8

The Combinatorics of Formal Power Series

Now that we have the technical machinery of formal power series at our disposal, we can resume our combinatorial agenda of studying infinite weighted sets. We will develop versions of the weighted sum and product rules in this setting. We will also explore the combinatorial significance of other operations on formal series, like composition and exponentiation. These techniques will be used to obtain deeper combinatorial information about objects studied earlier in the book, including trees, integer partitions, and set partitions.

8.1 Sum Rule for Infinite Weighted Sets

8.1. Definition: Admissible Weighted Sets and Generating Functions. Suppose S is a set with weight function $\mathrm{wt} : S \to \mathbb{N}$. The weighted set (S, wt) is called *admissible* iff for every $n \geq 0$, the set $S_n = \{z \in S : \mathrm{wt}(z) = n\}$ is finite. In this case, the *generating function* of the weighted set (S, wt) is the formal power series

$$G_S = G_{S,\mathrm{wt}} = \sum_{n=0}^{\infty} |S_n| x^n \in \mathbb{Q}[[x]].$$

Informally, this series represents $\sum_{z \in S} x^{\mathrm{wt}(z)}$.

8.2. Example. Every *finite* weighted set S is admissible. Furthermore, the generating function for such a set is a *polynomial* in x, since $|S_n| = 0$ for all large enough n. We studied generating functions of this type in Chapter 6.

8.3. Example. Let S be the set of all binary trees (with any number of vertices), and let $\mathrm{wt}(T)$ be the number of vertices in T for each tree $T \in S$. Then S_n consists of all binary trees with n vertices. Even without determining the precise cardinality of S_n (which we did in 2.36), one can check that S_n is finite for each $n \geq 0$. Thus, S is an admissible weighted set. In most applications, we will not have calculated the cardinality $|S_n|$ in advance — the whole point of using generating functions is to help solve problems like this! But in most situations of interest, it will follow routinely from the nature of the objects and weights that S_n is finite for all n. So we will often omit explicit proofs that the weighted sets under consideration are indeed admissible.

8.4. Example. Let (S, wt) be an admissible weighted set. Let T be any subset of S with the same weight function as S. Then $T_n \subseteq S_n$ for all n, so that (T, wt) is also admissible. Similarly, a finite disjoint union of admissible weighted sets is again admissible.

8.5. Theorem: Weight-Preserving Bijection Rule. Suppose (S, wt_1) and (T, wt_2) are two weighted sets such that there exists a *weight-preserving bijection* $f : S \to T$ (i.e., $\mathrm{wt}_2(f(s)) = \mathrm{wt}_1(s)$ for all $s \in S$). Then S is admissible iff T is admissible, and $G_S = G_T$.

Proof. Because f (and hence f^{-1}) are weight-preserving, f restricts to bijections $f_n : S_n \to T_n$ for each $n \geq 0$. So $|S_n| = |T_n|$ for all $n \geq 0$, which implies the desired conclusions. □

At last we are ready for the most general version of the sum rule.

8.6. Sum Rule for Infinite Weighted Sets. Suppose (S, wt) is an admissible weighted set that is the disjoint union of subsets $\{T_i : i \in I\}$, where the index set I is finite or equal to $\mathbb{N}$. Assume that for all $i \in I$ and all $x \in T_i$, $\mathrm{wt}_{T_i}(x) = \mathrm{wt}_S(x)$. Then

$$G_S = \sum_{i \in I} G_{T_i}.$$

Proof. Let us compute the coefficient of x^n on each side, for fixed $n \geq 0$. Write S_n (resp. $(T_i)_n$) for the set of objects in S (resp. T_i) of weight n. By assumption, S_n is a finite set which is the disjoint union of the (necessarily finite) sets $(T_i)_n$. Let I_n be the set of indices such that $(T_i)_n$ is nonempty; then I_n must be finite, since S_n is finite. By the ordinary sum rule for finite sets,

$$|S_n| = \sum_{i \in I_n} |(T_i)_n|.$$

The left side is the coefficient of x^n in G_S, while the right side is evidently the coefficient of x^n in $\sum_{i \in I} G_{T_i}$, since the summands corresponding to $i \notin I_n$ contribute zero to the coefficient of x^n. When $I = \mathbb{N}$, this argument also proves the convergence of the infinite sum of formal power series $\sum_{i \in I} G_{T_i}$, since the coefficient of x^n stabilizes once $i \geq \max\{j : j \in I_n\}$. □

8.2 Product Rule for Infinite Weighted Sets

In this section we prove two versions of the product rule for generating functions. The first version is designed for situations in which we build weighted objects by making a *finite* sequence of choices, as in Chapter 1. The second version extends this rule to certain infinite choice sequences, which leads to formulas involving infinite products of formal power series.

8.7. Informal Product Rule for Infinite Weighted Sets. Suppose (S, wt) is a weighted set; k is a fixed, finite positive integer; and (T_i, wt_i) are admissible weighted sets for $1 \leq i \leq k$. Suppose each $z \in S$ can be uniquely constructed by choosing $z_1 \in T_1$, then $z_2 \in T_2$, …, then $z_k \in T_k$, and then assembling these choices in some manner. Further suppose that

$$\mathrm{wt}(z) = \sum_{i=1}^{k} \mathrm{wt}(z_i) \qquad (8.1)$$

for all $z \in S$. Then (S, wt) is admissible, and

$$G_S = \prod_{i=1}^{k} G_{T_i}.$$

Proof. Recast in formal terms, our hypothesis is that there is a weight-preserving bijection from (S, wt) to the weighted set (T, wt) where $T = T_1 \times \cdots \times T_k$ and $\mathrm{wt}(z_1, \ldots, z_k) = \sum_{i=1}^{k} \mathrm{wt}(z_i)$. So it suffices to replace S by the Cartesian product set T. Furthermore, it suffices to prove the result when $k=2$, since the general case follows by induction as in 1.5. Fix $n \geq 0$; we are reduced to proving

$$G_{T_1 \times T_2}(n) = (G_{T_1} G_{T_2})(n).$$

The left side is the cardinality of the set $A = \{(t_1, t_2) \in T_1 \times T_2 : \mathrm{wt}_1(t_1) + \mathrm{wt}_2(t_2) = n\}$. Now A is the disjoint union of the sets $(T_1)_k \times (T_2)_{n-k}$, where $(T_1)_k$ (resp. $(T_2)_{n-k}$) is the finite set of objects in T_1 (resp. T_2) of weight k (resp. $n-k$), and k ranges from 0 to n. So A is a finite set (proving admissibility of $T_1 \times T_2$), and the ordinary sum and product rules for finite unweighted sets give

$$|A| = \sum_{k=0}^{n} |(T_1)_k| \cdot |(T_2)_{n-k}| = \sum_{k=0}^{n} G_{T_1}(k) \cdot G_{T_2}(n-k).$$

This sum is precisely the coefficient of x^n in $G_{T_1} G_{T_2}$, so we are done. □

The following technical device will allow us to obtain generating functions for objects that are built by making an *infinite* sequence of choices.

8.8. Definition: Restricted Cartesian Product. Suppose $\{(T_n, \mathrm{wt}_n) : n \geq 1\}$ is a countable collection of admissible weighted sets such that every T_n contains exactly one element of weight zero; call this element 1_n. Let $T = \prod_{n\geq 1}^{*} T_n$ be the set of all infinite sequences $(t_n : n \geq 1)$ such that $t_n \in T_n$ for all n and $t_n = 1_n$ for all but finitely many indices n. We make T into a (not necessarily admissible) weighted set by defining $\mathrm{wt}((t_n : n \geq 1)) = \sum_{n\geq 1} \mathrm{wt}(t_n)$; this sum is defined since all but finitely many summands are zero.

8.9. Product Rule for the Restricted Cartesian Product. Let $\{(T_n, \mathrm{wt}_n) : n \geq 1\}$ and $T = \prod_{n\geq 1}^{*} T_n$ be as in 8.8. If $\mathrm{ord}(G_{T_n} - 1) \to \infty$ as $n \to \infty$, then (T, wt) is admissible and

$$G_T = \prod_{n=1}^{\infty} G_{T_n}.$$

Proof. The condition on the orders of $G_{T_n} - 1$ ensures that the infinite product of formal series is defined (see 7.33(c)). Fix $m \geq 0$; let us compute $G_T(m)$. Choose N so that $n > N$ implies $\mathrm{ord}(G_{T_n} - 1) > m$. Consider an object $t = (t_n : n \geq 1)$ in T. If $t_n \neq 1_n$ for some $n > N$, then $\mathrm{wt}(t) \geq \mathrm{wt}_n(t_n) > m$, so this object does not contribute to the coefficient $G_T(m)$. So we need only consider objects where $t_n = 1_n$ for all $n > N$. Dropping all coordinates after position N gives a weight-preserving bijection between this set of objects and the weighted set $T_1 \times \cdots \times T_n$. We already know that the generating function for this weighted set is $\prod_{n=1}^{N} G_{T_n}$. So

$$G_T(m) = \left(\prod_{n=1}^{N} G_{T_n}\right)_m = \left(\prod_{n=1}^{\infty} G_{T_n}\right)_m;$$

the last equality holds since $\mathrm{ord}(G_{T_n} - 1) > m$ for $n > N$. This argument has shown that $G_T(m)$ is finite for each m, which is equivalent to admissibility of T. □

To apply this result, we start with some weighted set (S, wt) and describe an "infinite choice sequence" for building objects in S by choosing "building blocks" from the sets T_n. Each set T_n has a "dummy object" of weight zero. Any particular choice sequence must eventually terminate by choosing the dummy object for all sufficiently large n, but there is no fixed bound on the number of "non-dummy" choices we might make. This informal choice procedure amounts to giving a weight-preserving bijection from S to the restricted product $\prod_{n\geq 1}^{*} T_n$. We can then conclude that $G_S = \prod_{n\geq 1} G_{T_n}$, provided that the infinite product on the right side converges.

8.3 Generating Functions for Trees

This section illustrates the sum and product rules for infinite weighted sets by deriving the generating functions for various classes of trees.

8.10. Example: Binary Trees. Let S be the set of all binary trees, weighted by the number of vertices. By definition (see 2.36), every tree $t \in S$ is either empty or is an ordered triple $(\bullet, t_1, t_2)$, where t_1 and t_2 are binary trees. Let S_0 be the one-element set consisting of the empty binary tree, let $S^+ = S \sim S_0$ be the set of nonempty binary trees, and let $N = \{\bullet\}$ be a one-element set such that $\mathrm{wt}(\bullet) = 1$. By definition of the generating function for a weighted set, we have $G_{S_0} = x^0 = 1$ and $G_N = x^1 = x$. By the sum rule for infinite weighted sets,

$$G_S = G_{S_0} + G_{S^+} = 1 + G_{S^+}.$$

By the recursive definition of nonempty binary trees, we can uniquely construct every tree $t \in S^+$ by: (i) choosing the root node $\bullet \in N$; (ii) choosing the left subtree $t_1 \in S$; (iii) choosing the right subtree $t_2 \in S$; and assembling these choices to form the tree $t = (\bullet, t_1, t_2)$. It follows from the product rule for infinite weighted sets that

$$G_{S^+} = G_N G_S G_S = x G_S^2.$$

Writing F to denote the unknown generating function G_S, we conclude that F satisfies the equation

$$F = 1 + xF^2$$

in $\mathbb{Q}[[x]]$, which is equivalent to $xF^2 - F + 1 = 0$. Furthermore, $F(0) = 1$ since there is exactly one binary tree with zero vertices. In 7.80, we saw that this quadratic equation and initial condition has the unique solution

$$F = \frac{1 - \sqrt{1-4x}}{2x} = \sum_{n=0}^{\infty} \frac{1}{2n+1} \binom{2n+1}{n+1, n} x^n.$$

Taking the coefficient of x^n gives the number of binary trees with n nodes, which is the Catalan number C_n. A more combinatorial approach to this result was given in Chapter 2.

8.11. Example: Full Binary Trees. A binary tree is called *full* iff every vertex in the tree has either zero or two (nonempty) children. In the context of binary trees, a *leaf* is a vertex with zero children. Let S be the set of nonempty full binary trees, weighted by the number of leaves. We can write S as the disjoint union of $S_1 = \{(\bullet, \emptyset, \emptyset)\}$ and $S_{\geq 2} = S \sim S_1$. We can build an element t of $S_{\geq 2}$ by choosing any $t_1 \in S$ as the (nonempty) left subtree of the root, and then choosing any $t_2 \in S$ as the right subtree of the root. Note that $\mathrm{wt}(t) = \mathrm{wt}(t_1) + \mathrm{wt}(t_2)$ since the weight is the number of leaves. So, by the product rule, $G_{S_{\geq 2}} = G_S^2$. We see directly that $G_{S_1} = x$. The sum rule now gives the relation

$$G_S = x + G_S^2,$$

with $G_S(0) = 0$. Solving the quadratic $G_S^2 - G_S + x = 0$ by calculations analogous to those in 7.80, we find that

$$G_S = \frac{1 - \sqrt{1-4x}}{2} = xF,$$

where F is the generating function considered in the previous example. It follows that $G_S(n) = F(n-1) = C_{n-1}$ for all $n \geq 1$.

8.12. Example: Ordered Trees. Let S be the set of *ordered trees*, weighted by the number of vertices. We recall the recursive definition of ordered trees from 3.79. First, 0 is an ordered tree with one vertex. Second, for every integer $k \geq 1$, a tuple $t = (k, t_1, t_2, \ldots, t_k)$ such that each $t_i \in S$ is an ordered tree, and the number of vertices of t is $1 + \sum_{i=1}^{k} \mathrm{wt}(t_i)$. (Informally, t represents a tree whose root has k children, which are ordered from left to right, and where each child is itself an ordered tree.) All ordered trees arise by applying the two rules a finite number of times. The first rule can be considered a degenerate version of the second rule in which $k = 0$. Let us find the generating function G_S. First, write S as the disjoint union of sets $\{S_k : k \geq 0\}$ where S_k consists of all trees $t \in S$ such that the root node has k children. By the sum rule for infinite weighted sets,

$$G_S = \sum_{k=0}^{\infty} G_{S_k}.$$

(One can verify the admissibility hypothesis on S by noting that every tree in S_k has k or more leaves.) For each $k \geq 0$, a direct application of the product rule (with $k+1$ choices) shows that

$$G_{S_k} = x^1 \cdot \underbrace{G_S \cdot \ldots \cdot G_S}_{k} = x G_S^k.$$

(The x arises by choosing the root node from the set $\{\bullet\}$.) Substitution into the previous formula gives

$$G_S = \sum_{k=0}^{\infty} x G_S^k = \frac{x}{1 - G_S};$$

the last step is valid (using 7.41) since $G_S(0) = 0$. Doing algebra in the ring $\mathbb{Q}[[x]]$ leads to the relation $G_S^2 - G_S + x = 0$. This is the same equation that occurred in the previous example. So we conclude, as before, that

$$G_S = \frac{1 - \sqrt{1 - 4x}}{2} = \sum_{n \geq 1} C_{n-1} x^n.$$

Our generating function calculations have led us to the following (possibly unexpected) enumeration result: *the set of binary trees with n vertices, the set of full binary trees with $n+1$ leaves, and the set of ordered trees with $n+1$ vertices all have cardinality C_n*. Now that this result is in hand, it is natural to seek a *bijective proof* in which the three sets of objects are linked by explicitly defined bijections. Some methods for building such bijections from recursions were studied in Chapter 2. Here we are seeking weight-preserving bijections on infinite sets, which can be defined as follows.

Let S denote the set of all binary trees, and let T be the set of all nonempty full binary trees. We define a weight-preserving bijection $f : S \to T$ recursively by setting $f(\emptyset) = (\bullet, \emptyset, \emptyset)$ and

$$f((\bullet, t_1, t_2)) = (\bullet, f(t_1), f(t_2)).$$

See Figure 8.1. To see that the weights work, first note that the zero-vertex tree $\emptyset$ is mapped to the one-leaf tree $(\bullet, \emptyset, \emptyset)$. In the recursive formula, suppose t_1 and t_2 have a vertices and b vertices, respectively. By induction, $f(t_1)$ and $f(t_2)$ are nonempty full binary trees with $a+1$ leaves and $b+1$ leaves, respectively. It follows that f sends the tree $(\bullet, t_1, t_2)$ with $a+b+1$ vertices to a full binary tree with $(a+1)+(b+1) = (a+b+1)+1$ leaves, as desired. The inverse of f has an especially simple pictorial description: just erase all the leaves! This works since a nonempty full binary tree always has one more leaf vertex than internal (non-leaf) vertex (see 8.50).

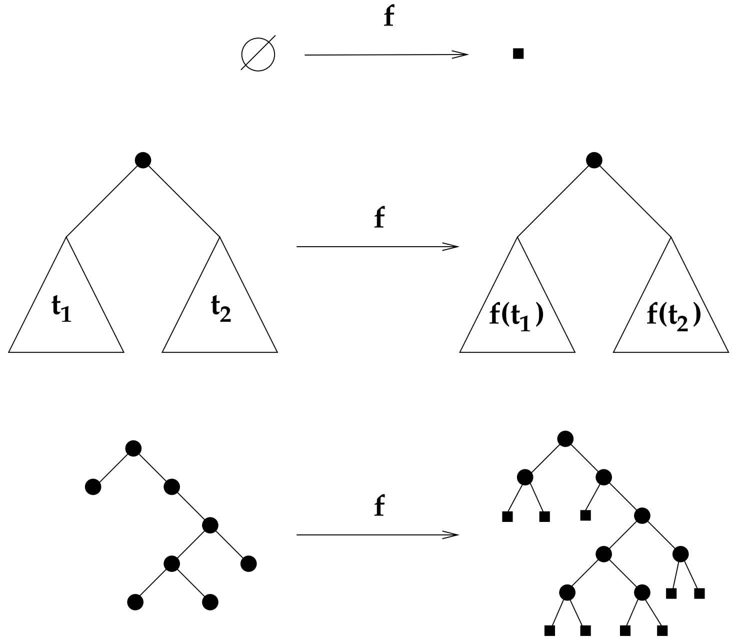

FIGURE 8.1
Bijection between binary trees and full binary trees.

Now let U be the set of all ordered trees. We define a weight-preserving bijection $g : T \to U$. First, $g((\bullet, \emptyset, \emptyset)) = 0$. Second, if $t = (\bullet, t_1, t_2) \in T$ with t_1 and t_2 nonempty, define $g(t)$ by starting with the ordered tree $g(t_1)$ and appending the ordered tree $g(t_2)$ as a new, rightmost child of the root node of $g(t_1)$. See Figure 8.2. More formally, if $g(t_1) = k\, u_1 \, \ldots \, u_k$, let $g(t) = (k+1)\, u_1 \, \ldots \, u_k \, g(t_2)$. As above, one may check that the number of vertices in $g(t)$ equals the number of leaves in t, as required.

8.13. Remark. These examples show that generating functions are a powerful algebraic tool for deriving enumeration results. However, once such results are found, it is often desirable to find direct combinatorial proofs that do not rely on generating functions. In particular, bijective proofs are more informative (and often more elegant) than algebraic proofs in the sense that they give us an explicit pairing between the objects in two sets.

8.4 Compositional Inversion Formulas

Let $F = \sum_{n \geq 1} F_n x^n$ be a formal series with $F(0) = 0$ and $F_1 \neq 0$. We have seen in 7.65 that there is a unique series $G = \sum_{n \geq 1} G_n x^n$ with $G(0) = 0$ and $G_1 \neq 0$ such that $F \bullet G = x = G \bullet F$. Our goal in this section is to find combinatorial and algebraic formulas for the coefficients of G.

Since $F_1 \neq 0$, we can write $F = x/R$ where $R = \sum_{n \geq 0} R_n x^n$ is a series with $R_0 \neq 0$ (see 7.40). We have $F \bullet G = x$ iff $G/(R \bullet G) = x$ iff $G = x(R \bullet G)$. It turns out that we can solve the equation $G = x(R \bullet G)$ by taking G to be the generating function for the set

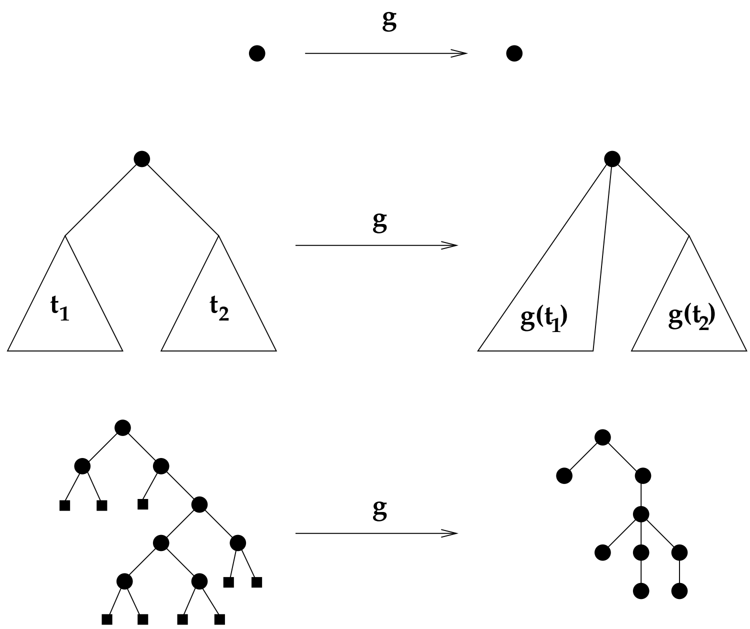

FIGURE 8.2
Bijection between full binary trees and ordered trees.

of ordered trees (or equivalently, terms), relative to a suitable weight function. This is the essence of the following combinatorial formula for G.

8.14. Theorem: Combinatorial Compositional Inversion. Let $F = x/R$ where $R = \sum_{n\geq 0} R_n x^n$ is a given series in $K[[x]]$ with $R_0 \neq 0$. Let T be the set of all terms (§3.13). Let the *weight* of a term $w = w_1 w_2 \cdots w_s \in T$ be $\mathrm{wt}(w) = R_{w_1} R_{w_2} \cdots R_{w_s} x^s$. Then $G = G_T = \sum_{w\in T} \mathrm{wt}(w)$ is the compositional inverse of F.

Proof. Note first that, for any two words v and w, $\mathrm{wt}(vw) = \mathrm{wt}(v)\,\mathrm{wt}(w)$. Also $G(0) = 0$, since every term has positive length. Now, by 3.85 we know that for every term $w \in T$, there exist a unique integer $n \geq 0$ and unique terms $t_1, \ldots, t_n \in T$ such that $w = nt_1t_2\cdots t_n$. For fixed n, we build such a term by choosing the symbol n (which has weight xR_n), then choosing terms $t_1 \in T$, $t_2 \in T$, …, $t_n \in T$. By the product rule for generating functions (which generalizes to handle the current weights), the generating function for terms starting with n is therefore xR_nG^n. Now by the sum rule, we conclude that

$$G = \sum_{n\geq 0} xR_nG^n = x(R \bullet G).$$

By the remarks preceding the theorem, this shows that $F \bullet G = x$, as desired. □

In 3.91 we gave a formula that counts all terms in a given anagram class $\mathcal{R}(0^{k_0}1^{k_1}2^{k_2}\cdots)$. Combining this formula with the previous result, we deduce the following algebraic recipe for the coefficients of G.

8.15. Theorem: Lagrange's Inversion Formula. Let $F = x/R$ where $R = \sum_{n\geq 0} R_n x^n$ is a given series in $K[[x]]$ with $R_0 \neq 0$. Let G be the compositional inverse of F. Then

$$G(n) = (R^n)_{n-1}/n = \frac{1}{n!}\left[\left(\frac{d}{dx}\right)^{n-1} R^n\right]_0 \quad (n \geq 1).$$

Proof. The second equality follows routinely from the definition of formal differentiation. As for the first, let T_n be the set of terms of length n. By 8.14, we know that

$$G(n) = \sum_{w \in T_n} \mathrm{wt}(w).$$

Let us group together summands on the right side corresponding to terms of length n that contain k_0 zeroes, k_1 ones, etc., where $\sum_{i\geq 0} k_i = n$. Each such term has weight $x^n R_0^{k_0} R_1^{k_1} \cdots$, and the number of such terms is $\frac{1}{n}\binom{n}{k_0,k_1,k_2,\ldots}$, provided that $k_0 = 1 + \sum_{i\geq 1}(i-1)k_i$ (3.91). Summing over all possible choices of the k_i, we get

$$G(n) = \sum_{\substack{(k_0,k_1,k_2,\ldots):\\ \sum_{i\geq 0} k_i = n,\ k_0 = 1+\sum_{i\geq 1}(i-1)k_i}} \frac{1}{n}\binom{n}{k_0,k_1,k_2,\ldots} R_0^{k_0} R_1^{k_1} R_2^{k_2} \cdots .$$

Now, in the presence of the condition $\sum_{i\geq 0} k_i = n$, the equation $k_0 = 1 + \sum_{i\geq 1}(i-1)k_i$ holds iff $\sum_{i\geq 0}(i-1)k_i = -1$ iff $\sum_{i\geq 0} ik_i = n-1$. So

$$G(n) = \sum_{\substack{(k_0,k_1,k_2,\ldots):\\ \sum_{i\geq 0} k_i = n,\ \sum_{i\geq 0} ik_i = n-1}} \frac{1}{n}\binom{n}{k_0,k_1,k_2,\ldots} R_0^{k_0} R_1^{k_1} R_2^{k_2} \cdots .$$

On the other hand, 7.11 gives

$$(R^n_{n-1})/n = \sum_{\substack{(k_0,k_1,k_2,\ldots):\\ \sum_{i\geq 0} k_i = n, \sum_{i\geq 0} ik_i = n-1}} \frac{1}{n}\binom{n}{k_0,k_1,k_2,\ldots} R_0^{k_0} R_1^{k_1} R_2^{k_2} \cdots .$$

The right sides agree, so we are done. ☐

8.16. Example. Let us use 8.15 to find the compositional inverse G of the formal series $F = x/e^x$. Here $R = e^x = \sum_{k\geq 0} x^k/k!$, so $R^n = e^{nx} = \sum_{k\geq 0}(n^k/k!)x^k$. It follows that

$$G(n) = (R^n)_{n-1}/n = \frac{n^{n-1}}{n \cdot (n-1)!} = \frac{n^{n-1}}{n!}.$$

Thus, $G = \sum_{n\geq 1}(n^{n-1}/n!)x^n$.

8.5 Generating Functions for Partitions

This section uses formal power series to prove some fundamental results involving integer partitions. Recall from §2.8 that Par denotes the set of all integer partitions. Our first result gives an infinite product formula for the partition generating function.

8.17. Theorem: Partition Generating Function.

$$\sum_{\mu \in \mathrm{Par}} x^{|\mu|} = \prod_{i=1}^{\infty} \frac{1}{1-x^i}.$$

Proof. The proof is an application of the infinite product rule 8.9. We build a typical partition $\mu \in \text{Par}$ by making an infinite sequence of choices, as follows. First, choose how many parts of size 1 will occur in μ. The possible choices here are $0, 1, 2, 3, \ldots$. The generating function for this choice (relative to area) is $1+x+x^2+x^3+\cdots = (1-x)^{-1}$. Second, choose how many parts of size 2 will occur in μ. Again the possibilities are $0, 1, 2, 3, \ldots$, and the generating function for this choice is $1+x^2+x^4+x^6+\cdots = (1-x^2)^{-1}$. Proceed similarly, choosing for every $i \geq 1$ how many parts of size i will occur in μ. The generating function for choice i is $\sum_{k=0}^{\infty}(x^i)^k = (1-x^i)^{-1}$. Multiplying the generating functions for all the choices gives the infinite product in the theorem.

Here is a more formal rephrasing of the proof just given. For each $i \geq 1$, let T_i be the set of all integer partitions ν (including the empty partition of zero) such that every part of ν is equal to i. As argued above, $G_{T_i} = (1-x^i)^{-1}$. Given any partition μ, write $\mu = (1^{a_1}2^{a_2}\cdots i^{a_i}\cdots)$ to indicate that μ has a_i parts equal to i for all i. (Note that $a_i = 0$ for large enough i.) Then the map $(1^{a_1}2^{a_2}\cdots i^{a_i}\cdots) \mapsto ((1^{a_1}), (2^{a_2}), \cdots, (i^{a_i}), \cdots)$ is a weight-preserving bijection from Par onto $\prod_{i\geq 1}^{*} T_i$. The result now follows directly from 8.9. $\square$

We can add another variable to the partition generating function to keep track of additional information. Recall that, for $\mu \in \text{Par}$, μ_1 is the length of the first (longest) part of μ, and $\ell(\mu)$ is the number of nonzero parts of μ.

8.18. Theorem: Enumerating Partitions by Area and Length.

$$\sum_{\mu\in\text{Par}} t^{\ell(\mu)}x^{|\mu|} = \prod_{i=1}^{\infty}\frac{1}{1-tx^i} = \sum_{\mu\in\text{Par}} t^{\mu_1}x^{|\mu|} \quad \text{in } \mathbb{Q}(t)[[x]].$$

Proof. To prove the first equality, we modify the preceding argument to take into account the t-weight. At stage i, suppose we choose k copies of the part i for inclusion in μ. This will increase $\ell(\mu)$ by k and increase $|\mu|$ by ki. So the generating function for the choice made at stage i is

$$\sum_{k\geq 0} t^k x^{ki} = \sum_{k\geq 0}(tx^i)^k = \frac{1}{1-tx^i}.$$

The result now follows from the product rule, as before. To prove the second equality, observe that conjugation is a bijection on Par that preserves area and satisfies $(\mu')_1 = \ell(\mu)$. $\square$

We can use variations of the preceding arguments to derive generating functions for various classes of integer partitions.

8.19. Theorem: Partitions with Odd Parts. Let OddPar be the set of integer partitions all of whose parts are odd. Then

$$\sum_{\mu\in\text{OddPar}} x^{|\mu|} = \prod_{k=1}^{\infty}\frac{1}{1-x^{2k-1}}.$$

Proof. Repeat the proof of 8.17, but now only make choices for the odd part lengths $1, 3, 5, 7$, etc. $\square$

8.20. Theorem: Partitions with Distinct Parts. Let DisPar be the set of integer partitions all of whose parts are distinct. Then

$$\sum_{\mu\in\text{DisPar}} x^{|\mu|} = \prod_{i=1}^{\infty}(1+x^i).$$

Proof. We build a partition $\mu \in \text{DisPar}$ via the following choice sequence. For each part length $i \geq 1$, either choose to not use that part in μ or to include that part in μ (note that the part is only allowed to occur once). The generating function for this choice is $1 + x^i$. The result now follows from the product rule 8.9. □

By comparing the generating functions in the last two theorems, we are led to the following unexpected result.

8.21. Theorem: OddPar **vs.** DisPar.

$$\sum_{\mu \in \text{OddPar}} x^{|\mu|} = \sum_{\nu \in \text{DisPar}} x^{|\nu|}.$$

Proof. We make the following calculation with formal power series:

$$\begin{aligned}
\sum_{\mu \in \text{OddPar}} x^{|\mu|} &= \prod_{k=1}^{\infty} \frac{1}{1 - x^{2k-1}} \\
&= \prod_{k=1}^{\infty} \frac{1}{1 - x^{2k-1}} \prod_{j=1}^{\infty} \frac{1}{1 - x^{2j}} \prod_{j=1}^{\infty} (1 - x^{2j}) \\
&= \prod_{i=1}^{\infty} \frac{1}{1 - x^{i}} \prod_{j=1}^{\infty} [(1 - x^j)(1 + x^j)] \\
&= \prod_{i=1}^{\infty} \frac{1}{1 - x^{i}} \prod_{j=1}^{\infty} (1 - x^j) \prod_{j=1}^{\infty} (1 + x^j) \\
&= \prod_{j=1}^{\infty} (1 + x^j) = \sum_{\nu \in \text{DisPar}} x^{|\nu|}.
\end{aligned}$$

The first and last equalities hold by the two preceding theorems. The penultimate equality uses a cancellation of infinite products that was formally justified in 7.42; the same example (with x replaced by x^2) justifies the second equality. The third and fourth equalities can be verified by similar methods; the reader should fill in the details here (cf. 7.140). □

8.6 Partition Bijections

We have just seen that the generating function for partitions into odd parts (relative to area) coincides with the generating function for partitions with distinct parts. We gave an *algebraic proof* of this result based on manipulation of infinite products in the formal power series ring $\mathbb{Q}[[x]]$. However, from the combinatorial standpoint, it is natural to ask for a *bijective proof* of the same result. We therefore seek an area-preserving bijection $F : \text{OddPar} \to \text{DisPar}$. Two such bijections are presented in this section.

8.22. Sylvester's Bijection. Define $F : \text{OddPar} \to \text{DisPar}$ as follows. Given $\mu \in \text{OddPar}$, draw a *centered* version of the Ferrers diagram of μ in which the middle boxes of the parts of μ are all drawn in the same column; see Figure 8.3. Note that each part of μ does have a middle box, because the part is odd. Label the columns in the centered diagram of μ as $-k, \ldots, -2, -1, 0, 1, 2, \ldots, k$ from left to right, so the center column is column 0. Label the rows $1, 2, 3, \ldots$ from top to bottom. We define $\nu = F(\mu)$ by dissecting the centered diagram

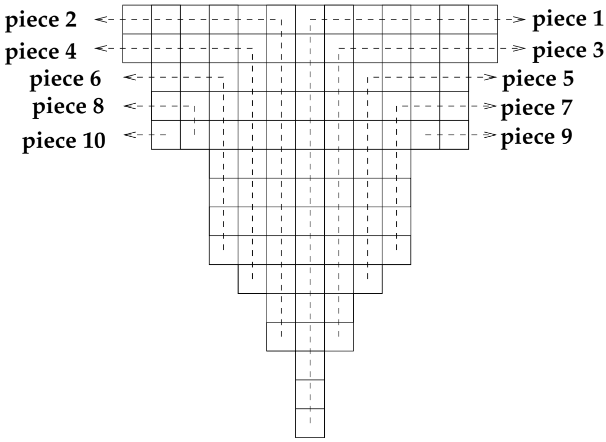

FIGURE 8.3
Sylvester's partition bijection.

of μ into a sequence of disjoint L-shaped pieces (described below), and letting the parts of ν be the number of cells in each piece. The first L-shaped piece consists of all cells in column 0 together with all cells to the right of column 0 in row 1. The second L-shaped piece consists of all cells in column -1 together with all cells left of column -1 in row 1. The third piece consists of the unused cells in column 1 (so row 1 is excluded) together with all cells right of column 1 in row 2. The fourth piece consists of the unused cells in column -2 together with all cells left of column -2 in row 2. We proceed similarly, working outwards in both directions from the center column, cutting off L-shaped pieces that alternately move up and right, then up and left (see Figure 8.3).

One may check geometrically that the size of each L-shaped piece is strictly less than the size of the preceding piece. It follows that $F(\mu) = \nu = (\nu_1 > \nu_2 > \cdots)$ is indeed an element of DisPar. Furthermore, since $|\mu|$ is the sum of the sizes of all the L-shaped pieces, the map $F : \text{OddPar} \to \text{DisPar}$ is area-preserving. We must also check that F is a *bijection* by constructing a map $G : \text{DisPar} \to \text{OddPar}$ that is the two-sided inverse of F.

To see how to define G, let us examine more closely the dimensions of the L-shaped pieces that appear in the definition of $F(\mu)$. Note that each L-shaped piece consists of a corner square, a "vertical portion" of zero or more squares below the corner, and a "horizontal portion" of zero or more squares to the left or right of the corner. Let y_0 be the number of cells in column 0 of the centered diagram of μ (so $y_0 = \ell(\mu)$). For all $i \geq 1$, let x_i be the number of cells in the horizontal portion of the $(2i-1)$th L-shaped piece for μ. For all $i \geq 0$, let y_i be the number of cells in the vertical portion of the $2i$th L-shaped piece for μ. For example, in Figure 8.3 we have $(y_0, y_1, y_2, \ldots) = (15, 11, 8, 6, 1, 0, 0, \ldots)$ and $(x_1, x_2, \ldots) = (6, 5, 3, 2, 1, 0, 0, \ldots)$. Note that for all $i \geq 1$, $y_{i-1} > y_i$ whenever $y_{i-1} > 0$, and $x_i > x_{i+1}$ whenever $x_i > 0$. Moreover, by the symmetry of the centered diagram of μ and the definition of F, we see that

$$\begin{aligned} \nu_1 &= y_0 + x_1, & \nu_2 &= x_1 + y_1, \\ \nu_3 &= y_1 + x_2, & \nu_4 &= x_2 + y_2, \\ \nu_5 &= y_2 + x_3, & \nu_6 &= x_3 + y_3, \end{aligned}$$

and, in general,

$$\nu_{2i-1} = y_{i-1} + x_i \quad (i \geq 1); \qquad \nu_{2i} = x_i + y_i \quad (i \geq 1). \tag{8.2}$$

To compute $G(\nu)$ for $\nu \in \text{DisPar}$, we need to solve the preceding system of equations for x_i and y_i, given the part lengths ν_j. Noting that ν_k, x_k, and y_k must all be zero for large enough indices k, we can solve for each variable by taking the alternating sum of all the given equations from some point forward. This forces us to define

$$\begin{aligned} y_i &= \nu_{2i+1} - \nu_{2i+2} + \nu_{2i+3} - \nu_{2i+4} + \cdots \qquad (i \geq 0); \\ x_i &= \nu_{2i} - \nu_{2i+1} + \nu_{2i+2} - \nu_{2i+3} + \cdots \qquad (i \geq 1). \end{aligned}$$

One verifies immediately that these choices of x_i and y_i do indeed satisfy the equations $\nu_{2i-1} = y_{i-1} + x_i$ and $\nu_{2i} = x_i + y_i$. Furthermore, because the nonzero parts of ν are distinct, the required inequalities ($y_{i-1} > y_i$ whenever $y_{i-1} > 0$, and $x_i > x_{i+1}$ whenever $x_i > 0$) also hold. Now that we know the exact shape of each L-shaped piece, we can fit the pieces together to recover the centered diagram of $\mu = G(\nu) \in \text{OddPar}$. For example, given $\nu = (9, 8, 5, 3, 1, 0, 0, \ldots)$, we compute

$$\begin{aligned} y_0 &= 9 - 8 + 5 - 3 + 1 = 4 \\ x_1 &= 8 - 5 + 3 - 1 = 5 \\ y_1 &= 5 - 3 + 1 = 3 \\ x_2 &= 3 - 1 = 2 \\ y_2 &= 1. \end{aligned}$$

Using this data to reconstitute the centered diagram, we find that $G(\nu) = (11, 7, 5, 3)$. In closing, we remark that bijectivity of F is equivalent to the fact that, for each $\nu \in \text{DisPar}$, the system of equations (8.2) has exactly one solution for the unknowns x_i and y_i.

8.23. Glaisher's Bijection. We define a map $H : \text{DisPar} \to \text{OddPar}$ as follows. Each integer $k \geq 1$ can be written uniquely in the form $k = 2^e c$, where $e \geq 0$ and c is odd. Given $\nu \in \text{DisPar}$, we replace each part k in ν by 2^e copies of the part c (where $k = 2^e c$, as above). Sorting the resulting odd numbers into decreasing order gives us an element $H(\nu)$ in OddPar such that $|H(\nu)| = |\nu|$. For example,

$$\begin{aligned} H((15, 12, 10, 8, 6, 3, 1)) &= \text{sort}((15, 3, 3, 3, 3, 5, 5, 1, 1, 1, 1, 1, 1, 1, 1, 3, 3, 3, 1)) \\ &= (15, 5, 5, 3, 3, 3, 3, 3, 3, 3, 1, 1, 1, 1, 1, 1, 1, 1, 1). \end{aligned}$$

The inverse map $K : \text{OddPar} \to \text{DisPar}$ is defined as follows. Consider a partition $\mu \in \text{OddPar}$. For each odd number c that appears as a part of μ, let $n = n(c) \geq 1$ be the number of times c occurs in μ. We can write n uniquely as a sum of distinct powers of 2 (this is the base-2 expansion of the integer n, cf. 5.5). Say $n = 2^{d_1} + 2^{d_2} + \cdots + 2^{d_s}$. We replace the n copies of c in μ by parts of size $2^{d_1}c$, $2^{d_2}c$, …, $2^{d_s}c$. These parts are distinct from one another (since the d_j's are distinct), and they are also distinct from the parts obtained in the same way from other odd values of c appearing as parts of μ. Sorting the parts thus gives a partition $K(\mu) \in \text{DisPar}$. For example,

$$K((7, 7, 7, 7, 7, 3, 3, 3, 3, 3, 3, 1, 1, 1)) = \text{sort}((28, 7, 12, 6, 2, 1)) = (28, 12, 7, 6, 2, 1).$$

It is readily verified that $H \circ K$ and $K \circ H$ are identity maps.

Glaisher's bijection generalizes to prove the following theorem.

8.24. Theorem: Glaisher's Partition Identity. For all $d \geq 2$ and $N \geq 0$, the number of partitions of N where no part repeats d or more times equals the number of partitions of N with no part divisible by d.

Proof. For fixed d, let A be the set of partitions where no part repeats d or more times, and let B be the set of partitions with no part divisible by d. It suffices to describe weight-preserving maps $H : A \to B$ and $K : B \to A$ such that $H \circ K$ and $K \circ H$ are identity maps. We define K by analogy with what we did above. Fix $\mu \in B$. For each c that appears as a part of μ, let $n = n(c)$ be the number of times this part occurs in μ. Write n in base d as

$$n = \sum_{k=0}^{s} a_k d^k \qquad (0 \leq a_k < d),$$

where n and $a_0, \ldots, a_s$ all depend on c. To construct $K(\mu)$, we replace the n copies of c in μ by a_0 copies of $d^0 c$, a_1 copies of $d^1 c$, …, a_k copies of $d^k c$, …, and a_s copies of $d^s c$. One checks that the resulting partition lies in A, using the fact that no part c of μ is divisible by d.

To compute $H(\nu)$ for $\nu \in A$, note that each part m in ν can be written uniquely in the form $m = d^k c$ for some $k \geq 0$ and some $c = c(m)$ not divisible by d. Adding up all such parts of ν that have the same value of c produces an expression of the form $\sum_{k\geq 0} a_k d^k c$, where $0 \leq a_k < d$ by definition of A. To get $H(\nu)$, we replace all these parts by $\sum_{k\geq 0} a_k d^k$ copies of the part c, for every possible c not divisible by d. Comparing the descriptions of H and K, one sees that these two maps are inverses. □

8.25. Remark: Rogers-Ramanujan Identities. A huge number of partition identities have been discovered, which are similar in character to the one we just proved. Two especially famous examples are the *Rogers-Ramanujan identities.* The first such identity says that, for all N, the number of partitions of N into parts congruent to 1 or 4 modulo 5 equals the number of partitions of N into distinct parts $\nu_1 > \nu_2 > \cdots > \nu_k > 0$ such that $\nu_i - \nu_{i+1} \geq 2$ for all $i < k$. The second identity says that, for all N, the number of partitions of N into parts congruent to 2 or 3 modulo 5 equals the number of partitions of N into distinct parts $\nu_1 > \nu_2 > \cdots > \nu_k > 0 = \nu_{k+1}$ such that $\nu_i - \nu_{i+1} \geq 2$ for all $i \leq k$. One can seek algebraic and/or bijective proofs for these and other identities. Proofs of both types are known for the Rogers-Ramanujan identities, but the bijective proofs are all quite complicated.

8.7 Euler's Pentagonal Number Theorem

We have seen that $\prod_{i\geq 1}(1 + x^i)$ is the generating function for partitions with distinct parts, whereas $\prod_{i\geq 1}(1 - x^i)^{-1}$ is the generating function for all integer partitions. This section investigates the infinite product $\prod_{i\geq 1}(1 - x^i)$, which is the multiplicative inverse for the partition generating function (see 7.42). The next theorem shows that expanding this product leads to a remarkable amount of cancellation of terms due to the minus signs (cf. 7.139).

8.26. Pentagonal Number Theorem.

$$\begin{aligned}\prod_{i=1}^{\infty}(1 - x^i) &= 1 + \sum_{n=1}^{\infty}(-1)^n [x^{n(3n-1)/2} + x^{n(3n+1)/2}] \\ &= 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - x^{35} - x^{40} + \cdots.\end{aligned}$$

Proof. Consider the set DisPar of integer partitions with distinct parts, weighted by area. For $\mu \in$ DisPar, define the *sign* of μ to be $(-1)^{\ell(\mu)}$. By modifying the argument in 8.20 to include these signs, we obtain

$$\prod_{i=1}^{\infty}(1-x^i) = \sum_{\mu \in \text{DisPar}} (-1)^{\ell(\mu)} x^{|\mu|}.$$

We now define an ingenious area-preserving, sign-reversing involution I on DisPar (due to Franklin). Given a partition $\mu = (\mu_1 > \mu_2 > \cdots > \mu_s) \in$ DisPar, let $a \geq 1$ be the largest index such that the part sizes $\mu_1, \mu_2, \ldots, \mu_a$ are consecutive integers, and let $b = \mu_s$ be the smallest part of μ. Figure 8.4 shows how a and b can be read from the Ferrers diagram of μ. For most partitions μ, we define I as follows. If $a < b$, let $I(\mu)$ be the partition obtained by decreasing the first a parts of μ by 1 and adding a new part of size a to the end of μ. If $a \geq b$, let $I(\mu)$ be the partition obtained by removing the last part of μ (of size b) and increasing the first b parts of μ by 1 each. See the examples in Figure 8.4. I is weight-preserving and sign-reversing, since $I(\mu)$ has either one more or one fewer part than μ. It is also routine to check that $I(I(\mu)) = \mu$. Thus we can cancel out all the pairs of objects $\{\mu, I(\mu)\}$.

It may seem at first glance that we have canceled *all* the objects in DisPar! However, there are some choices of μ where the definition of $I(\mu)$ in the previous paragraph fails to produce a partition with distinct parts. Consider what happens in the "overlapping" situation $a = \ell(\mu)$. If $b = a+1$ in this situation, the prescription for creating $I(\mu)$ leads to a partition whose smallest two parts both equal a. On the other hand, if $b = a$, the definition of $I(\mu)$ fails because there are not enough parts left to increment by 1 after dropping the smallest part of μ. In all other cases, the definition of I works even when $a = \ell(\mu)$. We see now that there are two classes of partitions that cannot be canceled by I (Figure 8.5). First, there are partitions of the form $(2n, 2n-1, \ldots, n+1)$, which have length n and area $n(3n+1)/2$, for all $n \geq 1$. Second, there are partitions of the form $(2n-1, 2n-2, \ldots, n)$, which have length n and area $n(3n-1)/2$, for all $n \geq 1$. Furthermore, the empty partition is not canceled by I. Adding up these signed, weighted objects gives the right side of the equation in the theorem. □

We can now deduce Euler's recursion for counting integer partitions that we stated in 2.48.

8.27. Theorem: Partition Recursion. For every $n \in \mathbb{Z}$, let $p(n)$ be the number of integer partitions of n. The numbers $p(n)$ satisfy the recursion

$$\begin{aligned} p(n) &= p(n-1)+p(n-2)-p(n-5)-p(n-7)+p(n-12)+p(n-15)-\cdots \\ &= \textstyle\sum_{k\geq 1}(-1)^{k-1}[p(n-k(3k-1)/2)+p(n-k(3k+1)/2)] \end{aligned} \tag{8.3}$$

for $n \geq 1$. The initial conditions are $p(0) = 1$ and $p(n) = 0$ for all $n < 0$.

Proof. We have proved the identities

$$\begin{aligned} \prod_{i\geq 1}\frac{1}{1-x^i} &= \sum_{\mu \in \text{Par}} x^{|\mu|} = \sum_{n\geq 0} p(n)x^n, \\ \prod_{i\geq 1}(1-x^i) &= 1+\sum_{k\geq 1}(-1)^k[x^{k(3k-1)/2}+x^{k(3k+1)/2}]. \end{aligned}$$

The product of the left sides of these two identities is 1, so the product of the right sides is

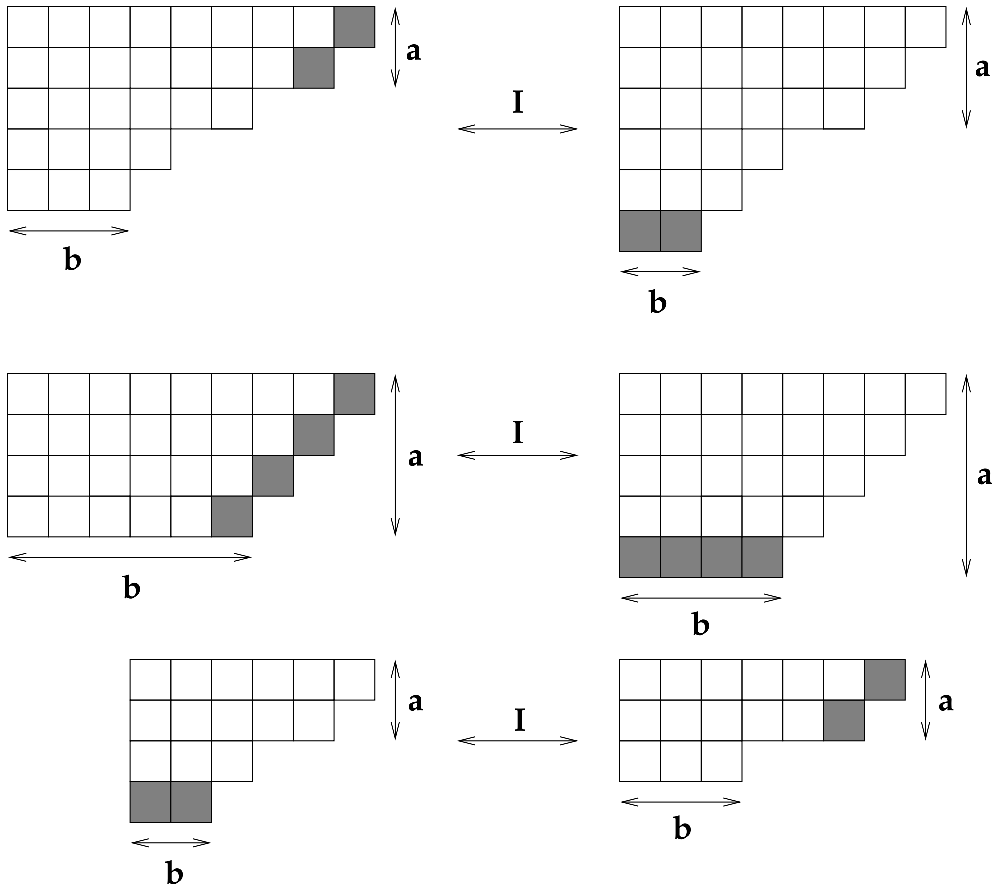

FIGURE 8.4
Franklin's partition involution.

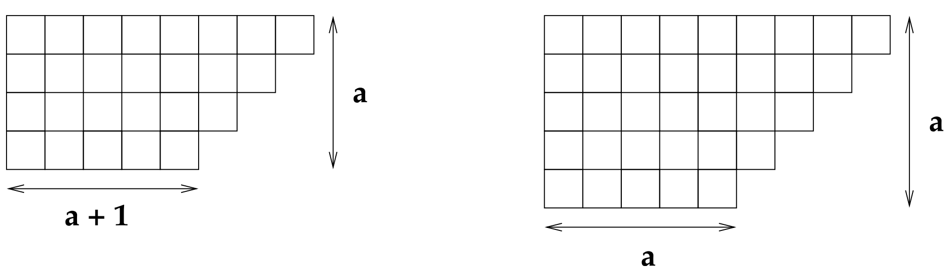

FIGURE 8.5
Fixed points of Franklin's involution.

also 1. Thus, for each $n \geq 1$, the coefficient of x^n in the product

$$\left(\sum_{n\geq 0} p(n)x^n\right) \cdot \left(1 + \sum_{k\geq 1}(-1)^k[x^{k(3k-1)/2} + x^{k(3k+1)/2}]\right)$$

is zero. This coefficient also equals $p(n) - p(n-1) - p(n-2) + p(n-5) + p(n-7) - \cdots$. Solving for $p(n)$ yields the recursion in the theorem. □

8.8 Stirling Numbers of the First Kind

We can often translate combinatorial recursions into generating functions for the objects in question. We illustrate this process by developing generating functions for the Stirling numbers of the first and second kind.

Recall from §3.6 that the signless Stirling number of the first kind (which we denote here by $c(n,k)$) counts the number of permutations of n objects whose functional digraphs consist of k disjoint cycles. These numbers satisfy $c(n,0) = 0$ for $n > 0$, $c(n,n) = 1$ for $n \geq 0$, and

$$c(n,k) = c(n-1,k-1) + (n-1)c(n-1,k) \qquad (0 < k < n).$$

We also set $c(n,k) = 0$ whenever $k < 0$ or $k > n$.

Define a polynomial $f_n = \sum_{k=0}^{n} c(n,k)t^k \in \mathbb{Q}[t]$ for each $n \geq 0$, and define the formal power series

$$F = \sum_{n\geq 0} \frac{f_n}{n!}x^n = \sum_{n\geq 0}\sum_{k=0}^{n} \frac{c(n,k)}{n!}t^k x^n \in \mathbb{Q}(t)[[x]].$$

As we will see, it is technically convenient to introduce the denominators $n!$ as done here. Note that the coefficient of $t^k x^n$ in F, namely $c(n,k)/n!$, is the *probability* that a randomly chosen permutation of n objects will have k cycles. The coefficient of x^0 in F is the polynomial $1 \in \mathbb{Q}[t]$.

We will use the recursion for $c(n,k)$ to prove the relation $(1-x)D_xF = tF$, where D_xF is the formal partial derivative of F with respect to x (§7.16). We first compute

$$\begin{aligned} D_xF &= \sum_{n\geq 0}\sum_{k=0}^{n} \frac{nc(n,k)}{n!}t^k x^{n-1} = \sum_{n\geq 0}\sum_{k=0}^{n} \frac{n[c(n-1,k-1) + (n-1)c(n-1,k)]}{n!}t^k x^{n-1} \\ &= \sum_{n\geq 0}\sum_{k=0}^{n} \frac{c(n-1,k-1)}{(n-1)!}t^k x^{n-1} + \sum_{n\geq 0}\sum_{k=0}^{n} \frac{c(n-1,k)}{(n-2)!}t^k x^{n-1}. \end{aligned}$$

In the first summation, let $m = n-1$ and $j = k-1$. After discarding zero terms, we see that

$$\sum_{n\geq 0}\sum_{k=0}^{n} \frac{c(n-1,k-1)}{(n-1)!}t^k x^{n-1} = t\sum_{m\geq 0}\sum_{j=0}^{m} \frac{c(m,j)}{m!}t^j x^m = tF.$$

On the other hand, letting $m = n-1$ in the second summation shows that

$$\sum_{n\geq 0}\sum_{k=0}^{n} \frac{c(n-1,k)}{(n-2)!}t^k x^{n-1} = x\sum_{m\geq 0}\sum_{k=0}^{m} c(m,k)t^k \frac{x^{m-1}}{(m-1)!} = xD_xF,$$

since $D_x(x^m/m!) = x^{m-1}/(m-1)!$. So we indeed have $D_xF = tF + xD_xF$, as claimed.

We now know a formal differential equation satisfied by F, together with the initial condition $F(0) = 1$. To find an explicit formula for F, we need only "solve for F" by techniques that may be familiar from calculus (cf. 7.176). However, one must remember that all our computations need to be justifiable at the level of *formal* power series. Let us work in the ring $R = K[[x]]$ where K is the field $\mathbb{Q}(t)$. We begin by writing the differential equation in the form

$$(1-x)D_xF = tF.$$

Since $1 - x \in R$ and $F \in R$ have nonzero constant terms, we can divide by these quantities (using 7.40) to arrive at

$$\frac{D_xF}{F} = \frac{t}{1-x}.$$

Now, $\log(F)$ is defined because $F(0) = 1$ (see 7.92), and the derivative of $\log(F)$ is $(D_xF)/F$ by 7.96. On the other hand, using 7.96 and 7.79, we see that $\frac{t}{1-x}$ is the formal derivative of $\log[(1-x)^{-t}]$. We therefore have

$$\frac{d}{dx}(\log(F)) = \frac{d}{dx}\left(\log[(1-x)^{-t}]\right).$$

Since both logarithms have constant term zero, we deduce

$$\log(F) = \log[(1-x)^{-t}]$$

using 7.130 or 7.132. Finally, taking the formal exponential of both sides and using 7.94 gives

$$F = (1-x)^{-t}. \tag{8.4}$$

Having discovered this formula for F, we can now give an independent verification of its correctness by invoking our earlier results on Stirling numbers and generalized powers:

$$\begin{aligned}
(1-x)^{-t} &= \sum_{n\geq 0} \frac{(-t)\downarrow_n}{n!}(-x)^n \quad \text{by 7.74}\\
&= \sum_{n\geq 0} \frac{(t)\uparrow_n}{n!}x^n \quad \text{by 2.76}\\
&= \sum_{n\geq 0} f_n \frac{x^n}{n!} \quad \text{by 2.78}\\
&= F.
\end{aligned}$$

8.9 Stirling Numbers of the Second Kind

In this section, we derive a generating function for Stirling numbers of the second kind. Recall from §2.9 that $S(n,k)$ is the number of set partitions of an n-element set into k nonempty blocks. We will study the formal power series

$$G = \sum_{n\geq 0}\sum_{k=0}^{n} \frac{S(n,k)}{n!} t^k x^n \in \mathbb{Q}(t)[[x]].$$

The following recursion will help us find a differential equation satisfied by G.

8.28. Theorem: Recursion for Stirling Numbers. For all $n \geq 0$ and $0 \leq k \leq n+1$,

$$S(n+1,k) = \sum_{i=0}^{n} \binom{n}{i} S(n-i,k-1).$$

The initial conditions are $S(0,0) = 1$ and $S(n,k) = 0$ whenever $k < 0$ or $k > n$.

Proof. Consider set partitions of $\{1,2,\ldots,n+1\}$ into k blocks such that the block containing $n+1$ has i other elements in it (where $0 \leq i \leq n$). To build such a set partition, choose the i elements that go in the block with $n+1$ in $\binom{n}{i}$ ways, and then choose a set partition of the remaining $n-i$ elements into $k-1$ blocks. The recursion now follows from the sum and product rules. (Compare to the proof of 2.53.) $\square$

8.29. Theorem: Differential Equation for G. The series $G = \sum_{n\geq 0}\sum_{k=0}^{n} \frac{S(n,k)}{n!} t^k x^n$ satisfies $G(0) = 1$ and

$$D_x G = te^x G.$$

Proof. The derivative of G with respect to x is

$$D_x G = \sum_{m\geq 0}\sum_{k=0}^{m} \frac{S(m,k)}{m!} t^k m x^{m-1} = \sum_{n\geq 0}\sum_{k=0}^{n+1} \frac{S(n+1,k)}{n!} t^k x^n,$$

where we have set $n = m-1$. Using 8.28 transforms this expression into

$$\sum_{n\geq 0}\sum_{k=0}^{n+1}\sum_{i=0}^{n} \frac{1}{n!}\binom{n}{i} S(n-i,k-1) t^k x^n = \sum_{n\geq 0}\sum_{k=0}^{n+1}\sum_{i=0}^{n} \frac{S(n-i,k-1)t^k x^{n-i}}{(n-i)!} \cdot \frac{x^i}{i!}.$$

Setting $j = k-1$, the formula becomes

$$t\sum_{n\geq 0}\sum_{j=0}^{n}\sum_{i=0}^{n} \frac{S(n-i,j)t^j x^{n-i}}{(n-i)!} \cdot \frac{x^i}{i!} = t\sum_{n\geq 0}\sum_{i=0}^{n}\sum_{j=0}^{n-i} \frac{S(n-i,j)t^j x^{n-i}}{(n-i)!} \cdot \frac{x^i}{i!}.$$

Finally, recalling 7.6, the last expression equals

$$t\left(\sum_{i\geq 0} \frac{x^i}{i!}\right) \cdot \left(\sum_{m\geq 0}\left[\sum_{j=0}^{m} \frac{S(m,j)}{m!} t^j\right] x^m\right) = te^x G. \qquad \square$$

8.30. Theorem: Generating Function for Stirling Numbers of the Second Kind.

$$\sum_{n\geq 0}\sum_{k=0}^{n} \frac{S(n,k)}{n!} t^k x^n = \exp[t(e^x - 1)] \in \mathbb{Q}(t)[[x]].$$

Proof. Call the left side G, as above. We proceed to solve the differential equation $D_x G = te^x G$ and initial condition $G(0) = 1$. The series G is invertible, having nonzero constant term, so we have $(D_x G)/G = te^x$. Formally integrating both sides with respect to x (cf. 7.130) leads to $\log(G) = te^x + c$, where $c \in \mathbb{Q}(t)$. As $\log(G)_0 = 0$ and $(te^x)_0 = t$, the constant of integration must be $c = -t$. Finally, exponentiating both sides shows that

$$G = \exp(te^x - t) = \exp[t(e^x - 1)]. \qquad \square$$

8.31. Theorem: Generating Function for $S(n,k)$ for fixed k. For all $k \geq 0$,

$$\sum_{n \geq k} S(n,k)\frac{x^n}{n!} = \frac{1}{k!}(e^x - 1)^k \in \mathbb{Q}[[x]].$$

Proof. We have

$$\exp[t(e^x - 1)] = \sum_{k \geq 0} t^k \frac{(e^x - 1)^k}{k!} \in \mathbb{Q}((x))[[t]].$$

Extracting the coefficient of t^k and using 8.30, we obtain the desired formula. □

8.10 The Exponential Formula

Many combinatorial structures can be decomposed into disjoint unions of "connected components." For example, set partitions consist of a collection of disjoint blocks; permutations can be regarded as a collection of disjoint cycles; and graphs are disjoint unions of connected graphs. The *exponential formula* allows us to compute the generating function for such structures from the generating function for the connected "building blocks" of which they are composed.

For each $k \geq 1$, let $\mathcal{C}_k$ be an admissible weighted set of "connected structures of size k," and let $C_k = \sum_{z \in \mathcal{C}_k} \mathrm{wt}(z) \in \mathbb{Q}[[t]]$. We introduce the generating function $C^* = \sum_{k \geq 1} \frac{C_k}{k!} x^k$ to encode information about all the sets $\mathcal{C}_k$ (cf. 7.168). Next, we must formally define a set of structures of size n that consist of disjoint unions of connected structures of various sizes summing to n. For every $n \geq 0$, let U_n be the set of pairs (S, f) such that: S is a set partition of $\{1, 2, \ldots, n\}$, and $f : S \to \bigcup_{k \geq 1} \mathcal{C}_k$ is a function such that $f(A) \in \mathcal{C}_{|A|}$ for all $A \in S$. This says that for every m-element block A of the set partition S, $f(A)$ is a connected structure of size m. Let $\mathrm{wt}(S, f) = \prod_{A \in S} \mathrm{wt}(f(A))$, and define

$$F = \sum_{n \geq 0} \left(\sum_{u \in U_n} \mathrm{wt}(u) \right) \frac{x^n}{n!}.$$

Note that $F(0) = 1$.

8.32. Theorem: Exponential Formula. With notation as above, $F = \exp(C^*)$.

Proof. By 7.92,

$$\exp(C^*) = \sum_{m \geq 0} \frac{1}{m!} \left(\sum_{k \geq 1} \frac{C_k}{k!} x^k \right)^m.$$

This series has constant term 1. For $n > 0$, the coefficient of x^n in $\exp(C^*)$ is (by 7.10)

$$\sum_{m=1}^{n} \frac{1}{m!} \sum_{k_1=1}^{n} \sum_{k_2=1}^{n} \cdots \sum_{k_m=1}^{n} \chi(k_1 + k_2 + \cdots + k_m = n) \frac{C_{k_1} C_{k_2} \cdots C_{k_m}}{k_1! k_2! \cdots k_m!}.$$

This coefficient can also be written

$$\frac{1}{n!} \sum_{m=1}^{n} \frac{1}{m!} \sum_{\substack{(k_1,\ldots,k_m): \\ k_i > 0, k_1 + \cdots + k_m = n}} \binom{n}{k_1, k_2, \ldots, k_m} C_{k_1} C_{k_2} \cdots C_{k_m}.$$

Comparing to the definition of F, we need to prove that

$$\sum_{u\in U_n}\mathrm{wt}(u)=\sum_{m=1}^{n}\frac{1}{m!}\sum_{\substack{(k_1,\dots,k_m):\\ k_i>0,k_1+\cdots+k_m=n}}\binom{n}{k_1,k_2,\dots,k_m}C_{k_1}C_{k_2}\cdots C_{k_m}.$$

Consider objects $(S,f)\in U_n$ for which $|S|=m$ (so that the object has m "connected components"). By the sum rule, it will suffice to prove

$$m!\sum_{\substack{(S,f)\in U_n:\\ |S|=m}}\mathrm{wt}(u)=\sum_{\substack{(k_1,\dots,k_m):\\ k_i>0,k_1+\cdots+k_m=n}}\binom{n}{k_1,k_2,\dots,k_m}C_{k_1}C_{k_2}\cdots C_{k_m}\qquad(1\le m\le n).$$

The left side counts pairs (T,f), where $T=(T_1,T_2,\dots,T_m)$ is an *ordered* set partition of $\{1,2,\dots,n\}$ into m blocks, and for each $i\le m$, $f(T_i)\in\mathcal{C}_{|T_i|}$. (We must multiply by $m!$ to pass from the set partition $\{T_1,\dots,T_m\}$ to the ordered set partition $(T_1,\dots,T_m)$.) The right side counts the same set of objects, as we see by the following counting argument. Let $(k_1,\dots,k_m)$ be the sizes of the blocks in the ordered list $T=(T_1,\dots,T_m)$. We can identify T with a word $w=w_1w_2\cdots w_n$ in $\mathcal{R}(1^{k_1}\cdots m^{k_m})$ by letting $w_i=j$ iff $i\in T_j$. It follows that there are $\binom{n}{k_1,k_2,\dots,k_m}$ choices for the ordered set partition T (see 1.46). Next, for $1\le i\le m$, we choose $f(T_i)\in\mathcal{C}_{k_i}$. The generating function for this choice is C_{k_i}. The formula on the right side now follows from the product and sum rules. □

The generating functions for Stirling numbers (derived in §8.8 and §8.9) are consequences of the exponential formula, as we now show.

8.33. Example: Bell Numbers and Stirling Numbers of the Second Kind. For each $k\ge 1$, let $\mathcal{C}_k$ consist of a single element of weight 1. Then $C^*=\sum_{k\ge1}1x^k/k!=e^x-1$. With this choice of the sets $\mathcal{C}_k$, an element $(S,f)\in U_n$ can be identified with the set partition S of an n-element set, since there is only one possible choice for the function f. Therefore, in this example,

$$F=\sum_{u\in U_n}\mathrm{wt}(u)\frac{x^n}{n!}=\sum_{n\ge0}\frac{B(n)}{n!}x^n,$$

where the Bell number $B(n)$ counts set partitions (see 2.51). The exponential formula now gives

$$\sum_{n\ge0}\frac{B(n)}{n!}x^n=\exp(e^x-1).$$

Intuitively, the unique element in $\mathcal{C}_k$ is the k-element set $\{1,2,\dots,k\}$ which is the prototypical example of a k-element block in a set partition. If we let this element have weight t (for all k), then $\mathrm{wt}(S,f)=t^{|S|}$ will encode the number of blocks in the set partition S. In this case, $C^*=t(e^x-1)$ and the exponential formula gives

$$\sum_{n\ge0}\sum_{k=0}^{n}\frac{S(n,k)}{n!}t^kx^n=\exp[t(e^x-1)],$$

in agreement with 8.30.

8.34. Example: Stirling Numbers of the First Kind. For each $k\ge1$, let $\mathcal{C}_k$ consist of all k-cycles on $\{1, 2, \ldots, k\}$, each having weight t. Since there are $(k-1)!$ such k-cycles, we have

$$C^* = \sum_{k\geq 1} \frac{t(k-1)!}{k!} x^k = t \sum_{k\geq 1} \frac{x^k}{k} = -t\log(1-x) = \log[(1-x)^{-t}]$$

(the last step used 7.97). Consider an element $(S, f) \in U_n$. If $A = \{i_1 < i_2 < \cdots < i_k\}$ is a block of S, then $f(A)$ is some k-cycle $(j_1, j_2, \ldots, j_k)$, where $j_1, \ldots, j_k$ is a rearrangement of $1, 2, \ldots, k$. By replacing the numbers $1, 2, \ldots, k$ in this k-cycle by $i_1, i_2, \ldots, i_k$, we obtain a k-cycle with vertex set A. Doing this for every $A \in S$ produces the functional digraph of a permutation of n elements. More formally, we have just defined a bijection from U_n to the set S_n of permutations of $\{1, 2, \ldots, n\}$. Note that $\mathrm{wt}(S, f) = t^{|S|} = t^c$, where c is the number of cycles in the permutation associated to (S, f). It follows from these observations and the exponential formula that

$$\sum_{n\geq 0} \sum_{k=0}^{n} \frac{c(n,k)}{n!} t^k x^n = \sum_{n\geq 0} \sum_{w\in S_n} \frac{1}{n!} t^{\mathrm{cyc}(w)} x^n = \exp(C^*) = (1-x)^{-t},$$

in agreement with (8.4).

In the next example, we use the inverse of the exponential formula to deduce the generating function C^* from knowledge of the generating function F.

8.35. Example: Connected Components of Graphs. For each $k \geq 1$, let $\mathcal{C}_k$ consist of all *connected* simple graphs on the vertex set $\{1, 2, \ldots, k\}$; let each such graph have weight 1. Direct computation of the generating function C^* is difficult. On the other hand, consider an object $(S, f) \in U_n$. Given a block $A = \{i_1 < i_2 < \cdots < i_k\}$ in S, $f(A)$ is a connected graph with vertex set $1, 2, \ldots, k$. Renaming these vertices to be $i_1, i_2, \ldots, i_k$ produces a connected graph with vertex set A. Doing this for every $A \in S$ produces an *arbitrary* simple graph with vertex set $\{1, 2, \ldots, n\}$. Thus, F is the generating function for such graphs, with x keeping track of the number of vertices. There are $2^{\binom{n}{2}}$ simple graphs on n vertices, since we can either include or exclude each of the $\binom{n}{2}$ possible edges. Accordingly,

$$\exp(C^*) = F = \sum_{n\geq 0} \frac{2^{\binom{n}{2}}}{n!} x^n$$

and so $C^* = \log(F)$. Extracting the coefficient of $x^n/n!$ on both sides leads to the exact formula

$$\sum_{m=1}^{n} \sum_{\substack{(k_1,\ldots,k_m):\\ k_i>0, k_1+\cdots+k_m=n}} \frac{(-1)^{m-1}}{m} \binom{n}{k_1, \ldots, k_m} 2^{\binom{k_1}{2}+\cdots+\binom{k_m}{2}} \tag{8.5}$$

for the number of connected simple graphs on n vertices.

Summary

- *Admissible Sets and Generating Functions.* A weighted set (T, wt) is admissible iff $T_n = \{z \in T : \mathrm{wt}(z) = n\}$ is finite for all $n \geq 0$. The generating function for an admissible weighted set is $G_{T,\mathrm{wt}} = \sum_{n=0}^{\infty} |T_n| x^n \in \mathbb{Q}[[x]]$. Two weighted sets have the same generating function iff there is a weight-preserving bijection between them.

- *Sum Rule for Generating Functions.* If an admissible weighted set S is a finite or countable disjoint union of subsets T_i, then each T_i is admissible and $G_S = \sum_i G_{T_i}$.

- *Finite Product Rule for Generating Functions.* If $T = T_1 \times \cdots \times T_k$ is a finite product of admissible weighted sets, with $\mathrm{wt}(z_1, \ldots, z_k) = \sum_i \mathrm{wt}(z_i)$, then T is admissible and $G_T = \prod_{i=1}^{k} G_{T_i}$.

- *Infinite Product Rule for Generating Functions.* Suppose $\{T_n : n \geq 1\}$ is a family of admissible weighted sets where each T_n has a unique element 1_n of weight zero and $\mathrm{ord}(G_{T_n} - 1) \to \infty$ as $n \to \infty$. Let $T = \prod_{n\geq 1}^{*} T_n$ be the set of sequences $z = (z_n : n \geq 1)$ with $z_n \in T_n$ and $z_n = 1_n$ for all large enough n, with $\mathrm{wt}(z) = \sum_n \mathrm{wt}(z_n)$. Then T is admissible and $G_T = \prod_{n=1}^{\infty} G_{T_n}$.

- *Generating Functions for Trees.* The formal series

$$\frac{1 - \sqrt{1-4x}}{2} = \sum_{n\geq 1} C_{n-1} x^n = \sum_{n\geq 1} \frac{1}{2n-1} \binom{2n-1}{n-1, n}$$

is the generating function for the following sets of weighted trees: (a) binary trees, weighted by number of vertices plus 1; (b) nonempty full binary trees, weighted by number of leaves; (c) ordered trees, weighted by number of vertices.

- *Compositional Inversion Formulas.* Given $F = x/R$ where $R \in K[[x]]$ and $R(0) \neq 0$, the unique formal series G such that $F \bullet G = x = G \bullet F$ is the generating function for the set of terms, where the weight of a term $w_1 \cdots w_n$ is $x^n R_{w_1} \cdots R_{w_n}$. Furthermore, $G(n) = (R^n)_{n-1}/n = [(d/dx)^{n-1} R^n]_0 / n!$ for $n \geq 1$.

- *Partition Generating Functions.* By building integer partitions row by row or column by column and using the product rule, one sees that

$$\sum_{\mu\in\mathrm{Par}} t^{\ell(\mu)} x^{|\mu|} = \prod_{i=1}^{\infty} \frac{1}{1-tx^i} = \sum_{\mu\in\mathrm{Par}} t^{\mu_1} x^{|\mu|};$$

$$\sum_{\mu\in\mathrm{OddPar}} x^{|\mu|} = \prod_{k=1}^{\infty} \frac{1}{1-x^{2k-1}} = \prod_{i=1}^{\infty} (1+x^i) = \sum_{\mu\in\mathrm{DisPar}} x^{|\mu|}.$$

Sylvester's bijection dissects the centered Ferrers diagram of $\mu \in \mathrm{OddPar}$ into L-shaped pieces that give a partition in DisPar. Glaisher's bijection replaces each part $k = 2^e c$ in a partition $\nu \in \mathrm{DisPar}$ (where $e \geq 0$ and c is odd) by 2^e copies of c, giving a partition in OddPar.

- *Pentagonal Number Theorem.* Franklin proved

$$\prod_{i=1}^{\infty} (1-x^i) = 1 + \sum_{n=1}^{\infty} (-1)^n [x^{n(3n-1)/2} + x^{n(3n+1)/2}]$$

by an involution on signed partitions with distinct parts. The map moves boxes between the "staircase" at the top of the partition and the bottom row; this move cancels all partitions except the "pentagonal" ones counted by the right side. Since $\prod_{i\geq 1}(1-x^i)$ is the inverse of the partition generating function, we deduce the partition recursion

$$p(n) = \sum_{k\geq 1} (-1)^{k-1} [p(n - k(3k-1)/2) + p(n - k(3k+1)/2)].$$

- *Generating Functions for Stirling Numbers.* By solving formal differential equations or using the exponential formula, one obtains

$$\sum_{n\geq 0}\sum_{k=0}^{n}\frac{c(n,k)}{n!}t^k x^n = (1-x)^{-t}; \qquad \sum_{n\geq 0}\sum_{k=0}^{n}\frac{S(n,k)}{n!}t^k x^n = \exp[t(e^x-1)].$$

  Hence, $\sum_{n\geq k} S(n,k)x^n/n! = (e^x-1)^k/k!$ for $k\geq 0$.

- *Exponential Formula.* Suppose T_k is a weighted set with $G_{T_k}\in\mathbb{Q}[[t]]$, for $k\geq 0$. Let $C^* = \sum_{k\geq 1}(x^k/k!)G_{T_k}\in\mathbb{Q}[[t,x]]$, and let U_n be the set of pairs (S,f) where S is a set partition of $\{1,2,\ldots,n\}$ and f is a function on S with $f(A)\in T_{|A|}$ for all $A\in S$. Let $\mathrm{wt}(S,f) = \prod_{A\in S}\mathrm{wt}(f(A))$ and $F = \sum_{n\geq 0}\sum_{(S,f)\in U_n}\mathrm{wt}(S,f)x^n/n!$. Then $F = \exp(C^*)$. Informally, if C^* is the exponential generating function for a set of connected building blocks, then $F=\exp(C^*)$ is the exponential generating function for the set of objects obtained by taking labeled disjoint unions of these building blocks.

Exercises

8.36. (a) Let W be the set of all words over a k-letter alphabet, weighted by length. Find the generating function G_W. (b) Show that $x(G'_W)$ is the generating function for the set of all nonempty words in which one letter has been underlined.

8.37. Let S be the set of k-element subsets of $\mathbb{N}$, weighted by the largest element in S. Find G_S. What happens if we weight a subset by its smallest element?

8.38. Fix $k\in\mathbb{N}^+$. Use the sum and product rules for weighted sets to find the generating function for the set of all compositions with k parts, weighted by the sum of the parts.

8.39. Compute the images of these partitions under Sylvester's bijection (see 8.22): (a) $(15,5^2,3^7,1^9)$; (b) $(7^5,3^6,1^3)$; (c) $(11,7,5,3)$; (d) (9^8); (e) $(2n-1,2n-3,\ldots,5,3,1)$. (The notation 5^2 means two parts equal to 5, etc.)

8.40. Compute the images of these partitions under the inverse of Sylvester's bijection: (a) $(15,12,10,8,6,3,1)$; (b) $(28,12,7,6,2,1)$; (c) $(11,7,5,3)$; (d) $(21,17,16,13,11,9,8,3,2,1)$; (e) $(n,n-1,\ldots,3,2,1)$.

8.41. Compute the images of these partitions under Glaisher's bijection (see 8.23): (a) $(9,8,5,3,1)$; (b) $(28,12,7,6,2,1)$; (c) $(11,7,5,3)$; (d) $(21,17,16,13,11,9,8,3,2,1)$.

8.42. Compute the images of these partitions under the inverse of Glaisher's bijection: (a) $(15,5^2,3^7,1^9)$; (b) $(13^2,11^3,7^4,5,3^2,1^3)$ (c) $(11,7,5,3)$; (d) (9^8); (e) (1^n).

8.43. Which partitions map to themselves under Glaisher's bijection? What about the generalized bijection in 8.24?

8.44. Let H and K be the maps in the proof of 8.24. (a) Find $H(25,17,17,10,9,6,6,5,2,2)$, for $d=3,4,5$. (b) Find $K(8^{10},7^7,2^{20},1^{30})$, for $d=3,5,6$.

8.45. Calculate the image of each partition under Franklin's involution (§8.7): (a) $(17,16,15,14,13,10,8,7,4)$; (b) $(17,16,15,14,13,10,8)$; (c) $(n,n-1,\ldots,3,2,1)$; (d) (n).

8.46. Find the generating function for the set of all integer partitions that satisfy each restriction below: (a) all parts are divisible by 3; (b) all parts are distinct and even; (c) odd parts appear at most twice; (d) each part is congruent to 1 or 4 mod 7; (e) for each $i > 0$, there are at most i parts of size i.

8.47. Give combinatorial interpretations for the coefficients in the following formal power series: (a) $\prod_{i\geq 1}(1-x^{5i})^{-1}$; (b) $\prod_{i\geq 0}(1+x^{6i+1})(1+x^{6i+5})$; (c) $\prod_{i\geq 2}(1-x^{2i}+x^{4i})/(1-x^i)$.

8.48. (a) Show that the first Rogers-Ramanujan identity (see 8.25) can be written $\prod_{n=0}^{\infty} \frac{1}{(1-x^{5n+1})(1-x^{5n+4})} = 1 + \sum_{k=1}^{\infty} \frac{x^{k^2}}{(1-x)(1-x^2)\cdots(1-x^k)}$. (b) Find a similar formulation of the second Rogers-Ramanujan identity. (c) Verify the Rogers-Ramanujan identities for partitions of $N = 12$ by explicitly listing all the partitions satisfying the relevant restrictions.

8.49. Give a detailed verification of the claim in 8.11 that the quadratic equation $G^2 - G + x = 0$ has the unique solution $G = (1 - \sqrt{1-4x})/2$ in $\mathbb{Q}[[x]]$.

8.50. Prove that a nonempty full binary tree with a leaves has $a - 1$ non-leaf vertices.

8.51. Let f be the bijection in Figure 8.1. Compute $f(T)$, where T is the binary tree in Figure 2.12.

8.52. Let g be the bijection shown in Figure 8.2. Verify that the number of vertices in $g(t)$ equals the number of leaves in t, for each full binary tree t.

8.53. (a) Describe the inverse of the bijection g shown in Figure 8.2. (b) Calculate the image of the ordered tree in Figure 3.18 under g^{-1}.

8.54. List all full binary trees with 5 leaves, and compute the image of each tree under the map g in Figure 8.2.

8.55. Verify that $(R^n)_{n-1}/n = ((d/dx)^{n-1}R^n)_0/n!$ for all $n \geq 1$ and $R \in K[[x]]$.

8.56. Give an algebraic proof of 8.24 using formal power series.

8.57. (a) Carefully verify that the maps H and K in 8.23 are two-sided inverses. (b) Repeat part (a) for the maps H and K in 8.24.

8.58. (a) Verify that the partition $(2n, 2n-1, \ldots, n+1)$ (one of the fixed points of Franklin's involution) has area $n(3n + 1)/2$. (b) Verify that the partition $(2n - 1, 2n - 2, \ldots, n)$ has area $n(3n - 1)/2$.

8.59. Carry out the computations showing how the equation $C^* = \log(F)$ leads to formula (8.5).

8.60. Rewrite (8.5) as a sum over partitions of n.

8.61. Use (8.5) to compute the number of connected simple graphs with vertex set: (a) $\{1, 2, 3, 4\}$; (b) $\{1, 2, 3, 4, 5\}$.

8.62. (a) Modify (8.5) to include a power of t that keeps track of the number of edges in the connected graph. (b) How many connected simple graphs with vertex set $\{1, 2, 3, 4, 5, 6\}$ have exactly seven edges?

8.63. (a) Find the generating function for the set of all Dyck paths, where the weight of a path ending at (n, n) is n. (b) A *marked* Dyck path is a Dyck path in which one step (north or east) has been circled. Find the generating function for marked Dyck paths.

8.64. Recall that $\sum_{n\geq 0}\sum_{k=0}^{n}\frac{S(n,k)}{n!}t^k x^n = \exp[t(e^x-1)]$. Use partial differentiation of this generating function to find generating functions for: (a) the set of set partitions, where one block in the partition has been circled; (b) the set of set partitions, where one element of one block in the partition has been circled.

8.65. (a) List all terms of length at most 5. (b) Use (a) and 8.14 to write down explicit formulas for the first five coefficients of the compositional inverse of x/R as combinations of the coefficients of R. (c) Use (b) to find the first five terms in the compositional inverse of $x/(1-3x+2x^2+5x^4)$.

8.66. Use 8.15 to compute the compositional inverse of the following formal series: (a) xe^{2x}; (b) $x-x^2$; (c) $x/(1+ax)$; (d) $x-4x^4+4x^7$.

8.67. Let S be the set of paths that start at $(0,0)$ and take horizontal steps (right 1, up 0), vertical steps (right 0, up 1), and diagonal steps (right 1, up 1). By considering the final step of a path, find an equation satisfied by G_S and solve for G_S, taking the weight of a path ending at (c,d) to be: (a) the number of steps in the path; (b) $c+d$; (c) c.

8.68. For fixed $k\geq 1$, find the generating function for integer partitions with: (a) k nonzero parts; (b) k nonzero distinct parts. (c) Deduce summation formulas for the infinite products $\prod_{i\geq 1}(1-x^i)^{-1}$ and $\prod_{i\geq 1}(1+x^i)$.

8.69. A *ternary tree* is either $\emptyset$ or a 4-tuple $(\bullet,t_1,t_2,t_3)$, where each t_i is itself a ternary tree. Find an equation satisfied by the generating function for ternary trees, weighted by number of vertices.

8.70. Let S be the set of ordered trees where every node has at most two children, weighted by the number of vertices. (a) Use the sum and product rules to find an equation satisfied by G_S. (b) Solve this equation for G_S. (c) How many trees in S have 7 vertices?

8.71. Find a formula for the number of simple digraphs with n vertices such that the graph obtained by erasing loops and ignoring the directions on the edges is connected.

8.72. Prove that the number of integer partitions of N in which no even part appears more than once equals the number of partitions of N in which no part appears 4 or more times.

8.73. Prove that the number of integer partitions of N that have no part equal to 1 and no parts that differ by 1 equals the number of partitions of N in which no part appears exactly once.

8.74. (a) Write down an infinite product that is the generating function for integer partitions with odd, distinct parts. (b) Show that the generating function for self-conjugate partitions (i.e., partitions such that $\lambda'=\lambda$) is $1+\sum_{k=1}^{\infty}x^{k^2}/((1-x^2)(1-x^4)\cdots(1-x^{2k}))$. (c) Find an area-preserving bijection between the sets of partitions in (a) and (b), and deduce an associated formal power series identity.

8.75. Evaluate $\sum_{k=1}^{\infty}x^k(1-x)^{-k}$.

8.76. How many integer partitions of n have the form $(i^j(i+1)^k)$ for some $i,j,k>0$?

8.77. Dobinski's Formula. Prove that the Bell numbers (see 2.51) satisfy $B(n) = e^{-1}\sum_{k=0}^{\infty}(k^n/k!)$ for $n\geq 0$.

8.78. Show that, for all $N\geq 0$, $(-1)^N|\,\mathrm{OddPar}\cap\mathrm{DisPar}\cap\mathrm{Par}(N)|$ equals $|\{\mu\in\mathrm{Par}(N):\ell(\mu)\text{ is even}\}|-|\{\mu\in\mathrm{Par}(N):\ell(\mu)\text{ is odd}\}|$.

8.79. (a) Use an involution on the set $\text{Par} \times \text{DisPar}$ to give a combinatorial proof of the identity $\prod_{n=1}^{\infty} \frac{1}{1-x^n} \prod_{n=1}^{\infty}(1-x^n) = 1$. (b) More generally, for $S \subseteq \mathbb{N}^+$, prove combinatorially that $\prod_{n \in S} \frac{1}{1-x^n} \prod_{n \in S}(1-x^n) = 1$.

8.80. (a) Find a bijection from the set of terms of length n to the set of binary trees with $n-1$ nodes. (b) Use (a) to formulate a version of 8.14 that expresses the coefficients in the compositional inverse of x/R as sums of suitably weighted binary trees.

8.81. (a) Find a bijection from the set of terms of length n to the set of Dyck paths ending at $(n-1, n-1)$. (b) Use (a) to formulate a version of 8.14 that expresses the coefficients in the compositional inverse of x/R as sums of suitably weighted Dyck paths.

8.82. Let $d(n,k)$ be the number of derangements in S_n with k cycles. Find a formula for $\sum_{n \geq 0} \sum_{k=0}^{n} \frac{d(n,k)}{n!} t^k x^n$.

8.83. Compute $\sum_{n \geq 0} d_n x^n / n!$, where d_n is the number of derangements of n objects.

8.84. Let $S_1(n,k)$ be the number of set partitions of $\{1, 2, \ldots, n\}$ into k blocks where no block consists of a single element. Find a formula for $\sum_{n \geq 0} \sum_{k=0}^{n} \frac{S_1(n,k)}{n!} t^k x^n$.

8.85. What is the generating function for set partitions in which all block sizes must belong to a given subset $T \subseteq \mathbb{N}^+$?

8.86. Let $A(n,k)$ be the number of ways to assign n people to k committees in such a way that each person belongs to exactly one committee, and each committee has one member designated as chairman. Find a formula for $\sum_{n \geq 0} \sum_{k=0}^{n} \frac{A(n,k)}{n!} t^k x^n$.

8.87. Involution Proof of Euler's Partition Recursion. (a) For fixed $n \geq 1$, prove $\sum_{j \in \mathbb{Z}} (-1)^j p(n - (3j^2 + j)/2) = 0$ by verifying that the following map I is a sign-reversing involution with no fixed points. The domain of I is the set of pairs (j, λ) with $j \in \mathbb{Z}$ and $\lambda \in \text{Par}(n - (3j^2 + j)/2)$. To define $I(j, \lambda)$, consider two cases. If $\ell(\lambda) + 3j \geq \lambda_1$, set $I(j, \lambda) = (j - 1, \mu)$ where μ is formed by preceding the first part of λ by $\ell(\lambda) + 3j$ and then decrementing all parts by 1. If $\ell(\lambda) + 3j < \lambda_1$, set $I(j, \lambda) = (j + 1, \nu)$ where ν is formed by deleting the first part of λ, incrementing the remaining nonzero parts of λ by 1, and appending an additional $\lambda_1 - 3j - \ell(\lambda) - 1$ parts equal to 1. (b) For $n = 21$, $j = 1$, $\lambda = (5, 5, 4, 3, 2)$, compute $I(j, \lambda)$ and verify that $I(I(j, \lambda)) = (j, \lambda)$.

8.88. We say that an integer partition λ *extends* a partition μ iff for all k, k occurs in λ at least as often as k occurs in μ; otherwise, λ *avoids* μ. Suppose $\{\mu^i : i \geq 1\}$ and $\{\nu^i : i \geq 1\}$ are two sequences of distinct, nonzero partitions such that for all finite $S \subseteq \mathbb{N}^+$, $\sum_{i \in S} |\mu^i| = \sum_{i \in S} |\nu^i|$. (a) Prove that for every N, the number of partitions of N that avoid every μ^i equals the number of partitions of N that avoid every ν^i. (b) Show how 8.24 can be deduced from (a). (c) Use (a) to prove that the number of partitions of N into parts congruent to 1 or 5 mod 6 equals the number of partitions of N into distinct parts not divisible by 3.

Notes

The applications of formal power series to combinatorial problems go well beyond the topics covered in this chapter. The texts [10, 127, 139] offer more detailed treatments of the uses of generating functions in combinatorics. Two more classical references are [90, 113]. For an introduction to the vast subject of partition identities, the reader may consult [5, 102]. Sylvester's bijection appears in [129], Glaisher's bijection in [54], and Franklin's involution in [44]. The Rogers-Ramanujan identities are discussed in [106, 117]; Garsia and Milne gave the first bijective proof of these identities [49, 50]. Our treatment of compositional inversion closely follows the presentation in [107].