b)!}{a!b!}.$$

Proof. We define a bijection between the set of integer partitions in the theorem statement and the set of all lattice paths from the origin to (a,b). We draw our partition diagrams

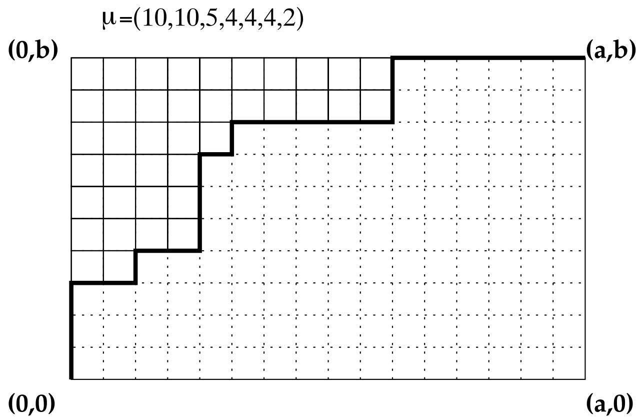


FIGURE 2.18
Counting partitions that fit in an $a \times b$ box.

in the box with corners $(0,0)$, $(a,0)$, $(0,b)$, and (a,b), as shown in Figure 2.18. Given a partition μ whose diagram fits in this box, the southeast boundary of $\mathrm{dg}(\mu)$ is a lattice path from the origin to (a,b). We call this lattice path the *frontier* of μ (which depends on a and b as well as μ). For example, if $a = 16$, $b = 10$, and $\mu = (10,10,5,4,4,4,2)$, we see from Figure 2.18 that the frontier of μ is

NNNEENEENNNENEEEEENNEEEEEE.

Conversely, given any lattice path ending at (a,b), the set of lattice squares northwest of this path in the box uniquely determines the diagram of an integer partition. We already know that the number of lattice paths from the origin to (a,b) is $\binom{a+b}{a,b}$, so the theorem follows. □

2.48. Remark: Euler's Partition Recursion. Our recursion for $p(n,k)$ gives a quick method for computing the quantities $p(n,k)$ and $p(n) = \sum_{k=1}^{n} p(n,k)$. One may ask whether the numbers $p(n)$ satisfy any recursion. In fact, Euler's study of the infinite product $\prod_{i=1}^{\infty}(1-x^i)$ leads to the following recursion for $p(n)$:

$$\begin{aligned} p(n) &= \sum_{m=1}^{\infty} (-1)^{m-1}[p(n-m(3m-1)/2) + p(n-m(3m+1)/2)] \\ &= p(n-1) + p(n-2) - p(n-5) - p(n-7) + p(n-12) + p(n-15) \\ &\quad -p(n-22) - p(n-26) + p(n-35) + p(n-40) - p(n-51) - p(n-57) + \cdots . \end{aligned}$$

The initial conditions are $p(0) = 1$ and $p(j) = 0$ for all $j < 0$. It follows that, for each fixed n, the recursive expression for $p(n)$ is really a finite sum, since the terms become zero once the argument to p becomes negative. For example, Figure 2.19 illustrates the calculation of $p(n)$ from Euler's recursion for $1 \le n \le 12$. We shall prove Euler's recursion later (see 8.27 and 8.87).

2.49. Remark: Hardy-Rademacher-Ramanujan Formula for $p(n)$. There exists an

$$\begin{aligned}
p(1) &= p(0) = 1\\
p(2) &= p(1)+p(0) = 1+1 = 2\\
p(3) &= p(2)+p(1) = 2+1 = 3\\
p(4) &= p(3)+p(2) = 3+2 = 5\\
p(5) &= p(4)+p(3)-p(0) = 5+3-1 = 7\\
p(6) &= p(5)+p(4)-p(1) = 7+5-1 = 11\\
p(7) &= p(6)+p(5)-p(2)-p(0) = 11+7-2-1 = 15\\
p(8) &= p(7)+p(6)-p(3)-p(1) = 15+11-3-1 = 22\\
p(9) &= p(8)+p(7)-p(4)-p(2) = 22+15-5-2 = 30\\
p(10) &= p(9)+p(8)-p(5)-p(3) = 30+22-7-3 = 42\\
p(11) &= p(10)+p(9)-p(6)-p(4) = 42+30-11-5 = 56\\
p(12) &= p(11)+p(10)-p(7)-p(5)+p(0) = 56+42-15-7+1 = 77.
\end{aligned}$$

FIGURE 2.19
Calculating $p(n)$ using Euler's recursion.

explicit, non-recursive formula for the number of partitions of n. The formula is

$$p(n) = \frac{1}{\pi\sqrt{2}}\sum_{k=1}^{\infty} A_k(n)\sqrt{k}\left[\frac{d}{dx}\frac{\sinh\left((\pi/k)\sqrt{\frac{2}{3}(x-\frac{1}{24})}\right)}{\sqrt{x-\frac{1}{24}}}\right]\Bigg|_{x=n},$$

where

$$A_k(n) = \sum_{\substack{1\le h\le k:\\ \gcd(h,k)=1}} \omega_{h,k}e^{-2\pi i n h/k},$$

and $\omega_{h,k}$ is a certain complex $24k$th root of unity. By estimating $p(n)$ by the first term of this series, one can deduce the following asymptotic formula for $p(n)$:

$$p(n) \sim \frac{1}{4n\sqrt{3}}\exp\left[\pi\sqrt{2n/3}\right].$$

We will not prove these results. For more details, consult Andrews [5, Chapter 5].

2.9 Set Partitions

2.50. Definition: Set Partitions. Let X be a set. A *set partition* of X is a collection P of nonempty, pairwise disjoint subsets of X whose union is X. Each element of P is called a *block* of the partition. The cardinality of P (which may be infinite) is called the *number of blocks* of the partition.

For example, if $X = \{1,2,3,4,5,6,7,8\}$, then

$$P = \{\{3,5,8\},\{1,7\},\{2\},\{4,6\}\}$$

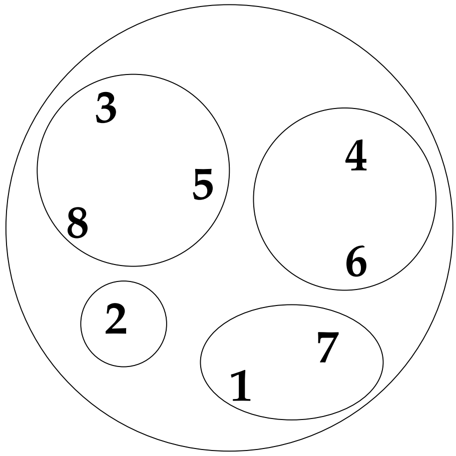


FIGURE 2.20
A picture of the set partition $\{\{3,5,8\},\{1,7\},\{2\},\{4,6\}\}$.

is a set partition of X with four blocks. Note that the ordering of the blocks in this list, and the ordering of the elements within each block, is irrelevant when deciding the equality of two set partitions. For instance,

$$\{\{6,4\},\{1,7\},\{2\},\{5,8,3\}\}$$

is the same set partition as the partition P mentioned above. It is convenient to visualize a set partition P by drawing the elements of X in a circle, and then drawing smaller circles enclosing the elements of each block of P. See Figure 2.20.

2.51. Definition: Stirling Numbers and Bell Numbers. Let $S(n,k)$ be the number of set partitions of $\{1,2,\dots,n\}$ into exactly k blocks. $S(n,k)$ is called a *Stirling number of the second kind*. Let $B(n)$ be the total number of set partitions of $\{1,2,\dots,n\}$. $B(n)$ is called a *Bell number*. One can check that $S(n,k)$ is the number of partitions of *any* given n-element set into k blocks; similarly for $B(n)$.

Stirling numbers and Bell numbers are not given by closed formulas involving factorials, binomial coefficients, etc. (although there are summation formulas and generating functions for these quantities). However, the Stirling numbers satisfy a recursion that can be used to compute $S(n,k)$ and $B(n)$ quite rapidly.

2.52. Theorem: Recursion for Stirling Numbers of the Second Kind. For all $n>0$ and $k>0$,

$$S(n,k)=S(n-1,k-1)+kS(n-1,k).$$

The initial conditions are $S(0,0)=1$, $S(n,0)=0$ for $n>0$, and $S(0,k)=0$ for $k>0$. Furthermore, $B(0)=1$ and $B(n)=\sum_{k=1}^{n} S(n,k)$ for $n>0$.

Proof. Fix $n,k>0$. Let A be the set of set partitions of $\{1,2,\dots,n\}$ into exactly k blocks. Let $A'=\{P\in A:\{n\}\in P\}$ and $A''=\{P\in A:\{n\}\not\in P\}$. A is the disjoint union of A' and A''. A' consists of those set partitions such that n is in a block by itself, while A'' consists of those set partitions such that n is in a block with some other elements. To build a typical partition $P\in A'$, we first choose an arbitrary set partition P_0 of $\{1,2,\dots,n-1\}$ into $k-1$ blocks in any of $S(n-1,k-1)$ ways. Then we add $\{n\}$ to P_0 to get P. To build

	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$	$B(n)$
$n=0:$	1	0	0	0	0	0	0	0	0	1
$n=1:$	0	1	0	0	0	0	0	0	0	1
$n=2:$	0	1	1	0	0	0	0	0	0	2
$n=3:$	0	1	3	1	0	0	0	0	0	5
$n=4:$	0	1	7	6	1	0	0	0	0	15
$n=5:$	0	1	15	25	10	1	0	0	0	52
$n=6:$	0	1	31	90	65	15	1	0	0	203
$n=7:$	0	1	63	301	350	140	21	1	0	877
$n=8:$	0	1	127	966	1701	1050	266	28	1	4140

FIGURE 2.21
Calculating $S(n,k)$ and $B(n)$ recursively.

a typical partition $P \in A''$, we first choose an arbitrary set partition P_1 of $\{1,2,\ldots,n-1\}$ into k blocks in any of $S(n-1,k)$ ways. Then we choose one of these k blocks and add n as a new member of that block. By the sum and product rules,

$$S(n,k) = |A| = |A'| + |A''| = S(n-1,k-1) + kS(n-1,k).$$

The initial conditions are immediate from the definitions (note that $P = \emptyset$ is the unique set partition of $X = \emptyset$). The formula for $B(n)$ follows from the sum rule. □

Figure 2.21 computes $S(n,k)$ and $B(n)$ for $n \le 8$ using the recursion from the last theorem. Note that each entry $S(n,k)$ in row n and column k is computed by taking the number immediately northwest and adding k times the number immediately above the given entry. The numbers $B(n)$ are found by adding the numbers in each row.

The Bell numbers also satisfy a nice recursion.

2.53. Theorem: Recursion for Bell Numbers. For all $n > 0$,

$$B(n) = \sum_{k=0}^{n-1} \binom{n-1}{k} B(n-1-k).$$

The initial condition is $B(0) = 1$.

Proof. For $n > 0$, we construct a typical set partition P counted by $B(n)$ as follows. Let k be the number of elements in the block of P containing n, not including n itself; thus, $0 \le k \le n-1$. To build P, first choose k elements from $\{1,2,\ldots,n-1\}$ that belong to the same block as n in any of $\binom{n-1}{k}$ ways. Then, choose an arbitrary set partition of the $n-1-k$ elements that do not belong to the same block as n; this choice can be made in any of $B(n-1-k)$ ways. The recursion now follows from the sum and product rules. □

For example, assuming that $B(m)$ is already known for $m < 8$ (cf. Figure 2.21), we calculate

$$\begin{aligned} B(8) &= \binom{7}{0}B(7) + \binom{7}{1}B(6) + \binom{7}{2}B(5) + \cdots + \binom{7}{7}B(0) \\ &= 1\cdot 877 + 7\cdot 203 + 21\cdot 52 + 35\cdot 15 + 35\cdot 5 + 21\cdot 2 + 7\cdot 1 + 1\cdot 1 \\ &= 4140. \end{aligned}$$

We close this section by reviewing the connection between set partitions and equivalence relations.

2.54. Definition: Types of Relations. Let X be any set. A *relation* on X is any subset of $X \times X$. If R is a relation on X and $x, y \in X$, we often write xRy as an abbreviation for $(x, y) \in R$. We read this symbol as "x is related to y under R." A relation R on X is *reflexive on* X iff xRx for all $x \in X$. R is *irreflexive on* X iff xRx is false for all $x \in X$. R is *symmetric* iff for all $x, y \in X$, xRy implies yRx. R is *antisymmetric* iff for all $x, y \in X$, xRy and yRx imply $x = y$. R is *transitive* iff for all $x, y, z \in X$, xRy and yRz imply xRz. R is an *equivalence relation on* X iff R is symmetric, transitive, and reflexive on X. If R is an equivalence relation and $x_0 \in X$, the *equivalence class of* x_0 *relative to* R is the set $[x_0]_R = \{y \in X : yRx_0\}$.

2.55. Theorem: Set Partitions vs. Equivalence Relations. Suppose X is a fixed set. Let $\mathcal{A}$ be the set of all set partitions of X, and let $\mathcal{B}$ be the set of all equivalence relations on X. There are canonical bijections $\phi : \mathcal{A} \to \mathcal{B}$ and $\phi' : \mathcal{B} \to \mathcal{A}$. If $P \in \mathcal{A}$, then the number of blocks of P equals the number of equivalence classes of $\phi(P)$. Hence, $S(n, k)$ is the number of equivalence relations on an n-element set having k equivalence classes, and $B(n)$ is the number of equivalence relations on an n-element set.

Proof. We sketch the proof, leaving certain details as exercises. Given a set partition $P \in \mathcal{A}$, define

$$\phi(P) = \{(x, y) \in X : \exists S \in P, x \in S \text{ and } y \in S\}.$$

In other words, $x\phi(P)y$ iff x and y belong to the same block of P. The reader should check that $\phi(P)$ is indeed an equivalence relation on X, i.e., that $\phi(P) \in \mathcal{B}$. Thus, ϕ is a well-defined function from $\mathcal{A}$ into $\mathcal{B}$.

Given an equivalence relation $R \in \mathcal{B}$, define

$$\phi'(R) = \{[x]_R : x \in X\}.$$

In other words, the blocks of $\phi'(R)$ are precisely the equivalence classes of R. The reader should check that $\phi'(R)$ is indeed a set partition of X, i.e., that $\phi'(R) \in \mathcal{A}$. Thus, ϕ' is a well-defined function from $\mathcal{B}$ into $\mathcal{A}$.

To complete the proof, the reader should check that ϕ and ϕ' are two-sided inverses of one another. In other words, prove that for all $P \in \mathcal{A}$, $\phi'(\phi(P)) = P$; and for all $R \in \mathcal{B}$, prove that $\phi(\phi'(R)) = R$. It follows that ϕ and ϕ' are bijections. $\square$

2.10 Surjections

Recall that a function $f : X \to Y$ is a *surjection* iff for every $y \in Y$, there exists $x \in X$ with $f(x) = y$.

2.56. Definition: $\mathrm{Surj}(n, k)$. Let $\mathrm{Surj}(n, k)$ denote the number of surjections from an n-element set onto a k-element set.

2.57. Theorem: Recursion for Surjections. For $n \geq k > 0$,

$$\mathrm{Surj}(n, k) = k\,\mathrm{Surj}(n-1, k-1) + k\,\mathrm{Surj}(n-1, k).$$

The initial conditions are $\mathrm{Surj}(n, k) = 0$ for $n < k$, $\mathrm{Surj}(0, 0) = 1$, and $\mathrm{Surj}(n, 0) = 0$ for $n > 0$.

Proof. Fix $n \geq k > 0$. Let us build a typical surjection $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, k\}$ by considering two cases. Case 1: $f(i) \neq f(n)$ for all $i < n$. In this case, we first choose $f(n)$ in k ways, and then we choose a surjection from $\{1, 2, \ldots, n-1\}$ onto $\{1, 2, \ldots, k\} \sim \{f(n)\}$ in $\mathrm{Surj}(n-1, k-1)$ ways. The total number of possibilities is $k\,\mathrm{Surj}(n-1, k-1)$.

Case 2: $f(n) = f(i)$ for some $i < n$. In this case, note that the restriction of f to $\{1, 2, \ldots, n-1\}$ is still surjective. Thus we can build f by first choosing a surjection from $\{1, 2, \ldots, n-1\}$ onto $\{1, 2, \ldots, k\}$ in $\mathrm{Surj}(n-1, k)$ ways, and then choosing $f(n) \in \{1, 2, \ldots, k\}$ in k ways. The total number of possibilities is $k\,\mathrm{Surj}(n-1, k)$. The recursion now follows from the sum rule.

The initial conditions are immediate, once we note that the function with graph $\emptyset$ is the unique surjection from $\emptyset$ onto $\emptyset$. □

Surjections are closely related to Stirling numbers of the second kind. Indeed, we have the following relation between $\mathrm{Surj}(n, k)$ and $S(n, k)$.

2.58. Theorem. For all $n, k \geq 0$, $\mathrm{Surj}(n, k) = k!S(n, k)$.

Proof. We give two proofs. *First Proof:* We argue by induction on n. The result holds when $n = 0$ and k is arbitrary, since $\mathrm{Surj}(0, k) = \chi(k = 0) = 0!S(0, k)$. Assume that $n > 0$ and that $\mathrm{Surj}(m, k) = k!S(m, k)$ for all k and all $m < n$. Using the recursions for $\mathrm{Surj}(n, k)$ and $S(n, k)$, we compute

$$\begin{aligned}
\mathrm{Surj}(n, k) &= k\,\mathrm{Surj}(n-1, k-1) + k\,\mathrm{Surj}(n-1, k) \\
&= k(k-1)!S(n-1, k-1) + k(k!)S(n-1, k) \\
&= k![S(n-1, k-1) + kS(n-1, k)] \\
&= k!S(n, k).
\end{aligned}$$

Second Proof: We prove the formula by a direct counting argument. To construct a surjection $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, k\}$, first choose a set partition P of $\{1, 2, \ldots, n\}$ into k blocks in any of $S(n, k)$ ways. Choose one of these blocks (in k ways), and let f map everything in this block to 1. Then choose a different block (in $k-1$ ways), and let f map everything in this block to 2. Continue similarly; at the last stage, there is 1 block left, and we let f map everything in this block to k. By the product rule,

$$\mathrm{Surj}(n, k) = S(n, k) \cdot k \cdot (k-1) \cdot \ldots \cdot 1 = k!S(n, k).$$ □

2.59. Example. To illustrate the second proof, suppose $n = 8$ and $k = 4$. In the first step, let us choose the partition $P = \{\{1, 4, 7\}, \{2\}, \{3, 8\}, \{5, 6\}\}$. In the next four steps, we choose a permutation of the four blocks of P, say

$$\{2\}, \{5, 6\}, \{3, 8\}, \{1, 4, 7\}.$$

Now we define the associated surjection f by setting

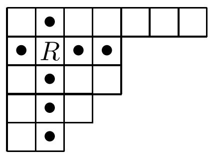
$$f(2) = 1,\ f(5) = f(6) = 2,\ f(3) = f(8) = 3,\ f(1) = f(4) = f(7) = 4.$$

2.11 Stirling Numbers and Rook Theory

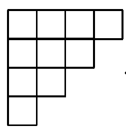
Recall that the Stirling numbers of the second kind (denoted $S(n, k)$) count the number of set partitions of an n-element set into k blocks. This section gives another combinatorial interpretation of these Stirling numbers. We show that $S(n, k)$ counts certain placements of rooks on a triangular chessboard. A slight variation of this setup leads us to introduce the (signless) Stirling numbers of the first kind. The relationship between the two kinds of Stirling numbers will be illuminated in the following section.

2.60. Definition: Ferrers Boards and Rooks. A *Ferrers board* is the diagram of an integer partition, viewed as a collection of unit squares as in §2.8. A *rook* is a chess piece that can occupy any of the squares in a Ferrers board. In chess, a rook can move any number of squares horizontally or vertically from its current position in a single move. A rook located in row i and column j of a Ferrers board *attacks* all squares in row i and all squares in column j.

For example, in the Ferrers board shown below, the rook R attacks all squares on the board marked with a dot (and its own square).

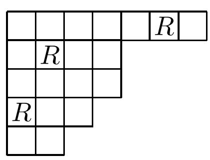


For each $n > 0$, let Δ_n denote the diagram of the partition $(n-1, n-2, \ldots, 3, 2, 1)$. Δ_n is a triangular Ferrers board with $n(n-1)/2$ total squares. For example,

$\Delta_5 =$  .

2.61. Definition: Non-attacking Rook Placements. A *placement* of k rooks on a given Ferrers board is a subset of k squares in the Ferrers board. These k squares represent the locations of k identical rooks on the board. A placement of rooks in a Ferrers board is called *non-attacking* iff no rook occupies a square attacked by another rook. Equivalently, all rooks in the placement occupy distinct rows and distinct columns of the board.

2.62. Example. The following diagram illustrates a non-attacking placement of 3 rooks on the Ferrers board corresponding to the partition $(7, 4, 4, 3, 2)$.



2.63. Theorem: Rook-Theoretic Interpretation of Stirling Numbers of the Second Kind. For $n > 0$ and $0 \leq k \leq n$, let $S'(n, k)$ denote the number of non-attacking placements of $n - k$ rooks on the Ferrers board Δ_n. If $n > 1$ and $0 < k < n$, then

$$S'(n, k) = S'(n-1, k-1) + kS'(n-1, k).$$

The initial conditions are $S'(n, 0) = 0$ and $S'(n, n) = 1$ for all $n > 0$. Therefore, $S'(n, k) = S(n, k)$, a Stirling number of the second kind.

Proof. Fix $n > 1$ with $0 < k < n$. Let A, B, C denote the set of placements counted by $S'(n, k)$, $S'(n-1, k-1)$, and $S'(n-1, k)$, respectively. Let A_0 consist of all rook placements in A with no rook in the top row, and let A_1 consist of all rook placements in A with one rook in the top row. A is the disjoint union of A_0 and A_1. Deleting the top row of the Ferrers board Δ_n produces the smaller Ferrers board Δ_{n-1}. It follows that deleting the (empty) top row of a rook placement in A_0 gives a bijection between A_0 and B (note that a placement in B involves $(n-1)-(k-1)=n-k$ rooks). On the other hand, we can build a typical rook placement in A_1 as follows. First, choose a placement of $n-k-1$ non-attacking rooks from the set C, and use this rook placement to fill the bottom $n-1$ rows of Δ_n. These rooks occupy $n-k-1$ distinct columns. This leaves $(n-1)-(n-k-1)=k$ columns in the top row in which we are allowed to place the final rook. By the product rule, $|A_1|=|C|k$. We conclude that

$$S'(n,k)=|A|=|A_0|+|A_1|=|B|+k|C|=S'(n-1,k-1)+kS'(n-1,k).$$

We cannot place n non-attacking rooks on the Ferrers board Δ_n (which has only $n-1$ columns), and hence $S'(n,0)=0$. On the other hand, for any $n>0$ there is a unique placement of zero rooks on Δ_n. This placement is non-attacking (vacuously), and hence $S'(n,n)=1$. Counting set partitions, we see that $S(n,0)=0$ and $S(n,n)=1$ for all $n>0$. Since $S'(n,k)$ and $S(n,k)$ satisfy the same recursion and initial conditions, a routine induction argument (cf. 2.34) shows that $S'(n,k)=S(n,k)$ for all n and k. □

2.64. Remark. We have given combinatorial proofs that the numbers $S'(n,k)$ and $S(n,k)$ satisfy the same recursion. We can link together these proofs to get a recursively defined bijection between rook placements and set partitions, using the ideas in 2.39. We can also directly define a bijection between rook placements and set partitions. We illustrate such a bijection via an example. Figure 2.22 displays a rook placement counted by $S'(8,3)$. We write the numbers 1 through n below the last square in each column of the diagram, as shown in the figure. We view these numbers as labeling both the rows and columns of the diagram; note that the column labels increase from left to right, while row labels decrease from top to bottom. The bijection between non-attacking rook placements π and set partitions P acts as follows. For all $j<i\leq n$, there is a rook in row i and column j of π iff i and j are consecutive elements in the same block of P (when the elements of the block are written in increasing order). For example, the rook placement π in Figure 2.22 maps to the set partition

$$P=\{\{1,3,4,5,7\},\{2,6\},\{8\}\}.$$

The set partition $\{\{2\},\{1,5,8\},\{4,6,7\},\{3\}\}$ maps to the rook placement shown in Figure 2.23.

One may check that a non-attacking placement of $n-k$ rooks on Δ_n corresponds to a set partition of n with exactly k blocks; furthermore, the rook placement associated to a given set partition is automatically non-attacking.

2.65. Definition: Wrooks and Stirling Numbers of the First Kind. A *wrook* (weak rook) is a new chess piece that attacks only the squares in its row. For all $n>0$ and $0\leq k\leq n$, let $s'(n,k)$ denote the number of placements of $n-k$ non-attacking wrooks on the Ferrers board Δ_n. The numbers $s'(n,k)$ are called *signless Stirling numbers of the first kind.* The numbers $s(n,k)=(-1)^{n-k}s'(n,k)$ are called *(signed) Stirling numbers of the first kind.* Another combinatorial definition of the Stirling numbers of the first kind will be given in §3.6. By convention, we set $s(0,0)=1=s'(0,0)$.

2.66. Theorem: Recursion for Signless Stirling Numbers of the First Kind. If $n>1$ and $0<k<n$, then

$$s'(n,k)=s'(n-1,k-1)+(n-1)s'(n-1,k).$$

The initial conditions are $s'(n,0)=\chi(n=0)$ and $s'(n,n)=1$.

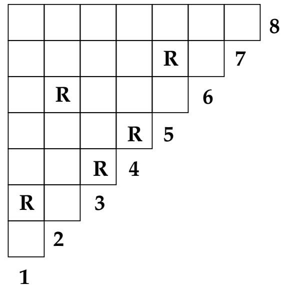


FIGURE 2.22
A rook placement counted by $S'(n,k)$, where $n=8$ and $k=3$.

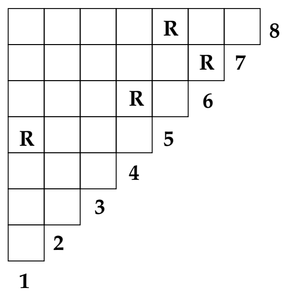


FIGURE 2.23
The rook placement associated to $\{\{2\},\{1,5,8\},\{4,6,7\},\{3\}\}$.

	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
$n=0:$	1	0	0	0	0	0	0	0
$n=1:$	0	1	0	0	0	0	0	0
$n=2:$	0	-1	1	0	0	0	0	0
$n=3:$	0	2	-3	1	0	0	0	0
$n=4:$	0	-6	11	-6	1	0	0	0
$n=5:$	0	24	-50	35	-10	1	0	0
$n=6:$	0	-120	274	-225	85	-15	1	0
$n=7:$	0	720	-1764	1624	-735	175	-21	1

FIGURE 2.24
Signed Stirling numbers of the first kind.

Proof. Fix $n > 1$ with $0 < k < n$. Let A, B, C denote the set of placements counted by $s'(n,k)$, $s'(n-1,k-1)$, and $s'(n-1,k)$, respectively. Write A as the disjoint union of A_0 and A_1, where A_i consists of all elements of A with i wrooks in the top row. As above, deletion of the empty top row gives a bijection from A_0 to B. On the other hand, we can build a typical wrook placement in A_1 as follows. First, choose the position of the wrook in the top row of Δ_n in $n-1$ ways. Second, choose any placement of $n-k-1$ non-attacking wrooks from the set C, and use this wrook placement to fill the bottom $n-1$ rows of Δ_n. These wrooks do not attack the wrook in the first row. By the sum and product rules,

$$s'(n,k) = |A| = |A_0| + |A_1| = |B| + (n-1)|C| = s'(n-1,k-1) + (n-1)s'(n-1,k). \quad \square$$

We can use the recursion and initial conditions to compute the (signed or unsigned) Stirling numbers of the first kind. See Figure 2.24, and compare to the computation of Stirling numbers of the second kind in Figure 2.21. There is a surprising relation between the two arrays of numbers in these figures. Specifically, for any fixed $n > 0$, consider the lower-triangular matrices $A = (s(i,j))_{1\leq i,j\leq n}$ and $B = (S(i,j))_{1\leq i,j\leq n}$. It turns out that A and B are inverse matrices! The reader may check this for small n using Figure 2.21 and Figure 2.24. We will prove this fact for all n in §2.13.

2.12 Linear Algebra Review

In the next few sections, and at other places later in the book, we will need to use some concepts from linear algebra such as vector spaces, bases, and linear independence. This section quickly reviews the definitions we will need; for a thorough treatment of linear algebra, the reader may consult texts such as Hoffman and Kunze [69].

2.67. Definition: Vector Spaces. Given a field F, a *vector space over F* consists of a set V, an addition operation $+ : V \times V \to V$, and a scalar multiplication operation $\cdot : F \times V \to V$, that satisfy the following axioms.

$$\begin{array}{ll}
\forall x, y \in V,\ x + y \in V & \text{(closure under addition)} \\
\forall x, y, z \in V,\ x + (y + z) = (x + y) + z & \text{(associativity of addition)} \\
\forall x, y \in V,\ x + y = y + x & \text{(commutativity of addition)} \\
\exists 0_V \in V, \forall x \in V, x + 0_V = x = 0_V + x & \text{(existence of additive identity)} \\
\forall x \in V, \exists -x \in V, x + (-x) = 0_V = (-x) + x & \text{(existence of additive inverses)} \\
\forall c \in F, \forall v \in V,\ c \cdot v \in V & \text{(closure under scalar multiplication)} \\
\forall c \in F, \forall v, w \in V,\ c \cdot (v + w) = (c \cdot v) + (c \cdot w) & \text{(left distributive law)} \\
\forall c, d \in F, \forall v \in V,\ (c + d) \cdot v = (c \cdot v) + (d \cdot v) & \text{(right distributive law)} \\
\forall c, d \in F, \forall v \in V,\ (cd) \cdot v = c \cdot (d \cdot v) & \text{(associativity of scalar multiplication)} \\
\forall v \in V,\ 1 \cdot v = v & \text{(identity property)}
\end{array}$$

When discussing vector spaces, elements of V are often called *vectors*, while elements of F are called *scalars*.

For example, the set $F^n = \{(x_1, \ldots, x_n) : x_i \in F\}$ is a vector space over F with operations

$$(x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n);$$

$$c(x_1, \ldots, x_n) = (cx_1, \ldots, cx_n) \quad (c, x_i, y_i \in F).$$

Similarly, the set of polynomials $a_0 + a_1x + \cdots + a_kx^k$, where all coefficients a_i come from F, is a vector space over F under the operations

$$\sum_{i\geq 0} a_ix^i + \sum_{i\geq 0} b_ix^i = \sum_{i\geq 0}(a_i + b_i)x^i; \quad c\sum_{i\geq 0} a_ix^i = \sum_{i\geq 0}(ca_i)x^i.$$

We consider two polynomials $\sum_{i\geq 0} a_ix^i$ and $\sum_{i\geq 0} b_ix^i$ to be *equal* iff $a_i = b_i$ for all i; see §7.3 for a more formal discussion of how to define polynomials.

2.68. Definition: Spanning Sets and Linear Combinations. A subset S of a vector space V over F *spans* V iff for every $v \in V$, there exists a *finite* list of vectors $v_1, \ldots, v_k \in S$ and scalars $c_1, \ldots, c_k \in F$ with $v = c_1v_1 + \cdots + c_kv_k$. Any expression of the form $c_1v_1 + \cdots + c_kv_k$ is called a *linear combination* of $v_1, \ldots, v_k$. A linear combination must be a *finite* sum of vectors.

2.69. Definition: Linear Independence. A list $(v_1, \ldots, v_k)$ of vectors in a vector space V over F is called *linearly dependent* iff there exist scalars $c_1, \ldots, c_k \in F$ such that $c_1v_1 + \cdots + c_kv_k = 0_V$ and at least one c_i is not zero. Otherwise, the list $(v_1, \ldots, v_k)$ is called *linearly independent.* A set $S \subseteq V$ (possibly infinite) is *linearly dependent* iff there is a finite list of *distinct* elements of S that is linearly dependent; otherwise, S is *linearly independent.*

2.70. Definition: Basis of a Vector Space. A *basis* of a vector space V is a set $S \subseteq V$ that is linearly independent and spans V.

For example, for any field F, define $e_i \in F^n$ to be the vector with 1_F in position i and 0_F in all other positions. Then $\{e_1, \ldots, e_n\}$ is a basis for F^n. Similarly, one may check that the infinite set $S = \{1, x, x^2, x^3, \ldots, x^n, \ldots\}$ is a basis for the vector space V of polynomials in x with coefficients in F. S spans V since every polynomial must be a *finite* linear combination of powers of x. The linear independence of S follows from the definition of polynomial equality: the only linear combination $c_01 + c_1x + c_2x^2 + \cdots$ that can equal the zero polynomial is the one where $c_0 = c_1 = c_2 = \cdots = 0_F$. We now state without proof some of the fundamental facts about spanning sets, linear independence, and bases.

2.71. Theorem: Linear Algebra Facts. Every vector space V over a field F has a basis (possibly infinite). Any two bases of V have the same cardinality, which is called the *dimension* of V and denoted $\dim(V)$. Given a basis of V, every $v \in V$ can be expressed in exactly one way as a linear combination of the basis elements. Any linearly independent set in V can be enlarged to a basis of V. Any spanning set for V contains a basis of V. A set $S \subseteq V$ with $|S| > \dim(V)$ must be linearly dependent. A set $T \subseteq V$ with $|T| < \dim(V)$ cannot span V.

For example, $\dim(F^n) = n$ for all $n \geq 1$, whereas the vector space of polynomials with coefficients in F has (countably) infinite dimension.

2.13 Stirling Numbers and Polynomials

In this section, we use our recursions for Stirling numbers (of both kinds) to give algebraic proofs of certain polynomial identities. We will see that these identities connect certain frequently used bases of the vector space of one-variable polynomials. This linear-algebraic interpretation of Stirling numbers will be used to show the inverse relation between the two triangular matrices of Stirling numbers (cf. Figures 2.21 and 2.24). The following section will give combinatorial proofs of the same identities using rook theory.

2.72. Theorem: Polynomial Identity for Stirling Numbers of the Second Kind. For all $n \geq 0$ and all real x,

$$x^n = \sum_{k=0}^{n} S(n,k)x(x-1)(x-2)\cdots(x-k+1). \tag{2.6}$$

Proof. We give an algebraic proof here; the next section gives a combinatorial proof using rook theory. Recall that $S(0,0) = 1$ and $S(n,k) = S(n-1,k-1) + kS(n-1,k)$ for $n \geq 1$ and $1 \leq k \leq n$. We prove the stated identity by induction on n. If $n = 0$, the right side is $S(0,0) = 1 = x^0$, so the identity holds. For the induction step, fix $n \geq 1$ and assume that $x^{n-1} = \sum_{k=0}^{n-1} S(n-1,k)x(x-1)\cdots(x-k+1)$. Multiplying both sides by $x = (x-k)+k$, we can write

$$\begin{aligned}
x^n &= \sum_{k=0}^{n-1} S(n-1,k)x(x-1)\cdots(x-k+1)(x-k) \\
&\quad + \sum_{k=0}^{n-1} S(n-1,k)x(x-1)\cdots(x-k+1)k \\
&= \sum_{j=0}^{n-1} S(n-1,j)x(x-1)\cdots(x-j) + \sum_{k=0}^{n-1} kS(n-1,k)x(x-1)\cdots(x-k+1).
\end{aligned}$$

In the first summation, replace j by $k-1$. The calculation continues:

$$\begin{aligned}
x^n &= \sum_{k=1}^{n} S(n-1,k-1)x(x-1)\cdots(x-k+1) \\
&\quad + \sum_{k=0}^{n-1} kS(n-1,k)x(x-1)\cdots(x-k+1) \\
&= \sum_{k=0}^{n} S(n-1,k-1)x(x-1)\cdots(x-k+1) \\
&\quad + \sum_{k=0}^{n} kS(n-1,k)x(x-1)\cdots(x-k+1) \text{ (since } S(n-1,-1) = S(n-1,n) = 0) \\
&= \sum_{k=0}^{n} [S(n-1,k-1) + kS(n-1,k)]x(x-1)\cdots(x-k+1) \\
&= \sum_{k=0}^{n} S(n,k)x(x-1)\cdots(x-k+1) \quad \text{(using the recursion for } S(n,k)). \quad \square
\end{aligned}$$

2.73. Theorem: Polynomial Identity for Signless Stirling Numbers of the First Kind. For all $n \geq 0$ and all real x,

$$x(x+1)(x+2)\cdots(x+n-1) = \sum_{k=0}^{n} s'(n,k)x^k. \tag{2.7}$$

Proof. Recall that $s'(0,0) = 1$ and $s'(n,k) = s'(n-1,k-1) + (n-1)s'(n-1,k)$ for $n \geq 1$ and $1 \leq k \leq n$. We use induction on n again. If $n = 0$, both sides of (2.7) evaluate to 1. For the induction step, fix $n \geq 1$ and assume that

$$x(x+1)(x+2)\cdots(x+n-2) = \sum_{k=0}^{n-1} s'(n-1,k)x^k.$$

Multiply both sides of this assumption by $x+n-1$ and compute:

$$\begin{aligned}
x(x+1)\cdots(x+n-1) &= \sum_{k=0}^{n-1} s'(n-1,k)x^k(x+n-1) \\
&= \sum_{k=0}^{n-1} s'(n-1,k)x^{k+1} + \sum_{k=0}^{n-1}(n-1)s'(n-1,k)x^k \\
&= \sum_{k=1}^{n} s'(n-1,k-1)x^k + \sum_{k=0}^{n-1}(n-1)s'(n-1,k)x^k \\
&= \sum_{k=0}^{n}[s'(n-1,k-1)+(n-1)s'(n-1,k)]x^k \\
&= \sum_{k=0}^{n} s'(n,k)x^k \quad \text{(using the recursion for } s'(n,k)). \quad \square
\end{aligned}$$

2.74. Theorem: Polynomial Identity for Signed Stirling Numbers of the First Kind. For all $n \geq 0$ and all real x,

$$x(x-1)(x-2)\cdots(x-n+1) = \sum_{k=0}^{n} s(n,k)x^k. \tag{2.8}$$

Proof. Replace x by $-x$ in (2.7) to obtain

$$(-x)(-x+1)(-x+2)\cdots(-x+n-1) = \sum_{k=0}^{n} s'(n,k)(-x)^k.$$

Factoring out -1's, we get

$$(-1)^n x(x-1)(x-2)\cdots(x-n+1) = \sum_{k=0}^{n}(-1)^k s'(n,k)x^k.$$

Moving the $(-1)^n$ to the right side and recalling that $s(n,k) = (-1)^{n+k}s'(n,k)$, the result follows. $\square$

2.75. Theorem: Summation Formulas for Stirling Numbers of the First Kind. For all $n \geq 1$ and $1 \leq k \leq n$, we have

$$s'(n,k) = \sum_{1\leq i_1<i_2<\cdots<i_{n-k}\leq n-1} i_1 i_2 \cdots i_{n-k}.$$

Proof. We give an algebraic proof and a combinatorial proof of this result.

Algebraic Proof. We apply the generalized distributive law to the left side of the identity

$$(x+0)(x+1)(x+2)\cdots(x+n-1) = \sum_{k=0}^{n} s'(n,k)x^k.$$

According to the distributive law, the left side expands to a sum of terms obtained by choosing either x or i from each factor (for $0 \leq i < n$) and multiplying the chosen terms together. To obtain a contribution to the coefficient of x^k, we must choose x exactly k times and choose a number i exactly $n-k$ times. Adding up all these contributions gives the coefficient of x^k, namely $s'(n,k)$. The term $i_1 i_2 \cdots i_{n-k}$ is the contribution from the choice sequence where we choose i_1 from the factor $(x+i_1)$, i_2 from the factor $(x+i_2)$, etc., and choose x from all factors $(x+i)$ with i different from all i_j's.

Combinatorial Proof. Recall that $s'(n,k)$ counts the number of placements of $n-k$ non-attacking wrooks on the triangular Ferrers board Δ_n. Since wrooks only attack cells in their rows, a placement is non-attacking iff all wrooks occupy distinct rows of Δ_n. Let us classify wrook placements based on which rows contain wrooks. Suppose the $n-k$ wrooks appear in the rows of lengths $i_1, i_2, \ldots, i_{n-k}$, where $1 \leq i_1 < i_2 < \cdots < i_{n-k} \leq n-1$. The product rule shows that the number of placements of wrooks in these rows is $i_1 i_2 \cdots i_{n-k}$. The formula in the theorem now follows from the sum rule. □

2.76. Definition: Special Bases for Polynomials. Let V be the vector space of all polynomials in one variable x with real coefficients. For any integer $n \geq 0$, introduce the *falling factorial* polynomials

$$(x)\!\downarrow_0 = 1, \quad (x)\!\downarrow_n = x(x-1)(x-2)\cdots(x-n+1).$$

Similarly, the *rising factorial* polynomials are defined by

$$(x)\!\uparrow_0 = 1, \quad (x)\!\uparrow_n = x(x+1)(x+2)\cdots(x+n-1).$$

The *monomial basis* of V is the indexed set $M = \{x^n : n \geq 0\}$. The *falling factorial basis* of V is $F = \{(x)\!\downarrow_n : n \geq 0\}$. The *rising factorial basis* of V is $R = \{(x)\!\uparrow_n : n \geq 0\}$.

It is a routine exercise to prove that any indexed collection of polynomials $\{p_n(x) : n \geq 0\}$ such that $\deg(p_n) = n$ for all n is a basis of V. Since x^n, $(x)\!\downarrow_n$, and $(x)\!\uparrow_n$ all have degree n, it follows that M, F, and R really are bases of V. Define $M_N = \{x^n : 0 \leq n \leq N\}$, and define F_N and R_N similarly. The three indexed collections M_N, F_N, and R_N are all bases of the vector space V_N of polynomials in x of degree at most N.

We can now recast the preceding theorems in the language of linear algebra. Recall that if $B = (v_1, \ldots, v_n)$ and $C = (w_1, \ldots, w_n)$ are two ordered bases of a finite-dimensional vector space W, the *transition matrix from B to C* is the unique $n \times n$ matrix $A = (a_{ij})$ such that

$$v_j = \sum_{i=1}^{n} a_{ij} w_i \qquad (1 \leq j \leq n). \tag{2.9}$$

This matrix is so named because if $v \in W$ has coordinates $[v]_B = (s_1, \ldots, s_n)^T$ relative to the basis B (i.e., $v = \sum_j s_j v_j$), then the coordinates of v relative to the basis C are given by $[v]_C = A[v]_B$. Thus, multiplication by A transforms coordinates relative to B into coordinates relative to C. From linear algebra, we know that A is invertible, and A^{-1} is none other than the transition matrix from C to B.

2.77. Theorem: Transition Matrices between Polynomial Bases. Fix $N \geq 0$, and write $M_N = (x^n : n \leq N)$, $F_N = ((x)\!\downarrow_n : n \leq N)$, and $R_N = ((x)\!\uparrow_n : n \leq N)$, as above.
(a) The matrix $\mathbf{S} = (S(n,k))_{0 \leq n,k \leq N}$ of Stirling numbers of the second kind is the transpose of the transition matrix from the basis M_N to the basis F_N of the vector space V_N of polynomials of degree at most N.
(b) The matrix $\mathbf{s}' = (s'(n,k))_{0 \leq n,k \leq N}$ of signless Stirling numbers of the first kind is the transpose of the transition matrix from the basis R_N to the basis M_N of V_N.
(c) The matrix $\mathbf{s} = (s(n,k))_{0 \leq n,k \leq N}$ of signed Stirling numbers of the first kind is the transpose of the transition matrix from the basis F_N to the basis M_N of V_N.
(d) The $(N+1) \times (N+1)$ matrices $\mathbf{S}$ and $\mathbf{s}$ are inverses of one another.

Proof. The first three statements follow from equations (2.6), (2.7), (2.8), and the definition of transition matrices (2.9). The final statement is a special case of the fact that the transition matrix from B to C is the inverse of the transition matrix from C to B. □

Part (d) of the theorem says that we have matrix identities $\mathbf{Ss} = \mathbf{I} = \mathbf{sS}$, where $\mathbf{I}$ is the $(N+1) \times (N+1)$ identity matrix. Writing out what this means entry by entry, we obtain the formulas

$$\sum_k S(i,k)s(k,j) = \chi(i=j) = \sum_k s(i,k)S(k,j) \qquad (i,j \geq 0).$$

A combinatorial proof of the second equality will be given later (see 4.6).

2.14 Combinatorial Proofs of Polynomial Identities

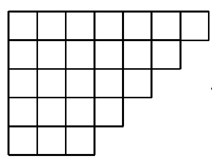
This section gives combinatorial proofs of some of the polynomial identities in the previous section. We use these proofs to illustrate a common technique in combinatorics in which we verify that a polynomial identity holds for *all* x by proving (combinatorially or otherwise) that the identity holds for sufficiently many particular values of the variable x.

Let us introduce this technique through a specific example.

2.78. Theorem: Stirling Numbers of the First Kind Revisited. For all *nonnegative integers* x and all $n > 0$, we have

$$x(x+1)(x+2)\cdots(x+n-1) = \sum_{k=0}^{n} s'(n,k)x^k. \qquad (2.10)$$

Proof. Recall that $s'(n,k)$ counts placements of $n-k$ non-attacking wrooks on the Ferrers board $\Delta_n = \mathrm{dg}(n-1, n-2, \ldots, 1, 0)$. Fix an integer $x \geq 0$ and consider the extended Ferrers board $\Delta_n(x) = \mathrm{dg}(x+n-1, x+n-2, \ldots, x+1, x)$. For example,

$\Delta_5(3) =$  .

Call the squares in the first x columns of this board *new* squares. Let A be the set of placements of n non-attacking wrooks on the board $\Delta_n(x)$. Note that every row of the board must be occupied by exactly one wrook. If we place the wrooks on the board one row at a time, working upwards from the bottom row, the product rule yields

$$|A| = x(x+1)(x+2)\cdots(x+n-1).$$

On the other hand, we can write A as the disjoint union of sets A_k, where A_k consists of those placements $\pi \in A$ in which exactly k wrooks occupy new squares. To build a placement $\pi \in A_k$, first place $n-k$ non-attacking wrooks in the old squares in $s'(n,k)$ ways. There are now k unused rows, each of which has x new squares, and k wrooks left to be placed. Visit each of these rows (say from top to bottom), and choose one of the x squares to be occupied by a wrook. By the product rule, $|A_k| = s'(n,k)x^k$. The sum rule now gives

$$|A| = \sum_{k=0}^{n} s'(n,k)x^k,$$

and the theorem follows. □

Comparing the combinatorial proof in 2.78 to the algebraic proof in 2.73, a subtle difference emerges: the combinatorial proof is valid only for *nonnegative integers* x, while the algebraic proof is valid for all *real* x (or even for formal polynomials in any polynomial ring $F[x]$, as defined in §7.3). The following result shows that our combinatorial proof is equally as good as the algebraic proof.

2.79. Theorem: Verifying Polynomial Identities. Suppose F is a field and p, q are two polynomials in $F[x]$. Say p and q have degree at most N. If $p(c) = q(c)$ for $N+1$ elements $c \in F$, then $p = q$ in the polynomial ring $F[x]$, and hence $p(c) = q(c)$ for *all* $c \in F$. In particular, if two real polynomials agree at each nonnegative integer, then the two polynomials are identical.

Proof. Consider the polynomial $p - q \in F[x]$. If this polynomial is nonzero, its degree is at most N. A well-known fact from algebra asserts that a nonzero polynomial of degree at most N has at most N roots in F (cf. 2.157 and 12.147). Our hypothesis says that $p - q$ has $N+1$ roots in F. Therefore, $p - q = 0$ in $F[x]$. Evaluating $p - q$ at any field element c, it follows that $p(c) = q(c)$. □

2.80. Example. To apply 2.79 to 2.78, fix n. The left side of (2.10), namely $p = x(x+1)\cdots(x+n-1)$, is a polynomial in x of degree n. The right side $q = \sum_{k=0}^{n} s'(n,k)x^k$ is also a real polynomial in x of degree n. The wrook-theoretic proof in 2.78 showed that $p(m) = q(m)$ for all integers $m \geq 0$. Hence, $p = q \in \mathbb{R}[x]$ and thus $p(r) = q(r)$ for all real r.

This type of argument involving 2.79 occurs so commonly that we will seldom spell out the details in the future. However, one must remember to check that both sides of the identity in question are *polynomials* in the variable x.

2.81. Theorem: Stirling Numbers of the Second Kind Revisited. For all integers $n > 0$ and all real x,

$$x^n = \sum_{k=0}^{n} S(n,k)x(x-1)(x-2)\cdots(x-k+1). \tag{2.11}$$

Proof. Both sides of the identity are polynomials in x, so it suffices to verify the identity when x is a nonnegative integer. Fix $x \geq 0$ and $n > 0$. Let A be the set of placements of n non-attacking rooks on the extended Ferrers board $\Delta_n(x) = \mathrm{dg}(x+n-1, x+n-2, \ldots, x+1, x)$. We can build a placement $\pi \in A$ by placing one rook in each row, working from bottom to top. The rook in the bottom row can go in any of x squares. The rook in the next row can go in any of $(x+1)-1 = x$ squares, since one column is attacked by the rook in the bottom row. In general, the rook located $i \geq 0$ rows above the bottom row can go in $(x+i)-i = x$ squares, since i distinct columns are already attacked by lower rooks when the time comes to place the rook in this row. The product rule therefore gives $|A| = x^n$.

On the other hand, A is the disjoint union of the sets A_k consisting of all $\pi \in A$ with exactly k rooks in the new squares and $n-k$ rooks in the old squares. (Recall that the new squares are the leftmost x columns of $\Delta_n(x)$.) To build $\pi \in A_k$, first place $n-k$ non-attacking rooks in Δ_n in any of $S(n,k)$ ways. There are now k unused rows of new squares, each of length x. Visit these rows from top to bottom (say), placing one rook in each row. There are x choices for the first rook, then $x-1$ choices for the second rook (since the first rook's column must be avoided), then $x-2$ choices for the third rook, etc. The product rule gives $|A_k| = S(n,k)x(x-1)(x-2)\cdots(x-k+1)$. Hence, by the sum rule,

$$|A| = \sum_{k=1}^{n} S(n,k)x(x-1)(x-2)\cdots(x-k+1).$$

The result follows by comparing our two formulas for $|A|$. $\square$

2.82. Remark. The proof technique of using extended boards such as $\Delta_n(x)$ can be reused to prove other results in rook theory. See §12.3.

We can also prove polynomial identities in several variables by verifying that the identity holds for sufficiently many values of the variables. As an example, we now present a combinatorial proof of the binomial theorem.

2.83. Combinatorial Binomial Theorem. For all integers $n \geq 0$,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \text{ in } \mathbb{R}[x,y].$$

Proof. Let $p = (x+y)^n \in \mathbb{R}[x,y]$ and $q = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \in \mathbb{R}[x,y]$. We first show that $p(i,j) = q(i,j)$ for all nonnegative integers $i, j \geq 0$. Fix such integers i and j. Consider an alphabet A consisting of i consonants and j vowels. Let B be the set of all n-letter words using letters from A. The product rule gives $|B| = (i+j)^n = p(i,j)$. On the other hand, B is the disjoint union of sets B_k (for $0 \leq k \leq n$) where B_k consists of the words in B with exactly k consonants. To build a word $w \in B_k$, first choose the positions for the k consonants out of the n available positions in $\binom{n}{k}$ ways. Choose the k consonants in these positions from left to right (i ways each), and then choose the $n-k$ vowels in the remaining positions from left to right (j ways each). The product rule gives $|B_k| = \binom{n}{k} i^k j^{n-k}$. Hence, the sum rule gives $|B| = q(i,j)$.

We complete the proof by invoking 2.79 twice. First, for each nonnegative integer $i \geq 0$, the polynomials $p(i,y)$ and $q(i,y)$ in $\mathbb{R}[y]$ agree for infinitely many values of the formal variable y. So, $p(i,y) = q(i,y)$ in $\mathbb{R}[y]$ and also in $\mathbb{R}(y)$ (the field of fractions of the polynomial ring $\mathbb{R}[y]$, cf. 7.44). Now regard p and q as elements of the polynomial ring $\mathbb{R}(y)[x]$. We have just shown that these polynomials (viewed as polynomials in the single variable x) agree in $\mathbb{R}(y)$ for infinitely many values of x. Hence, $p = q$ in $\mathbb{R}(y)[x]$, and so $p = q$ in $\mathbb{R}[x,y]$. This argument generalizes to polynomial identities in any number of variables (see 2.158), so we will omit the details in the future. $\square$

Summary

- *Generalized Distributive Law.* To multiply out a product of factors, where each factor is a sum of terms, choose one term from each factor, multiply these choices together, and add up the resulting products. Formally, this can be written:

$$\prod_{k=1}^{n} \left(\sum_{i_k \in I_k} x_{k,i_k} \right) = \sum_{(i_1,\ldots,i_n) \in I_1 \times \cdots \times I_n} \left(\prod_{k=1}^{n} x_{k,i_k} \right) \qquad \text{(all } x_{k,j} \text{ lie in a ring } R\text{).}$$

  If $A_1, \ldots, A_n$ are finite subsets of R, we can also write

$$\left(\sum_{w_1 \in A_1} w_1 \right) \cdot \left(\sum_{w_2 \in A_2} w_2 \right) \cdot \cdots \cdot \left(\sum_{w_n \in A_n} w_n \right) = \sum_{(w_1,w_2,\cdots,w_n) \in A_1 \times A_2 \times \cdots \times A_n} w_1 w_2 \cdots w_n.$$

- *Multinomial and Binomial Theorems.* In any ring R (possibly non-commutative),

$$(z_1 + z_2 + \cdots + z_s)^n = \sum_{\text{words } w \in \{1,\ldots,s\}^n} z_{w_1} z_{w_2} \cdots z_{w_n} \qquad (s, n \in \mathbb{N}^+, z_i \in R).$$

  If $z_i z_j = z_j z_i$ for all i, j, this becomes

$$(z_1 + z_2 + \cdots + z_s)^n = \sum_{n_1+n_2+\cdots+n_s=n} \binom{n}{n_1, n_2, \ldots, n_s} z_1^{n_1} z_2^{n_2} \cdots z_s^{n_s}.$$

  If $xy = yx$, we have the binomial theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

- *Combinatorial Proofs.* To prove a formula of the form $a = b$ combinatorially, one can define a set S of objects and give two counting arguments showing $|S| = a$ and $|S| = b$, respectively. To prove a polynomial identity $p(x) = q(x)$, it suffices to verify the identity for infinitely many values of the variable x (say for all nonnegative integers x). Similar comments apply to multivariable polynomial identities.

- *Identities for Binomial Coefficients.*

$$\binom{n}{k} = \binom{n}{n-k}; \quad \sum_{k=0}^{n} \binom{n}{k} = 2^n; \quad \sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}; \quad \binom{a+b}{a,b} = \sum_{k=0}^{a} \binom{k+b-1}{k, b-1};$$

$$\binom{a+b+c+1}{a, b+c+1} = \sum_{k=0}^{a} \binom{k+b}{k, b} \binom{a-k+c}{a-k, c} \quad \text{(Chu-Vandermonde formula)}$$

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} \quad \text{(Pascal recursion).}$$

- *Combinatorial Definitions.* An *integer partition of* n *into* k *parts* is a weakly decreasing list $\mu = (\mu_1, \mu_2 \ldots, \mu_k)$ of positive integers that sum to n. A *set partition of a set* S *into* k *blocks* is a set P of k nonempty, pairwise disjoint sets with union S. An *equivalence relation on* S is a reflexive, symmetric, transitive relation on S. The map that sends an equivalence relation to the set of its equivalence classes defines a bijection from the set of equivalence relations on S to the set of set partitions of S.

- *Notation for Combinatorial Objects.* Table 2.1 indicates notation used for counting some collections of combinatorial objects.

- *Recursions.* A collection of combinatorial objects can often be described *recursively*, by using smaller objects of the same kind to build larger objects. Induction arguments can be used to prove facts about recursively defined objects. Table 2.2 lists some recursions satisfied by the quantities in Table 2.1. In each case, the recursion together with appropriate *initial conditions* uniquely determine the quantities under consideration. If two collections of objects satisfy the same recursion and initial conditions, one can link together two combinatorial proofs of the recursion to obtain recursively defined bijections between the two collections.

TABLE 2.1
Notation for counting combinatorial objects.
Here f_n is a Fibonacci number; $B(n)$ is a Bell number; $S(n,k)$ is a Stirling number of the second kind; $s'(n,k)$ is a signless Stirling number of the first kind; $T(a,b)$ is a ballot number; $T_m(a,b)$ is an m-ballot number; and C_n is a Catalan number.

Notation	What it counts
$C(n,k) = \binom{n}{k}$	k-element subsets of an n-element set
$C(n; n_1, \ldots, n_s) = \binom{n}{n_1,\ldots,n_s}$	anagrams of the word $a_1^{n_1} \cdots a_s^{n_s}$
$L(a,b) = \binom{a+b}{a,b}$	lattice paths from $(0,0)$ to (a,b)
f_n	words in $\{0,1\}^n$ not containing 00
$M(n,k) = \binom{k+n-1}{k,n-1}$	k-element multisets of an n-element set
$B(n)$	set partitions of an n-element set; equivalence relations on an n-element set
$S(n,k)$	set partitions of $\{1,\ldots,n\}$ into k blocks; equiv. relations on $\{1,\ldots,n\}$ with k equiv. classes; placements of $n-k$ non-attacking rooks on Δ_n
$\mathrm{Surj}(n,k) = k!S(n,k)$	surjections from $\{1,\ldots,n\}$ to $\{1,\ldots,k\}$
$s'(n,k)$	placements of $n-k$ non-attacking wrooks on Δ_n; permutations of n objects with k cycles (§3.6)
$s(n,k) = (-1)^{n+k}s'(n,k)$	Stirling numbers of first kind (with signs)
$p(n)$	integer partitions of n
$p(n,k)$	integer partitions of n into k parts
$T(a,b) = \frac{b-a+1}{b+a+1}\binom{a+b+1}{a,b+1}$	lattice paths from $(0,0)$ to (a,b) that do not go below $y = x$ $(b \geq a)$
$T_m(a,b) = \frac{b-ma+1}{b+a+1}\binom{a+b+1}{a,b+1}$	lattice paths from $(0,0)$ to (a,b) that do not go below $y = mx$ $(m \in \mathbb{N}^+, b \geq ma)$
$C_n = T(n,n) = \frac{1}{n+1}\binom{2n}{n,n}$	Dyck paths ending at (n,n); binary trees with n vertices; 231-avoiding permutations; etc.

TABLE 2.2
Some combinatorial recursions.

$$
\begin{aligned}
C(n,k) &= C(n-1,k-1)+C(n-1,k)\\
C(n;n_1,\dots,n_s) &= \sum_{k=1}^{s} C(n-1;n_1,\dots,n_k-1,\dots,n_s)\\
f_n &= f_{n-1}+f_{n-2}\\
M(n,k) &= M(n-1,k)+M(n,k-1)\\
L(a,b) &= L(a-1,b)+L(a,b-1)\\
T(a,b) &= T(a-1,b)+T(a,b-1)\chi(b-1\ge a)\\
C_n &= \sum_{k=1}^{n} C_{k-1}C_{n-k}\\
p(n,k) &= p(n-1,k-1)+p(n-k,k)\\
p(n) &= \sum_{m\ge 1}(-1)^{m-1}\left[p\left(n-\frac{m(3m-1)}{2}\right)+p\left(n-\frac{m(3m+1)}{2}\right)\right]\\
S(n,k) &= S(n-1,k-1)+kS(n-1,k)\\
B(n) &= \sum_{k=0}^{n-1}\binom{n-1}{k}B(n-1-k)\\
\operatorname{Surj}(n,k) &= k\operatorname{Surj}(n-1,k-1)+k\operatorname{Surj}(n-1,k)\\
s'(n,k) &= s'(n-1,k-1)+(n-1)s'(n-1,k)
\end{aligned}
$$

- *Polynomial Identities for Stirling numbers.* Define *rising* and *falling* factorials by $(x){\uparrow_n}= x(x+1)(x+2)\cdots(x+n-1)$ and $(x){\downarrow_n}= x(x-1)(x-2)\cdots(x-n+1)$. The sets $\{x^n : n\ge 0\}$, $\{(x){\uparrow_n}: n\ge 0\}$, and $\{(x){\downarrow_n}: n\ge 0\}$ are all bases of the vector space of real polynomials in one variable. The Stirling numbers are the entries in the transition matrices between these bases. More specifically,

$$x^n=\sum_k S(n,k)(x){\downarrow_k};\quad (x){\uparrow_n}=\sum_k s'(n,k)x^k;\quad (x){\downarrow_n}=\sum_k s(n,k)x^k.$$

  So the matrices $\mathbf{S}=(S(n,k))_{n,k}$ and $\mathbf{s}=(s(n,k))_{n,k}$ are inverses of each other, i.e.,

$$\sum_k S(i,k)s(k,j)=\chi(i=j)=\sum_k s(i,k)S(k,j)\qquad (i,j\ge 0).$$

Exercises

2.84. Simplify the product $(B+C+H)(A+E+U)(R+T)$, where each letter denotes an arbitrary $n\times n$ real matrix.

2.85. Expand $(A+B+C)^3$, where A,B,C are $n\times n$ matrices.

2.86. Find the coefficient of $w^2x^3yz^3$ in $(w+x+y+z)^9$, assuming w, x, y, and z commute.

2.87. Expand $(3x-2)^5$ into a sum of monomials.

2.88. Find the constant term in $(2x-x^{-1})^6$.

2.89. Find the coefficient of x^3 in $(x^2+x+1)^4$.

2.90. Prove algebraically that $\sum_{k=0}^{n}(-1)^k\binom{n}{k} = \chi(n=0)$ and $\sum_{k=0}^{n} 2^k\binom{n}{k} = 3^n$ for all $n \geq 0$. Can you find combinatorial proofs?

2.91. Given $n \in \mathbb{N}^+$, evaluate $\sum_{0\leq j<k\leq n}\binom{n}{j}\binom{n}{k}$.

2.92. Given $m, n \in \mathbb{N}^+$, evaluate $\sum_{k_1+k_2+\cdots+k_m=n}(k_1!k_2!\cdots k_m!)^{-1}$.

2.93. Use Pascal's recursion to compute $\binom{9}{k}$ for $0 \leq k \leq 9$ and $\binom{10}{k}$ for $0 \leq k \leq 10$.

2.94. Give a proof of the recursion 2.27 for multinomial coefficients based on multidimensional lattice paths.

2.95. Compute the ballot numbers $T(a,7)$ for $0 \leq a \leq 7$ by drawing a picture.

2.96. For fixed $k \in \mathbb{N}^+$, let a_n be the number of n-letter words using the alphabet $\{0,1,\ldots,k\}$ that do not contain 00. Find a recursion and initial conditions for a_n.

2.97. How many words in $\{0,1,2\}^8$ do not contain 000?

2.98. How many lattice paths from $(1,1)$ to $(6,6)$ always stay weakly between the lines $y=2x/5$ and $y=5x/2$?

2.99. How many lattice paths go from $(1,1)$ to $(8,8)$ without ever passing through a point (p,q) such that p and q are both prime?

2.100. Show that C_n counts integer partitions μ such that $\mathrm{dg}(\mu) \subseteq \Delta_n$.

2.101. Draw pictures of all integer partitions of $n=6$ and $n=7$. Indicate which partitions are conjugates of one another.

2.102. Compute $p(8,3)$ by direct enumeration and by using a recursion.

2.103. Use Euler's recursion to compute $p(k)$ for $13, 14, 15, 16$ (see Figure 2.19).

2.104. (a) Write down all the set partitions and rook placements counted by $S(5,2)$. (b) List all the set partitions and equivalence relations counted by $B(4)$. (c) Draw all the wrook placements counted by $s'(4,2)$.

2.105. Compute $S(9,k)$ for $0 \leq k \leq 9$ and $S(10,k)$ for $0 \leq k \leq 10$ (use Figure 2.21).

2.106. Compute the Bell number $B(k)$ for $k = 9, 10, 11, 12$ (use Figure 2.21).

2.107. Compute $s(8,k)$ for $0 \leq k \leq 8$ (use Figure 2.24).

2.108. (a) Find a combinatorial proof of the formula $\sum_{i=1}^{n} i = n(n+1)/2$. (b) Can you prove $\sum_{i=1}^{n} i^2 = n(n+1)(2n+1)/6$ combinatorially?

2.109. Prove the identity $k\binom{n}{k} = n\binom{n-1}{k-1} = (n-k+1)\binom{n}{k-1}$ algebraically and combinatorially (where $1 \leq k \leq n$).

2.110. Suppose X is an n-element set. Count the number of: (a) relations on X; (b) reflexive relations on X; (c) irreflexive relations on X; (d) symmetric relations on X; (e) irreflexive and symmetric relations on X; (f) antisymmetric relations on X.

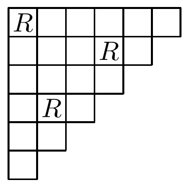
2.111. Let X be a nine-element set and Y a four-element set. (a) How many functions map X into Y? (b) How many functions map Y into X? (c) How many surjections are there from X onto Y? (d) How many injections are there from Y into X?

2.112. Verify equations (2.6), (2.7), and (2.8) by direct calculation for $n = 3$ and $n = 4$.

2.113. (a) Find the rook placement associated to the set partition

$$\{\{2, 5\}, \{1, 4, 7, 10\}, \{3\}, \{6, 8\}, \{9\}\}$$

by the bijection in 2.64. (b) Find the set partition associated to the following rook placement:



2.114. Let $f : \{1, 2, \ldots, 7\} \to \{1, 2, 3\}$ be the surjection given by $f(1) = 3$, $f(2) = 3$, $f(3) = 1$, $f(4) = 3$, $f(5) = 2$, $f(6) = 3$, $f(7) = 1$. In the second proof of 2.58, what choice sequence can be used to construct f?

2.115. How many compositions of 20 only use parts of sizes 1, 3, or 5?

2.116. Use the recursion 2.26 for multisets to prove by induction that the number of k-element multisets using an n-element alphabet is $M(n, k) = \frac{(k+n-1)!}{k!(n-1)!}$.

2.117. Given $a, b, c, n \in \mathbb{N}^+$ with $a + b + c = n$, prove combinatorially that $\binom{n}{a,b,c} = \sum_{k=0}^{c} \left[\binom{n-k-1}{a-1,b,c-k} + \binom{n-k-1}{a,b-1,c-k}\right]$.

2.118. Complete the proof of 2.31 by proving $T_m(a, b) = \frac{b-ma+1}{b+a+1}\binom{a+b+1}{a}$ by induction.

2.119. Show that $|S_n^\tau| = C_n$ for (a) $\tau = 132$; (b) $\tau = 213$; (c) $\tau = 312$. (d) Convert the binary tree in Figure 2.12 to a τ-avoiding permutation for each of these choices of τ.

2.120. (a) Let G_n be the set of lists of integers $(g_0, g_1, \ldots, g_{n-1})$ where $g_0 = 0$, each $g_i \geq 0$, and $g_{i+1} \leq g_i + 1$ for all $i < n - 1$. Prove that $|G_n| = C_n$. (b) For $m \in \mathbb{N}^+$, let $G_n^{(m)}$ be the set of lists of integers $(g_0, g_1, \ldots, g_{n-1})$ where $g_0 = 0$, each $g_i \geq 0$, and $g_{i+1} \leq g_i + m$ for all $i < n - 1$. Prove that $|G_n^{(m)}| = T_m(n, mn)$, the number of lattice paths from $(0, 0)$ to (n, mn) that never go below the line $y = mx$.

2.121. Consider the 231-avoiding permutation $w = 1\ 5\ 2\ 4\ 3\ 11\ 7\ 6\ 10\ 8\ 9$. Use recursive bijections based on the Catalan recursion to map w to objects of the following kinds: (a) a Dyck path; (b) a binary tree; (c) a 312-avoiding permutation (see 2.119); (d) an element of G_n (see 2.120).

2.122. Let π be the Dyck path NNENEENNNENNENNEEENENEEE. Use recursive bijections based on the Catalan recursion to map π to objects of the following kinds: (a) a binary tree; (b) a 231-avoiding permutation; (c) a 213-avoiding permutation (see 2.119).

2.123. Show that the number of possible rhyme schemes for an n-line poem using k different rhyme syllables is the Stirling number $S(n,k)$. (For example, ABABCDCDEFEFGG is a rhyme scheme with $n=14$ and $k=7$.)

2.124. (a) Find explicit formulas for $S(n,k)$ when $k = 1, 2, n-1$, and n. Prove your formulas using counting arguments. (b) Repeat part (a) for $\mathrm{Surj}(n,k)$.

2.125. Give a combinatorial proof of the identity $kS(n,k) = \sum_{j=1}^{n} \binom{n}{j} S(n-j,k-1)$, where $1 \le k \le n$.

2.126. Prove $C_n = \sum_{k\in\mathbb{N}:\ 0\le k\le n/2} T(k,n-k)^2$ for $n \ge 1$.

2.127. Consider lattice paths that can take unit steps up (N), down (S), left (W), or right (E), with self-intersections allowed. How many such paths begin and end at $(0,0)$ and have 10 steps?

2.128. Use 2.52, 2.63, and the ideas in 2.39 to give a recursive definition of a bijection between rook placements counted by $S'(n,k)$ and set partitions counted by $S(n,k)$. Is this bijection the same as the bijection described in 2.64?

2.129. Fix $n \in \mathbb{N}^+$, let μ be an integer partition of length $\ell(\mu) \le n$, and set $\mu_k = 0$ for $\ell(\mu) < k \le n$. Let $s'(\mu,k)$ be the number of placements of $n-k$ non-attacking wrooks on the board $\mathrm{dg}(\mu)$. (a) Find a summation formula for $s'(\mu,k)$ analogous to 2.75. (b) Prove that

$$(x+\mu_1)(x+\mu_2)\cdots(x+\mu_n) = \sum_{k=0}^{n} s'(\mu,k)x^k.$$

(c) For $n=7$ and $\mu=(8,5,3,3,1)$, find $s'(\mu,k)$ for $0 \le k \le 7$.

2.130. (a) Show that the Fibonacci number f_{n-1} (see 2.23) is the number of compositions of n in which every part has size 1 or 2. (b) Show that f_n is the number of subsets of $\{1,2,\dots,n\}$ that do not contain two consecutive integers. (c) Combine (b) with 1.113 to deduce a summation formula for f_n.

2.131. (a) Show that the sequence $a_n = f_{2n}$ (see 2.23) satisfies the recursion $a_n = 3a_{n-1} - a_{n-2}$ for $n \ge 2$. What are the initial conditions? (b) Show that a_n is the number of words in $\{A,B,C\}^n$ in which A is never immediately followed by B.

2.132. For $n \ge 0$, let a_n be the number of words in $\{1,2,\dots,k\}^n$ in which 1 is never immediately followed by 2. Find a recursion satisfied by the sequence $(a_n : n \ge 0)$, and prove it with a suitable bijection.

2.133. (a) Give algebraic and combinatorial proofs of the identity

$$x^n - 1 = (x-1)1 + (x-1)x + (x-1)x^2 + \cdots + (x-1)x^{n-1} \qquad (x \in \mathbb{R}).$$

(b) Deduce a formula for $\sum_{m=0}^{\infty} x^m$, valid for real numbers x with $|x| < 1$.

2.134. Define $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1}+F_{n-2}$ for all $n > 1$ (so $f_n = F_{n+2}$ for $n \ge 0$). Give algebraic or combinatorial proofs of the following formulas. (a) $F_n = (\phi^n - \psi^n)/\sqrt{5}$, where $\phi = (1+\sqrt{5})/2$, $\psi = (1-\sqrt{5})/2$. (b) $\sum_{k=0}^{n} F_k = F_{n+2} - 1$. (c) $\sum_{k=0}^{n-1} F_{2k+1} = F_{2n}$. (d) $\sum_{k=0}^{n} F_{2k} = F_{2n+1} - 1$.

2.135. Give a combinatorial proof of equation (2.11) by interpreting both sides as counting a suitable collection of functions.

2.136. Let $C_{n,k}$ be the number of Dyck paths of order n that end with exactly k east steps. Prove the recursion

$$C_{n,k} = \sum_{r=1}^{n-k} \binom{k-1+r}{k-1,r} C_{n-k,r}.$$

2.137. Let p be prime. Prove that $\binom{p}{k}$ is divisible by p for $0 < k < p$. Can you find a combinatorial proof?

2.138. *Fermat's Little Theorem* states that $a^p \equiv a \pmod{p}$ for $a \in \mathbb{N}^+$ and p prime. Prove this by expanding $a^p = (1+1+\cdots+1)^p$ using the multinomial theorem (cf. 2.137).

2.139. Ordered Set Partitions. An *ordered set partition* of a set X is a sequence $P = (T_1, T_2, \ldots, T_k)$ of distinct sets such that $\{T_1, T_2, \ldots, T_k\}$ is a set partition of X. Let $B_o(n)$ be the number of ordered set partitions of an n-element set. (a) Show $B_o(n) = \sum_{k=1}^{n} k!S(n,k)$ for $n \geq 1$. (b) Find a recursion relating $B_o(n)$ to values of $B_o(m)$ for $m < n$. (c) Compute $B_o(n)$ for $0 \leq n \leq 5$.

2.140. (a) Let $B_1(n)$ be the number of set partitions of an n-element set such that no block of the partition has size 1. Find a recursion and initial conditions for $B_1(n)$, and use these to compute $B_1(n)$ for $1 \leq n \leq 6$. (b) Let $S_1(n,k)$ be the number of set partitions as in (a) with k blocks. Find a recursion and initial conditions for $S_1(n,k)$.

2.141. Let $p_d(n,k)$ be the set of integer partitions of n with first part k and all parts *distinct*. Find a recursion and initial conditions for $p_d(n,k)$.

2.142. Let $p_o(n,k)$ be the set of integer partitions of n with first part k and all parts *odd*. Find a recursion and initial conditions for $p_o(n,k)$.

2.143. Let $q(n,k)$ be the number of integer partitions μ of length k and area n such that $\mu' = \mu$ (such partitions are called *self-conjugate*). Find a recursion and initial conditions for $q(n,k)$.

2.144. Verify the statement made in 2.76 that any indexed collection of polynomials $\{p_n(x) : n \geq 0\}$ such that $\deg(p_n) = n$ for all n is a basis for the real vector space of polynomials in one variable with real coefficients.

2.145. Verify the following statements about transition matrices from §2.13. (a) If B and C are ordered bases of W and A is the transition matrix from B to C, then $[v]_C = A[v]_B$ for all $v \in W$. (b) If A is the transition matrix from B to C, then A is invertible, and A^{-1} is the transition matrix from C to B.

2.146. Complete the proof of 2.55 by verifying that: (a) $\phi(P) \in \mathcal{B}$ for all $P \in \mathcal{A}$; (b) $\phi'(R) \in \mathcal{A}$ for all $R \in \mathcal{B}$; (c) $\phi \circ \phi' = \mathrm{id}_{\mathcal{B}}$; (d) $\phi' \circ \phi = \mathrm{id}_{\mathcal{A}}$.

2.147. Consider a product $x_1 \times x_2 \times \cdots \times x_n$ where the binary operation $\times$ is not necessarily associative. Show that the number of ways to parenthesize this expression is the Catalan number C_{n-1}. For example, the five possible parenthesizations when $n = 4$ are

$$(((x_1 \times x_2) \times x_3) \times x_4),\ ((x_1 \times x_2) \times (x_3 \times x_4)),\ (x_1 \times ((x_2 \times x_3) \times x_4)),$$

$$(x_1 \times (x_2 \times (x_3 \times x_4))),\ ((x_1 \times (x_2 \times x_3)) \times x_4).$$

2.148. Generalized Associative Law. Let $\times$ be an associative binary operation on a set S (so $(x \times y) \times z = x \times (y \times z)$ for all $x, y, z \in S$). Given $x_1, \ldots, x_n \in S$, recursively define $\prod_{i=1}^{1} x_i = x_1$ and $\prod_{i=1}^{n} x_i = (\prod_{i=1}^{n-1} x_i) \times x_n$. Prove that every parenthesization of the expression $x_1 \times x_2 \times \cdots \times x_n$ evaluates to $\prod_{i=1}^{n} x_i$ (use strong induction). This result justifies the omission of parentheses in expressions of this kind.

2.149. Generalized Commutative Law. Let $+$ be an associative and commutative binary operation on a set S. Prove that for any bijection $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ and any elements $x_1, \ldots, x_n \in S$,

$$x_1 + x_2 + \cdots + x_n = x_{f(1)} + x_{f(2)} + \cdots + x_{f(n)}.$$

2.150. For each positive integer n, let $\mathbb{Z}_n = \{0, 1, 2, \ldots, n-1\}$. Define binary operations $\oplus$ and $\otimes$ on $\mathbb{Z}_n$ by letting $a \oplus b = (a+b) \bmod n$ and $a \otimes b = (a \cdot b) \bmod n$, where $c \bmod n$ denotes the unique remainder in $\mathbb{Z}_n$ when c is divided by n. (a) Prove that $\mathbb{Z}_n$ with these operations is a commutative ring. (b) Prove that $\mathbb{Z}_n$ is a field iff n is a prime number.

2.151. Let R be a ring, and let $M_n(R)$ be the set of all $n \times n$ matrices with entries in R. Define matrix addition and multiplication as follows. Writing $A(i, j)$ for the i, j-entry of a matrix A, let $(A + B)(i, j) = A(i, j) + B(i, j)$ and $(AB)(i, j) = \sum_{k=1}^{n} A(i, k)B(k, j)$. Verify the ring axioms in 2.2 for $M_n(R)$. Show that this ring is non-commutative whenever $n > 1$ and $|R| > 1$. If R is finite, what is $|M_n(R)|$?

2.152. Let R be a ring, and let $R[x]$ be the set of all one-variable "formal" polynomials with coefficients in R. Define polynomial addition and multiplication as follows. If $p = \sum_{i \geq 0} a_i x^i$ and $q = \sum_{j \geq 0} b_j x^j$ with $a_i, b_j \in R$ (and $a_i = 0, b_j = 0$ for large enough i, j), set $p + q = \sum_i (a_i + b_i)x^i$ and $pq = \sum_k c_k x^k$, where $c_k = a_0 b_k + a_1 b_{k-1} + \cdots + a_k b_0$. Also, by definition, $p = q$ means $a_i = b_i$ for all $i \geq 0$. Verify the ring axioms in 2.2 for $R[x]$.

2.153. (a) Let R be the set of all functions $f : \mathbb{Z} \to \mathbb{Z}$. Given $f, g \in R$, define $f \oplus g : \mathbb{Z} \to \mathbb{Z}$ by $(f \oplus g)(n) = f(n) + g(n)$ for all $n \in \mathbb{Z}$, and define $f \circ g$ by $(f \circ g)(n) = f(g(n))$ (composition of functions). Show that $(R, \oplus, \circ)$ satisfies all the ring axioms in 2.2 except commutativity of multiplication and the left distributive law. (b) Let S be the set of $f \in R$ such that $f(m + n) = f(m) + f(n)$ for all $m, n \in \mathbb{Z}$. Prove that $(S, \oplus, \circ)$ is a ring.

2.154. Prove the binomial theorem 2.14 by induction on n. Mark each place in your proof where you use the hypothesis $xy = yx$.

2.155. Prove the commutative multinomial theorem 2.12 using the binomial theorem and induction on s.

2.156. Let f, g be smooth functions of x (which means f and g have derivatives of all orders). Recall the product rule: $D(fg) = D(f)g + fD(g)$, where D denotes differentiation with respect to x. (a) Prove that the nth derivative of fg is given by

$$D^n(fg) = \sum_{k=0}^{n} \binom{n}{k} D^k(f) D^{n-k}(g).$$

(b) Find and prove a similar formula for $D^n(f_1 f_2 \cdots f_s)$, where $f_1, \ldots, f_s$ are smooth functions.

2.157. (a) Given a field F, an element $c \in F$, and a polynomial $p \in F[x]$, show that $p(c) = 0$ iff $x - c$ divides p in $F[x]$. (b) Show that if $p \in F[x]$ is a nonzero polynomial with more than N roots in F, then $\deg(p) > N$. (c) Show that (b) can fail if F is a commutative ring that is not a field.

2.158. Let $p, q \in \mathbb{R}[x_1, \ldots, x_n]$ be multivariable polynomials such that $p(m_1, \ldots, m_n) = q(m_1, \ldots, m_n)$ for all $m_1, \ldots, m_n \in \mathbb{N}^+$. Prove that $p = q$.

2.159. (a) Give a combinatorial proof of the multinomial theorem 2.12, assuming that $z_1, z_2, \ldots, z_s$ are positive integers. (b) Deduce that this theorem is also valid for all $z_1, \ldots, z_s \in \mathbb{R}$.

2.160. Let A_n be the set of lattice paths from $(0,0)$ to (n,n) that take exactly one north step below the line $y=x$. What is $|A_n|$?

2.161. Prove: for $n \in \mathbb{N}$, $\sum_{k=0}^{n} \binom{2k}{k}\binom{2n-2k}{n-k} = 4^n$.

Notes

The book by Gould [58] contains an extensive, systematic list of binomial coefficient identities. More recently, Petkovsek, Wilf and Zeilberger [104] developed an algorithm, called the WZ-method, that can automatically evaluate many hypergeometric summations (which include binomial coefficient identities) or prove that such a summation has no closed form. For more information on hypergeometric series, see Koepf [80].

A wealth of information about integer partitions, including a discussion of the Hardy-Rademacher-Ramanujan formula 2.49, may be found in [5]. There is a vast literature on pattern-avoiding permutations; for more information on this topic, consult Bòna [15].

A great many combinatorial interpretations have been discovered for the Catalan numbers C_n. A partial list appears in Exercise 6.19 of Stanley [127, Vol. 2]; this list continues in the "Catalan addendum," which currently resides on the Internet at the following location:

`http://www-math.mit.edu/~rstan/ec/catadd.pdf`