Chapter 4

Relations[1]

4.1 RELATIONS

A fundamental idea in science as well as in everyday life is to see how two objects, items, or alternatives are related. We might say that a is bigger than b, a is louder than b, a is a brother of b, a is preferred to b, or a and b are equally talented. In this chapter we make precise the idea of a relation between objects, in particular a binary relation, and then note that the study of binary relations is closely related to the study of digraphs from Chapter 3. We pay special attention to those relations that define what are called order relations, and apply them to problems arising from such fields as computer science, economics, psychophysics, biology, and archaeology.

4.1.1 Binary Relations

Suppose that X and Y are sets. The *cartesian product* of X with Y, denoted $X \times Y$, is the set of all ordered pairs (a, b) where a is in X and b is in Y. A *binary relation* R on a set X is a subset of the cartesian product $X \times X$, that is, a set of ordered pairs (a_1, a_2) where a_1 and a_2 are in X. To emphasize the importance of the underlying set, we often speak of the binary relation (X, R) rather than just the binary relation R. If X is the set $\{1, 2, 3, 4\}$, examples of binary relations on X are given by

$$R = \{(1,1), (1,2), (2,1), (3,1), (3,2), (3,3), (3,4)\} \tag{4.1}$$

and

$$S = \{(1,2), (1,3), (1,4), (2,3), (2,4), (3,4)\}. \tag{4.2}$$

Looking back to Section 3.1.2, we see that binary relations are defined exactly the same way as digraphs. Recall that a digraph was defined as a pair (V, A) where

[1]This chapter may be omitted. Section 3.1.2 is suggested as a prerequisite. Alternatively, this chapter may be skipped at this point and returned to later. Ideas from this chapter are needed only in selected examples in a few sections of the book, in particular in parts of Chapters 8, 12, and 13.

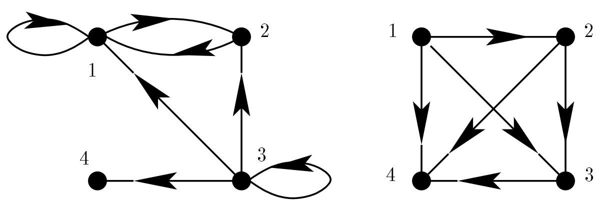

Figure 4.1: Digraph representations of binary relations (4.1) and (4.2).

V is a set and A is a set of ordered pairs of elements of V; this is just another way of saying that A is a binary relation on a set V; that is, A is a subset of the cartesian product $V \times V$. The digraphs corresponding to the binary relations defined from (4.1) and (4.2) can be seen in Figure 4.1. Note that since all digraphs in this book have finite vertex sets, we will only talk about digraphs of binary relations (X, R) for X a finite set.

In the case of a binary relation R on a set X, we shall usually write aRb to denote the statement that $(a, b) \in R$ or that there is an arc from a to b in the digraph of R. Thus, for example, if S is the relation[2] from (4.2), then $1S4$ and $2S3$ but not $3S1$. We shall also use $\sim aRb$ to denote the statement that (a, b) is not in R or that there is no arc from a to b in the digraph of R.

As the name suggests, a binary relation represents what it means for two elements to be related, and in what order. Binary relations arise very frequently from everyday language. For example, if X is the set of all people in the world, then the set

$$F = \{(a, b) : a \in X \text{ and } b \in X \text{ and } a \text{ is the father of } b\}$$

defines a binary relation on X, which we may call, by a slight abuse of language, "father of."

Example 4.1 Preference Suppose that X is any collection of alternatives among which you are choosing, for example, a menu of dinner items or a set of presidential candidates or a set of job candidates or a set of software packages. Suppose that

$$P = \{(a, b) \in X \times X : \text{ you strictly prefer } a \text{ to } b\}.$$

Then P may be called your relation of *strict preference* on the set X. Strict preference is to be distinguished from *weak preference*: The former means "better than" and the latter means "at least as good as." We will normally qualify preference as either being strict or weak. The relation (X, P) is widely studied in economics, political science, psychology, and other fields. To give a concrete example, suppose that you are considering preferences among alternative vacation destinations, your set of possible destinations is $X = \{$San Francisco, Los Angeles, New York, Boston, Miami, Atlanta, Phoenix$\}$, and your strict preference relation is given by

[2]We will often use the term "relation" to mean "binary relation." More generally, a relation is a subset of the cartesian product $X \times X \times \cdots \times X$.

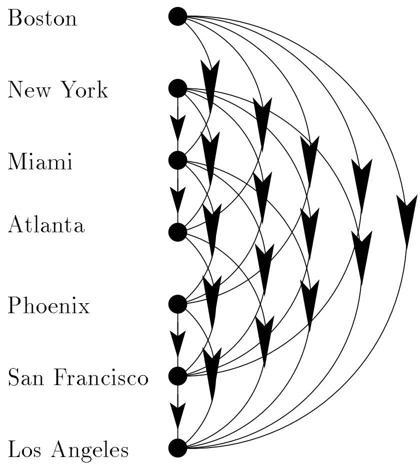

Figure 4.2: Preference digraph for data of Example 4.1.

$P = \{$(Boston, Atlanta), (Boston, Phoenix), (Boston, San Francisco), (Boston, Miami), (Boston, Los Angeles), (New York, Atlanta), (New York, Phoenix), (New York, San Francisco), (New York, Miami), (New York, Los Angeles), (San Francisco, Los Angeles), (Atlanta, San Francisco), (Atlanta, Los Angeles), (Miami, San Francisco), (Miami, Los Angeles), (Miami, Atlanta), (Miami, Phoenix), (Phoenix, Los Angeles), (Phoenix, San Francisco)$\}$. Thus, for example, you strictly prefer Miami to Atlanta. The digraph corresponding to this (X, P) is shown in Figure 4.2. ∎

Example 4.2 Psychophysical Scaling The study of the relationship between the physical properties of stimuli and their psychological properties is called *psychophysics*. In psychophysics, for instance, we try to relate the psychological response of loudness or brightness or sweetness to the physical properties of a sound, light, or food. (See Falmagne [1985] for an introduction to psychophysics from a mathematical point of view.) We often start by making comparisons. For example, if X is a set of sounds, such as coming from different airplanes at different distances from us, we might say that one "sounds louder than" another. If aLb means that "a sounds louder than b," then (X, L) is a binary relation. For example, let a be a Boeing 747 at 2000 feet, a' be a Boeing 747 at 3000 feet, b be a Boeing 757 at 2000 feet, b' be a Boeing 757 at 3000 feet, c be a Boeing 767 at 2000 feet, and c' be a Boeing 767 at 3000 feet. Suppose that

$$L = \{(a,a'),(a,b),(a,b'),(a,c),(a,c'),(a',b),(a',b'),(a',c'),(b,b'),(b,c'),(b',c'),(c,c')\}.$$

Then, for example, a sounds louder than c. The digraph corresponding to this (X, L) is shown in Figure 4.3. ∎

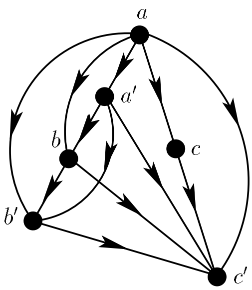

Figure 4.3: "Sounds louder than" digraph for psychophysical scaling. Digraph corresponds to relation (X, L) of Example 4.2.

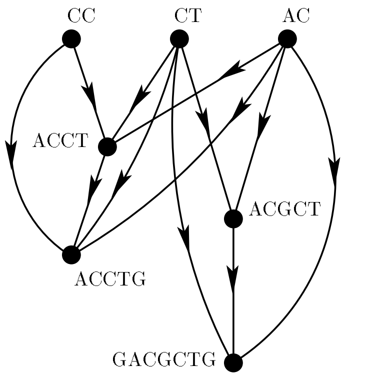

Figure 4.4: Digraph corresponding to relation (X, S) of Example 4.3.

Example 4.3 The Substring Problem In both biology and computer science, we deal with strings of symbols from some alphabet. We are often interested in whether one string appears as a consecutive substring of another string. This is very important in molecular biology, where we seek "patterns" in large molecular sequences such as DNA or RNA sequences, patterns being defined as small, consecutive substrings. We return to a related idea in Example 11.2 and Section 11.6.5.

Suppose that X is a collection of strings. Let us denote by aSb the observation that string a appears as a consecutive substring of string b. This defines a binary relation (X, S). To give a concrete example, let

$$X = \{\text{CC, CT, AC, ACCT, ACCTG, ACGCT, GACGCTG}\}.$$

Then we have (CC, ACCT) $\in S$, (ACCT, ACCTG) $\in S$, and so on. The corresponding digraph is shown in Figure 4.4.

The binary relation (X, S) plays an important role in computer science. One is often given a fixed set of strings and asked to determine whether a given string is a consecutive substring of any string in the set. The data structures known as suffix trees play an important role in solving this problem. See Gusfield [1997] for a discussion of this problem. ■

Example 4.4 Search Engines Web search engines, such as Google, use measures of relevance between a query and a web page. Thus, we start with a set X of web pages. Of course, because the Internet is constantly growing and changing, the set X is changing, but at any given instant, let us consider it fixed. One of the challenges for search engines is to find the set X at any given time. Let q be a query, x be a web page, and $r(x, q)$ be a measure of the relevance of web page x to query q. Another challenge for search engines is to figure out how to measure $r(x, q)$. Let us say that x is ranked over y if $r(x, q) > r(y, q)$. In this case, we write

Table 4.1: Web Pages x and Their Relevance r to a Query q

x	a	b	c	d	e	f
$r(x,q)$	7	8	5	7	8	4

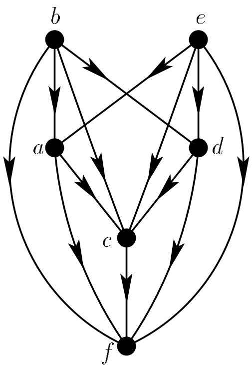

Figure 4.5: Digraph corresponding to relation (X, R) of Example 4.4.

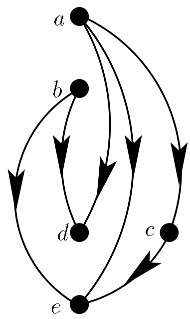

Figure 4.6: Precedence digraph for sequencing in archaeology. Digraph corresponds to relation (X, Q) of Example 4.5

xRy. Consider, for instance, the values of $r(x, q)$ in Table 4.1. Thus,

$$R = \{(b,a),(b,c),(b,d),(b,f),(e,a),(e,c),(e,d),(e,f),(a,c),(a,f),(d,c),(d,f),(c,f)\}.$$

The corresponding digraph is shown in Figure 4.5. ■

Example 4.5 Sequencing in Archaeology A common problem in many applied contexts involves placing items or individuals or events in a natural order based on some information about them. For instance, in archaeology, several types of pottery or other artifacts are found in different digs. We would like to place the artifacts in some order corresponding to when they existed in historical times. We know, for instance, that artifact a preceded artifact b in time. Can we reconstruct an order for the artifacts? This problem, known as the problem of *sequence dating* or *sequencing* or *seriation*, goes back to the work of Flinders Petrie [1899, 1901]. Some mathematical discussion of sequence dating can be found in Kendall [1963, 1969a,b] and Roberts [1976, 1979a]. To give a concrete example, suppose that X consists of five types of pottery, a, b, c, d, e, and we know that a preceded c, d, and e, b preceded d and e, and c preceded e. Then if xQy means that x preceded y, we have

$$Q = \{(a,c),(a,d),(a,e),(b,d),(b,e),(c,e)\}.$$

The digraph corresponding to (X, Q) is shown in Figure 4.6. ■

Table 4.2: Properties of Relations

A binary relation (X, R) is:	Provided that:
Reflexive	aRa, all $a \in X$
Nonreflexive	it is not reflexive
Irreflexive	$\sim aRa$, all $a \in X$
Symmetric	$aRb \Rightarrow bRa$, all $a, b \in X$
Nonsymmetric	it is not symmetric
Asymmetric	$aRb \Rightarrow \sim bRa$, all $a, b \in X$
Antisymmetric	aRb & $bRa \Rightarrow a = b$, all $a, b \in X$
Transitive	aRb & $bRc \Rightarrow aRc$, all $a, b, c \in X$
Nontransitive	it is not transitive
Negatively transitive	$\sim aRb$ & $\sim bRc \Rightarrow \sim aRc$, all $a, b, c \in X$ or $aRc \Rightarrow aRb$ or bRc, all $a, b, c \in X$
Strongly complete	aRb or bRa, all $a, b \in X$
Complete	aRb or bRa, all $a \neq b \in X$

4.1.2 Properties of Relations/Patterns in Digraphs

There are a number of properties that are common to many naturally occurring relations. In this section we discuss some of these properties and their representation when considering the digraphs of relations with these properties. These properties are summarized in Table 4.2.

A binary relation (X, R) is *reflexive* if for all $a \in X$, aRa. Thus, for example, if X is a set of numbers and R is the relation "equality" on X, then (X, R) is reflexive because a number is always equal to itself. However, if $X = \{1, 2, 3, 4\}$, the relation R from (4.1) is not reflexive, since $2R2$ (and $4R4$) does not hold. In this case the binary relation is called *nonreflexive*, which simply means "not reflexive." Again if $X = \{1, 2, 3, 4\}$, the relation S from (4.2) is nonreflexive, since $\sim 1S1$ (and $\sim 2S2$ and $\sim 3S3$ and $\sim 4S4$). When a binary relation is as nonreflexive as this relation, it is called *irreflexive*. That is, (X, R) is irreflexive if $\sim aRa$ for all $a \in X$. In this sense, the relation "father of" on a set of people is irreflexive. So are the relations (X, P), (X, L), (X, S), (X, R), (X, Q) of Examples 4.1, 4.2, 4.3, 4.4, and 4.5, respectively. What does this all mean with regard to digraphs? If a binary relation is reflexive, its digraph has loops at every vertex. In a nonreflexive relation, at least one loop is not present, and in an irreflexive relation, no loop is present. Thus, existence or nonexistence of these three properties can easily be discovered from the digraph of a relation. Consider the digraphs in Figure 4.7. We can quickly ascertain that digraph (a) is irreflexive and nonreflexive, digraph (b) is only nonreflexive, and digraph (c) is reflexive.

In Section 3.1.2 we defined a graph from a digraph by checking to see whether or not there is an arc from u to v whenever there is an arc from v to u. In the

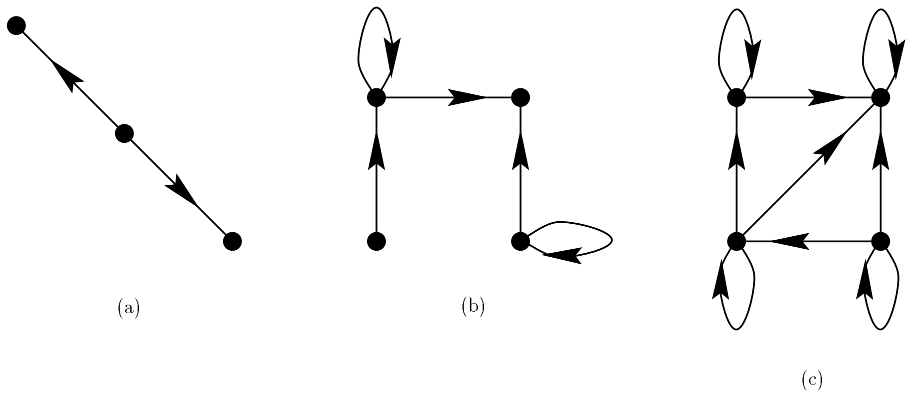

Figure 4.7: Examples of digraphs.

affirmative case we say the digraph is a graph and replace each pair of arcs between vertices by a single nondirected line and call it an edge. This condition of having an arc from u to v whenever there is an arc from v to u is exactly the condition of our next property. A binary relation (X, R) is called *symmetric* if for all $a, b \in X$,

$$aRb \Rightarrow bRa.$$

That is, (X, R) is symmetric if $(b, a) \in R$ whenever $(a, b) \in R$. So, by our discussion above, any graph represents a symmetric binary relation. (Note that a symmetric digraph may or may not have loops.) The relation "equality" on any set of numbers is symmetric. So is the relation "brother of" on the set of all males in the United States. However, the relation "brother of" on the set of all people in the United States is not symmetric, for if a is the brother of b, it does not necessarily follow that b is the brother of a. (Why?) This shows why it is important to speak of the underlying set when defining a relation and studying its properties.

Other examples of *nonsymmetric* (not symmetric) relations are the relation "father of" on the set of people of the world and the relations (X, P), (X, L), (X, S), (X, R), and (X, Q) of Examples 4.1, 4.2, 4.3, 4.4, and 4.5, respectively. These six relations are all highly nonsymmetric. In fact, they are called *asymmetric* because they satisfy the rule

$$aRb \Rightarrow \sim bRa.$$

Other asymmetric relations include the relation "greater than," $>$, on a set of real numbers, "strictly contained in," $\subsetneq$, on any collection of sets, and the relation S from (4.2) on the set $X = \{1, 2, 3, 4\}$. What properties of the corresponding digraph capture the idea that a relation is asymmetric? One interpretation is that the digraph of an asymmetric relation will have no loops and that for all vertices u and v,

$$d(u, v) + d(v, u) \neq 2. \tag{4.3}$$

Some relations (X, R) are not quite asymmetric, but are almost asymmetric in the sense that loops are allowed but for vertices $u \neq v$, Equation (4.3) holds. Let us say that (X, R) is *antisymmetric* if for all $a, b \in X$,

$$aRb \;\&\; bRa \Rightarrow a = b.$$

So, an antisymmetric digraph is like an asymmetric digraph which allows loops. In many examples, being antisymmetric versus asymmetric means that "equality of elements" is allowed. For example, the relation "greater than or equal to," $\geq$, on a set of real numbers, "contained in," $\subseteq$, on any collection of sets, and "at least as tall as" on any set of people no two of whom have the same height are three examples of antisymmetric relations. It is easy to show that every asymmetric binary relation is antisymmetric but the converse is false (Exercise 15).

A relation (X, R) is called *transitive* if for all $a, b, c \in X$, whenever aRb and bRc, then aRc. That is, (X, R) is transitive if for all $a, b, c \in X$,

$$aRb \;\&\; bRc \Rightarrow aRc.$$

Examples of transitive relations are the relations $=$ and $>$ on a set of real numbers, "implies" on a set of statements, and the relation (X, S) where $X = \{1, 2, 3, 4\}$ and S is given by (4.2). The relations (X, P), (X, L), (X, S), (X, R), and (X, Q) of Examples 4.1, 4.2, 4.3, 4.4, and 4.5, respectively, are all transitive. This is clearly the case for (X, S) and (X, R), which are defined very precisely, and it seems reasonable for the other three examples as well. Thus, it seems reasonable to assume that the relation of strict preference among alternative vacation destinations is always transitive, for if you prefer a to b and b to c, you should be expected to prefer a to c. Similarly, it seems reasonable to assume that the relation "sounds louder than" on a set of airplanes is always transitive and similarly for the relation "preceded" on a set of artifacts. As reasonable as these last three examples appear, only with empirical data may verification be obtained. In strict preferences arising in real applications, we sometimes find transitivity violated. If $X = \{1, 2, 3, 4\}$ and R is given by (4.1), then (X, R) is *nontransitive*, i.e, not transitive, because $2R1$ and $1R2$ but $\sim 2R2$. Another relation that is not transitive is the relation "father of" on the set of people in the world. How does one tell from a digraph whether or not its associated binary relation is transitive? To be transitive means that if there is an arc from vertex u to vertex v and an arc from vertex v to vertex w, then there will be an arc from vertex u to vertex w. Transitivity can also be defined by a restriction on the distance function for a digraph (see Exercise 28).

Our next property is similar to transitivity but in a negative sense. A binary relation (X, R) is called *negatively transitive* if for all $a, b, c \in X$, $\sim aRb$ and $\sim bRc$ imply that $\sim aRc$. A binary relation (X, R) is negatively transitive if the relation "not in R," defined on the set X, is transitive. To give an example, the relation $R =$ "greater than" on a set of real numbers is negatively transitive, for "not in R" is the relation "not greater than" or "less than or equal to," which is certainly transitive. It is easy to show that if $X = \{1, 2, 3, 4\}$, the relation S from (4.2) is negatively transitive. So are the relations (X, S) and (X, R) from Examples 4.3 and 4.4. Similarly, strict preference on a set of alternatives, "sounds louder than" on a set of sounds, and preceded on a set of artifacts are probably negatively transitive. (When

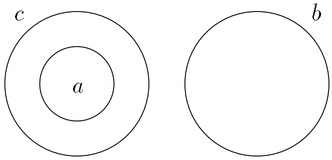

Figure 4.8: The relation "contained in" is not necessarily negatively transitive.

dealing with economics, political science, psychology, or psychophysics, terms like "probably" are often the best that can be expected.) Verifying negative transitivity can be annoyingly confusing. It is often easier to test the contrapositive (and hence equivalent) condition: For all $a, b, c \in X$, if aRc, then aRb or bRc. Using this notion, we see easily that the relation "greater than" is negatively transitive, for if $a > c$, then for all b, either $a > b$ or $b > c$. Similarly, one sees that "contained in" is not negatively transitive, for if a is contained in c, there may very well be a b so that a is not contained in b and b is not contained in c. (See Figure 4.8 for an example.) The relation "father of" on the set of all people in the world is not negatively transitive, nor is the relation (X, R) where $X = \{1, 2, 3, 4\}$ and R is given by (4.1). To see the latter, note that $2R1$ but not $2R4$ and not $4R1$.

Checking to see if a digraph is negatively transitive can be a nontrivial task. In the worst case, for every arc (u, v), two arcs must be searched for, (u, w) and (w, v), for every vertex w, $w \neq u, v$. Thus, it is possible that checking for negative transitivity could take $2a(n-2)$ searches, where n and a are the sizes of the vertex set and arc set of the digraph, respectively.

Let us say that a binary relation (X, R) is *strongly complete* if for all $a, b \in X$, aRb or bRa. Thus, $\geq$, "greater than or equal to," on a set of numbers is strongly complete. However, "strict containment" on a family of sets may not be as it is possible that for two sets, neither is strictly contained in the other. Similarly, "father of" on the set of all people in the world is not strongly complete. (Why?) The substring relation of Example 4.3 is also not strongly complete. One digraph test for strong completeness involves its underlying graph (see Exercise 6).

Notice that "greater than" on a set of numbers is not strongly complete since if $a = b$, then $\sim a > b$ and $\sim b > a$. This relation is almost strongly complete in the sense that for all $a \neq b$, aRb or bRa. A binary relation satisfying this condition is called *complete*. The relation R on $X = \{1, 2, 3, 4\}$ defined by Equation (4.1) is neither complete nor strongly complete (why?), but the relation S on X defined by Equation (4.2) is complete but not strongly complete. The relation (X, R) of Example 4.4 is neither complete nor strongly complete. (Again, see Exercise 6 for a digraph representation of the property of strong completeness.)

EXERCISES FOR SECTION 4.1

1. (a) Consider the binary relation "a divides b" on the set of positive integers. Which of the following properties does this relation have: Reflexive, irreflexive, symmetric, asymmetric, antisymmetric, transitive, negatively transitive?

   (b) Repeat part (a) for the relation "uncle of" on a set of people.

(c) For the relation "has the same weight as" on a set of mice.

(d) For the relation "feels smoother than" on a set of objects.

(e) For the relation "admires" on a set of people.

(f) For the relation "has the same blood type as" on a set of people.

(g) For the relation "costs more than" on a set of cars.

(h) For the relation (X, R) where X is the set of all bit strings and aRb means some proper suffix of a is a proper prefix of b. (A *proper suffix* of a string $b_1 b_2 \cdots b_n$ is a string of the form $b_i b_{i+1} \cdots b_n$, $n \geq i > 1$. A proper prefix is defined similarly.)

2. Show that the binary relation "brother of" on the set of all people in the world is not symmetric.

3. Show that the binary relation "father of" on the set of all people in the world is not strongly complete.

4. Find an example of "everyday language" that can be used to describe the binary relation of (4.2).

5. (Stanat and McAllister [1977]) We are given a library of documents comprising a set Y, and develop a set Z of "descriptors" (e.g., keywords) to describe the documents. Let $X = Y \cup Z$ and let aRb hold if descriptor b applies to document a. Document retrieval systems use the relation (X, R) to find relevant documents for users. Which of the properties of Exercise 1(a) are satisfied by the relation (X, R)?

6. (a) Describe a test for a digraph D's underlying graph that must be satisfied if and only if D is strongly complete.

   (b) How would your answer to part (a) change if strongly complete is replaced by complete?

7. If (X, R) is a binary relation, the *converse* of R is the relation R^{-1} on X defined by

$$aR^{-1}b \text{ iff } bRa.$$

   (a) Describe the digraph of R^{-1} as compared to R.

   (b) Identify the converse of the binary relation "uncle of" on the set of all people in Sweden.

8. If (X, R) is a binary relation, the *complement* of R is the relation R^c on X defined by

$$aR^cb \text{ iff } \sim aRb.$$

   (a) Describe the digraph of R^c as compared to R.

   (b) Prove or give a counterexample to the statement: If R is symmetric, then R^c is symmetric.

   (c) Identify the complement of the binary relation "father of" on the set of all people in Sri Lanka.

9. If (X, R) and (X, S) are binary relations, the *intersection* relation $R \cap S$ on X is defined by

$$R \cap S = \{(a, b) : aRb \text{ and } aSb\}.$$

(a) Identify $(X, R \cap S)$ when X = the set of all people in Ireland, R = "older than," and S = "father of." (Be as succinct as possible.)

(b) Draw the digraph of $(X, R \cap S)$ where $X = \{1, 2, 3, 4\}$ and R and S are as given in (4.1) and (4.2).

10. Suppose that (X, R) and (X, S) are binary relations. For each property listed in Exercise 1(a), assume that (X, R) and (X, S) each have the stated property. Then either prove that $(X, R \cap S)$ also has the property or give an example to show that it may not.
11. Repeat Exercise 10 for (X, R^{-1}).
12. Repeat Exercise 10 for (X, R^c).
13. Which of the properties in Exercise 1(a) hold for $(X, \emptyset)$?
14. Which of the properties in Exercise 1(a) hold for $(X, X \times X)$?
15. (a) Show that every asymmetric relation is antisymmetric.

    (b) Show that the converse of part (a) does not hold.
16. Prove that an asymmetric binary relation will be irreflexive.
17. Show that a binary relation is asymmetric and transitive if and only if it is irreflexive, antisymmetric, and transitive.
18. (a) Show that it is not possible for a binary relation to be both symmetric and asymmetric.

    (b) Show that it is possible for a binary relation to be both symmetric and antisymmetric.
19. Show that there are binary relations that are:

    (a) Transitive but not negatively transitive

    (b) Negatively transitive but not transitive

    (c) Neither negatively transitive nor transitive

    (d) Both negatively transitive and transitive
20. A binary relation (X, R) is an *equivalence relation* if it is reflexive, symmetric, and transitive. Which of the following binary relations (X, R) are equivalence relations?

    (a) $(Re, =)$ where Re is the set of real numbers

    (b) $(Re, \geq)$ (c) $(Re, >)$

    (d) X = a set of people, aRb iff a and b have the same weight

    (e) $X = \{0, 1, 2, \ldots, 22\}$, aRb iff $a \equiv b (\text{mod } 5)$

    (f) X is the collection of all finite sets of real numbers and aRb iff $a \cap b \neq \emptyset$

    (g) $X = \{(1,1), (2,3), (3,8)\}$ and $R = \{\langle(1,1),(1,1)\rangle, \langle(2,3),(2,3)\rangle, \langle(3,8),(3,8)\rangle, \langle(1,1),(2,3)\rangle, \langle(2,3),(1,1)\rangle\}$
21. If (X, R) and (X, S) are equivalence relations (see Exercise 20), is

    (a) $(X, R \cap S)$ where $R \cap S = \{(a, b) : aRb \text{ and } aSb\}$?

    (b) $(X, R \cup S)$ where $R \cup S = \{(a, b) : aRb \text{ or } aSb\}$?

(c) $(X, R/S)$ where $R/S = \{(a,b)|$ for some $c \in X, aRc$ and $cSb\}$?

22. To show that all the properties of an equivalence relation are needed, give an example of a binary relation that is:

    (a) Reflexive, symmetric, and not transitive

    (b) Reflexive, transitive, and not symmetric

    (c) Symmetric, transitive, and not reflexive

23. If (X, R) is an equivalence relation (see Exercise 20), let $C(a) = \{b \in X|aRb\}$. This is called the *equivalence class containing a*. For example, if $X = \{1,2,3,4,5\}$ and $R = \{(1,1),(1,2),(2,1),(2,2),(3,3),(4,4),(5,5)\}$, then (X, R) is an equivalence relation. The equivalence classes are $C(1) = C(2) = \{1,2\}$, $C(3) = \{3\}$, $C(4) = \{4\}$, $C(5) = \{5\}$.

    (a) Find all equivalence classes in equivalence relation (X, R) of Exercise 20(e).

    (b) Show that two equivalence classes $C(a)$ and $C(b)$ are either disjoint or identical.

    (c) Give an example of an equivalence relation with three distinct equivalence classes.

    (d) Give an example of an equivalence relation with two distinct equivalence classes, one of which has three elements and the other two.

24. Suppose that $X = Re$ and
$$aRb \text{ iff } a > b + 1.$$
Which of the properties in Exercise 1a hold for (X, R)?

25. Consider the binary relation (X, S) where $X = Re$ and
$$aSb \text{ iff } |a - b| \leq 1.$$
This relation is closely related to the binary relation (X, R) of Exercise 24. Which of the properties in Exercise 1(a) hold for (X, S)?

26. If (X, R) is a binary relation, the *symmetric complement* of R is the binary relation S on X defined by
$$aSb \text{ iff } (\sim aRb \;\&\; \sim bRa).$$
Note that if R is strict preference, then S is indifference; you are *indifferent* between two alternatives if and only if you prefer neither.

    (a) Show that the symmetric complement is always symmetric.

    (b) Show that if (X, R) is negatively transitive, the symmetric complement is transitive.

    (c) Show that the converse of part (b) is false.

    (d) If $X = Re$ and R is as defined in Exercise 25, find an inequality to describe the symmetric complement of R.

    (e) Identify the symmetric complement of the following relations:

        i. $(Re, >)$
        ii. $(Re, =)$

iii. (N, R), where N is the set of positive integers and xRy means that x does not divide y

27. Compute the number of binary relations on a set X having n elements.
28. Given a digraph D, prove that D is transitive if and only if $d(u, v) \neq 2$ whenever v is reachable from u.

4.2 ORDER RELATIONS AND THEIR VARIANTS

In this section we study the special binary relations known as order relations and their variants.

4.2.1 Defining the Concept of Order Relation

Example 4.6 Utility Functions Suppose that (X, P) is the strict preference relation of Example 4.1. In economics or psychology, we sometimes seek to reflect preferences by a numerical value so that the higher the value assigned to an object, the more preferred that object is. Thus, we might ask if we can assign a value $f(a)$ to each alternative a in X so that a is strictly preferred to b if and only if $f(a) > f(b)$:

$$aPb \Leftrightarrow f(a) > f(b). \tag{4.4}$$

If this can be done, f is sometimes called a *utility function* (*ordinal utility function*). Utility functions are very useful in decisionmaking applications because they give us a single numerical value on which to base our choices and they provide an *order* for the alternatives. We often choose courses of action that maximize our utility (or "expected" utility). In Example 4.1, a utility function satisfying (4.4) can be found. One example of such a function is

$$\begin{gathered} f(\text{Boston}) = 5, \quad f(\text{New York}) = 5, \quad f(\text{Miami}) = 4, \\ f(\text{Atlanta}) = 2, \quad f(\text{Phoenix}) = 2, \quad f(\text{San Francisco}) = 1, \\ f(\text{Los Angeles}) = 0. \end{gathered} \tag{4.5}$$

This gives us an order for X: Boston and New York tied for first, Miami next, then Atlanta and Phoenix tied, then San Francisco, finally Los Angeles. The relation (X, P) is both transitive and antisymmetric. (In fact, it is asymmetric.) Indeed, any binary relation (X, P) for which there is a function satisfying (4.4) is transitive and antisymmetric (even asymmetric). To see why, note that if aPb and bPc, then (4.4) implies that $f(a) > f(b)$ and $f(b) > f(c)$. Therefore, $f(a) > f(c)$ and, by (4.4), aPc. Antisymmetry is proven similarly.

The notion of utility goes back at least to the eighteenth century. Much of the original interest in this concept goes back to Jeremy Bentham [1789]. According to Bentham: "By utility is meant that property in any object, whereby it tends to produce benefit, advantage, pleasure, good, or happiness ... ." Bentham formulated procedures for measuring utility, for he thought that societies should strive for "the greatest good for the greatest number"—that is, maximum utility. The problem of

Table 4.3: Order Relations and their Variants[a]

	Relation Type:						
Defining Property:	Order Relation	Weak Order	Strict Weak Order	Linear Order	Strict Linear Order	Partial Order	Strict Partial Order
Reflexive						$\surd$	
Symmetric							
Transitive	$\surd$	$\surd$		$\surd$	$\surd$	$\surd$	$\surd$
Asymmetric			$\surd$		$\surd$		$\surd$
Antisymmetric	$\surd$			$\surd$		$\surd$	
Negatively transitive			$\surd$				
Strongly complete		$\surd$		$\surd$			
Complete					$\surd$		

[a]A given type of relation can satisfy more of these properties than those indicated. Only the defining properties are indicated.

how to measure utility is a complex one and much has been written about it. See, for example, Barberà, Hammond, and Seidel [2004], Fishburn [1970b], Keeney and Raiffa [1993], Luce [2000], or Roberts [1979b] for discussions. Utilities are used in numerous applications. They can help in business decisions such as when to buy or sell or which computer system to invest in, personal decisions such as where to live or which job offer to accept, in choice of public policy such as new environmental regulations or investments in homeland security initiatives, and so on. ■

A binary relation (X, R) satisfying transitivity and antisymmetry will be called an *order relation* and we say that X is *ordered* by R. Thus, for example, "contained in" on a family of sets is an order relation, as is "strictly contained in," and so is "descendant of" on the set of people in the world and $\geq$ on a set of numbers. Notice that in the digraph of Figure 4.2, every arc is drawn heading downward. If (X, R) is a transitive relation and its corresponding digraph has this kind of a drawing, with the possible exception of loops, then antisymmetry follows and it is an order relation. Notice that, by transitivity, (a, b) is in the relation whenever there is a path from a to b with each arc heading down.[3]

In this section we define a variety of order relations and relations that are closely related to order relations. We summarize the definitions in Table 4.3. The properties defining a type of order relation are not the only properties the relation has. However, in mathematics, we try to use a minimal set of properties in making a definition.

Figure 4.9 shows a number of different examples of order relations. Notice that even though there are ascending arcs, digraph (c) of Figure 4.9 is an order relation. A redrawing exists with all arcs heading down (see Exercise 23). In fact, it is not

[3]Up to now, the position of vertices in a graph or digraph was unimportant—only the adjacencies mattered. Now, position will matter. Actually, we have previously seen this idea when we introduced rooted trees, but the positioning of vertices was not emphasized in that discussion.

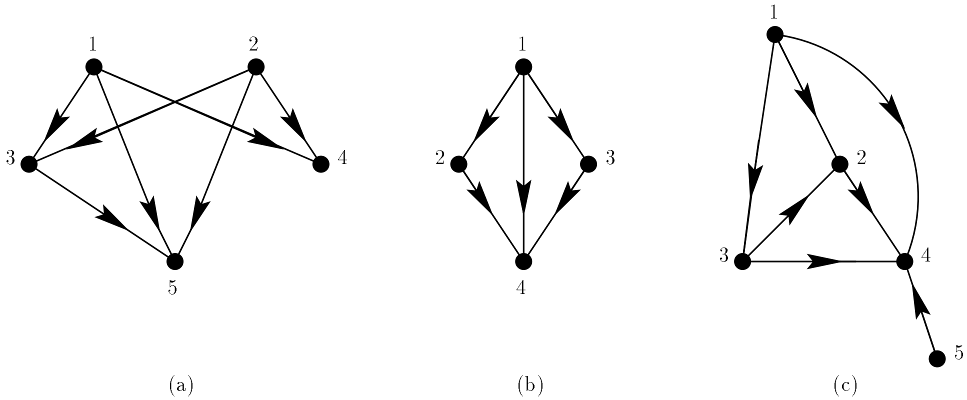

Figure 4.9: Examples of order relations.

hard to show the following:

Theorem 4.1 A transitive binary relation (X, R) is an order relation if and only if its digraph can be drawn so that all arcs (other than loops) head down.[4]

This shows that relations (X, P), (X, L), (X, S), (X, R), and (X, Q) whose corresponding digraphs are shown in Figures 4.2, 4.3, 4.4, 4.5, and 4.6, respectively, are order relations.

One interesting consequence of the two defining properties for order relations is the following theorem.

Theorem 4.2 The digraph of an order relation has no cycles (except loops).

Proof. Suppose that (X, R) is an order relation and that $C = a_1, a_2, \ldots, a_j, a_1$ is a cycle in the corresponding digraph. Thus, $a_1 \neq a_2$. Since (X, R) is transitive and antisymmetric and by the definition of C we know that $(a_1, a_2), (a_2, a_3), \ldots (a_j, a_1)$ are arcs in the digraph. Using the arcs (a_2, a_3) and (a_3, a_4), we apply transitivity to show that $(a_2, a_4) \in R$. Then, since $(a_2, a_4) \in R$ and $(a_4, a_5) \in R$, transitivity implies that $(a_2, a_5) \in R$. Continuing in this way, we conclude that $(a_2, a_j) \in R$. This plus $(a_j, a_1) \in R$ gives us $(a_2, a_1) \in R$. Thus, (a_1, a_2) and (a_2, a_1) are arcs of the digraph, which contradicts the fact that the digraph is antisymmetric, since $a_1 \neq a_2$. Q.E.D.

A binary relation for which we can find a numerical representation satisfying (4.4) has more properties than just transitivity and antisymmetry. For example, it is negatively transitive, to give just one example of another property. (Why?) In later subsections, we will define stronger types of order relations by adding properties that they are required to satisfy.

Besides transitivity and antisymmetry, many of the order relations that we will introduce will be either reflexive or irreflexive. These two possibilities are usually

[4] Although we do not make it explicit, this theorem assumes the hypothesis that X is finite.

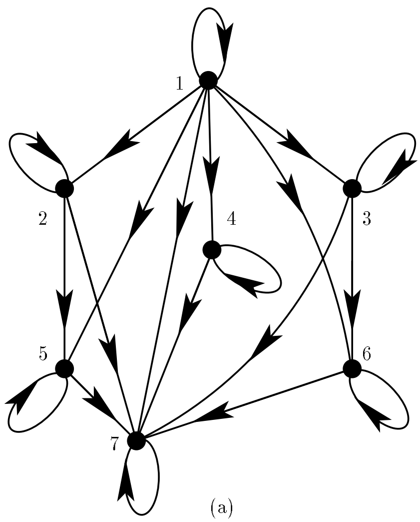

Figure 4.10: A digraph (a) that is a partial order and its associated diagram (b).

based on the context of the problem at hand. If we assume reflexivity, we call our order relation (X, R) a *partial order* or X a *partially ordered set* or *poset.*[5] Figure 4.10(a) shows a partial order. If irreflexivity is assumed, then the adjective "strict" will be used; (X, R) is a *strict partial order* if it is irreflexive, antisymmetric, and transitive. Figures 4.9 (a), (b), (c), 4.2, 4.3, 4.4, 4.5, and 4.6 show strict partial orders. A strict partial order is sometimes defined more succinctly as an asymmetric and transitive binary relation. This is because we have the following theorem.

Theorem 4.3 A binary relation is irreflexive, transitive, and antisymmetric if and only if it is transitive and asymmetric.

Proof. Suppose that (X, R) is irreflexive, transitive, and antisymmetric. Suppose that aRb and bRa. Then $a = b$ by antisymmetry. But aRa fails by irreflexivity. Thus, aRb and bRa cannot both hold and (X, R) is asymmetric.

Conversely, suppose that (X, R) is transitive and asymmetric. Then aRb and bRa cannot both hold, so (X, R) is (vacuously) antisymmetric. Moreover, asymmetry implies irreflexivity, since aRa implies that aRa and aRa, which cannot be the case by asymmetry. Q.E.D.

4.2.2 The Diagram of an Order Relation

Consider an order relation R on a set X. Since antisymmetry and transitivity are defining properties, these can be used to simplify the digraph of the relation.

[5] Some authors use "ordered set" and "partially ordered set" interchangeably. We shall make the distinction to allow for nonreflexive order relations.

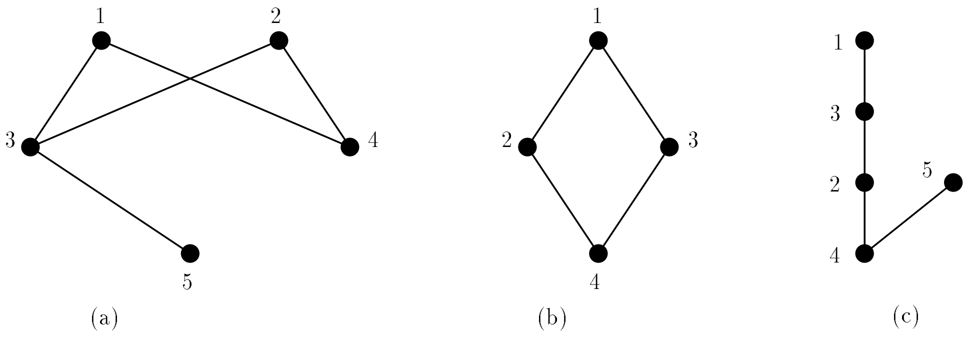

Figure 4.11: Diagrams of the order relations (a), (b), and (c) from Figure 4.9.

We illustrate with digraph (a) of Figure 4.10. Since all arcs (except loops) point downward, why not remove the arrowheads from the arcs, thus turning them into edges? In addition, this digraph is reflexive, so drawing a loop at each vertex is unnecessary. These two changes transform the digraph into a loopless graph. Next, consider the edges from 3 to 6, 6 to 7, and 3 to 7. Since transitivity is known to hold, the edge from 3 to 7 is unnecessary. In general, we can remove all edges that are implied by transitivity. This will simplify the digraph (graph) of the order relation dramatically. The graph produced by removing loops, arrowheads, and arcs implied by transitivity will be called the *diagram* (*order diagram* or *Hasse diagram*) of the order relation. Figure 4.10(b) shows the diagram of the order relation from Figure 4.10(a). In a diagram, aRb if there is a descending chain from a to b. For example, in Figure 4.10(b), $1R5$ and $3R7$ but $\sim 3R5$. The same kinds of simplifications can be made for the digraphs of order relations which are irreflexive. The graphs resulting from these same simplifications of the digraphs of an irreflexive order relation will also be called diagrams. The diagrams associated with digraphs (a), (b), and (c) of Figure 4.9 are shown in Figure 4.11.

Similarly, any loopless graph can be reduced to a diagram, i.e., an order relation, as long as no edge is horizontal. Edges (arcs) that can be assumed by transitivity in the graph can be removed. A diagram will never contain loops, so we will not be able to ascertain reflexivity of the relation from the diagram. Only through the definition of the relation or the context of the presentation can reflexivity be determined.

Consider an order relation R on a set X. We say that x *covers* y or xKy if xRy and there is no z for which xRz and zRy. The binary relation (X, K) is called the *cover relation* of (X, R). We can define the *cover graph* G_K associated with R as follows: (a) $V(G_K)$ is the set X; (b) $\{x, y\} \in E(G_K)$ if x covers y. Hence, the drawing of the cover graph associated with an order relation is actually the diagram of the order relation with y lower than x whenever x covers y. Alternatively, any diagram is the cover graph of an order relation. Consider diagram (b) of Figure 4.11. This diagram defines the cover graph of the relation (X, R) defined by Figure 4.9(b). Here, $K = \{(1,2), (1,3), (2,4), (3,4)\}$.

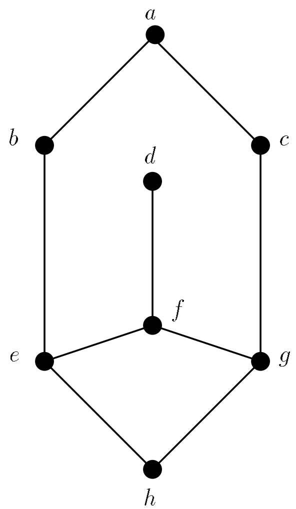

Figure 4.12: Another diagram.

Again, consider an order relation R on a set X. We will use xSy to mean that xRy and $x \neq y$. If xSy or ySx, then x and y are said to be *comparable* and we write xCy. Alternatively, x and y are said to be *incomparable*, written xIy, if neither xRy nor yRx.

The diagram implications of S, C, and I are also straightforward. xSy if and only if there is a descending chain from x to y. Either a descending chain from x to y or y to x means that xCy, while for $x \neq y$, xIy implies that either there is no chain between x and y or that the only chains are neither strictly ascending nor strictly descending. In the diagram of Figure 4.10(b) we see immediately that $3C7$, $3S7$, $\sim 2C6$, and $2I3$.

An element x is called *maximal* in an ordered set (X, R) if there is no $y \in X$ such that ySx. If there is only one maximal element, that element is called *maximum* and denoted $\widehat{1}$. We let $\max(X, R)$ be the set of maximal elements in (X, R). Similar definitions can be made for the terms *minimal*, *minimum*, and $\min(X, R)$. If a minimum element exists, it is denoted $\widehat{0}$. Consider the diagram of Figure 4.12. Here d is a maximal element, h is minimum, $\max(X, R) = \{a, d\}$, and $\min(X, R) = \{h\}$ (since h is minimum). While every order relation (on a finite set) has at least one minimal and at least one maximal element [see Exercise 24(a)], it may not necessarily have either a maximum or a minimum element.

4.2.3 Linear Orders

In many applications, we seek an order relation that "ranks" alternatives, i.e., it gives a first choice, second choice, third choice, etc. Such an order is an example of what we will call a strict linear order. Any time we need to "line up" a set of elements, we are creating a strict linear ordering for that set. The way patients are seen by a doctor at a clinic, when schoolchildren need to line up in single file, and the way television programs are scheduled for airing on a certain channel are all

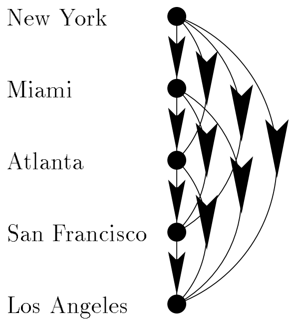

Figure 4.13: A generated subgraph of the digraph of Figure 4.2.

examples of strict linear orders. When we rank alternative political candidates or software packages according to "quality," we are aiming to produce a strict linear order. A binary relation is called a *strict linear order* if it is transitive, asymmetric, and complete. The order relation drawn in Figure 4.13 is an example of a strict linear order. (The terms *strict total order* and *strict simple order* are also used.) A strict linear order can and will satisfy more of our properties but only the three properties given are needed to define it. (It is left to the reader to show that none of the properties of the definition are superfluous; see Exercise 21.) In fact, a strict linear order will also be irreflexive, antisymmetric, and negatively transitive. Asymmetry implies both irreflexivity and antisymmetry (see proof of Theorem 4.3), while transitivity with completeness implies negative transitivity. (The proof of the latter is left to the reader; see Exercise 22.)

The prototype of strict linear orders is the relation $>$ on any set of real numbers. By Theorem 4.1 and the completeness property, in a strict linear order R on a finite set X, the diagram of R consists of the elements of X laid out on a vertical line, i.e., aRb if and only if a is above b. On the other hand, a diagram in the shape of a vertical line will always be a strict linear order if we assume irreflexivity (no loops); completeness is the only property that needs to be checked, since transitivity and asymmetry follow from the definition of order relations under the assumption of irreflexivity. To show completeness, consider any two elements $a \neq b$ in X. Since one must be above the other in a vertical line diagram, by the definition of a diagram, the higher element is "R" to the lower element. Therefore, it must be the case that either aRb or bRa. Since drawing strict linear orders (vertical line graphs) is unilluminating, we will use the notation of Trotter [1992] to describe strict linear orders succinctly. Let $L_S = [a_{i_1}, a_{i_2}, \ldots, a_{i_n}]$ denote the strict linear order S on the set $X = \{a_1, a_2, \ldots, a_n\}$, where $a_{i_j} S a_{i_k}$ whenever a_{i_j} precedes a_{i_k} in L_S. In this notation, the strict linear order of Figure 4.13 is given by [New York, Miami, Atlanta, San Francisco, Los Angeles].

Recall that the term "strict" in strict orders refers to the fact that the relation is irreflexive, whereas the "nonstrict" version is reflexive. We define a *linear order* by the antisymmetric, transitive, and strongly complete properties. It is simple to show that the same things can be said for linear orders as strict linear orders except for the fact that linear orders are reflexive. In particular, the prototype of linear orders is the relation $\geq$ on any set of real numbers. The strict linear order notation, $[a_{i_1}, a_{i_2}, \ldots, a_{i_n}]$, will also be used for a linear order R, with the only change being that $a_j R a_j$ holds for all j.

Example 4.7 Lexicographic Orders (Stanat and McAllister [1977]) Let Σ be a finite alphabet and let R be a strict linear order of elements of Σ. Let X be the set of strings from Σ and define the lexicographic (dictionary) order S on X as follows. First, we take xSy if x is a prefix of y. Second, suppose that $x = zu$, $y = zv$, and z is the longest prefix common to x and y. Then we take xSy if the first symbol of u precedes the first symbol of v in the strict linear order R. For example, if Σ is the alphabet $\{a, b, c, \ldots, z\}$ and R is the usual alphabetical order, then $abSabht$. Also, $abdtSabeab$ since we have $z = ab$, $u = dt$, $v = eab$, and the first symbol of u, i.e., d, precedes the first symbol of v, i.e., e. The binary relation (X, S) corresponds to the usual ordering used in dictionaries. It defines a strict linear order since it is transitive, asymmetric, and complete (see Exercise 15).

Let us continue with the case where Σ is the alphabet $\{a, b, c, \ldots, z\}$ and R is the usual alphabetical order. In the language of Section 4.2.2, if x is any element of X, x covers xa since $xSxa$ and there is no string z in X such that $xSzSxa$. On the other hand, no element covers xb. Why? (See Exercise 17.) This example has an infinite set X. A finite example arises if we take all strings in Σ of at most a given length. ■

4.2.4 Weak Orders

When we are ranking alternatives as first choice, second choice, and so on, we may want to allow ties. We next consider relations called weak orders that are like linear orders except that ties are allowed. A *weak order* is a binary relation that is transitive and strongly complete. Note that since antisymmetry is not assumed, a weak order may not be an order relation. Since antisymmetry is the one defining property of linear orders not necessarily assumed for weak orders, we can have aRb and bRa for $a \neq b$ for weak orders R. In this case, we can think of a and b as "tied" in R. Figure 4.14 shows a "diagram" of a typical weak order. This is not a diagram in the sense that we have defined diagrams. Here, each element has a horizontal level, all elements a and b at the same horizontal level satisfy aRb and bRa, and, otherwise, aRb if and only if a is at a higher level than b. Thus, in the weak order corresponding to Figure 4.14, R is given by

$$x_i R y_j \text{ iff } x = y \text{ or } x \text{ precedes } y \text{ in the alphabet,}$$

where x and y are a, b, c, d, e, or f. For example, $b_2 R e_3$ and $a_1 R f_2$. One can show (see Roberts [1976, 1979b]) that every weak order (on a finite set) arises this way.

It is sometimes useful to consider *strict weak orders*, binary relations that arise in the same way as weak orders except that for elements a and b at the same horizontal level, we do not have aRb. Thus, in a figure like Figure 4.14, we have $x_i R y_j$ if and only if x precedes y in the alphabet. The relation R defined in this way from such

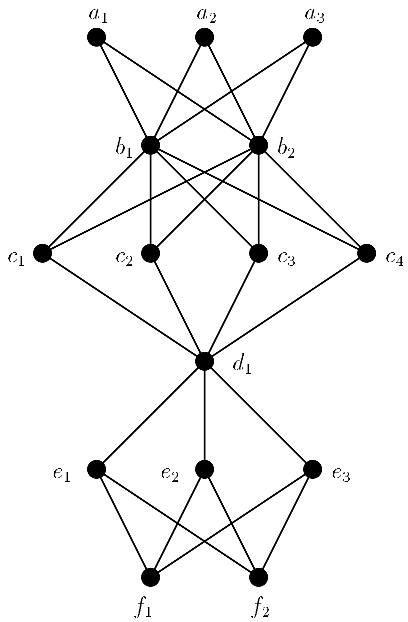

Figure 4.14: This figure defines a "typical" weak order/strict weak order.

a "diagram" is easily seen to be asymmetric and negatively transitive. Conversely, if X is finite, a relation (X, R) that is asymmetric and negatively transitive can be seen to come from a "diagram" like Figure 4.14 in this way (see Roberts [1976, 1979b]). A relation that is asymmetric and negatively transitive is called a *strict weak order*. It is simple to show that weak orders are to strict weak orders as linear orders are to strict linear orders. The only difference is for elements at the same horizontal level, i.e., "tied."

Note that whereas weak orders may not be order relations according to our definition, strict weak orders always are (why?). Note also that strict weak orders allow incomparable elements but only in a special way. The digraph of Figure 4.2 defines a strict weak order. It is easy to see that it can be redrawn with Boston and New York at the top level, then Miami, then Atlanta and Phoenix, then San Francisco, and, finally, Los Angeles. The levels are readily obtained from the function f of (4.5). Recall that function f is a utility function, defined from (4.4). Conversely, if a strict preference relation P is defined from a utility function f by (4.4), it is easy to see that it is asymmetric and negatively transitive, i.e., a strict weak order. The proof is left as an exercise (Exercise 20).

For the same reason that the digraph of Figure 4.2 defines a strict weak order, so does the digraph of Figure 4.5. If we let $f(x) = r(x, q)$ as in Example 4.4, then we have

$$xRy \Leftrightarrow r(x, q) > r(y, q) \Leftrightarrow f(x) > f(y),$$

which gives us the equivalent of (4.4).

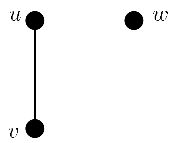

Figure 4.15: A diagram not found in a strict weak order.

Table 4.4: Preference Orderings (Strict Linear Orderings) for a Size 4 Stable Marriage Problem.

Men's Preferences	Women's Preferences
$m_1 : [w_1, w_2, w_3, w_4]$	$w_1 : [m_4, m_3, m_2, m_1]$
$m_2 : [w_2, w_1, w_4, w_3]$	$w_2 : [m_3, m_4, m_1, m_2]$
$m_3 : [w_3, w_4, w_1, w_2]$	$w_3 : [m_2, m_1, m_4, m_3]$
$m_4 : [w_4, w_3, w_2, w_1]$	$w_4 : [m_1, m_2, m_3, m_4]$

It is not hard to show that the digraphs of Figures 4.3 and 4.6 are not strict weak orders. Thus, a function satisfying (4.4) does not exist in either case. To see why, we consider the diagram in Figure 4.15. Notice that elements u and w are incomparable and elements w and v are incomparable. In particular, $\sim uRw$ and $\sim wRv$ follows. However, uRv. Thus, this diagram is not negatively transitive, nor can it be part of a larger negatively transitive diagram. This diagram is essentially the definition of not negatively transitive with regard to diagrams. An order relation is a strict weak order if and only if its diagram does not contain the diagram in Figure 4.15.

4.2.5 Stable Marriages[6]

Suppose that n men and n women are to be married to each other. Before we decide on how to pair up the couples, each man and each woman supplies a preference list of the opposite sex, a strict linear order. A *set of stable marriages* (or a *stable matching*) is a pairing (or matching) of the men and women so that no man and woman would both be better off (in terms of their preferences) by leaving their assigned partners and marrying each other. This problem and a number of its variations were introduced in Gale and Shapley [1962].

Consider the case where $n = 4$ and the preferences are given by the strict linear orders in Table 4.4. Note that $M_1 = \{m_1 - w_4, m_2 - w_3, m_3 - w_1, m_4 - w_2\}$ is a stable set of marriages. To see why, note that w_4 and w_3 married their first choice; so neither would be willing to leave their partner, m_1, m_2, respectively. Also, m_3 and m_4 are getting their first choice among the other women, namely, w_1 and w_2, respectively. In all, there are 10 stable matchings for this problem. Two other obvious stable matchings are $M_9 = \{m_1 - w_1, m_2 - w_2, m_3 - w_3, m_4 - w_4\}$ and

[6] This subsection is based on Gusfield and Irving [1989].

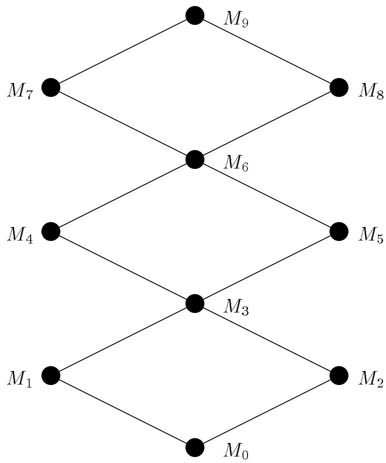

Figure 4.16: The partial order of the man-oriented dominance relation on stable matchings.

$M_0 = \{m_1 - w_4, m_2 - w_3, m_3 - w_2, m_4 - w_1\}$. These are both stable since each man (woman) got his (her) first choice. The full list of 10 stable matchings is

$$\begin{array}{lcl}
M_0 & = & \{m_1 - w_4, m_2 - w_3, m_3 - w_2, m_4 - w_1\} \\
M_1 & = & \{m_1 - w_4, m_2 - w_3, m_3 - w_1, m_4 - w_2\} \\
M_2 & = & \{m_1 - w_3, m_2 - w_4, m_3 - w_2, m_4 - w_1\} \\
M_3 & = & \{m_1 - w_3, m_2 - w_4, m_3 - w_1, m_4 - w_2\} \\
M_4 & = & \{m_1 - w_2, m_2 - w_4, m_3 - w_1, m_4 - w_3\} \\
M_5 & = & \{m_1 - w_3, m_2 - w_1, m_3 - w_4, m_4 - w_2\} \\
M_6 & = & \{m_1 - w_2, m_2 - w_1, m_3 - w_4, m_4 - w_3\} \\
M_7 & = & \{m_1 - w_2, m_2 - w_1, m_3 - w_3, m_4 - w_4\} \\
M_8 & = & \{m_1 - w_1, m_2 - w_2, m_3 - w_4, m_4 - w_3\} \\
M_9 & = & \{m_1 - w_1, m_2 - w_2, m_3 - w_3, m_4 - w_4\}.
\end{array}$$

Given this set of all 10 stable matchings, person x would prefer one stable matching M_i over another M_j if x prefers his/her partner in M_i to his/her partner in M_j. We can then define the *man-oriented dominance relation* as follows: M_i dominates M_j if every man prefers M_i to M_j or is indifferent between them. It is not hard to show that man-oriented dominance is a partial order. Its diagram is shown in Figure 4.16. (See Section 12.8 for a more complete treatment of the stable marriage problem.)

EXERCISES FOR SECTION 4.2

1. Which of the following are order relations?

   (a) $\subsetneq$ on the collection of subsets of $\{1, 2, 3, 4\}$

(b) (X, P), where $X = Re \times Re$ and

$$(a, b)P(s, t) \text{ iff } (a > s \text{ and } b > t)$$

(c) (X, Q), where X is a set of n-dimensional alternatives, $f_1, f_2, \ldots, f_n$ are real-valued scales on X, and Q is defined by

$$aQb \Leftrightarrow [f_i(a) > f_i(b) \text{ for each } i]$$

(d) (X, Q), where X is as in part (c) and

$$aQb \Leftrightarrow [f_i(a) \geq f_i(b) \text{ for each } i \text{ and } f_i(a) > f_i(b) \text{ for some } i]$$

2. Which of the binary relations in Exercise 1 are strict partial orders?
3. Which of the binary relations in Exercise 1 are linear orders?
4. Which of the binary relations in Exercise 1 are strict linear orders?
5. Which of the binary relations in Exercise 1 are weak orders?
6. Which of the binary relations in Exercise 1 are strict weak orders?
7. Which of the following are linear orders?

   (a) $\{(a,a),(b,b),(c,c),(d,d),(c,d),(c,b),(c,a),(d,b),(d,a),(b,a)\}$ on set $X = \{a, b, c, d\}$

   (b) $\{(a,a),(a,b),(a,d),(b,b),(b,d),(c,a),(c,b),(c,d),(d,d)\}$ on set $X = \{a,b,c,d\}$

   (c) $\{(a,a),(a,b),(a,c),(a,d),(b,b),(b,c),(b,d),(c,a),(c,c),(c,d),(d,d)\}$ on set $X = \{a, b, c, d\}$

   (d) $\{(\lambda,\lambda), (\lambda,\xi), (\lambda,\zeta), (\lambda,\varphi), (\xi,\xi), (\xi,\zeta), (\xi,\varphi), (\rho,\lambda), (\rho,\xi), (\rho,\rho), (\rho,\zeta), (\rho,\varphi), (\zeta,\zeta), (\zeta,\varphi), (\varphi,\varphi)\}$ on set $X = \{\lambda, \xi, \zeta, \varphi, \rho\}$

8. Which of the following are strict linear orders?

   (a) $\{(c,d),(c,b),(c,a),(d,b),(d,a),(b,a)\}$ on set $X = \{a,b,c,d\}$

   (b) $\{(a,b),(c,b),(a,d),(c,d),(b,d),(c,a),(b,c)\}$ on set $X = \{a,b,c,d\}$

   (c) $\{(\lambda,\delta),(\lambda,\xi),(\lambda,\zeta),(\lambda,\varphi),(\xi,\delta),(\xi,\zeta),(\xi,\varphi),(\xi,\xi),(\rho,\lambda),(\rho,\xi),(\rho,\delta),(\rho,\zeta), (\rho,\varphi), (\zeta,\delta), (\zeta,\varphi), (\varphi,\delta)\}$ on set $X = \{\lambda, \xi, \zeta, \varphi, \rho\}$

   (d) $\{(5,1),(5,2),(5,4),(1,2),(1,4),(3,5),(3,1),(3,2),(3,4),(2,4)\}$ on set $X = \{1,2,3,4,5\}$

9. Which of the following are weak orders?

   (a) $\{(a,a), (b,b), (c,c), (d,d), (b,a), (b,c), (b,d), (c,a), (c,d), (a,d)\}$ on set $X = \{a, b, c, d\}$

   (b) $\{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (1,7), (1,8), (1,9), (2,2), (2,5), (2,6), (2,7), (2,8), (2,9), (3,3), (3,5), (3,6), (3,7), (3,8), (3,9), (4,4), (4,5), (4,6), (4,7), (4,8), (4,9), (5,5), (5,7), (5,8), (5,9), (6,6), (6,7), (6,8), (6,9), (7,7), (7,9), (8,8), (8,9), (9,9)\}$ on set $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

   (c) $\{(\Gamma,\Delta), (\Delta,\Psi), (\Gamma,\Upsilon), (\Delta,\Upsilon), (\Gamma,\Omega), (\Upsilon,\Psi), (\Gamma,\Psi), (\Psi,\Omega), (\Delta,\Omega), (\Upsilon,\Omega)\}$ on set $X = \{\Gamma, \Delta, \Psi, \Upsilon, \Omega\}$

10. Which of the binary relations of Exercise 9 are strict weak orders?

11. Draw the diagrams corresponding to the strict partial orders of:

    (a) Figure 4.3 (b) Figure 4.4 (c) Figure 4.6

12. Draw the diagram for the strict weak order of Figure 4.5 in the same way as Figure 4.14, with "tied" elements at the same horizontal level.

13. Suppose that $X = \{1, 2, 3, 4\}$ and

$$R = \{(1,1), (2,2), (3,3), (4,4), (1,3), (1,4), (2,3), (2,4), (3,4)\}.$$

    (a) Show that (X, R) is a partial order.

    (b) Draw the diagram for (X, R).

    (c) Find (X, K), the cover relation associated with (X, R).

14. Suppose that $X = \{t, u, v, w, x, y, z\}$ and

$$R = \{(t,u), (t,v), (u,v), (w,v), (w,x), (w,y), (w,z), (x,v), (x,z), (y,v), (y,z), (z,v)\}.$$

    (a) Show that (X, R) is a strict partial order.

    (b) Draw the diagram for (X, R).

    (c) Find (X, K), the cover relation associated with (X, R).

15. Show that the binary relation (X, S) of Example 4.7 is:

    (a) Transitive (b) Asymmetric (c) Complete

16. From Example 4.7, write out S for the strict linear order (X, S), where X is the set of all strings of length at most 4 and:

    (a) $\Sigma = \{a, b\}$ if aRb (b) $\Sigma = \{a, b, c\}$ if aRb, aRc, and bRc

17. From Example 4.7, if $\Sigma = \{a, b, c, \ldots, z\}$ and R is the usual alphabetical order, explain why $xb \in X$ has no cover.

18. Suppose that $X = \{a, b, c, d, e, f\}$ and

$$R = \{(a,c), (a,f), (b,c), (b,f), (d,a), (d,b), (d,c), (d,e), (d,f), (e,c), (e,f)\}.$$

    (a) Show that (X, R) is a strict weak order.

    (b) Draw the diagram for (X, R).

19. Consider the strict linear order $L_S = [x_1, x_2, \ldots, x_n]$.

    (a) Find $L_{S^{-1}}$. (b) Find $L_S \cap L_{S^{-1}}$. (c) Find $L_S \cup L_{S^{-1}}$.

20. If (X, P) is a strict preference relation defined from a utility function f by (4.4), show that P is asymmetric and negatively transitive.

21. Prove that no two of the following three properties imply the third: transitive, complete, asymmetric.

22. Prove that a transitive and complete binary relation will be negatively transitive.

23. Redraw digraph (c) in Figure 4.9 to prove that it is an order relation. That is, redraw the digraph so that all arcs are descending.

24. (a) Prove that every order relation (on a finite set) has at least one maximal and one minimal element.

    (b) Prove that every (strict) linear order has a maximum and a minimum element.

25. (a) Is the converse R^{-1} of a strict partial order necessarily a strict partial order? (For the definition of converse, see Exercise 7, Section 4.1.)

    (b) Is the converse R^{-1} of a partial order necessarily a partial order?

26. Show that every strict weak order is a strict partial order.

27. If (X, R) is strict weak, define S on X by
$$aSb \text{ iff } (aRb \text{ or } a = b).$$
    Show that (X, S) is a partial order.

28. Draw the diagram of the converse (Exercise 7, Section 4.1) of the strict partial order defined by Figure 4.14.

29. (a) Is the converse of a strict weak order necessarily a strict weak order? Why?

    (b) Is the converse of a strict partial order necessarily a strict partial order? Why?

30. (a) Is the complement of a strict weak order necessarily a strict weak order? (For the definition of complement, see Exercise 8, Section 4.1.)

    (b) Is the complement of a strict partial order necessarily a strict partial order?

31. Prove that the man-oriented dominance relation of Section 4.2.5 is a partial order.

32. Consider the set of stable matchings of Section 4.2.5.

    (a) Write the definition for a woman-oriented dominance relation on a set of stable matchings.

    (b) Draw the diagram for the woman-oriented dominance relation on this set of stable matchings.

    (c) How does the diagram for man-oriented dominance compare with the diagram for woman-oriented dominance?

33. (a) Explain why
$$\{m_1 - w_1, m_2 - w_4, m_3 - w_2, m_4 - w_3\}$$
    is not a set of stable marriages for the preference orderings of Table 4.4.

    (b) How many sets of marriages are not stable for the preference orderings of Table 4.4?

    (c) Prove that $M_0, M_1, \ldots, M_9$ are the only sets of stable marriages for the preference orderings of Table 4.4.

4.3 LINEAR EXTENSIONS OF PARTIAL ORDERS

4.3.1 Linear Extensions and Dimension

Example 4.8 Mystery Novels and Linear Extensions When writing a good murder mystery novel, the author gives clues that help the reader figure out "who

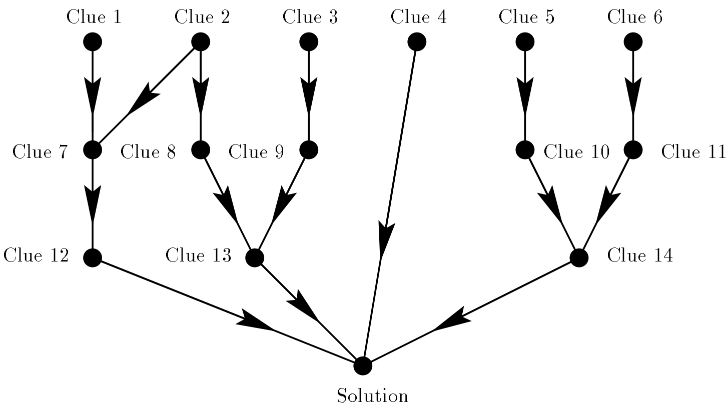

Figure 4.17: "Clue" digraph for a novel.

done it." Some clues depend on earlier clues to be understood. Consider Figure 4.17. The vertices of the digraph represent the clues and "Solution" in some novel, and the arcs represent dependency of one clue on another. For example, the arc from Clue 3 to Clue 9 represents the fact that Clue 3 is needed to understand Clue 9. Clearly, we can assume transitivity, so we do not have to draw all of the arcs between clues. Also, certainly antisymmetry holds, so our digraph defines an order relation. The arcs from some clues to the Solution vertex represent the fact that those clues are needed to figure out the mystery. Thus, by transitivity, all clues are needed to figure out the mystery.

The task facing the author is in what order to present the clues in the novel in a "coherent" way. By coherent we mean that no clue is presented until all of its dependent clues are given first. Depending on the digraph, there could be lots of ways to present the clues. Figure 4.18 shows four ways in which to present the clues for the digraph of Figure 4.17. Again, transitivity is assumed in these digraphs. The digraphs of Figure 4.18 illustrate the idea of a linear extension.

Consider strict partial orders R and S on the same set X. If aRb whenever aSb, then R is an *extension* of S. Put another way, R is an *extension* of S if all of the ordered pairs that define S are found in R. If R is a strict linear order, R is called a *linear extension* of S. The digraph of Figure 4.17 is in fact a strict partial order (if we recall that the arcs implied by transitivity are omitted) and each digraph of Figure 4.18 gives a linear extension of this order relation. A linear extension is just the thing the author of the mystery novel is searching for. Because of the linear nature in which the words of a book are written, the clues must be presented in the order given by a linear extension. ■

Example 4.9 Topological Sorting Linear extensions of strict partial orders arise naturally when we need to enter a strict partial order (X, R) into a computer. We enter the elements of X in a sequential order and want to enter them in such a way that the strict partial order is preserved. The process generalizes to finding a

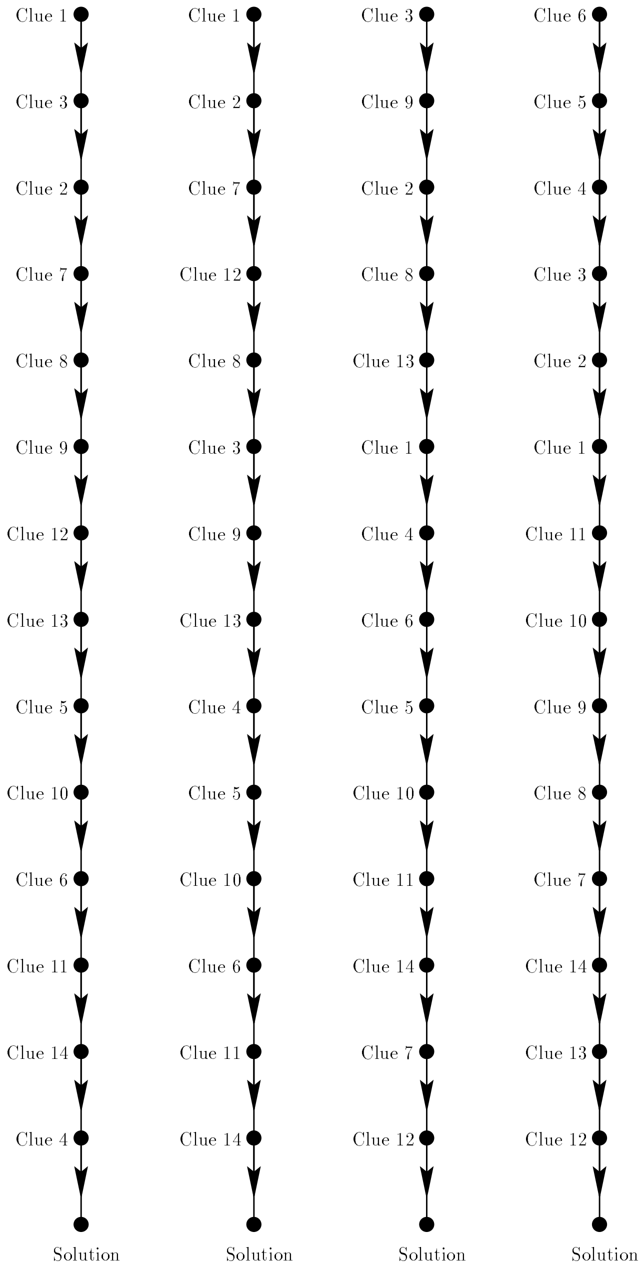

Figure 4.18: Possible clue presentation models.

way to label the vertices of an arbitrary digraph D with the integers $1, 2, \ldots, n$ so that every arc of D goes from a vertex with a smaller label to a vertex with a larger label. This process is called *topological sorting.* We return to this in Sections 11.6.2–11.6.4, where we describe applications of the idea to project planning, scheduling, and facilities design. ■

The following theorem says that linear extensions don't happen by chance.

Theorem 4.4 (Szpilrajn's Extension Theorem [1930]) Every strict partial order has a linear extension. Moreover, if (X, R) is a strict partial order and xIy for distinct $x, y \in X$ (i.e., x and y are incomparable), there is at least one linear extension L with xLy.

It is not hard to show that the following algorithm will find a linear extension of a given strict partial order. In fact, it can be used in such a way as to find all of the linear extensions of a strict partial order. The algorithm uses the notion of a suborder. If (X, R) is a strict partial order, D is its corresponding digraph, and $Y \subseteqq X$, the subgraph generated by Y is again the digraph of a strict partial order since it is asymmetric and transitive. We call this relation a *suborder* and denote it by (Y, R). (Technically speaking, this is not proper notation. We mean all ordered pairs in $Y \times Y$ that are contained in R.) Figure 4.13 is the suborder of the order relation of Figure 4.2 generated by the set

$$Y = \{\text{New York, Miami, Atlanta, San Francisco, Los Angeles}\}.$$

Algorithm 4.1: Linear Extension

Input: A strict partial order (X, R).
Output: A linear extension of (X, R).

Step 1. Set $m = |X|$.

Step 2. Find a minimal element x in (X, R).

Step 3. Let $X = X - x$, $R = R \cap \{(X - x) \times (X - x)\}$, and $b_m = x$.

Step 4. Decrease m by 1. If m is now 0, stop and output the digraph B whose vertex set is X and where (b_i, b_j) is an arc if $i < j$. If not, return to Step 2.

Here are a few remarks about the algorithm: (1) By Theorem 4.2, a strict partial order will always have a minimal element, which is needed in Step 2. (2) $X - x$ with R as redefined in Step 3 is the suborder we are denoting $(X - x, R)$. It is again a strict partial order and hence has a minimal element. (3) B is easily seen to be a linear extension of (X, R).

To illustrate Algorithm 4.1, consider the strict partial order in Figure 4.19. Step 1 sets $m = 4$ and we can choose the minimal element a_4 for Step 2. Then we consider the strict partial order with a_4 removed and let $b_4 = a_4$; this is Step 3. Next, we can choose the minimal element a_3, let $b_3 = a_3$, and consider the strict partial order with a_3 (and a_4) removed. We can then choose the minimal element

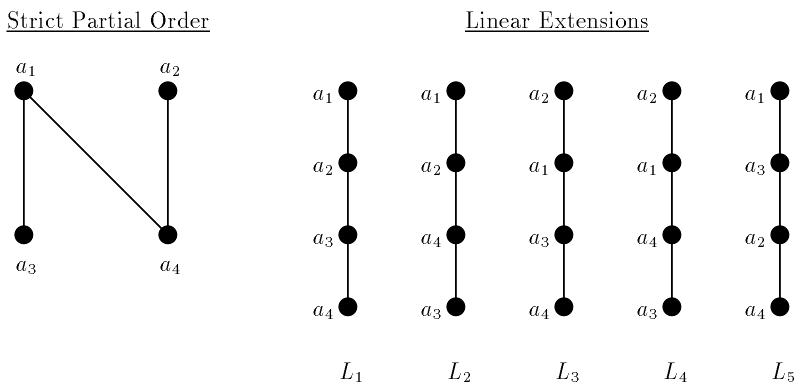

Figure 4.19: A strict partial order on four elements and its linear extensions.

a_2, let $b_2 = a_2$, and consider the strict partial order with a_2 (a_3 and a_4) removed. Finally, we choose the remaining (minimal) element a_1 and let $b_1 = a_1$. The digraph B defined in Step 4 corresponds to the linear extension L_1 in the figure.

Let us consider the set $\mathcal{F}$ of all linear extensions of a strict partial order (X, R). If xRy, the same must be true in every linear extension of $\mathcal{F}$. If xIy, there is at least one linear extension in $\mathcal{F}$ that has x above y and another that has y above x, by Szpilrajn's Extension Theorem. Thus, we get the following result.

Theorem 4.5 If (X, R) is a strict partial order and $\mathcal{F}$ is the set of linear extensions of (X, R), then

$$(X, R) = \bigcap_{L \in \mathcal{F}} L.$$

Since a strict partial order is the intersection of all its linear extensions, a natural question to ask is: How many linear extensions are needed before their intersection is the original strict partial order? Given a strict partial order (X, R), the size of the smallest set of linear extensions whose intersection is (X, R) is called the *dimension* of (X, R), written $\dim(X, R)$. (This idea was introduced by Dushnik and Miller [1941].) Figure 4.19 gives all five linear extensions, L_1, L_2, L_3, L_4, L_5, of the strict partial order (X, R) given there. (Recall that for a set of four elements, there are 4! orderings of the elements. In this case, only five of the $4! = 24$ possible orderings are linear extensions.) Since (X, R) is not a linear order, $\dim(X, R) > 1$, and by Theorem 4.5, $\dim(X, R) \leq 5$. It is easy to see that $L_4 \cap L_5$ equals (X, R). [For example, $a_3 I a_4$ in (X, R) and $a_3 S a_4$ in L_5 while $a_4 S a_3$ in L_4.] Therefore, $\dim(X, R) = 2$. Finding the dimension of an arbitrary order relation is not an easy problem. Given an order relation (X, R), Yannakakis [1982] showed that testing for $\dim(X, R) \leq t$ is NP-complete, for every fixed $t \geq 3$. See Trotter [1996] for a survey article on dimension. Due to the Yannakakis result, most of the results related to dimension take the form of bounds or exact values for specific classes of ordered sets. We present a few of them in the next section.

Example 4.10 Multidimensional Utility Functions Suppose that (X, P) is the strict preference relation of Example 4.1. In Example 4.6, we talked about a function f measuring the value $f(a)$ of each alternative a in X, with a strictly preferred to b if and only if the value of a is greater than the value of b; that is,

$$aPb \leftrightarrow f(a) > f(b).$$

What if we use several characteristics or dimensions, say value $f_1(a)$, beauty $f_2(a)$, quality $f_3(a)$, and so on? We might express strict preference for a over b if and only if a is better than b according to each characteristic; that is,

$$aPb \leftrightarrow [f_1(a) > f_1(b)] \,\&\, [f_2(a) > f_2(b)] \,\&\, \cdots \,\&\, [f_t(a) > f_t(b)]\,. \tag{4.6}$$

Such situations often arise in comparisons of software and hardware. For instance, we might rate two different software packages on the basis of cost, speed, accuracy (measured say by 0 = poor, 1 = fair, 2 = good, 3 = excellent), and ease of use (again using poor, fair, good, excellent). Then we might definitely strictly prefer one package to another if and only if it scores higher on each "dimension." (To be precise, this works only if we use 1/cost rather than cost. Why?) We might also limit our decision on packages to which no other package is strictly preferred. A detailed example is described by Fenton and Pfleeger [1997, p. 225].

If P is defined using (4.6), it is easy to see that (X, P) is a strict partial order. The converse problem is of importance in preference theory. Suppose that we are given a strict partial order (X, P). Can we find functions $f_1, f_2, \ldots, f_t$, each f_i assigning a real number to each a in X, so that (4.6) holds? In fact, if (X, P) has dimension t, with $P = \bigcap_{i=1}^{t} L_i$, we can define f_i so that

$$aL_ib \leftrightarrow f_i(a) > f_i(b).$$

Then (4.6) follows. Hence, we can always find $f_1, f_2, \ldots, f_t$ for sufficiently large t, and the smallest number t for which we can find t such functions is at most the dimension of P. In fact, the smallest number equals the dimension except if the dimension is 2, in which case we might be able to find one function f_1 satisfying (4.6). This occurs for example if (X, P) is a strict weak order. See Baker, Fishburn, and Roberts [1972] for a discussion of the connection between strict partial orders and the multidimensional model for preference given by (4.6). For more on multidimensional utility functions, see Keeney and Raiffa [1993] or Vincke [1992]. ∎

4.3.2 Chains and Antichains

In a strict partial order (X, R), a suborder that is also a strict linear order is called a *chain*. Thus, a chain is a suborder (Y, R) of (X, R) where any two distinct elements of Y are comparable. The *length* of a chain (Y, R) equals $|Y| - 1$, which is the same as the number of edges in the diagram of (Y, R). On the other hand, a suborder in which none of the elements are comparable is called an *antichain.* $L_1 = [a_1, c_2, d_1]$ and $L_2 = [a_3, b_1, c_2, d_1, e_1, f_2]$ are examples of chains in the strict partial order whose diagram is given in Figure 4.14, while (Y, R), where $Y = \{c_1, c_3, c_4\}$, is an example of an antichain. The terms "chain" and "antichain" sometimes refer only to the subset and not the suborder of the strict partial order. A chain or antichain is *maximal* if it is not part of a longer chain or antichain, respectively. Note that chain L_1 is not maximal but L_2 is maximal. Y is not a maximal antichain but $Y \cup \{c_2\}$ is maximal.

Two of the most famous theorems on the relationship between chains and antichains are Dilworth's Theorems.

Theorem 4.6 (Dilworth [1950]) If (X, R) is a strict partial order and the number of elements in a longest chain is j, there are j antichains, X_1, X_2, $\ldots, X_j$, such that

$$X = X_1 \cup X_2 \cup \cdots \cup X_j$$

and

$$X_i \cap X_k = \emptyset$$

for $i \neq k$.

Proof. Let $X_1 = \max(X, R)$, $X_2 = \max(X - X_1, R)$, $X_3 = \max(X - X_1 - X_2, R)$, $\ldots$. Continue this process until $X - X_1 - X_2 - \cdots - X_p = \emptyset$. By the definition of maximal elements, each X_i will be an antichain and

$$X_i \cap X_k = \emptyset$$

for $i \neq k$. To finish the proof we need to show that $j = p$.

Each element in a longest chain must be in a different antichain X_i. Thus, $p \geq j$. Consider X_i and X_{i+1}. For each element x in X_{i+1}, there is an element y in X_i such that yRx. Otherwise, x would have been maximal in $X - X_1 - X_2 - \cdots - X_{i-1}$ and would have been in X_i. Therefore, we can construct a chain using one element from each X_i, so $p \leq j$. Q.E.D.

The proof of the next theorem is similar and is left as an exercise (Exercise 19).

Theorem 4.7 (Dilworth [1950]) If (X, R) is a strict partial order and the size of a largest antichain is j, then there are j chains, $C_1, C_2, \ldots, C_j$, such that

$$X = C_1 \cup C_2 \cup \cdots \cup C_j$$

and

$$C_i \cap C_k = \emptyset$$

for $i \neq k$.

Example 4.11 Subset Containment Let Δ be the set $\{1, 2, \ldots, n\}$. Consider the strict partial order $\subsetneq$ on the set $S =$ subsets of Δ. What is the largest collection of subsets of Δ with the property that no member is contained in another? This is the same as asking for the largest antichain in $(S, \subsetneqq)$.

Clearly, the set of all subsets of a given size k, for any k, is an antichain of size $\binom{n}{k}$. Of these antichains, which is largest? It is easy to prove that the largest antichain of this type occurs when k is "half" of n (see Exercise 26 of Section 2.7). That is, for $0 \leq k \leq n$,

$$\binom{n}{k} \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}. \tag{4.7}$$

Thus, there exists an antichain of size $\binom{n}{\lfloor \frac{n}{2} \rfloor}$. Are there any bigger ones? Sperner [1928] proved that there weren't. Consider the maximal chains in $(S, \subsetneqq)$. A maximal chain is made up of the null set, a subset of size 1, a subset of size 2, ..., and finally the set itself. There are n choices for the subset of size 1. The subset of size 2 must contain the subset of size 1 that was picked previously. Thus, there are $n-1$ choices for the second element in the subset. Continuing to build up the maximal chain in this way, we see that there are $n!$ maximal chains. Suppose that A is an antichain. If $s \in A$ and $|s| = k$, then s belongs to $k!(n-k)!$ maximal chains. (Why? See Exercise 14.) Since at most one member of A belongs to any chain, we get

$$\sum_{s \in A} |s|!(n - |s|)! \leq n!.$$

Dividing through by $n!$ we get

$$\sum_{s \in A} \frac{1}{\binom{n}{|s|}} \leq 1. \tag{4.8}$$

Combining (4.7) and (4.8) yields

$$\sum_{s \in A} \frac{1}{\binom{n}{\lfloor \frac{n}{2} \rfloor}} \leq 1. \tag{4.9}$$

Since the left-hand side of (4.9) has no summands with s in them,

$$\sum_{s \in A} \frac{1}{\binom{n}{\lfloor \frac{n}{2} \rfloor}} = \frac{1}{\binom{n}{\lfloor \frac{n}{2} \rfloor}} \sum_{s \in A} 1 = \frac{1}{\binom{n}{\lfloor \frac{n}{2} \rfloor}} |A|.$$

Combining this with (4.9), we get

$$\frac{|A|}{\binom{n}{\lfloor \frac{n}{2} \rfloor}} \leq 1 \quad \text{or} \quad |A| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

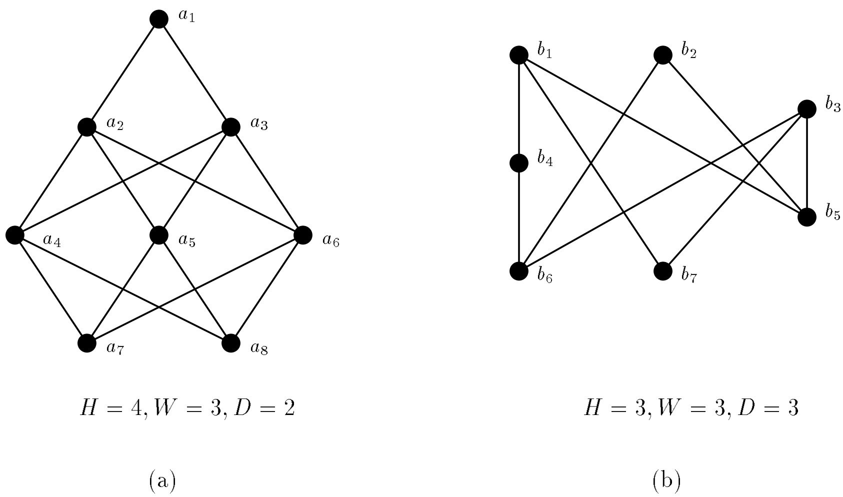

Figure 4.20: Strict partial orders and their height H, width W, and dimension D.

Therefore, all subsets of size $\left\lfloor \frac{n}{2} \right\rfloor$ form the largest antichain in the strict partial order $(S, \subsetneq)$: an antichain of size $\binom{n}{\left\lfloor \frac{n}{2} \right\rfloor}$. ■

The definitions of chain and antichain lead to parameters, *width* and *height*, that can be used to bound the dimension of a strict partial order. The *width* of (X, R), $W(X, R)$, is the size of a maximum-sized antichain, while the *height* of (X, R), $H(X, R)$, equals the length of a maximum-sized chain plus one. The strict partial orders in Figure 4.20 provide examples of these parameters.

The following lemma will be used to provide an upper bound on dimension of a strict partial order using its width.

Lemma 4.1 (Hiraguchi [1955]) Let (X, R) be a strict partial order and $C \subseteqq X$ be a chain. Then there is a linear extension L'_C of R such that xL'_Cy for every $x, y \in X$, $x \in C$, $y \notin C$, and xIy.

Proof. Using Algorithm 4.1 to produce L'_C, always choose an element not in C whenever possible. The linear extensions produced in these cases will certainly satisfy the necessary conditions of the theorem. Q.E.D.

Theorem 4.8 (Dilworth [1950]) Given a strict partial order (X, R),

$$\dim(X, R) \leq W(X, R).$$

Proof. Let $w = W(X, R)$. By Theorem 4.7, there exist w chains, $C_1, C_2, \ldots, C_w$, such that $X = C_1 \cup C_2 \cup \cdots \cup C_w$ and $C_i \cap C_k = \emptyset$, for $i \neq k$. We will use these

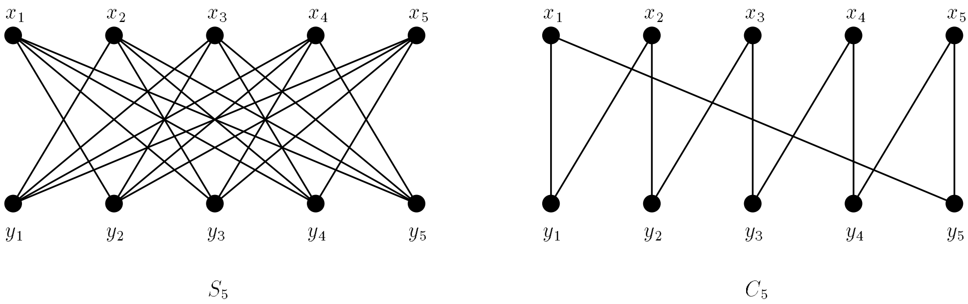

Figure 4.21: Strict partial orders S_5 and C_5.

chains to construct w linear extensions, $L_1, L_2, \ldots, L_w$, whose intersection equals (X, R).

Let L_i equal the linear extension L'_{C_i} from Lemma 4.1 using the chain C_i. Since the chains $C_1, C_2, \ldots, C_w$ are nonoverlapping, any element of X is in one and only one chain. If xIy, there must be distinct chains containing x and y. Call them C_j and C_k, respectively. Then L_j will have xL_jy and L_k will have yL_kx. This along with the fact that the linear extensions contain R assures that $(X, R) = \cap_{i=1}^{w} L_i$.
Q.E.D.

Consider the two strict partial orders S_5 and C_5 in Figure 4.21, each on a set of 10 elements. The width of S_5 equals 5. So, by Theorem 4.8, the dimension of S_5 is at most 5. In fact, it equals 5, showing that the bound of Theorem 4.8 can be attained. To prove this, we will show that no four linear extensions will intersect to produce S_5. Suppose that $L_1 \cap L_2 \cap L_3 \cap L_4 = S_5$. Note that x_iIy_i, for $i = 1, 2, 3, 4, 5$. Thus, y_i precedes x_i in at least one of the linear extensions. By the pigeonhole principle, at least two of these precedences must appear in the same linear extension. Without loss of generality, suppose that y_1 precedes x_1 and y_2 precedes x_2 in L_1. Since x_2 precedes y_1 and x_1 precedes y_2 in S_5, this must also be true in L_1. Therefore, in L_1, we have $y_1L_1x_1$, $x_1L_1y_2$, $y_2L_1x_2$, $x_2L_1y_1$, and transitivity gives us $y_1L_1y_1$, a contradiction.

S_5 is just one example of a whole class of strict partial orders which have equal dimension and width. Generalizing S_5, for $n \geq 3$, let $S_n = (X, R)$ be a strict partial order with $X = \{x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n\}$. The maximal elements of S_n are $\{x_1, x_2, \ldots, x_n\}$, the minimal elements of S_n are $\{y_1, y_2, \ldots, y_n\}$, both sets are antichains, and x_iRy_j if and only if $i \neq j$. S_n will have dimension n and width n. The proof is analogous to that used for S_5.

In other cases, the bound in Theorem 4.8 may be as far from attained as desired. The second strict partial order in Figure 4.21, C_5, has width 5 and dimension 3. The proofs of these facts are left to the exercises [Exercises 10(a) and 10(b)]. Baker, Fishburn, and Roberts [1972] used the term *crown* to refer to this type of strict partial order. In general, they defined a crown order relation $C_n = (X, R)$, $n \geq 3$, as follows. Let $X = \{x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n\}$. The maximal elements of C_n are $\{x_1, x_2, \ldots, x_n\}$, the minimal elements of C_n are $\{y_1, y_2, \ldots, y_n\}$, both sets are antichains, and x_iRy_j if and only if $i = j$ or $j \equiv i - 1 (\bmod n)$. C_n has dimension n and width 3. The proof of this fact is left to the exercises (Exercise 16). Other bounds for dimension are given in the exercises.

4.3.3 Interval Orders

The dimension of many important strict partial orders has been computed. We close this section by commenting on the dimension of one very important class of strict partial orders, the interval orders. To get an interval order, imagine again that there is a collection of alternatives among which you are choosing. You do not know the exact value for each alternative a but you estimate a range of possible values given by a closed interval $J(a) = [\alpha(a), \beta(a)]$. Then you prefer a to b if and only if you are sure that the value of a is greater than the value of b, that is, if and only if $\alpha(a) > \beta(b)$. It is easy to show (Exercise 20) that the corresponding digraph gives a strict partial order, i.e., that it is asymmetric and transitive. (In this digraph, the vertices are a family of closed real intervals, and there is an arc from interval $[a, b]$ to interval $[c, d]$ if and only if $a > d$.) Any strict partial order that arises this way is called an *interval order*. The notion of interval order is due to Fishburn [1970a]. He showed the following.

Theorem 4.9 A digraph $D = (V, A)$ is an interval order if and only if D has no loops and whenever $(a, b) \in A$ and $(c, d) \in A$, then either $(a, d) \in A$ or $(c, b) \in A$.

To illustrate Theorem 4.9, note that strict partial order C_5 of Figure 4.21 is not an interval order since (x_1, y_1) and (x_3, y_3) are arcs of C_5 but (x_1, y_3) and (x_3, y_1) are not.

In studying interval orders, which are somehow one-dimensional in nature, it came as somewhat of a surprise that their dimension as strict partial orders could be arbitrarily large. (This is the content of the following theorem, whose proof uses a generalization of the Ramsey theory results from Section 2.19.3.) This implies that if preferences arise in the very natural way that defines interval orders, we might need very many dimensions or characteristics to explain preference in the sense of (4.6) from Example 4.10.

Theorem 4.10 (Bogart, Rabinovitch, and Trotter [1976]) There are interval orders of arbitrarily high dimension.

EXERCISES FOR SECTION 4.3

1. Does the intersection of the four linear extensions in Figure 4.18 equal the order relation in Figure 4.17?
2. (a) Find two linear extensions whose intersection is the strict partial order (a) in Figure 4.20.

   (b) Find three linear extensions whose intersection is the strict partial order (b) in Figure 4.20.

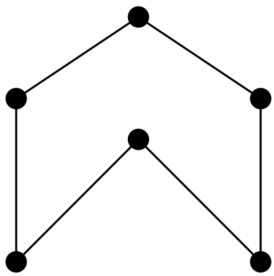

Figure 4.22: A strict partial order.

3. Six software packages (SP), A, B, C, D, E, and F, are rated on the basis of "1/cost," "speed," "accuracy," and "ease of use" using the functions f_1, f_2, f_3, and f_4, respectively. Suppose that the following data have been collected:

SP	f_1	f_2	f_3	f_4
A	1	1	2	2
B	0	0	1	0
C	3	3	2	3
D	3	2	3	3
E	0	1	2	1
F	2	3	3	2

   (a) Use (4.6) to determine the relation P of strict preference.

   (b) Define a new relation P' where $aP'b$ if and only if a scores higher than b on 3 out of 4 of the rating functions. Find P'.

4. Find the dimension of the following strict partial orders.

   (a) Figure 4.22 (b) Figure 4.23(a) (c) Figure 4.23(b)
   (d) Figure 4.23(c) (e) P_1 of Figure 4.24 (f) P_2 of Figure 4.24

5. (a) Give an example of a strict partial order that has exactly two linear extensions.

   (b) Is it possible to give an example of a strict partial order that has exactly three linear extensions?

   (c) If a strict partial order (X, R) has exactly two linear extensions, find $\dim(X, R)$.

6. (a) Use Algorithm 4.1 on the order relation S_5 in Figure 4.21 to find a linear extension. Show the results of each step of the algorithm.

   (b) How many linear extensions exist for S_5?

7. Find the family of linear extensions for the strict partial order (X, R), where $X = \{1, 2, 3, 4, 5, 6\}$ and $R = \{(1, 3), (1, 4), (2, 3), (2, 4), (3, 4), (1, 5), (2, 5), (3, 5), (4, 5), (1, 6), (2, 6), (3, 6), (4, 6)\}$.

8. Find the height and width for each of the strict partial orders in Figure 4.23.

9. (a) Let C be the chain $[\widehat{1}, d, a]$ in the strict partial order (a) of Figure 4.23. Find a linear extension L'_C as in Lemma 4.1.

   (b) Repeat part (a) with $C = [\widehat{1}, x, d, a]$.

   (c) Repeat part (a) with $C = [y, d, \widehat{0}]$ in the strict partial order (b) of Figure 4.23.

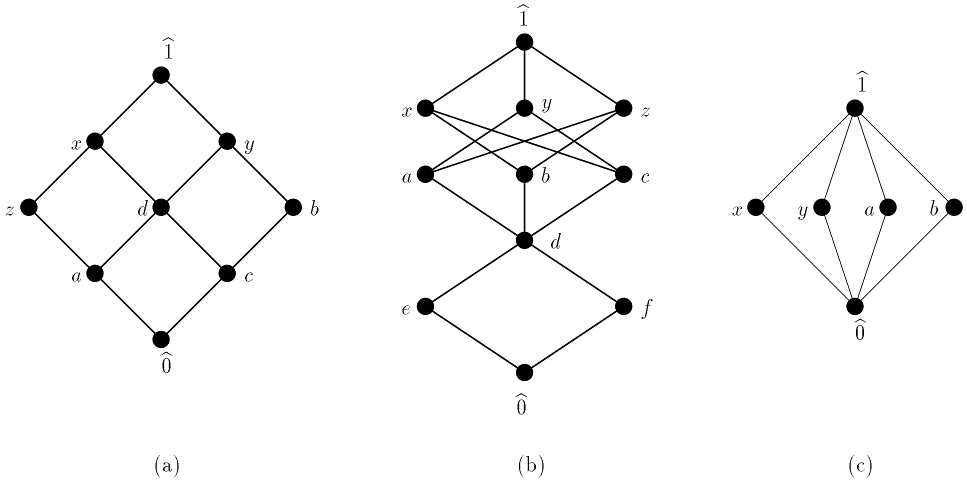

Figure 4.23: Strict partial orders (that are also lattices).

(d) Repeat part (a) with $C = [x]$ in the strict partial order (c) of Figure 4.23.

10. For the strict partial order C_5 of Figure 4.21:
    (a) Prove that the width is 5.
    (b) Prove that the dimension is 3.
    (c) Show that any suborder of C_5 has dimension 2.
11. If X is the set of all subsets of $\{1, 2, 3\}$ and R is the strict partial order $\subsetneqq$, show that (X, R) has dimension 3. (Komm [1948] proved that the strict partial order $\subsetneqq$ on the set of subsets of a set X has dimension $|X|$.)
12. Hiraguchi [1955] showed that if (X, R) is a strict partial order with $|X|$ finite and at least 4, then $\dim(X, R) \leq |X|/2$. Show that dimension can be less than $\lfloor |X|/2 \rfloor$.
13. Show that every strict weak order has dimension at most 2.
14. Suppose that A is an antichain in the subset containment order of Example 4.11. If $s \in A$ and $|s| = k$, prove that s belongs to $k!(n-k)!$ maximal chains.
15. Recall the definition of strict partial order S_n on page 269. Prove that S_n has dimension and width both equaling n.
16. Recall the definition of the crown strict partial order C_n on page 269. Prove that C_n has dimension 3 and width n.
17. Find "j" chains that satisfy Dilworth's Theorem (Theorem 4.7) for the strict partial order P_1 of Figure 4.24.
18. Find an antichain of size "j" that satisfies Dilworth's Theorem (Theorem 4.6) for the strict partial order P_2 of Figure 4.24.
19. Prove Theorem 4.7.
20. Show that if V is any set of closed intervals and there is an arc from $[a, b]$ to $[c, d]$ if and only if $a > d$, then the resulting digraph is a strict partial order.

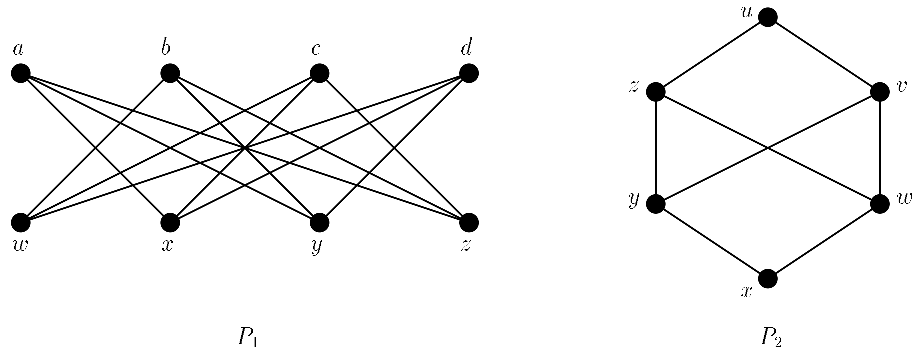

Figure 4.24: Two strict partial orders.

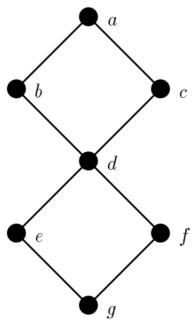

Figure 4.25: A planar strict partial order.

21. (a) Show that if a digraph is defined as in Exercise 20, the necessary condition of Theorem 4.9 is satisfied.

    (b) Use Theorem 4.9 to determine which of the digraphs of Figures 4.19–4.25 are interval orders.

22. (Trotter [1992]) A strict partial order is *planar* if its diagram can be drawn without edges crossing. [While there are planar strict partial orders with arbitrarily large dimension, Trotter [1992] cites a result of Baker showing that a lattice (as defined in Section 4.4) is planar if and only if its dimension is at most 2.]

    (a) Show that strict partial order P_1 in Figure 4.24 is a planar strict partial order by redrawing its diagram without edge crossings.

    (b) Unlike graphs, suborders of planar strict partial orders are not necessarily planar. Find a nonplanar suborder of the planar strict partial order P in Figure 4.25.

23. Recall that the Ramsey number $R(a, b)$ was defined in Section 2.19.3 and revisited in Section 3.8. Show that if a strict partial order has at least $R(a+1, b+1)$ vertices, then it either has a path of $a+1$ vertices or a set of $b+1$ vertices, no two of which are joined by arcs. (A famous theorem of Dilworth [1950] says that the same conclusion holds as long as the strict partial order has at least $ab + 1$ vertices.)

4.4 LATTICES AND BOOLEAN ALGEBRAS

4.4.1 Lattices

Let (X, R) be a strict partial order. Throughout this section we use R^* to denote the binary relation on X defined by $aR^*b \Leftrightarrow a = b$ or aRb. An *upper bound* of a subset $U \subseteq X$ is an element $a \in X$ such that aR^*x for all $x \in U$. If a is an upper bound of U and bRa for all other upper bounds b of U, then a is called the *least upper bound* of U, *lub* U. The terms $\sup U$, $\vee U$, and *join* of U are also used. Similarly, we define a *lower bound* of a subset $U \subseteq X$ to be an element $a \in X$ such that xR^*a for all $x \in U$. If a is a lower bound of U and aRb for all other lower bounds b of U, then a is called the *greatest lower bound* of U, *glb* U. The terms $\inf U$, $\wedge U$, and *meet* of U are also used. If U is only a pair of elements, say $\{x, y\}$, a is the glb of U, and b is the lub of U, we can write

$$a = x \wedge y, \quad b = x \vee y.$$

Consider the strict partial order P_2 in Figure 4.24. The set $\{y, w\}$ has no lub and the set $\{z, v\}$ has no glb. However, $\{z, v, y\}$ has a lub, namely u, while the glb of $\{y, w\}$ is x. Note that u is also the lub of $\{u, z, v\}$.

Example 4.12 Lexicographic Orders (Example 4.7 Revisited) (Stanat and McAllister [1977]) In Example 4.7 we introduced a lexicographic order (X, S). Consider the situation where Σ is the alphabet $\{a, b\}$ and R is the strict linear order defined by $R = \{(a, b)\}$. Then the set of strings of the form $a^m b = aa \cdots ab$ with m a's, $m \geq 0$, has no lub. That is because the only upper bounds are strings of the form $a^m = aa \cdots a$, with m a's and, for any $n > m$, $a^m S a^n$. ■

A *lattice* is a strict partial order in which every pair of elements has a glb and a lub. Sometimes lattices are defined as strict partial orders in which every nonempty subset of elements has a glb and a lub. These definitions are equivalent when dealing with strict partial orders on finite sets. (Why? See Exercise 4.) In what follows we consider only finite lattices, so either definition is acceptable.

Some examples of lattices are:

- The subsets of a given set ordered by strict containment, $\subsetneq$.
- The divisors of a given integer ordered by "proper divisor of."
- The transitive binary relations on a given set ordered by strict containment, $\subsetneq$.
- The stable matchings for a given set of preference orderings ordered by "dominance."

More examples can be found in Figure 4.23.

How many maximal elements does a lattice have? Consider all of the maximal elements of a lattice (X, R), i.e., $\max(X, R)$. Then a lub $\max(X, R)$ exists and

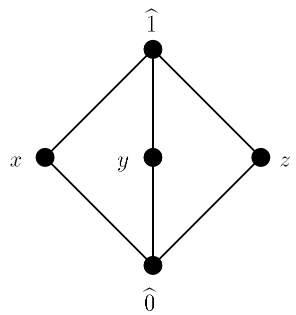

Figure 4.26: A nondistributive, complemented lattice.

must follow all of the elements in $\max(X, R)$. Therefore, $|\max(X, R)| = 1$. A similar argument applies to the minimal elements of (X, R). Thus, we have the following theorem.

Theorem 4.11 Every lattice has a maximum element and a minimum element.

As before, we denote by $\widehat{1}$ the maximum element of a lattice and by $\widehat{0}$ the minimum element. In the lattices of Figure 4.23, $\widehat{1}$ and $\widehat{0}$ are shown.

The following is a list of some basic properties of lattices whose proofs are left for the exercises. Consider a lattice (X, R). For all $a, b, c, d \in X$:

- If aRb and cRd, then
$$(a \wedge c)R(b \wedge d) \text{ and } (a \vee c)R(b \vee d) \quad (\textit{order preserving}).$$
- $a \vee b = b \vee a$ and $a \wedge b = b \wedge a$ (*commutative*).
- $(a \vee b) \vee c = a \vee (b \vee c)$ and $(a \wedge b) \wedge c = a \wedge (b \wedge c)$ (*associative*).
- $a \wedge (a \vee b) = a \vee (a \wedge b) = a$ (*absorptive*).
- $a \vee a = a \wedge a = a$ (*idempotent*).

Notice that a distributive property is not a part of this list. The reason is that not all lattices have such a property. Two *distributive properties* of interest are

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \tag{4.10}$$

and

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \tag{4.11}$$

For lattices, conditions (4.10) and (4.11) are equivalent (see Exercise 7). Lattices satisfying either of these conditions are called *distributive lattices.* For example, in Figure 4.26, $z \wedge (x \vee y) = z$ while $(z \wedge x) \vee (z \wedge y) = \widehat{0}$. Thus, this lattice is not distributive because condition (4.10) is violated.

Another property that lattices may or may not have is called complemented. An element x of a lattice with a maximum element $\widehat{1}$ and minimum element $\widehat{0}$ has a *complement* c if

$$x \vee c = \widehat{1} \text{ and } x \wedge c = \widehat{0}.$$

For example, in Figure 4.26, x has a complement in both y and z. Since $\widehat{0}$ and $\widehat{1}$ always have each other as complements in any lattice, it is the case that every element of this lattice has a complement. If every element of a lattice has a complement, the lattice is said to be *complemented*. A lattice that is both complemented and distributive is called a *Boolean algebra*. This area of lattice theory, Boolean algebras, has a number of important applications, such as to the theory of electrical switching circuits. See Gregg [1998] or Greenlaw and Hoover [1998]. We turn to it next.

4.4.2 Boolean Algebras

We have seen that an element x can have more than one complement. However, this cannot happen in a Boolean algebra.

Theorem 4.12 In a Boolean algebra, each element has one and only one complement.

Proof. Suppose that (X, R) is a Boolean algebra and that $x \in X$. Assume that x has two distinct complements y and z and use the distributive property to reach a contradiction. Details are left as an exercise (Exercise 13). Q.E.D.

If (X, R) is a Boolean algebra and $x \in X$, we let x' denote the complement of x.

Example 4.13 The $\{0, 1\}$-Boolean Algebra Let $X = \{0, 1\}$ and define R on X by $R = \{(1, 0)\}$. Then (X, R) defines a lattice with

$$0 \vee 0 = 0,\ 0 \vee 1 = 1 \vee 0 = 1 \vee 1 = 1,\ 0 \wedge 0 = 0 \wedge 1 = 1 \wedge 0 = 0,\ 1 \wedge 1 = 1. \quad (4.12)$$

We can summarize (4.12) with the following tables:

$$\begin{array}{c|cc} \vee & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 1 \end{array} \qquad \begin{array}{c|cc} \wedge & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 1 \end{array} \quad . \qquad (4.13)$$

We have $\widehat{1} = 1$, $\widehat{0} = 0$, and, moreover, the complement of 1 is 0 and the complement of 0 is 1, which shows that (X, R) is complemented. We can summarize the latter observation by the table

$$\begin{array}{c|c} ' & \\ \hline 0 & 1 \\ 1 & 0 \end{array} \quad . \qquad (4.14)$$

The distributive property is easy to demonstrate and is left to the reader (Exercise 9). Hence, (X, R) defines a Boolean algebra. ■

Example 4.14 Truth Tables Think of 0 as representing the idea that a statement is false (F) and 1 as representing the idea that a statement is true (T). We can think of $\vee$ as standing for the disjunction "or" and $\wedge$ as standing for the conjunction "and." We can replace the tables in (4.13) by the following *truth tables*:

$$\begin{array}{c|cc} \text{or} & \text{F} & \text{T} \\ \hline \text{F} & \text{F} & \text{T} \\ \text{T} & \text{T} & \text{T} \end{array} \qquad \begin{array}{c|cc} \text{and} & \text{F} & \text{T} \\ \hline \text{F} & \text{F} & \text{F} \\ \text{T} & \text{F} & \text{T} \end{array} \quad . \tag{4.15}$$

The first table corresponds to the fact that the statement "p or q" (sometimes written "$p \vee q$") is true if and only if either p is true, q is true, or both are true, while the second table corresponds to the fact that the statement "p and q" (sometimes written "$p \wedge q$") is true if and only if both p and q are true. For instance, using these truth tables, we can conclude that the following statements are true:

- $65 > 23$ or Washington, DC is the capital of the United States.
- $35 + 29 = 64$ and basketball teams play only 5 players at a time.

However, the statement

$$2 + 3 = 6 \quad \text{or} \quad 13 \text{ inches} = 1 \text{ foot}$$

is false. If complement $'$ corresponds to the negation "not," table (4.14) can be written as

$$\begin{array}{c|c} \text{not} & \\ \hline \text{F} & \text{T} \\ \text{T} & \text{F} \end{array} \quad . \tag{4.16}$$

Using (4.15) and (4.16), we can analyze situations in which complex statements are given. This analysis will take the form of a larger *truth table*. For instance, consider the statement "(p or q) and p" [sometimes written "$(p \vee q) \wedge p$"]. We can analyze this statement with a truth table as follows:

p	q	p or q	(p or q) and p
F	F	F	F
F	T	T	F
T	F	T	T
T	T	T	T

The first two columns give all combinations of F and T for statements p and q. The third column shows that "p or q" is true in the case where p, q is F,T or T,F or T,T, respectively, as given by the first table of (4.15). Now in the latter two cases, both "p or q" and p are T, which makes "(p or q) and p" T by the second table of (4.15). A similar analysis can be made of more complex statements. For instance, consider the statement "John lies and (Mary lies or John tells the truth)." Let p = "John lies" and q = "Mary lies." The truth table analysis of our statement gives us

p	q	p'	q or p'	p and (q or p')
F	F	T	T	F
F	T	T	T	F
T	F	F	F	F
T	T	F	T	T

This shows that our statement is true only in the case where both John and Mary lie.

Finally, consider the statement "p' or q." The truth table for this statement is given by

p	q	p' or q
F	F	T
F	T	T
T	F	F
T	T	T

(4.17)

This truth table also describes the logical meaning of the *conditional* statement "if p then q." For when p is true, "if p then q" can only be true when q is also true. When p is false, "if p then q" must be true since the "if" part of the statement is false. Since conditional statements arise often in various contexts, "$p \to q$" is used in place of "p' or q," i.e., "$p' \vee q$," for notational facility. ■

Example 4.15 Logic Circuits The Boolean algebra of Example 4.13 is critical in computer science. We can think of electrical networks as designed from wires that carry two types of voltages, "high" (1) or "low" (0). (Alternatively, we can think of switches that are either "on" or "off," respectively.) We can think of combining inputs with certain kinds of "gates." An *or-gate* takes two voltages x and y as inputs and outputs a voltage $x \vee y$, while an *and-gate* takes x and y as inputs and outputs voltage $x \wedge y$, where $\vee$ and $\wedge$ are defined by (4.13). For example, an and-gate turns two high voltages into a high voltage and turns one high voltage and one low voltage into a low voltage. An *inverter* receives voltage x as input and outputs voltage x', i.e., it turns a high voltage into a low one and a low voltage into a high one.

Figure 4.27 shows a schematic or-gate, and-gate, and inverter. Figure 4.28 shows a circuit diagram. Here, we receive three inputs x_1, x_2, x_3 and calculate the output $(x_1 \vee x_2) \wedge (x_2' \wedge x_3)$. We can think of the computer corresponding to this circuit diagram as calculating the switching function

$$f(x_1, x_2, x_3) = (x_1 \vee x_2) \wedge (x_2' \wedge x_3).$$

Using (4.13) and (4.14), we can calculate f as follows:

x_1	x_2	x_3	$(x_1 \vee x_2) \wedge (x_2' \wedge x_3)$
1	1	1	0
1	1	0	0
1	0	1	1
1	0	0	0
0	1	1	0
0	1	0	0
0	0	1	0
0	0	0	0

■

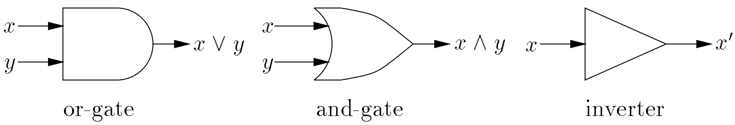

Figure 4.27: Schematic or-gate, and-gate, and inverter.

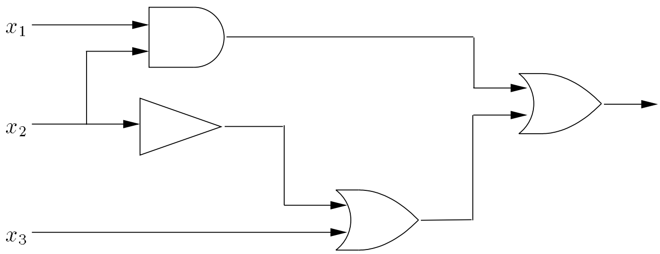

Figure 4.28: A circuit diagram that calculates $(x_1 \vee x_2) \wedge (x_2' \wedge x_3)$.

Example 4.16 Overhead Lighting Consider a room with an overhead light. Each of the three doorways leading to the room has a switch that controls the overhead light. Whenever one of the switches is changed, the light goes off if it was on and on if it was off. How are the switches wired? In most cases, the light and switches are connected by a type of electrical circuit called an *and-or circuit.* And-or circuits are logic circuits.

Switches in the room would correspond to the inputs of the and-or circuit, and the light would correspond to a lone output of the circuit. (More than one output corresponding to multiple lights could be present, but we consider only single-output circuits.) The light is on when the output of the circuit is 1 and off when it is 0. Suppose that the switches in our room are denoted by x, y, and z. Consider an and-or circuit that outputs $(x \wedge y \wedge z) \vee (x \wedge y' \wedge z') \vee (x' \wedge y \wedge z') \vee (x' \wedge y' \wedge z)$. All the possible switch voltages and their corresponding overhead light result are summarized by

x	y	z	$(x \wedge y \wedge z) \vee (x \wedge y' \wedge z') \vee (x' \wedge y \wedge z') \vee (x' \wedge y' \wedge z)$
1	1	1	1
1	1	0	0
1	0	1	0
1	0	0	1
0	1	1	0
0	1	0	1
0	0	1	1
0	0	0	0

(4.18)

Pick any row in (4.18). It is easy to check that a single change in x, y, or z will result in the output changing from high voltage to low voltage or low voltage to high voltage. Therefore, this and-or circuit is precisely what is needed for our overhead light example. ■

EXERCISES FOR SECTION 4.4

1. Consider the strict partial orders of Figure 4.24.
   (a) Does every pair of elements have a lub?
   (b) Does every pair of elements have a glb?
2. In each lattice of Figure 4.23, find:
   (a) lub $\{a, b, x\}$ (b) glb $\{a, b, x\}$
   (c) $x \vee y$ (d) $a \wedge b$
3. (a) Which of the strict partial orders in Figure 4.20 are lattices?
   (b) Which of the strict partial orders in Figure 4.21 are lattices?
4. Suppose that (X, R) is a strict partial order on a finite set X. Prove that every pair of elements of X has a glb and a lub if and only if every nonempty subset of X has a glb and a lub.
5. Let X be an arbitrary set and $\triangledown, \vartriangle$ be binary operations such that
   - $\triangledown$ and $\vartriangle$ are commutative.
   - $\triangledown$ and $\vartriangle$ are associative.
   - absorption holds with $\triangledown$ and $\vartriangle$.
   - $x \triangledown x = x \vartriangle x = x$.

   Prove that (X, R) is a strict partial order where xRy if $x \triangledown y = y$, for all $x, y \in X$.
6. Prove that every lattice is:
   (a) Commutative (b) Associative
   (c) Absorptive (d) Idempotent
7. For lattices, prove that conditions (4.10) and (4.11) are equivalent.
8. Suppose that (X, R) is a lattice and aRb and cRd.
   (a) Show that $(a \wedge c)R(b \wedge d)$. (b) Show that $(a \vee c)R(b \vee d)$.
9. Let (X, R) be the lattice in Example 4.13. Show that (X, R) is distributive.
10. Decide whether or not each lattice in Figure 4.23 is complemented.
11. Decide whether or not each lattice in Figure 4.23 is distributive.
12. Suppose that (X, R) is a distributive lattice. Show that if yRx, then $y \vee (x \wedge z) = x \wedge (y \vee z)$, for all $z \in X$. (Lattices with this property are called *modular.*)
13. Prove that in a Boolean algebra, each element has a unique complement. (*Hint:* Use the distributive property to show that a complement of an element must be unique.)
14. Construct the truth table for the following statements:
    (a) $p \wedge q'$ (b) $(p \wedge q) \vee (p' \wedge q')$
    (c) $(p \wedge q') \rightarrow q$ (d) $(p \wedge q) \rightarrow (p \wedge r)$

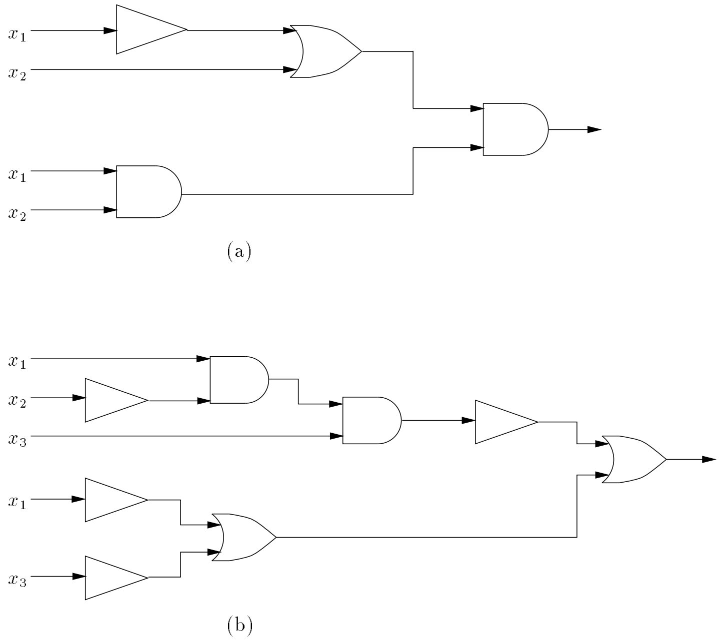

Figure 4.29: Two logic circuits.

15. Find a symbolic form and then construct the truth table for the following statements:

    (a) If Pete loves Christine, then Christine loves Pete.

    (b) Pete and Christine love each other.

    (c) It is not true that Pete loves Christine and Christine doesn't love Pete.

16. Two statements are said to be *equivalent* if one is true if and only if the other is true. One can demonstrate equivalence of two statements by constructing their truth tables and showing that there is a T in the corresponding rows. Use this idea to check if the following pairs of statements are equivalent:

    (a) $p' \wedge q'$ and $(p \vee q)'$ (b) $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$

    (c) $p \vee (p \wedge q)$ and p (d) $(p \rightarrow q) \rightarrow r$ and $p \rightarrow (q \rightarrow r)$

    (e) $(p \vee q) \rightarrow (p \wedge q)$ and $((p \vee q) \wedge (p \wedge q)') \rightarrow (p \vee q)'$

17. Consider the conditional statement $p \rightarrow q$ and these related statements: $q \rightarrow p$ (converse), $p' \rightarrow q'$ (inverse), and $q' \rightarrow p'$ (contrapositive). Which pairs of these four statements are equivalent? (See Exercise 16.)

18. Give the switching function for the following logic circuits:

    (a) Figure 4.29(a) (b) Figure 4.29(b)

19. Draw a logic circuit for the following switching functions:

    (a) $(p \wedge q) \vee (p' \vee q)'$ (b) $(p \vee (q \wedge r)) \wedge ((p \vee q) \wedge (p \vee r))$

(c) $(p \to q) \vee q$ [see (4.17)]

20. Find an and-or circuit to model an overhead light with two switches. (*Hint:* Consider the statement A $\vee$ B, where A and B are each one of $p \wedge q$, $p' \wedge q$, $p \wedge q'$, or $p' \wedge q'$.)

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