Chapter 2

Permutations and Combinations

Most readers of this book will have had some experience with simple counting problems, so the concepts "permutation" and "combination" are probably familiar. But the experienced counter knows that even rather simple-looking problems can pose difficulties in their solutions. While it is generally true that in order to learn mathematics one must *do* mathematics, it is especially so here—the serious student should attempt to solve a large number of problems.

In this chapter, we explore four general principles and some of the counting formulas that they imply. Each of these principles gives a complementary principle, which we also discuss. We conclude with an application of counting to finite probability.

2.1 Four Basic Counting Principles

The first principle[1] is very basic. It is one formulation of the principle that the whole is equal to the sum of its parts.

Let S be a set. A *partition* of S is a collection $S_1, S_2, \ldots, S_m$ of subsets of S such that each element of S is in exactly one of those subsets:

$$S = S_1 \cup S_2 \cup \cdots \cup S_m,$$

$$S_i \cap S_j = \emptyset, \quad (i \neq j).$$

Thus, the sets $S_1, S_2, \ldots, S_m$ are pairwise disjoint sets, and their union is S. The subsets $S_1, S_2, \ldots, S_m$ are called the *parts* of the partition. We note that by this definition a part of a partition may be empty, but usually there is no advantage in

[1]According to the *The Random House College Dictionary, Revised Edition*, 1997, a *principle* is (1) an accepted or professed rule of action or conduct, (2) a basic law, axiom, or doctrine. Our principles in this section are basic *laws* of mathematics and important *rules of action* for solving counting problems.

considering partitions with one or more empty parts. The *number of objects of a set* S is denoted by $|S|$ and is sometimes called the *size* of S.

Addition Principle. *Suppose that a set S is partitioned into pairwise disjoint parts $S_1, S_2, \ldots, S_m$. The number of objects in S can be determined by finding the number of objects in each of the parts, and adding the numbers so obtained:*

$$|S| = |S_1| + |S_2| + \cdots + |S_m|.$$

If the sets $S_1, S_2, \ldots, S_m$ are allowed to overlap, then a more profound principle, the inclusion–exclusion principle of Chapter 6, can be used to count the number of objects in S.

In applying the addition principle, we usually define the parts descriptively. In other words, we break up the problem into mutually exclusive cases that exhaust all possibilities. The art of applying the addition principle is to partition the set S to be counted into "manageable parts"—that is, parts which we can readily count. But this statement needs to be qualified. If we partition S into too many parts, then we may have defeated ourselves. For instance, if we partition S into parts each containing only one element, then applying the addition principle is the same as counting the number of parts, and this is basically the same as listing all the objects of S. Thus, a more appropriate description is that *the art of applying the addition principle is to partition the set S into not too many manageable parts.*

Example. Suppose we wish to find the number of different courses offered by the University of Wisconsin–Madison. We partition the courses according to the department in which they are listed. *Provided there is no cross-listing* (cross-listing occurs when the same course is listed by more than one department), the number of courses offered by the University equals the sum of the number of courses offered by each department. □

Another formulation of the addition principle in terms of choices is the following: *If an object can be selected from one pile in p ways and an object can be selected from a separate pile in q ways, then the selection of one object chosen from either of the two piles can be made in $p + q$ ways.* This formulation has an obvious generalization to more than two piles.

Example. A student wishes to take either a mathematics course or a biology course, but not both. If there are four mathematics courses and three biology courses for which the student has the necessary prerequisites, then the student can choose a course to take in $4 + 3 = 7$ ways. □

The second principle is a little more complicated. We state it for two sets, but it can also be generalized to any finite number of sets.

Multiplication Principle. *Let S be a set of ordered pairs (a, b) of objects, where the first object a comes from a set of size p, and for each choice of object a there are q*

choices for object b. Then the size of S is $p \times q$:

$$|S| = p \times q.$$

The multiplication principle is actually a consequence of the addition principle. Let $a_1, a_2, \ldots, a_p$ be the p different choices for the object a. We partition S into parts $S_1, S_2, \ldots, S_p$ where S_i is the set of ordered pairs in S with first object a_i, $(i = 1, 2, \ldots, p)$. The size of each S_i is q; hence, by the addition principle,

$$\begin{aligned} |S| &= |S_1| + |S_2| + \cdots + |S_p| \\ &= q + q + \cdots + q \quad (p\ q\text{'s}) \\ &= p \times q. \end{aligned}$$

Note how the basic fact—multiplication of whole numbers is just repeated addition—enters into the preceding derivation.

A second useful formulation of the multiplication principle is as follows: *If a first task has p outcomes and, no matter what the outcome of the first task, a second task has q outcomes, then the two tasks performed consecutively have* $p \times q$ *outcomes.*

Example. A student is to take two courses. The first meets at any one of 3 hours in the morning, and the second at any one of 4 hours in the afternoon. The number of schedules that are possible for the student is $3 \times 4 = 12$. □

As already remarked, the multiplication principle can be generalized to three, four, or any finite number of sets. Rather than formulate it in terms of n sets, we give examples for $n = 3$ and $n = 4$.

Example. Chalk comes in three different lengths, eight different colors, and four different diameters. How many different kinds of chalk are there?

To determine a piece of chalk of a specific type, we carry out three different tasks (it does not matter in which order we take these tasks): Choose a length, Choose a color, Choose a diameter. By the multiplication principle, there are $3 \times 8 \times 4 = 96$ different kinds of chalk. □

Example. The number of ways a man, woman, boy, and girl can be selected from five men, six women, two boys, and four girls is $5 \times 6 \times 2 \times 4 = 240$.

The reason is that we have four different tasks to carry out: select a man (five ways), select a woman (six ways), select a boy (two ways), select a girl (four ways). If, in addition, we ask for the number of ways one person can be selected, the answer is $5 + 6 + 2 + 4 = 17$. This follows from the addition principle for four piles. □

Example. Determine the number of positive integers that are factors of the number

$$3^4 \times 5^2 \times 11^7 \times 13^8.$$

The numbers 3, 5, 11, and 13 are prime numbers. By the *fundamental theorem of arithmetic*, each factor is of the form

$$3^i \times 5^j \times 11^k \times 13^l,$$

where $0 \leq i \leq 4$, $0 \leq j \leq 2$, $0 \leq k \leq 7$, and $0 \leq l \leq 8$. There are five choices for i, three for j, eight for k, and nine for l. By the multiplication principle, the number of factors is

$$5 \times 3 \times 8 \times 9 = 1080.$$

□

In the multiplication principle the q choices for object b may vary with the choice of a. The only requirement is that there be the *same number* q of choices, not necessarily the same choices.

Example. How many two-digit numbers have distinct and nonzero digits?

A two-digit number ab can be regarded as an ordered pair (a, b), where a is the tens digit and b is the units digit. Neither of these digits is allowed to be 0 in the problem, and the two digits are to be different. There are nine choices for a, namely $1, 2, \ldots, 9$. Once a is chosen, there are eight choices for b. If $a = 1$, these eight choices are $2, 3, \ldots, 9$, if $a = 2$, the eight choices are $1, 3, \ldots, 9$, and so on. What is important for application of the multiplication principle is that the number of choices is always 8. The answer to the questions is, by the multiplication principle, $9 \times 8 = 72$.

We can arrive at the answer 72 in another way. There are 90 two-digit numbers, $10, 11, 12, \ldots, 99$. Of these numbers, nine have a 0, (namely, $10, 20, \ldots, 90$) and nine have identical digits (namely, $11, 22, \ldots, 99$). Thus the number of two-digit numbers with distinct and nonzero digits equals $90 - 9 - 9 = 72$. □

The preceding example illustrates two ideas. One is that there may be more than one way to arrive at the answer to a counting question. The other idea is that to find the number of objects in a set A (in this case the set of two-digit numbers with distinct and nonzero digits) it may be easier to find the number of objects in a larger set U containing S (the set of all two-digit numbers in the preceding example) and then subtract the number of objects of U that do not belong to A (the two-digit numbers containing a 0 or identical digits). We formulate this idea as our third principle.

Subtraction Principle. Let A be a set and let U be a larger set containing A. Let

$$\overline{A} = U \setminus A = \{x \in U : x \notin A\}$$

be the *complement of* A *in* U. Then the number $|A|$ of objects in A is given by the rule

$$|A| = |U| - |\overline{A}|.$$

In applying the subtraction principle, the set U is usually some natural set consisting of all the objects under discussion (the so-called *universal set*). Using the subtraction principle makes sense only if it is easier to count the number of objects in U and in $\overline{A}$ than to count the number of objects in A.

Example. Computer passwords are to consist of a string of six symbols taken from the digits $0, 1, 2, \ldots, 9$ and the lowercase letters $a, b, c, \ldots, z$. How many computer passwords have a repeated symbol?

We want to count the number of objects in the set A of computer passwords with a repeated symbol. Let U be the set of all computer passwords. Taking the complement of A in U we get the set $\overline{A}$ of computer passwords with no repeated symbol. By two applications of the multiplication principle, we get

$$|U| = 36^6 = 2,176,782,336$$

and

$$|\overline{A}| = 36 \cdot 35 \cdot 34 \cdot 33 \cdot 32 \cdot 31 = 1,402,410,240.$$

Therefore,

$$|A| = |U| - |\overline{A}| = 2,176,782,336 - 1,402,410,240 = 774,372,096.$$

□

We now formulate the final principle of this section.

Division Principle. Let S be a finite set that is partitioned into k parts in such a way that each part contains the same number of objects. Then the number of parts in the partition is given by the rule

$$k = \frac{|S|}{\text{number of objects in a part}}.$$

Thus, we can determine the number of parts if we know the number of objects in S and the common value of the number of objects in the parts.

Example. There are 740 pigeons in a collection of pigeonholes. If each pigeonhole contains 5 pigeons, the number of pigeonholes equals

$$\frac{740}{5} = 148.$$

□

More profound applications of the division principle will occur later in this book. Now consider the next example.

Example. You wish to give your Aunt Mollie a basket of fruit. In your refrigerator you have six oranges and nine apples. The only requirement is that there must be at least one piece of fruit in the basket (that is, an empty basket of fruit is not allowed). How many different baskets of fruit are possible?

One way to count the number of baskets is the following: First, ignore the requirement that the basket cannot be empty. We can compensate for that later. What distinguishes one basket of fruit from another is the number of oranges and number of apples in the basket. There are 7 choices for the number of oranges $(0, 1, \ldots, 6)$ and 10 choices for the number of apples $(0, 1, \ldots, 9)$. By the multiplication principle, the number of different baskets is $7 \times 10 = 70$. Subtracting the empty basket, the answer is 69. Notice that if we had not (temporarily) ignored the requirement that the basket be nonempty, then there would have been 9 or 10 choices for the number of apples depending on whether or not the number of oranges was 0, and we could not have applied the multiplication principle directly. But an alternative solution is the following. Partition the nonempty baskets into two parts, S_1 and S_2, where S_1 consists of those baskets with no oranges and S_2 consists of those baskets with at least one orange. The size of S_1 is 9 $(1, 2, \ldots, 9$ apples) and the size of S_2 by the foregoing reasoning is $6 \times 10 = 60$. The number of possible baskets of fruit is, by the addition principle, $9 + 60 = 69$. □

We made an implicit assumption in the preceding example which we should now bring into the open. It was assumed in the solution that the oranges were indistinguishable from one another (an orange is an orange is an orange is ...) and that the apples were indistinguishable from one another. Thus, what mattered in making up a basket of fruit was *not* which apples and which oranges went into it but only the *number* of each type of fruit. If we distinguished among the various oranges and the various apples (one orange is perfectly round, another is bruised, a third very juicy, and so on), then the number of baskets would be larger. We will return to this example in Section 3.5.

Before continuing with more examples, we discuss some general ideas.

A great many counting problems can be classified as one of the following types:

(1) Count the number of *ordered* arrangements or *ordered* selections of objects

 (a) without repeating any object,

 (b) with repetition of objects permitted (but perhaps limited).

(2) Count the number of *unordered* arrangements or *unordered* selections of objects

 (a) without repeating any object,

 (b) with repetition of objects permitted (but perhaps limited).

Instead of distinguishing between nonrepetition and repetition of objects, it is sometimes more convenient to distinguish between selections from a set and a multiset. A *multiset* is like a set except that its members need not be distinct.[2] For example,

[2]Thus a multiset breaks one of the cardinal rules of sets, namely, elements are not repeated in sets; they are either in the set or not in the set. The set $\{a, a, b\}$ is the same as the set $\{a, b\}$ but not so for multisets.

we might have a multiset M with three a's, one b, two c's, and four d's, that is, 10 elements of 4 different types: 3 of type a, 1 of type b, 2 of type c, and 4 of type d. We shall usually indicate a multiset by specifying the number of times different types of elements occur in it. Thus, M shall be denoted by $\{3 \cdot a, 1 \cdot b, 2 \cdot c, 4 \cdot d\}$.[3] The numbers $3, 1, 2$, and 4 are the *repetition numbers* of the multiset M. A set is a multiset that has all repetition numbers equal to 1. To include the listed case (b) when there is no limit on the number of times an object of each type can occur (except for that imposed by the size of the arrangement), we allow infinite repetition numbers.[4] Thus, a multiset in which a and c each have an infinite repetition number and b and d have repetition numbers 2 and 4, respectively, is denoted by $\{\infty \cdot a, 2 \cdot b, \infty \cdot c, 4 \cdot d\}$. Arrangements or selections in (1) in which order is taken into consideration are generally called *permutations*, whereas arrangements or selections in (2) in which order is irrelevant are generally called *combinations*. In the next two sections we will develop some general formulas for the number of permutations and combinations of sets and multisets. But not all permutation and combination problems can be solved by using these formulas. It is often necessary to return to the basic addition, multiplication, subtraction, and division principles.

Example. How many odd numbers between 1000 and 9999 have distinct digits?

A number between 1000 and 9999 is an *ordered* arrangement of four digits. Thus we are asked to count a certain collection of permutations. We have four choices to make: a units, a tens, a hundreds, and a thousands digit. Since the numbers we want to count are odd, the units digit can be any one of $1, 3, 5, 7, 9$. The tens and the hundreds digit can be any one of $0, 1, \ldots, 9$, while the thousands digit can be any one of $1, 2, \ldots, 9$. Thus, there are five choices for the units digit. Since the digits are to be distinct, we have eight choices for the thousands digit, whatever the choice of the units digit. Then, there are eight choices for the hundreds digit, whatever the first two choices were, and seven choices for the tens digit, whatever the first three choices were. Thus, by the multiplication principle, the answer to the question is $5 \times 8 \times 8 \times 7 = 2240$. □

Suppose in the previous example we made the choices in a different order: First choose the thousands digit, then the hundreds, tens, and units. There are nine choices for the thousands digit, then nine choices for the hundreds digit (since we are allowed to use 0), eight choices for the tens digit, but now the number of choices for the units digit (which has to be odd) *depends* on the previous choices. If we had chosen no odd digits, the number of choices for the units digit would be 5; if we had chosen one odd digit, the number of choices for the units digit would be 4; and so on. Thus, we cannot invoke the multiplication principle if we carry out our choices in the reverse order. There are two lessons to learn from this example. One is that as soon as your

[3] If we wanted to follow standard set-theoretic notation, we could designate the multiset M using ordered pairs as $\{(a, 3), (b, 1), (c, 2), (d, 4)\}$.

[4] There are no circumstances in which we will have to worry about different sizes of infinity.

answer for the number of choices of one of the tasks is "it depends" (or some such words), the multiplication principle cannot be applied. The second is that there may not be a fixed order in which the tasks have to be taken, and by changing the order a problem may be more readily solved by the multiplication principle. A rule of thumb to keep in mind is to make the most restrictive choice first.

Example. How many integers between 0 and 10,000 have only one digit equal to 5?

Let S be the set of integers between 0 and 10,000 with only one digit equal to 5. *First solution*: We partition S into the set S_1 of one-digit numbers in S, the set S_2 of two-digit numbers in S, the set S_3 of three-digit numbers in S, and the set S_4 of four-digit numbers in S. There are no five-digit numbers in S. We clearly have

$$|S_1| = 1.$$

The numbers in S_2 naturally fall into two types: (1) the units digit is 5, and (2) the tens digit is 5. The number of the first type is 8 (the tens digit cannot be 0 nor can it be 5). The number of the second type is 9 (the units digit cannot be 5). Hence,

$$|S_2| = 8 + 9 = 17.$$

Reasoning in a similar way, we obtain

$$|S_3| = 8 \times 9 + 8 \times 9 + 9 \times 9 = 225, \text{ and}$$

$$|S_4| = 8 \times 9 \times 9 + 8 \times 9 \times 9 + 8 \times 9 \times 9 + 9 \times 9 \times 9 = 2673.$$

Thus,

$$|S| = 1 + 17 + 225 + 2673 = 2916.$$

Second solution: By including leading zeros (e.g., think of 6 as 0006, 25 as 0025, 352 as 0352), we can regard each number in S as a four-digit number. Now we partition S into the sets S'_1, S'_2, S'_3, S'_4 according to whether the 5 is in the first, second, third, or fourth position. Each of the four sets in the partition contains $9 \times 9 \times 9 = 729$ integers, and so the number of integers in S equals

$$4 \times 729 = 2916.$$

□

Example. How many different five-digit numbers can be constructed out of the digits 1, 1, 1, 3, 8?

Here we are asked to count permutations of a multiset with three objects of one type, one of another, and one of a third. We really have only two choices to make: which position is to be occupied by the 3 (five choices) and then which position is to be occupied by the 8 (four choices). The remaining three places are occupied by 1s. By the multiplication principle, the answer is $5 \times 4 = 20$.

If the five digits are 1, 1, 1, 3, 3, the answer is 10, half as many. □

These examples clearly demonstrate that mastery of the addition and multiplication principles is essential for becoming an expert counter.

2.2 Permutations of Sets

Let r be a positive integer. By an *r-permutation* of a set S of n elements, we understand an ordered arrangement of r of the n elements. If $S = \{a, b, c\}$, then the three 1-permutations of S are

$$a \quad b \quad c,$$

the six 2-permutations of S are

$$ab \quad ac \quad ba \quad bc \quad ca \quad cb,$$

and the six 3-permutations of S are

$$abc \quad acb \quad bac \quad bca \quad cab \quad cba.$$

There are no 4-permutations of S since S has fewer than four elements.

We denote by $P(n, r)$ the number of r-permutations of an n-element set. If $r > n$, then $P(n, r) = 0$. Clearly $P(n, 1) = n$ for each positive integer n. An n-permutation of an n-element set S will be more simply called a *permutation of S* or a *permutation of n elements*. Thus, a *permutation of a set S can be thought of as a listing of the elements of S in some order.* Previously we saw that $P(3, 1) = 3, P(3, 2) = 6$, and $P(3, 3) = 6$.

Theorem 2.2.1 *For n and r positive integers with $r \leq n$,*

$$P(n, r) = n \times (n-1) \times \cdots \times (n-r+1).$$

Proof. In constructing an r-permutation of an n-element set, we can choose the first item in n ways, the second item in $n-1$ ways, whatever the choice of the first item, . . . , and the rth item in $n-(r-1)$ ways, whatever the choice of the first $r-1$ items. By the multiplication principle the r items can be chosen in $n \times (n-1) \times \cdots \times (n-r+1)$ ways. □

For a nonnegative integer n, we define $n!$ (read *n factorial*) by

$$n! = n \times (n-1) \times \cdots \times 2 \times 1,$$

with the convention that $0! = 1$. We may then write

$$P(n,r) = \frac{n!}{(n-r)!}.$$

For $n \geq 0$, we define $P(n,0)$ to be 1, and this agrees with the formula when $r = 0$. The number of permutations of n elements is

$$P(n,n) = \frac{n!}{0!} = n!.$$

Example. The number of four-letter "words" that can be formed by using each of the letters a, b, c, d, e at most once is $P(5,4)$, and this equals $5!/(5-4)! = 120$. The number of five-letter words equals $P(5,5)$, which is also 120. □

Example. The so-called "15 puzzle" consists of 15 sliding unit squares labeled with the numbers 1 through 15 and mounted in a 4-by-4 square frame as shown in Figure 2.1. The challenge of the puzzle is to move from the initial position shown to any specified position. (That challenge is not the subject of this problem.) By a position, we mean an arrangement of the 15 numbered squares in the frame with one empty unit square. What is the number of positions in the puzzle (ignoring whether it is possible to move to the position from the initial one)?

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	

Figure 2.1

The problem is equivalent to determining the number of ways to assign the numbers $1, 2, \ldots, 15$ to the 16 squares of a 4-by-4 grid, leaving one square empty. Since we can assign the number 16 to the empty square, the problem is also equivalent to determining the number of assignments of the numbers $1, 2, \ldots, 16$ to the 16 squares, and this is $P(16,16) = 16!$.

What is the number of ways to assign the numbers $1, 2, \ldots, 15$ to the squares of a 6-by-6 grid, leaving 21 squares empty? These assignments correspond to the 15-permutations of the 36 squares as follows: To an assignment of the numbers $1, 2, \ldots, 15$ to 15 of the squares, we associate the 15-permutation of the 36 squares obtained by putting the square labeled 1 first, the square labeled 2 second, and so on. Hence the total number of assignments is $P(36,15) = 36!/21!$. □

Example. What is the number of ways to order the 26 letters of the alphabet so that no two of the vowels a, e, i, o, and u occur consecutively?

The solution to this problem (like so many counting problems) is straightforward once we see how to do it. We think of two main tasks to be accomplished. The first task is to decide how to order the consonants among themselves. There are 21 consonants, and so 21! permutations of the consonants. Since we cannot have two consecutive vowels in our final arrangement, the vowels must be in 5 of the 22 spaces before, between, and after the consonants. Our second task is to put the vowels in these places. There are 22 places for the a, then 21 for the e, 20 for the i, 19 for the o, and 18 for the u. That is, the second task can be accomplished in

$$P(22,5) = \frac{22!}{17!}$$

ways. By the multiplication principle, we determine that the number of ordered arrangements of the letters of the alphabet with no two vowels consecutive is

$$21! \times \frac{22!}{17!}.$$

□

Example. How many seven-digit numbers are there such that the digits are distinct integers taken from $\{1, 2, \ldots, 9\}$ and such that the digits 5 and 6 do not appear consecutively in either order?

We want to count certain 7-permutations of the set $\{1, 2, \ldots, 9\}$, and we partition these 7-permutations into four types: (1) neither 5 nor 6 appears as a digit; (2) 5, but not 6, appears as a digit; (3) 6, but not 5, appears as a digit; (4) both 5 and 6 appear as digits. The permutations of type (1) are the 7-permutations of $\{1, 2, 3, 4, 7, 8, 9\}$, and hence their number is $P(7, 7) = 7! = 5040$. The permutations of type (2) can be counted as follows: The digit equal to 5 can be any one of the seven digits. The remaining six digits are a 6-permutation of $\{1, 2, 3, 4, 7, 8, 9\}$. Hence there are $7P(7, 6) = 7(7!) = 35,280$ numbers of type (2). In a similar way we see that there are 35,280 numbers of type (3). To count the number of permutations of type (4), we partition the permutations of type (4) into three parts:

First digit equal to 5, and so second digit not equal to 6:

$$\underline{\ 5\ }\ \ \underline{\neq 6}\ \ \underline{\quad}\ \ \underline{\quad}\ \ \underline{\quad}\ \ \underline{\quad}\ \ \underline{\quad}\ .$$

There are five places for the 6. The other five digits constitute a 5-permutation of the 7 digits $\{1, 2, 3, 4, 7, 8, 9\}$. Hence, there are

$$5 \times P(7, 5) = \frac{5 \times 7!}{2!} = 12,600$$

numbers in this part.

Last digit equal to 5, and so next to last digit not equal to 6:

$$\underline{\quad}\ \underline{\quad}\ \underline{\quad}\ \underline{\quad}\ \underline{\quad}\ \underline{\neq 6}\ \underline{\ 5\ }\ .$$

By an argument similar to the preceding, we conclude that there are also 12,600 numbers in this part.

A digit other than the first or last is equal to 5:

$$\underline{\quad}\ \underline{\quad}\ \underline{\neq 6}\ \underline{\ 5\ }\ \underline{\neq 6}\ \underline{\quad}\ \underline{\quad}\ .$$

The place occupied by 5 is any one of the five interior places. The place for the 6 can then be chosen in four ways. The remaining five digits constitute a 5-permutation of the seven digits $\{1,2,3,4,7,8,9\}$. Hence, there are $5 \times 4 \times P(7,5) = 50,400$ numbers in this category. Thus, there are

$$2(12{,}600) + 50{,}400 = 75{,}600$$

numbers of types (4). By the addition principle, the answer to the problem posed is

$$5040 + 2(35{,}280) + 75{,}600 = 151{,}200.$$

The solution just given was arrived at by partitioning the set of objects we wanted to count into manageable parts, parts the number of whose objects we could calculate, and then using the addition principle. An alternative, and computationally easier, solution is to use the subtraction principle as follows. Let us consider the entire collection T of seven-digit numbers that can be formed by using distinct integers from $\{1,2,\ldots,9\}$. The set T then contains

$$P(9,7) = \tfrac{9!}{2!} = 181,440$$

numbers. Let S consist of those numbers in T in which 5 and 6 do not occur consecutively; so the complement $\overline{S}$ consists of those numbers in T in which 5 and 6 do occur consecutively. We wish to determine the size of S. If we can find the size of $\overline{S}$, then our problem is solved by the subtraction principle. How many numbers are there in $\overline{S}$? In $\overline{S}$, the digits 5 and 6 occur consecutively. There are six ways to position a 5 followed by a 6, and six ways to position a 6 followed by a 5. The remaining digits constitute a 5-permutation of $\{1,2,3,4,7,8,9\}$. So the number of numbers in $\overline{S}$ is

$$2 \times 6 \times P(7,5) = 30,240.$$

But then S contains $181,440 - 30,240 = 151,200$ numbers.

The permutations that we have just considered are more properly called *linear permutations*. We think of the objects as being arranged in a line. If instead of arranging objects in a line, we arrange them in a circle, the number of permutations is smaller. Think of it this way: Suppose six children are marching in a circle. In how many different ways can they form their circle? Since the children are moving, what matters are their positions relative to each other and not to their environment. Thus, it is natural to regard two circular permutations as being the same provided one can be brought to the other by a rotation, that is, by a circular shift. There are six linear permutations for each circular permutation. For example, the circular permutation

$$\begin{array}{ccc} & 1 & \\ 2 & & 6 \\ 3 & & 5 \\ & 4 & \end{array}$$

arises from each of the linear permutations

$$\begin{array}{ccc} 123456 & 234561 & 345612 \\ 456123 & 561234 & 612345 \end{array}$$

by regarding the last digit as coming before the first digit. Thus, there is a 6-to-1 correspondence between the linear permutations of six children and the circular permutations of the six children. Therefore, to find the number of circular permutations, we divide the number of linear permutations by 6. Hence, the number of circular permutations of the six children equals $6!/6 = 5!$.

Theorem 2.2.2 *The number of circular r-permutations of a set of n elements is given by*

$$\frac{P(n,r)}{r} = \frac{n!}{r \cdot (n-r)!}.$$

In particular, the number of circular permutations of n elements is $(n-1)!$.

Proof. A proof is essentially contained in the preceding paragraph and uses the division principle. The set of linear r-permutations can be partitioned into parts in such a way that two linear r-permutations correspond to the same circular r-permutation if and only if they are in the same part. Thus, the number of circular r-permutations equals the number of parts. Since each part contains r linear r-permutations, the number of parts is given by

$$\frac{P(n,r)}{r} = \frac{n!}{r \cdot (n-r)!}.$$

□

For emphasis, we remark that the preceding argument worked because each part contained the same number of r-permutations so that we could apply the division principle to determine the number of parts. If, for example, we partition a set of 10 objects into parts of sizes 2, 4, and 4, respectively, the number of parts cannot be obtained by dividing 10 by 2 or 4.

Another way to view the counting of circular permutations is the following: Suppose we wish to count the number of circular permutations of A, B, C, D, E, and F (the number of ways to seat A, B, C, D, E, and F around a table). Since we are free to rotate the people, any circular permutation can be rotated so that A is in a fixed position; think of it as the "head" of the table:

$$\begin{array}{ccc} & A & \\ D & & C \\ F & & B \\ & E & \end{array}$$

Now that A is fixed, the circular permutations of A, B, C, D, E, and F can be identified with the linear permutations of B, C, D, E, and F. (The preceding circular permutation is identified with the linear permutation $DFEBC$.) There are 5! linear permutations of B, C, D, E, and F, and hence 5! circular permutations of A, B, C, D, E, and F.

This way of looking at circular permutations is also useful when the formula for circular permutations cannot be applied directly.

Example. Ten people, including two who do not wish to sit next to one another, are to be seated at a round table. How many circular seating arrangements are there?

We solve this problem using the subtraction principle. Let the 10 people be $P_1, P_2, P_3, \ldots, P_{10}$, where P_1 and P_2 are the two who do not wish to sit together. Consider seating arrangements for 9 people $X, P_3, \ldots, P_{10}$ at a round table. There are 8! such arrangements. If we replace X by either P_1, P_2 or by P_2, P_1 in each of these arrangements, we obtain a seating arrangement for the 10 people in which P_1 and P_2 are next to one another. Hence using the subtraction principle, we see that the number of arrangements in which P_1 and P_2 are not together is $9! - 2 \times 8! = 7 \times 8!$.

Another way to analyze this problem is the following: First seat P_1 at the "head" of the table. Then P_2 cannot be on either side of P_1. There are 8 choices for the person on P_1's left, 7 choices for the person on P_1's right, and the remaining seats can be filled in 7! ways. Thus, the number of seating arrangements in which P_1 and P_2 are not together is

$$8 \times 7 \times 7! = 7 \times 8!.$$

□

As we did before we discussed circular permutations, we will continue to use permutation to mean "linear permutation."

Example. The number of ways to have 12 different markings on a rotating drum is $P(12, 12)/12 = 11!$. □

Example. What is the number of necklaces that can be made from 20 beads, each of a different color?

There are 20! permutations of the 20 beads. Since each necklace can be rotated without changing the arrangement of the beads, the number of necklaces is at most $20!/20 = 19!$. Since a necklace can also be turned over without changing the arrangement of the beads, the total number of necklaces, by the division principle, is $19!/2$. □

Circular permutations and necklaces are counted again in Chapter 14, in a more general context.

2.3 Combinations (Subsets) of Sets

Let S be a set of n elements. A *combination* of a set S is a term usually used to denote an unordered selection of the elements of S. The result of such a selection is a *subset* A of the elements of S: $A \subseteq S$. Thus a combination of S is a choice of a subset of S. As a result, the terms *combination* and *subset* are essentially interchangeable, and we shall generally use the more familiar *subset* rather than perhaps the more awkward *combination*, unless we want to emphasize the selection process.

Now let r be a nonnegative integer. By an *r-combination* of a set S of n elements, we understand an unordered selection of r of the n objects of S. The result of an r-combination is an *r-subset* of S, a subset of S consisting of r of the n objects of S. Again, we generally use "r-subset" rather than "r-combination."

If $S = \{a, b, c, d\}$, then

$$\{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}$$

are the four 3-subsets of S. We denote by $\binom{n}{r}$ the number of r-subsets of an n-element set.[5] Obviously,

$$\binom{n}{r} = 0 \qquad \text{if } r > n.$$

Also,

$$\binom{0}{r} = 0 \qquad \text{if } r > 0.$$

The following facts are readily seen to be true for each nonnegative integer n:

$$\binom{n}{0} = 1, \ \binom{n}{1} = n, \ \binom{n}{n} = 1.$$

In particular, $\binom{0}{0} = 1$. The basic formula for the number of r-subsets is given in the next theorem.

[5]Other common notations for these numbers are $C(n, r)$ and ${}_nC_r$.

Theorem 2.3.1 *For $0 \le r \le n$,*

$$P(n,r) = r!\binom{n}{r}.$$

Hence,

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

Proof. Let S be an n-element set. Each r-permutation of S arises in exactly one way as a result of carrying out the following two tasks:

(1) Choose r elements from S.

(2) Arrange the chosen r elements in some order.

The number of ways to carry out the first task is, by definition, the number $\binom{n}{r}$. The number of ways to carry out the second task is $P(r,r) = r!$. By the multiplication principle, we have $P(n,r) = r! \binom{n}{r}$. We now use our formula $P(n,r) = \frac{n!}{(n-r)!}$ and obtain

$$\binom{n}{r} = \frac{P(n,r)}{r!} = \frac{n!}{r!(n-r)!}. \qquad \square$$

Example. Twenty-five points are chosen in the plane so that no three of them are collinear. How many straight lines do they determine? How many triangles do they determine?

Since no three of the points lie on a line, every pair of points determines a unique straight line. Thus, the number of straight lines determined equals the number of 2-subsets of a 25-element set, and this is given by

$$\binom{25}{2} = \frac{25!}{2!23!} = 300.$$

Similarly, every three points determines a unique triangle, so that the number of triangles determined is given by

$$\binom{25}{3} = \frac{25!}{3!22!}.$$

$\square$

Example. There are 15 people enrolled in a mathematics course, but exactly 12 attend on any given day. The number of different ways that 12 students can be chosen is

$$\binom{15}{12} = \frac{15!}{12!3!}.$$

If there are 25 seats in the classroom, the 12 students could seat themselves in $P(25,12) = 25!/13!$ ways. Thus, there are

$$\binom{15}{12}P(25,12) = \frac{15!25!}{12!3!13!}$$

ways in which an instructor might see the 12 students in the classroom. □

Example. How many eight-letter words can be constructed by using the 26 letters of the alphabet if each word contains three, four, or five vowels? It is understood that there is no restriction on the number of times a letter can be used in a word.

We count the number of words according to the number of vowels they contain and then use the addition principle.

First, consider words with three vowels. The three positions occupied by the vowels can be chosen in $\binom{8}{3}$ ways; the other five positions are occupied by consonants. The vowel positions can then be completed in 5^3 ways and the consonant positions in 21^5 ways. Thus, the number of words with three vowels is

$$\binom{8}{3}5^3 21^5 = \frac{8!}{3!5!}5^3 21^5.$$

In a similar way, we see that the number of words with four vowels is

$$\binom{8}{4}5^4 21^4 = \frac{8!}{4!4!}5^4 21^4,$$

and the number of words with five vowels is

$$\binom{8}{5}5^5 21^3 = \frac{8!}{5!3!}5^5 21^3.$$

Hence, the total number of words is

$$\frac{8!}{3!5!}5^3 21^5 + \frac{8!}{4!4!}5^4 21^4 + \frac{8!}{5!3!}5^5 21^3.$$

□

The following important property is immediate from Theorem 2.3.1:

Corollary 2.3.2 *For $0 \le r \le n$,*

$$\binom{n}{r} = \binom{n}{n-r}.$$

□

The numbers $\binom{n}{r}$ have many important and fascinating properties, and Chapter 5 is devoted to some of these. For the moment, we discuss only two basic properties.

Theorem 2.3.3 **(Pascal's formula)** *For all integers n and k with $1 \leq k \leq n-1$,*

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

Proof. One way to prove this identity is to substitute the values of these numbers as given in Theorem 2.3.1 and then check that both sides are equal. We leave this straightforward verification to the reader.

A *combinatorial proof* can be obtained as follows: Let S be a set of n elements. We distinguish one of the elements of S and denote it by x. Let $S \setminus \{x\}$ be the set obtained from S by removing the element x. We partition the set X of k-subsets of S into two parts, A and B. In A we put all those k-subsets which do not contain x. In B we put all the k-subsets which do contain x. The size of X is $|X| = \binom{n}{k}$; hence, by the addition principle,

$$\binom{n}{k} = |A| + |B|.$$

The k-subsets in A are exactly the k-subsets of the set $S \setminus \{x\}$ of $n-1$ elements; thus, the size of A is

$$|A| = \binom{n-1}{k}.$$

A k-subset in B can always be obtained by adjoining the element x to a $(k-1)$-subset of $S \setminus \{x\}$. Hence, the size of B satisfies

$$|B| = \binom{n-1}{k-1}.$$

Combining these facts, we obtain

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

□

To illustrate the proof, let $n = 5$, $k = 3$, and $S = \{x, a, b, c, d\}$. Then the 3-subsets of S in A are

$$\{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}.$$

These are the 3-subsets of the set $\{a, b, c, d\}$. The 3-subsets S in B are

$$\{x, a, b\}, \{x, a, c\}, \{x, a, d\}, \{x, b, c\}, \{x, b, d\}, \{x, c, d\}.$$

Upon deletion of the element x in these 3-subsets, we obtain

$$\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\},$$

the 2-subsets of $\{a, b, c, d\}$. Thus,

$$\binom{5}{3} = 10 = 4 + 6 = \binom{4}{3} + \binom{4}{2}.$$

Theorem 2.3.4 *For $n \geq 0$,*

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n,$$

and the common value equals the number of subsets of an n-element set.

Proof. We prove this theorem by showing that both sides of the preceding equation count the number of subsets of an n-element set S, but in different ways. First we observe that every subset of S is an r-subset of S for some $r = 0, 1, 2, \ldots, n$. Since $\binom{n}{r}$ equals the number of r-subsets of S, it follows from the addition principle that

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}$$

equals the number of subsets of S.

We can also count the number of subsets of S by breaking down the choice of a subset into n tasks: Let the elements of S be $x_1, x_2, \ldots, x_n$. In choosing a subset of S, we have two choices to make for each of the n elements: x_1 either goes into the subset or it doesn't, x_2 either goes into the subset or it doesn't, . . . , x_n either goes into the subset or it doesn't. Thus, by the multiplication principle, there are 2^n ways we can form a subset of S. We now equate the two counts and complete the proof. □

The proof of Theorem 2.3.4 is an instance of obtaining an identity by counting the objects of a set (in this case the subsets of a set of n elements) in two different ways and setting the results equal to one another. This technique of "double counting" is a powerful one in combinatorics, and we will see several other applications of it.

Example. The number of 2-subsets of the set $\{1, 2, \ldots, n\}$ of the first n positive integers is $\binom{n}{2}$. Partition the 2-subsets according to the largest integer they contain. For each $i = 1, 2, \ldots, n$, the number of 2-subsets in which i is the largest integer is $i - 1$ (the other integer can be any of $1, 2, \ldots, i - 1$). Equating the two counts, we obtain the identity

$$0 + 1 + 2 + \cdots + (n - 1) = \binom{n}{2} = \frac{n(n-1)}{2}.$$

□

2.4 Permutations of Multisets

If S is a multiset, an *r-permutation* of S is an ordered arrangement of r of the objects of S. If the total number of objects of S is n (counting repetitions), then an n-permutation of S will also be called a *permutation of S*. For example, if $S = \{2 \cdot a, 1 \cdot b, 3 \cdot c\}$, then

$$acbc \qquad cbcc$$

are 4-permutations of S, while

$$abccca$$

is a permutation of S. The multiset S has no 7-permutations since $7 > 2 + 1 + 3 = 6$, the number of objects of S. We first count the number of r-permutations of a multiset S, each of whose repetition number is infinite.

Theorem 2.4.1 *Let S be a multiset with objects of k different types, where each object has an infinite repetition number. Then the number of r-permutations of S is k^r.*

Proof. In constructing an r-permutation of S, we can choose the first item to be an object of any one of the k types. Similarly, the second item can be an object of any one of the k types, and so on. Since all repetition numbers of S are infinite, the number of different choices for any item is always k and it does not depend on the choices of any previous items. By the multiplication principle, the r items can be chosen in k^r ways. □

An alternative phrasing of the theorem is: The number of r-permutations of k distinct objects, each available in unlimited supply, equals k^r. We also note that the conclusion of the theorem remains true if the repetition numbers of the k different types of objects of S are all at least r. The assumption that the repetition numbers are infinite is a simple way of ensuring that we never run out of objects of any type.

Example. What is the number of ternary numerals[6] with at most four digits?

The answer to this question is the number of 4-permutations of the multiset $\{\infty \cdot 0, \infty \cdot 1, \infty \cdot 2\}$ or of the multiset $\{4 \cdot 0, 4 \cdot 1, 4 \cdot 2\}$. By Theorem 2.4.1, this number equals $3^4 = 81$. □

We now count permutations of a multiset with objects of k different types, each with a finite repetition number.

Theorem 2.4.2 *Let S be a multiset with objects of k different types with finite repetition numbers $n_1, n_2, \ldots, n_k$, respectively. Let the size of S be $n = n_1 + n_2 + \cdots + n_k$. Then the number of permutations of S equals*

$$\frac{n!}{n_1! n_2! \cdots n_k!}.$$

[6] A *ternary numeral*, or base 3 numeral, is one arrived at by representing a number in terms of powers of 3. For instance, $46 = 1 \times 3^3 + 2 \times 3^2 + 0 \times 3^1 + 1 \times 3^0$, and so its ternary numeral is 1201.

Proof. We are given a multiset S having objects of k types, say $a_1, a_2, \ldots, a_k$, with repetition numbers $n_1, n_2, \ldots, n_k$, respectively, for a total of $n = n_1 + n_2 + \cdots + n_k$ objects. We want to determine the number of permutations of these n objects. We can think of it this way. There are n places, and we want to put exactly one of the objects of S in each of the places. We first decide which places are to be occupied by the a_1's. Since there are n_1 a_1's in S, we must choose a subset of n_1 places from the set of n places. We can do this in $\binom{n}{n_1}$ ways. We next decide which places are to be occupied by the a_2's. There are $n - n_1$ places left, and we must choose n_2 of them. This can be done in $\binom{n-n_1}{n_2}$ ways. We next find that there are $\binom{n-n_1-n_2}{n_3}$ ways to choose the places for the a_3's. We continue like this, and invoke the multiplication principle and find that the number of permutations of S equals

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n-n_1-n_2-\cdots-n_{k-1}}{n_k}.$$

Using Theorem 2.3.1, we see that this number equals

$$\frac{n!}{n_1!(n-n_1)!}\frac{(n-n_1)!}{n_2!(n-n_1-n_2)!}\frac{(n-n_1-n_2)!}{n_3!(n-n_1-n_2-n_3)!}\cdots$$
$$\cdots\frac{(n-n_1-n_2-\cdots-n_{k-1})!}{n_k!(n-n_1-n_2-\cdots-n_k)!},$$

which, after cancellation, reduces to

$$\frac{n!}{n_1!n_2!n_3!\cdots n_k!0!} = \frac{n!}{n_1!n_2!n_3!\cdots n_k!}.$$

□

Example. The number of permutations of the letters in the word MISSISSIPPI is

$$\frac{11!}{1!4!4!2!},$$

since this number equals the number of permutations of the multiset $\{1 \cdot M, 4 \cdot I, 4 \cdot S, 2 \cdot P\}$. □

If the multiset S has only two types, a_1 and a_2, of objects with repetition numbers n_1 and n_2, respectively, where $n = n_1 + n_2$, then according to Theorem 2.4.2, the number of permutations of S is

$$\frac{n!}{n_1!n_2!} = \frac{n!}{n_1!(n-n_1)!} = \binom{n}{n_1}.$$

Thus *we may regard* $\binom{n}{n_1}$ *as the number of* n_1*-subsets of a set of* n *objects, and also as the number of permutations of a multiset with two types of objects with repetition numbers* n_1 *and* $n - n_1$*, respectively.*

There is another interpretation of the numbers $\frac{n!}{n_1!n_2!\cdots n_k!}$ that occur in Theorem 2.4.2. This concerns the problem of partitioning a set of objects into parts of prescribed sizes *where the parts now have labels assigned to them.* To understand the implications of the last phrase, we offer the next example.

Example. Consider a set of the four objects $\{a, b, c, d\}$ that is to be partitioned into two sets, each of size 2. If the parts are not labeled, then there are three different partitions:

$$\{a, b\}, \{c, d\}; \quad \{a, c\}, \{b, d\}; \quad \{a, d\}, \{b, c\}.$$

Now suppose that the parts are labeled with different labels (e.g,. the colors red and blue). Then the number of partitions is greater; indeed, there are six, since we can assign the labels red and blue to each part of a partition in two ways. For instance, for the particular partition $\{a, b\}, \{c, d\}$ we have

$$\text{red box}\{a, b\}, \text{blue box}\{c, d\}$$

and

$$\text{blue box}\{a, b\}, \text{red box}\{c, d\}.$$

□

In the general case, we can label the parts $B_1, B_2, \ldots, B_k$ (thinking of color 1, color 2, ..., color k), and we also think of the parts as boxes. We then have the following result.

Theorem 2.4.3 *Let n be a positive integer and let $n_1, n_2, \ldots, n_k$ be positive integers with $n = n_1 + n_2 + \cdots + n_k$. The number of ways to partition a set of n objects into k labeled boxes in which Box 1 contains n_1 objects, Box 2 contains n_2 objects, ..., Box k contains n_k objects equals*

$$\frac{n!}{n_1!n_2!\cdots n_k!}.$$

If the boxes are not labeled, and $n_1 = n_2 = \cdots = n_k$, then the number of partitions equals

$$\frac{n!}{k!n_1!n_2!\cdots n_k!}.$$

Proof. The proof is a direct application of the multiplication principle. We have to choose which objects go into which boxes, subject to the size restrictions. We first choose n_1 objects for the first box, then n_2 of the remaining $n - n_1$ objects for the second box, then n_3 of the remaining $n - n_1 - n_2$ objects for the third box, ..., and finally $n - n_1 - \cdots - n_{k-1} = n_k$ objects for the kth box. By the multiplication principle, the number of ways to make these choices is

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n-n_1-n_2-\cdots-n_{k-1}}{n_k}.$$

As in the proof of Theorem 2.4.2, this gives

$$\frac{n!}{n_1!n_2!\cdots n_k!}.$$

If boxes are not labeled and $n_1 = n_2 = \cdots = n_k$, then the result has to be divided by $k!$. This is so because, as in the preceding example, for each way of distributing the objects into the k unlabeled boxes there are $k!$ ways in which we can now attach the labels $1, 2, \ldots, k$. Hence, using the division principle, we find that the number of partitions with unlabeled boxes is

$$\frac{n!}{k!n_1!n_2!\cdots n_k!}.$$

□

The more difficult problem of counting partitions in which the sizes of the parts are not prescribed is studied in Section 8.2.

We conclude this section with an example of a kind that we shall refer to many times in the remainder of the text.[7] The example concerns nonattacking rooks on a chessboard. Lest the reader be concerned that knowledge of chess is a prerequisite for the rest of the book, let us say at the outset that the only fact needed about the game of chess is that *two rooks can attack one another if and only if they lie in the same row or the same column of the chessboard.* No other knowledge of chess is necessary (nor does it help!). Thus, a set of nonattacking rooks on a chessboard simply means a collection of "pieces" called rooks that occupy certain squares of the board, and no two of the rooks lie in the same row or in the same column.

Example. How many possibilities are there for eight nonattacking rooks on an 8-by-8 chessboard?

An example of eight nonattacking rooks on an 8-by-8 board is the following:

					⊗		
							⊗
⊗							
		⊗					
				⊗			
						⊗	
	⊗						
			⊗				

We give each square on the board a pair (i, j) of coordinates. The integer i designates the row number of the square, and the integer j designates the column number

[7]It is the author's favorite kind of example to illustrate many ideas.

of the square. Thus, i and j are integers between 1 and 8. Since the board is 8-by-8 and there are to be eight rooks on the board that cannot attack one another, there must be exactly one rook in each row. Thus, the rooks must occupy eight squares with coordinates

$$(1, j_1), (2, j_2), \ldots, (8, j_8).$$

But there must also be exactly one rook in each column so that no two of the numbers $j_1, j_2, \ldots, j_8$ can be equal. More precisely,

$$j_1, j_2, \ldots, j_8$$

must be a permutation of $\{1, 2, \ldots, 8\}$. Conversely, if $j_1, j_2, \ldots, j_8$ is a permutation of $\{1, 2, \ldots, 8\}$, then putting rooks in the squares with coordinates $(1, j_1), (2, j_2), \ldots, (8, j_8)$ we arrive at eight nonattacking rooks on the board. Thus, we have a one-to-one correspondence between sets of 8 nonattacking rooks on the 8-by-8 board and permutations of $\{1, 2, \ldots, 8\}$. Since there are 8! permutations of $\{1, 2, \ldots, 8\}$, there are 8! ways to place eight rooks on an 8-by-8 board so that they are nonattacking.

We implicitly assumed in the preceding argument that the rooks were *indistinguishable* from one another, that is, they form a multiset of eight objects all of one type. Therefore, the only thing that mattered was which squares were occupied by rooks. If we have eight distinct rooks, say eight rooks each colored with one of eight different colors, then we have also to take into account which rook is in each of the eight occupied squares. Let us thus suppose that we have eight rooks of eight different colors. Having decided which eight squares are to be occupied by the rooks (8! possibilities), we now have also to decide what the color is of the rook in each of the occupied squares. As we look at the rooks from row 1 to row 8, we see a permutation of the eight colors. Hence, having decided which eight squares are to be occupied (8! possibilities), we then have to decide which permutation of the eight colors (8! permutations) we shall assign. Thus, the number of ways to have eight nonattacking rooks of eight different colors on an 8-by-8 board equals

$$8!8! = (8!)^2.$$

Now suppose that, instead of rooks of eight different colors, we have one red (R) rook, three blue (B) rooks, and four (Y) yellow rooks. It is assumed that rooks of the same color are indistinguishable from one another.[8] Now, as we look at the rooks from row 1 to row 8, we see a permutation of the colors of the multiset

$$\{1 \cdot R, 3 \cdot B, 4 \cdot Y\}.$$

The number of permutations of this multiset equals, by Theorem 2.4.2,

$$\frac{8!}{1!3!4!}.$$

[8] Put another way, the only way we can tell one rook from another is by color.

Thus, the number of ways to place one red, three blue, and four yellow rooks on an 8-by-8 board so that no rook can attack another equals

$$8!\frac{8!}{1!3!4!} = \frac{(8!)^2}{1!3!4!}.$$

□

The reasoning in the preceding example is quite general and leads immediately to the next theorem.

Theorem 2.4.4 *There are n rooks of k colors with n_1 rooks of the first color, n_2 rooks of the second color, . . . , and n_k rooks of the kth color. The number of ways to arrange these rooks on an n-by-n board so that no rook can attack another equals*

$$n!\frac{n!}{n_1!n_2!\cdots n_k!} = \frac{(n!)^2}{n_1!n_2!\cdots n_k!}.$$

□

Note that if the rooks all have different colors ($k = n$ and all $n_i = 1$), the formula gives $(n!)^2$ as an answer. If the rooks are all colored the same ($k = 1$ and $n_1 = n$), the formula gives $n!$ as an answer.

Let S be an n-element multiset with repetition numbers equal to $n_1, n_2, \ldots, n_k$, so that $n = n_1 + n_2 + \cdots + n_k$. Theorem 2.4.2 furnishes a simple formula for the number of n-permutations of S. If $r < n$, there is, in general, no simple formula for the number of r-permutations of S. Nonetheless a solution can be obtained by the technique of generating functions, and we discuss this in Chapter 7. In certain cases, we can argue as in the next example.

Example. Consider the multiset $S = \{3 \cdot a, 2 \cdot b, 4 \cdot c\}$ of nine objects of three types. Find the number of 8-permutations of S.

The 8-permutations of S can be partitioned into three parts:

(i) 8-permutations of $\{2 \cdot a, 2 \cdot b, 4 \cdot c\}$, of which there are

$$\frac{8!}{2!2!4!} = 420;$$

(ii) 8-permutations of $\{3 \cdot a, 1 \cdot b, 4 \cdot c\}$, of which there are

$$\frac{8!}{3!1!4!} = 280;$$

(iii) 8-permutations of $\{3 \cdot a, 2 \cdot b, 3 \cdot c\}$, of which there are

$$\frac{8!}{3!2!3!} = 560.$$

Thus, the number of 8-permutations of S is

$$420 + 280 + 560 = 1260.$$

□

2.5 Combinations of Multisets

If S is a multiset, then an *r-combination* of S is an unordered selection of r of the objects of S. Thus, an r-combination of S (more precisely, the result of the selection) is itself a multiset, a *submultiset* of S of size r, or, for short, an *r-submultiset.* If S has n objects, then there is only one n-combination of S, namely, S itself. If S contains objects of k different types, then there are k 1-combinations of S. Unlike when discussing combinations of sets, we generally use *combination* rather than *submultiset.*

Example. Let $S = \{2 \cdot a, 1 \cdot b, 3 \cdot c\}$. Then the 3-combinations of S are

$$\{2 \cdot a, 1 \cdot b\}, \quad \{2 \cdot a, 1 \cdot c\}, \quad \{1 \cdot a, 1 \cdot b, 1 \cdot c\},$$

$$\{1 \cdot a, 2 \cdot c\}, \quad \{1 \cdot b, 2 \cdot c\}, \quad \{3 \cdot c\}.$$

□

We first count the number of r-combinations of a multiset all of whose repetition numbers are infinite (or at least r).

Theorem 2.5.1 *Let S be a multiset with objects of k types, each with an infinite repetition number. Then the number of r-combinations of S equals*

$$\binom{r+k-1}{r} = \binom{r+k-1}{k-1}.$$

Proof. Let the k types of objects of S be $a_1, a_2, \ldots, a_k$ so that

$$S = \{\infty \cdot a_1, \infty \cdot a_2, \ldots, \infty \cdot a_k\}.$$

Any r-combination of S is of the form $\{x_1 \cdot a_1, x_2 \cdot a_2, \ldots, x_k \cdot a_k\}$, where $x_1, x_2, \ldots, x_k$ are nonnegative integers with $x_1 + x_2 + \cdots + x_k = r$. Conversely, every sequence $x_1, x_2, \ldots, x_k$ of nonnegative integers with $x_1 + x_2 + \cdots + x_k = r$ corresponds to an r-combination of S. Thus, the number of r-combinations of S equals the number of solutions of the equation

$$x_1 + x_2 + \cdots + x_k = r,$$

where $x_1, x_2, \ldots, x_k$ are nonnegative integers. We show that the number of these solutions equals the number of permutations of the multiset

$$T = \{r \cdot 1, (k-1) \cdot *\}$$

of $r + k - 1$ objects of two different types.[9] Given a permutation of T, the $k - 1$ $*$'s divide the r 1s into k groups. Let there be x_1 1s to the left of the first $*$, x_2 1s between the first and the second $*$, ..., and x_k 1s to the right of the last $*$. Then $x_1, x_2, \ldots, x_k$ are nonnegative integers with $x_1 + x_2 + \cdots + x_k = r$. Conversely, given nonnegative integers $x_1, x_2, \ldots, x_k$ with $x_1 + x_2 + \cdots + x_k = r$, we can reverse the preceding steps and construct a permutation of T.[10] Thus, the number of r-combinations of the multiset S equals the number of permutations of the multiset T, which by Theorem 2.4.2 is

$$\frac{(r+k-1)!}{r!(k-1)!} = \binom{r+k-1}{r}.$$

□

Another way of phrasing Theorem 2.5.1 is as follows: *The number of r-combinations of k distinct objects, each available in unlimited supply, equals*

$$\binom{r+k-1}{r}.$$

We note that Theorem 2.5.1 remains true if the repetition numbers of the k distinct objects of S are all at least r.

Example. A bakery boasts eight varieties of doughnuts. If a box of doughnuts contains one dozen, how many different options are there for a box of doughnuts?

It is assumed that the bakery has on hand a large number (at least 12) of each variety. This is a combination problem, since we assume the order of the doughnuts in a box is irrelevant for the purchaser's purpose. The number of different options for boxes equals the number of 12-combinations of a multiset with objects of 8 types, each having an infinite repetition number. By Theorem 2.5.1, this number equals

$$\binom{12+8-1}{12} = \binom{19}{12}.$$

□

Example. What is the number of nondecreasing sequences of length r whose terms are taken from $1, 2, \ldots, k$?

[9] Equivalently, the number of sequences of 0s and 1s of length $r + k - 1$ in which there are r 1s and $k - 1$ 0s.

[10] For example, if $k = 4$ and $r = 5$, then the permutation of $T = \{5 \cdot 1, 3 \cdot *\}$ given by $*111**11$ corresponds to the solution of $x_1 + x_2 + x_3 + x_4 = 5$ given by $x_1 = 0, x_2 = 3, x_3 = 0, x_4 = 2$.

The nondecreasing sequences to be counted can be obtained by first choosing an r-combination of the multiset

$$S = \{\infty \cdot 1, \infty \cdot 2, \ldots, \infty \cdot k\}$$

and then arranging the elements in increasing order. Thus, the number of such sequences equals the number of r-combinations of S, and hence, by Theorem 2.5.1, equals

$$\binom{r+k-1}{r}.$$

□

In the proof of Theorem 2.5.1, we defined a one-to-one correspondence between r-combinations of a multiset S with objects of k different types and the nonnegative integral solutions of the equation

$$x_1 + x_2 + \cdots + x_k = r.$$

In this correspondence, x_i represents the number of objects of the ith type that are used in the r-combination. Putting restrictions on the number of times each type of object is to occur in the r-combination can be accomplished by putting restrictions on the x_i. We give a first illustration of this in the next example.

Example. Let S be the multiset $\{10 \cdot a, 10 \cdot b, 10 \cdot c, 10 \cdot d\}$ with objects of four types, a, b, c, and d. What is the number of 10-combinations of S that have the property that each of the four types of objects occurs at least once?

The answer is the number of *positive* integral solutions of

$$x_1 + x_2 + x_3 + x_4 = 10,$$

where x_1 represents the number of a's in a 10-combination, x_2 the number of b's, x_3 the number of c's, and x_4 the number of d's. Since the repetition numbers all equal 10, and 10 is the size of the combinations being counted, we can ignore the repetition numbers of S. We then perform the changes of variable:

$$y_1 = x_1 - 1,\ y_2 = x_2 - 1,\ y_3 = x_3 - 1,\ y_4 = x_4 - 1$$

to get

$$y_1 + y_2 + y_3 + y_4 = 6,$$

where the y_i's are to be nonnegative. The number of nonnegative integral solutions of the new equation is, by Theorem 2.5.1,

$$\binom{6+4-1}{6} = \binom{9}{6} = 84. \qquad \square$$

Example. Continuing with the doughnut example following Theorem 2.5.1, we see that the number of different options for boxes of doughnuts containing at least one doughnut of each of the eight varieties equals

$$\binom{4+8-1}{4} = \binom{11}{4} = 330.$$

□

General lower bounds on the number of times each type of object occurs in the combination also can be handled by a change of variable. We illustrate this in the next example.

Example. What is the number of integral solutions of the equation

$$x_1 + x_2 + x_3 + x_4 = 20,$$

in which

$$x_1 \geq 3,\ x_2 \geq 1,\ x_3 \geq 0 \text{ and } x_4 \geq 5?$$

We introduce the new variables

$$y_1 = x_1 - 3,\ y_2 = x_2 - 1,\ y_3 = x_3,\ y_4 = x_4 - 5,$$

and our equation becomes

$$y_1 + y_2 + y_3 + y_4 = 11.$$

The lower bounds on the x_i's are satisfied if and only if the y_i's are nonnegative. The number of nonnegative integral solutions of the new equation, and hence the number of nonnegative solutions of the original equation, is

$$\binom{11+4-1}{11} = \binom{14}{11} = 364.$$

□

It is more difficult to count the number of r-combinations of a multiset

$$S = \{n_1 \cdot a_1, n_2 \cdot a_2, \ldots, n_k \cdot a_k\}$$

with k types of objects and general repetition numbers $n_1, n_2, \ldots, n_k$. The number of r-combinations of S is the same as the number of integral solutions of

$$x_1 + x_2 + \cdots + x_k = r,$$

where

$$0 \leq x_1 \leq n_1, \quad 0 \leq x_2 \leq n_2, \quad \ldots, \quad 0 \leq x_k \leq n_k.$$

We now have upper bounds on the x_i's, and these cannot be handled in the same way as lower bounds. In Chapter 6 we show how the inclusion–exclusion principle provides a satisfactory method for this case.

2.6 Finite Probability

In this section we give a brief and informal introduction to finite probability.[11] As we will see, it all reduces to counting, and so the counting techniques discussed in this chapter can be used to calculate probabilities.

The setting for finite probability is this: There is an *experiment* $\mathcal{E}$ which when carried out results in one of a finite set of outcomes. We assume that each outcome is *equally likely* (that is, no outcome is more likely to occur than any other); we say that the experiment is carried out *randomly*. The set of all possible outcomes is called the *sample space* of the experiment and is denoted by S. Thus S is a finite set with, say, n elements:

$$S = \{s_1, s_2, \ldots, s_n\}.$$

When $\mathcal{E}$ is carried out, each s_i has a 1 in n chance of occuring, and so we say that the probability of the outcome s_i is $1/n$, written

$$\text{Prob}(s_i) = \frac{1}{n}, \quad (i = 1, 2, \ldots, n).$$

An *event* is just a subset E of the sample space S, but it is usually given descriptively and not by actually listing all the outcomes in E.

Example. Consider the experiment $\mathcal{E}$ of tossing three coins, where each of the coins lands showing either Heads (H) or Tails (T). Since each coin can come up either H or T, the sample space of this experiment is the set S of consisting of the eight ordered pairs

$$(H, H, H), (H, H, T), (H, T, H), (H, T, T),$$

$$(T, H, H), (T, H, T), (T, T, H), (T, T, T),$$

where, for instance, (H, T, H) means that the first coin comes up as H, the second coin comes up as T, and the third coin comes up as H. Let E be the event that at least two coins come up H. Then

$$E = \{(H, H, H), (H, H, T), (H, T, H), (T, H, H)\},$$

Since E consists of four outcomes out of a possible eight outcomes, it is natural to assign to E the probability $4/8 = 1/2$. This is made more precise in the next definition. □

The *probability* of an event E in an experiment with a sample space S is defined to be the proportion of outcomes in S that belong to E; thus,

$$\text{Prob}(E) = \frac{|E|}{|S|}.$$

[11] As opposed to the continuous probability that is calculus based.

By this definition, the probability of an event E satisfies

$$0 \le \mathrm{Prob}(E) \le 1,$$

where $\mathrm{Prob}(E) = 0$ if and only if E is the empty event $\emptyset$ (the impossible event) and $\mathrm{Prob}(E) = 1$ if and only if E is the entire sample space S (the guaranteed event). Thus to compute the probability of an event E, we have to make two counts: count the number of outcomes in the sample space S and count the number of outcomes in the event E.

Example. We consider an ordinary deck of 52 cards with each card having one of 13 ranks $1, 2, \ldots, 10, 11, 12, 13$ and four suits Clubs (C), Diamonds (D), Hearts (H), and Spades (S). Usually, 11 is denoted as a *Jack*, 12 as a *Queen*, and 13 as a *King*. In addition, 1 has two roles: either as a 1 (low; below the 2) or as an *Ace* (high; above the King).[12] Consider the experiment $\mathcal{E}$ of drawing a card at random. Thus the sample space S is the set of 52 cards, each of which is assigned a probability of $1/52$. Let E be the event that the card drawn is a 5. Thus

$$E = \{(\mathrm{C}, 5), (\mathrm{D}, 5), (\mathrm{H}, 5), (\mathrm{S}, 5)\}.$$

Since $|E| = 4$ and $|S| = 52$, $\mathrm{Prob}(E) = 4/52 = 1/13$. □

Example. Let n be a positive integer. Suppose we choose a sequence $i_1, i_2, \ldots, i_n$ of integers between 1 and n at random. (1) What is the probability that the chosen sequence is a permutation of $1, 2, \ldots, n$? (2) What is the probability that the sequence contains exactly $n - 1$ different integers?

The sample space S is the set of all possible sequences of length n each of whose terms is one of the integers $1, 2, \ldots, n$. Hence $|S| = n^n$ because there are n choices for each of the n terms.

(1) The event E that the sequence is a permutation satisfies $|E| = n!$. Hence

$$\mathrm{Prob}(E) = \frac{n!}{n^n}.$$

(2) Let F be the event that the sequence contains exactly $n-1$ different integers. A sequence in F contains one repeated integer, and exactly one of the integers $1, 2, \ldots, n$ is missing in the sequence (so $n - 2$ other integers occur in the sequence). There are n choices for the repeated integer, and then $n - 1$ choices for the missing integer. The

[12] For those who are either unfamiliar with card games or don't like them, here is a more abstract description: An ordinary deck of 52 cards is, abstractly, just the collection of the 52 ordered pairs (x, y), where x is one of four "suits" C, D, H, and S, and y is one of the thirteen ranks $1, 2, \ldots, 13$, where the smallest rank 1 can also be used as the largest rank (so we can think of a circle with 1 following 13).

places for the repeated integer can be chosen in $\binom{n}{2}$ ways; the other $n-2$ integers can be put in the remaining $n-2$ places in $(n-2)!$ ways. Hence

$$|F| = n(n-1)\binom{n}{2}(n-2)! = \frac{n!^2}{2!(n-2)!},$$

and

$$\text{Prob}(F) = \frac{n!^2}{2!(n-2)!n^n}.$$

□

Example. Five identical rooks are placed at random in nonattacking positions on an 8-by-8 board. What is the probability that the rooks are both in rows $1,2,3,4,5$ and in columns $4,5,6,7,8$?

Our sample space S consist of all placements of five nonattacking rooks on the board and so

$$|S| = \binom{8}{5}^2 \cdot 5! = \frac{8!^2}{3!^2 5!}.$$

Let E be the event that the five rooks are in the rows and columns prescribed above. Then E has size 5!, since there are 5! ways to place five nonattacking rooks on a 5-by-5 board. Hence we have

$$\text{Prob}(E) = \frac{5!^2 3!^2}{8!^2} = \frac{1}{3136}.$$

□

Example. This is a multipart example relating to the card game Poker played with an ordinary deck of 52 cards. A poker hand consists of 5 cards. Our experiment $\mathcal{E}$ is to select a poker hand at random. Thus the sample space S consists of the $\binom{52}{5} = 2,598,960$ possible poker hands and each has the same chance as being selected, namely $1/2,598,960$.

(1) Let E be the event that the poker hand is a *full house*; that is, three cards of one rank and two cards of a different rank (suit doesn't matter). To compute the probability of E, we need to calculate $|E|$. How do we determine the number of full houses? We use the multiplication principle thinking of four tasks:

   (a) Choose the rank with three cards.

   (b) Choose the three cards of that rank i.e., their 3 suits.

   (c) Choose the rank with two cards.

   (d) Choose the two cards of that rank i.e., their 2 suits.

   The number of ways of carrying these tasks out is as follows:

   (a) 13

(b) $\binom{4}{3} = 4$

(c) 12 (after choice (a), 12 ranks remain)

(d) $\binom{4}{2} = 6$

Thus $|E| = 13 \cdot 4 \cdot 12 \cdot 6 = 3,744$ and

$$\Pr(E) = \frac{3,744}{2,598,960} \approx 0.0014.$$

(2) Let E be the event that the poker hand is a *straight*; that is, five cards of consecutive ranks (suit doesn't matter), keeping in mind that the 1 is also the Ace. To compute $|E|$, we think of two tasks:

(a) Choose the five consecutive ranks.

(b) Choose the suit of each of the ranks.

The number of ways of carrying out these two tasks is as follows:

(a) 10 (the straights can begin with any of $1, 2. \ldots, 10$)

(b) 4^5 (four possible suits for each rank)

Thus $|E| = 10 \cdot 4^5 = 10,240$ and

$$\Pr(E) = \frac{10,240}{2,598,960} \approx 0.0039.$$

(3) Let E be the event that the poker hand is a *straight flush*; that is, five cards of consecutive ranks, all of the same suit. Using the reasoning in (b), we see that $|E| = 10 \cdot 4 = 40$ and

$$\Pr(E) = \frac{40}{2,598,960} \approx 0.0000154.$$

(4) Let E be the event that the poker hand consists of *exactly two pairs*; that is, two cards of one rank, two cards of a different rank, and one card of an additionally different rank. Here we have to be a little careful since the first two mentioned ranks appear in the same way (as opposed to the full house, where there were three cards of one rank and two cards of a different rank). To compute $|E|$ in this case, we think of three tasks (not six if we had imitated (1)):

(a) Choose the two ranks occuring in the two pairs.

(b) Choose the two suits for each of these two ranks.

(c) Choose the remaining card.

The number of ways of carrying out these three tasks is as follows:

(a) $\binom{13}{2} = 78$

(b) $\binom{4}{2}\binom{4}{2} = 6 \cdot 6 = 36$

(c) 44

Thus $|E| = 78 \cdot 36 \cdot 44 = 123,552$, and

$$\Pr(E) = \frac{123,552}{2,598,960} \approx 0.048,$$

almost a 1 in 20 chance.

(5) Let E be the event that the poker hand contains at least one Ace. Here we use our subtraction principle. Let $\overline{E} = S \setminus E$ be the complementary event of a poker hand with no aces. Then $|\overline{E}| = \binom{48}{5} = 1,712,304$. Thus $|E| = |S| - |\overline{E}| = 2,598,960 - 1,712,304 = 886,656$, and

$$\begin{aligned}
\Pr(E) &= \frac{2,598,960 - 1,712,304}{2,598,960} \\
&= 1 - \frac{1,712,304}{2,598,960} \\
&= \frac{886,656}{2,598,960} \\
&\approx 0.34.
\end{aligned}$$

□

As we see in the calculation in (5), our subtraction principle in terms of probability becomes

$$\Pr(E) = 1 - \Pr(\overline{E}), \text{ equivalently, } \Pr(\overline{E}) = 1 - \Pr(E).$$

More probability calculations are given in the Exercises.

2.7 Exercises

1. For each of the four subsets of the two properties (a) and (b), count the number of four-digit numbers whose digits are either 1, 2, 3, 4, or 5:

   (a) The digits are distinct.

   (b) The number is even.

Note that there are four problems here: $\emptyset$ (no further restriction), {a} (property (a) holds), {b} (property (b) holds), {a, b} (both properties (a) and (b) hold).

2. How many orderings are there for a deck of 52 cards if all the cards of the same suit are together?

3. In how many ways can a poker hand (five cards) be dealt? How many different poker hands are there?

4. How many distinct positive divisors does each of the following numbers have?

   (a) $3^4 \times 5^2 \times 7^6 \times 11$

   (b) 620

   (c) 10^{10}

5. Determine the largest power of 10 that is a factor of the following numbers (equivalently, the number of terminal 0s, using ordinary base 10 representation):

   (a) 50!

   (b) 1000!

6. How many integers greater than 5400 have both of the following properties?

   (a) The digits are distinct.

   (b) The digits 2 and 7 do not occur.

7. In how many ways can four men and eight women be seated at a round table if there are to be two women between consecutive men around the table?

8. In how many ways can six men and six women be seated at a round table if the men and women are to sit in alternate seats?

9. In how many ways can 15 people be seated at a round table if B refuses to sit next to A? What if B only refuses to sit on A's right?

10. A committee of five people is to be chosen from a club that boasts a membership of 10 men and 12 women. How many ways can the committee be formed if it is to contain at least two women? How many ways if, in addition, one particular man and one particular woman who are members of the club refuse to serve together on the committee?

11. How many sets of three integers between 1 and 20 are possible if no two consecutive integers are to be in a set?

12. A football team of 11 players is to be selected from a set of 15 players, 5 of whom can play only in the backfield, 8 of whom can play only on the line, and 2 of whom can play either in the backfield or on the line. Assuming a football team has 7 men on the line and 4 men in the backfield, determine the number of football teams possible.

13. There are 100 students at a school and three dormitories, A, B, and C, with capacities 25, 35 and 40, respectively.

    (a) How many ways are there to fill the dormitories?

    (b) Suppose that, of the 100 students, 50 are men and 50 are women and that A is an all-men's dorm, B is an all-women's dorm, and C is co-ed. How many ways are there to fill the dormitories?

14. A classroom has two rows of eight seats each. There are 14 students, 5 of whom always sit in the front row and 4 of whom always sit in the back row. In how many ways can the students be seated?

15. At a party there are 15 men and 20 women.

    (a) How many ways are there to form 15 couples consisting of one man and one woman?

    (b) How many ways are there to form 10 couples consisting of one man and one woman?

16. Prove that
$$\binom{n}{r} = \binom{n}{n-r}$$
by using a combinatorial argument and not the values of these numbers as given in Theorem 3.3.1.

17. In how many ways can six indistinguishable rooks be placed on a 6-by-6 board so that no two rooks can attack one another? In how many ways if there are two red and four blue rooks?

18. In how many ways can two red and four blue rooks be placed on an 8-by-8 board so that no two rooks can attack one another?

19. We are given eight rooks, five of which are red and three of which are blue.

    (a) In how many ways can the eight rooks be placed on an 8-by-8 chessboard so that no two rooks can attack one another?

    (b) In how many ways can the eight rooks be placed on a 12-by-12 chessboard so that no two rooks can attack one another?

20. Determine the number of circular permutations of $\{0, 1, 2, \ldots, 9\}$ in which 0 and 9 are not opposite. (*Hint*: Count those in which 0 and 9 are opposite.)

21. How many permutations are there of the letters of the word ADDRESSES? How many 8-permutations are there of these nine letters?

22. A footrace takes place among four runners. If ties are allowed (even all four runners finishing at the same time), how many ways are there for the race to finish?

23. Bridge is played with four players and an ordinary deck of 52 cards. Each player begins with a hand of 13 cards. In how many ways can a bridge game start? (Ignore the fact that bridge is played in partnerships.)

24. A roller coaster has five cars, each containing four seats, two in front and two in back. There are 20 people ready for a ride. In how many ways can the ride begin? What if a certain two people want to sit in different cars?

25. A ferris wheel has five cars, each containing four seats in a row. There are 20 people ready for a ride. In how many ways can the ride begin? What if a certain two people want to sit in different cars?

26. A group of mn people are to be arranged into m teams each with n players.

    (a) Determine the number of ways if each team has a different name.

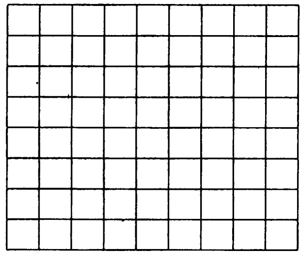
    (b) Determine the number of ways if the teams don't have names.

27. In how many ways can five indistinguishable rooks be placed on an 8-by-8 chessboard so that no rook can attack another and neither the first row nor the first column is empty?

28. A secretary works in a building located nine blocks east and eight blocks north of his home. Every day he walks 17 blocks to work. (See the map that follows.)

    (a) How many different routes are possible for him?

    (b) How many different routes are possible if the one block in the easterly direction, which begins four blocks east and three blocks north of his home, is under water (and he can't swim)? (*Hint*: Count the routes that use the block under water.)



29. Let S be a multiset with repetition numbers $n_1, n_2, \ldots, n_k$, where $n_1 = 1$. Let $n = n_2 + \cdots + n_k$. Prove that the number of circular permutations of S equals

$$\frac{n!}{n_2! \cdots n_k!}.$$

30. We are to seat five boys, five girls, and one parent in a circular arrangement around a table. In how many ways can this be done if no boy is to sit next to a boy and no girl is to sit next to a girl? What if there are two parents?

31. In a soccer tournament of 15 teams, the top three teams are awarded gold, silver, and bronze cups, and the last three teams are dropped to a lower league. We regard two outcomes of the tournament as the same if the teams that receive the gold, silver, and bronze cups, respectively, are identical and the teams which drop to a lower league are also identical. How many different possible outcomes are there for the tournament?

32. Determine the number of 11-permutations of the multiset

$$S = \{3 \cdot a, 4 \cdot b, 5 \cdot c\}.$$

33. Determine the number of 10-permutations of the multiset

$$S = \{3 \cdot a, 4 \cdot b, 5 \cdot c\}.$$

34. Determine the number of 11-permutations of the multiset

$$S = \{3 \cdot a, 3 \cdot b, 3 \cdot c, 3 \cdot d\}.$$

35. List all 3-combinations and 4-combinations of the multiset

$$\{2 \cdot a, 1 \cdot b, 3 \cdot c\}.$$

36. Determine the total number of combinations (of any size) of a multiset of objects of k different types with finite repetition numbers $n_1, n_2, \ldots, n_k$, respectively.

37. A bakery sells six different kinds of pastry. If the bakery has at least a dozen of each kind, how many different options for a dozen of pastries are there? What if a box is to contain at least one of each kind of pastry?

38. How many integral solutions of

$$x_1 + x_2 + x_3 + x_4 = 30$$

satisfy $x_1 \geq 2$, $x_2 \geq 0$, $x_3 \geq -5$, and $x_4 \geq 8$?

39. There are 20 identical sticks lined up in a row occupying 20 distinct places as follows:

$$||||||||||||||||||||.$$

Six of them are to be chosen.

(a) How many choices are there?

(b) How many choices are there if no two of the chosen sticks can be consecutive?

(c) How many choices are there if there must be at least two sticks between each pair of chosen sticks?

40. There are n sticks lined up in a row, and k of them are to be chosen.

(a) How many choices are there?

(b) How many choices are there if no two of the chosen sticks can be consecutive?

(c) How many choices are there if there must be at least l sticks between each pair of chosen sticks?

41. In how many ways can 12 indistinguishable apples and 1 orange be distributed among three children in such a way that each child gets at least one piece of fruit?

42. Determine the number of ways to distribute 10 orange drinks, 1 lemon drink, and 1 lime drink to four thirsty students so that each student gets at least one drink, and the lemon and lime drinks go to different students.

43. Determine the number of r-combinations of the multiset

$$\{1 \cdot a_1, \infty \cdot a_2, \ldots, \infty \cdot a_k\}.$$

44. Prove that the number of ways to distribute n different objects among k children equals k^n.

45. Twenty different books are to be put on five book shelves, each of which holds at least twenty books.

    (a) How many different arrangements are there if you only care about the number of books on the shelves (and not which book is where)?

    (b) How many different arrangements are there if you care about which books are where, but the order of the books on the shelves doesn't matter?

    (c) How many different arrangements are there if the order on the shelves does matter?

46. (a) There is an even number $2n$ of people at a party, and they talk together in pairs, with everyone talking with someone (so n pairs). In how many different ways can the $2n$ people be talking like this?

    (b) Now suppose that there is an odd number $2n+1$ of people at the party with everyone but one person talking with someone. How many different pairings are there?

47. There are $2n+1$ identical books to be put in a bookcase with three shelves. In how many ways can this be done if each pair of shelves together contains more books than the other shelf?

48. Prove that the number of permutations of m A's and at most n B's equals
$$\binom{m+n+1}{m+1}.$$

49. Prove that the number of permutations of at most m A's and at most n B's equals
$$\binom{m+n+2}{m+1} - 1.$$

50. In how many ways can five identical rooks be placed on the squares of an 8-by-8 board so that four of them form the corners of a rectangle with sides parallel to the sides of the board?

51. Consider the multiset $\{n \cdot a, 1, 2, 3, \ldots, n\}$ of size $2n$. Determine the number of its n-combinations.

52. Consider the multiset $\{n \cdot a, n \cdot b, 1, 2, 3, \ldots, n+1\}$ of size $3n+1$. Determine the number of its n-combinations.

53. Find a one-to-one correspondence between the permutations of the set $\{1, 2, \ldots, n\}$ and the towers $A_0 \subset A_1 \subset A_2 \subset \cdots \subset A_n$ where $|A_k| = k$ for $k = 0, 1, 2, \ldots, n$.

54. Determine the number of towers of the form $\emptyset \subseteq A \subseteq B \subseteq \{1, 2, \ldots, n\}$.

55. How many permutations are there of the letters in the words

    (a) TRISKAIDEKAPHOBIA (fear of the number 13)?

    (b) FLOCCINAUCINIHILIPILIFICATION (estimating something as worthless)?

    (c) PNEUMONOULTRAMICROSCOPICSILICOVOLCANOCONIOSIS (a lung disease caused by inhaling fine particles of silica)? (This word is, by some accounts, the longest word in the English language.)

    (d) DERMATOGLYPHICS (skin patterns or the study of them)? (This word is the (current) longest word in the English language that doesn't repeat a letter; another word of the same length is UNCOPYRIGHTABLE.[13])

56. What is the probability that a poker hand contains a *flush* (that is, five cards of the same suit)?

57. What is the probability that a poker hand contains exactly one pair (that is, a poker hand with exactly four different ranks)?

58. What is the probability that a poker hand contains cards of five different ranks but does not contain a flush or a straight?

59. Consider the deck of 40 cards obtained from an ordinary deck of 52 cards by removing the jacks (11s), queens (12s), and kings (13s), where now the 1 (ace) can be used to follow a 10. Compute the probabilities for the various poker hands described in the example in Section 3.6.

60. A bagel store sells six different kinds of bagels. Suppose you choose 15 bagels at random. What is the probability that your choice contains at least one bagel of each kind? If one of the kinds of bagels is Sesame, what is the probability that your choice contains at least three Sesame bagels?

61. Consider an 9-by-9 board and nine rooks of which five are red and four are blue. Suppose you place the rooks on the board in nonattacking positions at random. What is the probability that the red rooks are in rows $1, 3, 5, 7, 9$? What is the probability that the red rooks are both in rows $1, 2, 3, 4, 5$ and in columns $1, 2, 3, 4, 5$?

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[13] Anu Garg: *The Dord, the Diglot, and An Avocado or Two*, Plume, Penguin Group, New York (2007).

62. Suppose a poker hand contains seven cards rather than five. Compute the probabilities of the following poker hands:

    (a) a seven-card straight
    (b) four cards of one rank and three of a different rank
    (c) three cards of one rank and two cards of each of two different ranks
    (d) two cards of each of three different ranks, and a card of a fourth rank
    (e) three cards of one rank and four cards of each of four different ranks
    (f) seven cards each of different rank

63. Four (standard) dice (cubes with 1, 2, 3, 4, 5, 6, respectively, dots on their six faces), each of a different color, are tossed, each landing with one of its faces up, thereby showing a number of dots. Determine the following probabilities:

    (a) The probability that the total number of dots shown is 6
    (b) The probability that at most two of the dice show exactly one dot
    (c) The probability that each die shows at least two dots
    (d) The probability that the four numbers of dots shown are all different
    (e) The probability that there are exactly two different numbers of dots shown

64. Let n be a positive integer. Suppose we choose a sequence $i_1, i_2, \ldots, i_n$ of integers between 1 and n at random.

    (a) What is the probability that the sequence contains exactly $n-2$ different integers?
    (b) What is the probability that the sequence contains exactly $n-3$ different integers?