Chapter 4

The Principle of Inclusion and Exclusion

4.1. Introduction

The addition principle (AP) was stated at the beginning of Chapter 1. Its simplest form may be addressed as follows:

> If A and B are finite sets such that $A \cap B = \emptyset$,
> then $|A \cup B| = |A| + |B|$.

What is the corresponding equality for $|A \cup B|$ if $A \cap B \neq \emptyset$? If $A \cap B \neq \emptyset$, then in the counting of $|A|$ and $|B|$, the elements in $A \cap B$ are counted exactly twice. Thus we have (see also Figure 4.1.1):

$$|A \cup B| = |A| + |B| - |A \cap B| \tag{4.1.1}$$

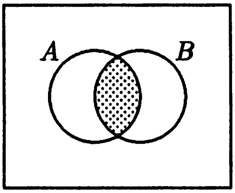


Figure 4.1.1.

As we have seen in the previous chapters, in solving certain more complicated counting problems, the sets whose elements are to be enumerated are usually divided into appropriate disjoint subsets so that (AP) can be directly applied. However, the task of dividing a set into such disjoint subsets may not be easy. Formula (4.1.1) provides us with a more flexible way:

Express the given set as $A \cup B$, where A and B need not be disjoint, and then count $|A|$, $|B|$ and $|A \cap B|$ independently. The 'inclusion' of $|A|$ and $|B|$ and the 'exclusion' of $|A \cap B|$ in Formula (4.1.1) will automatically give us the desired result for $|A \cup B|$.

Formula (4.1.1) is the simplest form of the so-called *Principle of Inclusion and Exclusion* (PIE), which we are going to study in the chapter. In Section 2 below, we shall extend formula (4.1.1) for 2 sets to a formula for n sets, $n \geq 2$. A much more general result which includes the latter as a special case will be established in Section 3. Applications of this general result to various classical enumeration problems will be discussed in the remaining sections.

4.2. The Principle

To begin with, let us see how we can apply (4.1.1) to get a corresponding formula for $|A \cup B \cup C|$, where A, B and C are three arbitrary finite sets.

Observe that

$$\begin{aligned}
|A \cup B \cup C| &= |(A \cup B) \cup C| \quad \text{(associative law)} \\
&= |A \cup B| + |C| - |(A \cup B) \cap C| \quad \text{(by (4.1.1))} \\
&= |A \cup B| + |C| - |(A \cap C) \cup (B \cap C)| \quad \text{(distributive law)} \\
&= |A| + |B| - |A \cap B| + |C| \\
&\quad - (|A \cap C| + |B \cap C| - |(A \cap C) \cap (B \cap C)|) \quad \text{(by (4.1.1))} \\
&= (|A| + |B| + |C|) - (|A \cap B| + |A \cap C| + |B \cap C|) \\
&\quad + |A \cap B \cap C|.
\end{aligned}$$

Thus we have:

$$|A \cup B \cup C| = (|A|+|B|+|C|)-(|A \cap B|+|A \cap C|+|B \cap C|)+|A \cap B \cap C|. \tag{4.2.1}$$

As a matter of fact, we have the following general result.

Theorem 4.2.1. (PIE) *For any q finite sets $A_1, A_2, ..., A_q$, $q \geq 2$,*

$$\begin{aligned}
&|A_1 \cup A_2 \cup \cdots \cup A_q| \\
&= \sum_{i=1}^{q} |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k| \\
&\quad - \cdots + (-1)^{q+1} |A_1 \cap A_2 \cap \cdots \cap A_q|. \quad \blacksquare
\end{aligned} \tag{4.2.2}$$

Theorem 4.2.1 can be proved by induction on q. This proof is left to the reader (see Problem 4.7). Instead, we shall prove a more general result in Section 3 which includes Theorem 4.2.1 as a special case.

Example 4.2.1. Let $S = \{1, 2, ..., 500\}$. Find the number of integers in S which are divisible by 2, 3 or 5.

Before solving the problem, two observations are in order.

(1) For each $n \in \mathbf{N}$, the number of integers in S which are divisible by (or multiples of) n is given by $\lfloor \frac{500}{n} \rfloor$.

(2) For $a, b, c \in \mathbf{N}$, c is divisible by both a and b if and only if c is divisible by the LCM of a and b.

Bearing these in mind, you will find it easy to follow the solution given below.

Solution. For each $k \in \mathbf{N}$, let

$$B_k = \{x \in S \mid x \text{ is divisible by } k\}.$$

Thus, our aim is to find $|B_2 \cup B_3 \cup B_5|$.

To apply (PIE), we first need to perform the following computations.

By observation (1),

$$|B_2| = \left\lfloor \frac{500}{2} \right\rfloor = 250,$$

$$|B_3| = \left\lfloor \frac{500}{3} \right\rfloor = 166,$$

and

$$|B_5| = \left\lfloor \frac{500}{5} \right\rfloor = 100.$$

By observations (1) and (2),

$$|B_2 \cap B_3| = |B_6| = \left\lfloor \frac{500}{6} \right\rfloor = 83,$$

$$|B_2 \cap B_5| = |B_{10}| = \left\lfloor \frac{500}{10} \right\rfloor = 50,$$

$$|B_3 \cap B_5| = |B_{15}| = \left\lfloor \frac{500}{15} \right\rfloor = 33,$$

and
$$|B_2 \cap B_3 \cap B_5| = |B_{30}| = \left\lfloor \frac{500}{30} \right\rfloor = 16.$$

Now, by (PIE),

$$\begin{aligned}|B_2 \cup B_3 \cup B_5| &= (|B_2|+|B_3|+|B_5|)-(|B_2 \cap B_3|+|B_2 \cap B_5|+|B_3 \cap B_5|)\\ &\quad + |B_2 \cap B_3 \cap B_5|\\ &= (250+166+100)-(83+50+33)+16\\ &= 366. \quad \blacksquare\end{aligned}$$

4.3. A Generalization

In Example 4.2.1, we have counted the number of integers in $S = \{1, 2, ..., 500\}$ which are divisible by at least one of the integers 2, 3, 5. We may ask further related questions. For instance, how many integers are there in S which are divisible by

(1) none of 2, 3, 5?
(2) exactly one of 2, 3, 5?
(3) exactly two of 2, 3, 5?
(4) all of 2, 3, 5?

For easy reference, we show in Figure 4.3.1 the desired sets corresponding to the above questions.

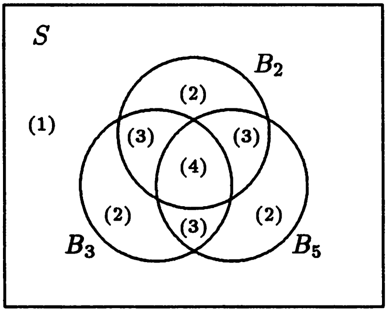


Figure 4.3.1.

The above questions cannot be solved directly by Theorem 4.2.1. In this section, we shall establish a general result which enables us to provide direct solutions to questions of this type.

Before proceeding any further, we first note the following. Let S be a given universal set. Then any subset A of S induces a *property* P such that, for any $x \in S$,

$$x \text{ possesses the property } P \quad \Leftrightarrow \quad x \in A.$$

For example, if $S = \{1, 2, ..., 10\}$ and $A = \{1, 2, 3, 4\}$, then $A = \{x \in S \mid x \text{ possesses } P\}$, where the property P may be taken to be "< 5". On the other hand, any property P of elements of S determines a *subset* A of S such that the elements of A are precisely those possessing P. For instance, if $S = \{1, 2, ..., 10\}$ and P is the property of being "divisible by 3", then P determines the subset $\{3, 6, 9\}$ of S. In view of this, it is reasonable for us to use the term "properties" to replace "subsets" in the following generalization. An advantage of this replacement is that the statements or formulae concerned can be substantially simplified.

In what follows, let S be an n-element universal set, and let $P_1, P_2, ..., P_q$ be q properties for the elements of S, where $q \geq 1$. It should be clear that a property may be possessed by none, some or all elements of S; and on the other hand, an element of S may have none, some or all of the properties.

For integer m with $0 \leq m \leq q$, let $E(m)$ denote the number of elements of S that possess *exactly* m of the q properties; and for $1 \leq m \leq q$, let $\omega(P_{i_1}P_{i_2}\cdots P_{i_m})$ denote the number of elements of S that possess the properties $P_{i_1}, P_{i_2}, \ldots, P_{i_m}$, and let

$$\omega(m) = \sum(\omega(P_{i_1}P_{i_2}\cdots P_{i_m}),$$

where the summation is taken over all m-combinations $\{i_1, i_2, \ldots, i_m\}$ of $\{1, 2, \ldots, q\}$.

We also define $\omega(0)$ to be $|S|$; i.e., $\omega(0) = |S| = n$. The result that we wish to establish is the following generalized principle of inclusion and exclusion (GPIE).

Theorem 4.3.1. **(GPIE)** *Let S be an n-element set and let $\{P_1, P_2, \ldots, P_q\}$ be a set of q properties for elements of S. Then for each $m = 0, 1, 2, ..., q$,*

$$\begin{aligned} E(m) &= \omega(m) - \binom{m+1}{m}\omega(m+1) + \binom{m+2}{m}\omega(m+2) \\ &\quad - \cdots + (-1)^{q-m}\binom{q}{m}\omega(q) \\ &= \sum_{k=m}^{q}(-1)^{k-m}\binom{k}{m}\omega(k). \end{aligned} \tag{4.3.1}$$

We illustrate this result with the following example.

Example 4.3.1. Let $S = \{1, 2, ..., 14\}$, and let P_1, P_2, P_3 and P_4 be 4 given properties. Assume that an element $j \in S$ possesses the property P_i if and only if the (i, j) entry in Table 4.3.1 is indicated by a tick "$\surd$".

S	1	2	3	4	5	6	7	8	9	10	11	12	13	14
P_1	$\surd$			$\surd$				$\surd$	$\surd$			$\surd$		
P_2	$\surd$	$\surd$	$\surd$	$\surd$		$\surd$		$\surd$		$\surd$	$\surd$		$\surd$	
P_3	$\surd$		$\surd$					$\surd$	$\surd$		$\surd$			$\surd$
P_4	$\surd$		$\surd$					$\surd$		$\surd$	$\surd$			$\surd$

Table 4.3.1.

Consider the property P_1. Since there are 5 elements of S (namely, 1, 4, 8, 9 and 12) having this property, we have $\omega(P_1) = 5$. Consider the properties P_2 and P_3. Since there are 4 elements of S (namely, 1, 3, 8 and 11) having both P_2 and P_3, we have $\omega(P_2P_3) = 4$. Checking through the table, we have the following data:

$\omega(P_1) = 5$, $\omega(P_2) = 9$, $\omega(P_3) = 6$, $\omega(P_4) = 6$;
$\omega(P_1P_2) = 3$, $\omega(P_1P_3) = 3$, $\omega(P_1P_4) = 2$,
$\omega(P_2P_3) = 4$, $\omega(P_2P_4) = 5$, $\omega(P_3P_4) = 5$;
$\omega(P_1P_2P_3) = 2$, $\omega(P_1P_2P_4) = 2$, $\omega(P_1P_3P_4) = 2$, $\omega(P_2P_3P_4) = 4$;
$\omega(P_1P_2P_3P_4) = 2$.

Thus, by definition,

$$\begin{aligned}
\omega(0) &= |S| = 14,\\
\omega(1) &= \sum_{i=1}^{4}\omega(P_i) = 5+9+6+6 = 26,\\
\omega(2) &= \sum_{i<j}\omega(P_iP_j) = 3+3+2+4+5+5 = 22,\\
\omega(3) &= \sum_{i<j<k}\omega(P_iP_jP_k) = 2+2+2+4 = 10,\\
\omega(4) &= \omega(P_1P_2P_3P_4) = 2.
\end{aligned}$$

On the other hand, by scrutinizing the table again, we find

$E(0) = 2$ (elements 5 and 7), $E(1) = 4$ (elements 2, 6, 12, 13),
$E(2) = 4$ (elements 4, 9, 10, 14), $E(3) = 2$ (elements 3, 11),
$E(4) = 2$ (elements 1, 8).

Suppose $m = 0$. Observe that the RHS of identity (4.3.1) is

$$\begin{aligned}
&\omega(0) - \omega(1) + \omega(2) - \omega(3) + \omega(4)\\
&= 14 - 26 + 22 - 10 + 2 = 2,
\end{aligned}$$

which agrees with $E(0)$.

We leave it to the reader to verify identity (4.3.1) for $m = 1, 2, 3$. ∎

We are now ready to prove Theorem 4.3.1.

Proof. We shall show that each $x \in S$ contributes the same "count", either 0 or 1, to each side of the equality (4.3.1).

Let $x \in S$ be given. Assume that x possesses *exactly* t properties.

Case 1. $t < m$. Clearly, x contributes a count of 0 to both sides.

Case 2. $t = m$. In this case, x is counted once in $E(m)$. On the other hand, x contributes 1 to $\omega(m)$ but 0 to $\omega(r)$ for $r > m$. Thus x contributes a count of 1 to both sides.

Case 3. $t > m$. Now, x contributes a count of 0 to $E(m)$. On the other hand, x is counted

$$\binom{t}{m} \quad \text{times in} \quad \omega(m),$$

$$\binom{t}{m+1} \quad \text{times in} \quad \omega(m+1),$$
$$\vdots$$
$$\binom{t}{t} \quad \text{times in} \quad \omega(t);$$

but x contributes 0 to $\omega(r)$ for $r > t$.

Thus the count that x contributes to RHS is given by

$$\lambda = \binom{t}{m} - \binom{m+1}{m}\binom{t}{m+1} + \binom{m+2}{m}\binom{t}{m+2} - \cdots + (-1)^{t-m}\binom{t}{m}\binom{t}{t}.$$

It remains to show that the value of λ is equal to 0. Indeed, since

$$\binom{n}{k}\binom{k}{r} = \binom{n}{r}\binom{n-r}{k-r}$$

by identity (2.1.5), it follows that

$$\begin{aligned}\lambda &= \binom{t}{m} - \binom{t}{m}\binom{t-m}{1} + \binom{t}{m}\binom{t-m}{2} - \cdots + (-1)^{t-m}\binom{t}{m}\binom{t-m}{t-m}\\ &= \binom{t}{m}\left\{1 - \binom{t-m}{1} + \binom{t-m}{2} - \cdots + (-1)^{t-m}\binom{t-m}{t-m}\right\},\end{aligned}$$

which is 0 by identity (2.3.2). The proof is thus complete. ∎

As we shall see later, Theorem 4.3.1 is particularly useful when $m = 0$.

Corollary 1. $E(0) = \omega(0) - \omega(1) + \omega(2) - \cdots + (-1)^q\omega(q)$ (4.3.2)

$$= \sum_{k=0}^{q}(-1)^k\omega(k). \quad ∎$$

Corollary 2. *Let $A_1, A_2, ..., A_q$ be any q subsets of a finite set S. Then*

$$\begin{aligned}&|\bar{A}_1 \cap \bar{A}_2 \cap \cdots \cap \bar{A}_q|\\ &= |S| - \sum_{i=1}^{q}|A_i| + \sum_{i<j}|A_i \cap A_j| - \sum_{i<j<k}|A_i \cap A_j \cap A_k| + \cdots\\ &\quad + (-1)^q|A_1 \cap A_2 \cap \cdots \cap A_q|,\end{aligned}$$

where $\bar{A}_i$ denotes the complement of A_i in S (i.e., $\bar{A}_i = S \backslash A_i$).

Proof. For each $i = 1, 2, \ldots, q$, define a property P_i by saying that an element x of S possesses P_i if and only if $x \in A_i$. Then

$$\omega(0) = |S|, \qquad \omega(1) = \sum_{i=1}^{q} |A_i|,$$

$$\omega(2) = \sum_{i<j} |A_i \cap A_j|, \ldots, \omega(q) = \left|\bigcap_{i=1}^{q} A_i\right|,$$

and

$$E(0) = \left|\bigcap_{i=1}^{q} \bar{A}_i\right|.$$

With these, Corollary 2 now follows from Corollary 1. ∎

We leave it to the reader to show that Theorem 4.2.1 now follows from Corollary 2. (see Problem 4.7).

In solving some complicated enumeration problems in which several properties are given, students may make mistakes by 'under-counting' or 'over-counting' in the problems. The significance of applying (PIE) or (GPIE) is this: We split such a problem into some simpler sub-problems, and the principle itself will automatically take care of the under-counting or over-counting. This point will be illustrated in many examples given in the remaining sections.

Historically, Theorem 4.2.1 was discovered by A. de Moivre in 1718 and its corresponding result in probability theory was found by H. Poincaré in 1896. The formulae given in Corollaries 1 and 2 to Theorem 4.3.1 were obtained independently by D.A. da Silva in 1854 and J.J. Sylvester in 1883. The probabilistic form of Theorem 4.3.1 was established by C. Jordan in 1927. For more details about the history and further generalizations of (PIE), the reader may refer to Takács [T].

4.4. Integer Solutions and Shortest Routes

In this section, we give two examples, one on integer solutions of linear equations and the other on shortest routes in rectangular grids, to illustrate how (GPIE) could be applied.

Example 4.4.1. Find the number of nonnegative integer solutions to the equation

$$x_1 + x_2 + x_3 = 15 \tag{1}$$

where $x_1 \le 5$, $x_2 \le 6$ and $x_3 \le 7$.

We learned in Chapter 1 that the number of nonnegative integer solutions to (1) (without any further condition) is given by

$$\binom{15+3-1}{15} = \binom{17}{2}.$$

The upper bounds "$x_1 \le 5$, $x_2 \le 6$, $x_3 \le 7$" imposed in the problem make it not so straightforward. How can we cope with it? Notice that the problem of counting integer solutions of a linear equation is easy if the values of the variables are not bounded above. This then suggests that we may tackle the problem indirectly by considering the "complements" of the given upper bounds, which are "$x_1 \ge 6, x_2 \ge 7, x_3 \ge 8$". In the following solution, we shall see how this idea can be incorporated with (GPIE) to solve the problem.

Solution. Let S be the set of nonnegative integer solutions of (1), namely $S = \{(a_1, a_2, a_3) \mid a_i \in \mathbf{N}^*, a_1 + a_2 + a_3 = 15\}$. We define 3 properties for elements of S. An element (a_1, a_2, a_3) in S is said to possess the property

$$P_1 \Leftrightarrow a_1 \ge 6,$$

$$P_2 \Leftrightarrow a_2 \ge 7,$$

and

$$P_3 \Leftrightarrow a_3 \ge 8.$$

Thus the number of integer solutions to (1) satisfying the requirement is the number of elements of S which possess none of the properties P_1, P_2 and P_3. This number is exactly $E(0)$ in (GPIE).

We now apply Corollary 1 to Theorem 4.3.1 to determine $E(0)$. First of all, we need to find $\omega(i)$, $i = 0, 1, 2, 3$. Observe that

$$\omega(0) = |S| = \binom{15+3-1}{15} = \binom{17}{2};$$

$$\omega(1) = \omega(P_1) + \omega(P_2) + \omega(P_3)$$

$$= \binom{(15-6)+3-1}{2} + \binom{(15-7)+3-1}{2}$$
$$+ \binom{(15-8)+3-1}{2}$$
$$= \binom{11}{2} + \binom{10}{2} + \binom{9}{2};$$
$$\omega(2) = \omega(P_1P_2) + \omega(P_1P_3) + \omega(P_2P_3)$$
$$= \binom{(15-6-7)+3-1}{2} + \binom{(15-6-8)+3-1}{2}$$
$$+ \binom{(15-7-8)+3-1}{2}$$
$$= \binom{4}{2} + \binom{3}{2} + \binom{2}{2};$$

and $\qquad \omega(3) = \omega(P_1P_2P_3) = 0.$

Hence, $E(0) = \omega(0) - \omega(1) + \omega(2) - \omega(3) = \binom{17}{2} - \binom{11}{2} - \binom{10}{2} - \binom{9}{2} + \binom{4}{2} + \binom{3}{2} + \binom{2}{2}$. ∎

Remarks. (1) To apply Theorem 4.3.1, we first have to know what the universal set S is in the problem, and then define "appropriate" properties for elements of S. In the above example, the properties we introduced are "complements" of the requirements: $x_1 \leq 5$, $x_2 \leq 6$, $x_3 \leq 7$ so that the desired integer solutions are the elements of S which possess "none" of the properties. Thus $E(0)$ is our required answer. The remaining task of computing $E(0)$ by formula is just a mechanical one.

(2) Using (GPIE) often enables us to obtain a more direct solution to a problem, as illustrated in the example above. However, this does not imply that the solution is always simple. For instance, we are able to provide a simpler solution to the problem in Example 4.4.1. We first introduce new variables t_1, t_2 and t_3 by putting

$$t_1 = 5 - x_1,$$
$$t_2 = 6 - x_2,$$
and
$$t_3 = 7 - x_3.$$

Then the equation $x_1 + x_2 + x_3 = 15$ becomes $t_1 + t_2 + t_3 = 3$ and the constraints $x_1 \leq 5$, $x_2 \leq 6$ and $x_3 \leq 7$ become $0 \leq t_1 \leq 5$, $0 \leq t_2 \leq 6$ and

$0 \le t_3 \le 7$, which are no constraints at all, since any nonnegative integers t_1, t_2 and t_3 satisfying the equation above do not exceed 3. So the required answer is $\binom{3+2}{2} = \binom{5}{2}$, which is the same as $E(0)$ in the above solution.

Example 4.4.2. Figure 4.4.1 shows a 11 by 6 rectangular grid with 4 specified segments AB, CD, EF and GH. Find the number of shortest routes from O to P in each of the following cases:

(i) All the 4 segments are deleted;

(ii) Each shortest route must pass through exactly 2 of the 4 segments.

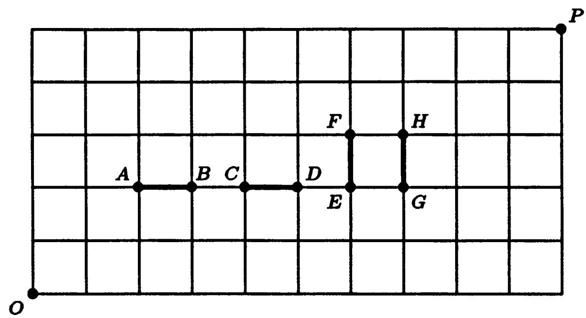


Figure 4.4.1.

We shall apply (GPIE) to solve this problem. First of all, we have to identify the universal set S. Since we are now dealing with shortest routes, it is natural that we take S to be the set of shortest routes from O to P (without any further condition). Note that from what we learned in Chapter 1, $|S| = \binom{10+5}{5} = \binom{15}{5}$. The next task is to define "appropriate" properties for elements of S. What properties should we introduce?

Solution. Let S be the set of shortest routes from O to P in the grid of Figure 4.4.1. We define 4 properties P_i $(i = 1, 2, 3, 4)$ as follows: A shortest route from O to P is said to possess the property

$$
\begin{aligned}
P_1 &\Leftrightarrow \text{ it passes through } AB;\\
P_2 &\Leftrightarrow \text{ it passes through } CD;\\
P_3 &\Leftrightarrow \text{ it passes through } EF;\\
P_4 &\Leftrightarrow \text{ it passes through } GH.
\end{aligned}
$$

(i) ***All the 4 segments are deleted.***

In this case, a desired shortest route is one that does not pass through any of the 4 segments; i.e., one that satisfies none of the 4 properties. Thus, the number of desired shortest routes is given by $E(0)$.

Observe that

$$\begin{aligned}
\omega(0) &= |S| = \binom{15}{5};\\
\omega(1) &= \omega(P_1)+\omega(P_2)+\omega(P_3)+\omega(P_4)\\
&= \binom{4}{2}\binom{10}{3}+\binom{6}{2}\binom{8}{3}+\binom{8}{2}\binom{6}{2}+\binom{9}{2}\binom{5}{2};\\
\omega(2) &= \omega(P_1P_2)+\omega(P_1P_3)+\omega(P_1P_4)+\omega(P_2P_3)+\omega(P_2P_4)+\omega(P_3P_4)\\
&= \binom{4}{2}\binom{8}{3}+\binom{4}{2}\binom{6}{2}+\binom{4}{2}\binom{5}{2}+\binom{6}{2}\binom{6}{2}+\binom{6}{2}\binom{5}{2}+0;\\
\omega(3) &= \omega(P_1P_2P_3)+\omega(P_1P_2P_4)+\omega(P_1P_3P_4)+\omega(P_2P_3P_4)\\
&= \binom{4}{2}\binom{6}{2}+\binom{4}{2}\binom{5}{2}+0+0;
\end{aligned}$$

and $\omega(4) = \omega(P_1P_2P_3P_4) = 0.$

Thus

$$\begin{aligned}
E(0) &= \omega(0) - \omega(1) + \omega(2) - \omega(3) + \omega(4)\\
&= \binom{15}{5} - \binom{4}{2}\binom{10}{3} - \binom{6}{2}\binom{8}{3} - \binom{8}{2}\binom{6}{2} - \binom{9}{2}\binom{5}{2}\\
&\quad + \binom{4}{2}\binom{8}{3} + \binom{4}{2}\binom{6}{2} + \binom{4}{2}\binom{5}{2} + \binom{6}{2}\binom{6}{2} + \binom{6}{2}\binom{5}{2}\\
&\quad - \binom{4}{2}\binom{6}{2} - \binom{4}{2}\binom{5}{2}.
\end{aligned}$$

(ii) ***Each shortest route passes through exactly 2 of the 4 segments.***

The required number of shortest routes is $E(2)$, which is equal to

$$\begin{aligned}
&\omega(2)-\binom{3}{2}\omega(3)+\binom{4}{2}\omega(4)\\
&= \binom{4}{2}\binom{8}{3}+\binom{4}{2}\binom{6}{2}+\binom{4}{2}\binom{5}{2}+\binom{6}{2}\binom{6}{2}+\binom{6}{2}\binom{5}{2}\\
&\quad -\binom{3}{2}\left\{\binom{4}{2}\binom{6}{2}+\binom{4}{2}\binom{5}{2}\right\}. \quad \blacksquare
\end{aligned}$$

4.5. Surjective Mappings and Stirling Numbers of the Second Kind

As shown in Section 2.6, the number of surjective (onto) mappings from $\mathbf{N}_n$ to $\mathbf{N}_m$ (where $n, m \in \mathbf{N}$) is given by $m!S(n,m)$, where $S(n,m)$, the Stirling number of the 2nd kind, is defined as the number of ways of distributing n distinct objects into m identical boxes so that no box is empty. In Section 1.7, we gave the values of $S(n,m)$ in some special cases. Now, we shall apply Corollary 1 to Theorem 4.3.1 to derive a general formula for the number of surjective mappings from $\mathbf{N}_n$ to $\mathbf{N}_m$ which, in turn, gives rise to a formula for $S(n,m)$.

Theorem 4.5.1. *Let $F(n,m)$, $n, m \in \mathbf{N}$, denote the number of surjective mappings from $\mathbf{N}_n$ to $\mathbf{N}_m$. Then*

$$F(n,m) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} (m-k)^n. \tag{4.5.1}$$

Remark. Evidently, $F(n,m)$ can also be regarded as the number of ways of distributing n distinct objects into m distinct boxes so that no box is empty.

Proof. Let S be the set of mappings from $\mathbf{N}_n$ to $\mathbf{N}_m$. We define m properties P_i ($i = 1, 2, ..., m$) for members of S as follows: For each $i = 1, 2, ..., m$, a mapping $f : \mathbf{N}_n \to \mathbf{N}_m$ is said to possess P_i if and only if $i \notin f(\mathbf{N}_n)$ (i.e., the element i of $\mathbf{N}_m$ is not contained in the image of $\mathbf{N}_n$ under f).

It then follows that a mapping $f : \mathbf{N}_n \to \mathbf{N}_m$ is surjective if and only if f possesses none of the properties P_i ($i = 1, 2, ..., m$). We therefore have $F(n,m) = E(0)$.

Observe that

$$\omega(0) = |S| = m^n;$$

$$\omega(1) = \sum_{i=1}^{m} \omega(P_i) = \binom{m}{1}(m-1)^n;$$

and in general, for each k with $0 \le k \le m$,

$$\omega(k) = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le m} \omega(P_{i_1} P_{i_2} \cdots P_{i_k}) = \binom{m}{k}(m-k)^n.$$

Thus, by Corollary 1 to Theorem 4.3.1,

$$\begin{aligned} F(n,m) &= E(0) \\ &= \sum_{k=0}^{m}(-1)^k\omega(k) \\ &= \sum_{k=0}^{m}(-1)^k\binom{m}{k}(m-k)^n, \end{aligned}$$

as desired. ∎

Since $F(n,m) = m!S(n,m)$, we thus have:

Corollary 1. *For $n, m \in \mathbf{N}$,*

$$S(n,m) = \frac{1}{m!}\sum_{k=0}^{m}(-1)^k\binom{m}{k}(m-k)^n. \quad ∎$$

The following results about the Stirling numbers of the 2nd kind were stated in Section 1.7: For $n, m \in \mathbf{N}$,

(1) $S(n,m) = 0$ if $n < m$;

(2) $S(n,n) = 1$;

(3) $S(n,n-1) = \binom{n}{2}$; and

(4) $S(n,n-2) = \binom{n}{3} + 3\binom{n}{4}$.

Combining these with Corollary 1 to Theorem 4.5.1, we obtain some nontrivial identities involving alternating sums.

Corollary 2. *For $n, m \in \mathbf{N}$,*

(1) $\sum_{k=0}^{m}(-1)^k\binom{m}{k}(m-k)^n = 0$ if $n < m$; (4.5.2)

(2) $\sum_{k=0}^{n}(-1)^k\binom{n}{k}(n-k)^n = n!$; (4.5.3)

(3) $\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}(n-1-k)^n = (n-1)!\binom{n}{2}$; (4.5.4)

(4) $\sum_{k=0}^{n-2}(-1)^k\binom{n-2}{k}(n-2-k)^n = (n-2)!\left\{\binom{n}{3} + 3\binom{n}{4}\right\}$. (4.5.5)

Remark. The identities (4.5.2) and (4.5.3) are usually referred to as Euler's formula. For their applications and relations to other combinatorial notions, the reader may read the interesting article [G] by H.W. Gould.

4.6. Derangements and A Generalization

In 1708, the Frenchman Pierre Rémond de Montmort (1678-1719) posed the following problem. Suppose two decks, A, and B, of cards are given. The cards of A are first laid out in a row, and those of B are then placed at random, one at the top on each card of A such that 52 pairs of cards are formed. Find the probability that no 2 cards are the same in each pair. This problem is known as "le problème des rencontres" (in French, 'rencontres' means 'match').

The essential part of the above problem is to find, given the layout of cards of A, the number of ways of placing the cards of B such that no 'match' can occur. This can naturally be generalized as follows: Find the number of permutations $a_1a_2\cdots a_n$ of $\mathbf{N}_n$ such that $a_i \neq i$ for each $i = 1, 2, ..., n$. We call such a permutation a *derangement* (nothing is in its right place) of $\mathbf{N}_n$, and we denote by D_n the number of derangements of $\mathbf{N}_n$. Thus "le problème des rencontres" is essentially the problem of enumerating D_n for $n = 52$. The general problem for arbitrary n was later solved by N. Bernoulli and P.R. Montmort in 1713.

In 1983, Hanson, Seyffarth and Weston [HSW] introduced the following notion, which is a generalization of derangements. For $1 \leq r \leq n$, recall that an r-permutation of $\mathbf{N}_n$ is an arrangement $a_1a_2\cdots a_r$ of r elements of $\mathbf{N}_n$ in a row. An r-permutation $a_1a_2\cdots a_r$ of $\mathbf{N}_n$ is said to have a *fixed point* at i ($i = 1, 2, ..., r$) if $a_i = i$. For $0 \leq k \leq r$, let $D(n, r, k)$ denote the number of r-permutations of $\mathbf{N}_n$ that have exactly k fixed points. Thus, $D_n = D(n, n, 0)$. Our aim here is to find a formula for $D(n, r, k)$ by (GPIE).

Example 4.6.1. The 24 ($= P_3^4$) 3-permutations of $\mathbf{N}_4$ are classified into 4 groups according to the number of fixed points as shown in Table 4.6.1.

Theorem 4.6.1. [HSW] *For integers n, r, k such that $n \geq r \geq k \geq 0$ and $r \geq 1$,*

$$D(n,r,k) = \frac{\binom{r}{k}}{(n-r)!}\sum_{i=0}^{r-k}(-1)^i\binom{r-k}{i}(n-k-i)!\,. \tag{4.6.1}$$

Number of Fixed Points	3-permutations of $\mathbf{N}_4$	$D(4,3,k)$
$k=0$	231, 312, 214, 241, 412, 314, 341, 431, 234, 342, 432	11
1	132, 213, 321, 142, 421, 134, 413, 243, 324	9
2	124, 143, 423	3
3	123	1

Table 4.6.1

Proof. Let S be the set of r-permutations of $\mathbf{N}_n$. We define r properties for elements of S as follows:

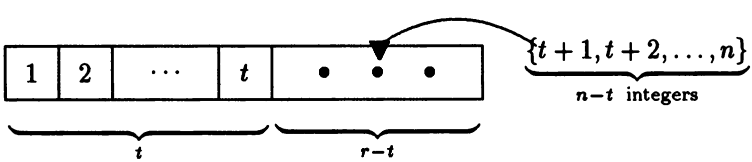
An element $a_1a_2\cdots a_r$ in S is said to possess the property P_i if and only if $a_i = i$, where $i = 1, 2, ..., r$.

It thus follows by definition that

$$D(n, r, k) = E(k).$$

Observe that for $0 \le t \le r$,

$$\omega(P_1P_2\cdots P_t) = \binom{n-t}{r-t}\cdot(r-t)! = \frac{(n-t)!}{(n-r)!}.$$



Likewise, $\omega(P_{i_1}P_{i_2}\cdots P_{i_t}) = \omega(P_1P_2\cdots P_t) = \frac{(n-t)!}{(n-r)!}$, for any t-element subset $\{i_1, i_2, ..., i_t\}$ of $\{1, 2, ..., r\}$. Thus,

$$\omega(t) = \sum_{1\le i_1<i_2<\cdots<i_t\le r} \omega(P_{i_1}P_{i_2}\cdots P_{i_t}) = \binom{r}{t}\frac{(n-t)!}{(n-r)!}.$$

By (GPIE),

$$\begin{aligned}
D(n,r,k) = E(k) &= \sum_{i=0}^{r-k}(-1)^i\binom{k+i}{k}\omega(k+i)\\
&= \sum_{i=0}^{r-k}(-1)^i\binom{k+i}{k}\binom{r}{k+i}\frac{(n-k-i)!}{(n-r)!}\\
&= \frac{1}{(n-r)!}\sum_{i=0}^{r-k}(-1)^i\binom{r}{k}\binom{r-k}{k+i-k}(n-k-i)!\\
&= \frac{\binom{r}{k}}{(n-r)!}\sum_{i=0}^{r-k}(-1)^i\binom{r-k}{i}(n-k-i)!,
\end{aligned}$$

as desired. ∎

Some interesting identities involving the $D(n,r,k)$'s, that can be found in [HSW], are listed below.

(1) $D(n,r,k) = \binom{r}{k}D(n-k,r-k,0)$; (4.6.2)

(2) $D(n,r,k) = D(n-1,r-1,k-1) + (n-1)D(n-1,r-1,k)$
$+(r-1)\{D(n-2,r-2,k) - D(n-2,r-2,k-1)\}$,
where $D(n,r,-1)$ is defined to be 0; (4.6.3)

(3) $D(n,n,k) = nD(n-1,n-1,k) + (-1)^{n-k}\binom{n}{k}$; (4.6.4)

(4) $\binom{k}{t}D(n,r,k) = \binom{r}{t}D(n-t,r-t,k-t)$, $t \geq 0$; (4.6.5)

(5) $D(n,r,k) = rD(n-1,r-1,k) + D(n-1,r,k)$,
where $r < n$; (4.6.6)

(6) $D(n,n-r,0) = \sum_{i=0}^{r}\binom{r}{i}D(n-i,n-i,0)$. (4.6.7)

Since $D_n = D(n,n,0)$, by Theorem 4.6.1, we have

$$\begin{aligned}
D_n &= \binom{n}{0}\sum_{i=0}^{n}(-1)^i\binom{n}{i}(n-i)!\\
&= \sum_{i=0}^{n}(-1)^i\frac{n!}{i!(n-i)!}(n-i)! = n!\sum_{i=0}^{n}\frac{(-1)^i}{i!}.
\end{aligned}$$

Corollary. *For any $n \in \mathbf{N}$,*

(i) $D_n = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right)$; (4.6.8)

(ii) $\lim_{n\to\infty} \frac{D_n}{n!} = e^{-1} \simeq 0.367$. ∎ (4.6.9)

We state below two useful identities involving D_n that are, respectively, special cases of identities (4.6.3) and (4.6.4).

(7) $D_n = (n-1)(D_{n-1} + D_{n-2})$; (4.6.10)

(8) $D_n = nD_{n-1} + (-1)^n$; (4.6.11)

The reader may refer to the article by Karl [Kr] for different types of generalization of derangements.

For reference, the first 10 values of D_n are given in Table 4.6.2.

n	1	2	3	4	5	6	7	8	9	10
D_n	0	1	2	9	44	265	1854	14833	133496	1334961

Table 4.6.2.

4.7. The Sieve of Eratosthenes and Euler φ-function

In this section, we present two classical problems in number theory that can be solved by (GPIE).

A number $n \geq 2$ is said to be *composite* if n is not a prime. The number "1" is neither a prime nor a composite number. Just like "bricks" which can be combined together to build a "wall", primes can be combined to form any natural number greater than 1 by multiplication. This can be seen in the following result which is so important to the study of integers that it is called the *Fundamental Theorem of Arithmetic*:

For every $n \in \mathbf{N}$, $n > 1$, there exist primes

$$p_1 < p_2 < \cdots < p_k$$

and positive integers $m_1, m_2, ..., m_k$ such that

$$n = p_1^{m_1} p_2^{m_2} \cdots p_k^{m_k} = \prod_{i=1}^{k} p_i^{m_i};$$

and such a factorization is unique if we disregard the order of primes.

The first example that we shall discuss here is concerned with the counting of primes. Given $n \in \mathbf{N}$, $n \geq 2$, how many primes are there between 2 and n inclusive? We shall solve this problem by applying Corollary 1 to Theorem 4.3.1. But first of all, we need to introduce a special device for distinguishing primes from composite numbers. This device, which was discovered by the Greek mathematician Eratosthenes (276-194 B.C.) who lived in Alexandra around 2000 years ago, is known as the *Sieve of Eratosthenes.*

The Sieve of Eratosthenes

Write down the numbers $2, 3, ..., n$ in order. Keep the first prime "2" and cross off all other multiples of 2. Keep the first of the remaining integers greater than "2" (i.e., the prime "3") and cross off all other multiples of "3" that remain. Keep the first of the remaining integers greater then "3" (i.e., the prime "5") and cross off all other multiples of "5" that remain. This procedure is repeated until the first of the currently remaining integers is greater than $\sqrt{n}$. The numbers on the list that are not crossed off are the primes between 1 and n.

$$\begin{array}{cccccccccccc}
\boxed{2} & \boxed{3} & \cancel{4} & \boxed{5} & \cancel{6} & \boxed{7} & \cancel{8} & \cancel{9} & \cancel{10} & 11 & \cancel{12} & 13 \\
\cancel{14} & \cancel{15} & \cancel{16} & 17 & \cancel{18} & 19 & \cancel{20} & \cancel{21} & \cancel{22} & 23 & \cancel{24} & \cancel{25} \\
\cancel{26} & \cancel{27} & \cancel{28} & 29 & \cancel{30} & 31 & \cancel{32} & \cancel{33} & \cancel{34} & \cancel{35} & \cancel{36} & 37 \\
\cancel{38} & \cancel{39} & \cancel{40} & 41 & \cancel{42} & 43 & \cancel{44} & \cancel{45} & \cancel{46} & 47 & \cancel{48} &
\end{array}$$

Example 4.7.1. To find the primes between 2 and 48 inclusive by the Sieve of Eratosthenes, we first write down the number 2, 3, ..., 48 in order:

We then keep 2 and cross off all multiples of 2 (i.e., 4, 6, 8, ..., 48). The first of the remaining integers (i.e., "3") is kept and all multiples of 3 that remain (i.e., 9, 15, 21, 27, 33, 39, 45) are crossed off. We then keep "5", the first remaining integer and cross off the multiples of 5 that remain (i.e., 25, 35). The procedure terminates now, since the first remaining integer is "7", which is greater than $\sqrt{48}$. The numbers that are not crossed off are:

$$2,\ 3,\ 5,\ 7,\ 11,\ 13,\ 17,\ 19,\ 23,\ 29,\ 31,\ 37,\ 41,\ 43,\ 47$$

which are the required primes. Note that there are 15 primes altogether. ∎

Remark. In the above procedure, we actually cross off the multiples of the primes not exceeding $\sqrt{n}$ (i.e., 2, 3, 5,) except the primes themselves. The reason we do not have to proceed beyond $\sqrt{n}$ is this: If a number, say k, $2 \le k \le n$, is a multiple of a prime p, where $p > \sqrt{n}$, then k must be a multiple of a smaller prime p', where $p' \le \sqrt{n}$, and so k has already been crossed off.

Using the idea behind the above procedure, we now illustrate by an example how (GPIE) can be used to compute the number of primes from 1 to n, for a given $n \in \mathbf{N}$.

Example 4.7.2. Find the number of primes between 1 and 48 inclusive.

Solution. Let $S = \{1, 2, ..., 48\}$. There are 3 primes not exceeding $\sqrt{48}$, namely, 2, 3 and 5. We define 3 corresponding properties P_1, P_2, P_3 as follows: A number $x \in S$ is said to possess property

$$\begin{aligned}
P_1 &\Leftrightarrow 2|x; \\
P_2 &\Leftrightarrow 3|x; \\
P_3 &\Leftrightarrow 5|x.
\end{aligned}$$

It follows from the sieve that the desired number of primes is equal to

$$E(0) \ + \ 3 \ - \ 1,$$

because the 3 primes "2, 3, 5" not counted in $E(0)$ must be included, whereas, the number "1" counted in $E(0)$ must be excluded.
Observe that

$$\begin{aligned}
\omega(0) &= |S| = 48; \\
\omega(1) &= \omega(P_1) + \omega(P_2) + \omega(P_3) \\
&= \left\lfloor \frac{48}{2} \right\rfloor + \left\lfloor \frac{48}{3} \right\rfloor + \left\lfloor \frac{48}{5} \right\rfloor = 24 + 16 + 9 = 49; \\
\omega(2) &= \omega(P_1P_2) + \omega(P_1P_3) + \omega(P_2P_3) \\
&= \left\lfloor \frac{48}{6} \right\rfloor + \left\lfloor \frac{48}{10} \right\rfloor + \left\lfloor \frac{48}{15} \right\rfloor = 8 + 4 + 3 = 15;
\end{aligned}$$

and
$$\omega(3) = \omega(P_1P_2P_3) = \left\lfloor \frac{48}{30} \right\rfloor = 1.$$

Thus $E(0) = \omega(0) - \omega(1) + \omega(2) - \omega(3) = 48 - 49 + 15 - 1 = 13$ and the desired number of primes is $E(0) + 3 - 1 = 15$. ∎

We shall now discuss our second example. For $a, b \in \mathbf{N}$, let (a, b) denote the HCF of a and b. Thus $(8, 15) = 1$ while $(9, 15) = 3$. We say that a is *coprime* to b (and vice versa) if $(a, b) = 1$. Around 1760, in his attempt to generalize a result of Fermat's in number theory, the Swiss mathematician Leonard Euler (1707-1783) introduced the following notion. For $n \in \mathbf{N}$, let $\varphi(n)$ denote the number of integers between 1 and n which are coprime to n. Table 4.7.1 shows those integers x, $1 \leq x \leq n$, which are coprime to n, and the values of $\varphi(n)$ for $n \leq 15$. The function $\varphi(n)$, now known as the *Euler φ-function*, plays a significant role in many enumeration problems in number theory and modern algebra. As seen in Table 4.7.1, the values of $\varphi(n)$ are rather irregularly distributed except when n is a prime. Mathematicians had been interested in finding a general formula of $\varphi(n)$, and it really took some time before the following result was established.

n	integers x such that $1 \le x \le n$ and $(x,n)=1$	$\varphi(n)$
1	1	1
2	1	1
3	1,2	2
4	1,3	2
5	1,2,3,4	4
6	1,5	2
7	1,2,3,4,5,6	6
8	1,3,5,7	4
9	1,2,4,5,7,8	6
10	1,3,7,9	4
11	1,2,3,4,5,6,7,8,9,10	10
12	1,5,7,11	4
13	1,2,3,4,5,6,7,8,9,10,11,12	12
14	1,3,5,9,11,13	6
15	1,2,4,7,8,11,13,14	8

Table 4.7.1.

Example 4.7.3. Let $n \in \mathbf{N}$, and let

$$n = p_1^{m_1} p_2^{m_2} \cdots p_k^{m_k}$$

be its prime factorization as stated in the fundamental theorem of arithmetic. Show that

$$\varphi(n) = n\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\cdots\left(1-\frac{1}{p_k}\right) = n\prod_{i=1}^{k}\left(1-\frac{1}{p_i}\right). \quad (4.7.1)$$

The following two observations are useful in the proof below.

(i) Let $x \in \mathbf{N}$ such that $x \le n$, where n is given in Example 4.7.3. Then $(x,n)=1$ if and only if $p_i \not| x$ for all $i = 1, 2, ..., k$.

(ii) For real numbers $r_1, r_2, ..., r_k$,

$$(1-r_1)(1-r_2)\cdots(1-r_k) = 1 - \sum_{i=1}^{k} r_i + \sum_{i<j} r_i r_j - \sum_{i<j<\ell} r_i r_j r_\ell + \cdots + (-1)^k r_1 r_2 \cdots r_k.$$

Proof of (4.7.1). Let $S = \{1, 2, ..., n\}$. Corresponding to the k primes $p_1, p_2, ..., p_k$ in the factorization of n, we define k properties $P_1, P_2, ..., P_k$ as follows: An element $x \in S$ is said to possess

$$P_i \Leftrightarrow p_i | x, \quad \text{where } i = 1, 2, ..., k.$$

It follows from the observation (i) above that $x \in S$ is coprime to n if and only if x possesses none of the properties $P_1, P_2, ..., P_k$. Consequently, we have

$$\varphi(n) = E(0).$$

Observe that $\omega(0) = |S| = n$; and for $1 \leq t \leq k - 1$,

$$\begin{aligned}\omega(t) &= \sum_{i_1<i_2<\cdots<i_t} \omega(P_{i_1} P_{i_2} \cdots P_{i_t}) = \sum_{i_1<i_2<\cdots<i_t} \left\lfloor \frac{n}{p_{i_1} p_{i_2} \cdots p_{i_t}} \right\rfloor \\ &= \sum_{i_1<i_2<\cdots<i_t} \frac{n}{p_{i_1} p_{i_2} \cdots p_{i_t}}\end{aligned}$$

and $\omega(k) = \left\lfloor \dfrac{n}{p_1 p_2 \cdots p_k} \right\rfloor = \dfrac{n}{p_1 p_2 \cdots p_k}.$

Hence

$$\begin{aligned}\varphi(n) &= E(0) \\ &= n - \sum_{i=1}^{k} \frac{n}{p_i} + \sum_{i<j} \frac{n}{p_i p_j} - \sum_{i<j<l} \frac{n}{p_i p_j p_l} + \cdots + (-1)^k \frac{n}{p_1 p_2 \cdots p_k} \\ &= n \left(1 - \sum \frac{1}{p_i} + \sum \frac{1}{p_i p_j} - \sum \frac{1}{p_i p_j p_l} + \cdots + (-1)^k \frac{1}{p_1 p_2 \cdots p_k}\right) \\ &= n \left(1 - \frac{1}{p_1}\right) \left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_k}\right) \quad \text{(by observation (ii))} \\ &= n \prod_{i=1}^{k} \left(1 - \frac{1}{p_i}\right). \quad \blacksquare\end{aligned}$$

It is noted that from expression (4.7.1), $\varphi(n)/n$ is independent of the powers m_i's of the primes in the factorization of n. For some interesting properties and generalizations of $\varphi(n)$, and historical remarks on the above result, the reader may read [D, p113-158]. To end this section, we state the following beautiful identity involving Euler φ-function, due to Smith (1875):

$$\begin{vmatrix} (1,1) & (1,2) & \cdots & (1,n) \\ (2,1) & (2,2) & \cdots & (2,n) \\ \vdots & \vdots & \ddots & \vdots \\ (n,1) & (n,2) & \cdots & (n,n) \end{vmatrix} = \varphi(1)\varphi(2)\cdots\varphi(n),$$

where (a,b) is the HCF of a and b.

4.8. The 'Probléme des Ménages'

At the end of Section 1.3, we stated the following problem, known as the probléme des ménages (in French, 'ménages' means 'married couples'):

> *How many ways are there to seat n married couples, $n \geq 3$, around a table such that men and women alternate and each woman is not adjacent to her husband?*

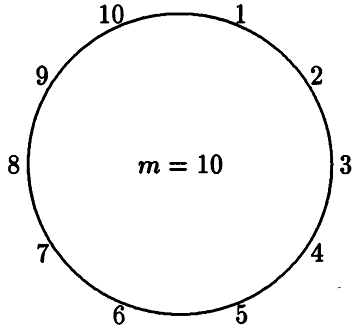
This famous problem was raised and popularized by E. Lucas in his book [L] published in 1891. In fact, an equivalent problem was first posed by P.G. Tait much earlier in 1876 and was settled by A. Cayley and T. Muir independently in 1877.

In this section, we shall apply Theorem 4.3.1 to solve the above problem in a more general way. Before doing this, let us first study a problem due to I. Kaplansky [Kp].

Example 4.8.1. Suppose that the numbers $1, 2, ..., m$ $(m \geq 3)$ are placed in order around a circle as shown below. For $0 \leq k \leq \lfloor \frac{m}{2} \rfloor$, let $\alpha(k)$ denote the number of k-element subsets of $\mathbf{N}_m$ in which no two elements are adjacent around the table. Show that

$$\alpha(k) = \frac{m}{k}\binom{m-k-1}{k-1}. \tag{4.8.1}$$

For instance, if $m = 10$ and $k = 4$, then $\{1,3,6,9\}$ and $\{3,5,8,10\}$ are such subsets while $\{1,6,8,10\}$ and $\{2,6,7,9\}$ are not. Note that if $\lfloor \frac{m}{2} \rfloor < k \leq m$, no such k-element subsets can exist. When $k = 3$, Example 4.8.1 is the same as Problem 1.34. If the term "circle" is replaced by "row" in Example 4.8.1, the problem is identical to that in Example 1.5.3.



Proof. For each $i = 1, 2, ..., m$, let α_i denote the number of such k-element subsets of $\mathbf{N}_m$ which contain "i". By symmetry, $\alpha_1 = \alpha_2 = \cdots = \alpha_m$. We now count α_1.

If B is such a k-element subset of $\mathbf{N}_m$ containing "1", then by the hypothesis, $2, m \notin B$, and thus the remaining $k - 1$ elements of B must be chosen from $\{3, 4, ..., m - 1\}$ such that no two are adjacent (in a row). Hence, by the result of Example 1.5.3,

$$\alpha_1 = \binom{(m-3)-(k-1)+1}{k-1} = \binom{m-k-1}{k-1}$$

and so $\sum_{i=1}^{m} \alpha_i = m\binom{m-k-1}{k-1}$.

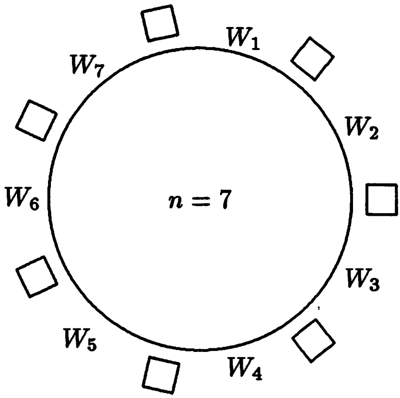
Since $\sum_{i=1}^{m} \alpha_i = k \cdot \alpha(k)$ (why?), it follows that

$$\alpha(k) = \frac{1}{k}\sum_{i=1}^{m} \alpha_i = \frac{m}{k}\binom{m-k-1}{k-1}. \quad \blacksquare$$

We are now in a position to establish the following result.

Example 4.8.2. There are n married couples ($n \geq 3$) to be seated in the $2n$ chairs around a table. Suppose that the n wives have already been seated such that there is one and only one empty chair between two adjacent wives as shown below. Let $M(n, r)$, $0 \leq r \leq n$, denote the number of ways to seat the n husbands in the remaining chairs such that exactly r husbands are adjacent to their own wives. Show that

$$M(n,r) = \sum_{k=r}^{n} (-1)^{k-r}\binom{k}{r}\frac{2n}{2n-k}\binom{2n-k}{k}(n-k)! \qquad (4.8.2)$$



Proof. Let S be the set of all arrangements of the n husbands $H_1, H_2, ..., H_n$. We define $2n$ properties $P_1, P_2, ..., P_{2n}$ as follows: An arrangement in S is said to possess the property:

$$\begin{array}{lcl}
P_1 & \Leftrightarrow & H_1 \text{ sits to the right of his wife } W_1; \\
P_2 & \Leftrightarrow & H_1 \text{ sits to the left of his wife } W_1; \\
P_3 & \Leftrightarrow & H_2 \text{ sits to the right of his wife } W_2; \\
P_4 & \Leftrightarrow & H_2 \text{ sits to the left of his wife } W_2; \\
 & \vdots & \\
P_{2n-1} & \Leftrightarrow & H_n \text{ sits to the right of his wife } W_n; \\
P_{2n} & \Leftrightarrow & H_n \text{ sits to the left of his wife } W_n.
\end{array}$$

It is important to note that P_i and P_{i+1} cannot hold at the same time, for each $i = 1, 2, ..., 2n$, where P_{2n+1} is defined as P_1. Thus

$$\omega(P_i P_{i+1}) = 0 \quad \text{for each } i = 1, 2, .., 2n. \tag{1}$$

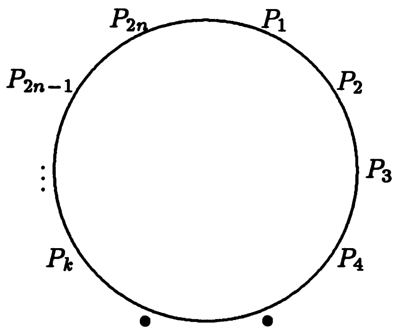
If we arrange the $2n$ properties in order around a circle as shown below, then by (1)

$$\omega(P_{i_1} P_{i_2} \cdots P_{i_k}) = 0,$$

if the k-element subset $\{P_{i_1}, P_{i_2}, \ldots, P_{i_k}\}$ contains 2 adjacent members around the circle.

This implies, in particular, that

(i) For $n < k \le 2n$, $\omega(P_{i_1} P_{i_2} \cdots P_{i_k}) = 0$ (and so $\omega(k) = 0$); (2)



(ii) For $1 \le k \le n$,

$$\begin{aligned}\omega(k) &= \sum_{1 \le i_1 < i_2 < \cdots < i_k \le 2n} \omega(P_{i_1} P_{i_2} \cdots P_{i_k}) \\ &= \underbrace{\frac{2n}{k}\binom{2n-k-1}{k-1}}_{\boxed{\text{By (4.8.1)}}} \bullet \underbrace{(n-k)!}_{\boxed{\text{number of ways to seat the remaining } n-k \text{ husbands}}} \qquad (3)\end{aligned}$$

Finally, we have

$$\begin{aligned}M(n,r) &= E(r) \\ &= \sum_{k=r}^{2n-r} (-1)^{k-r}\binom{k}{r}\omega(k) \quad \text{(by GPIE)} \\ &= \sum_{k=r}^{n} (-1)^{k-r}\binom{k}{r}\frac{2n}{k}\binom{2n-k-1}{k-1}(n-k)! \quad \text{(by (2) and (3))} \\ &= \sum_{k=r}^{n} (-1)^{k-r}\binom{k}{r}\frac{2n}{2n-k}\binom{2n-k}{k}(n-k)!,\end{aligned}$$

as required. ∎

The case when $r = 0$ gives the solution to the 'problème des ménages'. The formula for this special case, i.e.,

$$M(n,0) = \sum_{k=0}^{n} (-1)^k \frac{2n}{2n-k}\binom{2n-k}{k}(n-k)! \qquad (4.8.3)$$

was discovered by J. Touchard in 1934. The idea used in the above proof is due to I. Kaplansky [Kp]. For a different proof of formula (4.8.3) (but still based on (PIE)), the reader may refer to Bogart and Doyle [BD].

For simplicity, let $M_n = M(n,0)$. Two interesting identities involving M_n's are given below:

$$(n-2)M_n = n(n-2)M_{n-1} + nM_{n-2} + 4(-1)^{n+1} \tag{4.8.4}$$

$$\sum_{k=0}^{n} \binom{2n}{k} M_{n-k} = n!, \quad \text{where } M_0 = 1 \text{ and } M_1 = -1. \tag{4.8.5}$$

As pointed out in [Kp], the following limit follows from identity (4.8.3):

$$\lim_{n\to\infty} \frac{M_n}{n!} = e^{-2}. \tag{4.8.6}$$

To end this chapter, we give in Table 4.8.1 the values of M_n for $2 \le n \le 10$.

n	2	3	4	5	6	7	8	9	10
M_n	0	1	2	13	80	579	4738	43387	439792

Table 4.8.1

Exercise 4

1. A group of 100 students took examinations in Chinese, English and Mathematics. Among them, 92 passed Chinese, 75 English and 63 Mathematics; at most 65 passed Chinese and English, at most 54 Chinese and Mathematics, and at most 48 English and Mathematics. Find the largest possible number of the students that could have passed all the three subjects.

2. (a) Let A, B and C be finite sets. Show that

   (i) $|\bar{A} \cap B| = |B| - |A \cap B|$;

   (ii) $|\bar{A} \cap \bar{B} \cap C| = |C| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$.

   (b) Find the number of integers in the set $\{1, 2, \ldots, 10^3\}$ which are not divisible by 5 nor by 7 but are divisible by 3.

3. Find the number of integers in the set $\{1, 2, \ldots, 120\}$ which are divisible by exactly 'm' of the integers: 2, 3, 5, 7, where $m = 0, 1, 2, 3, 4$. Find also the number of primes which do not exceed 120.

4. How many positive integers n are there such that n is a divisor of at least one of the numbers 10^{40}, 20^{30}? (Putnum 1983)

5. Find the number of positive divisors of at least one of the numbers: 10^{60}, 20^{50}, 30^{40}.

6. Find the number of integers in each of the following sets which are not of the form n^2 or n^3, where $n \in \mathbf{N}$:
   (i) $\{1, 2, \ldots, 10^4\}$,
   (ii) $\{10^3, 10^3 + 1, \ldots, 10^4\}$.

7. Prove Theorem 4.2.1 by
   (a) induction on q;
   (b) Corollary 2 to Theorem 4.3.1.

8. A year is a *leap* year if it is either (i) a multiple of 4 but not a multiple of 100, or (ii) a multiple of 400. For example, 1600 and 1924 were leap years while 2200 will not be. Find the number of leap years between 1000 and 3000 inclusive.

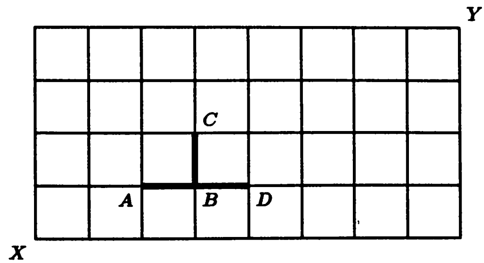
9. Each of n boys attends a school gathering with both of his parents. In how many ways can the $3n$ people be divided into groups of three such that each group contains a boy, a male parent and a female parent, and no boy is with both of his parents in his group?

10. A man has 6 friends. At dinner in a certain restaurant, he has met each of them 12 times, every two of them 6 times, every three of them 4 times, every four of them 3 times, every five twice and all six only once. He has dined out 8 times without meeting any of them. How many times has he dined out altogether?

11. Three identical black balls, four identical red balls and five identical white balls are to be arranged in a row. Find the number of ways that this can be done if all the balls with the same colour do not form a single block.

12. How many arrangements of $a, a, a, b, b, b, c, c, c$ are there such that
    (i) no three consecutive letters are the same?
    (ii) no two consecutive letters are the same?

13. Find the number of shortest routes from corner X to corner Y in the following rectangular grid if the segments AB, BC and BD are all deleted.



14. Find the number of integer solutions to the equation

$$x_1 + x_2 + x_3 = 28$$

where $3 \le x_1 \le 9$, $0 \le x_2 \le 8$ and $7 \le x_3 \le 17$.

15. Find the number of integer solutions to the equation

$$x_1 + x_2 + x_3 = 40$$

where $6 \le x_1 \le 15$, $5 \le x_2 \le 20$ and $10 \le x_3 \le 25$.

16. Find the number of integer solutions to the equation

$$x_1 + x_2 + x_3 + x_4 = 20$$

where $1 \le x_1 \le 5, 0 \le x_2 \le 7, 4 \le x_3 \le 8$ and $2 \le x_4 \le 6$.

17. Let $k, n, r \in \mathbf{N}$. Show that the number of integer solutions to the equation

$$x_1 + x_2 + \cdots + x_n = r$$

such that $0 \le x_i \le k$ for each $i = 1, 2, \ldots, n$ is given by

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} \binom{r - (k+1)i + n - 1}{n - 1}.$$

18. Let $k, n, r \in \mathbf{N}$. Show that the number of integer solutions to the equation

$$x_1 + x_2 + \cdots + x_n = r$$

such that $1 \le x_i \le k$ for each $i = 1, 2, \ldots, n$ is given by

$$\sum_{i=0}^{n}(-1)^i\binom{n}{i}\binom{r-ki-1}{n-1}.$$

19. Find the number of ways of arranging n couples $\{H_i, W_i\}$, $i = 1, 2, \ldots, n$, in a row such that H_i is not adjacent to W_i for each $i = 1, 2, \ldots, n$.

20. Let $p, q \in \mathbf{N}$ with p odd. There are pq beads of q different colours: $1, 2, \ldots, q$ with exactly p beads in each colour. Assuming that beads of the same colour are identical, in how many ways can these beads be put in a string in such a way that
    (i) beads of the same colour must be in a single block?
    (ii) beads of the same colour must be in two separated blocks?
    (iii) beads of the same colour must be in at most two blocks?
    (iv) beads of the same colour must be in at most two blocks and the size of each block must be at least 2?

21. (a) Find the number of ways of distributing r identical objects into n distinct boxes such that no box is empty, where $r \ge n$.
    (b) Show that

$$\sum_{i=0}^{n-1}(-1)^i\binom{n}{i}\binom{r+n-i-1}{r} = \binom{r-1}{n-1},$$

where $r, n \in \mathbf{N}$ with $r \ge n$.

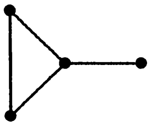
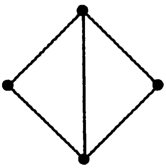
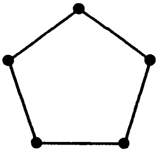
22. (a) Let B be a subset of A with $|A| = n$ and $|B| = m$. Find the number of r-element subsets of A which contain B as a subset, where $m \le r \le n$.
    (b) Show that for $m, r, n \in \mathbf{N}$ with $m \le r \le n$,

$$\binom{n-m}{n-r} = \sum_{i=0}^{m}(-1)^i\binom{m}{i}\binom{n-i}{r}.$$

23. (a) For $n \in \mathbf{N}$, find the number of binary sequences of length n which do not contain '01' as a block.
    (b) Show that

$$n+1 = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor}(-1)^i\binom{n-i}{i}2^{n-2i}.$$

24. In each of the following configurations, each vertex is to be coloured by one of the λ different colours. It how many ways can this be done if any two vertices which are joined by a line segment must be coloured by different colours?

(i)  (ii)  (iii) 

25. n persons are to be allocated to q distinct rooms. Find the number of ways that this can be done if only m of the q rooms have exactly k persons each, where $1 \leq m \leq q$ and $mk \leq n$.

26. Suppose that $A = \{k \cdot x_1, k \cdot x_2, \ldots, k \cdot x_n\}$ is a multiset, where $k, n \in \mathbf{N}$. For $m \in \mathbf{N}^*$ with $m \leq n$, let $\alpha(m)$ denote the number of ways to arrange the members of A in a row such that the number of blocks containing all the k elements of the same type in the arrangement is exactly m. Show that

$$\alpha(m) = \frac{(-1)^m}{(k!)^n}\binom{n}{m}\sum_{i=m}^{n}(-1)^i\binom{n-m}{i-m}(k!)^i \cdot \{kn - i(k-1)\}!$$

27. Prove identities (4.6.2)–(4.6.7).
([HSW]; for (4.6.7), see E. T. H. Wang, E2947, *Amer. Math. Monthly*, **89** (1982), 334.)

28. For $n \in \mathbf{N}$, let C_n denote the number of permutations of the set $\{1, 2, \ldots, n\}$ in which k is never followed immediately by $k+1$ for each $k = 1, 2, \ldots, n-1$.
(i) Find C_n;
(ii) Show that $C_n = D_n + D_{n-1}$ for each $n \in \mathbf{N}$.

29. Let $m, n \in \mathbf{N}$ with $m < n$. Find, in terms of D_k's, the number of derangements $a_1 a_2 \ldots a_n$ of $\mathbf{N}_n$ such that

$$\{a_1, a_2, \ldots, a_m\} = \{1, 2, \ldots, m\}.$$

30. Let $m, n \in \mathbf{N}$ with $n \geq 2m$. Find the number of derangements $a_1 a_2 \ldots a_n$ of $\mathbf{N}_n$ such that

$$\{a_1, a_2, \ldots, a_m\} = \{m+1, m+2, \ldots, 2m\}$$

in each of the following cases:

(i) $n = 2m$;

(ii) $n = 2m + 1$;

(iii) $n = 2m + r$, $r \geq 2$.

31. Apply identity (4.6.8) to prove identities (4.6.10) and (4.6.11).

32. Given $n \in \mathbf{N}$, show that D_n is even iff n is odd.

33. Let $D_n(k) = D(n, n, k)$. Show that

(i) $D_n(k) = \binom{n}{k} D_{n-k}$;

(ii) $\binom{n}{1} D_1 + \binom{n}{2} D_2 + \cdots + \binom{n}{n} D_n = n!$;

(iii) $(k+1)D_{n+1}(k+1) = (n+1)D_n(k)$.

34. Let $D_n(k)$ be the number of permutations of the set $\{1, 2, \ldots, n\}, n \geq 1$, which have exactly k fixed points (i.e., $D_n(k) = D(n, n, k)$). Prove that

$$\sum_{k=0}^{n} k \cdot D_n(k) = n!\,.$$

(IMO, 1987/1)

35. Let $D_n(k)$ denote $D(n, n, k)$. Show that

$$D_n(0) - D_n(1) = (-1)^n$$

for each $n \in \mathbf{N}$.

36. Let $D_n(k)$ denote $D(n, n, k)$. Prove that

$$\sum_{k=0}^{n} (k-1)^2 D_n(k) = n!.$$

(West Germany MO, 1987)

37. Let $D_n(k)$ denote $D(n, n, k)$. Prove that

$$\sum_{k=r}^{n} k(k-1)\cdots(k-r+1)D_n(k) = n!,$$

where $r, n \in \mathbf{N}^*$ with $r \leq n$. (D. Hanson, *Crux Mathematicorum*, **15**(5) (1989), 139.)

38. (a) Without using equality (4.7.1), show that

    (i) the Euler φ-function is a multiplicative function; that is, $\varphi(mn) = \varphi(m)\varphi(n)$ whenever $m, n \in \mathbf{N}$ with $(m, n) = 1$.

    (ii) for a prime p and an integer $i \geq 1$,

$$\varphi(p^i) = p^i - p^{i-1}.$$

    (b) Derive equality (4.7.1) from (i) and (ii).

39. (i) Compute $\varphi(100)$ and $\varphi(300)$.

    (ii) Show that $\varphi(m)\big|\varphi(n)$ whenever $m\big|n$.

40. Show that for $n \in \mathbf{N}$,

$$\sum(\varphi(d) \mid d \in \mathbf{N}, d|n) = n.$$

41. Let $m, n \in \mathbf{N}$ with $(m, n) = h$. Show by using equality (4.7.1) that

$$\varphi(mn) \cdot \varphi(h) = \varphi(m) \cdot \varphi(n) \cdot h.$$

42. Show that for $n \in \mathbf{N}$ with $n \geq 3$, $\varphi(n)$ is always even.

43. Let $n \in \mathbf{N}$ with $n \geq 2$. Show that if n has exactly k distinct prime factors, then

$$\varphi(n) \geq n \cdot 2^{-k}.$$

44. Let $n \in \mathbf{N}$ with $n \geq 2$. Show that if n has exactly k distinct odd prime factors, then

$$2^k\big|\varphi(n).$$

45. Does there exist an $n \in \mathbf{N}$ such that $\varphi(n) = 14$? Justify your answer.

46. For $n \in \mathbf{N}$, show that

$$\varphi(2n) = \begin{cases} \varphi(n) & \text{if } n \text{ is odd} \\ 2\varphi(n) & \text{if } n \text{ is even.} \end{cases}$$

47. For $m, r, q \in \mathbf{N}$ with $m \leq r \leq q$, let

$$A(m, r) = \sum_{k=m}^{r} (-1)^{k-m} \binom{k}{m} \omega(k).$$

Thus Theorem 4.3.1 says that $E(m) = A(m, q)$. Prove that

(i) if m and r have the same parity (i.e., $m \equiv r \pmod 2$), then

$$E(m) \leq A(m, r);$$

(ii) if m and r have different parities, then

$$E(m) \geq A(m, r);$$

(iii) strict inequality in (i) (resp., (ii)) holds iff $\omega(t) > 0$ for some t with $r < t \leq q$.

(See K. M. Koh, Inequalities associated with the principle of inclusion and exclusion, *Mathematical Medley*, Singapore Math. Soc. **19** (1991), 43-52.)

48. Prove the following Bonferroni inequality:

$$\sum_{k=j}^{q} (-1)^{k-j} \omega(k) \geq 0$$

for each $j = 0, 1, \ldots, q$.

49. (i) Let $A_1, A_2, \ldots, A_n$ be n finite sets. Show that

$$\left| \bigcup_{k=1}^{n} A_k \right| \geq \sum_{k=1}^{n} |A_k| - \sum_{1 \leq i < j \leq n} |A_i \cap A_j|.$$

(ii) Apply (i) to prove the following (see Example 1.5.4): A permutation of n couples $\{H_1, W_1, H_2, W_2, \ldots, H_n, W_n\}$ $(n \geq 1)$ in a row is said to have property P if at least one couple H_i and $W_i (i = 1, 2, \ldots, n)$ are adjacent in the row. Show that for each n there are more permutations with property P than without.

50. Let $B_0 = 1$ and for $r \in \mathbf{N}$, let $B_r = \sum_{k=1}^{r} S(r, k)$. The number B_r is called the rth *Bell* number (see Section 1.7). Show that

(i) Corollary 1 to Theorem 4.5.1 can be written as

$$S(r, k) = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} j^r,$$

where $r, k \in \mathbf{N}$;

(ii) $B_r = e^{-1} \sum_{j=0}^{\infty} \frac{j^r}{j!}$.

51. For $n \in \mathbf{N}^*$ and $r \in \mathbf{N}$, let

$$a_n = \sum_{i=0}^{n} (-1)^i \frac{r}{i+r} \binom{n}{i}.$$

Show that

$$a_n = \frac{n}{n+r} a_{n-1}.$$

Deduce that

$$a_n = \frac{1}{\binom{n+r}{r}}.$$

52. We follow the terminology given in Theorem 4.3.1. For $1 \le m \le q$, let $L(m)$ denote the number of elements of S that possess *at least* m of the q properties. Show that

$$L(m) = \sum_{k=m}^{q} (-1)^{k-m} \binom{k-1}{m-1} \omega(k).$$

Note. One possible proof is to follow the argument given in the proof of Theorem 4.3.1 and to apply the identity given in the preceding problem.

53. For $k = 1, 2, \ldots, 1992$, let A_k be a set such that $|A_k| = 44$. Assume that $|A_i \cap A_j| = 1$ for all $i, j \in \{1, 2, \ldots, 1992\}$ with $i \ne j$. Evaluate $\left| \bigcup_{k=1}^{1992} A_k \right|$.

54. Twenty-eight random draws are made from the set

$$\{1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, J, K, L, U, X, Y, Z\}$$

containing 20 elements. What is the probability that the sequence

$$CUBAJULY1987$$

occurs in that order in the chosen sequence? (Belgium, 1987)

55. A sequence of 35 random draws, one at a time with replacement, is made from the set of the English alphabet:

$$\{A, B, C, \ldots, X, Y, Z\}.$$

What is the probability that the string

$$MERRYCHRISTMAS$$

occurs as a block in the sequence?

56. In a group of 1990 people, each person has at least 1327 friends. Show that there are 4 people in the group such that every two of them are friends (assuming that friendship is a mutual relationship). (Proposed by France at the 31st IMO.)

57. Let $\mathbf{C}$ be the set of complex numbers, and let $S = \{z \in \mathbf{C} \mid |z| = 1\}$. For each mapping $f : S \to S$ and $k \in \mathbf{N}$, define the mapping $f^k : S \to S$ by $f^k(z) = \underbrace{f(f(\cdots(f}_{k}(z))\cdots))$. An element $w \in S$ is called an *n-periodic point* $(n \in \mathbf{N})$ *of* f if

$$f^i(w) \neq w \text{ for all } i = 1, 2, \ldots, n-1, \text{ but } f^n(w) = w.$$

Suppose $f : S \to S$ is a mapping defined by

$$f(z) = z^m \quad (m \in \mathbf{N}).$$

Find the number of 1989-periodic points of f. (Chinese Math. Competition, 1989)

58. For $m, n \in \mathbf{N}$, let $\mathcal{M}$ be the set of all $m \times n$ $(0,1)$-matrices. Let

$$\mathcal{M}_r = \{M \in \mathcal{M} \mid M \text{ has at least one zero row}\}$$

and

$$\mathcal{M}_c = \{M \in \mathcal{M} \mid M \text{ has at least one zero column}\}.$$

Show that the number of matrices in $(\mathcal{M} \backslash \mathcal{M}_r) \cap \mathcal{M}_c$ is given by

$$\sum_{i=1}^{n} (-1)^{i-1} \binom{n}{i} (2^{n-i} - 1)^m.$$

(C. J. Everett and P. R. Stein, *Discrete Math.* **6** (1973), 29.)

59. For $n, m \in \mathbf{N}$ with $m \leq n$, let $P_n(m)$ denote the number of permutations of $\{1, 2, \ldots, n\}$ for which m is the first number whose position is left unchanged. Thus $P_n(1) = (n-1)!$ and $P_n(2) = (n-1)! - (n-2)!$. Show that

(i) $P_n(m) = \sum_{i=0}^{m-1} (-1)^i \binom{m-1}{i} (n-1-i)!$;

(ii) $P_n(m+1) = P_n(m) - P_{n-1}(m)$ for each $m = 1, 2, \ldots, n-1$.

(see Problem 979, *Maths. Magazine*, **50** (1977), 269-270)

60. Let P be a nonempty, finite set with p members, and Q be a finite set with q members. Let $N_k(p,q)$ be the number of binary relations of cardinality k with domain P and range Q. (Equivalently, $N_k(p,q)$ is the number of $p \times q$ matrices of 0's and 1's with exactly k entries equal to 1 and no row or column identically 0.) Compute $\sum_{k=1}^{pq}(-1)^{k-1}N_k(p,q)$. (Proposed by S. Leader, see *Amer. Math. Monthly*, **80** (1973), 84)

61. Let D_n and M_n denote the derangement number and the ménage number respectively. Prove or disprove that the sequence $\{M_n/D_n\}, n = 4,5,6,\ldots$ is monotonically increasing and $\lim_{n\to\infty}(M_n/D_n) = 1/e$. (Proposed by E. T. H. Wang, see *Amer. Math. Monthly*, **87** (1980), 829-830.)

62. Show that for $n \in \mathbf{N}$ and $r \in \mathbf{N}^*$,

$$\sum_{k=0}^{n} k^r \binom{n}{k} D_{n-k} = n! \sum_{m=0}^{\min\{r,n\}} S(r,m).$$

Deduce that for $n \geq r$,

$$\sum_{k=0}^{n} k^r \binom{n}{k} D_{n-k} = B_r \cdot n!,$$

where B_r is the rth Bell number. (See *Amer. Math. Monthly*, **94** (1987), 187-189)

63. Let $S = \{1,2,3,\ldots,280\}$. Find the smallest integer n such that each n-element subset of S contains at least 5 numbers which are pairwise relatively prime. (IMO, 1991/3)

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