Chapter 8

The Pólya Theory of Counting[1]

8.1 EQUIVALENCE RELATIONS

8.1.1 Distinct Configurations and Databases

In this book we have paid a great deal of attention to counting different kinds of configurations. Increasingly, in many fields of science and many areas of scientific application, configurations of various kinds are stored in massive databases. The configurations might be very complex. In medical decisionmaking, we maintain large databases of medical images. In molecular biology, we maintain huge databases of protein structures. In environmental modeling, we keep massive databases of environmental features. Telecommunications and credit card companies maintain gigantic databases of calling and consumption patterns to help discover fraud. The configurations stored in these massive databases are often complex geometrical objects, or objects with a variety of dimensions or properties. The sheer size of the databases encountered makes searching, retrieval, and even just organization a daunting problem. It is sometimes useful to count the number of configurations of a certain kind to help estimate the length of a search in a database. One of the problems encountered is to decide whether or not two configurations are the same. In this chapter we develop techniques for counting the number of distinct configurations of a certain kind. These techniques, of course, make heavy use of the ideas involved in determining precisely whether or not two configurations are the same. Hence, we begin the chapter by studying what it means to say that two things are the same. In the examples in the chapter, it will be much easier to make precise the notion of "sameness" than it is in the examples just described. We will consider notions of sameness for organic molecules, colored (graph-theoretical) trees, switching functions, weak orders, and so on. Because the methods of combinatorics

[1]This chapter should be omitted in an elementary course. In many places, it requires comfort with algebra.

are increasingly important in newer, less precisely defined situations, one can hope that methods such as those described in this chapter will be applicable.

8.1.2 Definition of Equivalence Relations

Suppose that V is a set and S is a set of ordered pairs of elements of V. In Section 4.1.1 we called S a (binary) *relation* on V. For instance, if $V = \{1, 2, 3\}$ and $S = \{(1, 2), (2, 3)\}$, then S is a relation on V. We write aSb if the ordered pair (a, b) is in S. Thus, in our example, $1S2$ but not $2S1$ and not $1S3$.

Suppose that V is a set of configurations and that for a, b in V we write aSb to mean that a and b are the same. Then the relation S should have the following properties (previously defined in Section 4.1.2):

Reflexivity: For all a in V, aSa. (Any configuration is the same as itself.)
Symmetry: For all a, b in V, if aSb, then bSa. (If a is the same as b, then b is the same as a.)
Transitivity: For all a, b, c in V, if aSb and bSc, then aSc. (If a is the same as b and b is the same as c, then a is the same as c.)

If S satisfies these three properties, it is called an *equivalence relation*.

We now give several other examples of equivalence relations. Let V be the set of people in New Jersey, and let aSb mean that a and b have the same height. Then S defines an equivalence relation. Let V be the set of all people in the United States, and let aSb mean that a and b have the same birthdate. Then S is an equivalence relation. Let V be the set of all people in the United States and let aSb mean that a is the father of b. Then S does not define an equivalence relation: It is not reflexive, not symmetric, and not transitive. Let V be the set of real numbers and let aSb mean that $a \leq b$. Then S is not an equivalence relation: It is reflexive and transitive but not symmetric.

Let us give some more complicated examples that illustrate the basic problems we discuss in this chapter.

Example 8.1 Coloring a 2×2 Array Let us consider a 2×2 array in which each block is occupied or not. We color the block black if it is occupied and color it white or leave it uncolored otherwise. Figure 8.1 shows several such colorings. Let V be the collection of all such colorings. Let us suppose that we allow rotation[2] of the array by $0°, 90°, 180°$, or $270°$. We consider colored arrays a and b the same, and write aSb, if b can be obtained from a by one of the rotations in question. Then S defines an equivalence relation. To illustrate, note that Figure 8.1 shows some pairs of arrays that are considered in the relation S. To see why S is an equivalence relation, note that it is reflexive because a can be obtained from a by a $0°$ rotation. It is symmetric because if b can be obtained from a by a rotation of x degrees, then a can be obtained from b by a rotation of $360 - x$ degrees. Finally, it is transitive

[2] All rotations in this chapter are counterclockwise unless noted otherwise.

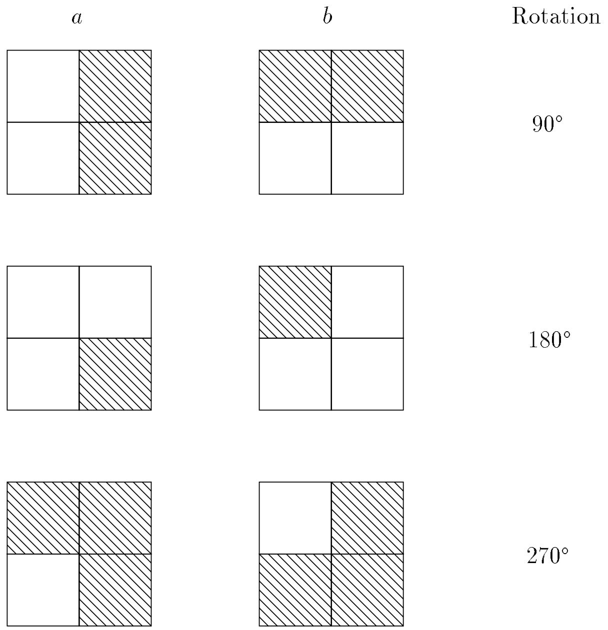

Figure 8.1: Colorings of a 2×2 array. In each case aSb for b is obtained from a by a rotation as indicated.

because if b can be obtained from a by a rotation and c from b by a rotation, then c can be obtained from a by following the first rotation by the second. (It is assumed that a rotation by $360 + x$ degrees is equivalent to a rotation by x degrees.) ■

Example 8.2 Necklaces Suppose that an open necklace consists of a string of k beads, each being either blue or red. Thus, a typical necklace of three beads can be represented by a string such as bbr or brb. A necklace is not considered to have a designated front end, so two such necklaces x and y are considered the same, and we write xSy, if x equals y or if y can be obtained from x by reversing. Thus, bbr is the same as rbb. S defines an equivalence relation. The verification is left to the reader (Exercise 4). ■

Example 8.3 Switching Functions (Example 2.4 Revisited) Recall from Example 2.4 that a switching function of n variables is a function that assigns to every bit string of length n a number 0 or 1. These functions arise in computer engineering. Recall from our discussion in Example 2.4 that certain switching functions are considered equivalent or the same. To make this precise, suppose that T and U are the two switching functions defined by Table 8.1. It is easy to see that $T(x_1x_2) = U(x_2x_1)$ for all bit strings x_1x_2. Thus, T can be obtained from U simply by reordering the input, interchanging the two positions. In this sense, T

Table 8.1: Two Switching Functions, T and U

Bit string x	$T(x)$	$U(x)$
00	1	1
01	0	1
10	1	0
11	1	1

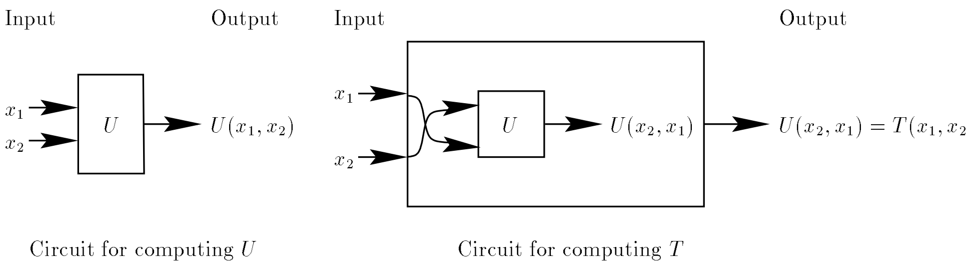

Figure 8.2: A circuit for computing T can be obtained from a circuit for computing U.

and U can be considered equivalent. Indeed, for all practical purposes, they are. For suppose that we can design an electronic circuit which computes U. Then we can design one to compute T, as shown in Figure 8.2, where the circuit computing U is shown as a black box. In general, we consider two switching functions, T and U of two variables the same, and write TSU, if either $T = U$ or $T(x_1x_2) = U(x_2x_1)$ for all bit strings x_1x_2. Then S is an equivalence relation. We leave the proof to the reader (Exercise 5). In what follows, we generalize this concept of equivalence to switching functions of more than two variables. In Section 2.1 we noted that there were many switching functions, even of four variables. Hence, it is impractical to compile a manual listing, for each switching function of n variables, the best corresponding electronic circuit. However, it is not necessary to include every switching function in such a manual, but only enough switching functions so that every switching function of n variables is equivalent to one of the included ones. Counting the number of switching functions required was an historically important problem in computer science (see Section 2.1) and we shall show how to make this computation. ■

Example 8.4 Coloring Trees[3] Let T be a fixed tree, for instance, the binary tree of seven vertices shown in Figure 8.3. Color each vertex of T black or white, and do not distinguish left from right. Let V be the collection of all colorings of T. Let aSb mean that a and b are considered the same, that is, if b can be obtained

[3] This example is from Reingold, Nievergelt, and Deo [1977].

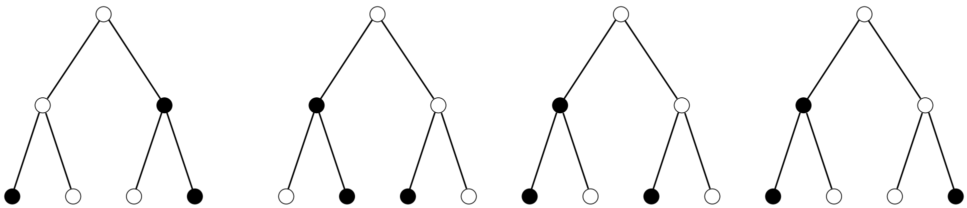

Figure 8.3: Four equivalent colorings of the binary tree of seven vertices.

from a by interchanging left and right subtrees. We shall define this more precisely below. However, since we do not distinguish left from right, it should be clear that all colorings in Figure 8.3 are considered the same. It follows from our general results below that S defines an equivalence relation. ■

Example 8.5 Organic Molecules[4] One of the historically important motivations for the theory developed in this chapter was the desire to count distinct organic molecules in chemistry. Consider the set V of molecules of the form shown in Figure 8.4, where C is a carbon atom and each X can be either CH_3 (methyl), C_2H_5 (ethyl), H (hydrogen), or Cl (chlorine). A typical such molecule is CH_2Cl_2, which has two hydrogen atoms and two chlorine atoms. We can model such a molecule using a regular tetrahedron, a figure consisting of four equilateral triangles that meet at six edges and four corners, as in Figure 8.5. The carbon atom is thought of as being at the center of this tetrahedron and the four components labeled X are at the corners labeled a, b, c, and d. Two such molecules x and y are considered the same, and we write xSy, if y can be obtained from x by one of the following 12 symmetries of the tetrahedron: no change; a rotation by 120° or 240° around a line connecting a vertex and the center of its opposite face (there are eight of these rotations); or a rotation by 180° around a line connecting the midpoints of opposite edges (there are three of these rotations). Figures 8.6 and 8.7 illustrate the second and third kinds of symmetries. ■

Example 8.6 Number of Weak Orders (Example 5.29 Revisited) Recall from Section 4.2 that Figure 4.14 shows a typical weak order R on a set A. Each element has a horizontal level, all elements a and b at the same horizontal level satisfy aRb and bRa, and otherwise, aRb iff a is at a higher level than b. We shall consider two weak orders on a set A to be the same if they have the same number of levels and the same number of elements at corresponding levels. For example, the first two weak orders shown in Figure 8.8 are considered the same. The first and third weak orders are, in fact, identical as weak orders since they have the same set of ordered pairs, $\{(a, c)$, (a, d), (a, e), (a, f), (a, g), (a, h), (b, c), (b, d), (b, e), (b, f), (b, g), (b, h), (c, d), (c, e), (c, f), (c, g), (c, h), (d, g), (d, h), (e, g), (e, h), (f, g),

[4] This example is from Liu [1968]. For more extensive treatment of chemical compounds from the point of view of this chapter, see Pólya and Read [1987].

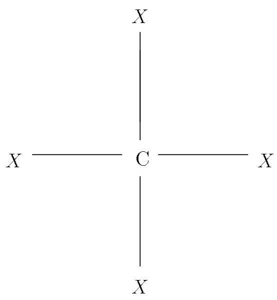

Figure 8.4: An organic molecule.

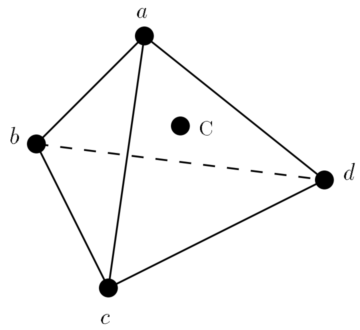

Figure 8.5: A regular tetrahedron.

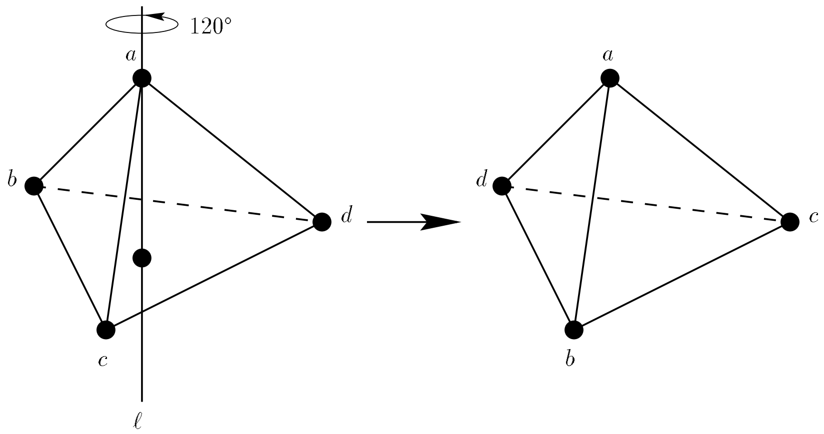

Figure 8.6: Rotation by 120° around a line ℓ connecting vertex a and the center of the opposite face.

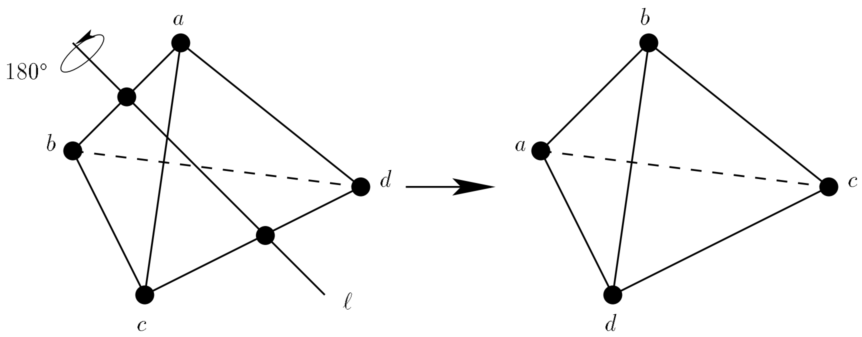

Figure 8.7: Rotation by 180° around a line ℓ connecting the midpoints of opposite edges.

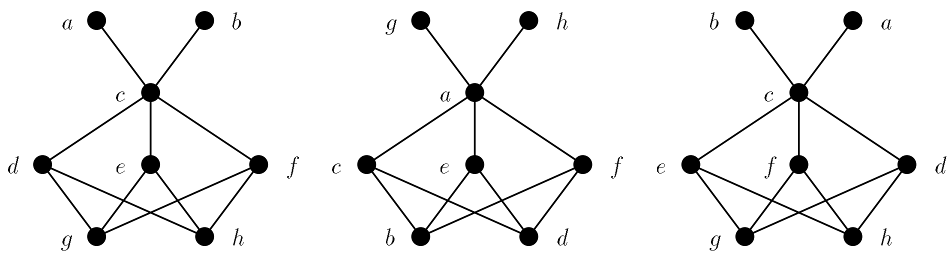

Figure 8.8: Three weak orders considered the same.

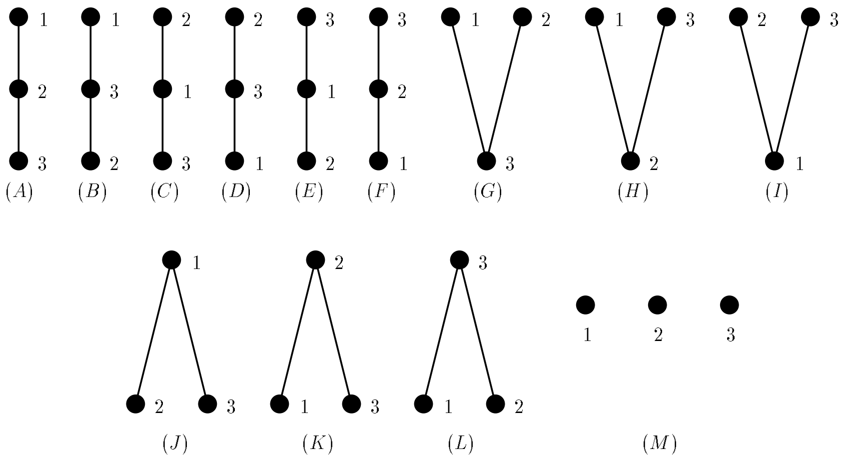

Figure 8.9: All weak orders on $\{1, 2, 3\}$.

$(f, h)\}$. It is easy to see that if aSb means that a and b are the same, S defines an equivalence relation among weak orders. Figure 8.9 shows all possible weak orders on $\{1, 2, 3\}$. Note that, for example, ASD, GSI, JSL. ■

8.1.3 Equivalence Classes

An equivalence relation S on V divides the elements of V into classes called *equivalence classes.* Specifically, if a is any element of V, the *equivalence class containing* a, $C(a)$, consists of all elements b such that aSb, i.e., $C(a) = \{b \in V : aSb\}$. By reflexivity, aSa, so every element of V is in some equivalence class; in particular, $a \in C(a)$. Moreover, for all a, b in V, either $C(a) = C(b)$ or $C(a)$ and $C(b)$ are disjoint. For suppose that x is in both $C(a)$ and $C(b)$. Then aSx and bSx. By symmetry, aSx and xSb. Transitivity now implies that aSb. This shows that $C(a) = C(b)$. For if y is in $C(b)$, then bSy. Now aSb and bSy imply aSy, so y is in $C(a)$. Thus, $C(b) \subseteqq C(a)$. Similarly, we can show that $C(a) \subseteqq C(b)$. Thus, $C(a) = C(b)$. If we now think of $C(a)$ and $C(b)$ as being the same if they have the

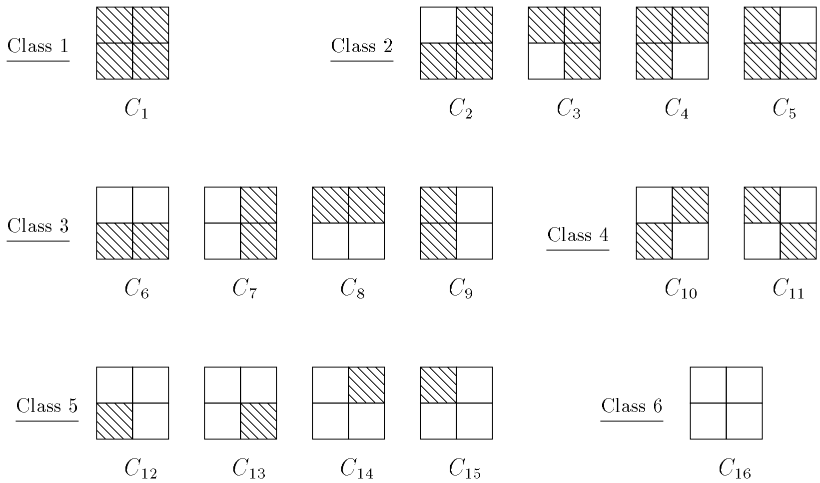

Figure 8.10: Equivalence classes of black-white colorings of the 2×2 array.

same members, we have the following theorem.

Theorem 8.1 If S is an equivalence relation, every element is in one and only one equivalence class.

To illustrate this result, note that in Example 8.2, if the necklaces have length 2, there are four kinds of necklaces, bb, br, rb, and rr. The second and third are equivalent. Thus, for instance, $C(bb) = \{bb\}$ and $C(br) = \{br, rb\}$. There are three distinct equivalence classes, $\{bb\}$, $\{br, rb\}$, and $\{rr\}$.

In Example 8.1, there are six equivalence classes. These are shown in Figure 8.10.

EXERCISES FOR SECTION 8.1

1. In each of the following cases, is S an equivalence relation on V? If not, determine which of the properties of an equivalence relation hold.

   (a) $V =$ real numbers, aSb iff $a = b$.

   (b) $V =$ real numbers, aSb iff $a \neq b$.

   (c) $V =$ real numbers, aSb iff a divides evenly into b.

   (d) $V =$ all subsets of $\{1, 2, \ldots, n\}$, aSb iff a and b have the same number of elements.

   (e) V as in part (d), aSb iff a and b overlap.

   (f) $V =$ all people in the world, aSb iff a is a sibling of b.

   (g) $V =$ all people in the United States, aSb iff a and b have the same blood type.

   (h) $V = \{1, 2, 3, 4\}$, $S = \{(1, 1), (2, 2), (3, 4), (4, 3), (1, 3), (3, 1)\}$.

(i) $V = \{w, x, y, z\}$, $S = \{(x, x),\ (y, y),\ (z, z),\ (w, w),\ (x, z),\ (z, x),\ (x, w),\ (w, x),\ (z, w),\ (w, z)\}$.

(j) V = all residents of California, aSb iff a and b live within 10 miles of each other.

2. Suppose that V is the set of bit strings of length 4, and aSb holds if and only if a and b have the same number of 1's. Is (V, S) an equivalence relation?

3. For each of the following equivalence relations, identify all equivalence classes.

(a) $V = \{a, b, c, d\}, S = \{(a, a), (b, b), (c, c), (d, d), (a, b), (b, a), (c, d), (d, c)\}$.

(b) $V = \{u, v, w\}$, $S = \{(u, u),\ (v, v),\ (w, w),\ (u, v),\ (v, u),\ (v, w),\ (w, v),\ (u, w),\ (w, u)\}$.

(c) $V = \{1, 2, 3, 4\}$, $S = \{(1, 1),\ (2, 2),\ (3, 3),\ (4, 4),\ (1, 2),\ (2, 1),\ (2, 3),\ (3, 2),\ (1, 3),\ (3, 1)\}$.

(d) V = the set of all positive integers, aSb iff $|a - b|$ is an even number.

(e) $V = \{1, 2, 3, \ldots, 15\}$, aSb iff $a \equiv b(\text{mod } 3)$.

4. Show that S of Example 8.2 is an equivalence relation.

5. Show that S of Example 8.3 is an equivalence relation among switching functions of two variables.

6. In Example 8.2, identify the equivalence classes of necklaces of length 3.

7. In Example 8.2, identify the equivalence classes of necklaces of length 2 if each bead can be one of three colors: blue, red, or purple.

8. In Example 8.1, suppose that we can use any of three colors: black (b), white (w), or red (r). Describe all equivalence classes of colorings.

9. In Example 8.1, suppose that we allow not only rotations but also reflections in either a vertical, a horizontal, or a diagonal line. (The latter would switch the colors assigned to two diagonally opposite cells.) Identify all equivalence classes of colorings. (Only two colors are used, black and white.)

10. In Example 8.6, identify the equivalence classes of the weak orders of Figure 8.9.

11. In Example 8.6, describe the equivalence classes of weak orders on $\{1, 2, 3, 4\}$.

12. In Example 8.3, identify all equivalence classes of switching functions of two variables.

13. For each tree of Figure 8.11, draw all trees that are equivalent to it in the sense of Example 8.4.

14. The *complement* x' of a bit string x is obtained from x by interchanging all 0's and 1's. For instance, if $x = 00110$, then $x' = 11001$. Suppose that we consider two switching functions T and U of n variables the same if $T = U$ or $T(x) = U(x')$ for every bit string x. Describe all equivalence classes of switching functions under this sameness relation if $n = 3$.

15. Suppose that V is the set of all colorings of the binary tree of Figure 8.12 in which each vertex gets one of the colors black or white. Find all equivalence classes of colorings if two colorings are considered the same if one can be obtained from the other by interchanging the colors of the vertices labeled 1 and 2.

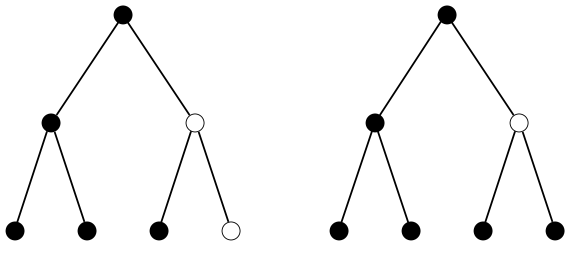

Figure 8.11: Trees for Exercise 13, Section 8.1.

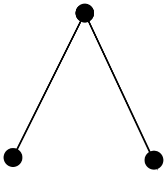

Figure 8.12: Tree for Exercise 15, Section 8.1.

16. In Example 8.5, suppose that a molecule has three hydrogen atoms and one nonhydrogen. How many other molecules with three hydrogens and one nonhydrogen are considered the same as this one?
17. Consider a square and let V be the set of all colorings of its vertices using colors red and blue. For colorings f and g, let fSg hold if g can be obtained from f by rotating the square by $0°$, $90°$, $180°$, or $270°$. Show that (V, S) is an equivalence relation and find all equivalence classes.
18. Generalizing Exercise 17, let V be the set of all colorings of the vertices of a regular p-gon using colors in $\{1, 2, \ldots, n\}$, and let fSg hold for colorings f and g if g can be obtained from f by rotating the p-gon through one of the angles $k(360/p)$ for $k = 0, 1, \ldots, p-1$. Count the number of equivalence classes if:

    (a) $p = 5$, $n = 2$ (b) $p = 6$, $n = 3$ (c) $p = 12$, $n = 2$

19. Repeat Exercise 17 with fSg holding if g can obtained from f by rotating the square by $0°$, $90°$, $180°$, or $270°$ or by reflecting about a line joining opposite corners of the square or by reflecting about a line joining midpoints of opposite sides of the square.
20. Consider the set $A = \{1, 2, \ldots, n\}$.

    (a) How many binary relations are possible on A?
    (b) How many reflexive relations are possible on A?
    (c) How many symmetric relations are possible on A?
    (d) How many transitive relations are possible on A?

(e) How many equivalence relations are possible on A when $n = 4$?

21. Suppose that V is the set of unlabeled graphs of n vertices and that aSb iff a and b are isomorphic.

    (a) Show that S is an equivalence relation on V.

    (b) Find one unlabeled graph from each equivalence class if $n = 3$.

22. Let E_n equal the number of equivalence relations on a set $A = \{1, 2, \ldots, n\}$. Show that E_n satisfies the following recurrence:

$$E_n = \sum_{i=0}^{n-1} \binom{n-1}{i} E_i, \qquad n \geq 1.$$

8.2 PERMUTATION GROUPS

8.2.1 Definition of a Permutation Group

In studying examples such as Examples 8.1–8.5, we make heavy use of the notion of a permutation. Recall that a permutation of a set $A = \{1, 2, ..., n\}$ is an ordering of the elements of A. The permutation that sends 1 to a_1, 2 to a_2, and so on, can be written as

$$\begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ a_1 & a_2 & a_3 & \cdots & a_n \end{pmatrix},$$

or as $a_1 a_2 \cdots a_n$ for short. Thus, the permutation 132 stands for

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}.$$

Similarly, the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}$$

is written as 3142.

A permutation of A can also be thought of as a function from A onto itself. This function must be one-to-one. Thus, the permutation

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

can be thought of as the function $\pi : \{1, 2, 3\} \to \{1, 2, 3\}$ defined by $\pi(1) = 1$, $\pi(2) = 3$, $\pi(3) = 2$. Similarly, if A is any finite set, any one-to-one function from A into A can be thought of as a permutation of A; we simply identify elements of A with the integers $1, 2, \ldots, n$. For instance, suppose that $A = \{a, b, c, d\}$ and $f(a) = b$, $f(b) = c$, $f(c) = d$, $f(d) = a$. If $a = 1, b = 2, c = 3, d = 4$, f can be thought of as the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}.$$

Suppose that π_1 and π_2 are permutations of the set A. We can define the *product* or *composition*, $\pi_1 \circ \pi_2$, of the permutations π_1 and π_2 as the permutation that first permutes by the permutation π_2 and then permutes the resulting arrangement by the permutation π_1. For instance, if

$$\pi_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix} \quad \text{and} \quad \pi_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix},$$

then

$$\pi_1 \circ \pi_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix}.$$

For 1 is sent to 2 by π_2, which is sent to 2 by π_1, so the composition sends 1 to 2. That is, $\pi_1 \circ \pi_2(1) = \pi_1(\pi_2(1)) = \pi_1(2) = 2$. Similarly, 2 is sent to 1 by π_2 and 1 to 4 by π_1, so the composition sends 2 to 4; and so on.

Let X be the collection of all permutations of the set A. Note that this collection of permutations satisfies the following conditions:

Condition **G1** (*Closure*). If $\pi_1 \in X$ and $\pi_2 \in X$, then $\pi_1 \circ \pi_2 \in X$.
Condition **G2** (*Associativity*). If $\pi_1, \pi_2, \pi_3 \in X$, then

$$\pi_1 \circ (\pi_2 \circ \pi_3) = (\pi_1 \circ \pi_2) \circ \pi_3.$$

Condition **G3** (*Identity*). There is an element $I \in X$, called the *identity*, so that for each $\pi \in X$,

$$I \circ \pi = \pi \circ I = \pi.$$

Condition **G4** (*Inverse*). For each $\pi \in X$, there is a $\pi^{-1} \in X$, called the *inverse* of π, so that

$$\pi \circ \pi^{-1} = \pi^{-1} \circ \pi = I.$$

To verify these conditions, note, for example, that **G3** follows by taking I to be the permutation

$$\begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & 2 & \cdots & n \end{pmatrix}.$$

Also, **G4** holds if we take π^{-1} to be the permutation that reverses what π does. For example, if

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix}$$

then

$$\pi^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \end{pmatrix}.$$

G1 has been tacitly assumed above. Its verification and that of **G2** are straightforward.

If X is any set and $\circ$ defines a product[5] on elements of X, the pair $G = (X, \circ)$ is called a *group* if the four properties G1, G2, G3, G4 hold. Let us give some examples of groups. If X is the positive real numbers and $a \circ b$ means $a \times b$, the pair $(X, \circ)$ is a group. Axiom **G1** holds because $a \times b$ is always a positive real number if a and b are positive reals. Axiom **G2** holds because $a \times (b \times c) = (a \times b) \times c$. Axiom **G3** holds because we take I to be 1. Axiom **G4** holds because we take a^{-1} to be $1/a$.

Another example of a group is $(X, \circ)$, where X is all the real numbers and $a \circ b$ is defined to be $a + b$. The identity element for Axiom **G3** is the number 0 and the inverse of element a is $-a$. Note that the real numbers where $a \circ b$ is defined to be $a \times b$ do not define a group. For the only possible identity is 1. But then the number 0 does not have an inverse: There is no number 0^{-1} so that $0 \times 0^{-1} = 1$.

We shall be interested in groups of permutations, or *permutation groups.* We have observed that the collection of *all* permutations of $A = \{1, 2, \ldots, n\}$ defines a group. This permutation group is called the *symmetric group.* Another example of a permutation group consists of the following three permutations of the set $\{1, 2, 3\}$:

$$\pi_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \quad \pi_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \quad \pi_3 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}. \tag{8.1}$$

It is left to the reader (Exercise 4) to verify that the group axioms are satisfied here.

Often, the symmetries of physical objects or configurations define groups, and hence the theory of groups is very important in modern physics. To give an example, consider the symmetries of the 2×2 array studied in Example 8.1: namely, the rotations by $0°, 90°, 180°$, and $270°$. These symmetries define a group if we take $a \circ b$ to mean first perform symmetry b and then perform symmetry a. For instance, if a is rotation by $90°$ and b is rotation by $180°$, then $a \circ b$ is rotation by $270°$.

This group of symmetries can be thought of as a permutation group, each symmetry permuting $\{1, 2, 3, 4\}$. To see why, let us label the four cells in the 2×2 array as in the first part of Figure 8.13. Then Figure 8.13 shows the resulting labeling from the different symmetries. We can think of this labeling as corresponding to a permutation that takes the label i into the label j. For example, we can think of the $90°$ rotation as the permutation that takes 1 into 4, 2 into 1, 3 into 2, and 4 into 3, that is, the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}.$$

The permutations corresponding to the other rotations are also shown in Figure 8.13.

Suppose that A is any finite set and f is any one-to-one function from A into A. Then as we have observed before, f can be thought of as a permutation of A. If X is a collection of such functions and $\circ$ is the composition of functions and $G = (X, \circ)$

[5] Technically, a product $\circ$ is a function that assigns to each pair of elements a and b of X, another element (of X) denoted $a \circ b$. (Note that we can either define $a \circ b$ to always be an element of X, or make explicit condition **G1**.)

0° rotation

2	1
3	4

$\pi_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$

90° rotation

1	4
2	3

$\pi_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$

180° rotation

4	3
1	2

$\pi_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$

270° rotation

3	2
4	1

$\pi_4 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$

Figure 8.13: The permutations corresponding to the rotations of the 2×2 array.

is a group, we can think of G as a permutation group. For instance, suppose that $A = \{a, b, c\}$ and f, g, and h are defined as follows:

$$\begin{array}{lll} f(a) = a, & f(b) = b, & f(c) = c; \\ g(a) = b, & g(b) = c, & g(c) = a; \\ h(a) = c, & h(b) = a, & h(c) = b. \end{array}$$

Then f, g, and h are one-to-one functions. It is easy to show that if $X = \{f, g, h\}$, then $(X, \circ)$ is a group. It is a permutation group. Indeed, if we take $a = 1$, $b = 2$, and $c = 3$, then f, g, and h are the permutations π_1, π_2, and π_3 of (8.1), so this is exactly the permutation group that we encountered earlier using different notation.

8.2.2 The Equivalence Relation Induced by a Permutation Group

Suppose that $G = (X, \circ)$ is a permutation group on a set A. We will sometimes use $\pi \in G$ to mean $\pi \in X$. We can define a sameness relation S on A by saying that

$$aSb \text{ iff there is a permutation } \pi \text{ in } G \text{ such that } \pi(a) = b; \tag{8.2}$$

that is, π takes a into b. For instance, if $A = \{1, 2, 3\}$ and G consists of the three permutations of (8.1), then $1S2$ because $\pi_2(1) = 2$ and $3S2$ because $\pi_3(3) = 2$. It is easy to see that for this S, aSb for all a, b.

Theorem 8.2 If G is a permutation group on a set A, then S as defined in (8.2) defines an equivalence relation on A.

Proof. We have to show that S satisfies reflexivity, symmetry, and transitivity. Since the identity permutation I is in G, $I(a) = a$ for all $a \in A$, so aSa holds for all a. Thus, reflexivity holds. If aSb, there is π in G so that $\pi(a) = b$. Now π^{-1} is in G and $\pi^{-1}(b) = a$. We conclude that bSa. Thus, symmetry holds. Finally, suppose that aSb and bSc. Then there are π_1 and π_2 in G so that $\pi_1(b) = c$ and $\pi_2(a) = b$. Then $\pi_1 \circ \pi_2(a) = c$, so aSc follows. Q.E.D.

The relation S will be called the *equivalence relation induced by the permutation group G*.

Let us give several more examples. If G is the group of permutations of $\{1,2,3,4\}$ shown in Figure 8.13, that is, the group of rotations of the 2×2 array, then aSb for all a, b in $\{1,2,3,4\}$. Thus, S has one equivalence class, $\{1,2,3,4\}$. Next, suppose that $A=\{1,2,3\}$ and G consists of the permutations

$$\pi_1=\begin{pmatrix}1&2&3\\1&2&3\end{pmatrix}\quad\text{and}\quad\pi_2=\begin{pmatrix}1&2&3\\3&2&1\end{pmatrix}.$$

Then G is a group (Exercise 3). Moreover, the equivalence classes under S are $\{1,3\}$ and $\{2\}$. Recall that $C(a)$, the equivalence class containing a, consists of all b in A such that aSb, equivalently all b in A such that $\pi(a)=b$ for some π in G. Thus,

$$C(a)=\{\pi(a):\pi\in G\}.$$

In the special case of a permutation group, $C(a)$ is sometimes called the *orbit* of a. In the example we have just given,

$$C(1)=\{\pi_1(1),\pi_2(1)\}=\{1,3\}$$

is the orbit of 1.

In counting the number of distinct configurations, we shall be interested in counting the number of (distinct) equivalence classes under a sameness relation. One way to count is simply to compute all the equivalence classes and enumerate them. But this is often impractical. In the next section we present a method for counting the number of equivalence classes without enumeration.

8.2.3 Automorphisms of Graphs

Let H be a fixed, unlabeled graph.[6] An *automorphism* of H is a permutation π of the vertices of H so that if $\{x,y\}\in E(H)$, then $\{\pi(x),\pi(y)\}\in E(H)$. To use the terminology of Section 3.1.3, an automorphism is an isomorphism from a graph into itself. Consider, for example, the graph of Figure 8.14. We can define an automorphism by labeling the vertices as $1,2,3,4,5$ as shown and taking $\pi(1)=4,\pi(2)=5,\pi(3)=1,\pi(4)=2,\pi(5)=3$. This is the same as a rotation by $144°$. A second automorphism is obtained by taking $\pi=\begin{pmatrix}1&2&3&4&5\\1&5&4&3&2\end{pmatrix}$. This is the same as a reflection through the line joining vertex 1 to the midpoint of edge $\{3,4\}$.

As a second example, consider the graph known as $K_{1,3}$ and shown in Figure 8.15. A vertex labeling using the integers 1, 2, 3, 4 is shown. One example of an automorphism of $K_{1,3}$ is obtained by rotating the labeled figure by $120°$ clockwise. This corresponds to the permutation $\begin{pmatrix}1&2&3&4\\1&3&4&2\end{pmatrix}$. Reflection through the edge $\{1,2\}$ produces the automorphism $\begin{pmatrix}1&2&3&4\\1&2&4&3\end{pmatrix}$. There are six automorphisms

[6] In this chapter, we reserve G for a group and use H for a graph.

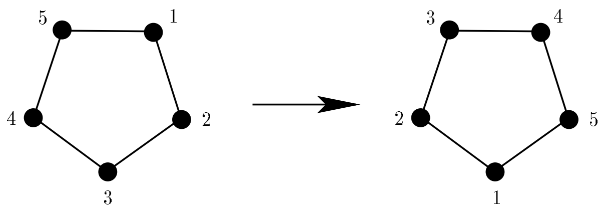

Figure 8.14: An automorphism is given by $\pi(1) = 4, \pi(2) = 5, \pi(3) = 1, \pi(4) = 2, \pi(5) = 3$.

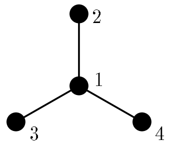

Figure 8.15: The graph $K_{1,3}$.

in all: the identity, the ones obtained by rotations of 120° and 240°, and the ones obtained by reflections through the three edges. How do we know that there are no others? Clearly, every automorphism must take 1 into 1. Thus, we need to look for permutations of $\{1, 2, 3, 4\}$ that take 1 into 1, and there are $3! = 6$ of them. The sameness relation of (8.2) gives two equivalence classes, $\{1\}$ and $\{2, 3, 4\}$.

Theorem 8.3 The set of all automorphisms of a graph is a permutation group.

Proof. Left to the reader (Exercise 26). Q.E.D.

We shall use the notation $\mathrm{Aut}(H)$ for the automorphism group of H. Thus, for instance, $\mathrm{Aut}(K_n)$ is the symmetric group on $\{1, 2, \ldots, n\}$, since all permutations of the vertices of K_n define automorphisms. For more on automorphism groups of graphs, see, for example, Cameron [1983] and Gross and Yellen [1999].

EXERCISES FOR SECTION 8.2

1. Write each of the following permutations in the form

$$\begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ a_1 & a_2 & a_3 & \cdots & a_n \end{pmatrix}.$$

(a) 7263541 (b) 52413 (c) 67583412

2. Find $\pi_1 \circ \pi_2$ if π_1 and π_2 are as follows:

(a) $\pi_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}, \quad \pi_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix}$

(b) $\pi_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}, \quad \pi_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{pmatrix}$

(c) $\pi_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 5 & 4 \end{pmatrix}$, $\pi_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 5 & 1 & 3 \end{pmatrix}$

(d) $\pi_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 4 & 5 & 6 & 2 \end{pmatrix}$, $\pi_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 1 & 3 & 6 & 5 \end{pmatrix}$

3. Suppose that $A = \{1, 2, 3\}$ and X is the set of permutations

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}.$$

If $\circ$ is composition, show that $(X, \circ)$ is a group.

4. Suppose that $A = \{1, 2, 3\}$ and X is the set of the three permutations given in Equation (8.1). Show that X defines a group under composition.
5. For each of the following X and $\circ$, check which of the four axioms for a group hold.

(a) X = the permutations $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}$ and $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 3 & 2 & 1 \end{pmatrix}$ and $\circ$ = composition

(b) X = the permutations $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}$ and $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 2 & 4 & 5 \end{pmatrix}$ and $\circ$ = composition

(c) $X = \{0, 1\}$ and $\circ$ is defined by the following rules: $0 \circ 0 = 0, 0 \circ 1 = 1, 1 \circ 0 = 1, 1 \circ 1 = 1$

(d) X = rational numbers, $\circ$ = addition

(e) X = rational numbers, $\circ$ = multiplication

(f) X = negative real numbers, $\circ$ = addition

(g) X = all 2×2 matrices of real numbers, $\circ$ = matrix multiplication

6. Show that the set of functions $\{f, g\}$ is a group of permutations on $A = \{x, y, u, v\}$ if $f(x) = v, f(y) = u, f(u) = y, f(v) = x$, and $g(x) = x, g(y) = y, g(u) = u, g(v) = v$.
7. Is the conclusion of Exercise 6 still true if f is redefined by $f(x) = y, f(y) = x, f(u) = v, f(v) = u$?
8. Suppose that $A = \{1, 2, 3, 4, 5, 6\}$ and G is the following group of permutations:

$$\left\{ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix} \right\}.$$

If S is the equivalence relation induced by G:

(a) Is $1S2$? (b) Is $3S5$? (c) Is $5S6$?

9. In Exercise 8, find the orbits $C(1)$ and $C(4)$.
10. If A and G are as follows, find the equivalence classes under the equivalence relation S induced by G.

(a)
$$\begin{aligned} A &= \{1, 2, 3, 4, 5\}, \\ G &= \left\{ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 3 & 2 & 1 \end{pmatrix} \right\} \end{aligned}$$

(b) $A = \{1,2,3,4,5,6\}$,

$$G = \left\{\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 3 & 4 & 5 & 6 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 4 & 3 & 5 & 6 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 6 & 5 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 4 & 3 & 5 & 6 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 3 & 4 & 6 & 5 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 4 & 3 & 6 & 5 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 4 & 3 & 6 & 5 \end{pmatrix}\right\}$$

(c) $A = \{1,2,3,4,5\}$,

$$G = \left\{\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 2 & 4 & 5 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 2 & 3 & 4 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 2 & 4 & 1 \end{pmatrix}\right\}$$

11. Consider the collection of all symmetries of the 2×2 array described in Exercise 9, Section 8.1. If π_1 and π_2 are the following symmetries, find $\pi_1 \circ \pi_2$.

    (a) π_1 = rotation by 90°, π_2 = reflection in a horizontal line
    (b) π_1 = reflection in a vertical line, π_2 = rotation by 180°
    (c) π_1 = rotation by 270°, π_2 = reflection in a vertical line
    (d) π_1 = rotation by 180°, π_2 = reflection in the diagonal going from lower left to upper right

12. Continuing with Exercise 11, describe the following symmetries as permutations of $\{1,2,3,4\}$:

    (a) Reflection in a horizontal line
    (b) Reflection in a vertical line
    (c) Reflection in a diagonal going from lower left to upper right
    (d) Reflection in a diagonal going from upper left to lower right

13. Continuing with Exercise 11, is the collection of all the symmetries (rotations and reflections) a group?
14. In Example 8.1, suppose that we can use any of c colors in any of the squares.

    (a) How many distinct colorings are possible with only rotations allowed?
    (b) How many distinct colorings are possible with rotations and reflections, in either a vertical, horizontal, or diagonal line, allowed?

15. Find the permutation corresponding to each automorphism of graph $K_{1,3}$ that is not described in the text.
16. In the situation of Exercise 17, Section 8.1, find $\pi_1 \circ \pi_2$ if π_1 and π_2 are rotations by 180° and 270°, respectively.
17. In the situation of Exercise 19, Section 8.1, find $\pi_1 \circ \pi_2$ if π_1 and π_2 are:

    (a) Rotation by 180° and reflection about a horizontal line joining midpoints of opposite sides, respectively
    (b) Reflection about a vertical line joining midpoints of opposite sides and rotation by 90°, respectively
    (c) Reflection about a positively sloped diagonal line (/) joining opposite corners and reflection about a horizontal line joining midpoints of opposite sides, respectively

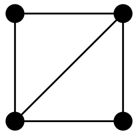

Figure 8.16: $K_4 - K_2$.

18. In the situation of Exercise 19, Section 8.1, is the collection of all symmetries described a group?
19. If π_1 and π_2 are permutations, $\pi_1 \circ \pi_2$ may not equal $\pi_2 \circ \pi_1$. (Thus, we say that the product of permutations is not necessarily commutative.)

    (a) Demonstrate this with $\pi_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{pmatrix}$ and $\pi_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix}$.

    (b) Find two symmetries π_1 and π_2 of the 2×2 array (π_1 and π_2 can be rotations or reflections) such that $\pi_1 \circ \pi_2 \neq \pi_2 \circ \pi_1$.

20. Show that for all prime numbers p, the set of integers $\{1, 2, \ldots, p-1\}$ with $\circ$ equal to multiplication modulo p forms a group.
21. In Exercise 20, do we still get a group if p is not a prime? Why?
22. Suppose that G is a permutation group. Fix a permutation σ in G. If π_1 and π_2 are in G, we say that $\pi_1 S \pi_2$ if $\pi_1 = \sigma^{-1} \circ (\pi_2 \circ \sigma)$. Show that S is an equivalence relation.
23. (a) Find $\text{Aut}(L_4)$, where L_4 is the chain of four vertices.

    (b) Find $\text{Aut}(Z_4)$, where Z_4 is the circuit of four vertices.

    (c) Find $\text{Aut}(K_4 - K_2)$, where $K_4 - K_2$ is the graph shown in Figure 8.16.
24. Describe $\text{Aut}(Z_n)$ and find the number of automorphisms of Z_n.
25. The graph $K_{m,n}$ has m vertices in one class, n vertices in a second class, and edges between all pairs of vertices in different classes. $K_{1,3}$ of Figure 8.15 is a special case.

    (a) If $m \neq n$, describe $\text{Aut}(K_{m,n})$ and find the number of automorphisms of $K_{m,n}$.

    (b) Repeat part (a) for $m = n$.

26. Prove Theorem 8.3.

8.3 BURNSIDE'S LEMMA

8.3.1 Statement of Burnside's Lemma

In this section we present a method for counting the number of (distinct) equivalence classes under the equivalence relation induced by a permutation group. Suppose that G is a group of permutations of a set A. An element a in A is said to be *invariant* (or *fixed*) under a permutation π of G if $\pi(a) = a$. Let $\text{Inv}(\pi)$ be the number of elements of A that are invariant under π.

Theorem 8.4 (Burnside's Lemma[7]) Let G be a group of permutations of a set A and let S be the equivalence relation on A induced by G. Then the number of equivalence classes in S is given by

$$\frac{1}{|G|}\sum_{\pi\in G}\mathrm{Inv}(\pi).$$

To illustrate this theorem, let us first consider the set $A = \{1, 2, 3\}$ and the group G of permutations of A defined by Equation (8.1). Then $\mathrm{Inv}(\pi_1) = 3$ since 1, 2, and 3 are invariant under π_1, and $\mathrm{Inv}(\pi_2) = \mathrm{Inv}\ (\pi_3) = 0$, since no element is invariant under either π_2 or π_3. Hence, the number of equivalence classes under the induced equivalence relation S is given by $\frac{1}{3}(3 + 0 + 0) = 1$. This is correct, since aSb holds for all a, $b \in A$. There is just one equivalence class, $\{1, 2, 3\}$.

To give a second example, suppose that $A = \{1, 2, 3, 4\}$ and G consists of the following permutations:

$$\begin{aligned}
\pi_1 &= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}, & \pi_2 &= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix}, \\
\pi_3 &= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix}, & \pi_4 &= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}.
\end{aligned} \tag{8.3}$$

It is easy to check that G is a group. Now $\mathrm{Inv}(\pi_1) = 4$, $\mathrm{Inv}\ (\pi_2) = 2$, $\mathrm{Inv}(\pi_3) = 2$, $\mathrm{Inv}(\pi_4) = 0$, and the number of equivalence classes under the induced equivalence relation S is $\frac{1}{4}(4 + 2 + 2 + 0) = 2$. This is correct since the two equivalence classes are $\{1,\ 2\}$ and $\{3,\ 4\}$ (Exercise 1).

As a third example, consider the set A of all weak orders on $\{1, 2, 3\}$, as shown in Figure 8.9. Every permutation π of $\{1, 2, 3\}$ induces a permutation π^* of A by replacing each element i in a weak order by $\pi(i)$. For instance, if $\pi = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$, then $\pi^*(J) = K$ for weak orders J and K shown in Figure 8.9. The set of all π^* defines a permutation group G of A. There are six permutations in G, one corresponding to each of $\pi_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$, $\pi_2 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$, $\pi_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$, $\pi_4 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$, $\pi_5 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$, $\pi_6 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$. If we use the sameness relation defined in Example 8.6, then $\mathrm{Inv}(\pi_1^*) = 13$ since π_1^* leaves invariant all 13 weak orders of Figure 8.9. $\mathrm{Inv}(\pi_2^*) = 3$ since π_2^* leaves invariant weak orders I, J, and M. Similarly, $\mathrm{Inv}(\pi_3^*) = \mathrm{Inv}(\pi_6^*) = 3$. Finally, $\mathrm{Inv}(\pi_4^*) = \mathrm{Inv}(\pi_5^*) = 1$ since π_4^* and π_5^* leave only M invariant. By Burnside's Lemma, the number of equivalence classes of weak orders is given by

$$\frac{1}{6}[13 + 3 + 3 + 1 + 1 + 3] = 4.$$

[7] This version of the lemma is a simple consequence of the crucial lemma given by Burnside [1911] and is usually called Burnside's Lemma.

The equivalence classes are given by $\{A, B, C, D, E, F\}$, $\{G, H, I\}$, $\{J, K, L\}$, and $\{M\}$.

As a fourth example, consider the automorphism group $G = \mathrm{Aut}(K_{1,3})$. From our discussion in Section 8.2.3, $|G| = 6$. $\mathrm{Inv}(\pi) = 4$ for the identity π, $\mathrm{Inv}(\pi) = 1$ for the rotations by 120° and 240°, and $\mathrm{Inv}(\pi) = 2$ for the three reflections through an edge. Thus,

$$\frac{1}{|G|}\sum_{\pi\in G}\mathrm{Inv}(\pi) = \frac{1}{6}[4+1+1+2+2+2] = 2,$$

which agrees with our conclusion that $\{1\}$ and $\{2, 3, 4\}$ are the equivalence classes under S.

In Section 8.4 we shall see how to apply Burnside's Lemma to examples such as Examples 8.1–8.5.

8.3.2 Proof of Burnside's Lemma[8]

We now present a proof of Burnside's Lemma. Suppose that G is a group of permutations on a set A. For each $a \in A$, let $\mathrm{St}(a)$, the *stabilizer* of a, be the set of all permutations in G under which a is invariant, i.e., $\mathrm{St}(a) = \{\pi \in G : \pi(a) = a\}$. Let $C(a)$ be the orbit of a, the equivalence class containing a under the induced equivalence relation S, that is, the set of all b such that $\pi(a) = b$ for some π in G. To illustrate, suppose that $A = \{1, 2, 3\}$ and G is defined by Equation (8.1). Then $C(2) = \{\pi_1(2), \pi_2(2), \pi_3(2)\} = \{1, 2, 3\}$. Also, $\mathrm{St}(2) = \{\pi_1\}$.

Lemma 8.1 Suppose that G is a group of permutations on a set A and a is in A. Then

$$|\mathrm{St}(a)| \cdot |C(a)| = |G|.$$

Proof. Suppose that $C(a) = \{b_1, b_2, \ldots, b_r\}$. Then there is a permutation π_1 that sends a to b_1. (There may be other permutations that send a to b_1, but we pick one such.) There is also a permutation π_2 that sends a to b_2, a permutation π_3 that sends a to b_3, and so on. Let $P = \{\pi_1, \pi_2, \ldots, \pi_r\}$. Note that $|P| = |C(a)|$. We shall show that every permutation π in G can be written in exactly one way as the product of a permutation in P and a permutation in $\mathrm{St}(a)$. It then follows by the product rule that $|G| = |P| \cdot |\mathrm{St}(a)| = |C(a)| \cdot |\mathrm{St}(a)|$.

Given π in G, note that $\pi(a) = b_k$, some k. Thus, $\pi(a) = \pi_k(a)$, so $\pi_k^{-1} \circ \pi$ leaves a invariant. Thus, $\pi_k^{-1} \circ \pi$ is in $\mathrm{St}(a)$. But

$$\pi_k \circ (\pi_k^{-1} \circ \pi) = (\pi_k \circ \pi_k^{-1}) \circ \pi = I \circ \pi = \pi,$$

so π is the product of a permutation in P and a permutation in $\mathrm{St}(a)$.

Next, suppose that π can be written in two ways as a product of a permutation in P and a permutation in $\mathrm{St}(a)$; that is, suppose that $\pi = \pi_k \circ \gamma = \pi_l \circ \delta$, where γ, δ are in $\mathrm{St}(a)$. Now $(\pi_k \circ \gamma)(a) = b_k$ and $(\pi_l \circ \delta)(a) = b_l$. Since $\pi_k \circ \gamma = \pi_l \circ \delta$, b_k must equal b_l, so $k = l$. Thus, $\pi_k \circ \gamma = \pi_k \circ \delta$, and by multiplying by π_k^{-1}, we conclude that $\gamma = \delta$. Q.E.D.

[8] This subsection may be omitted.

To illustrate this lemma, let $A = \{1, 2, 3\}$ and let G be defined by (8.1). By our computation above, $C(2) = \{1, 2, 3\}$ and $\mathrm{St}(2) = \{\pi_1\}$ Thus,

$$|G| = 3 = (1) \cdot (3) = |\mathrm{St}(2)| \cdot |C(2)|.$$

To complete the proof of Burnside's Lemma, we show that if $A = \{1, 2, \ldots, n\}$ and $G = \{\pi_1, \pi_2, \ldots \pi_m\}$, then

$$\mathrm{Inv}(\pi_1) + \mathrm{Inv}(\pi_2) + \cdots + \mathrm{Inv}(\pi_m) = |\mathrm{St}(1)| + |\mathrm{St}(2)| + \cdots + |\mathrm{St}(n)|.$$

This is true because both sides of this equation count the number of ordered pairs (a, π) such that $\pi(a) = a$. It then follows by Lemma 8.1 that

$$\frac{1}{|G|}[\mathrm{Inv}(\pi_1) + \mathrm{Inv}(\pi_2) + \cdots + \mathrm{Inv}(\pi_m)] = \frac{1}{|C(1)|} + \frac{1}{|C(2)|} + \cdots + \frac{1}{|C(n)|}. \quad (8.4)$$

Note that x is always in $C(x)$, since $I(x) = x$. Thus, by Theorem 8.1, $C(x) = C(y)$ iff x is in $C(y)$. Hence, if $C(x) = \{b_1, b_2, \ldots, b_k\}$, there are exactly k equivalence classes $C(b_1), C(b_2), \ldots, C(b_k)$ that equal $C(x)$. It follows that we may split the equivalence classes up into groups such as $\{C(b_1), C(b_2), \ldots, C(b_k)\}$, each group being a list of identical equivalence classes. Note that $|C(b_i)| = k$. Thus,

$$\frac{1}{|C(b_1)|} + \frac{1}{|C(b_2)|} + \cdots + \frac{1}{|C(b_k)|} = \frac{1}{k} + \frac{1}{k} + \cdots + \frac{1}{k} = 1.$$

It follows that the sum on the right-hand side of (8.4) will count the number of distinct equivalence classes, so this number is also given by the left-hand side of (8.4). Burnside's Lemma follows.

To illustrate the proof, suppose that $A = \{1, 2, 3, 4\}$ and G is given by the four permutations of (8.3). Then $C(1) = \{1, 2\}, C(2) = \{1, 2\}, C(3) = \{3, 4\}$, and $C(4) = \{3, 4\}$. Thus,

$$\frac{1}{|C(1)|} + \frac{1}{|C(2)|} = 1,$$
$$\frac{1}{|C(3)|} + \frac{1}{|C(4)|} = 1,$$

and

$$\frac{1}{|C(1)|} + \frac{1}{|C(2)|} + \frac{1}{|C(3)|} + \frac{1}{|C(4)|} = 2,$$

the number of equivalence classes.

EXERCISES FOR SECTION 8.3

1. Verify that the four permutations of (8.3) define a group and that the equivalence classes under the induced equivalence relation are $\{1, 2\}$ and $\{3, 4\}$.
2. For each automorphism a in $\mathrm{Aut}(K_{1,3})$, calculate $\mathrm{St}(a)$, $C(a)$, and verify Lemma 8.1.

3. Verify the proof of Burnside's Lemma for $\mathrm{Aut}(K_{1,3})$ by calculating
$$\frac{1}{|C(1)|}+\frac{1}{|C(2)|}+\frac{1}{|C(3)|}+\frac{1}{|C(4)|}.$$
4. In Exercise 8, Section 8.2, use Burnside's Lemma to find the number of equivalence classes under S.
5. In each case of Exercise 10, Section 8.2, use Burnside's Lemma to find the number of equivalence classes under S and check by computing the equivalence classes.
6. For each case of Exercise 10, Section 8.2, let $a = 1$.

   (a) Find $\mathrm{St}(a)$. (b) Find $C(a)$. (c) Verify Lemma 8.1.
7. Repeat Exercise 6 with $a = 3$.
8. For each case of Exercise 10, Section 8.2, check that
$$\frac{1}{|C(1)|}+\frac{1}{|C(2)|}+\cdots$$
gives the number of equivalence classes under S.
9. For every automorphism a of the graph of Figure 8.14, calculate $\mathrm{St}(a)$ and $C(a)$ and verify Lemma 8.1.
10. For the graph H of Figure 8.14, verify the proof of Burnside's Lemma for $\mathrm{Aut}(H)$ by calculating
$$\frac{1}{|C(1)|}+\frac{1}{|C(2)|}+\frac{1}{|C(3)|}+\frac{1}{|C(4)|}+\frac{1}{|C(5)|}$$
and comparing to the number of equivalence classes under S.
11. Use Burnside's Lemma to calculate the number of equivalence classes of weak orders on $\{1, 2, 3, 4\}$ if sameness is defined as in Example 8.6.
12. Use Burnside's Lemma to compute the number of distinct ways to seat 5 negotiators in fixed chairs around a circular table if rotating seat assignments around the circle is not considered to change the seating arrangement.
13. Suppose that
$$A = \{1, 2, 3, 4\},\quad G = \left\{\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix}\right\}.$$
Is $|\mathrm{St}(1)| \cdot |C(1)| = |G|$? Explain what happened.
14. Suppose that we label the n vertices of a graph H with the labels $1, 2, \ldots, n$. Any labeling of H can be thought of as a permutation of $\{1, 2, \ldots, n\}$, if we start with a fixed labeling. For instance, if we start with the original labeling shown in Figure 8.14, the new labeling shown in Figure 8.17 corresponds to the permutation
$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 5 & 3 & 4 \end{pmatrix}.$$
Every automorphism π of H can be thought of as taking any labeling σ of H into another labeling: We simply use the labeling $\pi \circ \sigma$. Thus, the number of distinct labelings of H corresponds to the number of equivalence classes in the equivalence relation induced on the set of permutations of $\{1, 2, \ldots, n\}$ by the automorphism group $\mathrm{Aut}(H)$.

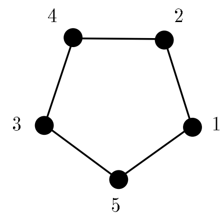

Figure 8.17: A new labeling of the graph of Figure 8.14.

(a) Show that the number of distinct labelings is given by $n!/|\mathrm{Aut}(H)|$.

(b) If L_4 is the chain of four vertices, find the number of distinct labelings from the result in part (a) and check by enumerating the labelings.

(c) Repeat for Z_4, the circuit of length 4.

(d) Repeat for $K_{1,3}$, the graph of Figure 8.15.

15. Using the methods of this section, do Exercise 25 from the Additional Exercises for Chapter 2.

8.4 DISTINCT COLORINGS

8.4.1 Definition of a Coloring

Suppose that D is a collection of objects. A *coloring of* D assigns a color to each object in D. In this sense, if D is the vertex set of a graph, a coloring simply assigns a color to each vertex, independent of the rule used in Chapter 3 that if x and y are joined by an edge, they must get different colors. A coloring can be thought of as a function $f : D \to R$, where R is the set of colors. If D has n elements and R has m elements, there are m^n colorings of D.

In Example 8.1, the set D is the set of four boxes in the 2×2 array, and the set R is the set {black, white}. In Example 8.2, the set D can be thought of as the integers $1, 2, \ldots, k$, representing the k spaces for beads, and R is the set $\{b, r\}$. In Example 8.3, D is the set of bit strings of length n, and R is the set $\{0, 1\}$. In Example 8.4, D is the set of vertices of the tree of seven vertices, and R = {black, white}. Finally, in Example 8.5, the set D consists of the vertices a, b, c, d of the regular tetrahedron, and the set R is the set $\{\mathrm{CH_3}, \mathrm{C_2H_5}, \mathrm{H}, \mathrm{Cl}\}$.

Every graph G on the vertex set $V = \{1, 2, \ldots, p\}$ can be thought of as a coloring. Take D to be the set of all 2-element subsets of V, R to be $\{0, 1\}$, and let $f(\{i, j\})$ be 1 if $\{i, j\} \in E(G)$ and 0 otherwise. Exercises 21 and 22 exploit this idea to compute the number of distinct (nonisomorphic) graphs of p vertices.

In all of our examples, we also allow certain permutations of the elements of D. These permutations define a group G. In particular, in Example 8.1 we allow the four rotations given in Figure 8.13, which define a group G of permutations. In

Example 8.2, in the case of two beads, the permutations are

$$\pi_1 = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \quad \text{and} \quad \pi_2 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

Note that π_1 and π_2 define a group—this is the group G. More generally, if there are k beads, the group G is the group of the two permutations

$$\begin{pmatrix} 1 & 2 & \cdots & k \\ 1 & 2 & \cdots & k \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 2 & \cdots & k \\ k & k-1 & \cdots & 1 \end{pmatrix}.$$

In Example 8.5, the permutations in the group G correspond to the symmetries of the regular tetrahedron that were described in Section 8.1. We return to this example in Section 8.5, where we describe a simple way to represent these permutations.

What is the group in Example 8.4? Suppose that we start with the first labeled tree shown in Figure 8.18, that with π_1 under it. Then any other labeling of this tree corresponds to a permutation of $\{1, 2, \ldots, 7\}$. Not every permutation of $\{1, 2, \ldots, 7\}$ corresponds to a labeling which is considered equivalent in the sense that left and right have been interchanged. Figure 8.18 shows all labeled trees obtained from the first one by interchanging left and right. For instance, the labeled tree with π_2 under it is obtained by interchanging vertices 4 and 5, and the labeled tree with π_3 under it is obtained by interchanging vertices 6 and 7. The labeled tree with π_4 under it is obtained by interchanging the subtree T_1 generated by vertices $2, 4, 5$ with the subtree T_2 generated by vertices $3, 6, 7$. The labeled tree with π_5 under it is obtained by interchanging both 4 and 5 and 6 and 7. The labeled tree with π_6 under it is obtained by first interchanging subtrees T_1 and T_2 and then interchanging vertices 6 and 7; and so on. These eight trees correspond to the legitimate permutations of the elements of D, the members of the group G. The permutation corresponding to each labeled tree is also shown in Figure 8.18. Note that

$$\pi_4 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 3 & 2 & 6 & 7 & 4 & 5 \end{pmatrix},$$

because vertex 1 stays unchanged, vertex 2 is changed to vertex 3 and vertex 3 to vertex 2, and so on. Verification that G is a group is left to the reader (Exercise 19). It is not hard to show that the permutations we have described are exactly the automorphisms of the first tree of Figure 8.18.

We shall discuss Example 8.3, the switching functions, shortly. In Examples 8.1–8.5, we are interested in determining whether or not two colorings are distinct and in counting the number of equivalence classes of colorings. However, the equivalence relation of two colorings being the same is not the same as the equivalence relation S induced by the permutation group G. For S is a relation on the set D, not on the set of colorings of D. Thus, direct use of Burnside's Lemma would not help us to count the number of equivalence classes of colorings. In Section 8.4.2 we discuss how to define the appropriate equivalence relation.

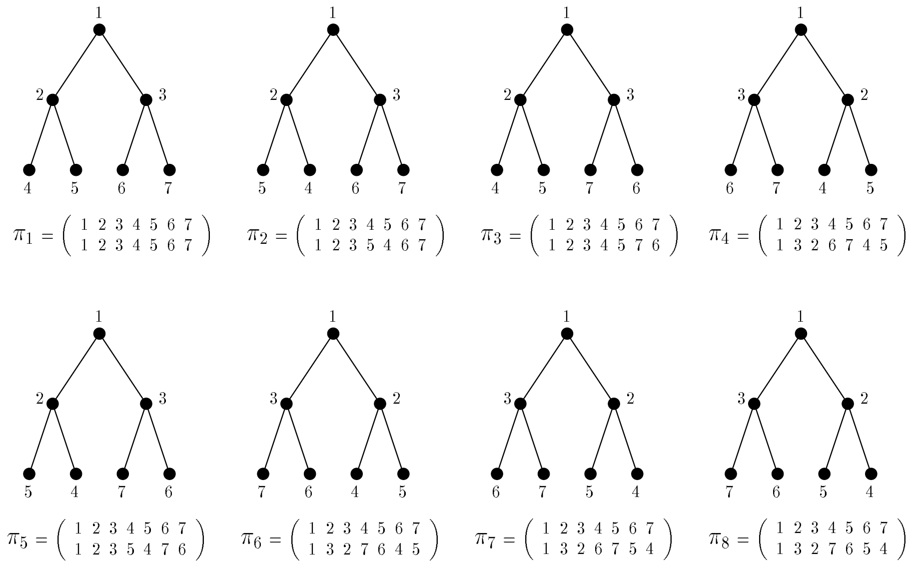

Figure 8.18: A labeled tree and the eight labeled subtrees obtained from it by interchanging left and right.

8.4.2 Equivalent Colorings

Suppose that $C(D, R)$ is the set of all colorings of D using colors in R and that G is a group of permutations of the set D and π is in G. Corresponding to π is a permutation π^* of $C(D, R)$. π^* takes each coloring in $C(D, R)$ into another coloring. If f is a coloring, the new coloring $\pi^* f$ is defined by taking $(\pi^* f)(a)$ to be $f(\pi(a))$. That is, $\pi^* f$ assigns to a the same color that f assigns to $\pi(a)$. In Example 8.1, π^* takes a given coloring C_i of the 2×2 array into another one. For instance, if π_2 is the $90°$ rotation of the array, as shown in Figure 8.13, let us compute π_2^*. If $C_1, C_2, \ldots$ are as in Figure 8.10, first note that $C_1(x) =$ black, all x, so $(\pi_2^* C_1)(a) = C_1(\pi_2(a)) =$ black. That means that $(\pi_2^* C_1)(a) =$ black, all a, so $\pi_2^* C_1$ is the coloring C_1. Next, since $4 = \pi_2(1), (\pi_2^* C_2)(1) = C_2(\pi_2(1)) = C_2(4) =$ black. Also, $(\pi_2^* C_2)(2) = C_2(\pi_2(2)) = C_2(1) =$ black, $(\pi_2^* C_2)(3) = C_2(\pi_2(3)) = C_2(2) =$ white, and $(\pi_2^* C_2)(4) = C_2(\pi_2(4)) = C_2(3) =$ black. Thus, $\pi_2^* C_2$ is the same as the coloring C_3. Similarly, $\pi_2^* C_3 = C_4, \pi_2^* C_4 = C_5$, and so on. In sum, the permutation π_2^* is given by

$$\pi_2^* = \begin{pmatrix} C_1 & C_2 & C_3 & C_4 & C_5 & C_6 & C_7 & C_8 & C_9 & C_{10} & C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_1 & C_3 & C_4 & C_5 & C_2 & C_7 & C_8 & C_9 & C_6 & C_{11} & C_{10} & C_{13} & C_{14} & C_{15} & C_{13} & C_{16} \end{pmatrix}. \quad (8.5)$$

Similarly, if π_1 is the $0°$ rotation, π_3 is the $180°$ rotation, and π_4 is the $270°$ rotation, then

$$\pi_1^* = \begin{pmatrix} C_1 & C_2 & C_3 & C_4 & C_5 & C_6 & C_7 & C_8 & C_9 & C_{10} & C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_1 & C_2 & C_3 & C_4 & C_5 & C_6 & C_7 & C_8 & C_9 & C_{10} & C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \end{pmatrix}, \quad (8.6)$$

$$\pi_3^* = \begin{pmatrix} C_1 & C_2 & C_3 & C_4 & C_5 & C_6 & C_7 & C_8 & C_9 & C_{10} & C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_1 & C_4 & C_5 & C_2 & C_3 & C_8 & C_9 & C_6 & C_7 & C_{10} & C_{11} & C_{14} & C_{15} & C_{12} & C_{13} & C_{16} \end{pmatrix}, \quad (8.7)$$

and

$$\pi_4^* = \begin{pmatrix} C_1 & C_2 & C_3 & C_4 & C_5 & C_6 & C_7 & C_8 & C_9 & C_{10} & C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_1 & C_5 & C_2 & C_3 & C_4 & C_9 & C_6 & C_7 & C_8 & C_{11} & C_{10} & C_{15} & C_{12} & C_{13} & C_{14} & C_{16} \end{pmatrix}. \quad (8.8)$$

Thus, the group G of permutations of D corresponds to a group G^* of permutations of $C(D,R)$; $G^* = \{\pi^* : \pi \in G\}$. (Why is G^* a group?) Note that G and G^* have the same number of elements. Moreover, if S^* is the equivalence relation induced by G^*, then S^* is the sameness relation in which we are interested. Under S^*, two colorings f and g are considered equivalent if for some permutation π of D, $g = \pi^* f$, that is, for all a in D, $g(a) = f(\pi(a))$.

In Example 8.2, suppose that

$$\pi = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

Let $f(1) = r, f(2) = b, g(1) = b, g(2) = r$. Then $g(a) = f(\pi(a))$ for all a, so fS^*g. This just says that the two colorings rb and br are equivalent. Henceforth, to distinguish equivalence classes under S from those under S^*, we shall refer to the latter equivalence classes as *patterns*. We are interested in computing the number of distinct patterns. This can be done by applying Burnside's Lemma to G^*.

In Example 8.1, $\text{Inv}(\pi_1^*) = 16$, $\text{Inv}(\pi_2^*) = 2$, $\text{Inv}(\pi_3^*) = 4$, and $\text{Inv}(\pi_4^*) = 2$, since π_1^* leaves all 16 colorings C_i invariant, π_2^* and π_4^* leave only C_1 and C_{16} invariant, and π_3^* leaves C_1, C_{10}, C_{11}, and C_{16} invariant. Thus, the number of equivalence classes under S^* is given by $\frac{1}{4}(16 + 2 + 4 + 2) = 6$, which agrees with Figure 8.10. [We could have computed $\text{Inv}(\pi_i^*)$ directly without first computing π_i^*. For instance, π_3^* leaves invariant only those colorings that agree in boxes 1 and 3 and agree in boxes 2 and 4. Since in such a coloring there are 2 choices for the color for boxes 1 and 3 and 2 choices for the color for boxes 2 and 4, there are $2^2 = 4$ choices for the coloring. Similarly, π_2^* leaves invariant only those colorings that agree in all four boxes, since each box must get the same color as the one $90°$ away in a clockwise direction. Thus, there are only 2 such colorings.]

In the case of the necklaces (Example 8.2), if

$$\pi_1 = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \quad \text{and} \quad \pi_2 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix},$$

then

$$\pi_1^* = \begin{pmatrix} bb & br & rb & rr \\ bb & br & rb & rr \end{pmatrix} \quad \text{and} \quad \pi_2^* = \begin{pmatrix} bb & br & rb & rr \\ bb & rb & br & rr \end{pmatrix}. \quad (8.9)$$

Note that $\text{Inv}(\pi_1^*) = 4$ and $\text{Inv}(\pi_2^*) = 2$, so that the number of patterns (equivalence classes) under S^* is given by $\frac{1}{2}(4+2) = 3$. This agrees with our earlier observation that the patterns are $\{bb\}, \{br, rb\}$, and $\{rr\}$.

In the tree colorings (Example 8.4), note that there are 2^7 tree colorings in all: Each vertex of the tree can get one of the two colors. Suppose that π_i^* is the permutation of tree colorings that corresponds to the permutation π_i of labelings shown in Figure 8.18. It is impractical to write out π_i^*. However, note that $\pi_1^* = I^*$ leaves invariant all 2^7 tree colorings, so $\text{Inv}(\pi_1^*) = 2^7 = 128$. Also, permutation π_2 interchanges vertices 4 and 5. Thus, π_2^* leaves invariant exactly those colorings that color vertices 4 and 5 the same, that is, 2^6 colorings. Thus, $\text{Inv}\ (\pi_2^*) = 2^6 = 64$. Similarly, $\text{Inv}(\pi_3^*) = 2^6 = 64$, $\text{Inv}(\pi_4^*) = 2^4 = 16$, $\text{Inv}(\pi_5^*) = 2^5 = 32$, $\text{Inv}(\pi_6^*) = 2^3 = 8$, $\text{Inv}(\pi_7^*) = 2^3 = 8$, and $\text{Inv}(\pi_8^*) = 2^4 = 16$. Thus, the number of patterns or the number of distinct colorings is

$$\frac{1}{8}(128 + 64 + 64 + 16 + 32 + 8 + 8 + 16) = 42.$$

8.4.3 Graph Colorings Equivalent under Automorphisms

Suppose that we wish to color the vertices of graph $K_{1,3}$ of Figure 8.15 using the colors green (G), black (B), or white (W), with no requirement that two vertices joined by an edge get different colors. We shall consider two such colorings equivalent if one can be obtained from the other by an automorphism. Any two colorings with vertex 1 of Figure 8.15 getting color G and the others all getting different colors are equivalent, since either a rotation by 120° or 240° or a reflection about the edge $\{1, 2\}$, $\{1, 3\}$, or $\{1, 4\}$ can be used to map one such coloring into another. These colorings are shown in Figure 8.19. The rotations also show that all colorings with vertex 1 getting G, two of the other vertices getting B, and the last getting W are equivalent. These colorings are shown in Figure 8.20. Similarly, for any choice of two distinct colors from {G, B, W}, $X \neq Y$, neither equal to G, there is a pattern (set of equivalent colorings) with vertex 1 getting G, two other vertices getting the first chosen color X from $\{X, Y\}$, and the last vertex getting the other chosen color Y from $\{X, Y\}$. There are $3 \times 2 = 6$ ways to choose the two distinct colors X and Y, so 6 patterns (equivalence classes) of this kind. There are also 3 patterns (equivalence classes) consisting of one coloring each, with vertex 1 getting color G and the other vertices all getting the same color (possibly, G). Hence, there are 10 equivalence classes of colorings in all in which vertex 1 gets color G. Repeating for the other two choices for color of vertex 1, we find 30 patterns (equivalence classes) under automorphism in all.

We can also obtain this result from Burnside's Lemma. All in all, there are 3^4 colorings of the vertices of $K_{1,3}$. Suppose that π_i^* is the permutation of graph colorings that corresponds to the automorphism π_i of $K_{1,3}$. The identity automorphism π_1 leaves all such colorings invariant, so $|\text{Inv}\,(\pi_1^*)| = 3^4$. The reflection through edge $\{1, 2\}$, π_2, leaves invariant those colorings in which vertices 3 and 4 get the same color. There are 3^3 such colorings, so $|\text{Inv}\,(\pi_2^*)| = 3^3$. Similarly, $|\text{Inv}\,(\pi_3^*)| = |\text{Inv}\,(\pi_4^*)| = 3^3$ if π_3 and π_4 are the reflections through the edges $\{1, 3\}$ and $\{1, 4\}$, respectively. A rotation through 120°, π_5, leaves invariant those colorings in which vertices 2, 3, and 4 get the same color. Thus, $|\text{Inv}\,(\pi_5^*)| = 3^2$. Similarly,

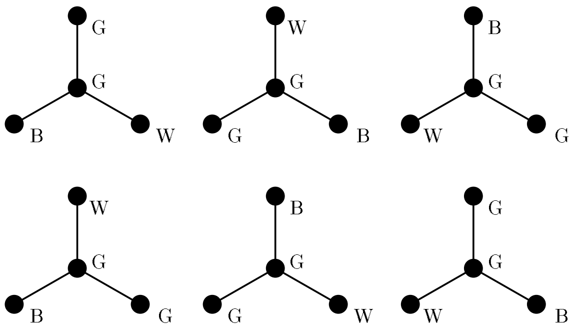

Figure 8.19: Pattern (equivalence class) of colorings of $K_{1,3}$ with vertex 1 of Figure 8.15 getting color G and vertices 2, 3, 4 of Figure 8.15 getting distinct colors from {G, B, W}.

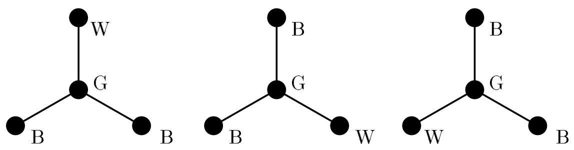

Figure 8.20: Pattern (equivalence class) of colorings of $K_{1,3}$ with vertex 1 of Figure 8.15 getting color G and vertices 2, 3, 4 of Figure 8.15 having two colored B and one colored W.

$|\text{Inv}(\pi_6^*)| = 3^2$ for π_6, the rotation through 240°. There are 6 permutations π_i^* in all, so Burnside's Lemma shows that the number of patterns (equivalence classes) of colorings under automorphism is

$$\frac{1}{6}\left[3^4 + 3^3 + 3^3 + 3^3 + 3^2 + 3^2\right] = \frac{1}{6}[180] = 30.$$

8.4.4 The Case of Switching Functions[9]

Let us now apply the theory we have been developing to the case of switching functions, Example 8.3. If there are two variables, we considered two such functions T and U the same if $T = U$ or $T(x_1x_2) = U(x_2x_1)$. This idea generalizes as follows: Two switching functions T and U of n variables are considered the same if there is a permutation π of $\{1, 2, \ldots, n\}$ so that

$$T(x_1x_2\cdots x_n) = U(x_{\pi(1)}x_{\pi(2)}\cdots x_{\pi(n)}). \tag{8.10}$$

In the case $n = 2$, the two possible π are

$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

[9] This subsection may be omitted.

Table 8.2: Switching Functions T and U Satisfying (8.10) for π as in (8.11)

Bit string x	$T(x)$	$U(x)$
000	1	1
001	0	0
010	1	0
011	1	0
100	0	1
101	0	1
110	1	1
111	0	0

If $n = 3$, an example of two switching functions satisfying (8.10) with

$$\pi = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \tag{8.11}$$

is given in Table 8.2. That (8.10) should correspond to sameness or equivalence makes sense. For if (8.10) holds, a circuit design for T can be obtained from one for U, in a manner analogous to Figure 8.2. Alternative sameness relations also make sense for computer engineering. We explore them in the exercises.

How does this sameness relation fit into the formal structure we have developed? D here is the set B_n of bit strings of length n. Let π be any permutation of $\{1, 2, \ldots, n\}$, and let S_n be the group of all permutations of $\{1, 2, \ldots, n\}$. Then a bit string $x_1 x_2 \cdots x_n$ can be looked at as a coloring of $\{1, 2, \ldots, n\}$ using the colors 0 and 1. The corresponding group of permutations of colorings is S_n^*. This is the group G of our theory. The group G^* is the group $(S_n^*)^*$. $G^* = (S_n^*)^*$ consists of permutations $(\pi^*)^*$ for all π in S_n. How does $(\pi^*)^*$ work? First, note that $\pi^*(x_1 x_2 \cdots x_n) = x_{\pi(1)} x_{\pi(2)} \cdots x_{\pi(n)}$. Note that S_n^* is a group of permutations of the collection of bit strings. $(S_n^*)^*$ consists of permutations of colorings of bit strings. But a coloring of bit strings using colors $0, 1$ is a switching function. Note that by definition, if U is a switching function,

$$[(\pi^*)^* U](x_1 x_2 \cdots x_n) = U[\pi^*(x_1 x_2 \cdots x_n)] = U\left(x_{\pi(1)} x_{\pi(2)} \cdots x_{\pi(n)}\right).$$

Thus, if $T = (\pi^*)^* U$, (8.10) follows.

Let $n = 2$. Then $S_n = \{\pi_1, \pi_2\}$, where

$$\pi_1 = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \quad \text{and} \quad \pi_2 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

The permutations in $G = S_n^*$ are

$$\pi_1^* = \begin{pmatrix} 00 & 01 & 10 & 11 \\ 00 & 01 & 10 & 11 \end{pmatrix} \quad \text{and} \quad \pi_2^* = \begin{pmatrix} 00 & 01 & 10 & 11 \\ 00 & 10 & 01 & 11 \end{pmatrix}. \tag{8.12}$$

Table 8.3: The 16 Switching Functions of Two Variables

Bit string	$T(x)$															
x	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8	T_9	T_{10}	T_{11}	T_{12}	T_{13}	T_{14}	T_{15}	T_{16}
00	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
01	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
10	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
11	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1

There are $2^{2^2} = 16$ switching functions of two variables. These are shown as $T_1, T_2, \ldots, T_{16}$ of Table 8.3. Then the permutations in $G^* = (S_n^*)^*$ are

$$(\pi_1^*)^* = \begin{pmatrix} T_1 & T_2 & T_3 & T_4 & T_5 & T_6 & T_7 & T_8 & T_9 & T_{10} & T_{11} & T_{12} & T_{13} & T_{14} & T_{15} & T_{16} \\ T_1 & T_2 & T_3 & T_4 & T_5 & T_6 & T_7 & T_8 & T_9 & T_{10} & T_{11} & T_{12} & T_{13} & T_{14} & T_{15} & T_{16} \end{pmatrix} \quad (8.13)$$

and

$$(\pi_2^*)^* = \begin{pmatrix} T_1 & T_2 & T_3 & T_4 & T_5 & T_6 & T_7 & T_8 & T_9 & T_{10} & T_{11} & T_{12} & T_{13} & T_{14} & T_{15} & T_{16} \\ T_1 & T_2 & T_5 & T_6 & T_3 & T_4 & T_7 & T_8 & T_9 & T_{10} & T_{13} & T_{14} & T_{11} & T_{12} & T_{15} & T_{16} \end{pmatrix}. \quad (8.14)$$

Note that $(\pi_2^*)^*U$ is the function T which does on 01 what U does on 10, and on 10 what U does on 01, and otherwise agrees with U. We have $\text{Inv}((\pi_1^*)^*) = 16$ and $\text{Inv}((\pi_2^*)^*) = 8$, and the number of equivalence classes or patterns of switching functions of two variables is given by $\frac{1}{2}(16 + 8) = 12$. The number of patterns of switching functions of three variables can similarly be shown to be 80, the number of patterns of switching functions of four variables can be shown to be 3,984, and the number of patterns of switching functions of five variables can be shown to be 37,333,248 (see Harrison [1965] or Prather [1976]). By allowing other symmetries (such as interchange of 0 and 1 in the domain or range) of a switching function—see Exercises 23 and 24—we can further reduce the number of equivalence classes. In fact, the number can be reduced to 222 if $n = 4$ (Harrison [1965], Stone [1973]). This gives a small enough number so that for $n = 4$ it is reasonable to prepare a catalog of optimal circuit for realizing switching functions which contains a representative of each equivalence class.

EXERCISES FOR SECTION 8.4

1. Suppose that $D = \{a, b, c\}$ and $R = \{1, 2\}$. Find all colorings in $C(D, R)$.
2. How many colorings (not necessarily distinct) are there for the vertices of a cube if the set of allowable colors is {red, green, blue}?
3. How many allowable colorings (not necessarily distinct) are there for the vertices of a regular tetrahedron if six colors are available?
4. In Example 8.1, check (8.5), (8.7), and (8.8) by computing:

   (a) $(\pi_2^* C_4)(3)$ (b) $(\pi_3^* C_5)(2)$ (c) $(\pi_4^* C_{11})(4)$

5. In Example 8.2, check (8.9) by computing:

   (a) $(\pi_2^* br)(1)$ (b) $(\pi_2^* br)(2)$ (c) $(\pi_2^* rr)(1)$

6. Suppose that $D = \{1,2,3,4\}, R = \{1,2\}$, and G consists of the permutations in Equation (8.3).

   (a) Suppose that f and g are the following colorings. $f(a) = 1$, all a, and that $g(1) = g(2) = 2$, $g(3) = g(4) = 1$. Is fS^*g?

   (b) Suppose that $f(1) = f(3) = 2, f(2) = f(4) = 1$ and $g(1) = g(2) = 2$, $g(3) = g(4) = 1$. Is fS^*g?

   (c) Find π_2^*. (d) Find π_3^*. (e) Find π_4^*.

   (f) Find $\mathrm{Inv}(\pi_2^*)$. (g) Find $\mathrm{Inv}(\pi_3^*)$. (h) Find $\mathrm{Inv}(\pi_4^*)$.

   (i) Find S. (j) Find S^*.

7. Repeat Exercise 6 [except parts (e) and (h)] if G consists of

$$\pi_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \quad \pi_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix}, \quad \pi_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix}.$$

8. In Example 8.2, suppose that $k = 2$ and that three colors are available: red (r), blue (b), and purple (p). Find π_2^* if

$$\pi_2 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

9. In Example 8.2, suppose that $k = 3$ and that two colors are available: red (r) and blue (b).

   (a) Find G^*.

   (b) Find the number of distinct necklaces using Burnside's Lemma.

   (c) Check your answer by enumerating the distinct necklaces.

10. In Exercise 15, Section 8.1, find:

   (a) D (b) R (c) G

   (d) G^* (e) The number of distinct colorings

11. If graph $K_{1,3}$ of Figure 8.15 is colored using colors from {G, B, W}, find the following patterns:

   (a) That containing the coloring with vertex 1 colored W and vertices 2, 3, 4 colored G, B, B, respectively

   (b) That containing the coloring with all vertices colored W

12. If graph $K_{1,3}$ is colored using colors from {G, B, W, P}, find the number of patterns:

   (a) By describing them (b) Using Burnside's Lemma

13. If the graph Z_4 is colored using colors from {G, B, W}, describe all patterns and count them using Burnside's Lemma.

14. If the graph $K_4 - K_2$ of Figure 8.16 is colored using colors from {B, W}, describe all patterns and count them using Burnside's Lemma.

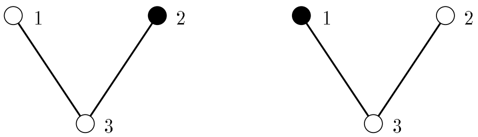

Figure 8.21: Colored weak orders.

15. In Example 8.4, verify that:
    (a) $\text{Inv}(\pi_3^*) = 2^6$ (b) $\text{Inv}(\pi_4^*) = 2^4$
    (c) $\text{Inv}(\pi_5^*) = 2^5$ (d) $\text{Inv}(\pi_6^*) = 2^3$
16. In Example 8.3, verify (8.14) by computing:
    (a) $(\pi_2^*)^* T_3$ (b) $(\pi_2^*)^* T_{12}$ (c) $(\pi_2^*)^* T_{15}$
17. In the situation of Exercise 9, Section 8.1:
    (a) Find G^*.
    (b) Use Burnside's Lemma to compute the number of distinct colorings.
    (c) Check your answer by comparing the enumeration of equivalence classes you gave as your answer in Section 8.1.
18. Find the number of distinct ways to 2-color a 4×4 array that can rotate by $0°$ or $180°$.
19. Show that the eight permutations in Figure 8.18 define a group.
20. Suppose that we consider a weak order on $\{1, 2, 3\}$ and color each element of $\{1, 2, 3\}$ dark or light. Then we distinguish weak order A from weak order B in Figure 8.21. Count the number of distinct colored weak orders if, independent of coloring, we use the notion of sameness of Example 8.6.
21. Suppose that $V = \{1, 2, \ldots, p\}$. Recall that there is a one-to-one correspondence between graphs (V, E) on the vertex set V and functions that assign 0 or 1 to each 2-element subset of V. The idea is that

$$f(\{i, j\}) = 1 \text{ iff } \{i, j\} \in E.$$

The function f is a coloring of the set D of all 2-element subsets of V, using the colors 0 and 1.

(a) If $p = 3$, find all such functions f and their corresponding graphs.
(b) If H and H' are two graphs on V and f and f' are their corresponding functions, show that H and H' are isomorphic iff there is a permutation π on D so that for all $\{i, j\}$ in D, $f(\{i, j\}) = f'(\pi(\{i, j\}))$, that is, so that f and f' are equivalent.
(c) Let G be the group of all permutations π of D. If $p = 3$, write down all the elements of G and compute $\text{Inv}(\pi_i^*)$ for all π_i in G.
(d) Use Burnside's Lemma to determine the number of distinct (nonisomorphic) graphs of three vertices, and verify your result by identifying the classes of equivalent (isomorphic) graphs.

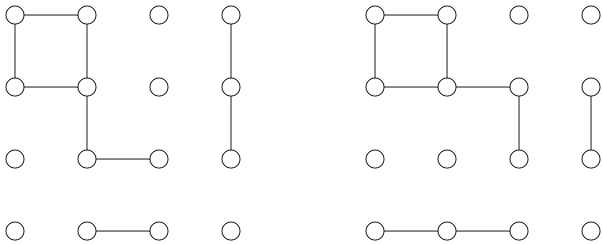

Figure 8.22: Sample interconnection patterns for chips.

22. Repeat parts (a), (c), and (d) of Exercise 21 for $p = 4$.

23. Suppose that D is the collection of all bit strings of length 3, and let G be the group that consists of the identity permutation and the permutation that complements a string by interchanging 0 and 1 (see Exercise 14, Section 8.1). Find the number of distinct switching functions (number of distinct colorings using the colors 0 and 1), that is, the number of equivalence classes in the equivalence relation induced by G^*. Note that in this case we have no S_n.

24. Suppose that two switching functions are considered equivalent if one can be obtained from the other by permuting or complementing the variables as in Exercise 23 or both. Find the number of distinct switching functions of two variables.

25. (Reingold, Nievergelt, and Deo [1977]) A manufacturer of integrated circuits makes chips that have 16 elements arranged in a 4×4 array. These elements are interconnected between some adjacent horizontal or vertical elements. Figure 8.22 shows some sample interconnection patterns. A photomask of the interconnection pattern is used to deposit interconnections on a chip. Two patterns are considered the same if the same photomask could be used for each. For instance, by flipping the photomask over on a diagonal, it can be used for both the interconnection patterns shown in Figure 8.22. Thus, they are considered the same. How many photomasks are required in order to lay out all possible interconnection patterns? Formulate this problem as a coloring problem by defining an appropriate D, R, and G. However, do not attempt to compute G^* or to solve the problem completely with the tools developed so far.

8.5 THE CYCLE INDEX

8.5.1 Permutations as Products of Cycles

It gets rather messy to apply Burnside's Lemma to many counting problems. For instance, it gets rather long and complicated to compute the permutations in the group G^*. We shall develop alternative procedures, procedures that will also allow us to get more information than provided by Burnside's Lemma.

The permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix}$$

cycles the numbers around, sending 1 to $2, 2$ to $3, 3$ to $4, 4$ to 5, and 5 to 1. It can be abbreviated by writing simply (12345). More generally, $(a_1 a_2 \cdots a_{m-1} a_m)$ will represent the permutation that takes a_1 to a_2, a_2 to $a_3, \ldots, a_{m-1}$ to a_m, and a_m to a_1. This is called a *cyclic permutation.* For instance,

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

is a cyclic permutation (132). Now consider the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 1 & 6 & 3 & 2 & 4 \end{pmatrix}.$$

This consists of two cycles, (152) and (364). We say that

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 1 & 6 & 3 & 2 & 4 \end{pmatrix}$$

is the *product* of these two cycles, and write it as (152)(364). It is the product of these cycles in the same sense as taking the product of two permutations, if we think of a cycle like (152) as leaving 3, 4, and 6 fixed. In the same way,

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 3 & 5 & 6 & 4 \end{pmatrix}$$

is the *product* of three cycles, (12)(3)(456), where (3) means that 3 is mapped into itself. In this product, the three cycles are *disjoint* in the sense that no two of them involve the same element.

We now show that every permutation of $\{1, 2, \ldots, n\}$ can be written as the product of *disjoint* cycles, with each element i of $1, 2, \ldots, n$ appearing in some cycle. To see why, let us take the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 4 & 8 & 6 & 7 & 2 & 1 & 3 \end{pmatrix}.$$

Take the element 1. It goes to 5. In turn, 5 goes to 7, and 7 to 1, so we have a cycle (157). Take the first element not in this cycle, 2. It goes to 4, which goes to 6, which goes to 2, so we have a cycle (246). Take the first element not in either of these cycles. It is 3. Now 3 goes to 8, which goes to 3, so we have a cycle (38). Thus, the original permutation is the product of the three disjoint cycles (157)(246)(38). Similar reasoning applies to any permutation.

Could a permutation π be written in two different ways as the product of disjoint cycles (with every element in some cycle)? The answer is yes if we consider $(a_1 a_2 \cdots a_m)$ and $(a_i a_{i+1} \cdots a_m a_1 a_2 \cdots a_{i-1})$ as different. However, we consider them the same, since they correspond to the same permutation. Thus, (123) and (231) and (312) are all the same. Moreover, the order in which we write the cycles does not matter in the product of disjoint cycles. For example, we consider (12)(345) and (345)(12) to be the same. Suppose that

$$(xyz\cdots)\cdots(abc\cdots) = \pi = (uvw\cdots)\cdots(\alpha\beta\gamma\cdots).$$

If these two ways of writing the permutation are different, there must be a number k such that the cycle containing k on the left is different from the cycle containing k on the right. We can write these two cycles with k first. Then whatever π takes k into must be next in each cycle, and whatever π takes this into must be third in each cycle, and so on. Thus, the two cycles must be the same. To summarize, we have the following result.

Theorem 8.5 Every permutation of $\{1, 2, \ldots, n\}$ can be written in exactly one way as the product of disjoint cycles with every element of $\{1, 2, \ldots, n\}$ appearing in some cycle.

We shall call the unique way of writing a permutation described in Theorem 8.5 the *cycle decomposition* of the permutation.

8.5.2 A Special Case of Pólya's Theorem

We are now ready to present another result about counting equivalence classes, which is a special case of the main theorem we are aiming for. Suppose that $\mathrm{cyc}(\pi)$ counts the number of cycles in the unique cycle decomposition of the permutation π. For instance, if $\pi = (12)(3)(456)$, then $\mathrm{cyc}(\pi) = 3$.

Theorem 8.6 (A Special Case of Pólya's Theorem) Suppose that G is a group of permutations of the set D and $C(D, R)$ is the set of colorings of elements of D using colors in R, a set of m elements. Then the number of distinct colorings in $C(D, R)$ (the number of equivalence classes or patterns in the equivalence relation S^* induced by G^*) is given by

$$\frac{1}{|G|}\left[m^{\mathrm{cyc}(\pi_1)} + m^{\mathrm{cyc}(\pi_2)} + \cdots + m^{\mathrm{cyc}(\pi_k)}\right],$$

where $G = \{\pi_1, \pi_2, \ldots, \pi_k\}$.

Note that this theorem allows us to compute the number of distinct colorings without first computing G^*. We prove the theorem in Section 8.5.6.

To illustrate this theorem, let us reconsider the case of the 2×2 arrays, Example 8.1. There are four permutations in G, the four rotations $\pi_1, \pi_2, \pi_3, \pi_4$ shown in Figure 8.13. These have the following cycle decompositions: $\pi_1 = (1)(2)(3)(4), \pi_2 = (1432), \pi_3 = (13)(24), \pi_4 = (1234)$. Thus, $\mathrm{cyc}(\pi_1) = 4$, $\mathrm{cyc}(\pi_2) = 1$, $\mathrm{cyc}(\pi_3) = 2$, and $\mathrm{cyc}(\pi_4) = 1$. The number of distinct colorings (number of patterns) is given by $\frac{1}{4}(2^4 + 2^1 + 2^2 + 2^1) = 6$, which agrees with our earlier computation.

In Example 8.2, with necklaces of k beads, we can write the two permutations in G as

$$\pi_1 = (1)(2)\cdots(k)$$

and as

$$\pi_2 = (1\ k)\,(2\ k-1)\,(3\ k-2)\cdots\left(\frac{k}{2}\ \frac{k}{2}+1\right)$$

if k is even and as

$$\pi_2 = (1\ k)\,(2\ k-1)\,(3\ k-2)\cdots\left(\frac{k-1}{2}\ \frac{k+3}{2}\right)\left(\frac{k+1}{2}\right)$$

if k is odd. For instance, if k is 4, $\pi_2 = (14)(23)$. If k is 5, $\pi_2 = (15)(24)(3)$. It follows that $\mathrm{cyc}(\pi_1) = k$ and $\mathrm{cyc}(\pi_2) = k/2$ if k is even and $(k+1)/2$ if k is odd. Hence, the number of distinct necklaces is $\frac{1}{2}(2^k + 2^{k/2})$ if k is even and $\frac{1}{2}(2^k + 2^{(k+1)/2})$ if k is odd. For instance, the case $k = 2$ gives us 3, which agrees with our earlier computation. The case $k = 3$ gives us 6, which is left to the reader to check (Exercise 8). If there are three different colors of beads and k is even, we would have $\frac{1}{2}(3^k + 3^{k/2})$ distinct necklaces. For instance, for $k = 2$, we would have 6 distinct necklaces. If the colors of beads are r, b, and p, the 6 equivalence classes are $\{rb, br\}, \{rp, pr\}, \{bp, pb\}, \{rr\}, \{bb\}, \{pp\}$.

In Example 8.4, the tree colorings, G is given by the eight permutations π_1, π_2, ..., π_8 of Figure 8.18. We have

$$\begin{array}{rclrcl}
\pi_1 & = & (1)(2)(3)(4)(5)(6)(7), & \pi_2 & = & (1)(2)(3)(45)(6)(7), \\
\pi_3 & = & (1)(2)(3)(4)(5)(67), & \pi_4 & = & (1)(23)(46)(57), \\
\pi_5 & = & (1)(2)(3)(45)(67), & \pi_6 & = & (1)(23)(4756), \\
\pi_7 & = & (1)(23)(4657), & \pi_8 & = & (1)(23)(47)(56).
\end{array}$$

Then the number of distinct tree colorings is given by

$$\frac{1}{8}(2^7 + 2^6 + 2^6 + 2^4 + 2^5 + 2^3 + 2^3 + 2^4) = 42,$$

which is what we computed earlier.

8.5.3 Graph Colorings Equivalent under Automorphisms Revisited

Let us return to the colorings of the vertices of graph $K_{1,3}$ as discussed in Section 8.4.3. It is easy to see that

$$\pi_1^* = (1)(2)(3)(4),\quad \pi_2^* = (1)(2)(3\ 4),\quad \pi_3^* = (1)(3)(2\ 4),$$
$$\pi_4^* = (1)(4)(2\ 3),\quad \pi_5^* = (1)(3\ 4\ 2),\quad \pi_6^* = (1)(4\ 2\ 3).$$

Thus, if we color using colors G, B, W, the number of distinct colorings is given by

$$\frac{1}{6}\left[3^{\mathrm{cyc}(\pi_1^*)} + 3^{\mathrm{cyc}(\pi_2^*)} + 3^{\mathrm{cyc}(\pi_3^*)} + 3^{\mathrm{cyc}(\pi_4^*)} + 3^{\mathrm{cyc}(\pi_5^*)} + 3^{\mathrm{cyc}(\pi_6^*)}\right]$$

$$= \frac{1}{6}\left[3^4 + 3^3 + 3^3 + 3^3 + 3^2 + 3^2\right] = 30.$$

This agrees with our earlier computation.

8.5.4 The Case of Switching Functions[10]

In Example 8.3, the switching functions, we have to consider the elements of $G = H^*$. In case we consider switching functions of two variables, we have $G = \{\pi_1^*, \pi_2^*\}$, where

$$\pi_1 = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \quad \text{and} \quad \pi_2 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

By our computation in Section 8.4.4, π_1^* and π_2^* are given by (8.12). Now we can think of π_1^* and π_2^* as acting on $\{1,2,3,4\}$, in which case $\pi_1^* = (1)(2)(3)(4)$ and $\pi_2^* = (1)(23)(4)$, so $\mathrm{cyc}(\pi_1^*) = 4$, $\mathrm{cyc}(\pi_2^*) = 3$, and the number of distinct switching functions is $\frac{1}{2}(2^4 + 2^3) = 12$, which agrees with our earlier computation.

8.5.5 The Cycle Index of a Permutation Group

It will be convenient to summarize the cycle structure of the permutations in a permutation group in a manner analogous to generating functions. Suppose that π is a permutation with b_1 cycles of length 1, b_2 cycles of length 2, ... in its unique cycle decomposition. Then if $x_1, x_2, \ldots$ are placeholders and k is at least the length of the longest cycle in the cycle decomposition of π, we can encode π by using the expression $x_1^{b_1}x_2^{b_2}\cdots x_k^{b_k}$. Moreover, we can encode an entire permutation group G by taking the sum of these expressions for members of G divided by the number of permutations of G. That is, if k is the length of the longest cycle in the cycle decomposition[11] of any π of G, we write

$$P_G(x_1, x_2, \ldots, x_k) = \frac{1}{|G|}\sum_{\pi \in G} x_1^{b_1}x_2^{b_2}\cdots x_k^{b_k}$$

and call $P_G(x_1, x_2, \ldots, x_k)$ the *cycle index* of G. For instance, consider Example 8.4. Then, to use the notation of Figure 8.18, $\pi_6 = (1)(23)(4756)$, and its corresponding code is $x_1x_2x_4$. Also, $\pi_4 = (1)(23)(46)(57)$, and it is encoded as $x_1x_2^3$. By a similar analysis, the cycle index for the group of permutations is

$$\begin{aligned} P_G(x_1, x_2, \ldots, x_8) = \ & \frac{1}{8}\left[x_1^7 + x_1^5x_2 + x_1^5x_2 + x_1x_2^3 + \right. \\ & \left. x_1^3x_2^2 + x_1x_2x_4 + x_1x_2x_4 + x_1x_2^3\right]. \end{aligned} \tag{8.15}$$

Note that if π is a permutation with corresponding code $x_1^{b_1}x_2^{b_2}\cdots x_k^{b_k}$, then $\mathrm{cyc}(\pi) = b_1 + b_2 + \cdots + b_k$ and $x_1^{b_1}x_2^{b_2}\cdots x_k^{b_k}$ is $m^{\mathrm{cyc}(\pi)}$ if all x_i are taken to be m. Hence, Theorem 8.6 can be restated as follows:

Corollary 8.6.1 Suppose that G is a group of permutations of the set D and that $C(D,R)$ is the set of colorings of elements of D using colors in R, a set

[10] This subsection may be omitted.

[11] We can also take $k = |D|$ and note that the length of the longest cycle in the cycle decomposition of any π of G is at most k.

of m elements. Then the number of distinct colorings in $C(D, R)$ is given by $P_G(m, m, \ldots, m)$.

It is this version of the result that we generalize in Section 8.6.

To make use of this result here, we return to Example 8.4. Here $m = 2$. We let $x_1 = x_2 = \cdots = x_8 = 2$ in (8.15), obtaining

$$P_G(2, 2, \ldots, 2) = \frac{1}{8}(2^7 + 2^6 + 2^6 + 2^4 + 2^5 + 2^3 + 2^3 + 2^4) = 42,$$

which agrees with our earlier result about the number of distinct colorings.

Let us now consider Example 8.5. In Section 8.1 we identified 12 different symmetries of the tetrahedron. These can be thought of as permutations of the letters a, b, c, d of Figure 8.5. The identity symmetry is the permutation $(a)(b)(c)(d)$, which has cycle structure code x_1^4. The $120°$ rotation about the line joining vertex a to the middle of the face determined by b, c, and d corresponds to the permutation $(a)(bdc)$, which can be encoded as x_1x_3 (see Figure 8.6). All eight $120°$ and $240°$ rotational symmetries have similar structure and coding. Finally, the rotation by $180°$ about the line connecting the midpoints of edges ab and cd corresponds to the permutation $(ab)(cd)$, which has the coding x_2^2. The other two $180°$ rotations have similar coding. Thus, the cycle index is given by

$$P_G(x_1, x_2, x_3) = \frac{1}{12}(x_1^4 + 8x_1x_3 + 3x_2^2).$$

We seek a coloring of the set $D = \{a, b, c, d\}$ using the four colors contained in the set $R = \{\mathrm{CH_3}, \mathrm{C_2H_5}, \mathrm{H}, \mathrm{Cl}\}$. Hence, $m = 4$, and the number of distinct colorings (number of distinct molecules) is given by

$$P_G(m, m, m) = \frac{1}{12}\left[4^4 + 8(4)(4) + 3(4)^2\right] = 36.$$

8.5.6 Proof of Theorem 8.6[12]

We shall apply Burnside's Lemma (Theorem 8.4) to G^* in order to prove Theorem 8.6. Since $|G| = |G^*|$, it suffices to show that $m^{\mathrm{cyc}(\pi)} = \mathrm{Inv}(\pi^*)$. Let π be in G. We try to compute $\mathrm{Inv}(\pi^*)$. Note that an element of $C(D, R)$ is left invariant by π^* iff in the corresponding permutation π of D, all the elements of D in each cycle of π receive the same color. For instance, suppose that $\pi = (12)(345)(67)(8)$. Let f be the coloring such that $f(1) = f(2) =$ black, $f(3) = f(4) = f(5) =$ white, $f(6) = f(7) =$ red, and $f(8) =$ blue. Then clearly $\pi^* f$ is the same coloring.

In sum, to find a coloring that is left invariant by π^*, we compute the cycle decomposition of π and color each element in a cycle with the same color. Now π has $\mathrm{cyc}(\pi)$ different cycles in its cycle decomposition, and we have m choices for the common color of each cycle. Hence, there are $m^{\mathrm{cyc}(\pi)}$ different colorings left invariant by π^*. In short, $\mathrm{Inv}(\pi^*) = m^{\mathrm{cyc}(\pi)}$.

[12] This subsection may be omitted.

EXERCISES FOR SECTION 8.5

1. Find the cycle decomposition of each permutation of Exercise 1, Section 8.2.
2. Find the cycle decomposition for all permutations π_i arising from the situation of Exercise 9, Section 8.1.
3. Compute $\text{cyc}(\pi)$ for every permutation in parts (a), (b), and (c) of Exercise 10, Section 8.2.
4. For every permutation of parts (a), (b), (c) of Exercise 10, Section 8.2, encode the permutation as $x_1^{b_1} x_2^{b_2} \cdots x_k^{b_k}$.
5. For each group of permutations in Exercise 10, Section 8.2, compute the cycle index.
6. Given D, R, and G as in Exercise 6, Section 8.4:

   (a) Find the number of distinct colorings by Theorem 8.6.

   (b) Repeat using Corollary 8.6.1.

7. Repeat Exercise 6 for D, R, and G as in Exercise 7, Section 8.4.
8. In Example 8.2, check that if $k = 3$, there are six distinct necklaces.
9. In Example 8.2, use Theorem 8.6 to find the number of distinct necklaces if:

   (a) The number of colors is 2 and k is 4

   (b) The number of colors is 2 and k is 5

   (c) The number of colors is 3 and k is 3

   (d) The number of colors is 3 and k is 4

10. Repeat Exercise 9 using Corollary 8.6.1.
11. (a) In Exercise 9, Section 8.1, use Theorem 8.6 to find the number of equivalence classes.

    (b) Repeat using Corollary 8.6.1.
12. (a) In Exercise 15, Section 8.1, use Theorem 8.6 to find the number of distinct colorings.

    (b) Repeat using Corollary 8.6.1.
13. (a) In Exercise 8, Section 8.1, use Theorem 8.6 to find the number of distinct colorings.

    (b) Repeat using Corollary 8.6.1.
14. In Example 8.1, for each $\pi_i, i = 1, 2, 3, 4$, verify that $\text{Inv}(\pi_i^*) = m^{\text{cyc}(\pi_i)}$.
15. Continuing with Exercise 23 of Section 8.2, if equivalence of two graph colorings is defined as in Section 8.4.3, find the number of distinct colorings of:

    (a) Graph L_4 with 2 colors

    (b) Graph L_4 with 3 colors

    (c) Graph Z_4 with 2 colors

    (d) Graph Z_4 with 3 colors

    (e) Graph $K_4 - K_2$ with 2 colors

    (f) Graph $K_4 - K_2$ with 3 colors
16. Repeat Exercise 20, Section 8.4, using the methods of this section.
17. (a) Use Theorem 8.6 to find the number of nonisomorphic graphs of $p = 3$ vertices. (See Exercise 21, Section 8.4.)

    (b) Repeat using Corollary 8.6.1.
18. Repeat Exercise 17 for $p = 4$.

19. (a) In Exercise 23, Section 8.4, use Theorem 8.6 to find the number of distinct switching functions.
    (b) Repeat using Corollary 8.6.1.
20. Repeat Exercise 19 for Exercise 24, Section 8.4.
21. If the definition of sameness of Exercise 24, Section 8.4, is adopted, find the number of distinct switching functions of three variables given that the cycle index for the appropriate group of permutations is

$$\frac{1}{12}\left[x_1^8 + 4x_2^4 + 2x_1^2x_3^2 + 2x_2x_6 + 3x_1^4x_2^2\right].$$

22. Consider a cube in 3-space. There are eight vertices. The following symmetries correspond to permutations of these vertices. Encode each of these symmetries in the form $x_1^{b_1}x_2^{b_2}\cdots x_k^{b_k}$ and compute the cycle index of the group G of all the permutations corresponding to these symmetries.
    (a) The identity symmetry
    (b) Rotations by 180° around lines connecting the centers of opposite faces (there are three)
    (c) Rotations by 90° or 270° around lines connecting the centers of opposite faces (there are six)
    (d) Rotations by 180° around lines connecting the midpoints of opposite edges (there are six)
    (e) Rotations by 120° around lines connecting opposite vertices (there are eight)
23. In Exercise 22, find the number of distinct ways of coloring the vertices of the cube with two colors, red and blue.
24. Complete the solution of Exercise 25, Section 8.4.
25. A *transposition* is a cycle (ij). Show that every permutation is the product of transpositions. (*Hint:* It suffices to show that every cycle is the product of transpositions.)
26. Continuing with Exercise 25, write (123456) as a product of transpositions.
27. Write each permutation of Exercise 1, Section 8.2, as the product of transpositions.
28. Show that a permutation can be written as a product of transpositions in more than one way.
29. Although a permutation can be written in more than one way as a product of transpositions, it turns out that every way of writing the permutation as such a product either includes an even number of transpositions or an odd number. (For a proof, see Exercise 31.) A permutation, therefore, can be called *even* if every way of writing it as a product of transpositions uses an even number of transpositions, and *odd* otherwise.
    (a) Identify all even permutations of $\{1, 2, 3\}$.
    (b) Show that the collection of even permutations of $\{1, 2, \ldots, n\}$ forms a group.
    (c) Does the collection of odd permutations of $\{1, 2, \ldots, n\}$ form a group?
30. The number of permutations of $\{1, 2, \ldots, n\}$ with code $x_1^{b_1}x_2^{b_2}\cdots x_k^{b_k}$ is given by the formula

$$\frac{n!}{b_1!b_2!\cdots b_n!1^{b_1}2^{b_2}\cdots n^{b_n}}.$$

This is called *Cauchy's formula.*

(a) Verify this formula for $n = 5, b_1 = 3, b_2 = 1, b_3 = b_4 = b_5 = 0$.

(b) Verify this formula for $n = 3$ and all possible codes.

31. Suppose that

$$D_n = (2-1)(3-2)(3-1)(4-3)(4-2)(4-1)\cdots(n-1). \tag{8.16}$$

If π is a permutation of $\{1, 2, \ldots, n\}$, define πD_n from D_n by replacing the term $(i - j)$ in (8.16) by the term $(\pi(i) - \pi(j))$.

(a) Find D_5.

(b) Find πD_5 if

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 5 & 2 & 1 \end{pmatrix}.$$

(c) Show that if π is a transposition, $\pi D_n = -D_n$.

(d) Conclude from part (c) that if π is the product of an even number of transpositions, $\pi D_n = D_n$, and if π is the product of an odd number of transpositions, $\pi D_n = -D_n$.

(e) Conclude that a permutation cannot both be written as the product of an even number of transpositions and the product of an odd number of transpositions.

8.6 PÓLYA'S THEOREM

8.6.1 The Inventory of Colorings

We may be interested in counting not just the number of distinct colorings, but the number of distinct colorings of a certain kind. For instance, in Example 8.1, we might be interested in counting the number of distinct 2-colorings of the 2×2 array in which exactly two black colors are used; in Example 8.5, we might be interested in counting the number of distinct molecules with at least one hydrogen atom; and so on. We shall now present a general result for answering questions of this type.

Let $D = \{a_1, a_2, \ldots, a_n\}$ be the set of objects to be colored and $R = \{r_1, r_2, \ldots, r_m\}$ be the set of colors. We shall distinguish colorings by assigning a *weight* $w(r)$ to each color r. This weight can be either a symbol or a number.

If we have assigned weights to the colors, we can assign a *weight to a coloring.* It is defined to be the product of the weights of the colors assigned to the elements of D. To illustrate this, suppose that $R = \{x, y, z\}$ and $w(x) = 1, w(y) = 5, w(z) = 7$. Suppose that the objects being colored are the seven vertices of the first binary tree of Figure 8.18, and we color vertices $1, 2, 4, 6$ with color x, vertices $3, 7$ with color y, and vertex 5 with color z. Then the weight of this coloring is $w(x)^4 w(y)^2 w(z) = (1)^4(5)^2 7 = 175$. If $w(x) = r, w(y) = g$, and $w(z) = b$, the weight of the coloring is $r^4 g^2 b$. We shall see below how the weight of a coloring encodes the coloring in a very useful way.

Suppose now that K is a set of colorings. The sum of the weights of colorings in K is called the *inventory* of K. For instance, suppose that $D = \{a, b, c, d\}, R =$

$\{x, y, z\}$, and $w(x) = r, w(y) = g, w(z) = b$. Let colorings f_1, f_2, and f_3 in $C(D, R)$ be defined as follows:

$$\begin{array}{llll} f_1(a) = x, & f_1(b) = y, & f_1(c) = y, & f_1(d) = z, \\ f_2(a) = z, & f_2(b) = z, & f_2(c) = x, & f_2(d) = z, \\ f_3(a) = x, & f_3(b) = z, & f_3(c) = y, & f_3(d) = x. \end{array}$$

Let $W(f_i)$ be the weight of coloring f_i. Then $W(f_1) = w(x)w(y)w(y)w(z) = rg^2b$, and, similarly, $W(f_2) = rb^3$ and $W(f_3) = r^2gb$. The inventory of the set $K = \{f_1, f_2, f_3\}$ is given by $rg^2b + rb^3 + r^2gb$. If all the weights of colors are different symbols, the weight of a coloring represents the distribution of colors used. For instance, $W(f_1) = rg^2b$ shows that f_1 used color x once, color y twice, and color z once. The inventory of a set of colorings summarizes the distribution of colors in the different colorings in the set. This is like a generating function.

Now suppose that G is a group of permutations of the set D and that f and g are two equivalent colorings in $C(D, R)$. Then as observed in Section 8.4.2, there is a π in G so that for all a in D, $g(a) = f(\pi(a))$ If $D = \{a_1, a_2, \ldots, a_n\}$, then

$$W(f) = w[f(a_1)]w[f(a_2)] \cdots w[f(a_n)] \tag{8.17}$$

and

$$W(g) = w[g(a_1)]w[g(a_2)] \cdots w[g(a_n)]. \tag{8.18}$$

Since π is a permutation, the set $\{a_1, a_2, \ldots, a_n\}$ has exactly the same elements as the set $\{\pi(a_1), \pi(a_2), \ldots, \pi(a_n)\}$. Thus, (8.17) implies that

$$W(f) = w[f(\pi(a_1))]w[(\pi(a_2))] \cdots w[f(\pi(a_n))]. \tag{8.19}$$

But since $g(a) = f(\pi(a))$, (8.18) and (8.19) imply that $W(f) = W(g)$. Thus, we have shown the following.

Theorem 8.7 If colorings f and g are equivalent, they have the same weight.

As a result of this theorem, we can speak of the *weight of an equivalence class of colorings* or, what is the same, the *weight of a pattern*. This is the weight of any coloring in this class. We shall also be able to speak of the *inventory of a set of patterns* or of a set of equivalence classes, the *pattern inventory*, as the sum of the weights of the patterns in the set. For instance, let us consider Example 8.1, the colorings of the 2×2 arrays. There are six patterns of colorings, as shown in Figure 8.10. Let the color black have weight b and the color white have weight w. Then the coloring of class 1 of Figure 8.10 has weight b^4, all the colorings of class 2 have weight b^3w, all of class 3 have weight b^2w^2, all of class 4 have weight b^2w^2, all of class 5 have weight bw^3, and the coloring of class 6 has weight w^4. Note that two different equivalence classes can have the same weight. The pattern inventory is given by

$$b^4 + b^3w + 2b^2w^2 + bw^3 + w^4. \tag{8.20}$$

We find that there is one equivalence class using four black colors, one using three black and one white, two using two blacks and two whites, and so on. This information can be read directly from the pattern inventory. If we simply wanted to find the number of patterns, we would proceed as we did with generating functions in Chapter 5, and take all the weights to be 1. Here, setting $b = w = 1$ in (8.20) gives us 6, the number of patterns. If we wanted to find the number of patterns using no black, we would set $w(\text{black}) = 0$ and $w(\text{white}) = 1$, or, equivalently, let $b = 0$ and $w = 1$ in (8.20). The result is 1. There is only one pattern with no black, that corresponding to the term w^4 in the pattern inventory. We shall now seek a method for computing the pattern inventory without knowing the equivalence classes.

8.6.2 Computing the Pattern Inventory

Theorem 8.8 (Pólya's Theorem[13]) Suppose that G is a group of permutations on a set D and $C(D, R)$ is the collection of all colorings of D using colors in R. If w is a weight assignment on R, the pattern inventory of colorings in $C(D, R)$ is given by

$$P_G\left(\sum_{r\in R} w(r), \sum_{r\in R}[w(r)]^2, \sum_{r\in R}[w(r)]^3, \ldots, \sum_{r\in R}[w(r)]^k\right),$$

where $P_G(x_1, x_2, x_3, \ldots, x_k)$ is the cycle index.

Note that Corollary 8.6.1 is a special case of this theorem in which $w(r) = 1$ for all r in R. To illustrate the theorem, let us return to Example 8.1, the 2×2 arrays, one more time. Note that G consists of the permutations $\pi_1 = (1)(2)(3)(4), \pi_2 = (1432), \pi_3 = (13)(24)$, and $\pi_4 = (1234)$. Thus, the cycle index is given by

$$P_G(x_1, x_2, x_3, x_4) = \frac{1}{4}(x_1^4 + 2x_4 + x_2^2).$$

Now let us assign a weight b to a black coloring and a weight w to a white coloring. Then $R = \{\text{black, white}\}$ and

$$\begin{array}{ll} \sum_{r\in R} w(r) = b + w, & \sum_{r\in R}[w(r)]^2 = b^2 + w^2, \\ \sum_{r\in R}[w(r)]^3 = b^3 + w^3, & \sum_{r\in R}[w(r)]^4 = b^4 + w^4. \end{array}$$

By Pólya's Theorem, the pattern inventory is given by taking $P_G(x_1, x_2, x_3, x_4)$ and substituting $\sum_{r\in R} w(r)$ for x_1, $\sum_{r\in R}[w(r)]^2$ for x_2, and so on. Thus, the pattern inventory is

$$\frac{1}{4}[(b + w)^4 + 2(b^4 + w^4) + (b^2 + w^2)^2]. \qquad (8.21)$$

[13] Pólya's fundamental theorem was first presented in his classic paper (Pólya [1937]). The result was anticipated by Redfield [1927], but few people understood Redfield's results and Pólya was unaware of them. A generalization of Pólya's Theorem can be found in de Bruijn [1959]–for an exposition of this, see, for example, Liu [1968]. A well-known exposition of Pólya theory is in a paper by de Bruijn [1964]. For other expositions, see Bogart [1999], Brualdi [1999], or Tucker [1995].

b	b
r	w

b	b
w	r

b	r
w	b

Figure 8.23: Examples of the three different patterns of colorings of the 2×2 array using two blacks, one white, and one red.

By using the binomial expansion (Theorem 2.7), we can expand out (8.21) and obtain (8.20), which was our previous description of the pattern inventory.

Suppose that we allow three colors in coloring the 2×2 array: black, white, and red. If we let $w(\text{red}) = r$, we find that the pattern inventory is

$$\frac{1}{4}[(b+w+r)^4 + 2(b^4+w^4+r^4) + (b^2+w^2+r^2)^2] = b^4 + w^4 + r^4 + b^3w + w^3b$$
$$+b^3r + r^3b + w^3r + r^3w + 2b^2w^2 + 2b^2r^2 + 2w^2r^2 + 3b^2wr + 3w^2br + 3r^2wb.$$

We see, for instance, that there are three patterns with two blacks, one white, and one red. One example of each of these patterns is shown in Figure 8.23. Any other pattern using two black, one white, and one red can be obtained from one of these by rotation. The number of patterns is obtained by substituting $b = w = r = 1$ into the pattern inventory. Notice how once having computed the cycle index, we can apply it easily to do a great many different counting procedures without having to repeat computation of the index.

Let us next consider Example 8.5, the organic molecules. We have already noted that

$$P_G(x_1, x_2, x_3) = \frac{1}{12}(x_1^4 + 8x_1x_3 + 3x_2^2).$$

Suppose that we want to find the number of distinct molecules (patterns) containing at least one chlorine atom. It is a little easier to compute first the number of patterns having no chlorine atoms. This can be obtained by assigning the weight of 1 to each color CH_3, C_2H_5, and H, and the weight of 0 to the color Cl. Then for all $k \geq 1$,

$$\sum_{r \in R} [w(r)]^k = [w(\mathrm{CH_3})]^k + [w(\mathrm{C_2H_5})]^k + [w(\mathrm{H})]^k + [w(\mathrm{Cl})]^k = 1+1+1+0 = 3.$$

It follows that the pattern inventory is given by $\frac{1}{12}(3^4 + 8(3)(3) + 3(3)^2) = 15$. Since we have previously calculated that there are 36 patterns in all, the number with at least one chlorine atom is $36 - 15 = 21$.

Continuing with this example, suppose that we assign a weight of 1 to each color except Cl and a weight of c to Cl. Then the pattern inventory is given by

$$\frac{1}{12}[(c+3)^4 + 8(c+3)(c^3+3) + 3(c^3+3)^2] = c^4 + 3c^3 + 6c^2 + 11c + 15.$$

We conclude that there is one pattern consisting of four chlorine atoms, while three patterns consist of three chlorine atoms, six of two chlorine atoms, 11 of one chlorine atom, and 15 of no chlorine atoms.

Next, let us return to the graph colorings of $K_{1,3}$ using colors in the set {G, B, W} as discussed in Sections 8.4.4 and 8.5.4. How many distinct colorings use no G's? The longest cycle in a permutation π_i^* has length 3, and from the results of Section 8.5.4, we see that

$$P_G(x_1, x_2, x_3) = \frac{1}{6}(x_1^4 + 3x_1^2 x_2 + 2x_1 x_3).$$

Letting $w(\mathrm{G}) = g$, $w(\mathrm{B}) = b$, $w(\mathrm{W}) = w$, we get

$$\begin{aligned} P_G(g, b, w) &= \frac{1}{6}\Big[(g+b+w)^4 + 3\,(g+b+w)^2\left(g^2+b^2+w^2\right) \\ &\quad + 2\,(g+b+w)\left(g^3+b^3+w^3\right)\Big]. \end{aligned} \tag{8.22}$$

We can answer our question by setting $g = 0$, $b = w = 1$ in (8.22), getting

$$\frac{1}{6}\,[16 + 24 + 8] = 8.$$

Thus, there are 8 distinct colorings of $K_{1,3}$ using no G. The reader should check this. What if we want to know how many colorings have exactly two W's? We can set $g = b = 1$ in (8.22) and calculate the pattern inventory

$$\frac{1}{6}\left[(2+w)^4 + 3\,(2+w)^2\left(2+w^2\right) + 2\,(2+w)\left(2+w^3\right)\right].$$

Simplifying, we get

$$8 + 10w + 7w^2 + 4w^3 + w^4. \tag{8.23}$$

The number of distinct colorings with two W's is given by the coefficient of w^2, i.e., 7. The reader should check this.

8.6.3 The Case of Switching Functions[14]

Next let us turn to Example 8.3, the switching functions, and take $n = 2$. Then G consists of the permutations π_1^* and π_2^* given by (8.12). As before, it is natural to think of π_1^* as (1)(2)(3)(4) and π_2^* as (1)(23)(4). Hence,

$$P_G(x_1, x_2) = \frac{1}{2}(x_1^4 + x_1^2 x_2).$$

Setting $w(0) = a$ and $w(1) = b$, we find that $\sum_{r\in R}[w(r)]^k = a^k + b^k$. Thus, the pattern inventory is given by

$$\frac{1}{2}[(a+b)^4 + (a+b)^2(a^2+b^2)] = a^4 + 3a^3 b + 4a^2 b^2 + 3ab^3 + b^4.$$

The term $3a^3b$ indicates that there are three patterns of switching functions which assign three 0's and one 1. The reader might wish to identify these patterns.

[14] This subsection may be omitted.

8.6.4 Proof of Pólya's Theorem[15]

We now present a proof of Pólya's Theorem. We proceed by a series of lemmas. Throughout, let us assume that $R = \{1, 2, \ldots, m\}$.

Lemma 8.2 Suppose that D is divided up into disjoint sets $D_1, D_2, \ldots, D_p$. Let C be the subset of $C(D, R)$ that consists of all colorings f with the property that if a and b are both in D_i, some i, then $f(a) = f(b)$. Then the inventory of the set C is given by

$$\left[w(1)^{|D_1|} + w(2)^{|D_1|} + \cdots + w(m)^{|D_1|}\right] \times \left[w(1)^{|D_2|} + w(2)^{|D_2|} + \cdots \right. \\ \left. + \, w(m)^{|D_2|}\right] \times \cdots \times \left[w(1)^{|D_p|} + w(2)^{|D_p|} + \cdots + w(m)^{|D_p|}\right]. \tag{8.24}$$

Proof. Multiplying out (8.24), we get terms such as

$$w(i_1)^{|D_1|} w(i_2)^{|D_2|} \cdots w(i_p)^{|D_p|}.$$

This is the weight of the coloring that gives color i_1 to all elements of D_1, color i_2 to all elements of D_2, and so on. Thus, (8.24) gives the sum of the weights of colorings that color all of D_i the same color. Q.E.D.

Lemma 8.3 Suppose that $G^* = \{\pi_1^*, \pi_2^*, \ldots\}$ is a group of permutations of $C(D, R)$. For each π^* in G^*, let $\bar{w}(\pi^*)$ be the sum of the weights of all colorings f in $C(D, R)$ left invariant by π^*. Suppose that $C_1, C_2, \ldots$ are the equivalence classes of colorings and $w(C_i)$ is the common weight of all f in C_i. Then

$$w(C_1) + w(C_2) + \cdots = \frac{1}{|G^*|}[\bar{w}(\pi_1^*) + \bar{w}(\pi_2^*) + \cdots]. \tag{8.25}$$

Note that if all weights are 1, Lemma 8.3 reduces to Burnside's Lemma.

Proof of Lemma 8.3. The sum on the right-hand side of (8.25) adds up for each π^* the weights of all colorings f left fixed by π^*. Thus, $w(f)$ is added in here exactly the number of times it is left invariant by some π^*. This is, to use the terminology of Section 8.3.2, the number of elements in the stabilizer of f, $\mathrm{St}(f)$. By Lemma 8.1 of Section 8.3.2, $|\mathrm{St}(f)| = |G^*|/|C(f)|$, where $C(f)$ is the equivalence class containing f. Therefore, if $C(D, R) = \{f_1, f_2, \ldots\}$, the right-hand side of (8.25) is given by

$$\frac{1}{|G^*|}[w(f_1) \cdot |\mathrm{St}(f_1)| + w(f_2) \cdot |\mathrm{St}(f_2)| + \cdots] = \\ \frac{1}{|G^*|}\left[w(f_1)\frac{|G^*|}{|C(f_1)|} + w(f_2)\frac{|G^*|}{|C(f_2)|} + \cdots\right],$$

which equals

$$\frac{w(f_1)}{|C(f_1)|} + \frac{w(f_2)}{|C(f_2)|} + \cdots. \tag{8.26}$$

[15] This subsection may be omitted.

If we add up the terms $w(f_i)/|C(f_i)|$ for f_i in equivalence class C_j, we get $w(C_j)$, since each $w(f_i) = w(C_j)$ and since $|C(f_i)| = |C_j|$. Thus, (8.26) equals $w(C_1) + w(C_2) + \cdots$. Q.E.D.

We are now ready to complete the proof of Pólya's Theorem. In (8.25) of Lemma 8.3, the left-hand side is the pattern inventory. Recall that $\bar{w}(\pi^*)$ is the sum of the weights of the colorings f left invariant by π^*. Suppose that the permutation π has cycles $D_1, D_2, \ldots, D_p$ in its cycle decomposition. Note that a coloring f is left invariant by π^* iff $f(a) = f(b)$ whenever a and b are in the same D_i. Thus, by Lemma 8.2, (8.24) gives the inventory or the sum of the weights of the set of colorings left invariant by π^*, i.e., (8.24) gives $\bar{w}(\pi^*)$. Each term in (8.24) is of the form

$$[w(1)]^j + [w(2)]^j + \cdots + [w(m)]^j = \sum_{r \in R} [w(r)]^j, \tag{8.27}$$

where $j = |D_i|$. Thus, a term (8.27) occurs in (8.24) as many times as $|D_i|$ equals j, that is, as many times as π has a cycle of length j. We denoted this as b_j in Section 8.5.5 when we defined the cycle index. Hence, $\bar{w}(\pi^*)$ or (8.24) can be rewritten as

$$\left[\sum_{r \in R} [w(r)]^1\right]^{b_1} \left[\sum_{r \in R} [w(r)]^2\right]^{b_2} \cdots.$$

Therefore, the right-hand side of (8.25) becomes

$$P_G\left(\sum_{r \in R} [w(r)]^1, \sum_{r \in R} [w(r)]^2, \ldots\right).$$

This proves Pólya's Theorem.

EXERCISES FOR SECTION 8.6

1. Find the weight of each coloring in column a in Figure 8.1 if $w(\text{black}) = 3$ and $w(\text{white}) = 4$.
2. If $w(1) = x$ and $w(2) = y$, find the weights of colorings f and g in parts (a) and (b) of Exercise 6, Section 8.4.
3. Suppose that K consists of the colorings C_2, C_8, C_{10}, and C_{14} of Figure 8.10. If $w(\text{black}) = b$ and $w(\text{white}) = w$, find the inventory of the collection K.
4. Suppose that K consists of the switching functions T_2, T_3, T_8, T_{10}, and T_{15} of Table 8.3. Find the inventory of K if $w(0) = a$ and $w(1) = b$.
5. In the situation of Exercise 9, Section 8.1, find the pattern inventory if $w(\text{black}) = b$ and $w(\text{white}) = w$.
6. In the situation of Exercise 7, Section 8.4, find the pattern inventory if $w(1) = \alpha$ and $w(2) = \beta$.
7. Use Pólya's Theorem to compute the number of distinct four-bead necklaces, where each bead has one of three colors.

8. In Example 8.4, suppose that we have four possible colors for the vertices. Use Pólya's Theorem to find the number of distinct colorings of the tree.
9. In Example 8.5, find the number of distinct molecules with no CH_3's.
10. How many four-bead necklaces are there in which each bead is one of the colors b, r, or p, and there is at least one p?
11. Consider colorings of $K_{1,3}$ with colors G, B, W.
    (a) Check that there are exactly 8 distinct colorings using no G, by showing the colorings.
    (b) Check that there are exactly 7 distinct colorings with exactly two W's, by showing the colorings.
    (c) Find the number of distinct colorings with exactly one W.
12. Find the number of distinct colorings of the following graphs, which were introduced in Section 8.2, Exercise 23, using the colors G, B, W and exactly one G.
    (a) L_4 (b) Z_4 (c) $K_4 - K_2$
13. Find the number of distinct colorings of the graphs in Exercise 12 using the colors G, B, W and exactly three B's.
14. Find the number of distinct switching functions of two variables that have at least one 1 in the range, that is, which assign 1 to at least one bit string.
15. In Exercise 23, Section 8.4, find the number of distinct switching functions of three variables that have at least one 1 in the range.
16. In Example 8.5, find the number of distinct molecules that have at least one Cl atom and at least one H atom.
17. Use Pólya's Theorem to compute the number of nonisomorphic graphs with:
    (a) Three vertices (b) Three vertices and two edges
    (c) Three vertices and at least one edge (d) Four vertices
    (e) Four vertices and three edges (f) Four vertices and at least two edges
    (See Exercises 21 and 22, Section 8.4.) For further applications of Pólya's Theorem to graph theory, see Chartrand and Lesniak [1996], Gross and Yellen [1999], Harary [1969], or Harary and Palmer [1973].
18. The vertices of a cube are to be colored and five colors are available: red, blue, green, yellow, and purple. Count the number of distinct colorings in which at least one green and one purple are used (see Exercises 22 and 23, Section 8.5).
19. Let $D = \{a, b, c, d\}$, $R = \{0, 1\}$, let G consist of the permutations $(1)(2)(3)(4)$, $(12)(34)$, $(13)(24)$, $(14)(23)$, and take $w(0) = 1, w(1) = x$.
    (a) Find $C(D, R)$. (b) Find G^*.
    (c) Find all equivalence classes of colorings under G^*.
    (d) Find the weights of all equivalence classes under G^*.
    (e) Let e_i be the number of colors of weight x^i and let $e(x)$ be the ordinary generating function of the e_i; that is, $e(x) = \sum_{i=0}^{\infty} e_i x^i$. Compute $e(x)$.

(f) Let E_j be the number of patterns of weight x^j and let $E(x)$ be the ordinary generating function of the E_j; that is, $E(x) = \sum_{j=0}^{\infty} E_j x^j$. Compute $E(x)$.

(g) Show that for $e(x)$ and $E(x)$ as computed in parts (e) and (f),

$$E(x) = P_G[e(x), e(x^2), e(x^3), \ldots]. \tag{8.28}$$

20. Repeat Exercise 19 for $D = \{a, b, c\}$, $R = \{0, 1\}$, G the set of permutations (1)(2)(3) and (12)(3), and $w(0) = x^2, w(1) = x^7$.

21. Generalizing the results of Exercises 19 and 20, suppose that for every $r, w(r) = x^p$ for some nonnegative integer p. Let $e(x)$ and $E(x)$ be defined as in Exercise 19. Show that, in general, (8.28) holds.

REFERENCES FOR CHAPTER 8

BOGART, K. P., *Introductory Combinatorics*, 3rd ed., Academic Press, San Diego, 1999.

BRUALDI, R. A., *Introductory Combinatorics*, 3rd ed., Prentice Hall, Upper Saddle River, NJ, 1999.

BURNSIDE, W., *Theory of Groups of Finite Order*, 2nd ed., Cambridge University Press, Cambridge, 1911. (Reprinted by Dover, New York, 1955.)

CAMERON, P. J., "Automorphism Groups of Graphs," in L. W. Beineke and R. J. Wilson (eds.), *Selected Topics in Graph Theory, 2*, Academic Press, London, 1983, 89–127.

CHARTRAND, G., and LESNIAK, L., *Graphs and Digraphs*, 3rd ed., CRC Press, Boca Raton, 1996.

DE BRUIJN, N. G., "Generalization of Pólya's Fundamental Theorem in Enumerative Combinatorial Analysis," *Ned. Akad. Wet., Proc. Ser. A 62, Indag. Math., 21* (1959), 59–79.

DE BRUIJN, N. G., "Pólya's Theory of Counting," in E. F. Beckenbach (ed.), *Applied Combinatorial Mathematics*, Wiley, New York, 1964, 144–184.

GROSS, J., and YELLEN, J., *Graph Theory and Its Applications*, CRC Press, Boca Raton, FL, 1999.

HARARY, F., *Graph Theory*, Addison-Wesley, Reading, MA, 1969.

HARARY, F., and PALMER, E. M., *Graphical Enumeration*, Academic Press, NY, 1973.

HARRISON, M. A., *Introduction to Switching and Automata Theory*, McGraw-Hill, New York, 1965.

LIU, C. L., *Introduction to Combinatorial Mathematics*, McGraw-Hill, New York, 1968.

PÓLYA, G., "Kombinatorische Anzahlbestimmungen für Gruppen, Graphen und chemische Verbindungen," *Acta Math., 68* (1937), 145–254.

PÓLYA, G., and READ, R. C., *Combinatorial Enumeration of Groups, Graphs, and Chemical Compounds*, Springer-Verlag, New York, 1987.

PRATHER, R. E., *Discrete Mathematical Structures for Computer Science*, Houghton Mifflin, Boston, 1976.

REDFIELD, J. H., "The Theory of Group-Reduced Distributions," *Amer. J. Math., 49* (1927), 433–455.

REINGOLD, E. M., NIEVERGELT, J., and DEO, N., *Combinatorial Algorithms: Theory and Practice*, Prentice Hall, Englewood Cliffs, NJ, 1977.

STONE, H. S., *Discrete Mathematical Structures and Their Applications*, Science Research Associates, Chicago, 1973.

TUCKER, A. C., *Applied Combinatorics*, 3rd ed., Wiley, New York, 1995.