Chapter 5

Generating Functions and Their Applications[1]

5.1 EXAMPLES OF GENERATING FUNCTIONS

Much of combinatorics is devoted to developing tools for counting. We have seen that it is often important to count the number of arrangements or patterns, but in practice it is impossible to list all of these arrangements. Hence, we need tools to help us in counting. In the next four chapters we present a number of tools that are useful in counting. One of the most powerful tools that we shall present is the notion of the generating function. This chapter is devoted to generating functions.

Often in combinatorics, we seek to count a quantity a_k that depends on an input or a parameter, say k. This is true, for instance, if a_k is the number of steps required to perform a computation if the input has size k. We can formalize the dependence on k by speaking of a sequence of unknown values, $a_0, a_1, a_2, \ldots, a_k, \ldots$. We seek to determine the kth term in this sequence. Generating functions provide a simple way to "encode" a sequence such as $a_0, a_1, a_2, \ldots, a_k, \ldots$, which can readily be "decoded" to find the terms of the sequence. The trick will be to see how to compute the encoding or generating function for the sequence without having the sequence. Then we can decode to find a_k. The method will enable us to determine the unknown quantity a_k in an indirect but highly effective manner.

The method of generating functions that we shall present is an old one. Its roots are in the work of De Moivre around 1720, it was developed by Euler in 1748 in connection with partition problems, and it was treated extensively in the late eighteenth and early nineteenth centuries by Laplace, primarily in connection with probability theory. In spite of its long history, the method continues to have

[1] In an elementary treatment, this chapter should be omitted. Chapters 5 and 6 are the only chapters that make use of the calculus prerequisites, except for assuming a certain level of "mathematical maturity" that comes from taking a calculus course.

widespread application, as we shall see. For a more complete treatment of generating functions, see Lando [2003], MacMahon [1960], Riordan [1980], Srivastava and Manocha [1984], or Wilf [2006]. (See also Riordan [1964].)

5.1.1 Power Series

In this chapter we use a fundamental idea from calculus, the notion of power series. The results about power series we shall need are summarized in this subsection. The reader who wants more details, including proofs of these results, can consult most any calculus book.

A *power series* is an infinite series of the form $\sum_{k=0}^{\infty} a_k x^k$. Such an infinite series always converges for $x = 0$. Either it does not converge for any other value of x, or there is a positive number R (possibly infinite) so that it converges for all x with $|x| < R$. In the latter case, the largest such R is called the *radius of convergence*. In the former case, we say that 0 is the radius of convergence. A power series $\sum_{k=0}^{\infty} a_k x^k$ can be thought of as a function of x, $f(x)$, which is defined for those values of x for which the infinite sum converges and is computed by calculating that infinite sum. In most of this chapter we shall not be concerned with matters of convergence. We simply assume that x has been chosen so that $\sum_{k=0}^{\infty} a_k x^k$ converges.[2]

Power series arise in calculus in the following way. Suppose that $f(x)$ is a function which has derivatives of all orders for all x in an interval containing 0. Then

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots. \tag{5.1}$$

The power series on the right-hand side of (5.1) converges for some values of x, at least for $x = 0$. The power series is called the *Maclaurin series expansion* for f or the *Taylor series expansion* for f about $x = 0$.

Some of the most famous and useful Maclaurin expansions are the following:

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \cdots, \text{ for } |x| < 1, \tag{5.2}$$

$$e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots, \text{ for } |x| < \infty, \tag{5.3}$$

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots, \text{ for } |x| < \infty, \tag{5.4}$$

$$\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots, \text{ for } |x| < 1. \tag{5.5}$$

[2]This assumption can be made more precise by thinking of $\sum_{k=0}^{\infty} a_k x^k$ as simply a formal expression, a *formal power series*, rather than as a function, and by performing appropriate manipulations on these formal expressions. For details of this approach, see Niven [1969].

To show, for instance, that (5.3) is a special case of (5.1), it suffices to observe that if $f(x) = e^x$, then $f^{(k)}(x) = e^x$ for all k, and $f^{(k)}(0) = 1$. Readers should check for themselves that Equations (5.2), (5.4), and (5.5) are also special cases of (5.1).

One of the reasons that power series are so useful is that they can easily be added, multiplied, divided, composed, differentiated, or integrated. We remind the reader of these properties of power series by formulating several general principles.

Principle 1. Suppose that $f(x) = \sum_{k=0}^{\infty} a_k x^k$ and $g(x) = \sum_{k=0}^{\infty} b_k x^k$. Then $f(x)+g(x)$, $f(x)g(x)$, and $f(x)/g(x)$ can be computed by, respectively, adding term by term, multiplying out, or using long division. [This is true for division only if $g(x)$ is not zero for the values of x in question.] Specifically,

$$\begin{aligned}
f(x)+g(x) &= \sum_{k=0}^{\infty}(a_k+b_k)x^k \\
&= (a_0+b_0)+(a_1+b_1)x+(a_2+b_2)x^2+\cdots, \\
f(x)g(x) &= a_0\sum_{k=0}^{\infty} b_k x^k + a_1 x \sum_{k=0}^{\infty} b_k x^k + a_2 x^2 \sum_{k=0}^{\infty} b_k x^k + \cdots \\
&= a_0(b_0+b_1x+b_2x^2+\cdots)+a_1x(b_0+b_1x+b_2x^2+\cdots) \\
&\quad + a_2x^2(b_0+b_1x+b_2x^2+\cdots)+\cdots, \\
\frac{f(x)}{g(x)} &= \frac{a_0}{\sum_{k=0}^{\infty} b_k x^k} + \frac{a_1 x}{\sum_{k=0}^{\infty} b_k x^k} + \frac{a_2 x^2}{\sum_{k=0}^{\infty} b_k x^k} + \cdots \\
&= \frac{a_0}{b_0+b_1x+b_2x^2+\cdots} + \frac{a_1 x}{b_0+b_1x+b_2x^2+\cdots} \\
&\quad + \frac{a_2x^2}{b_0+b_1x+b_2x^2+\cdots} + \cdots.
\end{aligned}$$

If the power series for $f(x)$ and $g(x)$ both converge for $|x| < R$, so do $f(x)+g(x)$ and $f(x)g(x)$. If $g(0) \neq 0$, then $f(x)/g(x)$ converges in some interval about 0.

For instance, using (5.2) and (5.3), we have

$$\begin{aligned}
\frac{1}{1-x}+e^x &= (1+x+x^2+x^3+\cdots)+\left(1+x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\cdots\right) \\
&= (1+1)+(1+1)x+\left(1+\frac{1}{2!}\right)x^2+\left(1+\frac{1}{3!}\right)x^3+\cdots \\
&= \sum_{k=0}^{\infty}\left(1+\frac{1}{k!}\right)x^k.
\end{aligned}$$

Also,

$$\begin{aligned}
\frac{1}{1-x}e^x &= (1+x+x^2+x^3+\cdots)\left(1+x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\cdots\right) \\
&= 1\left(1+x+\frac{1}{2!}x^2+\cdots\right)+x\left(1+x+\frac{1}{2!}x^2+\cdots\right) \\
&\quad + x^2\left(1+x+\frac{1}{2!}x^2+\cdots\right)+\cdots
\end{aligned}$$

$$= \quad 1 + 2x + \frac{5}{2}x^2 + \cdots .$$

Power series are also easy to compute under composition of functions.

Principle 2. If $f(x) = g(u(x))$ and if we know that $g(u) = \sum_{k=0}^{\infty} a_k u^k$, we have $f(x) = \sum_{k=0}^{\infty} a_k [u(x)]^k$.[3]

Thus, setting $u = x^4$ in Equation (5.5) gives us

$$\ln(1 + x^4) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^{4k}.$$

Principle 2 generalizes to the situation where we have a power series for $u(x)$.

Principle 3. If a power series $f(x) = \sum_{k=0}^{\infty} a_k x^k$ converges for all $|x| < R$ with $R > 0$, the derivative and antiderivative of $f(x)$ can be computed by differentiating and integrating term by term. Namely,

$$\frac{df}{dx}(x) = \frac{d}{dx}\left(\sum_{k=0}^{\infty} a_k x^k\right) = \sum_{k=0}^{\infty} \frac{d}{dx}(a_k x^k) = \sum_{k=1}^{\infty} k a_k x^{k-1} \tag{5.6}$$

and

$$\int_0^x f(t)\, dt = \int_0^x \left(\sum_{k=0}^{\infty} a_k t^k\right) dt = \sum_{k=0}^{\infty} \int_0^x a_k t^k \, dt = \sum_{k=0}^{\infty} \frac{1}{k+1} a_k x^{k+1}. \tag{5.7}$$

The power series in (5.6) and (5.7) also converge for $|x| < R$.

For instance, since

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\left[\frac{1}{1-x}\right],$$

we see from (5.2) and (5.6) that

$$\frac{1}{(1-x)^2} = \sum_{k=1}^{\infty} k x^{k-1} = 1 + 2x + 3x^2 + 4x^3 + \cdots .$$

5.1.2 Generating Functions

Suppose that we are interested in computing the kth term in a sequence (a_k) of numbers. We shall use the convention that (a_k) refers to the sequence and a_k, written without parentheses, to the kth term. The *(ordinary) generating function* for the sequence (a_k) is defined to be

$$G(x) = \sum_{k=0}^{\infty} a_k x^k = a_0 x^0 + a_1 x^1 + a_2 x^2 + \cdots . \tag{5.8}$$

[3] If the power series for $g(u)$ converges for $|u| < S$ and $|u(x)| < S$ whenever $|x| < R$, then the power series for $f(x)$ converges for all $|x| < R$.

The sum is finite if the sequence is finite and infinite if the sequence is infinite. In the latter case, we will think of x as having been chosen so that the sum in (5.8) converges.

Example 5.1 Suppose that $a_k = \binom{n}{k}$, for $k = 0, 1, \ldots, n$. Then the ordinary generating function for the sequence (a_k) is

$$G(x) = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n.$$

By the binomial expansion (Theorem 2.7),

$$G(x) = (1 + x)^n.$$

The advantage of what we have done is that we have expressed $G(x)$ in a simple, closed form (encoded form). Knowing this simple form for $G(x)$, one can now possibly derive a_k simply by remembering this closed form for $G(x)$ and decoding, that is, expanding out, and searching for the coefficient of x^k. Even more useful is the fact that, as we have observed before, often we are able to find $G(x)$ without knowing a_k and then to solve for a_k by expanding out. ■

Example 5.2 Suppose that $a_k = 1$, for $k = 0, 1, 2, \ldots$. Then

$$G(x) = 1 + x + x^2 + \cdots.$$

By Equation (5.2),

$$G(x) = \frac{1}{1 - x}$$

provided that $|x| < 1$. Again the reader will note the closed form for $G(x)$. ■

Example 5.3 Often we will know the generating function but not the sequence. We will try to "recover" the sequence from the generating function. For example, suppose that

$$G(x) = \sum_{k=0}^{\infty} a_k x^k$$

and we know

$$G(x) = \frac{x^2}{1 - x}$$

What is a_k? Using Equation (5.2), we have for $|x| < 1$,

$$\begin{aligned} G(x) &= x^2\left[\frac{1}{1 - x}\right] \\ &= x^2(1 + x + x^2 + \cdots) \\ &= x^2 + x^3 + x^4 + \cdots. \end{aligned}$$

Hence,

$$(a_k) = (0, 0, 1, 1, 1, \ldots).$$

In this chapter and Chapter 6 we study a variety of techniques for expanding out $G(x)$ to obtain the desired sequence (a_k). ■

Example 5.4 Suppose that $a_k = 1/k!$, for $k = 0, 1, 2, \ldots$. Then

$$G(x) = \frac{1}{0!} + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots.$$

By Equation (5.3), $G(x) = e^x$ for all values of x. ■

Example 5.5 Suppose that $G(x) = x\sin(x^2)$ is the ordinary generating function for the sequence (a_k). To find a_k, use Equation (5.4), substitute x^2 for x, and multiply by x, to find that

$$\begin{aligned} G(x) &= x\left[x^2 - \frac{1}{3!}(x^2)^3 + \frac{1}{5!}(x^2)^5 - \cdots\right] \\ &= x^3 - \frac{1}{3!}x^7 + \frac{1}{5!}x^{11} - \cdots. \end{aligned}$$

Thus, we see that a_k is the kth term of the sequence

$$\left(0, 0, 0, 1, 0, 0, 0, -\frac{1}{3!}, 0, 0, 0, \frac{1}{5!}, 0, \ldots\right).$$

■

Example 5.6 Suppose that $G(x) = \cos x$ is the ordinary generating function for the sequence (a_k). Since $G(x)$ has derivatives of all orders, we can expand out in a Maclaurin series, by calculating $f^{(k)}(0)$ for all k, and we see that

$$G(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}x^{2k} = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots. \tag{5.9}$$

The verification of this is left as an exercise. An alternative approach is to observe that $G(x) = d(\sin x)/dx$, and so to use Equation (5.4). Then we see that

$$\begin{aligned} G(x) &= \frac{d}{dx}(\sin x) \\ &= \frac{d}{dx}\left[x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots\right] \\ &= \frac{d}{dx}[x] + \frac{d}{dx}\left[-\frac{1}{3!}x^3\right] + \frac{d}{dx}\left[\frac{1}{5!}x^5\right] + \cdots \\ &= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots, \end{aligned}$$

which agrees with Equation (5.9). ■

Example 5.7 If $(a_k) = (1, 1, 1, 0, 1, 1, \ldots)$, the ordinary generating function is given by

$$\begin{aligned} G(x) &= 1 + x + x^2 + x^4 + x^5 + \cdots \\ &= (1 + x + x^2 + x^3 + x^4 + x^5 + \cdots) - x^3 \\ &= \frac{1}{1-x} - x^3. \end{aligned}$$ ■

Example 5.8 If $(a_k) = (1/2!, 1/3!, 1/4!, \ldots)$, the ordinary generating function is given by

$$\begin{aligned} G(x) &= \frac{1}{2!} + \frac{1}{3!}x + \frac{1}{4!}x^2 + \cdots \\ &= \frac{1}{x^2}\left(\frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots\right) \\ &= \frac{1}{x^2}\left(e^x - 1 - x\right). \end{aligned}$$ ■

Example 5.9 The Number of Labeled Graphs In Section 3.1.3 we counted the number $L(n, e)$ of labeled graphs with n vertices and e edges, $n \geq 2, e \leq C(n, 2)$. If n is fixed and we let $a_k = L(n, k), k = 0, 1, \ldots, C(n, 2)$, let us consider the generating function

$$G_n(x) = \sum_{k=0}^{C(n,2)} a_k x^k.$$

Note that in Section 3.1.3 we computed $L(n, e) = C(C(n, 2), e)$. Hence, if $r = C(n, 2)$,

$$G_n(x) = \sum_{k=0}^{r} C(r, k)x^k.$$

By the binomial expansion (Theorem 2.7), we have

$$G_n(x) = (1 + x)^r = (1 + x)^{C(n,2)}, \tag{5.10}$$

which is a simple way to summarize our knowledge of the numbers $L(n, e)$. In particular, from (5.10) we can derive a formula for the number $L(n)$ of labeled graphs of n vertices. For

$$L(n) = \sum_{k=0}^{r} C(r, k),$$

which is $G_n(1)$. Thus, taking $x = 1$ in (5.10) gives us

$$L(n) = 2^{C(n,2)},$$

which is the result we derived in Section 3.1.3. ■

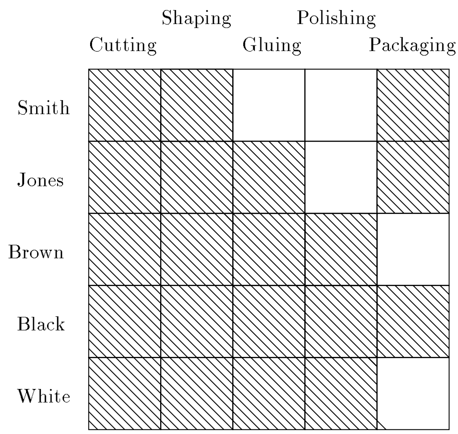


Figure 5.1: A board corresponding to a job assignment problem.

Example 5.10 Rook Polynomials: Job Assignments and Storing Computer Programs Suppose that B is any $n \times m$ board such as those in Figures 5.1 and 5.2, with certain squares forbidden and others acceptable, the acceptable ones being darkened. Let $r_k(B)$ be the number of ways to choose k acceptable (darkened) squares, no two of which lie in the same row and no two of which lie in the same column. We can think of B as part of a chess board. A *rook* is a piece that can travel either horizontally or vertically on the board. Thus, one rook is said to be able to *take* another if the two are in the same row or the same column. We wish to place k rooks on B in acceptable squares in such a way that no rook can take another. Thus, $r_k(B)$ counts the number of ways k nontaking rooks can be placed in acceptable squares of B.

The 5×5 board in Figure 5.1 arises from a job assignment problem. The rows correspond to workers, the columns to jobs, and the i, j position is darkened if worker i is suitable for job j. We wish to determine the number of ways in which each worker can be assigned to one job, no more than one worker per job, so that a worker only gets a job to which he or she is suited. It is easy to see that this is equivalent to the problem of computing $r_5(B)$.

The 5×7 board in Figure 5.2 arises from a problem of storing computer programs. The i, j position is darkened if storage location j has sufficient storage capacity for program i. We wish to assign each program to a storage location with sufficient storage capacity, at most one program per location. The number of ways this can be done is again given by $r_5(B)$. We shall compute $r_5(B)$ in these two examples in the text and exercises of Section 7.1.

The expression

$$R(x, B) = r_0(B) + r_1(B)x + r_2(B)x^2 + \cdots$$

is called the *rook polynomial* for the board B. The rook polynomial is indeed a polynomial in x, since $r_k(B) = 0$ for k larger than the number of acceptable squares.

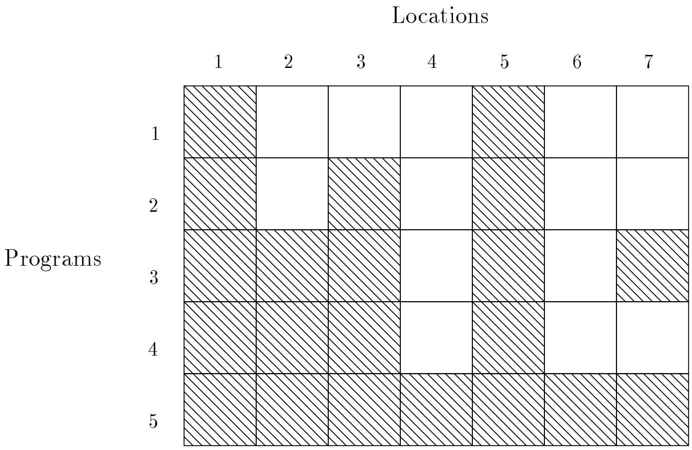


Figure 5.2: A board corresponding to a computer storage problem.

The rook polynomial is just the ordinary generating function for the sequence

$$(r_0(B), r_1(B), r_2(B), \ldots).$$

As with generating functions in general, we shall find methods for computing the rook polynomial without explicitly computing the coefficients $r_k(B)$, and then we shall be able to compute these coefficients from the polynomial.

To give some examples, consider the two boards B_1 and B_2 of Figure 5.3. In board B_1, there is one way to place no rooks (this will be the case for any board), two ways to place one rook (use either darkened square), one way to place two rooks (use both darkened squares), and no way to place more than two rooks. Thus,

$$R(x, B_1) = 1 + 2x + x^2.$$

In board B_2, there is again one way to place no rooks, four ways to place one rook (use any darkened square), two ways to place two rooks (use the diagonal squares or the nondiagonal squares), and no way to place more than two rooks. Thus,

$$R(x, B_2) = 1 + 4x + 2x^2.$$

■

EXERCISES FOR SECTION 5.1

1. For each of the following functions, find its Maclaurin expansion by computing the derivatives $f^{(k)}(0)$.

   (a) $\cos x$ (b) e^{3x} (c) $\sin(2x)$ (d) $x^3 + 4x + 7$

   (e) $x^2 + e^x$ (f) xe^x (g) $\ln(1 + 4x)$ (h) $x \sin x$

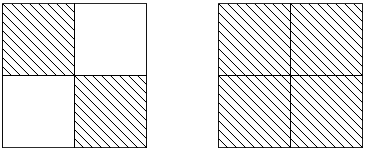

B_1 B_2

Figure 5.3: Two boards.

2. For each of the following functions, use known Maclaurin expansions to find the Maclaurin expansion, by adding, composing, differentiating, and so on.

(a) $x^3 + \dfrac{1}{1-x}$ (b) $x^2\dfrac{1}{1-x}$ (c) $\sin(x^4)$ (d) $\dfrac{1}{4-x}$

(e) $\sin(x^2+x+1)$ (f) $\dfrac{1}{(1-x)^3}$ (g) $5e^x + e^{3x}$ (h) $\ln(1-x)$

(i) $\ln(1+3x)$ (j) $x^3\sin(x^5)$ (k) $\ln(1+x^2)$ (l) $\dfrac{1}{1-2x}e^x$

3. For the following sequences, find the ordinary generating function and simplify if possible.

(a) $(1,1,1,0,0,\ldots)$ (b) $(1,0,2,3,4,0,0,\ldots)$ (c) $(5,5,5,\ldots)$

(d) $(1,0,0,1,1,\ldots)$ (e) $(0,0,1,1,1,1,1,\ldots)$ (f) $(0,0,4,4,4,\ldots)$

(g) $(1,1,2,1,1,1,\ldots)$ (h) $(a_k) = \left(\dfrac{3}{k!}\right)$ (i) $(a_k) = \left(\dfrac{3^k}{k!}\right)$

(j) $\left(0,0,\dfrac{1}{2!},\dfrac{1}{3!},\dfrac{1}{4!},\cdots\right)$ (k) $\left(1,1,\dfrac{1}{2!},\dfrac{1}{3!},\dfrac{1}{4!},\cdots\right)$ (l) $\left(\dfrac{1}{3!},\dfrac{1}{4!},\dfrac{1}{5!},\cdots\right)$

(m) $\left(3,-\dfrac{3}{2},\dfrac{3}{3},-\dfrac{3}{4},\cdots\right)$ (n) $(1,0,1,0,1,0,\ldots)$ (o) $(0,1,0,3,0,5,\ldots)$

(p) $\left(2,0,-\dfrac{2}{3!},0,\dfrac{2}{5!},\cdots\right)$ (q) $\left(1,-1,\dfrac{1}{2!},-\dfrac{1}{3!},\dfrac{1}{4!},\cdots\right)$ (r) $\left(0,-1,\dfrac{1}{2},-\dfrac{1}{3},\dfrac{1}{4},\cdots\right)$

4. Find the sequence whose ordinary generating function is given as follows:

(a) $(x+5)^2$ (b) $(1+x)^4$ (c) $\dfrac{x^3}{1-x}$

(d) $\dfrac{1}{1-3x}$ (e) $\dfrac{1}{1+8x}$ (f) e^{6x}

(g) $1+\dfrac{1}{1-x}$ (h) $5+e^{2x}$ (i) $x\sin x$

(j) $x^3+x^4+e^{2x}$ (k) $\dfrac{1}{1-x^2}$ (l) $2x+e^{-x}$

(m) e^{-2x} (n) $\sin 3x$ (o) $x^2\ln(1+2x)+e^x$

(p) $\dfrac{1}{1+x^2}$ (q) $\cos 3x$ (r) $\dfrac{1}{(1+x)^2}$

5. Suppose that the ordinary generating function for the sequence (a_k) is given as follows. In each case, find a_3.

(a) $(x-7)^3$ (b) $\dfrac{14}{1-x}$ (c) $\ln(1-2x)$ (d) e^{5x}

6. In each case of Exercise 5, find a_4.
7. Professor Jones wants to teach Calculus I or Linear Algebra, Professor Smith wants to teach Linear Algebra or Combinatorics, and Professor Green wants to teach Calculus I or Combinatorics. Each professor can be assigned to teach at most one course, with no more than one professor per course, and a professor only gets a course that he or she wants to teach. Set up a generating function and use it to answer the following questions.
   (a) In how many ways can we assign one professor to a course?
   (b) In how many ways can we assign two professors to courses?
   (c) In how many ways can we assign three professors to courses?
8. Suppose that worker a is suitable for jobs 3, 4, 5, worker b is suitable for jobs 2, 3, and worker c is suitable for jobs 1, 5. Also, each worker can be assigned to at most one job, no more than one worker per job, and a worker only gets a job to which he or she is suited. Set up a generating function and use it to answer the following questions.
   (a) In how many ways can we assign one worker to a job?
   (b) In how many ways can we assign two workers to jobs?
   (c) In how many ways can we assign three workers to jobs?
9. Suppose that T_n is the number of rooted (unlabeled) trees of n vertices. The ordinary generating function $T(x) = \sum T_n x^n$ is given by
$$T(x) = x + x^2 + 2x^3 + 4x^4 + 9x^5 + 20x^6 + \cdots.$$
(Riordan [1980] computes T_n for $n \leq 26$. More values of T_n are now known.) Verify the coefficients of x, x^2, x^3, x^4, x^5, and x^6.
10. Let $M(n, a)$ be the number of labeled digraphs with n vertices and a arcs and let $M(n)$ be the number of labeled digraphs with n vertices (see Section 3.1.3). If n is fixed, let $b_k = M(n, k)$ and let
$$D_n(x) = \sum_{k=0}^{n(n-1)} b_k x^k.$$
   (a) Find a simple, closed-form expression for $D_n(x)$.
   (b) Use this expression to derive a formula for $M(n)$.
11. Suppose that $c_k = R(n, k)$ is the number of labeled graphs with a certain property P and having n vertices and k edges, and $R(n)$ is the number of labeled graphs with property P and n vertices. Suppose that
$$G_n(x) = \sum_{k=0}^{\infty} c_k x^k$$
is the ordinary generating function and we know that $G_n(x) = (1 + x + x^2)^n$. Find $R(n)$.

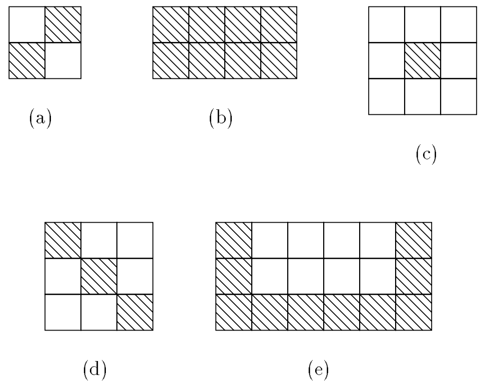


Figure 5.4: Boards for Exercise 13, Section 5.1.

1	2	3	4	5
2	3	1	5	4

Figure 5.5: A 2×5 Latin rectangle.

12. Suppose that $d_k = S(n, k)$ is the number of labeled digraphs with a certain property Q and having n vertices and k arcs, and $S(n)$ is the number of labeled digraphs with property Q and n vertices and at least two arcs. Let the ordinary generating function be given by

$$H_n(x) = \sum_{k=0}^{\infty} d_k x^k.$$

Suppose we know that $H_n(x) = (1 + x^2)^{n+5}$. Find $S(n)$.

13. Compute the rook polynomial for each of the boards of Figure 5.4.

14. Compute the rook polynomial for the $n \times n$ chess board with all squares darkened if n is

(a) 3 (b) 4 (c) 6 (d) 8.

15. A *Latin rectangle* is an $r \times s$ array with entries $1, 2, \ldots, n$, so that no two entries in any row or column are the same. A Latin square (Example 1.1) is a Latin rectangle with $r = s = n$. One way to build a Latin square is to build it up one row at a time, adding rows successively to Latin rectangles. In how many ways can we add a third row to the Latin rectangle of Figure 5.5? Set this up as a rook polynomial problem by observing what symbols can still be included in the jth column. You do not have to solve this problem.

16. *Use rook polynomials* to count the number of permutations of $1, 2, 3, 4$ in which 1 is not in the second position, 2 is not in the fourth position, and 3 is not in the first or fourth position.

17. Show that if board B' is obtained from board B by deleting rows or columns with no darkened squares, then $r_k(B) = r_k(B')$.

5.2 OPERATING ON GENERATING FUNCTIONS

A sequence defines a unique generating function and a generating function defines a unique sequence; we will be able to pass back and forth between sequences and generating functions. It will be useful to compile a "library" of basic generating functions and their corresponding sequences. Our list can start with the generating functions $1/(1-x)$, e^x, $\sin x$, $\ln(1+x)$, whose corresponding sequences can be derived from Equations (5.2)–(5.5). By operating on these generating functions as in Section 5.1.1, namely by adding, multiplying, dividing, composing, differentiating, or integrating, we can add to our basic list. In this section we do so.

In this section we observe how the various operations on generating functions relate to operations on the corresponding sequences. We start with some simple examples. Suppose that (a_k) is a sequence with ordinary generating function $A(x) = \sum_{k=0}^{\infty} a_k x^k$. Then multiplying $A(x)$ by x corresponds to shifting the sequence one place to the right and starting with 0. For $xA(x) = \sum_{k=0}^{\infty} a_k x^{k+1}$ is the ordinary generating function for the sequence $(0, a_0, a_1, a_2, \ldots)$. Similarly, multiplying $A(x)$ by $1/x$ and subtracting a_0/x corresponds to shifting the sequence one place to the left and deleting the first term, for

$$\frac{1}{x}A(x) - \frac{a_0}{x} = \sum_{k=0}^{\infty} a_k x^{k-1} - \frac{a_0}{x} = \sum_{k=1}^{\infty} a_k x^{k-1} = \sum_{k=0}^{\infty} a_{k+1} x^k.$$

To illustrate these two results, note that since $A(x) = e^x$ is the ordinary generating function for the sequence

$$\left(1, 1, \frac{1}{2!}, \frac{1}{3!}, \cdots\right),$$

xe^x is the ordinary generating function for the sequence

$$\left(0, 1, 1, \frac{1}{2!}, \frac{1}{3!}, \cdots\right),$$

and $(1/x)e^x - 1/x$ is the ordinary generating function for the sequence

$$\left(1, \frac{1}{2!}, \frac{1}{3!}, \cdots\right).$$

Similarly, by Equation (5.6), differentiating $A(x)$ with respect to x corresponds to multiplying the kth term of (a_k) by k and shifting by one place to the left. Thus, we saw in Section 5.1.1 that since

$$\frac{1}{(1-x)^2} = \frac{d}{dx}\left[\frac{1}{1-x}\right],$$

$1/(1-x)^2$ is the ordinary generating function for the sequence $(1, 2, 3, \ldots)$.

Suppose that (a_k) and (b_k) are sequences with ordinary generating functions $A(x) = \sum_{k=0}^{\infty} a_k x^k$ and $B(x) = \sum_{k=0}^{\infty} b_k x^k$, respectively. Since two power series can be added term by term, we see that $C(x) = A(x) + B(x)$ is the ordinary generating function for the sequence (c_k) whose kth term is $c_k = a_k + b_k$. This sequence (c_k) is called the *sum* of (a_k) and (b_k) and is denoted $(a_k) + (b_k)$. Thus,

$$\frac{1}{1-x} + e^x$$

is the ordinary generating function for

$$(1, 1, 1, \ldots) + \left(1, 1, \frac{1}{2!}, \frac{1}{3!}, \cdots\right) = \left(2, 2, 1 + \frac{1}{2!}, 1 + \frac{1}{3!}, \cdots\right).$$

From the point of view of combinatorics, the most interesting case arises from multiplying two generating functions. Suppose that

$$C(x) = A(x)B(x), \tag{5.11}$$

where $A(x), B(x)$, and $C(x)$ are the ordinary generating functions for the sequences $(a_k), (b_k)$, and (c_k), respectively. Does it follow that $c_k = a_k b_k$ for all k? Let $A(x) = 1 + x$ and $B(x) = 1 + x$. Then $C(x) = A(x)B(x)$ is given by $1 + 2x + x^2$. Now $(c_k) = (1, 2, 1, 0, 0, \ldots)$ and $(a_k) = (b_k) = (1, 1, 0, 0, \ldots)$, so $c_0 = a_0 b_0$ but $c_1 \neq a_1 b_1$. Thus, $c_k = a_k b_k$ does not follow from (5.11). What we can observe is that if we multiply $A(x)$ by $B(x)$, we obtain a term $c_k x^k$ by combining terms $a_j x^j$ from $A(x)$ with terms $b_{k-j} x^{k-j}$ from $B(x)$. Thus, for all k,

$$c_k = a_0 b_k + a_1 b_{k-1} + \cdots + a_{k-1} b_1 + a_k b_0. \tag{5.12}$$

This is easy to check in the case where both $A(x)$ and $B(x)$ are $1 + x$. Note that if $k = 0$, (5.12) says that $c_0 = a_0 b_0$. Note also that (5.12) for all k implies (5.11). If (5.12) holds for all k, we say the sequence (c_k) is the *convolution* of the two sequences (a_k) and (b_k), and we write $(c_k) = (a_k) * (b_k)$. Our results are summarized as follows.

Theorem 5.1 Suppose that $A(x), B(x)$, and $C(x)$ are, respectively, the ordinary generating functions for the sequences $(a_k), (b_k)$, and (c_k). Then

(a) $C(x) = A(x) + B(x)$ if and only if $(c_k) = (a_k) + (b_k)$.

(b) $C(x) = A(x)B(x)$ if and only if $(c_k) = (a_k) * (b_k)$.

Example 5.11 Suppose that $b_k = 1$, for all k. Then (5.12) becomes

$$c_k = a_0 + a_1 + \cdots + a_k.$$

The generating function $B(x)$ is given by $B(x) = 1/(1-x) = (1-x)^{-1}$. Hence, by Theorem 5.1,

$$C(x) = A(x)(1-x)^{-1}.$$

This is the generating function for the sum of the first k terms of a series. For instance, suppose that (a_k) is the sequence $(0,1,1,0,0,\ldots)$. Then $A(x) = x + x^2$ and

$$\begin{aligned} C(x) &= (x+x^2)[1+x+x^2+\cdots] \\ &= x + 2x^2 + 2x^3 + 2x^4 + \cdots. \end{aligned}$$

We conclude that

$$(x+x^2)(1-x)^{-1}$$

is the ordinary generating function for the sequence (c_k) given by $(0,1,2,2,\ldots)$. This can be checked by noting that

$$a_0 = 0,\ a_0 + a_1 = 1,\ a_0 + a_1 + a_2 = 2,\ a_0 + a_1 + a_2 + a_3 = 2,\ \ldots. \qquad \blacksquare$$

Example 5.12 If $A(x)$ is the generating function for the sequence (a_k), then $A^2(x)$ is the generating function for the sequence (c_k) where

$$c_k = a_0 a_k + a_1 a_{k-1} + \cdots + a_{k-1} a_1 + a_k a_0.$$

This result will also be useful in the enumeration of chemical isomers by counting trees in Section 6.4. In particular, if $a_k = 1$ for all k, then $A(x) = (1-x)^{-1}$. It follows that

$$C(x) = A^2(x) = (1-x)^{-2}$$

is the generating function for (c_k) where $c_k = k+1$. We have obtained this result before by differentiating $(1-x)^{-1}$. $\blacksquare$

Example 5.13 Suppose that

$$G(x) = \frac{1+x+x^2+x^3}{1-x}$$

is the ordinary generating function for a sequence (a_k). Can we find a_k? We can write

$$G(x) = (1+x+x^2+x^3)(1-x)^{-1}.$$

Now $1+x+x^2+x^3$ is the ordinary generating function for the sequence

$$(b_k) = (1,1,1,1,0,0,\ldots)$$

and $(1-x)^{-1}$ is the ordinary generating function for the sequence

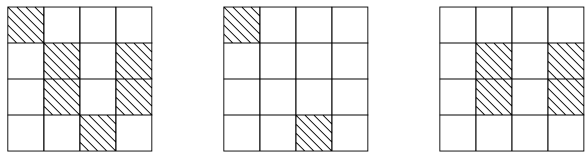
$$(c_k) = (1,1,1,\ldots).$$

Thus, $G(x)$ is the ordinary generating function for the convolution of these two sequences, that is,

$$a_k = b_0 c_k + b_1 c_{k-1} + \cdots + b_k c_0 = b_0 + b_1 + \cdots + b_k.$$

It is easy to show from this that

$$(a_k) = (1,2,3,4,4,\ldots). \qquad \blacksquare$$



B $\qquad$ B_I $\qquad$ B_J

Figure 5.6: B_I and B_J decompose B.

Example 5.14 A Reduction for Rook Polynomials In computing rook polynomials, it is frequently useful to reduce a complicated computation to a number of smaller ones, a trick we have previously encountered in connection with chromatic polynomials in Section 3.4. Exercise 17 of Section 5.1 shows one such reduction. Here we present another. Suppose that I is a set of darkened squares in a board B and B_I is the board obtained from B by lightening the darkened squares not in I. Suppose that the darkened squares of B are partitioned into two sets I and J so that no square in I lies in the same row or column as any square of J. In this case, we say that B_I and B_J *decompose* B. Figure 5.6 illustrates this situation.

If B_I and B_J decompose B, then since all acceptable squares fall in B_I or B_J, and no rook of I can take a rook of J, or vice versa, to place k nontaking rooks on B, we place p nontaking rooks on B_I, and then $k-p$ nontaking rooks on B_J, for some p. Thus,

$$\begin{aligned} r_k(B) &= r_0(B_I)r_k(B_J) + r_1(B_I)r_{k-1}(B_J) + \\ &\quad \cdots + r_p(B_I)r_{k-p}(B_J) + \cdots + r_k(B_I)r_0(B_J). \end{aligned}$$

This implies that the sequence $(r_k(B))$ is simply the convolution of the two sequences $(r_k(B_I))$ and $(r_k(B_J))$. Hence,

$$R(x, B) = R(x, B_I)R(x, B_J). \qquad \blacksquare$$

EXERCISES FOR SECTION 5.2

1. In each of the following, the function is the ordinary generating function for a sequence (a_k). Find this sequence.

   (a) $x \ln(1+x)$ (b) $\dfrac{1}{x}\sin x$ (c) $x^4 \ln(1+x)$

   (d) $\dfrac{1}{x^4}\sin x$ (e) $\dfrac{5}{1-x} + x^3 + 3x + 4$ (f) $\dfrac{x}{1-7x} + \dfrac{4}{1-x}$

   (g) $\dfrac{1}{1-x^2} + 6x + 5$ (h) $\dfrac{1}{2}\left(e^x - e^{-x}\right)$

2. For each of the following functions $A(x)$, suppose that $B(x) = xA'(x)$. Find the sequence for which $B(x)$ is the ordinary generating function.

(a) $\dfrac{1}{1-x}$ (b) e^{3x} (c) $\cos x$ (d) $\ln(1+x)$

3. In each of the following, find a formula for the convolution of the two sequences.

   (a) $(1, 1, 1, \ldots)$ and $(1, 1, 1, \ldots)$

   (b) $(1, 1, 1, \ldots)$ and $(0, 1, 2, 3, \ldots)$

   (c) $(1, 1, 0, 0, \ldots)$ and $(0, 1, 2, 3, \ldots)$

   (d) $(1, 2, 4, 0, 0, \ldots)$ and $(1, 2, 3, 4, 0, \ldots)$

   (e) $(1, 0, 1, 0, 0, 0, \ldots)$ and $(2, 4, 6, 8, \ldots)$

   (f) $(0, 0, 0, 1, 0, 0, \ldots)$ and $(8, 9, 10, 11, \ldots)$

4. In each of the following, the function is the ordinary generating function for a sequence (a_k). Find this sequence.

   (a) $\left(\dfrac{5}{1-x}\right)\left(\dfrac{3}{1-x}\right)$ (b) $\dfrac{1}{1-x}\ln(1+2x)$ (c) $\dfrac{x^3+x^5}{1-x}$

   (d) $\dfrac{x^2-3x}{1-x}+x$ (e) $(1+x)^q$, where q is a positive integer (f) $xe^{3x}+(1+x)^2$

5. If $G(x) = [1/(1-x)]^2$ is the ordinary generating function for the sequence (a_k), find a_4.

6. If $A(x) = (1-5x^2)(1+2x+3x^2+4x^3+\cdots)$ is the ordinary generating function for the sequence (a_k), find a_{11}.

7. Suppose that $A(x)$ is the ordinary generating function for the sequence (1, 3, 9, 27, 81, ...) and $B(x)$ is the ordinary generating function for the sequence (b_k). Find (b_k) if $B(x)$ equals

   (a) $A(x)+x$ (b) $A(x)+\dfrac{1}{1-x}$ (c) $2A(x)$

8. In each of the following cases, suppose that $B(x)$ is the ordinary generating function for (b_k) and $A(x)$ is the ordinary generating function for (a_k). Find an expression for $B(x)$ in terms of $A(x)$.

   (a) $b_k = \begin{cases} a_k & \text{if } k \neq 3 \\ 11 & \text{if } k = 3 \end{cases}$ (b) $b_k = \begin{cases} a_k & \text{if } k \neq 0, 4 \\ 2 & \text{if } k = 0 \\ 1 & \text{if } k = 4 \end{cases}$

9. Suppose that

$$a_k = \begin{cases} b_k & \text{if } k \neq 0, 2 \\ 4 & \text{if } k = 0 \\ 1 & \text{if } k = 2. \end{cases}$$

   Find an expression for $A(x)$, the ordinary generating function for the sequence (a_k), in terms of $B(x)$, the ordinary generating function for the sequence (b_k), if $b_0 = 2$ and $b_2 = 0$.

10. Find a simple, closed-form expression for the ordinary generating function of the following sequences (a_k).

(a) $a_k = k + 3$ (b) $a_k = 8k$ (c) $a_k = 3k + 4$

11. Make use of derivatives to find the ordinary generating function for the following sequences (b_k).

(a) $b_k = k^2$ (b) $b_k = k(k+1)$ (c) $b_k = (k+1)\frac{1}{k!}$

12. Suppose that $A(x)$ is the ordinary generating function for the sequence (a_k) and $b_k = a_{k+1}$. Find the ordinary generating function for the sequence (b_k).

13. Suppose that

$$a_k = \begin{cases} \sum_{i=0}^{k-2} b_i b_{k-2-i} & \text{if } k \geq 2 \\ 0 & \text{if } k = 0 \text{ or } k = 1. \end{cases}$$

Suppose that $A(x)$ is the ordinary generating function for (a_k) and $B(x)$ is the ordinary generating function for (b_k). Find an expression for $A(x)$ in terms of $B(x)$.

14. Suppose that $A(x)$ is the ordinary generating function for the sequence (a_k) and the sequence (b_k) is defined by taking

$$b_k = \begin{cases} 0 & \text{if } k < i \\ a_{k-i} & \text{if } k \geq i. \end{cases}$$

Find the ordinary generating function for the sequence (b_k) in terms of $A(x)$.

15. Find an ordinary generating function for the sequence whose kth term is

$$a_k = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{k!}.$$

16. (a) Use the reduction result of Example 5.14 to compute the rook polynomial of the board (d) of Figure 5.4.

(b) Generalize to an $n \times n$ board with all squares on the diagonal darkened.

17. Use the result of Exercise 17, Section 5.1, to find the rook polynomials of B_I and B_J of Figure 5.6. Then use the reduction result of Example 5.14 to compute the rook polynomial of the board B of Figure 5.6.

5.3 APPLICATIONS TO COUNTING

5.3.1 Sampling Problems

Generating functions will help us in counting. To illustrate how this will work, we first consider sampling problems where the objects being sampled are of different types and objects of the same type are indistinguishable. In the language of Section 2.9, we consider sampling without replacement. For instance, suppose that there are three objects, a, b, and c, and each one can be chosen or not. How many selections are possible? Let a_k be the number of ways to select k objects. Let $G(x)$ be the generating function $\sum a_k x^k$. Now it is easy to see that $a_k = \binom{3}{k}$, and hence

$$G(x) = \binom{3}{0}x^0 + \binom{3}{1}x^1 + \binom{3}{2}x^2 + \binom{3}{3}x^3. \tag{5.13}$$

Let us calculate $G(x)$ another way. We can either pick no a's *or* one a, *and* no b's *or* one b, *and* no c's *or* one c. Let us consider the schematic product

$$[(ax)^0 + (ax)^1][(bx)^0 + (bx)^1][(cx)^0 + (cx)^1], \tag{5.14}$$

where addition and multiplication correspond to the words "or" and "and," respectively, which are italicized in the preceding sentence. (Recall the sum rule and the product rule of Chapter 2.) The expression (5.14) becomes

$$(1 + ax)(1 + bx)(1 + cx),$$

which equals

$$1 + (a + b + c)x + (ab + ac + bc)x^2 + abcx^3. \tag{5.15}$$

Notice that the coefficient of x lists the ways to get one object: It is a, or b, or c. The coefficient of x^2 lists the ways to get two objects: It is a and b, or a and c, or b and c. The coefficient of x^3 lists the ways to get three objects, and the coefficient of x^0 (namely, 1) lists the number of ways to get no objects. If we set $a = b = c = 1$, the coefficient of x^k will count the number of ways to get k objects, that is, a_k. Hence, setting $a = b = c = 1$ in (5.15) gives rise to the generating function

$$G(x) = 1 + 3x + 3x^2 + x^3,$$

which is what we calculated in (5.13).

The same technique works in problems where it is not immediately clear what the coefficients in $G(x)$ are. Then we can calculate $G(x)$ by means of this technique and calculate the appropriate coefficients from $G(x)$.

Example 5.15 Suppose that we have three types of objects, a's, b's, and c's. Suppose that we can pick either 0, 1, or 2 a's, then 0 or 1 b, and finally 0 or 1 c. How many ways are there to pick k objects? The answer is not $\binom{4}{k}$. For example, 2 a's and 1 b is not considered the same as 1 a, 1 b, and 1 c. However, picking the first a and also b is considered the same as picking the second a and also b: The a's are indistinguishable. We want the number of distinguishable ways to pick k objects. Suppose that b_k is the desired number of ways. We shall try to calculate the ordinary generating function $G(x) = \sum b_k x^k$. The correct expression to consider here is

$$[(ax)^0 + (ax)^1 + (ax)^2][(bx)^0 + (bx)^1][(cx)^0 + (cx)^1], \tag{5.16}$$

since we can pick either 0, or 1, or 2 a's, and 0 or 1 b, and 0 or 1 c. The expression (5.16) reduces to

$$(1 + ax + a^2x^2)(1 + bx)(1 + cx),$$

which equals

$$1 + (a + b + c)x + (ab + bc + ac + a^2)x^2 + (abc + a^2b + a^2c)x^3 + a^2bcx^4. \tag{5.17}$$

As in Example 5.14, the coefficient of x^3 gives the ways of obtaining three objects: a, b, and c; or 2 a's and b; or 2 a's and c. The same thing holds for the other coefficients. Again, taking $a = b = c = 1$ in (5.17) gives the generating function

$$G(x) = 1 + 3x + 4x^2 + 3x^3 + x^4.$$

The coefficient of x^k is b_k. For example, $b_2 = 4$. (The reader should check why this is so.) ■

In general, suppose that we have p types of objects, with n_1 indistinguishable objects of type 1, n_2 of type 2, ..., n_p of type p. Let c_k be the number of distinguishable ways of picking k objects if we can pick any number of objects of each type. The ordinary generating function is given by $G(x) = \sum c_k x^k$. To calculate this, we consider the product

$$\left[(a_1x)^0 + (a_1x)^1 + \cdots + (a_1x)^{n_1}\right] \times \left[(a_2x)^0 + (a_2x)^1 + \cdots + (a_2x)^{n_2}\right] \times \cdots \\ \times \left[(a_px)^0 + (a_px)^1 + \cdots + (a_px)^{n_p}\right].$$

Setting $a_1 = a_2 = \cdots = a_p = 1$, we obtain

$$G(x) = (1 + x + x^2 + \cdots + x^{n_1})(1 + x + x^2 + \cdots + x^{n_2}) \cdots (1 + x + x^2 + \cdots + x^{n_p}).$$

The number c_k is given by the coefficient of x^k in $G(x)$. Thus, we have the following theorem.

Theorem 5.2 Suppose that we have p types of objects, with n_i indistinguishable objects of type $i, i = 1, 2, \ldots, p$. The number of distinguishable ways of picking k objects if we can pick any number of objects of each type is given by the coefficient of x^k in the ordinary generating function

$$G(x) = (1 + x + x^2 + \cdots + x^{n_1})(1 + x + x^2 + \cdots + x^{n_2}) \cdots (1 + x + x^2 + \cdots + x^{n_p}).$$

Example 5.16 Indistinguishable Men and Women Suppose that we have m (indistinguishable) men and n (indistinguishable) women. If we can choose any number of men and any number of women, Theorem 5.2 implies that the number of ways we can choose k people is given by the coefficient of x^k in

$$G(x) = (1 + x + \cdots + x^m)(1 + x + \cdots + x^n). \tag{5.18}$$

Note: This coefficient is not

$$\binom{m+n}{k}$$

because, for example, having 3 men and $k - 3$ women is the same no matter which men and women you pick. Now $1 + x + \cdots + x^m$ is the generating function for the sequence

$$(a_k) = (1, 1, \ldots, 1, 0, 0, \ldots)$$

and $1 + x + \cdots + x^n$ is the generating function for the sequence

$$(b_k) = (1, 1, \ldots, 1, 0, 0, \ldots),$$

where a_k is 1 for $k = 0, 1, \ldots, m$ and b_k is 1 for $k = 0, 1, \ldots, n$. It follows from (5.18) that $G(x)$ is the generating function for the convolution of the two sequences (a_k) and (b_k). We leave it to the reader (Exercise 11) to compute this convolution in general. For example, if $m = 8$ and $n = 7$, the number of ways we can choose $k = 9$ people is given by

$$a_0b_9 + a_1b_8 + \cdots + a_9b_0 = 0 + 0 + 1 + \cdots + 1 + 1 + 0 = 7.$$

As a check on the answer, we note that the seven ways are the following: 2 men and 7 women, 3 men and 6 women, ..., 8 men and 1 woman. ∎

Example 5.17 A Sampling Survey In doing a sampling survey, suppose that we have divided the possible men to be interviewed into various categories, such as teachers, doctors, lawyers, and so on, and similarly for the women. Suppose that in our group we have two men from each category and one woman from each category, and suppose that there are q categories. How many distinguishable ways are there of picking a sample of k people? We now want to distinguish people of the same gender if and only if they belong to different categories. The generating function for the number of ways to choose k people is given by

$$\begin{aligned} G(x) &= \underbrace{(1 + x + x^2)(1 + x + x^2)\cdots(1 + x + x^2)}_{q \text{ terms}}\underbrace{(1 + x)(1 + x)\cdots(1 + x)}_{q \text{ terms}} \\ &= (1 + x + x^2)^q(1 + x)^q. \end{aligned}$$

The individual terms of $G(x)$ come from the fact that in each of the q categories we choose either 0, 1, or 2 men to be sampled, thus giving a $(1 + x + x^2)$ term for each category; and in addition, we choose either 0 or 1 woman, so we have a $(1+x)$ term for each of the q categories. The number of ways to select k people is the coefficient of x^k in $G(x)$. For instance, if $q = 2$, then

$$G(x) = x^6 + 4x^5 + 8x^4 + 10x^3 + 8x^2 + 4x + 1.$$

For example, there are 10 ways to pick 3 people. The reader might wish to identify those 10 ways. In general, if there are m categories of men and n categories of women, and there are p_i men in category $i, i = 1, 2, \ldots, m$, and q_j women in category $j, j = 1, 2, \ldots, n$, the reader might wish to express the ordinary generating function for the number of ways to select k people. ∎

Example 5.18 Another Survey Suppose that our survey team is considering three households. The first household has two people living in it (let us call them "a's"). The second household has one person living in it (let us call that person a "b"). The third household has one person living in it (let us call that person a "c").

In how many ways can k people be selected if we either choose none of the members of a given household or all of the members of that household? Let us again think of the product and sum rule: We can either pick 0 or 2 a's, and either 0 or 1 b, and either 0 or 1 c. Hence, we consider the expression

$$[(ax)^0 + (ax)^2][(bx)^0 + (bx)^1][(cx)^0 + (cx)^1].$$

This becomes

$$(1 + a^2x^2)(1 + bx)(1 + cx) = 1 + (b + c)x + (bc + a^2)x^2 + (a^2b + a^2c)x^3 + a^2bcx^4.$$

The ways of choosing two people are given by the coefficient of x^2: We can choose b and c or we can choose two a's. Setting $a = b = c = 1$, we obtain the generating function

$$G(x) = 1 + 2x + 2x^2 + 2x^3 + x^4.$$

The number of ways of choosing k people is given by the coefficient of x^k. ■

Example 5.19 Suppose that we have p different kinds of objects, each in (for all practical purposes) infinite supply. How many ways are there of picking a sample of k objects? The answer is given by the coefficient of x^k in the ordinary generating function

$$\begin{aligned} G(x) &= \underbrace{(1 + x + x^2 + \cdots)(1 + x + x^2 + \cdots)\cdots(1 + x + x^2 + \cdots)}_{p \text{ terms}} \qquad (5.19) \\ &= (1 + x + x^2 + \cdots)^p \\ &= (1 - x)^{-p} \qquad [\text{by } (5.2)]. \end{aligned}$$

(An alternative approach, leading ultimately to the same result, is to apply Theorem 5.2 with each $n_i = k$.) We shall want to develop ways of finding the coefficient of x^k given an expression for $G(x)$ such as (5.19). To do so, we introduce the Binomial Theorem in the next section. ■

Example 5.20 Integer Solutions of Equations How many integer solutions are there to the equation

$$b_1 + b_2 + b_3 = 14,$$

if $b_i \geq 0$ for $i = 1, 2, 3$? Since each b_i can take on the value 0 or 1 or 2 or ..., the answer is given by the coefficient of x^{14} in the ordinary generating function

$$\begin{aligned} G(x) &= \left(1 + x + x^2 + \cdots\right)\left(1 + x + x^2 + \cdots\right)\left(1 + x + x^2 + \cdots\right) \\ &= \left(1 + x + x^2 + \cdots\right)^3 \\ &= (1 - x)^{-3}, \end{aligned}$$

by Example 5.19. ■

Example 5.21 Coding Theory (Example 2.27 Revisited) In Example 2.27 we considered the problem of checking codewords after transmission. Here we consider the common case of codewords as bit strings. Suppose that we check for three kinds of errors: addition of a digit (0 or 1), deletion of a digit (0 or 1), and reversal of a digit (0 to 1 or 1 to 0). In how many ways can we find 30 errors? To answer this question, we can assume that two errors of the same type are indistinguishable. Moreover, for all practical purposes, each type of error is in infinite supply. Thus, we seek the coefficient of x^{30} in the ordinary generating function

$$G(x) = (1 + x + x^2 + \cdots)(1 + x + x^2 + \cdots)(1 + x + x^2 + \cdots) = (1 - x)^{-3}.$$

This is a special case of (5.19).

Suppose next that we do not distinguish among the three types of errors, but we do keep a record of whether an error occurred in the first codeword sent, the second codeword sent, and so on. In how many different ways can we find 30 errors if there are 100 codewords? This is the question we addressed in Example 2.27. To answer this question, we note that there are 100 types of errors, one type for each codeword. We can either assume that the number of possible errors per codeword is, for all practical purposes infinite, or that it is bounded by 30, or that it is bounded by the number of digits per codeword. In the former case, we consider the ordinary generating function

$$G(x) = (1 + x + x^2 + \cdots)^{100} = (1 - x)^{-100}$$

and look for the coefficient of x^{30}. In the latter two cases, we simply end the terms in $G(x)$ at an appropriate power. The coefficient of x^{30} will, of course, be the same regardless, provided that at least 30 digits appear per codeword. ■

Example 5.22 Sicherman Dice Suppose that a standard pair of dice are rolled. What are the probabilities for the various outcomes for this roll? Since each die can take on the value 1, 2, 3, 4, 5, or 6, we consider the ordinary generating function

$$\begin{aligned} p(x) &= \left(x + x^2 + x^3 + x^4 + x^5 + x^6\right)\left(x + x^2 + x^3 + x^4 + x^5 + x^6\right) \\ &= \left(x^2 + 2x^3 + 3x^4 + 4x^5 + 5x^6 + 6x^7 + 5x^8 + 4x^9 + 3x^{10} + 2x^{11} + x^{12}\right). \end{aligned}$$

Note that the coefficient of x^j is the number of ways to obtain a roll of j. Also, the sum of the coefficients, 36, gives the total number of different rolls possible with two dice. So, for example, the probability of rolling a 5 is $4/36$.

What about other dice? Does there exist a different pair of six-sided dice which yield the same outcome probabilities as standard dice? We will only consider dice with positive, integer labels. Let $a_1, a_2, a_3, a_4, a_5, a_6$ be the values on one die and $b_1, b_2, b_3, b_4, b_5, b_6$ be the values on the other die. If another pair exists, it must be the case that

$$\begin{aligned} &\left(x + x^2 + x^3 + x^4 + x^5 + x^6\right)\left(x + x^2 + x^3 + x^4 + x^5 + x^6\right) = \\ &\quad \left(x^{a_1} + x^{a_2} + x^{a_3} + x^{a_4} + x^{a_5} + x^{a_6}\right)\left(x^{b_1} + x^{b_2} + x^{b_3} + x^{b_4} + x^{b_5} + x^{b_6}\right). \quad (5.20) \end{aligned}$$

The left-hand side of (5.20), $p(x)$, factors into

$$x^2(1+x)^2(1+x+x^2)^2(1-x+x^2)^2.$$

Since polynomials with integer coefficients always factor uniquely (a standard fact from any abstract algebra course), the right-hand side of (5.20) must also factor this way. Therefore,

$$f(x) = (x^{a_1} + x^{a_2} + x^{a_3} + x^{a_4} + x^{a_5} + x^{a_6}) = x^{c_1}(1+x)^{c_2}(1+x+x^2)^{c_3}(1-x+x^2)^{c_4}$$

and

$$g(x) = \left(x^{b_1} + x^{b_2} + x^{b_3} + x^{b_4} + x^{b_5} + x^{b_6}\right) = x^{d_1}(1+x)^{d_2}(1+x+x^2)^{d_3}(1-x+x^2)^{d_4},$$

where $c_i + d_i = 2$, for $i = 1, 2, 3, 4$.

To find c_i and d_i (and hence a_i and b_i), first consider

$$f(0) = (0^{a_1} + 0^{a_2} + 0^{a_3} + 0^{a_4} + 0^{a_5} + 0^{a_6}).$$

This equals 0, as does $g(0)$, so the x term must be present in the factorizations of both $f(x)$ and $g(x)$. Therefore, c_1 and d_1 both must equal 1. Next, consider $f(1) = (1^{a_1} + 1^{a_2} + 1^{a_3} + 1^{a_4} + 1^{a_5} + 1^{a_6}) = 6$ or $1^{c_1}2^{c_2}3^{c_3}1^{c_4}$. So, $c_2 = c_3 = 1$. A similar analysis and result, $d_2 = d_3 = 1$, is also true for $g(x)$. Finally, we consider c_4 and d_4. Either $c_4 = d_4 = 1$, which yields a standard pair of dice, or $c_4 = 0, d_4 = 2$ (or vice versa). In the latter case, after multiplying polynomials, we see that

$$f(x) = x + x^2 + x^2 + x^3 + x^3 + x^4, \quad g(x) = x + x^3 + x^4 + x^5 + x^6 + x^8$$

and

$$f(x)g(x) = \left(x^2 + 2x^3 + 3x^4 + 4x^5 + 5x^6 + 6x^7 + 5x^8 + 4x^9 + 3x^{10} + 2x^{11} + x^{12}\right).$$

This pair of dice, one labeled 1, 2, 2, 3, 3, 4 and the other labeled 1, 3, 4, 5, 6, 8, are called *Sicherman dice*.[4] Sicherman dice are the *only* alternative pair of dice (with positive integer labels) that yield the same outcome probabilities as standard dice! ■

Example 5.23 Partitions of Integers Recall from Section 2.10.5 that a *partition* of a positive integer k is a collection of positive integers that sum to k. For instance, the integer 4 has the partitions $\{1,1,1,1\}$, $\{1,1,2\}$, $\{2,2\}$, $\{1,3\}$, and $\{4\}$. Suppose that $p(k)$ is the number of partitions of the integer k. Thus, $p(4) = 5$. Exercises 12–16 investigate partitions of integers, using the techniques of this section. The idea of doing so goes back to Euler in 1748. For a detailed discussion of partitions, see most number theory books: for instance, Bressoud and Wagon [2000] or Hardy and Wright [1980]. See also Berge [1971], Cohen [1978], or Tomescu [1985]. ■

[4] George Sicherman first considered and solved this alternative dice problem whose outcome probabilities are the same as standard dice. See Gardner [1978] for a discussion of this problem.

Example 5.24 A Crucial Observation Underlying the Binary Arithmetic of Computing Machines: Partitions into Distinct Integers Let $p^*(k)$ be the number of ways to partition the integer k into distinct integers. Thus, $p^*(7) = 5$, using the partitions $\{7\}, \{1,6\}, \{2,5\}, \{3,4\}$, and $\{1,2,4\}$. A crucial observation underlying the binary arithmetic that pervades computing machines is that every integer k can be partitioned into distinct integers that are powers of 2. For example, $7 = 2^0 + 2^1 + 2^2$ and $19 = 2^0 + 2^1 + 2^4$. We ask the reader to prove this in Exercise 16(c), using methods of this section. ■

5.3.2 A Comment on Occupancy Problems

In Section 2.10 we considered occupancy problems, problems of distributing balls to cells. The second part of Example 5.21 involves an occupancy problem: We have 30 balls (errors) and 100 cells (codewords). This is a special case of the occupancy problem where we have k indistinguishable balls and we wish to distribute them among p distinguishable cells. If we put no restriction on the number of balls in each cell, it is easy to generalize the reasoning in Example 5.21 and show that the number of ways to distribute the k balls into the p cells is given by the coefficient of x^k in the ordinary generating function (5.19). By way of contrast, if we allow no more than n_i balls in the ith cell, we see easily that the number of ways is given by the coefficient of x^k in the ordinary generating function of Theorem 5.2.

EXERCISES FOR SECTION 5.3

1. In each of the following, set up the appropriate generating function. Do not calculate an answer but indicate what you are looking for: for example, the coefficient of x^{10}.

   (a) A survey team wants to select at most 3 male students from Michigan, at most 3 female students from Brown, at most 2 male students from Stanford, and at most 2 female students from Rice. In how many ways can 5 students be chosen to interview if only Michigan and Stanford males and Brown and Rice females can be chosen, and 2 students of the same gender from the same school are indistinguishable?

   (b) In how many ways can 5 letters be picked from the letters a, b, c, d if b, c, and d can be picked at most once and a, if picked, must be picked 4 times?

   (c) In making up an exam, an instructor wants to use at least 3 easy problems, at least 3 problems of medium difficulty, and at least 2 hard problems. She has limited the choice to 7 easy problems, 6 problems of medium difficulty, and 4 hard problems. In how many ways can she pick 11 problems? (The order of the exam problems will be decided upon later.) (*Note:* Do not distinguish 2 easy problems from each other, or 2 problems of medium difficulty, or 2 hard problems.)

   (d) In how many ways can 8 binary digits be picked if each must be picked an even number of times?

(e) How many ways are there to choose 12 voters from a group of 6 Republicans, 6 Democrats, and 7 Independents, if we want at least 4 Independents and any two voters of the same political persuasion are indistinguishable?

(f) A Geiger counter records the impact of five different kinds of radioactive particles over a period of 5 minutes. How many ways are there to obtain a count of 20?

(g) In checking the work of a communication device, we look for 4 types of transmission errors. In how many ways can we find 40 errors?

(h) In part (g), suppose that we do not distinguish the types of errors, but we do keep a record of the day on which an error occurred. In how many different ways can we find 40 errors in 100 days?

(i) How many ways are there to distribute 12 indistinguishable balls into 8 distinguishable cells?

(j) Repeat part (i) if no cell can be empty.

(k) If 14 standard dice are rolled, how many ways are there for the total to equal 30?

(l) A survey team divides the possible people to interview into 6 groups depending on age, and independently into 5 groups depending on geographic location. In how many ways can 10 people be chosen to interview, if 2 people are distinguished only if they belong to different age groups, live in different geographic locations, or are of opposite gender?

(m) Find the number of ways to make change for a dollar using coins (pennies, nickels, dimes, and/or quarters).

(n) Find the number of solutions to the equation

$$x_1 + x_2 + x_3 = 14$$

in which each x_i is a nonnegative integer and $x_i \leq 7$.

(o) Find the number of solutions to the equation

$$x_1 + x_2 + x_3 = 20$$

in which $x_i \geq 0$ for all i, and x_1 odd, $2 \leq x_2 \leq 5$, and x_3 prime.

2. Suppose that we wish to build up an RNA chain which has length 5 and uses U, C, and A each at most once and G arbitrarily often. How many ways are there to choose the bases (not order them)? Answer this question by setting up a generating function and computing an appropriate coefficient.

3. A customer wants to buy six pieces of fruit, including at most two apples, at most two oranges, at most two pears, and at least one but at most two peaches. How many ways are there to buy six pieces of fruit if any two pieces of fruit of the same type, for example, any two peaches, are indistinguishable?

4. Suppose that there are p kinds of objects, with n_i indistinguishable objects of the ith kind. Suppose that we can pick all or none of each kind. Set up a generating function for computing the number of ways to choose k objects.

5. Consider the voting situation of Exercise 7, Section 2.15.

(a) If all representatives of a province vote alike, set up a generating function for calculating the number of ways to get k votes. (Getting a vote from a representative of province A is considered different from getting a vote from a representative of province B, and so on.)

(b) Repeat part (a) if representatives of a province do not necessarily vote alike.

6. Consider the following basic groups of foods from the Food Guide Pyramid: breads, fruits, vegetables, meat, milk, and fats. A dietician wants to choose a daily menu in a cafeteria by choosing 10 items (the limit of the serving space), with at least one item from each category.

   (a) How many ways are there of choosing the basic menu if items in the same group are treated as indistinguishable and it is assumed that there are, for all practical purposes, an arbitrarily large number of different foods in each group? Answer this by setting up a generating function. Do not do the computation.

   (b) How would the problem be treated if items in a group were treated as distinguishable and there are, say, 30 items in each group?

7. Suppose that there are p kinds of objects, each in infinite supply. Let a_k be the number of distinguishable ways of choosing k objects if only an even number (including 0) of each kind of object can be taken. Set up a generating function for a_k.

8. In a presidential primary involving all of the New England states on the same day, a presidential candidate would receive a number of electoral votes from each state proportional to the number of voters who voted for that candidate in each state. For example, a candidate who won 50 percent of the vote in Maine and no votes in the other states would receive two electoral votes (or two state votes) in the convention, since Maine has four electoral votes. Votes would be translated into integers by rounding—there are no fractional votes. Set up a generating function for computing the number of ways that a candidate could receive 25 electoral votes. (The electoral votes of the states in the region as of the 2000 census are as follows: Connecticut, 7; Maine, 4; Massachusetts, 12; New Hampshire, 4; Rhode Island, 4; Vermont, 3.)

9. In Example 5.17 we showed that if $q = 2$, there are 10 ways to pick 3 people. What are they?

10. (Gallian [2002]) Suppose that you have an 18-sided die which is labeled 1, 2, 2, 3, 3, 3, 4, 4, 4, 5, 5, 5, 6, 6, 6, 7, 7, 8.

    (a) Find the labels of a 2-sided die so that this die, when rolled with the 18-sided die, has the same outcome probabilities as a standard pair of 6-sided dice.

    (b) Extending part (a), find the labels of a 4-sided die so that this die, when rolled with the 18-sided die, has the same outcome probabilities as a standard pair of 6-sided dice.

11. Find a general formula for the convolution of the two sequences (a_k) and (b_k) found in Example 5.16.

12. The next five exercises investigate partitions of integers, as defined in Examples 5.23 and 5.24. Let $p(k)$ be the number of partitions of integer k and let

$$G(x) = \frac{1}{(1-x)(1-x^2)(1-x^3)(1-x^4)\cdots}.$$

Show that $G(x)$ is the ordinary generating function for the sequence $(p(k))$.

13. Recall that $p^*(k)$ is the number of ways to partition the integer k into distinct integers.
    (a) Find $p^*(8)$. (b) Find $p^*(11)$.
    (c) Find an ordinary generating function for $(p^*(k))$.
14. Let $p_o(k)$ be the number of ways to partition integer k into not necessarily distinct odd integers.
    (a) Find $p_o(7)$. (b) Find $p_o(8)$. (c) Find $p_o(11)$.
    (d) Find a generating function for $(p_o(k))$.
15. Let $p^*(k)$ and $p_o(k)$ be defined as in Exercises 13 and 14, respectively. Show that $p^*(k) = p_o(k)$.
16. (a) Show that for $|x| < 1$,
    $$(1-x)(1+x)(1+x^2)(1+x^4)(1+x^8)\cdots(1+x^{2^k})\cdots = 1.$$
    (b) Deduce that for $|x| < 1$,
    $$1 + x + x^2 + x^3 + \cdots = (1+x)(1+x^2)(1+x^4)(1+x^8)\cdots(1+x^{2^k})\cdots.$$
    (c) Conclude that any integer can be written uniquely in binary form, that is, as a sum $a_0 2^0 + a_1 2^1 + a_2 2^2 + \cdots$, where each a_i is 0 or 1. (This conclusion is a crucial one underlying the binary arithmetic that pervades computing machines.)
17. Find the number of integer solutions to the following equations/inequalities:
    (a) $b_1 + b_2 + b_3 + b_4 + b_5 + b_6 = 15$, with $0 \leq b_i \leq 3$ for all i.
    (b) $b_1 + b_2 + b_3 = 15$, with $0 \leq b_i$ for all i and b_1 odd, b_2 even, and b_3 prime.
    (c) $b_1 + b_2 + b_3 + b_4 = 20$, with $2 \leq b_1 \leq 4$ and $4 \leq b_i \leq 7$ for $2 \leq i \leq 4$.
    (d) $b_1 + b_2 + b_3 + b_4 \leq 10$, with $0 \leq b_i$ for all i. (*Hint*: Include a "slack" variable b_5 to create an equality.)
18. Solve the question of Example 5.20 using the method of occupancy problems from Section 2.10.
19. Using the techniques from Example 5.22, find the number of pairs of 4-sided dice which have the same outcome probabilities as a "standard" pair of 4-sided dice. ("Standard" 4-sided dice are labeled 1, 2, 3, 4.)

5.4 THE BINOMIAL THEOREM

In order to expand out the generating function of Equation (5.19), it will be useful to find the Maclaurin series for the function $f(x) = (1+x)^u$, where u is an arbitrary real number, positive or negative, and not necessarily an integer. We have

$$\begin{aligned} f'(x) &= u(1+x)^{u-1} \\ f''(x) &= u(u-1)(1+x)^{u-2} \\ &\vdots \\ f^{(r)}(x) &= u(u-1)\cdots(u-r+1)(1+x)^{u-r}. \end{aligned}$$

Thus, by Equation (5.1), we have the following theorem.

Theorem 5.3 (Binomial Theorem)

$$(1+x)^u = 1 + ux + \frac{u(u-1)}{2!}x^2 + \cdots + \frac{u(u-1)\cdots(u-r+1)}{r!}x^r + \cdots. \quad (5.21)$$

One can prove that the expansion (5.21) holds for $|x| < 1$. This expansion can be written succinctly by introducing the *generalized binomial coefficient*

$$\binom{u}{r} = \begin{cases} \dfrac{u(u-1)\cdots(u-r+1)}{r!} & \text{if } r > 0 \\ 1 & \text{if } r = 0, \end{cases}$$

which is defined for any real number u and nonnegative integer r. Then (5.21) can be rewritten as

$$(1+x)^u = \sum_{r=0}^{\infty} \binom{u}{r} x^r. \quad (5.22)$$

If u is a positive integer n, then $\binom{u}{r}$ is 0 for $r > u$, and 5.22 reduces to the binomial expansion (Theorem 2.7).

Example 5.25 Computing Square Roots Before returning to generating functions, let us give one quick application of the binomial theorem to the computation of square roots. We have

$$(1+x)^{1/2} = 1 + \frac{1}{2}x + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}x^2 + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!}x^3 + \cdots, \quad (5.23)$$

if $|x| < 1$. Let us use this result to compute $\sqrt{30}$. Note that $|29| \geq 1$, so we cannot use (5.23) directly. However,

$$\sqrt{30} = \sqrt{25+5} = 5\sqrt{1+.2},$$

so we can apply (5.23) with $x = .2$. This gives us

$$\sqrt{30} = 5\left[1 + \frac{1}{2}(.2) - \frac{1}{8}(.2)^2 + \frac{1}{16}(.2)^3 - \cdots\right] \approx 5.4775.$$

■

Returning to Example 5.19, let us apply the Binomial Theorem to find the coefficient of x^k in the expansion of

$$G(x) = (1 + x + x^2 + \cdots)^p.$$

Note that provided that $|x| < 1$,

$$G(x) = \left(\frac{1}{1-x}\right)^p,$$

using the identity (5.2). $G(x)$ can be rewritten as

$$G(x) = (1-x)^{-p}.$$

We can now apply the binomial theorem with $-x$ in place of x and with $u = -p$. Then we have

$$G(x) = \sum_{r=0}^{\infty} \binom{-p}{r} (-x)^r.$$

For $k > 0$, the coefficient of x^k is

$$\binom{-p}{k}(-1)^k = \frac{(-p)(-p-1)\cdots(-p-k+1)}{k!}(-1)^k,$$

which equals

$$\begin{aligned}
\frac{p(p+1)\cdots(p+k-1)}{k!} &= \frac{(p+k-1)(p+k-2)\cdots p}{k!} \\
&= \frac{(p+k-1)!}{k!(p-1)!} \\
&= \binom{p+k-1}{k}.
\end{aligned}$$

Since $\binom{p+0-1}{0} = 1$, $\binom{p+k-1}{k}$ gives the coefficient of x^0 also. Hence, we have proved the following theorem.

Theorem 5.4 If there are p types of objects, the number of distinguishable ways to choose k objects if we are allowed unlimited repetition of each type is given by

$$\binom{-p}{k}(-1)^k = \binom{p+k-1}{k}.$$

Corollary 5.4.1 Suppose that p is a fixed positive integer. Then the ordinary generating function for the sequence (c_k), where

$$c_k = \binom{p+k-1}{k},$$

is given by

$$C(x) = \left(\frac{1}{1-x}\right)^p = (1-x)^{-p}.$$

Proof. This is a corollary of the proof. Q.E.D.

Note that the case $p = 2$ is covered in Example 5.12. In this case,

$$c_k = \binom{2+k-1}{k} = \binom{k+1}{k} = k+1.$$

Note that in terms of occupancy problems (Section 5.3.2), Theorem 5.4 says that the number of ways to place k indistinguishable balls into p distinguishable cells, with no restriction on the number of balls in a cell, is given by

$$\binom{p+k-1}{k}.$$

We have already seen this result in Theorem 2.4.

Example 5.26 The Pizza Problem (Example 2.20 Revisited) In a restaurant that serves pizza, suppose that there are nine types of toppings (see Example 2.20). If a pizza can have at most one kind of topping, how many ways are there to sell 100 pizzas? It is reasonable to assume that each topping is, for all practical purposes, in infinite supply. Now we have $p = 10$ types of toppings, including the topping "nothing but cheese." Then by Theorem 5.4, the number of distinguishable ways to pick $k = 100$ pizzas is given by $\binom{109}{100}$. ■

Example 5.27 Rating Computer Systems Alternative computer systems are rated on different benchmarks, with an integer score of 1 to 6 possible on each benchmark. In how many ways can the total of the scores on three benchmarks add up to 12? To answer this question, let us think of the score on each benchmark as being chosen as 1, 2, 3, 4, 5, or 6 points. Hence, the generating function to consider is

$$G(x) = (x + x^2 + \cdots + x^6)^3.$$

Note that we start with x rather than 1 (or x^0) because there must be at least one point chosen. We take the third power because there are three benchmarks and we want the coefficient of x^{12}. How can this be found? The answer uses the identity

$$1 + x + x^2 + \cdots + x^s = \frac{1 - x^{s+1}}{1 - x}. \tag{5.24}$$

Then we note that

$$\begin{aligned} G(x) &= [x(1 + x + x^2 + \cdots + x^5)]^3 \\ &= x^3 \left[\frac{1 - x^6}{1 - x}\right]^3 \\ &= x^3(1 - x^6)^3(1 - x)^{-3}. \end{aligned} \tag{5.25}$$

We already know that $C(x) = (1 - x)^{-p}$ is the generating function for the sequence (c_k) where

$$c_k = \binom{p + k - 1}{k}.$$

Here

$$c_k = \binom{3 + k - 1}{k}.$$

The expression $B(x) = x^3(1 - x^6)^3$ may be expanded out using the binomial *expansion*, giving us

$$\begin{aligned} B(x) &= x^3[1 - 3x^6 + 3x^{12} - x^{18}] \\ &= x^3 - 3x^9 + 3x^{15} - x^{21}. \end{aligned}$$

Hence, $B(x)$ is the generating function for the sequence

$$(b_k) = (0,0,0,1,0,0,0,0,0,-3,0,0,0,0,0,3,0,0,0,0,0,-1,0,0,\ldots).$$

It follows from (5.25) that $G(x)$ is the generating function for the convolution (a_k) of the sequences (b_k) and (c_k). We wish to find the coefficient a_{12} of x^{12}. This is obtained as

$$\begin{aligned}
a_{12} &= b_0c_{12} + b_1c_{11} + b_2c_{10} + \cdots + b_{12}c_0 \\
&= b_3c_9 + b_9c_3 \\
&= 1 \cdot \binom{3+9-1}{9} - 3 \cdot \binom{3+3-1}{3} \\
&= \binom{11}{9} - 3\binom{5}{3} \\
&= 25.
\end{aligned}$$

■

Example 5.28 List *T*-Colorings Chromatic polynomials from Section 3.4 are used to find the number of distinct colorings that are possible for a given graph using a fixed number of colors. Here we use generating functions to find the number of distinct list T-colorings (see Examples 3.20 and 3.22) of an unlabeled graph G. A *list T-coloring* of G is a T-coloring in which the color assigned to a vertex x belongs to the list $L(x)$ associated with x. We will assume that $T = \{0,1,2,\ldots,r\}$, $G = K_n$, and each vertex submits the list $\{1,2,\ldots,l\}$ as possible colors.

As an example, suppose that $r = 1$, $n = 5$, and $l = 16$. Consider a list T-coloring of K_5 using the colors 1, 3, 5, 8, 10. Each pair of colors differ by at least two in absolute value and $1 \leq 1 < 3 < 5 < 8 < 10 \leq 16$. This set of inequalities gives rise to a set of differences, namely, $1-1$, $3-1$, $5-3$, $8-5$, $10-8$, $16-10$ or 0, 2, 2, 3, 2, 6, and these differences sum to 15. Note that except for the first and last differences, none of the differences could equal 0 or 1 since $T = \{0,1\}$ and the only difference of 0 or 1 could occur as the first difference (which it did with a 0 in this case) or the last difference.

Next, consider a different sequence of nonnegative integers that sum to 15, say, 3, 3, 2, 2, 3, 2. These could be considered the set of differences from the set of inequalities $1 \leq 4 < 7 < 9 < 11 < 14 \leq 16$ and thus a list T-coloring of K_5 using the colors 4, 7, 9, 11, 14. Note that all absolute differences among consecutive integers in this increasing list of five integers are greater than 1.

Thus, there is easily seen to be a one-to-one correspondence between the list T-colorings to be counted and integer solutions to

$$b_1 + b_2 + b_3 + b_4 + b_5 + b_6 = 15,$$

where $b_1, b_6 \geq 0$ and $b_2, b_3, b_4, b_5 \geq 2$. Therefore, using the ideas of Example 5.20, we see that the number of list T-colorings in this example is the coefficient of x^{15}

in

$$\begin{aligned} f(x) &= \left(1+x+x^2+\cdots\right)^2\left(x^2+x^3+\cdots\right)^4 \\ &= x^8\left(1-x\right)^{-6} \end{aligned}$$

or the coefficient of x^7 in $(1-x)^{-6}$. This coefficient of x^7 equals

$$\binom{-6}{7}(-1)^7 = \binom{6+7-1}{7} = \binom{12}{7} = 792.$$

■

Example 5.29 Number of Weak Orders We can use generating functions to count the number of distinct weak orders (X, R) on an n-element set X. Recall that weak orders have levels and the elements on each level are equivalent. That is, if $x, y \in X$, then xRy for each x at a higher level than y and xRy and yRx if x, y are at the same level. Consider a weak order with w levels. There must be at least one element on each level. Therefore, the number of weak orders on n elements with w levels is the coefficient of x^n in the generating function

$$\begin{aligned} G(x) &= \underbrace{(x+x^2+\cdots)(x+x^2+\cdots)\cdots(x+x^2+\cdots)}_{w\text{ terms}} \\ &= (x+x^2+\cdots)^w \\ &= x^w(1+x+x^2+\cdots)^w \\ &= x^w(1-x)^{-w}. \end{aligned}$$

This is then the coefficient of x^{n-w} in $(1-x)^{-w}$, which is

$$\binom{-w}{n-w}(-1)^{n-w} = \binom{w+(n-w)-1}{n-w} = \binom{n-1}{n-w}.$$

Since the number of levels w can range from 1 to n, the total number of distinct weak orders on n elements equals

$$\binom{n-1}{n-1} + \binom{n-1}{n-2} + \cdots + \binom{n-1}{n-n} = \sum_{w=1}^{n} \binom{n-1}{n-w}.$$

By Theorem 2.8 [and from Pascal's triangle (Section 2.7)] this sum is given by 2^{n-1}.

■

EXERCISES FOR SECTION 5.4

1. Use the binomial theorem to find the coefficient of x^3 in the expansion of:
   (a) $\sqrt[4]{1+x}$ (b) $(1+x)^{-3}$
   (c) $(1-x)^{-4}$ (d) $(1+5x)^{3/4}$

2. Find the coefficient of x^7 in the expansion of:
   (a) $(1-x)^{-4}x^3$ (b) $(1-x)^{-2}x^8$
   (c) $x^2(1+x^2)^3(1-x)^{-3}$ (d) $(1+x)^{1/2}x^5$

3. If $(1+x)^{1/3}$ is the ordinary generating function for the sequence (a_k), find a_2.

4. Do the calculation to solve Exercise 6(a) of Section 5.3.

5. Use Theorem 5.4 to compute the number of ways to pick six letters if a and b are the only letters available. Check your answer by writing out all the ways.

6. How many ways are there to choose 11 personal computers if 6 different manufacturers' models are available?

7. How many ways are there to choose 50 shares of stock if 100 shares each of four different companies are available?

8. A fruit fly is classified as either dominant, hybrid, or recessive for eye color. Ten fruit flies are to be chosen for an experiment. In how many different ways can the genotypes (classifications) dominant, hybrid, and recessive be chosen if you are interested only in the number of dominants, number of hybrids, and number of recessives?

9. Five different banks offer certificates of deposit (CDs) that can only be purchased in multiples of \$1,000. If an investor has \$10,000, in how many different ways can she invest in CDs?

10. Suppose that there are six different kinds of fruit available, each in (theoretically) infinite supply. How many different fruit baskets of 10 pieces of fruit are there?

11. A person drinks one can of beer an evening, choosing one of six different brands. How many different ways are there in which to drink beer over a period of a week if as many cans of a given brand are available as necessary, any two such cans are interchangeable, and we do not distinguish between drinking brand x on Monday and drinking brand x on Tuesday?

12. In studying defective products in a factory, we classify defects found by the day of the week in which they are found. In how many different ways can we classify 10 defective products?

13. Suppose that there are p different kinds of objects, each in infinite supply. Let a_k be the number of distinguishable ways to pick k of the objects if we must pick at least one of each kind.

    (a) Set up a generating function for a_k.

    (b) The sequence (a_k) is the convolution of the sequence (b_k) whose generating function is x^p and a sequence (c_k). Find c_k.

    (c) Find a_k for all k.

14. In Exercise 7 of Section 5.3, solve for a_k. (*Hint:* Set $y = x^2$ in the generating function.)

15. Suppose that $B(x)$ is the ordinary generating function for the sequence (b_k). Let

$$Sb_k = b_0 + b_1 + \cdots + b_k$$

and

$$S^2(b_k) = S(Sb_k) = \sum_{j=0}^{k}(b_0 + b_1 + \cdots + b_j).$$

In general, let $a_k = S^p(b_k) = S(S^{p-1}(b_k))$. Then we can show that

$$a_k = b_k + pb_{k-1} + \cdots + \binom{p+j-1}{j} b_{k-j} + \cdots + \binom{p+k-1}{k} b_0.$$

(a) Verify this for $p = 2$.

(b) If $A(x)$ is the ordinary generating function for (a_k), find an expression for $A(x)$ in terms of $B(x)$.

16. Consider the list T-coloring problem of Example 5.28.

 (a) Solve this problem when:

  i. $r = 1$, $n = 5$, and $l = 22$

  ii. $r = 1$, $n = 4$, and $l = 16$

  iii. $r = 2$, $n = 5$, and $l = 16$

 (b) Find the general solution for any positive integers r, n, and l.

17. Consider the weak order counting problem of Example 5.29. How many distinct weak orders on 10 elements have:

 (a) 4 elements at the highest level?

 (b) at most 4 elements at the highest level?

 (c) an even number of elements at every level?

18. Let p_n^r be the number of partitions of the integer n into *exactly* r parts where order counts. For example, there are 10 partitions of 6 into exactly 4 parts where order matters, namely,

$$\begin{array}{ccccc} \{3,1,1,1\}, & \{1,3,1,1\}, & \{1,1,3,1\}, & \{1,1,1,3\}, & \{2,2,1,1\}, \\ \{2,1,2,1\}, & \{2,1,1,2\}, & \{1,2,2,1\}, & \{1,2,1,2\}, & \{1,1,2,2\}. \end{array}$$

 (a) Set up an ordinary generating function for p_n^r.

 (b) Solve for p_n^r.

19. A polynomial in the three variables u, v, w is called *homogeneous* if the total degree of each term $\alpha u^i v^j w^k$ is the same, that is, if $i + j + k$ is constant. For instance,

$$3v^4 + 2uv^2w + 4vw^3$$

is homogeneous with each term having total degree 4. What is the largest number of terms possible in a polynomial of three variables that is homogeneous of total degree n?

20. (a) Show that p_n^r as defined in Exercise 18 is the maximum number of terms in a homogeneous polynomial in r variables and having total degree n in which each term has each variable with degree at least 1.

 (b) Use this result and the result of Exercise 18 to answer the question in Exercise 19.

21. Three people each roll a die once. In how many ways can the score add up to 9?

5.5 EXPONENTIAL GENERATING FUNCTIONS AND GENERATING FUNCTIONS FOR PERMUTATIONS

5.5.1 Definition of Exponential Generating Function

So far we have used ordinary generating functions to count the number of combinations of objects—we use the word *combination* because order does not matter. Let us now try to do something similar if order does matter and we are counting permutations. Recall that $P(n, k)$ is the number of k-permutations of an n-set. The ordinary generating function for $P(n, k)$ with n fixed is given by

$$\begin{aligned} G(x) &= P(n,0)x^0 + P(n,1)x^1 + P(n,2)x^2 + \cdots + P(n,n)x^n \\ &= \sum_{k=0}^{n} \frac{n!}{(n-k)!} x^k. \end{aligned}$$

Unfortunately, there is no good way to simplify this expression. Had we been dealing with combinations, and the number of ways $C(n, k)$ of choosing k elements out of an n-set, we would have been able to simplify, for we would have had the expression

$$C(n,0)x^0 + C(n,1)x^1 + C(n,2)x^2 + \cdots + C(n,n)x^n, \tag{5.26}$$

which by the binomial expansion simplifies to $(1 + x)^n$. By Theorem 2.1,

$$P(n,r) = C(n,r)P(r,r) = C(n,r)r!.$$

Hence, the equivalence of (5.26) to $(1 + x)^n$ can be rewritten as

$$P(n,0)\frac{x^0}{0!} + P(n,1)\frac{x^1}{1!} + P(n,2)\frac{x^2}{2!} + \cdots + P(n,n)\frac{x^n}{n!} = (1+x)^n. \tag{5.27}$$

The number $P(n, k)$ is the coefficient of $x^k/k!$ in the expansion of $(1 + x)^n$.

This suggests the following idea. If (a_k) is any sequence, the *exponential generating function* for the sequence is the function

$$\begin{aligned} H(x) &= a_0\frac{x^0}{0!} + a_1\frac{x^1}{1!} + a_2\frac{x^2}{2!} + \cdots + a_k\frac{x^k}{k!} + \cdots \\ &= \sum_k a_k \frac{x^k}{k!}. \end{aligned}$$

As with the ordinary generating function, we think of x as being chosen so that the sum converges.[5]

[5] As mentioned in the footnote on page 286, we can make this precise using the notion of formal power series.

To give an example, if $a_k = 1$, for $k = 0, 1, \ldots$, then, using Equation (5.3), we see that the exponential generating function is

$$\begin{aligned} H(x) &= 1 \cdot \frac{x^0}{0!} + 1 \cdot \frac{x^1}{1!} + 1 \cdot \frac{x^2}{2!} + \cdots \\ &= 1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \cdots \\ &= e^x. \end{aligned}$$

To give another example, if $a_k = P(n, k)$, we have shown in (5.27) that the exponential generating function is $(1+x)^n$. To give still one more example, suppose that α is any real number and (a_k) is the sequence $(1, \alpha, \alpha^2, \alpha^3, \ldots)$. Then the exponential generating function for (a_k) is

$$\begin{aligned} H(x) &= \sum_{k=0}^{\infty} \alpha^k \frac{x^k}{k!} \\ &= \sum_{k=0}^{\infty} \frac{(\alpha x)^k}{k!} \\ &= e^{\alpha x}. \end{aligned}$$

Just as with ordinary generating functions, we will want to go back and forth between sequences and exponential generating functions.

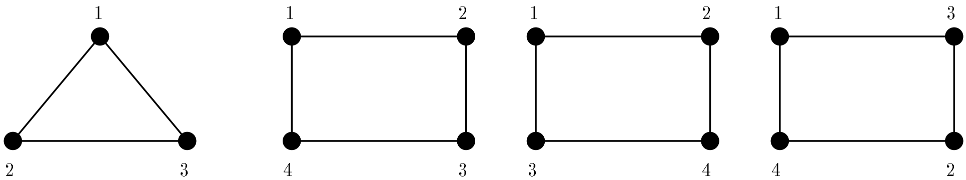
Example 5.30 Eulerian Graphs A connected graph will be called *eulerian* if every vertex has even degree. Eulerian graphs will be very important in a variety of applications discussed in Chapter 11. Harary and Palmer [1973] and Read [1962] show that if u_n is the number of labeled, connected eulerian graphs of n vertices, the exponential generating function $U(x)$ for the sequence (u_n) is given by

$$U(x) = x + \frac{x^3}{3!} + \frac{3x^4}{4!} + \frac{38x^5}{5!} + \cdots.$$

Thus, there is one labeled, connected eulerian graph of three vertices and there are three of four vertices. These are shown in Figure 5.7. ■

5.5.2 Applications to Counting Permutations

Example 5.31 A code can use three different letters, a, b, or c. A sequence of five or fewer letters gives a codeword. The codeword can use at most one b, at most one c, and at most three a's. How many possible codewords are there of length k, with $k \leq 5$? Note that order matters in a codeword. For example, codewords aab and aba are different, whereas previously, when considering subsets, $\{a, a, b\}$ and $\{a, b, a\}$ were the same. So we are interested in counting permutations rather than



(a) (b)

Figure 5.7: The labeled, connected eulerian graphs of (a) three vertices and (b) four vertices.

combinations. However, taking a hint from our previous experience, let us begin by counting combinations, the number of ways of getting k letters if it is possible to pick at most one b, at most one c, and at most three a's. The ordinary generating function is calculated by taking

$$(1 + ax + a^2x^2 + a^3x^3)(1 + bx)(1 + cx),$$

which equals

$$1 + (a + b + c)x + (bc + a^2 + ab + ac)x^2 + (a^3 + abc + a^2b + a^2c)x^3 +$$
$$(a^2bc + a^3b + a^3c)x^4 + a^3bcx^5.$$

The coefficient of x^k gives the ways of obtaining k letters. For example, three letters can be obtained as follows: 3 a's, a and b and c, 2 a's and b, or 2 a's and c. If we make a choice of a and b and c, there are 3! corresponding permutations:

$$abc, \quad acb, \quad bac, \quad bca, \quad cab, \quad cba.$$

For the 3 a's choice, there is only one corresponding permutation: aaa. For the 2 a's and b choice, there are 3 permutations:

$$aab, \quad aba, \quad baa.$$

From our general formula of Theorem 2.6 we see why this is true: The number of distinguishable permutations of 3 objects with 2 of one type and 1 of another is given by

$$\frac{3!}{2!1!}.$$

In general, if we have n_1 a's, n_2 b's, and n_3 c's, the number of corresponding permutations is

$$\frac{n!}{n_1!n_2!n_3!}.$$

In particular, in our schematic, the proper information for the ways to obtain codewords if three letters are chosen is given by

$$\frac{3!}{3!}a^3 + \frac{3!}{1!1!1!}abc + \frac{3!}{2!1!}a^2b + \frac{3!}{2!1!}a^2c. \tag{5.28}$$

Setting $a = b = c = 1$ would yield the proper count of number of such codewords of three letters. We can obtain (5.28) and the other appropriate coefficients by the trick of using

$$\frac{(ax)^p}{p!} = \frac{a^p}{p!}x^p$$

instead of $a^p x^p$ to derive our schematic generating function. In our example, we have

$$\left(1 + \frac{a}{1!}x + \frac{a^2}{2!}x^2 + \frac{a^3}{3!}x^3\right)\left(1 + \frac{b}{1!}x\right)\left(1 + \frac{c}{1!}x\right),$$

which equals

$$1 + \left(\frac{a}{1!} + \frac{b}{1!} + \frac{c}{1!}\right)x + \left(\frac{bc}{1!1!} + \frac{a^2}{2!} + \frac{ab}{1!1!} + \frac{ac}{1!1!}\right)x^2 +$$
$$\left(\frac{a^3}{3!} + \frac{abc}{1!1!1!} + \frac{a^2b}{2!1!} + \frac{a^2c}{2!1!}\right)x^3 + \left(\frac{a^2bc}{2!1!1!} + \frac{a^3b}{3!1!} + \frac{a^3c}{3!1!}\right)x^4 + \frac{a^3bc}{3!1!1!}x^5. \quad (5.29)$$

This is still not a satisfactory schematic—compare the coefficients of x^3 to the expression in (5.28). However, the schematic works if we consider this as an exponential generating function, and choose the coefficient of $x^k/k!$. For the expression (5.29) is equal to

$$1 \; + \; 1!\left(\frac{a}{1!} + \frac{b}{1!} + \frac{c}{1!}\right)\frac{x}{1!} \; + \; 2!\left(\frac{bc}{1!1!} + \frac{a^2}{2!} + \frac{ab}{1!1!} + \frac{ac}{1!1!}\right)\frac{x^2}{2!} \; +$$
$$3!\left(\frac{a^3}{3!} \; + \; \frac{abc}{1!1!1!} + \frac{a^2b}{2!1!} + \frac{a^2c}{2!1!}\right)\frac{x^3}{3!} \; + \; \cdots. \quad (5.30)$$

Setting $a = b = c = 1$ and taking the coefficient of $x^k/k!$ gives the appropriate number of codewords (permutations). For example, the number of length 3 is

$$3!\left(\frac{1}{3!} + 1 + \frac{1}{2!} + \frac{1}{2!}\right) = 13.$$

The corresponding codewords are the six we have listed with a, b, and c, the three with 2 a's and 1 b, the three with 2 a's and 1 c, and the one with 3 a's. ■

The analysis in Example 5.31 generalizes as follows.

Theorem 5.5 Suppose that we have p types of objects, with n_i indistinguishable objects of type i, $i = 1, 2, \ldots, p$. The number of distinguishable permutations of length k with up to n_i objects of type i is the coefficient of $x^k/k!$ in the exponential generating function

$$\left(1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{n_1}}{n_1!}\right)\left(1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{n_2}}{n_2!}\right)\cdots\left(1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{n_p}}{n_p!}\right).$$

Example 5.32 RNA Chains To give an application of this result, let us consider the number of 2-link RNA chains if we have available up to 3 A's, up to 3 G's, up to 2 C's, and up to 1 U. Since order matters, we seek an exponential generating function. This is given by

$$\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}\right)^2\left(1+x+\frac{x^2}{2!}\right)(1+x),$$

which turns out to equal

$$1+4x+\frac{15}{2}x^2+\frac{53}{6}x^3+\cdots.$$

Here, the coefficient of x^2 is $15/2$, so the coefficient of $x^2/2!$ is $2!(15/2)=15$. Thus, there are 15 such chains. They are AA, AG, AC, AU, GA, GG, GC, GU, CA, CG, CC, CU, UA, UG, and UC, that is, all but UU. Similarly, the number of 3-link RNA chains made up from these available bases is the coefficient of $x^3/3!$, or $3!(53/6)=53$. The reader can readily check this result. ■

Example 5.33 RNA Chains Continued To continue with Example 5.32, suppose that we wish to find the number of RNA chains of length k if we assume an arbitrarily large supply of each base. The exponential generating function is given by

$$\begin{aligned}\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)^4 &= (e^x)^4\\ &= e^{4x}\\ &= \sum_{k=0}^{\infty}\frac{(4x)^k}{k!}\\ &= \sum_{k=0}^{\infty}4^k\frac{x^k}{k!}.\end{aligned}$$

Thus, the number in question is given by 4^k. This agrees with what we already concluded in Chapter 2, by a simple use of the product rule.

Let us make one modification here, namely, to count the number of RNA chains of length k if the number of U links is even. The exponential generating function is given by

$$H(x)=\left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots\right)\left(1+x+\frac{x^2}{2!}+\cdots\right)^3.$$

Now the second term in $H(x)$ is given by $(e^x)^3=e^{3x}$. It is also not hard to show that the first term is given by

$$\frac{1}{2}(e^x+e^{-x}).$$

Thus,

$$\begin{aligned}
H(x) &= \frac{1}{2}(e^x + e^{-x})(e^{3x}) \\
&= \frac{1}{2}(e^{4x} + e^{2x}) \\
&= \frac{1}{2}\left[\sum_{k=0}^{\infty} 4^k \frac{x^k}{k!} + \sum_{k=0}^{\infty} 2^k \frac{x^k}{k!}\right] \\
&= \sum_{k=0}^{\infty} \left(\frac{4^k + 2^k}{2}\right) \frac{x^k}{k!}.
\end{aligned}$$

We conclude that the number of RNA chains in question is

$$\frac{4^k + 2^k}{2}.$$

To check this, note for example that if $k = 2$, this number is 10. The 10 chains are UU, GG, GA, GC, AG, AA, AC, CG, CA, and CC. ■

5.5.3 Distributions of Distinguishable Balls into Indistinguishable Cells[6]

Recall from Section 2.10.4 that the Stirling number of the second kind, $S(n,k)$, is defined to be the number of distributions of n distinguishable balls into k indistinguishable cells with no cell empty. Here we shall show that

$$S(n,k) = \frac{1}{k!}\sum_{i=0}^{k}(-1)^i \binom{k}{i}(k-i)^n. \tag{5.31}$$

Let us first consider the problem of finding the number $T(n,k)$ of ways to put n distinguishable balls into k distinguishable cells labeled $1, 2, \ldots, k$, with no cell empty. Note that

$$T(n,k) = k!S(n,k), \tag{5.32}$$

since we obtain a distribution of n distinguishable balls into k distinguishable cells with no cell empty by finding a distribution of n distinguishable balls into k indistinguishable cells with no cell empty and then labeling (ordering) the cells. Next we compute $T(n,k)$. Suppose that ball i goes into cell $C(i)$. We can encode the distribution of balls into distinguishable cells by giving a sequence $C(1)C(2)\cdots C(n)$. This is an n-permutation from the k-set $\{1, 2, \ldots, k\}$ with each label j in the k-set used at least once. Thus, $T(n,k)$ is the number of such permutations, and for fixed k, the exponential generating function for $T(n,k)$ is therefore given by

$$H(x) = \left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)^k = (e^x - 1)^k.$$

[6] This subsection may be omitted.

$T(n, k)$ is given by the coefficient of $x^n/n!$ in the expansion of $H(x)$. By the binomial expansion (Theorem 2.7),

$$H(x) = \sum_{i=0}^{k} \binom{k}{i} (-1)^i e^{(k-i)x}.$$

Substituting $(k-i)x$ for x in the power series (5.3) for e^x, we obtain

$$\begin{aligned} H(x) &= \sum_{i=0}^{k} \binom{k}{i} (-1)^i \sum_{n=0}^{\infty} \frac{1}{n!}(k-i)^n x^n \\ &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^n. \end{aligned}$$

Finding the coefficient of $x^n/n!$ in the expansion of $H(x)$, we have

$$T(n, k) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^n. \tag{5.33}$$

Now Equations (5.32) and (5.33) give us Equation (5.31).

EXERCISES FOR SECTION 5.5

1. For each of the following sequences (a_k), find a simple, closed-form expression for the exponential generating function.

   (a) $(5, 5, 5, \ldots)$ (b) $a_k = 3^k$ (c) $(1, 0, 0, 1, 1, \ldots)$
   (d) $(0, 0, 1, 1, \ldots)$ (e) $(1, 0, 1, 0, 1, \ldots)$ (f) $(2, 1, 2, 1, 2, 1, \ldots)$

2. For each of the following functions, find a sequence for which the function is the exponential generating function.

   (a) $4 + 4x + 4x^2 + 4x^3 + \cdots$ (b) $\dfrac{3}{1-x}$ (c) $x^2 + 5e^x$
   (d) $x^2 + 4x^3 + x^5$ (e) e^{6x} (f) $5e^x$
   (g) $e^{2x} + e^{5x}$ (h) $(1 + x^2)^n$ (i) $\dfrac{1}{1-6x}$

3. A graph is said to be *even* if every vertex has even degree. If e_k is the number of labeled, even graphs of k vertices, Harary and Palmer [1973] show that the exponential generating function $E(x)$ for the sequence (e_k) is given by

$$E(x) = x + \frac{x^2}{2!} + \frac{2x^3}{3!} + \frac{8x^4}{4!} + \cdots.$$

   Verify the coefficients of $x^3/3!$ and $x^4/4!$. (Note that e_k can be derived from Exercise 11, Section 11.3, and the results of Section 3.1.3.)

4. In Example 5.32, check by enumeration that there are 53 3-link RNA chains made up from the available bases.

5. Find the number of 3-link RNA chains if the available bases are 2 A's, 3 G's, 3C's, and 1 U. Check your answer by enumeration.
6. In each of the following, set up the appropriate generating function, but do not calculate an answer. Indicate what you are looking for, for example, the coefficient of x^8.
   (a) How many codewords of three letters can be built from the letters a, b, c, and d if b and d can only be picked once?
   (b) A codeword consists of at least one of each of the digits 0, 1, 2, 3, and 4, and has length 6. How many such codewords are there?
   (c) How many 11-digit numbers consist of at most four 0's, at most three 1's, and at most four 2's?
   (d) In how many ways can $3n$ letters be selected from $2n$ A's, $2n$ B's, and $2n$ C's?
   (e) If n is a fixed even number, find the number of n-digit words generated from the alphabet $\{0, 1, 2, 3\}$ in each of which the number of 0's and the number of 1's is even and the number of 2's is odd.
   (f) In how many ways can a total of 100 be obtained if 50 dice are rolled?
   (g) Ten municipal bonds are each to be rated as A, AA, or AAA. In how many different ways can the ratings be assigned?
   (h) In planning a schedule for the next 20 days at a job, in how many ways can one schedule the 20 days using at most 5 vacation days, at most 5 personal days, and at most 15 working days?
   (i) Suppose that with a type A coin, you get 1 point if the coin turns up heads and 2 points if it turns up tails. With a type B coin, you get 2 points for a head and 3 points for a tail. In how many ways can you get 12 points if you toss 3 type A coins and 5 type B coins?
   (j) In how many ways can 200 identical terminals be divided among four computer rooms so that each room will have 20 or 40 or 60 or 80 or 100 terminals?
   (k) One way for a ship to communicate with another visually is to hang a sequence of colored flags from a flagpole. The meaning of a signal depends on the order of the flags from top to bottom. If there are available 5 red flags, 4 green ones, 4 yellow ones, and 1 blue one, how many different signals are possible if 12 flags are to be used?
   (l) In part (k), how many different signals are possible if at least 12 flags are to be used?
7. (a) Find the number of RNA chains of length k if the number of A's is odd.
   (b) Illustrate for $k = 2$.
8. Find the number of RNA chains of length 2 with an even number of U's or an odd number of A's.
9. Suppose that there are p different kinds of objects, each in infinite supply. Let a_k be the number of permutations of k objects chosen from these objects. Find a_k explicitly by using exponential generating functions.
10. In how many ways can 60 identical terminals be divided among two computer rooms so that each room will have 20 or 40 terminals?

11. If order matters, find an exponential generating function for the number of partitions of integer k (Example 5.23 and Exercise 18, Section 5.4).
12. Find a simple, closed-form expression for the exponential generating function if we have p types of objects, each in infinite supply, and we wish to choose k objects, at least one of each kind, and order matters.
13. Find a simple, closed-form expression for the exponential generating function if we have p types of objects, each in infinite supply, and we wish to choose k objects, with an even number (including 0) of each kind, and order matters.
14. Find the number of codewords of length k from an alphabet $\{a, b, c, d, e\}$ if b occurs an odd number of times.
15. Find the number of codewords of length 3 from an alphabet $\{1, 2, 3, 4, 5, 6\}$ if 1, 3, 4, and 6 occur an even number of times.
16. Compute $S(4, 2)$ and $T(4, 2)$ from Equations (5.31) and (5.33), respectively, and check your answers by listing all the appropriate distributions.
17. Exercises 17–20 investigate combinations of exponential generating functions. Suppose that $A(x)$ and $B(x)$ are the exponential generating functions for the sequences (a_k) and (b_k), respectively. Find an expression for the kth term c_k of the sequence (c_k) whose exponential generating function is $C(x) = A(x) + B(x)$.
18. Repeat Exercise 17 for $C(x) = A(x)B(x)$.
19. Find a_3 if the exponential generating function for (a_k) is:

    (a) $e^x(1 + x)^6$ (b) $\dfrac{e^{3x}}{1 - x}$ (c) $\dfrac{x^2}{(1 - x)^2}$

20. Suppose that $a_{n+1} = (n + 1)b_n$, with $a_0 = b_0 = 1$. If $A(x)$ is the exponential generating function for the sequence (a_n) and $B(x)$ is the exponential generating function for the sequence (b_n), derive a relation between $A(x)$ and $B(x)$.

5.6 PROBABILITY GENERATING FUNCTIONS[7]

The simple idea of a generating function has interesting uses in the study of probability. In fact, the first complete treatment of generating functions was by Laplace in his *Théorie Analytique des Probabilités* (Paris, 1812), and much of the motivation for the development of generating functions came from probability. Suppose that after an experiment is performed, it is known that one and only one of a (finite or countably infinite) set of possible events will occur. Let p_k be the probability that the kth event occurs, $k = 0, 1, 2, \ldots$. (Of course, this notation does not work if there is a continuum of possible events.) The ordinary generating function

$$G(x) = \sum p_k x^k \tag{5.34}$$

[7] This section may be omitted without loss of continuity. Although it is essentially self-contained, the reader with some prior exposure to probability theory, at least at the level of a "finite math" book such as Goodman and Ratti [1992] or Kemeny, Snell, and Thompson [1974], will get more out of this.

is called the *probability generating function.* [Note that (5.34) converges at least for $|x| \leq 1$, since $p_0 + p_1 + \cdots + p_k + \cdots = 1$.] We shall see that probability generating functions are extremely useful in evaluating experiments, in particular in analyzing roughly what we "expect" the outcomes to be.

Example 5.34 Coin Tossing Suppose that the experiment is tossing a fair coin. Then the events are heads (H) and tails (T), with p_0, probability of H, equal to 1/2, and p_1, probability of T, equal to 1/2. Hence, the probability generating function is

$$G(x) = \frac{1}{2} + \frac{1}{2}x. \qquad \blacksquare$$

Example 5.35 Bernoulli Trials In Bernoulli trials there are n independent repeated trials of an experiment, with each trial leading to a success with probability p and a failure with probability $q = 1-p$. The experiment could be a test to see if a product is defective or nondefective, a test for the presence or absence of a disease, or a decision about whether to accept or reject a candidate for a job. If S stands for success and F for failure, a typical outcome in $n = 5$ trials is a sequence like $SSFSF$ or $SSFFF$. The probability that in n trials there will be k successes is given by

$$b(k, n, p) = C(n, k)p^k q^{n-k},$$

as is shown in any standard book on probability theory (such as Feller [1968], Parzen [1992], or Ross [1997]), or on finite mathematics (such as Goodman and Ratti [1992] or Kemeny, Snell, and Thompson [1974]). The probability generating function for the number of successes in n trials is given by

$$\begin{aligned} G(x) &= \sum_{k=0}^{n} b(k, n, p)x^k \\ &= \sum_{k=0}^{n} C(n, k)p^k q^{n-k} x^k. \end{aligned}$$

By the binomial expansion (Theorem 2.7), we have

$$G(x) = (px + q)^n. \qquad \blacksquare$$

Let us note some simple results about probability generating functions.

Theorem 5.6 If G is a probability generating function, then

$$G(1) = 1.$$

Proof. Since the outcomes are mutually exclusive and exhaustive by assumption, we have

$$p_0 + p_1 + \cdots + p_k + \cdots = 1. \qquad \text{Q.E.D.}$$

Corollary 5.6.1

$$\sum_{k=0}^{n} C(n,k)p^k q^{n-k} = 1.$$

Proof. In Bernoulli trials (Example 5.35), set $G(1) = 1$. Q.E.D.

Corollary 5.6.1 may also be proved directly from the binomial expansion, noting that

$$(p+q)^n = 1^n.$$

Suppose that in an experiment, if the kth event occurs, we get k dollars (or k units of some reward). Then the expression $E = \sum kp_k$ is called the *expected value* or the *expectation*. It is what we expect to "win" on the average if the experiment is repeated many times, and we expect 0 dollars a fraction p_0 of the time, 1 dollar a fraction p_1 of the time, and so on. For a more detailed discussion of expected value, see any elementary book on probability theory or on finite mathematics. Note that the expected value is defined only if the sum $\sum kp_k$ converges. If the sum does converge, we say that the expected value *exists*. We can have the same expected value in an experiment that always gives 1 dollar and in an experiment that gives 0 dollars with probability $\frac{1}{2}$ and 2 dollars with probability $\frac{1}{2}$. However, there is more variation in outcomes in the second experiment. Probability theorists have introduced the concept of variance to measure this variation. Specifically, the *variance* V is defined to be

$$V = \sum_k k^2 p_k - \left(\sum_k kp_k\right)^2. \tag{5.35}$$

See a probability book such as Feller [1968], Parzen [1992], or Ross [1997] for a careful explanation of this concept. Variance is defined only if the sums in (5.35) converge. In case they do converge, we say that the variance *exists*. In the first experiment mentioned above,

$$V = \left[1^2(1)\right] - [1(1)]^2 = 0.$$

In the second experiment mentioned above,

$$V = \left[0^2\left(\frac{1}{2}\right) + 2^2\left(\frac{1}{2}\right)\right] - \left[0\left(\frac{1}{2}\right) + 2\left(\frac{1}{2}\right)\right]^2 = 1.$$

Hence, the variance is higher in the second experiment. We shall see how the probability generating function allows us to calculate expected value and variance.

Differentiating (5.34) with respect to x leads to the equation

$$G'(x) = \sum kp_k x^{k-1}.$$

Hence, if $G'(x)$ converges for $x = 1$, that is, if $\sum kp_k$ converges, then

$$G'(1) = \sum kp_k. \tag{5.36}$$

If the kth event gives value k dollars or units, the expression on the right-hand side of (5.36) is the expected value.

Theorem 5.7 Suppose that $G(x)$ is the probability generating function and the kth event gives value k. If the expected value exists, $G'(1)$ is the expected value.

Let us apply Theorem 5.7 to the case of Bernoulli trials. We have

$$\begin{aligned} G(x) &= (px+q)^n \\ G'(x) &= n(px+q)^{n-1}p \\ G'(1) &= np(p+q)^{n-1} \\ &= np(1)^{n-1} \\ &= np. \end{aligned}$$

Thus, the expected number of successes in n trials is np. The reader who recalls the "standard" derivation of this fact should be pleased at how simple this derivation is. To illustrate the result, we note that in $n = 100$ tosses of a fair coin, the probability of a head (success) is $p = .5$ and the expected number of heads is $np = 50$.

Example 5.36 Chip Manufacturing A company manufacturing computer chips estimates that one chip in every 10,000 manufactured is defective. If an order comes in for 100,000 chips, what is the expected number of defective chips in that order? Assuming that defects appear independently, we have an example of Bernoulli trials and we see that the expected number is $(100,000)\dfrac{1}{10,000} = 10$. ■

Example 5.37 Packet Transmission In data transmission, the probability that a transmitted "packet" is lost is 1 in 1000. What is the expected number of packets transmitted before one is lost? Assuming that packet loss is independent from packet to packet, we have Bernoulli trials. We can try to calculate the probability that the *first success* occurs on trial k and then compute the expected value of the first success. We ask the reader to investigate this in Exercise 7. ■

The next theorem is concerned with variance. Its proof is left to the reader (Exercise 9).

Theorem 5.8 Suppose that $G(x)$ is the probability generating function and the kth event has value k. If the variance V exists, V is given by $V = G''(1) + G'(1) - [G'(1)]^2$.

Applying Theorem 5.8 to Bernoulli trials, we have

$$G''(x) = n(n-1)p^2(px+q)^{n-2}.$$

Also,

$$
\begin{aligned}
G'(1) &= np \\
G''(1) &= n(n-1)p^2.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
V &= G''(1) + G'(1) - [G'(1)]^2 \\
&= n(n-1)p^2 + np - n^2p^2 \\
&= np - np^2 \\
&= np(1-p) \\
&= npq.
\end{aligned}
$$

This gives npq for the variance, a well-known formula.

EXERCISES FOR SECTION 5.6

1. In each of the following situations, find a simple, closed-form expression for the probability generating function, and use this to compute expected value and variance.
   (a) $p_0 = p_1 = p_2 = \frac{1}{3}, p_k = 0$ otherwise.
   (b) $p_4 = \frac{2}{5}, p_7 = \frac{3}{5}, p_k = 0$ otherwise.
   (c) $p_1 = \frac{1}{4}, p_2 = \frac{1}{4}, p_4 = \frac{1}{2}, p_k = 0$ otherwise.
2. A company manufacturing small engines estimates that two engines in every 100,000 manufactured are defective. If an order comes in for 100 engines, what is the expected number of defective engines in the order?
3. A certain disease is thought to be noncontagious. A researcher estimates the disease to be found in 1 in every 50 people. What is the expected number of people the researcher has to examine before finding a person with the disease? What is the variance?
4. For fixed positive number λ, the *Poisson distribution* with parameter λ has

$$p_k = \frac{e^{-\lambda}\lambda^k}{k!}, \qquad k = 0, 1, 2, \ldots .$$

   (a) Find a simple, closed-form expression for the probability generating function.
   (b) Use the methods of generating functions to find the expected value and the variance.
5. (Daniel [1995]) Gibbons, Clark, and Fawcett [1990] studied the monthly distribution of adolescent suicides in Cook County, Illinois, between 1977 and 1987. They found that it closely followed a Poisson distribution with parameter $\lambda = 2.75$.
   (a) Find the probability that if a month is selected at random, it will have four adolescent suicides.

(b) Find the expected number of suicides per month.

(c) Find the variance.

6. (Daniel [1995]) Suppose that a large number of samples are taken from a pond and the average number of aquatic organisms of a given kind found in a sample is 2. Assuming that the number of organisms follows a Poisson distribution, find the probability that the next sample drawn will have three or four such organisms.
7. In Bernoulli trials, suppose that we compute the probability that the first success occurs on trial k. The probability is given by $p_k = 0$, $k = 0$ (assuming that we start with trial 1), and $p_k = (1-p)^{k-1}p$, $k > 0$. The probabilities p_k define the *geometric distribution.* Repeat Exercise 4 for this distribution and apply the results to the question in Example 5.37.
8. Fix a positive integer m. In Bernoulli trials, the probability that the mth success takes place on trial $k + m$ is given by

$$p_k = \binom{k+m-1}{k} q^k p^m.$$

The probabilities p_k define the *negative binomial distribution.*

(a) Show that the probability generating function $G(x)$ for the negative binomial distribution p_k is given by

$$G(x) = \frac{p^m}{(1-qx)^m}.$$

(b) Compute expected value and variance.

9. Prove Theorem 5.8.

5.7 THE COLEMAN AND BANZHAF POWER INDICES[8]

In Section 2.15 we introduced the notion of a simple game and the Shapley-Shubik power index. Here, we shall define two alternative power indices and discuss how to use generating functions to calculate them. We defined the *value* $v(S)$ of a coalition S to be 1 if S is winning and 0 if S is losing. Coleman [1971] defines the power of player i as

$$P_i^C = \frac{\sum_S [v(S) - v(S - \{i\})]}{\sum_S v(S)}. \tag{5.37}$$

In calculating this measure, one takes the sums over all coalitions S. The term

$$v(S) - v(S - \{i\})$$

is 1 if removal of i changes S from winning to losing, and it is 0 otherwise. (It cannot be -1, since we assumed that a winning coalition can never be contained in a losing one.) Thus, P_i^C is the number of winning coalitions from which removal of i leads to

[8] This section may be omitted without loss of continuity.

a losing coalition divided by the number of winning coalitions, or the proportion of winning coalitions in which i's defection is critical. This index avoids the seemingly extraneous notion of order that underlies the computation of the Shapley-Shubik index.

It is interesting to note that the Shapley-Shubik index p_i^S can be calculated by a formula similar to (5.37). For Shapley [1953] proved that

$$p_i^S = \sum_S \{\gamma(s)[v(S) - v(S - \{i\})] : S \text{ such that } i \in S\}, \tag{5.38}$$

where

$$s = |S| \quad \text{and} \quad \gamma(s) = \frac{(s-1)!(n-s)!}{n!}.$$

(See Exercise 15, Section 2.15.)

A variant of the Coleman power index is the Banzhaf index (Banzhaf [1965]), defined as

$$P_i^B = \frac{\sum_S [v(S) - v(S - \{i\})]}{\sum_{j=1}^n \sum_S [v(S) - v(S - \{j\})]}. \tag{5.39}$$

This index has the same numerator as Coleman's, while the denominator sums the numerators for all players j. Thus, P_i^B is the number of critical defections of player i divided by the total number of critical defections of all players, or player i's proportion of all critical defections.[9]

To give an example, let us consider the game $[51; 49, 48, 3]$. Here, the winning coalitions are $\{1,2,3\}, \{1,2\}, \{1,3\}, \{2,3\}$. Player 1's defection is critical to $\{1,2\}$ and $\{1,3\}$, so we have the Coleman index

$$P_1^C = \tfrac{2}{4} = \tfrac{1}{2}.$$

Similarly, each player's defection is critical to two coalitions, so

$$\begin{aligned} P_2^C &= \tfrac{2}{4} = \tfrac{1}{2} \\ P_3^C &= \tfrac{2}{4} = \tfrac{1}{2}. \end{aligned}$$

Note that in the Coleman index, the powers P_i^C may not add up to 1. It is the relative values that count. The Banzhaf index is given by

$$\begin{aligned} P_1^B &= \tfrac{2}{6} = \tfrac{1}{3} \\ P_2^B &= \tfrac{2}{6} = \tfrac{1}{3} \\ P_3^B &= \tfrac{2}{6} = \tfrac{1}{3}. \end{aligned}$$

[9] For a unifying framework for the Shapley-Shubik, Banzhaf, and Coleman indices, see Straffin [1980]. For a survey of the literature of the Shapley-Shubik index, see Shapley [1981]. For one on the Banzhaf and Coleman indices, see Dubey and Shapley [1979]. For applications of all three indices, see Brams, Lucas, and Straffin [1983], Brams, Schotter, and Schwödianer [1979], Lucas [1981, 1983], and Johnston [1995]. See also Section 2.15.5.

These two indices agree with the Shapley-Shubik index in saying that all three players have equal power. It is not hard to give examples where these indices may differ from that of Shapley-Shubik (see Exercise 2). (When a number of ways to measure something have been introduced, and they can differ, how do we choose among them? One approach is to lay down conditions or axioms that a reasonable measure should satisfy. We can then test different measures to see if they satisfy the axioms. One set of axioms, which is satisfied only by the Shapley-Shubik index, is due to Shapley [1953]; see Owen [1995], Dubey [1975], Myerson [1997], Roberts [1976], or Shapley [1981]. Another set of axioms, which is satisfied only by the Banzhaf index, is due to Dubey and Shapley [1979]; also see Owen [1978a,b] and Straffin [1980]. Felsenthal and Machover [1995] survey and expand on the axiomatic approaches to power indices. The axiomatic approach is probably the most reasonable procedure to use in convincing legislators or judges to use one measure over another, for legislators can then decide whether they like certain general conditions, rather than argue about a procedure. Incidentally, it is the Banzhaf index that has found use in the courts, in one-person, one-vote cases; see Lucas [1983].)

Generating functions can be used to calculate the numerator of P_i^C and P_i^B in case we have a weighted majority game $[q; v_1, v_2, \ldots, v_n]$. (Exercise 3 asks the reader to describe how to find the denominator of the former. The denominator of the latter is trivial to compute if all the numerators are known.) Suppose that player i has v_i votes. His defection will be critical if it comes from a coalition with q votes, or $q+1$ votes, or $\ldots$, or $q+v_i-1$ votes. His defection in these cases will lead to a coalition with $q-v_i$ votes, or $q-v_i+1$ votes, or $\ldots$, or $q-1$ votes. Suppose that $a_k^{(i)}$ is the number of coalitions with exactly k votes and not containing player i. Then the number of coalitions in which player i's defection is critical is given by

$$a_{q-v_i}^{(i)} + a_{q-v_i+1}^{(i)} + \cdots + a_{q-1}^{(i)} = \sum_{k=q-v_i}^{q-1} a_k^{(i)}. \tag{5.40}$$

This expression can be substituted for

$$\sum_S [v(S) - v(S - \{i\})]$$

in the computation of the Coleman or Banzhaf indices, provided that we can calculate the numbers $a_k^{(i)}$. Brams and Affuso [1976] point out that the numbers $a_k^{(i)}$ can be found using ordinary generating functions. To form a coalition, player j contributes either 0 votes or v_j votes. Hence, the ordinary generating function for the $a_k^{(i)}$ is given by

$$\begin{aligned} G^{(i)}(x) &= (1+x^{v_1})(1+x^{v_2})\cdots(1+x^{v_{i-1}})(1+x^{v_{i+1}})\cdots(1+x^{v_n}) \\ &= \Pi_{j\neq i}(1+x^{v_j}). \end{aligned}$$

The number $a_k^{(i)}$ is given by the coefficient of x^k.

Let us consider the weighted majority game $[4; 1, 2, 4]$ as an example. We have

$$\begin{aligned} G^{(1)}(x) &= (1+x^2)(1+x^4) &= 1+x^2+x^4+x^6 \\ G^{(2)}(x) &= (1+x)(1+x^4) &= 1+x+x^4+x^5 \\ G^{(3)}(x) &= (1+x)(1+x^2) &= 1+x+x^2+x^3. \end{aligned}$$

Thus, for example, $a_4^{(i)}$ is the coefficient of x^4 in $G^{(1)}(x)$, i.e., it is 1. There is one coalition not containing player 1 that has exactly four votes: namely, the coalition consisting of the third player alone. Using (5.40), we obtain

$$\begin{aligned} \textstyle\sum_S[v(S)-v(S-\{1\})] &= a_{4-1}^{(1)} = a_3^{(1)} = 0 \\ \textstyle\sum_S[v(S)-v(S-\{2\})] &= a_{4-2}^{(2)} + a_{4-2+1}^{(2)} = a_2^{(2)} + a_3^{(2)} = 0 \\ \textstyle\sum_S[v(S)-v(S-\{3\})] &= a_{4-4}^{(3)} + a_{4-4+1}^{(3)} + a_{4-4+2}^{(3)} + a_{4-4+3}^{(3)} \\ &= a_0^{(3)} + a_1^{(3)} + a_2^{(3)} + a_3^{(3)} \\ &= 4. \end{aligned}$$

This immediately gives us

$$\begin{aligned} P_1^B &= \tfrac{0}{4} &= 0 \\ P_2^B &= \tfrac{0}{4} &= 0 \\ P_3^B &= \tfrac{4}{4} &= 1. \end{aligned}$$

According to the Banzhaf index, player 3 has all the power. This makes sense: No coalition can be winning without him. The Coleman index and Shapley-Shubik index give rise to the same values. Computation is left to the reader.

EXERCISES FOR SECTION 5.7

1. Calculate the Banzhaf and Coleman power indices for each of the following games, using generating functions to calculate the numerators. Check your answer using the definitions of these indices.
   (a) [51; 51, 48, 1] (b) [51; 49, 47, 4] (c) [51; 40, 30, 20, 10]
   (d) [20; 1, 10, 10, 10] (e) [102; 80, 40, 80, 20]
   (f) The Australian government "game" (Section 2.15.1): [5; 1, 1, 1, 1, 1, 1, 3]
   (g) The Board of Supervisors, Nassau County, NY, 1964: [59; 31, 31, 21, 28, 2, 2]
2. Give an example of a game where:
   (a) The Banzhaf and Coleman power indices differ
   (b) The Banzhaf and Shapley-Shubik power indices differ
   (c) The Coleman and Shapley-Shubik power indices differ
   (d) All three of these indices differ
3. Describe how to find $\sum_s v(S)$ by generating functions.

4. Use the formula of Equation (5.38) to calculate the Shapley-Shubik power index of each of the weighted majority games in Exercises 1(a)–(e).
5. (a) Explain how you could use generating functions to compute the Shapley-Shubik power index.
   (b) Apply your results to the games in Exercise 1.

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