Chapter 6

Recurrence Relations

6.1. Introduction

Let us begin our discussion with the following counting problem. Figure 6.1.1 shows a $1 \times n$ rectangle $ABCD$, that is to be fully paved by two types of tiles of different sizes: 1×1 and 1×2. What is the number of ways that this can be done?

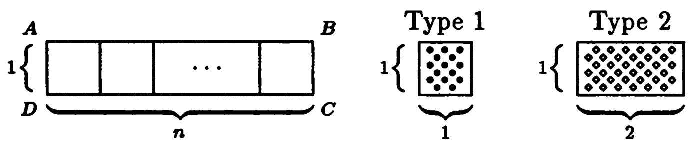
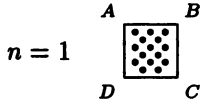
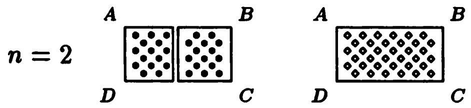


Figure 6.1.1.

Well, after some thought, we may find that it is not so easy to get a direct answer to the problem. Also, it seems that the methods we learnt in the previous chapters are not of much help. Let us therefore use a different approach. First of all, we consider some very special cases. When $n = 1$, it is clear that there is one and only one way to pave the 1×1 rectangle:

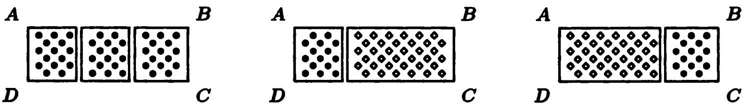


When $n = 2$, it is also easy to see that there are exactly 2 ways to pave the 1×2 rectangle:



When $n = 3$, there are 3 different ways as shown below:

$n = 3$



For convenience, let a_n denote the required number of ways to pave the $1 \times n$ rectangle $ABCD$. As shown above, we have

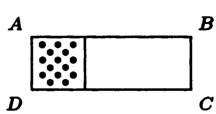
$$a_1 = 1,\ a_2 = 2,\ a_3 = 3.$$

If you proceed as before, you will find that

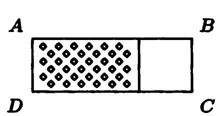
$$a_4 = 5,\ a_5 = 8,\ \text{etc.}$$

However, up to this stage, we do not see any direct way of solving the problem. Let us go back to analyze the case when $n = 3$. Paving the rectangle $ABCD$ from left to right, there are 2 possibilities for the first step:

(i) a tile of type 1 is used;



or (ii) a tile of type 2 is used.

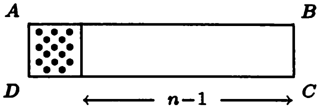


We now have a crucial observation. In case (i), a 1×2 rectangle is left behind, while in case (ii), a 1×1 rectangle is left behind. As noted earlier, there are $a_2 = 2$ ways to complete the paving in the former case, and $a_1 = 1$ way in the latter case. Thus by (AP),

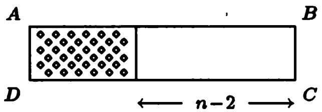
$$a_3 = a_2 + a_1.$$

Can this relation for a_3 be extended to an arbitrary term a_n, $n \geq 3$? To see this, we follow the same argument as we did before and obtain two cases for the first step:

(i)



(ii)



In case (i), there are by definition a_{n-1} ways to complete the paving and in case (ii), there are a_{n-2} ways to do so. Thus by (AP),

$$a_n = a_{n-1} + a_{n-2}, \quad \text{for } n \geq 3. \tag{6.1.1}$$

Although, up till now, we have not been able to find a formula $f(n)$ for a_n, we should be content with the relation (6.1.1), which at least enables us to compute a_n indirectly from the preceding numbers a_{n-2} and a_{n-1} (thus $a_4 = a_2 + a_3 = 2 + 3 = 5$, $a_5 = a_3 + a_4 = 3 + 5 = 8$, and so on).

The relation (6.1.1) governing the sequence (a_n) is called a *recurrence relation* for the sequence (a_n). In general, given a sequence $(a_n) = (a_0, a_1, a_2, \ldots)$ of numbers, a *recurrence relation* for (a_n) is an equation which relates the nth term a_n to some of its predecessors in the sequence. Thus the following are some more examples of recurrence relations:

$$a_n = a_{n-1} + 1$$
$$a_n - 5a_{n-1} + 6a_{n-2} = 0$$
$$a_n + 7a_{n-1} + 12a_{n-2} = 2^n$$
$$n(n-1)a_n = (n-1)(n-2)a_{n-1} - (n-3)a_{n-2}$$
$$a_n = \frac{a_{n-1}}{(2n-1)a_{n-1} + 1}$$

To initiate the computation for the terms of a recurrence relation, we need to know the values of some terms of the sequence (a_n). They are called *initial conditions* of the recurrence relation. For instance, to compute a_n

in the recurrence relation (6.1.1), we need to find out a_1 and a_2 before we can proceed. The values $a_1 = 1$ and $a_2 = 2$ in this example are the initial conditions.

The ***solution*** of a recurrence relation is an expression $a_n = g(n)$, where $g(n)$ is a function of n, which satisfies the recurrence relation. For instance, the expression

$$a_n = n$$

is the solution of the recurrence relation $a_n = a_{n-1}+1$ with initial condition $a_1 = 1$, since "$a_n = n$" satisfies the recurrence relation ($a_n = n = (n-1)+1 = a_{n-1} + 1$).

In combinatorics there are many problems that, like the above paving problem, may not be easily or directly enumerated, but could be well handled using the notion of recurrence relations. For each of these problems, deriving a recurrence relation is the first important step towards its solution. In this chapter, we shall gain, through various examples, some experience of deriving recurrence relations. We shall also learn some standard methods of solving (i.e., finding solutions of) certain families of "well-behaved" recurrence relations.

6.2. Two Examples

In this section, we introduce two counting problems that can be solved with the help of recurrence relations. We begin with a famous problem, known as the *Tower of Hanoi*, that was first formulated and studied by the French mathematician Edouard Lucas (1842-1891) in 1883.

Example 6.2.1. A tower of n circular discs of different sizes is stacked on one of the 3 given pegs in decreasing size from the bottom, as shown in Figure 6.2.1. The task is to transfer the entire tower to another peg by a sequence of moves under the following conditions:

(i) each move carries exactly one disc, and

(ii) no disc can be placed on top of a smaller one.

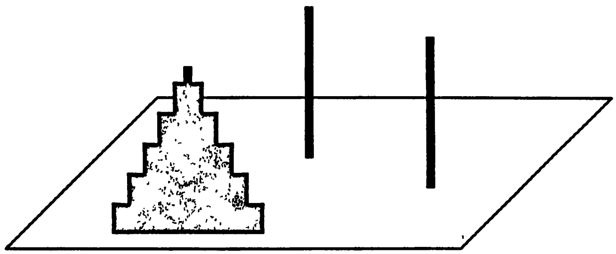


Figure 6.2.1.

For $n \geq 1$, let a_n denote the *minimum* number of moves needed to accomplish the task with n discs. Show that

$$a_1 = 1,\ a_2 = 3,\ a_3 = 7$$

and
$$a_n = 2a_{n-1} + 1 \tag{6.2.1}$$

for $n \geq 2$. Solve also the recurrence relation (6.2.1).

Solution. Obviously, $a_1 = 1$. For $n = 2$, the 3 moves shown in Figure 6.2.2 accomplish the task. It is clear that any two moves are not enough to do so. Thus $a_2 = 3$.

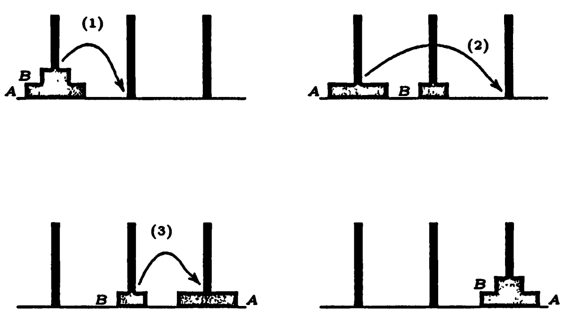


Figure 6.2.2.

For $n = 3$, the 7 moves shown in Figure 6.2.3 do the job. Any 6 moves are not sufficient to do so (why?). Thus $a_3 = 7$.

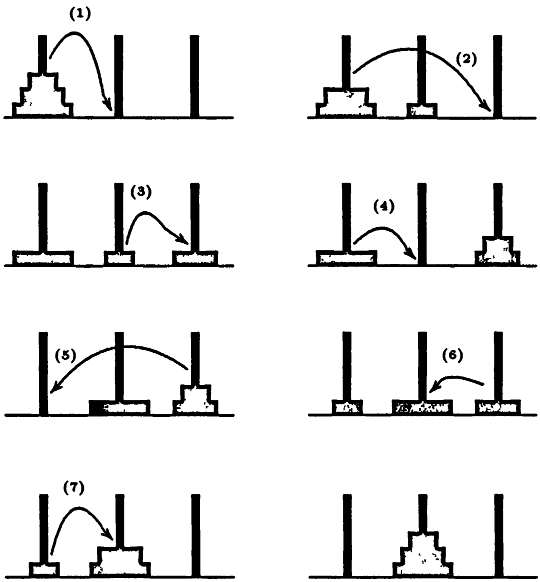


Figure 6.2.3.

We shall now consider the general term a_n, $n \geq 2$. The given task with n discs can be accomplished via the following main steps:

1° Transfer the top $(n-1)$ discs from the original peg to a different peg (see the first 3 diagrams in Figure 6.2.3);

2° Move the largest disc from the original peg to the only empty peg (see the 4th and 5th diagram in Figure 6.2.3); and

3° Transfer the $(n-1)$ discs from the peg accomplished in step 1° to the peg that the largest disc is currently placed (see the last 4 diagrams in Figure 6.2.3).

The number of moves required in steps 1°, 2° and 3° are, respectively, a_{n-1}, 1, and a_{n-1}. Thus, we have provided a way with

$$a_{n-1} + 1 + a_{n-1} = 2a_{n-1} + 1$$

moves to transfer the tower with n discs from one peg to another. By the definition of a_n, we have

$$a_n \leq 2a_{n-1} + 1 . \tag{1}$$

On the other hand, we notice that for any way of transfering the tower with n discs, the largest disc at the bottom has to be moved to an empty peg at some point, and this is possible only when step 1° has been performed. To complete the task, step 3° has to be done somehow. Thus, any way of transfering requires at least $2a_{n-1}+1$ moves; i.e.,

$$a_n \geq 2a_{n-1} + 1. \tag{2}$$

Combining (1) and (2), we obtain

$$a_n = 2a_{n-1} + 1,$$

as required.

Finally, we shall solve the recurrence relation (6.2.1) with the initial condition $a_1 = 1$ as follows: for $n \geq 2$,

$$\begin{aligned} a_n &= 2a_{n-1} + 1 \\ &= 2(2a_{n-2} + 1) + 1 \\ &= 2^2 a_{n-2} + 2 + 1 \\ &= 2^2(2a_{n-3} + 1) + 2 + 1 \\ &= 2^3 a_{n-3} + 2^2 + 2 + 1 \\ &\ \vdots \\ &= 2^{n-1}a_1 + 2^{n-2} + 2^{n-3} + \cdots + 2 + 1 \\ &= 2^{n-1} + 2^{n-2} + \cdots + 2 + 1 \\ &= \frac{1(2^n - 1)}{2 - 1} \\ &= 2^n - 1. \end{aligned}$$

Thus, $a_n = 2^n - 1$ is the required solution. ∎

Remarks. (1) The above method of obtaining a solution is often referred to as the *backward substitution*.

(2) The recurrence relation (6.2.1) is a special example of the following recurrence relation:

$$a_n = pa_{n-1} + q \quad \text{with } a_0 = r \tag{6.2.2}$$

where p, q and r are arbitrary constants. It can be proved (see Problem 6.12) that the solution of (6.2.2) is given by

$$a_n = \begin{cases} r + qn & \text{if } p = 1, \\ rp^n + \frac{(p^n-1)q}{p-1} & \text{if } p \neq 1. \end{cases}$$

Recall that in Example 2.5.3, we evaluated the number $g(n)$ of parallelograms contained in the nth subdivision of an equilateral triangle, and found that $g(n) = 3\binom{n+3}{4}$. In what follows, we shall give another proof of this result by the method of recurrence relation.

Example 6.2.2. Let a_n denote the number of parallelograms contained in the nth subdivision of an equilateral triangle. Find a recurrence relation for a_n and solve the recurrence relation.

Solution. Let ABC of Figure 6.2.4 be a given equilateral triangle. For convenience, call a point of intersection of any 2 line segments in the nth subdivision of $\triangle ABC$ a *node*. Thus there are altogether

$$1 + 2 + 3 + \cdots + (n+2) = \frac{1}{2}(n+2)(n+3)$$

nodes in the nth subdivision of $\triangle ABC$.

Clearly, $a_1 = 3$. For $n \geq 2$, observe that every parallelogram of the nth subdivision of $\triangle ABC$ contains *either* no node on BC as a vertex *or* at least one node on BC as a vertex. Thus, if we let X be the set of parallelograms of the latter case, then we have

$$a_n = a_{n-1} + |X|. \tag{1}$$

We shall now count $|X|$ indirectly. Let Y be the set of pairs $\{u, v\}$ of nodes such that u is on BC, v is not on BC, and u, v are both not contained on a common line segment. Define a correspondence from X to Y as follows: given a parallelogram in X, let u and v be the opposite vertices of the

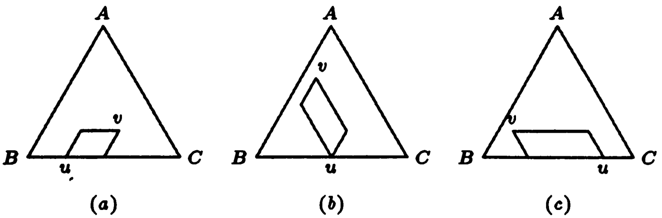


Figure 6.2.4.

parallelogram at which the angles of the parallelogram are acute (see Figure 6.2.4).

It is evident that this correspondence defines a bijection from X to Y. Thus, by (BP),

$$|X| = |Y|\,. \tag{2}$$

Since

(i) there are $n+2$ nodes u on BC,

(ii) there are $1+2+\cdots+(n+1) = \frac{1}{2}(n+1)(n+2)$ nodes v *not* on BC, and

(iii) u and v are *not* on a common line segment,

it follows, by (CP), that

$$\begin{aligned} |Y| &= (n+2)\cdot\frac{1}{2}(n+1)(n+2) - (n+2)(n+1) \\ &= \frac{1}{2}(n+1)(n+2)(n+2-2) \\ &= \frac{1}{2}n(n+1)(n+2) \\ &= 3\binom{n+2}{3}. \end{aligned} \tag{3}$$

Combining (1), (2) and (3), we arrive at the following recurrence relation

$$a_n = a_{n-1} + 3\binom{n+2}{3} \tag{6.2.3}$$

for $n \geq 2$.

To solve the recurrence relation (6.2.3), we apply the backward substitution to obtain the following:

$$\begin{aligned}
a_n &= a_{n-1} + 3\binom{n+2}{3} \\
&= a_{n-2} + 3\binom{n+1}{3} + 3\binom{n+2}{3} \\
&\ \vdots \\
&= a_1 + 3\binom{4}{3} + \cdots + 3\binom{n+1}{3} + 3\binom{n+2}{3} \\
&= 3\left\{\binom{3}{3} + \binom{4}{3} + \cdots + \binom{n+1}{3} + \binom{n+2}{3}\right\} \\
&= 3\binom{n+3}{4} \qquad \text{(by identity (2.5.1)).}
\end{aligned}$$

Thus, $a_n = 3\binom{n+3}{4}$ is the required solution. ∎

Note. The recurrence relation (6.2.3) is not a special case of the recurrence relation (6.2.2) since the term $3\binom{n+2}{3}$ in (6.2.3) depends on "n", and is thus not a constant.

6.3. Linear Homogeneous Recurrence Relations

In Examples 6.2.1 and 6.2.2, we solved the recurrence relations (6.2.1) and (6.2.3) using the method of backward substitution. Of course, not every recurrence relation can be solved in this way. In this and the next section, we shall introduce a general method that enables us to solve a class of recurrence relations, called *linear* recurrence relations. For the first step towards this end, we consider a sub-class of linear recurrence relations, called linear *homogeneous* recurrence relations.

Let (a_n) be a given sequence of numbers. A recurrence relation of the form:

$$c_0 a_n + c_1 a_{n-1} + \cdots + c_r a_{n-r} = 0 \tag{6.3.1}$$

where the c_i's are constants with $c_0, c_r \neq 0$, and $1 \leq r \leq n$, is called an *rth order linear homogeneous* recurrence relation for the sequence (a_n).

For instance, the recurrence relations

$$a_n = a_{n-1} + a_{n-2}$$

and

$$a_n - 2a_{n-1} + 3a_{n-2} - 5a_{n-3} = 0$$

are, respectively, linear homogeneous recurrence relations of 2nd and 3rd order.

Replacing the terms "a_i" by "x^i", $i = n, n-1, ..., n-r$, in (6.3.1), we obtain the following equation in "x":

$$c_0x^n + c_1x^{n-1} + c_2x^{n-2} + \cdots + c_rx^{n-r} = 0$$

or

$$c_0x^r + c_1x^{r-1} + c_2x^{r-2} + \cdots + c_{r-1}x + c_r = 0 \qquad (6.3.2)$$

The equation (6.3.2) is called the *characteristic equation* of (6.3.1). Any root of the equation (6.3.2) is called a *characteristic root* of the recurrence relation (6.3.1).

The notion of characteristic roots of a linear homogeneous recurrence relation plays a key role in the solution of the recurrence relation, which is shown in the following two results.

(I) If $\alpha_1, \alpha_2, ..., \alpha_r$ are the *distinct* characteristic roots of the recurrence relation (6.3.1), then

$$a_n = A_1(\alpha_1)^n + A_2(\alpha_2)^n + \cdots + A_r(\alpha_r)^n,$$

where the A_i's are constants, is the general solution of (6.3.1)

(II) $\alpha_1, \alpha_2, ..., \alpha_k$ $(1 \leq k \leq r)$ are the *distinct* characteristic roots of (6.3.1) such that α_i is of multiplicity m_i, $i = 1, 2, ..., k$, then the general solution of (6.3.1) is given by

$$\begin{aligned} a_n = {} & (A_{11} + A_{12}n + \cdots A_{1m_1}n^{m_1-1})(\alpha_1)^n \\ & + (A_{21} + A_{22}n + \cdots A_{2m_2}n^{m_2-1})(\alpha_2)^n \\ & + \quad \cdots \\ & + (A_{k1} + A_{k2}n + \cdots A_{km_k}n^{m_k-1})(\alpha_k)^n, \end{aligned}$$

where the A_{ij}'s are constants.

Result (I) is a clearly a special case of result (II). The proofs of these results can be found in many standard books in combinatorics (see, for instance, Roberts [12], pp. 210-213). Here, we shall only give a number of examples to illustrate the use of these results.

We first solve the recurrence relation obtained in the problem of paving a rectangle that is discussed in Section 1.

Example 6.3.1. Solve the recurrence relation

$$a_n = a_{n-1} + a_{n-2} \tag{6.1.1}$$

given that $a_0 = 1$ and $a_1 = 1$.

Notes. (1) Since the recurrence relation (6.1.1) is of 2nd order, we need two initial conditions to solve it.

(2) The original initial conditions for the paving problem are "$a_1 = 1$ and $a_2 = 2$". We replace "$a_2 = 2$" by "$a_0 = 1$" here simply for the ease of computation that could be seen later. Such a replacement does not affect the solution since $a_0 = 1$, $a_1 = 1$ and $a_2 = 2$ satisfy the recurrence relation (6.1.1).

Solution. The recurrence relation (6.1.1) may be written as

$$a_n - a_{n-1} - a_{n-2} = 0.$$

Its characteristic equation is

$$x^2 - x - 1 = 0,$$

and its characteristic roots are

$$\alpha_1 = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \alpha_2 = \frac{1-\sqrt{5}}{2}.$$

Thus, by result (I), the general solution of (6.1.1) is given by

$$a_n = A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n, \tag{1}$$

where A and B are constants to be determined.

The initial conditions $a_0 = a_1 = 1$ imply that

$$\begin{cases} A + B = 1 \\ A\left(\frac{1+\sqrt{5}}{2}\right) + B\left(\frac{1-\sqrt{5}}{2}\right) = 1, \end{cases}$$

i.e.,

$$\begin{cases} A + B = 1 \\ (A + B) + \sqrt{5}(A - B) = 2. \end{cases} \tag{2}$$

Solving the system (2) for A and B gives

$$\begin{cases} A = \frac{1+\sqrt{5}}{2\sqrt{5}} \\ B = -\frac{1-\sqrt{5}}{2\sqrt{5}}\,. \end{cases} \tag{3}$$

(Determining A and B using the initial conditions $a_1 = 1$ and $a_2 = 2$ is more tedious than the above. This is the advantage of using $a_0 = a_1 = 1$.)

By substituting (3) into (1), we obtain the desired solution of (6.1.1):

$$a_n = \left(\frac{1+\sqrt{5}}{2\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2\sqrt{5}}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n;$$

i.e.,

$$a_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]. \tag{6.3.3}$$

for all $n \geq 0$. ∎

From the recurrence relation (6.1.1) and the initial conditions $a_0 = a_1 = 1$, we obtain the first few terms of the sequence (a_n) as shown below:

$$1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ 144,\ \ldots$$

These numbers were named *Fibonacci numbers* by the French mathematician Edouard Locus (1842-1891), as they arose from a famous problem – the rabbit problem (see Problem 6.51) that was contained in the book "Liber Abaci" (1202) written by one of the great mathematical innovators of the Middle Ages, Leonardo Fibonacci (1175–1230) of Pisa. The beautiful formula (6.3.3) for the nth Fibonacci numbers is called the *Binet* formula after the French mathematician Jacques-Phillipe-Marie Binet (1786-1856). This formula was also derived independently by de Moivre (1667-1754) and D. Bernoulli (1700-1782). Locus had done a great deal of work on Fibonacci numbers, and the following variation of numbers:

$$2,\ 1,\ 3,\ 4,\ 7,\ 11,\ 18,\ 29,\ 47,\ 76,\ 123,\ 199,\ \ldots$$

bears his name.

In 1963, the American mathematician Verner E. Hoggatt, Jr. and his associates set up an organization, called the *Fibonacci Association* at the University of Santa Clara, in California, U.S.A.. Since then, the Association has been organizing a series of Fibonacci conferences, including the First International Conference on Fibonacci Numbers and Their Applications held in Patras, Greece. The Association has even been publishing an international mathematical journal, called the *Fibonacci Quarterly* for the promotion of all kinds of research related to Fibonacci numbers.

Fibonacci numbers and their related results can now be found in many branches of mathematics such as Geometry, Number Theory, Combinatorics, Linear Algebra, Numerical Analysis, Probability and Statistics, and in other disciplines outside of mathematics like Architectural Designs, Biology, Chemistry, Physics, Engineering, and so on. For those who wish to find out more about these numbers, the following books: Vorobyov [Vo], Hoggart [Hg], Vajda [Va], and the article by Honsberger (see [Hn], p.102-138) are recommended.

Example 6.3.2. Solve the recurrence relation

$$a_n - 7a_{n-1} + 15a_{n-2} - 9a_{n-3} = 0, \tag{6.3.4}$$

given that $a_0 = 1, a_1 = 2$ and $a_2 = 3$.

Solution. The characteristic equation of (6.3.4) is

$$x^3 - 7x^2 + 15x - 9 = (x-3)^2(x-1) = 0$$

and thus the characteristic roots of (6.3.4) are

$$\alpha_1 = 3 \quad \text{(of multiplicity 2)}$$
$$\text{and} \quad \alpha_2 = 1.$$

By result (II), the general solution of (6.3.4) is given by

$$\begin{aligned} a_n &= (A + Bn)(3)^n + C(1)^n; \\ \text{i.e.,}\quad a_n &= (A + Bn)3^n + C, \end{aligned} \tag{1}$$

where A, B and C are constants to be determined.
The initial conditions

$$a_0 = 1,\ a_1 = 2, \text{ and } a_2 = 3$$

imply that

$$\begin{cases} A & & +C & = 1 \\ 3A & +\ 3B & +C & = 2 \\ 9A & +18B & +C & = 3. \end{cases} \tag{2}$$

Solving the system (2) yields

$$A = 1, \quad B = -\frac{1}{3} \quad \text{and} \quad C = 0. \tag{3}$$

It follows from (1) and (3) that the required solution of (6.3.4) is given by

$$\begin{aligned} a_n &= (1 - \frac{n}{3})3^n \\ \text{or}\quad a_n &= (3 - n)3^{n-1} \quad \text{for } n \geq 0. \ \blacksquare \end{aligned}$$

In solving a polynomial equation, it is possible to obtain complex roots. In this case, it is sometimes convenient to express such roots in trigonometric form. We also note that if $\alpha = a + bi$ is a complex roots of a *real* polynomial equation $P(x) = 0$ (i.e., all the coefficients of $P(x)$ are real), then its conjugate $\bar{\alpha} = a - bi$ is also a root of $P(x) = 0$; i.e., complex roots of $P(x) = 0$ always occur in conjugate pairs. An example of this type is given below.

Example 6.3.3. Solve the recurrence relation

$$a_n = 2(a_{n-1} - a_{n-2}) \tag{6.3.5}$$

given that $a_0 = 1$ and $a_1 = 0$.

Solution. The characteristic equation of (6.3.5) is

$$x^2 - 2x + 2 = 0$$

and its roots are

$$\alpha = 1 + i \quad \text{and} \quad \bar{\alpha} = 1 - i.$$

Expressing α and $\bar{\alpha}$ in trigonometric form, we have

$$\begin{aligned} \alpha &= \sqrt{2}(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}) \\ \text{and} \quad \bar{\alpha} &= \sqrt{2}(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}). \end{aligned}$$

Thus the general solution of (6.3.5) is given by

$$\begin{aligned} a_n &= A(\alpha)^n + B(\bar{\alpha})^n \\ &= (\sqrt{2})^n \left\{A(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}) + B(\cos\frac{n\pi}{4} - i\sin\frac{n\pi}{4})\right\} \\ &= (\sqrt{2})^n(C\cos\frac{n\pi}{4} + D\sin\frac{n\pi}{4}), \end{aligned}$$

where $C = A + B$ and $D = i(A - B)$ are constants to be determined.

The initial conditions $a_0 = 1$ and $a_1 = 0$ imply that

$$\begin{cases} C = 1 \\ \sqrt{2}(\frac{\sqrt{2}}{2}C + \frac{\sqrt{2}}{2}D) = 0; \end{cases}$$

i.e. $C = 1$ and $D = -1$.

Thus the required solution of (6.3.5) is

$$a_n = (\sqrt{2})^n(\cos\frac{n\pi}{4} - \sin\frac{n\pi}{4})$$

for $n \geq 0$. ∎

6.4. General Linear Recurrence Relations

Let (a_n) be a given sequence of numbers. A recurrence relation of the form

$$c_0 a_n + c_1 a_{n-1} + \cdots + c_r a_{n-r} = f(n) \tag{6.4.1}$$

where the c_i's are constants with $c_0, c_r \neq 0$, $1 \leq r \leq n$, and f is a function of n, is called an *rth order linear recurrence relation* for the sequence (a_n).

Thus a linear homogeneous recurrence relation is a linear recurrence relation of the form (6.4.1) in which $f(n) = 0$ for all n. While the recurrence relations

$$a_n - 2a_{n-1} = 1 \tag{6.2.1}$$

$$a_n - a_{n-1} = 3\binom{n+2}{3} \tag{6.2.3}$$

are examples of linear recurrence relations of first order, the recurrence relation

$$a_n + 7a_{n-1} + 12a_{n-2} = 2^n$$

is a 2nd order linear recurrence relation.

How can we solve a linear recurrence relation of the form (6.4.1)? A way to do so is given below:

<u>Step 1°</u>. Find the general solution $a_n^{(h)}$ of the linear homogeneous recurrence relation obtained from (6.4.1)

$$c_0 a_n + c_1 a_{n-1} + \cdots + c_r a_{n-r} = 0.$$

<u>Step 2°</u>. Find a particular solution $a_n^{(p)}$ of (6.4.1).

<u>Step 3°</u>. The general solution of (6.4.1) is given by

$$a_n = a_n^{(h)} + a_n^{(p)}. \tag{6.4.2}$$

Remarks. (1) Those who are familiar with the theory of differential equations may note the analogy between the above method of solving general linear recurrence relations and a corresponding method of solving linear differential equations.

(2) The method of finding $a_n^{(h)}$ in Step 1° has been discussed in the preceding section.

(3) There is no general way of finding $a_n^{(p)}$ in Step 2°. However, $a_n^{(p)}$ could be found by inspection if the function 'f' in (6.4.1) is relatively simple. For instance, if $f(n)$ is a polynomial in n or an exponential function of n, then $a_n^{(p)}$ could be chosen as a function of similar type. We shall further elaborate this point through the following examples.

Example 6.4.1. Solve the recurrence relation

$$a_n - 3a_{n-1} = 2 - 2n^2 \tag{6.4.3}$$

given that $a_0 = 3$.

Solution. First of all, we find $a_n^{(h)}$. The characteristic equation of

$$a_n - 3a_{n-1} = 0$$

is $x - 3 = 0$, and its root is $\alpha = 3$. Thus

$$a_n^{(h)} = A \cdot 3^n, \tag{1}$$

where A is a constant.

Next, we find $a_n^{(p)}$. Since $f(n) = 2 - 2n^2$ is a polynomial in n of degree 2, we let

$$a_n^{(p)} = Bn^2 + Cn + D \tag{2}$$

where B, C and D are constants.

Since $a_n^{(p)}$ satisfies (6.4.3), we have

$$(Bn^2 + Cn + D) - 3\{B(n-1)^2 + C(n-1) + D\} = 2 - 2n^2.$$

Equating the coefficients of n^2, n and the constant terms, respectively, on both sides, we obtain:

$$\begin{cases} B - 3B = -2 \\ C + 6B - 3C = 0 \\ D - 3B + 3C - 3D = 2. \end{cases} \tag{3}$$

Solving the system (3) yields

$$B = 1,\ C = 3 \text{ and } D = 2. \tag{4}$$

It follows from (2) and (4) that

$$a_n^{(p)} = n^2 + 3n + 2.$$

By (6.4.2), the general solution of (6.4.3) is given by

$$\begin{aligned} a_n &= a_n^{(h)} + a_n^{(p)} \\ &= A \cdot 3^n + n^2 + 3n + 2. \end{aligned} \tag{5}$$

Putting $a_0 = 3$ (initial condition) in (5) gives

$$3 = A + 2, \qquad \text{i.e.,} \quad A = 1.$$

Hence the required solution of (6.4.3) is

$$a_n = 3^n + n^2 + 3n + 2, \quad n \geq 0. \ \blacksquare$$

Example 6.4.2. Solve the recurrence relation

$$a_n - 3a_{n-1} + 2a_{n-2} = 2^n \tag{6.4.4}$$

given that $a_0 = 3$ and $a_1 = 8$.

Solution. The characteristic equation of $a_n - 3a_{n-1} + 2a_{n-2} = 0$ is $x^2 - 3x + 2 = 0$, and its roots are 1 and 2. Thus

$$a_n^{(h)} = A(1)^n + B(2)^n = A + B2^n. \tag{1}$$

Corresponding to $f(n) = 2^n$, we may choose

$$a_n^{(p)} = C2^n.$$

However, as the term 2^n has appeared in (1), we need to multiply it by 'n', and set

$$a_n^{(p)} = Cn2^n. \tag{2}$$

(For a more general way to choose such particular solutions, the reader is referred to Table 6.4.1.)
Since $a_n^{(p)}$ satisfies (6.4.4), we have:

$$Cn2^n - 3C(n-1)2^{n-1} + 2C(n-2)2^{n-2} = 2^n,$$

which implies that $C = 2$.

Thus, by (2),

$$a_n^{(p)} = n2^{n+1}$$

and so the general solution of (6.4.4) is

$$\begin{aligned} a_n &= a_n^{(h)} + a_n^{(p)} \\ &= A + B2^n + n2^{n+1}. \end{aligned} \tag{3}$$

It follows from (3) and the initial conditions $a_0 = 3$ and $a_1 = 8$ that

$$\begin{cases} A + B = 3 \\ A + 2B + 4 = 8. \end{cases} \tag{4}$$

Solving the system (4) gives

$$A = 2 \quad \text{and} \quad B = 1.$$

Thus the required solution of (6.4.4) is

$$a_n = 2 + 2^n + n2^{n+1} \quad \text{for } n \geq 0. \quad \blacksquare$$

To end this section, we give in Table 6.4.1 more precise forms of $a_n^{(p)}$ for some special functions $f(n)$ in different situations.

In addition, we would like to point out that if $f(n)$ is a sum of an exponential function $f_1(n)$ and a polynomial $f_2(n)$, then $a_n^{(p)}$ can be chosen as the sum of the two particular solutions corresponding to $f_1(n)$ and $f_2(n)$.

6.5. Two Applications

In this section, we shall apply what we have learnt in the preceding two sections to solve two counting problems: one on the number of colourings of a certain map and the other on the evaluation of determinants of certain matrices.

	$f(n)$	$a_n^{(p)}$
	Exponential function	
(i)	Ak^n k is not a characteristic root	Bk^n
(ii)	Ak^n k is a characteristic root of multiplicity m	$Bn^m k^n$
	Polynomial	
(i)	$\sum_{i=0}^t p_i n^i$ 1 is not a characteristic root	$\sum_{i=0}^t q_i n^i$
(ii)	$\sum_{i=0}^t p_i n^i$ 1 is a characteristic root of multiplicity m	$n^m \sum_{i=0}^t q_i n^i$
	Special combined function	
(i)	$An^t k^n$ k is not a characteristic root	$\left(\sum_{i=0}^t q_i n^i\right) k^n$
(ii)	$An^t k^n$ k is a characteristic root of multiplicity m	$n^m \left(\sum_{i=0}^t q_i n^i\right) k^n$

Table 6.4.1

Example 6.5.1. The n sectors, $n \geq 1$, of the circle of Figure 6.5.1 are to be coloured by k distinct colours, where $k \geq 3$, in such a way that each sector is coloured by one colour and any two adjacent sectors must be coloured by different colours. Let a_n denote the number of ways this can be done.

(i) Evaluate a_1, a_2 and a_3.

(ii) Find a recurrence relation for (a_n), $n \geq 4$, and solve the recurrence relation.

Solution. (i) Evidently, we have

$$a_1 = k,$$

and

$$a_2 = k(k-1).$$

As shown in Figure 6.5.2, we have

$$a_3 = k(k-1)(k-2).$$

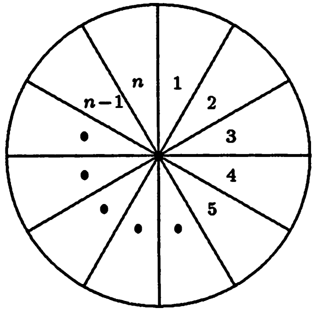


Figure 6.5.1.

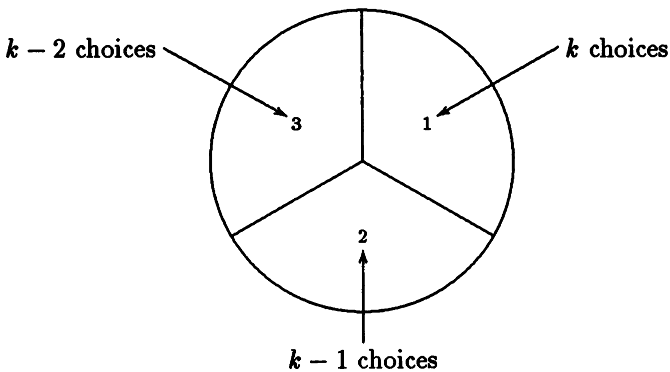


Figure 6.5.2.

(ii) It is no longer true that $a_4 = k(k-1)(k-2)(k-3)$. For $n \geq 3$, we shall obtain a recurrence relation for (a_n) in an indirect way. Imagine that the circle of Figure 6.5.1 is cut along the boundary separating sectors 1 and n as shown in Figure 6.5.3.

It is much easier now to count the number of ways to colour the n sectors of Figure 6.5.3 subject to the given conditions. The number of ways this can be done is clearly

$$\begin{array}{rccccl} & k & (k-1) & (k-1) & \cdots & (k-1) = k(k-1)^{n-1}. \\ & \uparrow & \uparrow & \uparrow & & \uparrow \\ \text{number of choices of sectors:} & 1 & 2 & 3 & & n \end{array}$$

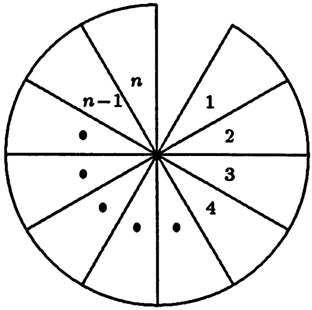


Figure 6.5.3.

These $k(k-1)^{n-1}$ ways of colourings can be divided into two groups:

(1) those colourings in which sectors 1 and n receive different colours;

(2) those colourings in which sectors 1 and n receive same colours.

Observe that the colourings in group (1) are precisely the colourings of the n sectors of Figure 6.5.1. On the other hand, there is a bijection between the colourings in group (2) and the colourings of the $(n-1)$ sectors of Figure 6.5.4.

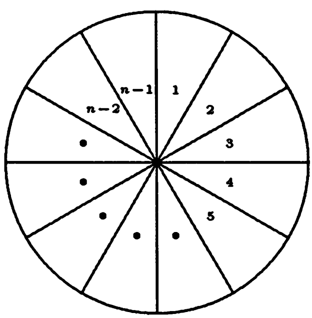


Figure 6.5.4

Thus, by (AP) and (BP), we have:

$$a_n + a_{n-1} = k(k-1)^{n-1} \tag{6.5.1}$$

where $n = 3, 4, \ldots$.

Finally, we solve the recurrence relation (6.5.1). The characteristic root of $a_n + a_{n-1} = 0$ is $\alpha = -1$, and so

$$a_n^{(h)} = A(-1)^n,$$

where A is a constant.

Since $f(n) = k(k-1)^{n-1}$, let

$$a_n^{(p)} = B(k-1)^{n-1}.$$

As $a_n^{(p)}$ satisfies (6.5.1), we have:

$$B(k-1)^{n-1} + B(k-1)^{n-2} = k(k-1)^{n-1};$$

i.e.,

$$Bk(k-1)^{n-2} = k(k-1)^{n-1}.$$

Thus $B = k-1$ and so $a_n^{(p)} = (k-1)^n$. Hence the general solution of (6.5.1) is

$$\begin{aligned} a_n &= a_n^{(h)} + a_n^{(p)} \\ &= A(-1)^n + (k-1)^n. \end{aligned}$$

Since $a_3 = k(k-1)(k-2)$, it follows that

$$k(k-1)(k-2) = -A + (k-1)^3,$$

and so

$$\begin{aligned} A &= (k-1)^3 - k(k-1)(k-2) \\ &= (k-1)(k^2 - 2k + 1 - k^2 + 2k) \\ &= k-1. \end{aligned}$$

Consequently, the required solution is

$$a_n = (-1)^n(k-1) + (k-1)^n$$

for all $n \geq 3$ and $k \geq 3$. ∎

Example 6.5.2. The $n \times n$ determinant a_n is defined for $n \geq 1$ by

$$a_n = \begin{vmatrix} p & p-q & 0 & 0 & \cdots & 0 & 0 \\ q & p & p-q & 0 & \cdots & 0 & 0 \\ 0 & q & p & p-q & \cdots & 0 & 0 \\ 0 & 0 & q & p & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & p & p-q \\ 0 & 0 & 0 & 0 & \cdots & q & p \end{vmatrix}$$

where p and q are distinct nonzero constants. Find a recurrence relation for (a_n), and solve the recurrence relation.

Note. In the following solution, we assume that the reader is familiar with the cofactor expansion of a determinant.

Solution. By applying the cofactor expansion of a_n along the first row, it follows that

$$a_n = p\underbrace{\left.\begin{vmatrix} p & p-q & 0 & \cdots & 0 & 0 \\ q & p & p-q & \cdots & 0 & 0 \\ 0 & q & p & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & p & p-q \\ 0 & 0 & 0 & \cdots & q & p \end{vmatrix}\right\}}_{n-1} n-1$$

$$- (p-q)\underbrace{\left.\begin{vmatrix} q & p-q & 0 & \cdots & 0 & 0 \\ 0 & p & p-q & \cdots & 0 & 0 \\ 0 & q & p & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & p & p-q \\ 0 & 0 & 0 & \cdots & q & p \end{vmatrix}\right\}}_{n-1} n-1$$

Now by applying the cofactor expansion of the second determinant along the first column, we have

$$a_n = pa_{n-1} - (p-q)qa_{n-2}, \tag{6.5.2}$$

which is the desired recurrence relation.

The characteristic equation of (6.5.2) is

$$x^2 - px + (p-q)q = 0$$

and its roots are $p-q$ and q.

Case 1. $p - q \neq q$.

In this case, as the roots are distinct, the general solution is

$$a_n = A(p-q)^n + Bq^n \tag{1}$$

To find A and B, we first evaluate a_1 and a_2. Clearly,

$$a_1 = p$$

and

$$a_2 = \begin{vmatrix} p & p-q \\ q & p \end{vmatrix} = p^2 - q(p-q) = p^2 + q^2 - pq.$$

We define a_0 as the number such that a_0, a_1 and a_2 satisfy the recurrence relation (6.5.2). Thus

$$a_2 = pa_1 - (p-q)qa_0,$$

and so

$$p^2 + q^2 - pq = p^2 - (p-q)qa_0,$$

which implies that

$$a_0 = 1.$$

Letting $a_0 = 1$ and $a_1 = p$ in (1) gives

$$\begin{cases} A + B = 1 \\ A(p-q) + Bq = p. \end{cases} \tag{2}$$

Solving the system (2) yields

$$A = \frac{p-q}{p-2q} \quad \text{and} \quad B = \frac{-q}{p-2q}.$$

Note that $p - 2q \neq 0$ in this case. Hence the desired solution is

$$a_n = \frac{(p-q)^{n+1} - q^{n+1}}{p-2q}.$$

Case 2. $p - q = q$.

In this case, q is a root of multiplicity 2, and the general solution is

$$a_n = (A + Bn)q^n. \tag{3}$$

Since $a_0 = 1$ and $a_1 = p$, it follows that

$$\begin{cases} A = 1 \\ (A+B)q = p = 2q; \end{cases}$$

i.e.,

$$A = B = 1.$$

Thus

$$a_n = (1+n)q^n$$

is the required solution of (6.5.2) in this case.

We conclude that

$$a_n = \begin{cases} \frac{(p-q)^{n+1} - q^{n+1}}{p-2q} & \text{if } p \neq 2q, \\ (1+n)q^n & \text{if } p = 2q. \end{cases} \blacksquare$$

6.6. A System of Linear Recurrence Relations

In the preceding three sections, we learned how to solve a *single* linear recurrence relation for a given sequence (a_n). In this section, we shall proceed one step further to consider systems of linear recurrence relations for two given sequences (a_n) and (b_n), that are of the form:

$$\begin{cases} a_n = pa_{n-1} + qb_{n-1} \\ b_n = rb_{n-1} + sa_{n-1}, \end{cases} \tag{6.6.1}$$

where p, q, r and s are arbitrary constants.

Recall that the method of *substitution* is an easy and standard method of solving "systems of equations". This method can similarly be used to solve systems of recurrence relations of the form (6.6.1).

Example 6.6.1. Solve the system of recurrence relations

$$\begin{cases} a_n + 2a_{n-1} - 4b_{n-1} = 0 & (1) \\ b_n + 5a_{n-1} - 7b_{n-1} = 0 & (2) \end{cases}$$

given that $a_1 = 4$ and $b_1 = 1$.

Solution. From (1),

$$b_{n-1} = \frac{1}{4}(a_n + 2a_{n-1}) \tag{3}$$

Substituting into (2) gives

$$\frac{1}{4}(a_{n+1} + 2a_n) + 5a_{n-1} - 7\{\frac{1}{4}(a_n + 2a_{n-1})\} = 0$$

or
$$a_{n+1} - 5a_n + 6a_{n-1} = 0. \tag{4}$$

The characteristic equation of (4) is

$$x^2 - 5x + 6 = 0$$

and its roots are 2 and 3.
Thus the general solution of (4) is

$$a_n = A2^n + B3^n \tag{5}$$

where A and B are constants.

Substituting into (3) gives

$$\begin{aligned} b_n &= \frac{1}{4}(a_{n+1} + 2a_n) \\ &= \frac{1}{4}(A2^{n+1} + B3^{n+1} + 2A2^n + 2B3^n) \\ &= \frac{1}{4}(4A2^n + 5B3^n). \end{aligned} \tag{6}$$

As $a_1 = 4$ and $b_1 = 1$, we have

$$\begin{cases} 2A + 3B = 4 \\ 2A + \frac{15}{4}B = 1. \end{cases} \tag{7}$$

Solving (7) gives

$$A = 8 \quad \text{and} \quad B = -4.$$

Hence

$$\begin{cases} a_n = 2^{n+3} - 4 \cdot 3^n \\ b_n = 2^{n+3} - 5 \cdot 3^n \end{cases}$$

for $n \geq 1$, are the required solutions. ∎

We shall now see how a system of linear recurrence relations can be set up to solve an IMO problem.

Example 6.6.2. **(IMO, 1979/6)** Let A and E be opposite vertices of a regular octagon. A frog starts jumping at vertex A. From any vertex of the octagon except E, it may jump to either of the two adjacent vertices. When it reaches vertex E, the frog stops and stays there. Let a_n be the number of distinct paths of exactly n jumps ending at E. Prove that

$$a_{2n-1} = 0$$

and $$a_{2n} = \frac{1}{\sqrt{2}}\{(2+\sqrt{2})^{n-1} - (2-\sqrt{2})^{n-1}\}, \quad n = 1, 2, 3, \ldots .$$

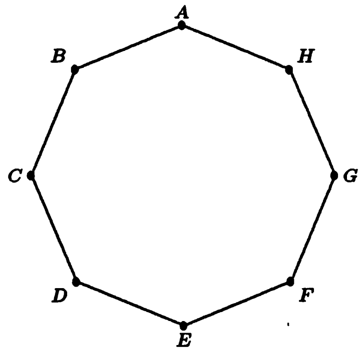


Figure 6.6.1.

Solution. The given regular octagon is shown in Figure 6.6.1. Since the number of edges in any path joining A and E is even, it is impossible to reach E from A in an odd number of jumps. Thus $a_{2n-1} = 0$, for all $n \geq 1$.

It is obvious that $a_2 = 0$. Also, as $ABCDE$ and $AHGFE$ are the only 2 different paths of 4 jumps from A to E, we have $a_4 = 2$.

To find a recurrence relation for a_{2n}, we introduce a new supplementary sequence (b_n) as follows: For each $n \geq 1$, let b_n be the number of paths of exactly n jumps from C (or G) to E.

Starting at A, there are 4 ways for the frog to move in the first 2 jumps, namely,

$$A \to B \to A, \quad A \to H \to A$$
$$A \to B \to C, \quad A \to H \to G.$$

Thus, by definitions of (a_n) and (b_n),

$$a_{2n} = 2a_{2n-2} + 2b_{2n-2}. \tag{1}$$

On the other hand, starting at C, there are 3 ways for the frog to move in the next 2 jumps if it does not stop at E, namely,

$$C \to B \to C, \quad C \to D \to C, \quad C \to B \to A.$$

Thus,

$$b_{2n} = 2b_{2n-2} + a_{2n-2}. \tag{2}$$

We shall now solve the system of linear recurrence relations (1) and (2). From (1),

$$b_{2(n-1)} = \frac{1}{2}a_{2n} - a_{2(n-1)}. \tag{3}$$

Substituting (3) into (2) gives

$$\frac{1}{2}a_{2(n+1)} - a_{2n} = a_{2n} - 2a_{2(n-1)} + a_{2(n-1)}$$

or

$$a_{2(n+1)} - 4a_{2n} + 2a_{2(n-1)} = 0. \tag{4}$$

Let $d_n = a_{2n}$. Then (4) may be written as

$$d_{n+1} - 4d_n + 2d_{n-1} = 0. \tag{5}$$

The characteristic equation of (5) is

$$x^2 - 4x + 2 = 0,$$

and its roots are $2 \pm \sqrt{2}$. Thus the general solution of (4) is

$$a_{2n} = d_n = A(2+\sqrt{2})^n + B(2-\sqrt{2})^n, \tag{6}$$

for $n \geq 1$, where A and B are constants.

To find A and B, we use the initial conditions $d_1 = a_2 = 0$ and $d_2 = a_4 = 2$. We define d_0 to be the number such that d_0, d_1 and d_2 satisfy (5). Thus

$$d_2 - 4d_1 + 2d_0 = 0,$$

and so

$$d_0 = \frac{1}{2}(4d_1 - d_2) = -1.$$

From (6) and the initial conditions that $d_0 = -1$ and $d_1 = 0$, it follows that

$$\begin{cases} A + B = -1 \\ A(2+\sqrt{2}) + B(2-\sqrt{2}) = 0, \end{cases} \tag{7}$$

Solving (7) gives

$$A = \frac{1}{\sqrt{2}}\left(\frac{1}{2+\sqrt{2}}\right) \quad \text{and} \quad B = -\frac{1}{\sqrt{2}}\left(\frac{1}{2-\sqrt{2}}\right).$$

Thus, by (6), the required solution is

$$a_{2n} = \frac{1}{\sqrt{2}}\left\{(2+\sqrt{2})^{n-1} - (2-\sqrt{2})^{n-1}\right\}, \quad \text{for } n \geq 1. \blacksquare$$

6.7. The Method of Generating Functions

In Chapter 5, we introduced the concept of a generating function for a given sequence and witnessed that, as a mathematical tool, generating functions are very powerful in solving combinatorial problems. In this section, we shall discuss how generating functions can be used also to help solve certain recurrence relations. We note that the method of generating functions could incorporate the initial conditions to find both $a_n^{(h)}$ and $a_n^{(p)}$ simultaneously when it is used to solve linear recurrence relations.

Example 6.7.1. Solve the recurrence relation

$$a_n - 2a_{n-1} = 2^n \tag{6.7.1}$$

given that $a_0 = 1$.

Solution. Let $A(x)$ be the generating function for the sequence (a_n). Then

$$A(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n + \cdots$$
$$-2xA(x) = -2a_0x - 2a_1x^2 - 2a_2x^3 - \cdots - 2a_{n-1}x^n - \cdots .$$

By taking their sum, we have

$$\begin{aligned}(1-2x)A(x) &= a_0 + (a_1 - 2a_0)x + (a_2 - 2a_1)x^2 + \cdots \\ &\quad + (a_n - 2a_{n-1})x^n + \cdots .\end{aligned}$$

Since $a_0 = 1$ and the sequence (a_n) satisfies (6.7.1), it follows that

$$\begin{aligned}(1-2x)A(x) &= 1 + 2x + 2^2x^2 + \cdots + 2^nx^n + \cdots \\ &= \frac{1}{1-2x} .\end{aligned}$$

Thus

$$A(x) = \left(\frac{1}{1-2x}\right)^2 = \sum_{r=0}^{\infty}(r+1)(2x)^r$$

and hence

$$a_n = (n+1)2^n,$$

for $n \geq 0$. ∎

As shown in the above solution, in applying the method of generating functions to solve a recurrence relation for a sequence (a_n), we first form the generating function $A(x)$ for the sequence (a_n). By using the given recurrence relation and initial conditions, we obtain $A(x)$ through some algebraic manipulations. Finally, the solution of recurrence relation is obtained by taking $a_n = g(n)$ where $g(n)$ is the coefficient of x^n in the expansion of $A(x)$.

Example 6.7.2. Solve, by the method of generating function, the recurrence relation

$$a_n - 5a_{n-1} + 6a_{n-2} = 5^n, \tag{6.7.2}$$

given that $a_0 = 0$ and $a_1 = 1$.

Solution. Let $A(x)$ be the generating function for the sequence (a_n). Then

$$\begin{aligned} A(x) &= a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots \\ -5xA(x) &= -5a_0x - 5a_1x^2 - 5a_2x^3 - 5a_3x^4 - \cdots \\ 6x^2A(x) &= 6a_0x^2 + 6a_1x^3 + 6a_2x^4 + \cdots \end{aligned}$$

and so

$$\begin{aligned} A(x)(1-5x+6x^2) &= x + \sum_{i=2}^{\infty}(a_i - 5a_{i-1} + 6a_{i-2})x^i \\ &= x + \sum_{i=2}^{\infty} 5^i x^i \\ &= x + \frac{(5x)^2}{1-5x} \\ &= \frac{25x^2 + x - 5x^2}{1-5x} \\ &= \frac{20x^2+x}{1-5x}. \end{aligned}$$

Thus,

$$A(x) = \frac{20x^2+x}{(1-5x)(1-2x)(1-3x)} = \frac{B}{1-5x} + \frac{C}{1-2x} + \frac{D}{1-3x}. \tag{1}$$

It follows from (1) that

$$20x^2 + x = B(1-2x)(1-3x) + C(1-5x)(1-3x) + D(1-5x)(1-2x). \tag{2}$$

Putting $x = 0$ in (2) gives

$$B + C + D = 0. \tag{3}$$

Putting $x = \frac{1}{2}$ in (2) gives

$$5 + \frac{1}{2} = C(1-\frac{5}{2})(1-\frac{3}{2}) = (-\frac{3}{2})(-\frac{1}{2})C,$$

or

$$C = \frac{22}{3} .$$

Putting $x = \frac{1}{3}$ in (2) gives

$$\frac{20}{9} + \frac{1}{3} = D(1 - \frac{5}{3})(1 - \frac{2}{3}) = D(-\frac{2}{3})(\frac{1}{3}) = \frac{-2}{9} D$$

or

$$D = -\frac{23}{2} .$$

Thus, by (3), we have

$$B = -C - D = \frac{25}{6} .$$

Hence,

$$\begin{aligned} A(x) &= \frac{25}{6}\left(\frac{1}{1-5x}\right) + \frac{22}{3}\left(\frac{1}{1-2x}\right) - \frac{23}{2}\left(\frac{1}{1-3x}\right) \\ &= \frac{25}{6}\sum_{i=0}^{\infty}(5x)^i + \frac{22}{3}\sum_{i=0}^{\infty}(2x)^i - \frac{23}{2}\sum_{i=0}^{\infty}(3x)^i \end{aligned}$$

and so the required solution is

$$a_n = \frac{25}{6} \cdot 5^n + \frac{22}{3} \cdot 2^n - \frac{23}{2} \cdot 3^n,$$

for $n \geq 1$. ∎

Finally, we use the method of generating functions to solve a system of linear recurrence relations.

Example 6.7.3. Solve the system of recurrence relations

$$\begin{cases} a_n + 2a_{n-1} + 4b_{n-1} = 0 \\ b_n - 4a_{n-1} - 6b_{n-1} = 0, \end{cases} \tag{6.7.3}$$

given that $a_0 = 1$ and $b_0 = 0$.

Solution. Let $A(x)$ be the generating function for (a_n) and $B(x)$ the generating function for (b_n). Then

$$\begin{array}{rcllll} A(x) & = & 1 & +a_1x & +a_2x^2 & +a_3x^3+\cdots \\ B(x) & = & & b_1x & +b_2x^2 & +b_3x^3+\cdots \\ 2xA(x) & = & & 2x & +2a_1x^2 & +2a_2x^3+\cdots \\ 4xB(x) & = & & & 4b_1x^2 & +4b_2x^3+\cdots \end{array}$$

and so

$$\begin{aligned}(1+2x)A(x)+4xB(x) &= 1+(a_1+2)x+(a_2+2a_1+4b_1)x^2 \\ &\quad +(a_3+2a_2+4b_2)x^3+\cdots \\ &= 1,\end{aligned}$$

by the first recurrence relation of (6.7.3).

Also,

$$\begin{array}{rcllllr} -4xA(x) & = & -4x & -4a_1x^2 & -4a_2x^3 & -4a_3x^4 & -\cdots \\ -6xB(x) & = & & -6b_1x^2 & -6b_2x^3 & -6b_3x^4 & -\cdots, \end{array}$$

and so

$$\begin{aligned}B(x)-4xA(x)-6xB(x) &= (b_1-4)x+(b_2-4a_1-6b_1)x^2 \\ &\quad +(b_3-4a_2-6b_2)x^3+\cdots,\end{aligned}$$

i.e.,

$$(1-6x)B(x)-4xA(x)=0,$$

by the second recurrence relation of (6.7.3). Thus we have

$$\begin{cases}(1+2x)A(x)+4xB(x)=1 \\ -4xA(x)+(1-6x)B(x)=0\end{cases} \tag{1}$$

Solving the system (1) for $A(x)$ and $B(x)$ gives

$$A(x)=\frac{1-6x}{(1-2x)^2} \quad \text{and} \quad B(x)=\frac{4x}{(1-2x)^2}.$$

Now,

$$A(x)=(1-6x)\sum_{r=0}^{\infty}(r+1)(2x)^r$$

and

$$B(x)=4x\sum_{r=0}^{\infty}(r+1)(2x)^r.$$

Hence, by determining the coefficients of x^n in $A(x)$ and $B(x)$ respectively, we obtain the desired solutions:

$$a_n=(n+1)2^n-6n2^{n-1}=2^n(1-2n)$$

and

$$b_n=4n\cdot 2^{n-1}=n2^{n+1},$$

for all $n\geq 1$. ∎

6.8. A Nonlinear Recurrence Relation and Catalan Numbers

We begin with a famous combinatorial problem of dissecting a polygon by means of nonintersecting diagonals into triangles, that was studied by Euler and others about 200 years ago. A way of solving the problem leads to an important class of nonlinear recurrence relations. These nonlinear recurrence relations can be solved by the method of generating functions.

Example 6.8.1. A *triangulation* of an n-gon P_n, $n \geq 3$, is a subdivision of P_n into triangles by means of nonintersecting diagonals of P_n. Let $a_0 = 1$ and for $n \geq 1$, let a_n be the number of different triangulations of an $(n+2)$-gon. Show that for $n \geq 1$,

$$a_n = a_0 a_{n-1} + a_1 a_{n-2} + a_2 a_{n-3} + \cdots + a_{n-2} a_1 + a_{n-1} a_0, \qquad (6.8.1)$$

and solve the recurrence relation.

Solution. It is obvious that $a_1 = 1$. For $n = 2, 3$, the respective triangulations of P_4 and P_5 are shown in Figure 6.8.1, which show that

$$a_2 = 2 \quad \text{and} \quad a_2 = 5.$$

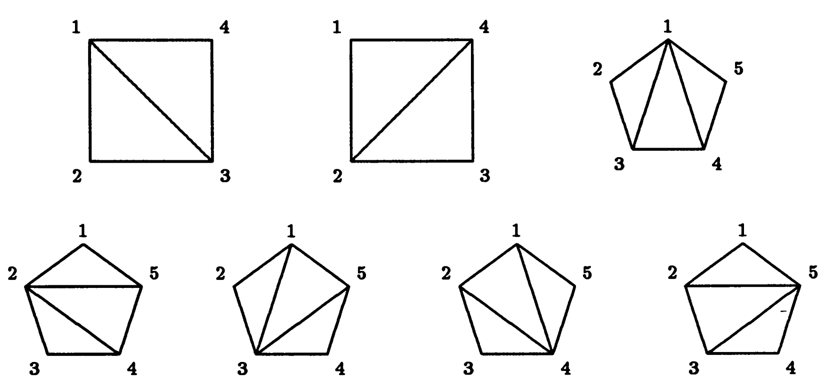


Figure 6.8.1.

To establish the recurrence relation (6.8.1), we form the $(n+2)$-gon P_{n+2} of Figure 6.8.2 and fix a triangulation of this P_{n+2}, where $n \geq 2$. (Note that the recurrence relation (6.8.1) holds trivially when $n = 1$.) Choose an arbitrary side, say $[1, n+2]$, which joins vertices 1 and $n+2$ of P_{n+2}. Clearly, $[1, n+2]$ belongs to a unique triangle of this triangulation. Denote the 3rd vertex of this triangle by r ($r = 2, 3, \ldots, n+1$), and the triangle by $\Delta(1, n+2, r)$ (see Figure 6.8.2). Observe that $\Delta(1, n+2, r)$ divides P_{n+2} into 2 smaller polygons:

the r-gon (1) and the $(n+3-r)$-gon (2)

of Figure 6.8.2. By definition, the r-gon (1) can be triangulated in a_{r-2} ways while the $(n+3-r)$-gon (2) in a_{n+1-r} ways independently. Thus, by (MP), the number of different triangulations of P_{n+2} in which $\Delta(1, n+2, r)$ occurs is

$$a_{r-2}a_{n+1-r}.$$

As the value of r ranges from 2 to $n+1$, it follows by (AP) that

$$a_n = \sum_{r=2}^{n+1} a_{r-2}a_{n+1-r},$$

i.e.,

$$a_n = \sum_{k=0}^{n-1} a_k a_{n-1-k}, \tag{6.8.1}$$

for all $n \geq 1$.

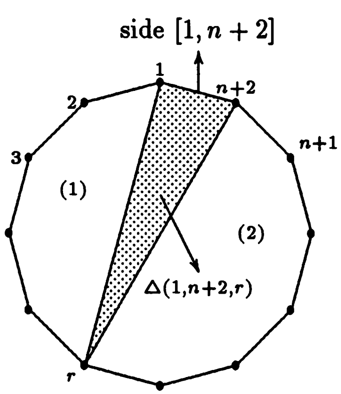


Figure 6.8.2.

We shall now solve (6.8.1) by the method of generating functions.

Let $A(x) = \sum_{n=0}^{\infty} a_n x^n$ be the generating function for (a_n). Then

$$\begin{aligned} A(x) - a_0 &= \sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} \left(\sum_{k=0}^{n-1} a_k a_{n-1-k} \right) x^n \\ &= (a_0 a_0)x + (a_0 a_1 + a_1 a_0)x^2 + (a_0 a_2 + a_1 a_1 + a_2 a_0)x^3 + \cdots \\ &= (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots)(a_0 x + a_1 x^2 + a_2 x^3 + \cdots) \\ &= x(A(x))^2. \end{aligned}$$

Thus, as $a_0 = 1$,

$$x(A(x))^2 - A(x) + 1 = 0.$$

Solving for $A(x)$ yields

$$\begin{aligned} A(x) &= \frac{1 \pm \sqrt{1-4x}}{2x} \\ &= \frac{1}{2x}\{1 \pm (1-4x)^{\frac{1}{2}}\}. \end{aligned} \tag{1}$$

The coefficient of x^n $(n \geq 1)$ in $(1-4x)^{\frac{1}{2}}$ is

$$\begin{aligned} \binom{\frac{1}{2}}{n}(-4)^n &= \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)\cdots(\frac{1}{2}-n+1)}{n!}(-4)^n \\ &= (-4)^n (\frac{1}{2})^n (-1)^{n-1} \frac{1 \cdot 1 \cdot 3 \cdot 5 \cdots (2n-3)}{n!} \\ &= -2^n \frac{(2n-2)!}{n! 2 \cdot 4 \cdot 6 \cdots (2n-2)} \\ &= -2^n \frac{(2n-2)!}{n! 2^{n-1} \cdot (n-1)!} \\ &= -\frac{2}{n} \frac{(2n-2)!}{(n-1)!(n-1)!} \\ &= -\frac{2}{n} \binom{2(n-1)}{n-1}. \end{aligned}$$

Since $a_n \geq 1$, it follows from (1) that

$$\begin{aligned} A(x) &= \frac{1}{2x}\{1 - (1-4x)^{\frac{1}{2}}\} \\ &= \frac{1}{2x} \sum_{n=1}^{\infty} \frac{2}{n} \binom{2(n-1)}{n-1} x^n. \end{aligned}$$

Hence, for $n \geq 1$,

$$a_n = \frac{1}{2} \cdot \frac{2}{n+1}\binom{2n}{n} = \frac{1}{n+1}\binom{2n}{n}. \blacksquare$$

Recall that the numbers $\frac{1}{n+1}\binom{2n}{n}$, called the *Catalan* numbers, was introduced in Example 2.6.4 as the number of increasing mappings $\alpha : \mathbf{N}_n \to \mathbf{N}_{n-1}$ with the property that $\alpha(a) < a$ for each a in $\mathbf{N}_n$. The above problem of dissecting an n-gon was first studied by Johann Andreas von Seguer (1704-1777) and then independently by L. Euler (1707-1783). Some other examples which give rise to these numbers are shown in Table 6.8.1. For more examples, references and generalizations of these interesting numbers, the reader may refer to Cohen [4], Breckenridge et.al. [B], Gardner [Ga], Gould [Go], Guy [Gu], Hilton and Pedersen [HP] and Chu [C].

6.9. Oscillating Permutations and an Exponential Generating Function

In this final section, we shall introduce another interesting problem studied by the French combinatorist D. André (1840-1917) in 1879 (see André [A1, A2] and also Honsberger [Hn], 69-75) on a special kind of permutations. The notion of generating functions has just been used in Sections 7 and 8 to help solve certain recurrence relations. We shall see here how this problem gives rise to a system of nonlinear recurrence relations and how the notion of exponential generating functions can be used to solve the recurrence relations.

Let S be a set of n natural numbers. A permutation $e_1e_2\cdots e_n$ of S is said to be *oscillating* if the following condition is satisfied:

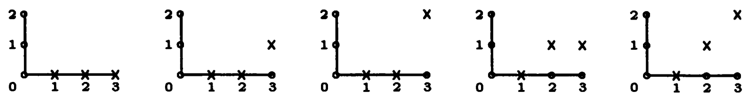
$$e_1 < e_2 > e_3 < e_4 > e_5 < \cdots$$

where

$$\begin{cases} e_{n-1} > e_n & \text{if } n \text{ is odd} \\ e_{n-1} < e_n & \text{if } n \text{ is even.} \end{cases}$$

The *length* of a permutation $e_1e_2\cdots e_n$ of n numbers is defined to be n. Table 6.9.1 shows all oscillating permutations of length n of $\mathbf{N}_n$ $(= \{1, 2, ..., n\})$, for each $n = 1, 2, 3, 4, 5$.

The numbers of increasing mappings $\alpha : \{1, 2, 3\} \to \{0, 1, 2\}$ such that $\alpha(a) < a$



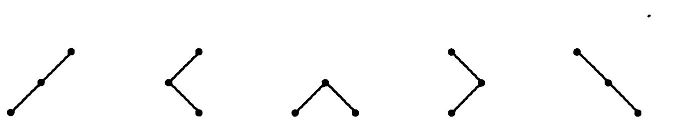
The number of ways of putting brackets in $x_1x_2x_3x_4$

$(((x_1x_2)x_3)x_4)$ $((x_1(x_2x_3))x_4)$ $((x_1x_2)(x_3x_4))$ $(x_2((x_2x_3)x_4))$ $(x_1(x_2(x_3x_4)))$

The number of sequences with 3 1's and 3 (-1)'s such that each partial sum is nonnegative

$(1, 1, 1, -1, -1, -1)$, $(1, 1, -1, 1, -1, -1)$, $(1, 1, -1, -1, 1, -1)$,
$(1, -1, 1, 1, -1, -1)$, $(1, -1, 1, -1, 1, -1)$

The number of rooted binary trees with 3 nodes



The number of ways to pair off 6 points on a circle by nonintersecting chords

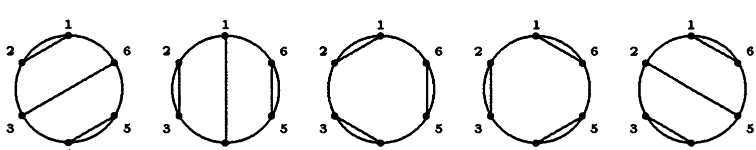


Table 6.8.1.

Length	Oscillating Permutations	No. of such permutations
1	1	1
2	12	1
3	132,231	2
4	1324,1423,2314,2413,3412	5
5	13254,14253,14352,15243, 15342,23154,24153,24351, 25143,25341,34152,34251, 35142,35241,45132,45231	16

Table 6.9.1

The problem that André considered is:

"What is the number of oscillating permutations of $\mathbf{N}_n$ for each $n \geq 1$?"

Before presenting André's solution, we have the following simple but useful observation.

Observation. Let $S = \{s_1, s_2, ..., s_n\}$ be any set of n distinct natural numbers. The number of oscillating permutations of S is the same as the number of oscillating permutations of $\mathbf{N}_n$.

This observation simply says that as far as the number of oscillating permutations of S is concerned, only the size $|S|$ of S matters and the magnitudes of numbers in S are not important. The observation can easily be verified by establishing a bijection between the oscillating permutations of S and those of $\mathbf{N}_n$. For instance, if $S = \{3, 5, 6, 9\}$, then from the correspondence:

$$3 \leftrightarrow 1,\ 5 \leftrightarrow 2,\ 6 \leftrightarrow 3,\ 9 \leftrightarrow 4,$$

one can establish the following bijection:

$$3659 \leftrightarrow 1324$$

$$3956 \leftrightarrow 1423$$

$$5639 \leftrightarrow 2314$$

$$5936 \leftrightarrow 2413$$

$$6935 \leftrightarrow 3412.$$

For convenience, given an oscillating permutation $e_1e_2\cdots e_n$, we underline e_i if e_i is smaller than its adjacent number (or numbers). Thus, we have

$$\begin{array}{l}\underline{1}3\underline{2}4\\ \underline{3}5\underline{1}4\underline{2}\\ \underline{2}4\underline{1}6\underline{3}5\\ \underline{2}7\underline{1}5\underline{4}6\underline{3}\ ;.\end{array}$$

Clearly, with this convention, an oscillating permutation $e_1e_2\cdots e_n$ ends up with an underlined e_n iff n is odd.

According to the parity of n, André introduced the following two sequences (a_n) and (b_n): For $n \geq 0$,

$$a_n = \begin{cases} \text{the no. of oscillating permutations of } \mathbf{N}_n & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even,} \end{cases}$$

$$b_n = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ 1 & \text{if } n = 0, \\ \text{the no. of oscillating permutations of } \mathbf{N}_n & \text{if } n \geq 2 \text{ is even.} \end{cases}$$

It follows readily that for each $n \geq 1$, $a_n + b_n$ counts the number of oscillating permutations of $\mathbf{N}_n$.

Claim 1. For odd n,

$$a_n = \sum_{k=0}^{n-1} \binom{n-1}{k} a_k a_{n-1-k}. \tag{6.9.1}$$

Proof. Let $\underline{e_1}e_2\underline{e_3}\cdots e_{n-1}\underline{e_n}$ (n odd) be an oscillating permutation of $\mathbf{N}_n$ and let $e_{k+1} = n$. Clearly, $e_{k+1}(= n)$ splits the permutation into 2 oscillating permutations of odd length k and $n-1-k$ respectively, where $k = 1, 3, 5, \ldots, n-2$:

$$\underbrace{(\underline{e_1}e_2\underline{e_3}\cdots e_{k-1}\underline{e_k})}_{\substack{\text{odd}\\ \text{(left)}}}\, e_{k+1}\, \underbrace{(\underline{e_{k+2}}\cdots e_{n-1}\underline{e_n})}_{\substack{\text{odd}\\ \text{(right)}}}$$

Since there are

(i) $\binom{n-1}{k}$ ways of choosing k numbers from $\mathbf{N}_{n-1}$ (to be arranged on the left),

(ii) a_k ways to form an oscillating permutation of odd length k on the left (see observation), and

(iii) a_{n-1-k} ways to form oscillating permutations of odd length $n-1-k$ on the right,

and since the value of k ranges from $1, 3, 5, \ldots$ to $n-2$, we have

$$a_n = \binom{n-1}{1} a_1 a_{n-2} + \binom{n-1}{3} a_3 a_{n-4} + \cdots + \binom{n-1}{n-2} a_{n-2} a_1.$$

As $a_n = 0$ if n is even, it follows that

$$a_n = \sum_{k=0}^{n-1} \binom{n-1}{k} a_k a_{n-1-k},$$

as required. ∎

Claim 2. For even n,

$$b_n = \sum_{k=0}^{n-1} \binom{n-1}{k} a_k b_{n-1-k} \tag{6.9.2}$$

Proof. Let $\underline{e_1} e_2 \cdots \underline{e_k} e_{k+1} \underline{e_{k+2}} \cdots \underline{e_{n-1}} e_n$ (n even) be an oscillating permutation of $\mathbf{N}_n$ where $e_{k+1} = n$. Again, $e_{k+1} (= n)$ splits the permutation into an oscillating permutation of odd length k on the left and an oscillating permutation of even length $n-1-k$ on the right, where $k = 1, 3, 5, \ldots, n-1$:

$$\underbrace{(\underline{e_1} e_2 \underline{e_3} \cdots e_{k-1} \underline{e_k})}_{\substack{\text{odd} \\ \text{(left)}}} \; e_{k+1} \; \underbrace{(\underline{e_{k+2}} e_{k+3} \cdots \underline{e_{n-1}} e_n)}_{\substack{\text{even} \\ \text{(right)}}}$$

Since there are

(i) $\binom{n-1}{k}$ ways of choosing k numbers from $\mathbf{N}_{n-1}$,

(ii) a_k ways to form an oscillating permutation of odd length k on the left, and

(iii) b_{n-1-k} ways to form oscillating permutations of even length $n-1-k$ on the right, it follows similarly that

$$b_n = \sum_{k=0}^{n-1} \binom{n-1}{k} a_k b_{n-1-k},$$

as desired. ∎

We are now ready to use the notion of exponential generating function to solve the recurrence relations (6.9.1) and (6.9.2). First, let

$$A(x) = \sum_{k=0}^{\infty} a_k \frac{x^k}{k!}$$

be the exponential generating function for the sequence (a_n).

Claim 3. $1 + A(x)^2 = A'(x)$.

Proof. Indeed,

$$\begin{aligned}
&1 + A(x)^2 \\
&= 1 + (a_0 + a_1\frac{x}{1!} + a_2\frac{x^2}{2!} + a_3\frac{x^3}{3!} + \cdots)^2 \\
&= 1 + a_0a_0 + (a_0a_1 + a_1a_0)x + \left(\frac{a_0a_2}{2!} + \frac{a_1a_1}{1!1!} + \frac{a_2a_0}{2!}\right)x^2 \\
&\qquad + (\frac{a_0a_3}{3!} + \frac{a_1a_2}{1!2!} + \frac{a_2a_1}{2!1!} + \frac{a_3a_0}{3!})x^3 + \cdots,
\end{aligned}$$

and the coefficient of $\frac{x^{n-1}}{(n-1)!}$ $(n \geq 2)$ in this expansion is

$$\begin{aligned}
&(n-1)!\left(\frac{a_0a_{n-1}}{(n-1)!} + \frac{a_1a_{n-2}}{1!(n-2)!} + \cdots + \frac{a_{n-1}a_0}{(n-1)!}\right) \\
&= \sum_{k=0}^{n-1} \frac{(n-1)!}{k!(n-1-k)!} a_k a_{n-1-k} \\
&= \sum_{k=0}^{n-1} \binom{n-1}{k} a_k a_{n-1-k} \\
&= a_n,
\end{aligned}$$

by (6.9.1) and that $a_n = 0$ if n is even.

On the other hand,

$$A'(x) = a_1 + a_2\frac{x}{1!} + a_3\frac{x^2}{2!} + a_4\frac{x^3}{3!} + \cdots$$

and the coefficient of $\frac{x^{n-1}}{(n-1)!}$ $(n \geq 2)$ in this expansion is a_n. Further, the constant term in $1 + A(x)^2$ is $1 + a_0 a_0 = 1$, which is the same as $a_1 = 1$, the constant term in $A'(x)$. We thus conclude that

$$1 + A(x)^2 = A'(x).$$

Let $B(x) = \sum_{k=0}^{\infty} b_k \frac{x^k}{k!}$ be the exponential generating function for the sequence (b_n).

Claim 4. $A(x)B(x) = B'(x)$.

Proof. Indeed,

$$\begin{aligned}
&A(x)B(x)\\
&= (a_0 + a_1\frac{x}{1!} + a_2\frac{x^2}{2!} + a_3\frac{x^3}{3!} + \cdots)(b_0 + b_1\frac{x}{1!} + b_2\frac{x^2}{2!} + b_3\frac{x^3}{3!} + \cdots)\\
&= a_0b_0 + (\frac{a_0b_1}{1!} + \frac{a_1b_0}{1!})x + (\frac{a_0b_2}{2!} + \frac{a_1b_1}{1!1!} + \frac{a_2b_0}{2!})x^2 + \cdots
\end{aligned}$$

and the coefficient of $\frac{x^{n-1}}{(n-1)!}$ $(n \geq 2)$ in this expansion is

$$\begin{aligned}
&(n-1)!\left(\frac{a_0b_{n-1}}{(n-1)!} + \frac{a_1b_{n-2}}{1!(n-2)!} + \cdots + \frac{a_{n-1}b_0}{(n-1)!}\right)\\
&= \sum_{k=0}^{n-1} \binom{n-1}{k} a_k b_{n-1-k}\\
&= b_n,
\end{aligned}$$

by (6.9.2) and the definition that $a_{k-1} = b_k = 0$ if k is odd.

On the other hand,

$$B'(x) = b_1 + b_2\frac{x}{1!} + b_3\frac{x^2}{2!} + \cdots$$

and the coefficient of $\frac{x^{n-1}}{(n-1)!}$ $(n \geq 2)$ in this expansion is b_n.

Further, the constant terms in $A(x)B(x)$ and $B'(x)$ are, respectively, $a_0b_0 = 0$ and $b_1 = 0$, which are equal. We thus conclude that

$$A(x)B(x) = B'(x). \quad \blacksquare$$

Our final task is to determine the generating functions $A(x)$ and $B(x)$ from Claim 3 and Claim 4.

By Claim 3, we have

$$\int \frac{A'(x)}{1+A(x)^2}dx = \int dx$$

and so $\tan^{-1} A(x) = x + C$.
When $x = 0$, $C = \tan^{-1} A(0) = \tan^{-1}(a_0) = \tan^{-1} 0 = 0$. Thus

$$\tan^{-1} A(x) = x$$

and

$$A(x) = \tan x \tag{1}$$

By Claim 4 and (1), we have

$$\frac{B'(x)}{B(x)} = A(x) = \tan x$$

and so

$$\int \frac{B'(x)}{B(x)}dx = \int \tan x dx$$

or

$$\ln B(x) = \ln(\sec x) + C.$$

When $x = 0$,

$$\begin{aligned} C &= \ln B(0) - \ln(\sec 0) \\ &= \ln \frac{B(0)}{\sec 0} = \ln \frac{b_0}{1} \\ &= \ln 1 = 0. \end{aligned}$$

Thus $\ln B(x) = \ln(\sec x)$ and so

$$B(x) = \sec x\,. \tag{2}$$

Consequently, the desired number of oscillating permutations of $\mathbf{N}_n$ is $a_n + b_n$, which is the coefficient of $\frac{x^n}{n!}$ in $\tan x + \sec x$.

The first 11 terms of the series expansion of $\tan x$ and $\sec x$ are given below:

$$\tan x = \frac{x^1}{1!} + 2\cdot\frac{x^3}{3!} + 16\cdot\frac{x^5}{5!} + 272\cdot\frac{x^7}{7!} + 7936\cdot\frac{x^9}{9!} + \cdots$$

$$\sec x = 1 + \frac{x^2}{2!} + 5\cdot\frac{x^4}{4!} + 61\cdot\frac{x^6}{6!} + 1385\cdot\frac{x^8}{8!} + 50521\cdot\frac{x^{10}}{10!} + \cdots$$

and the first 11 terms of $a_n + b_n$ are

1, 1, 1, 2, 5, 16, 61, 272, 1385, 7936, 50521.

Exercise 6

1. Solve
$$a_n = 3a_{n-1} - 2a_{n-2},$$
given that $a_0 = 2$ and $a_1 = 3$.

2. Solve
$$a_n - 6a_{n-1} + 9a_{n-2} = 0,$$
given that $a_0 = 2$ and $a_1 = 3$.

3. Solve
$$a_n = \frac{1}{2}(a_{n-1} + a_{n-2}),$$
given that $a_0 = 0$ and $a_1 = 1$.

4. Solve
$$a_n - 4a_{n-1} + 4a_{n-2} = 0,$$
given that $a_0 = -\frac{1}{4}$ and $a_1 = 1$.

5. Solve
$$2a_n = a_{n-1} + 2a_{n-2} - a_{n-3},$$
given that $a_0 = 0$, $a_1 = 1$ and $a_2 = 2$.

6. Solve
$$a_n - 6a_{n-1} + 11a_{n-2} - 6a_{n-3} = 0,$$
given that $a_0 = \frac{1}{3}$, $a_1 = 1$ and $a_2 = 2$.

7. Solve
$$a_n = -a_{n-1} + 16a_{n-2} - 20a_{n-3},$$
given that $a_0 = 0$, $a_1 = 1$ and $a_2 = -1$.

8. Find the general solution of the recurrence relation
$$a_n + a_{n-1} - 3a_{n-2} - 5a_{n-3} - 2a_{n-4} = 0.$$

9. Solve
$$a_n = \frac{1}{2}a_{n-1} - 3,$$
given that $a_0 = 2(3 + \sqrt{3})$.

10. Solve

$$a_n - 3a_{n-1} = 3 \cdot 2^n - 4n,$$

given that $a_1 = 2$.

11. Solve

$$a_n - a_{n-1} = 4n - 1,$$

given that $a_0 = 1$.

12. Solve

$$a_n = pa_{n-1} + q,$$

given that $a_0 = r$, where p, q and r are constants.

13. Let (a_n) be a sequence of numbers such that
    (i) $a_0 = 1$, $a_1 = \frac{3}{5}$ and
    (ii) the sequence $(a_n - \frac{1}{10}a_{n-1})$ is a geometric progression with common ratio $\frac{1}{2}$.

    Find a general formula for a_n, $n \geq 0$.

14. Solve

$$a_n^4 a_{n-1} = 10^{10},$$

given that $a_0 = 1$ and $a_n > 0$ for all n.

15. A sequence (a_n) of positive numbers satisfies

$$a_n = 2\sqrt{a_{n-1}}$$

with the initial condition $a_0 = 25$. Show that $\lim_{n\to\infty} a_n = 4$.

16. A sequence (a_n) of numbers satisfies

$$\left(\frac{a_n}{a_{n-1}}\right)^2 = \frac{a_{n-1}}{a_{n-2}}$$

with the initial conditions $a_0 = \frac{1}{4}$ and $a_1 = 1$. Solve the recurrence relation.

17. Solve

$$a_n + 3a_{n-1} = 4n^2 - 2n + 2^n,$$

given that $a_0 = 1$.

18. Solve

$$a_n - 2a_{n-1} + 2a_{n-2} = 0,$$

given that $a_0 = 1$ and $a_1 = 2$.

19. Solve

$$a_n - 4a_{n-1} + 4a_{n-2} = 2^n,$$

given that $a_0 = 0$ and $a_1 = 3$.

20. Solve

$$a_n - a_{n-1} - 2a_{n-2} = 4n,$$

given that $a_0 = -4$ and $a_1 = -5$.

21. Solve

$$a_n + a_{n-1} - 2a_{n-2} = 2^{n-2},$$

given that $a_0 = a_1 = 0$.

22. Solve

$$a_n - 3a_{n-1} + 2a_{n-2} = 2^n,$$

given that $a_0 = 0$ and $a_1 = 5$.

23. Solve

$$a_n + 5a_{n-1} + 6a_{n-2} = 3n^2,$$

given that $a_0 = 0$ and $a_1 = 1$.

24. Let (a_n) be a sequence of numbers satisfying the recurrence relation

$$pa_n + qa_{n-1} + ra_{n-2} = 0$$

with the initial conditions $a_0 = s$ and $a_1 = t$, where p, q, r, s, t are constants such that $p + q + r = 0$, $p \neq 0$ and $s \neq t$. Solve the recurrence relation.

25. Let (a_n) be a sequence of numbers satisfying the recurrence relation

$$a_n = \frac{pa_{n-1} + q}{ra_{n-1} + s}$$

where p, q, r and s are constants with $r \neq 0$.

(i) Show that

$$ra_n + s = p + s + \frac{qr - ps}{ra_{n-1} + s}. \tag{1}$$

(ii) By the substitution $ra_n + s = \frac{b_{n+1}}{b_n}$, show that (1) can be reduced to the second order linear homogeneous recurrence relation for (b_n):

$$b_{n+1} - (p+s)b_n + (ps - qr)b_{n-1} = 0.$$

26. Solve

$$a_n = \frac{3a_{n-1}}{2a_{n-1}+1},$$

given that $a_o = \frac{1}{4}$.

27. Solve

$$a_n = \frac{3a_{n-1}+1}{a_{n-1}+3},$$

given that $a_o = 5$.

28. A sequence (a_n) of numbers satisfies the condition

$$(2 - a_n)a_{n+1} = 1, \qquad n \geq 1.$$

Find $\lim_{n\to\infty} a_n$.

29. For $n \in \mathbf{N}$, recall that D_n is the number of derangements of the set $\mathbf{N}_n$. Prove by a combinatorial argument that

$$D_n = (n-1)(D_{n-1} + D_{n-2}).$$

30. For $n \in \mathbf{N}$, let a_n denote the number of ternary sequences of length n in which no two 0's are adjacent. Find a recurrence relation for (a_n) and solve the recurrence relation.

31. Let $C_0, C_1, C_2, \ldots$ be the sequence of circles in the Cartesian plane defined as follows:

(1) C_0 is the circle $x^2 + y^2 = 1$,

(2) for $n = 0, 1, 2, \ldots$, the circle C_{n+1} lies in the upper half-plane and is tangent to C_n as well as to both branches of the hyperbola $x^2 - y^2 = 1$.

Let a_n be the radius of C_n.

(i) Show that $a_n = 6a_{n-1} - a_{n-2}, \quad n \geq 2$.

(ii) Deduce from (i) that a_n is an integer and

$$a_n = \frac{1}{2}\left[(3+2\sqrt{2})^n + (3-2\sqrt{2})^n\right].$$

(Proposed by B. A. Reznick, see *Amer. Math. Monthly*, **96** (1989), 262.)

32. The $n \times n$ determinant a_n is defined by

$$a_n = \begin{vmatrix} p+q & pq & 0 & 0 & \cdots & 0 & 0 \\ 1 & p+q & pq & 0 & \cdots & 0 & 0 \\ 0 & 1 & p+q & pq & \cdots & 0 & 0 \\ 0 & 0 & 1 & p+q & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & p+q & pq \\ 0 & 0 & 0 & 0 & \cdots & 1 & p+q \end{vmatrix}$$

where p and q are nonzero constants. Find a recurrence relation for (a_n), and solve the recurrence relation.

33. Consider the following $n \times n$ determinant:

$$a_n = \begin{vmatrix} pq+1 & q & 0 & 0 & \cdots & 0 \\ p & pq+1 & q & 0 & \cdots & 0 \\ 0 & p & pq+1 & q & \cdots & 0 \\ 0 & 0 & p & pq+1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & q \\ 0 & 0 & 0 & 0 & \cdots & pq+1 \end{vmatrix}$$

where p and q are nonzero constants. Find a recurrence relation for (a_n), and solve the recurrence relation.

34. Given $n \in \mathbf{N}$, find the number of n-digit positive integers which can be formed from $1, 2, 3, 4$ such that 1 and 2 are not adjacent.

35. A $2 \times n$ rectangle ($n \in \mathbf{N}$) is to be paved with 1×2 identical blocks and 2×2 identical blocks. Let a_n denote the number of ways that can be done. Find a recurrence relation for (a_n), and solve the recurrence relation.

36. For $n \in \mathbf{N}$, let a_n denote the number of ways to pave a $3 \times n$ rectangle $ABCD$ with 1×2 identical dominoes. Clearly, $a_n = 0$ if n is odd. Show that

$$a_{2r} = \frac{1}{2\sqrt{3}}\left\{(\sqrt{3}+1)(2+\sqrt{3})^r + (\sqrt{3}-1)(2-\sqrt{3})^r\right\},$$

where $r \in \mathbf{N}$. (Proposed by I. Tomescu, see *Amer. Math. Monthly*, **81** (1974), 522-523.)

37. Solve the system of recurrence relations:

$$\begin{cases} a_{n+1} = a_n - b_n \\ b_{n+1} = a_n + 3b_n, \end{cases}$$

given that $a_0 = -1$ and $b_0 = 5$.

38. Solve the system of recurrence relations:

$$\begin{cases} a_n + 2a_{n-1} + 4b_{n-1} = 0 \\ b_n - 4a_{n-1} - 6b_{n-1} = 0, \end{cases}$$

given that $a_0 = 1$ and $b_0 = 0$.

39. Solve the system of recurrence relations:

$$\begin{cases} 10a_n = 9a_{n-1} - 2b_{n-1} \\ 5b_n = -a_{n-1} + 3b_{n-1}, \end{cases}$$

given that $a_0 = 4$ and $b_0 = 3$.

40. Solve the system of recurrence relations:

$$\begin{cases} 3a_n - 2a_{n-1} - b_{n-1} = 0 \\ 3b_n - a_{n-1} - 2b_{n-1} = 0, \end{cases}$$

given that $a_0 = 2$ and $b_0 = -1$.

41. Let (a_n) and (b_n) be two sequences of positive numbers satisfying the recurrence relations:

$$\begin{cases} a_n^2 = a_{n-1}b_n \\ b_n^2 = a_n b_{n-1} \end{cases}$$

with the initial conditions $a_0 = \frac{1}{8}$ and $b_0 = 64$. Show that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n,$$

and find the common limit.

42. For $n \in \mathbf{N}^*$, let a_n, b_n, c_n and d_n denote the numbers of binary sequences of length n satisfying the respective conditions:

	Number of 0's	Number of 1's
a_n	even	even
b_n	even	odd
c_n	odd	even
d_n	odd	odd

(i) Show that

$$a_n = b_{n-1} + c_{n-1},$$
$$b_n = a_{n-1} + d_{n-1} = c_n,$$
$$d_n = b_{n-1} + c_{n-1}.$$

(ii) Let $A(x)$, $B(x)$, $C(x)$ and $D(x)$ be, respectively, the generating functions of the sequences (a_n), (b_n), (c_n) and (d_n). Show that

$$A(x) = \frac{1-2x^2}{1-4x^2},$$
$$B(x) = C(x) = \frac{x}{1-4x^2},$$
$$D(x) = \frac{2x^2}{1-4x^2}.$$

(iii) Deduce from (ii) that

$$a_n = (-2)^{n-2} + 2^{n-2} \quad (n \geq 1),$$
$$b_n = c_n = -(-2)^{n-2} + 2^{n-2} \quad (n \geq 0),$$
$$d_n = (-2)^{n-2} + 2^{n-2} \quad (n \geq 1).$$

43. Three given sequences $(a_n), (b_n)$ and (c_n) satisfy the following recurrence relations:

$$a_{n+1} = \frac{1}{2}(b_n + c_n - a_n),$$
$$b_{n+1} = \frac{1}{2}(c_n + a_n - b_n),$$

and

$$c_{n+1} = \frac{1}{2}(a_n + b_n - c_n),$$

with the initial conditions $a_0 = p$, $b_0 = q$ and $c_0 = r$, where p, q, r are positive constants.

(i) Show that $a_n = \frac{1}{3}(p+q+r)(\frac{1}{2})^n + (-1)^n\frac{1}{3}(2p-q-r)$ for all $n \geq 0$.

(ii) Deduce that if $a_n > 0$, $b_n > 0$ and $c_n > 0$ for all $n \geq 0$, then $p = q = r$.

44. For $n \in \mathbf{N}$, let F_n denote the nth Fibonacci number. Thus

$$F_1 = 1, \quad F_2 = 1, \quad F_3 = 2, \quad F_4 = 3, \quad F_5 = 5, \quad F_6 = 8, \quad \ldots$$

and by Example 6.3.1,

$$F_{n+2} = F_n + F_{n+1}$$

and

$$F_n = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right\}.$$

Show that

(i) $\sum_{r=1}^{n} F_r = F_{n+2} - 1;$

(ii) $\sum_{r=1}^{n} F_{2r} = F_{2n+1} - 1;$

(iii) $\sum_{r=1}^{n} F_{2r-1} = F_{2n};$

(iv) $\sum_{r=1}^{n} (-1)^{r+1} F_r = (-1)^{n+1} F_{n-1} + 1.$

45. Show that for $m, n \in \mathbf{N}$ with $n \geq 2$,

(i) $F_{m+n} = F_m F_{n-1} + F_{m+1} F_n;$

(ii) $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix};$

(iii) $F_{n+1} F_{n-1} - F_n^2 = (-1)^n;$

(iv) $F_{n+1}^2 = 4F_n F_{n-1} + F_{n-2}^2, \quad n \geq 3;$

(v) $(F_n, F_{n+1}) = 1$, where (a, b) denotes the HCF of a and b.

Remark. In general, $(F_m, F_n) = F_{(m,n)}$. Also, $F_m | F_n$ iff $m | n$.

46. Show that for $n \geq 2$

(i) $F_n^2 + F_{n-1}^2 = F_{2n-1};$

(ii) $F_{n+1}^2 - F_{n-1}^2 = F_{2n};$

(iii) $F_{n+1}^3 + F_n^3 - F_{n-1}^3 = F_{3n}.$

47. Show that

(i) $\sum_{r=1}^{n} F_r^2 = F_n F_{n+1},$

(ii) $\sum_{r=1}^{2n-1} F_r F_{r+1} = F_{2n}^2$,

(iii) $\sum_{r=1}^{2n} F_r F_{r+1} = F_{2n+1}^2 - 1$.

48. Show that for $n \in \mathbf{N}^*$,

$$F_{n+1} = \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-r}{r}.$$

49. Show that for $m, n \in \mathbf{N}$,

$$\sum_{r=0}^{n} \binom{n}{r} F_{m+r} = F_{m+2n}.$$

50. Show that

$$\lim_{n\to\infty} \frac{F_n}{F_{n+1}} = \frac{\sqrt{5}-1}{2} \approx 0.618.$$

Note. The constant $\frac{\sqrt{5}-1}{2}$ is called the *golden number.*

51. Beginning with a pair of baby rabbits, and assuming that each pair gives birth to a new pair each month starting from the 2nd month of its life, find the number a_n of pairs of rabbits at the end of the nth month. (Fibonacci, Liber Abaci, 1202.)

52. Show that for $n \in \mathbf{N}^*$,

$$F_{n+1} = \begin{vmatrix} 1 & -1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 1 & -1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 1 & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & -1 \\ 0 & 0 & 0 & 0 & \dots & 1 & 1 \end{vmatrix}.$$

53. A man wishes to climb an n-step staircase. Let a_n denote the number of ways that this can be done if in each step he can cover either one step or two steps. Find a recurrence relation for (a_n).

54. Given $n \in \mathbf{N}$, find the number of binary sequences of length n in which no two 0's are adjacent.

55. For $n \in \mathbf{N}$ with $n \geq 2$, let a_n denote the number of ways to express n as a sum of positive integers greater than 1, taking order into account. Find a recurrence relation for (a_n) and determine the value of a_n.

56. Find the number of subsets of $\{1, 2, \ldots, n\}$, where $n \in \mathbf{N}$, that contain no consecutive integers. Express your answer in terms of a Fibonacci number.

57. Prove that
$$\sum_{j=0}^{n} \frac{\binom{n}{2j-n-1}}{5^j} = \frac{1}{2}(0.4)^n F_n.$$
(Proposed by S. Rabinowitz, see *Crux Mathematicorum*, **10** (1984), 269.)

58. Call an ordered pair (S, T) of subsets of $\{1, 2, \ldots, n\}$ *admissible* if $s > |T|$ for each $s \in S$, and $t > |S|$ for each $t \in T$. How many admissible ordered pairs of subsets of $\{1, 2, \ldots, 10\}$ are there? Prove your answer. (Putnam, 1990)

59. For each $n \in \mathbf{N}$, let a_n denote the number of natural numbers N satisfying the following conditions: the sum of the digits of N is n and each digit of N is taken from $\{1, 3, 4\}$. Show that a_{2n} is a perfect square for each $n = 1, 2, \ldots$. (Chinese Math. Competition, 1991)

60. Find a recurrence relation for a_n, the number of ways to place parentheses to indicate the order of multiplication of the n numbers $x_1 x_2 x_3 \ldots x_n$, where $n \in \mathbf{N}$.

61. For $n \in \mathbf{N}$, let b_n denote the number of sequences of $2n$ terms:
$$z_1, z_2, \ldots, z_{2n},$$
where each z_i is either 1 or -1 such that
(1) $\sum_{i=1}^{2n} z_i = 0$ and
(2) $\sum_{i=1}^{k} z_i \geq 0$ for each $k = 1, 2, \ldots, 2n-1$.
(i) Find b_n for $n = 1, 2, 3$.
(ii) Establish a bijection between the set of all sequences of $2n$ terms as defined above and the set of all parenthesized expressions of the $n+1$ numbers $x_1 x_2 \ldots x_n x_{n+1}$.

62. For $n \in \mathbf{N}$, let a_n denote the number of ways to pair off $2n$ distinct points on the circumference of a circle by n nonintersecting chords. Find a recurrence relation for (a_n).

63. Let $p(x_1, x_2, \ldots, x_n)$ be a polynomial in n variables with constant term 0, and let $\#(p)$ denote the number of distinct terms in p after terms with like exponents have been collected. Thus for example $\#((x_1+x_2)^5) = 6$. Find a formula for $\#(q_n)$ where

$$q_n = x_1(x_1 + x_2)(x_1 + x_2 + x_3)\cdots(x_1 + \cdots + x_n).$$

(Proposed by J. O. Shallit, see *Amer. Math. Monthly*, **93** (1986), 217-218.)

64. Find the total number of ways of arranging in a row the $2n$ integers $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ with the restriction that for each i, a_i precede b_i, a_i precede a_{i+1} and b_i precede b_{i+1}. (Proposed by E. Just, see *Amer. Math. Monthly*, **76** (1969), 419-420.)

65. Mr. Chen and Mr. Lim are the two candidates taking part in an election. Assume that Mr. Chen receives m votes and Mr. Lim receives n votes, where $m, n \in \mathbf{N}$ with $m > n$. Find the number of ways that the ballots can be arranged in such a way that when they are counted, one at a time, the number of votes for Mr. Chen is always more than that for Mr. Lim.

66. For $n \in \mathbf{N}$, let a_n denote the number of mappings $f : \mathbf{N}_n \to \mathbf{N}_n$ such that if $j \in f(\mathbf{N}_n)$, then $i \in f(\mathbf{N}_n)$ for all i with $1 \le i \le j$.

(i) Find the values of a_1, a_2 and a_3 by listing all such mappings f.

(ii) Show that

$$a_n = \sum_{k=1}^{n} \binom{n}{k} a_{n-k}.$$

(iii) Let $A(x)$ be the exponential generating function for (a_n), where $a_0 = 1$. Show that

$$A(x) = \frac{1}{2 - e^x}.$$

(iv) Deduce that

$$a_n = \sum_{r=0}^{\infty} \frac{r^n}{2^{r+1}}.$$

67. Define S_0 to be 1. For $n \geq 1$, let S_n be the number of $n \times n$ matrices whose elements are nonnegative integers with the property that $a_{ij} = a_{ji}$ $(i, j = 1, 2, \ldots, n)$ and where $\sum_{i=1}^{n} a_{ij} = 1$, $(j = 1, 2, \ldots, n)$. Prove
   (a) $S_{n+1} = S_n + nS_{n-1}$,
   (b) $\sum_{n=0}^{\infty} S_n \frac{x^n}{n!} = e^{x+\frac{x^2}{2}}$.
   (Putnam, 1967)

68. A sequence (a_n) of numbers satisfies the following conditions:
   (1) $a_1 = \frac{1}{2}$ and
   (2) $a_1 + a_2 + \cdots + a_n = n^2 a_n$, $n \geq 1$.
   Determine the value of a_n.

69. What is the sum of the greatest odd divisors of the integers $1, 2, 3, \ldots, 2^n$, where $n \in \mathbf{N}$? (West German Olympiad, 1982)
   (*Hint*: Let a_n be the sum of the greatest odd divisors of $1, 2, 3, \ldots, 2^n$. Show that $a_n = a_{n-1} + 4^{n-1}$.)

70. Let d_n be the determinant of the $n \times n$ matrix in which the element in the ith row and the jth column is the absolute value of the difference of i and j. Show that
   $$d_n = (-1)^{n-1}(n-1)2^{n-2}.$$
   (Putnam, 1969)

71. A sequence (a_n) of natural numbers is defined by $a_1 = 1$, $a_2 = 3$ and
   $$a_n = (n+1)a_{n-1} - na_{n-2} \qquad (n \geq 2).$$
   Find all values of n such that $11 | a_n$.

72. A sequence (a_n) of positive numbers is defined by
   $$a_n = \frac{1}{16}\left(1 + 4a_{n-1} + \sqrt{1 + 24a_{n-1}}\right)$$
   with $a_0 = 1$. Find a general formula for a_n.

73. A sequence (a_n) of numbers is defined by
   $$2a_n = 3a_{n-1} + \sqrt{5a_{n-1}^2 + 4} \quad (n \geq 1)$$
   with $a_0 = 0$. Show that for all $m \geq 1$, $1992 \nmid a_{2m+1}$.

74. Solve the recurrence relation

$$na_n = (n-2)a_{n-1} + (n+1),$$

given that $a_0 = 0$.

75. Solve the recurrence relation

$$n(n-1)a_n - (n-2)^2 a_{n-2} = 0,$$

given that $a_0 = 0$ and $a_1 = 1$.

76. A sequence (a_n) of numbers satisfies the recurrence relation

$$(a_n - a_{n-1})f(a_{n-1}) + g(a_{n-1}) = 0$$

with the initial condition $a_0 = 2$, where

$$f(x) = 3(x-1)^2 \quad \text{and} \quad g(x) = (x-1)^3.$$

Solve the recurrence relation.

77. A sequence (a_n) of numbers satisfies the recurrence relation

$$n(n-1)a_n = (n-1)(n-2)a_{n-1} - (n-3)a_{n-2}$$

with the initial conditions $a_0 = 1$ and $a_1 = 2$.
Find the value of

$$\sum_{k=0}^{1992} \frac{a_k}{a_{k+1}}.$$

78. Let $a(n)$ be the number of representations of the positive integer n as the sums of 1's and 2's taking order into account. For example, since

$$\begin{aligned} 4 &= 1+1+2 = 1+2+1 = 2+1+1 \\ &= 2+2 = 1+1+1+1, \end{aligned}$$

then $a(4) = 5$. Let $b(n)$ be the number of representations of n as the sum of integers greater than 1, again taking order into account and counting the summand n. For example, since $6 = 4+2 = 2+4 = 3+3 = 2+2+2$, we have $b(6) = 5$. Show that for each n, $a(n) = b(n+2)$. (Putnam, 1957)

79. Show that the sum of the first n terms in the binomial expansion of $(2-1)^{-n}$ is $\frac{1}{2}$, where $n \in \mathbf{N}$. (Putnam, 1967)

80. Prove that there exists a unique function f from the set $\mathbf{R}^+$ of positive real numbers to $\mathbf{R}^+$ such that

$$f(f(x)) = 6x - f(x)$$

and $f(x) > 0$ for all $x > 0$. (Putnam, 1988)

81. Let $T_0 = 2$, $T_1 = 3$, $T_2 = 6$, and for $n \geq 3$,

$$T_n = (n+4)T_{n-1} - 4nT_{n-2} + (4n-8)T_{n-3}.$$

The first few terms are

$$2, 3, 6, 14, 40, 152, 784, 5168, 40576.$$

Find, with proof, a formula for T_n of the form $T_n = A_n + B_n$, where (A_n) and (B_n) are well-known sequences. (Putnam, 1990)

82. Let $\{a_n\}$ and $\{b_n\}$ denote two sequences of integers defined as follows:

$$a_0 = 1, \;\; a_1 = 1, \;\; a_n = a_{n-1} + 2a_{n-2} \quad (n \geq 2),$$
$$b_0 = 1, \;\; b_1 = 7, \;\; b_n = 2b_{n-1} + 3b_{n-2} \quad (n \geq 2).$$

Thus, the first few terms of the sequences are:

$$a : 1, 1, 3, 5, 11, 21, \ldots$$
$$b : 1, 7, 17, 55, 161, 487, \ldots$$

Prove that, except for the "1", there is no term which occurs in both sequences. (USA MO, 1973)

83. The sequence $\{x_n\}$ is defined as follows: $x_1 = 2$, $x_2 = 3$, and

$$x_{2m+1} = x_{2m} + x_{2m-1}, \quad m \geq 1$$
$$x_{2m} = x_{2m-1} + 2x_{2m-2}, \quad m \geq 2.$$

Determine x_n (as a function of n). (Austrian MO, 1983)

84. Determine the number of all sequences $(x_1, x_2, \ldots, x_n)$, with $x_i \in \{a, b, c\}$ for $i = 1, 2, \ldots, n$ that satisfy $x_1 = x_n = a$ and $x_i \neq x_{i+1}$ for $i = 1, 2, \ldots, n-1$. (18th Austrian MO)

85. The sequence x_1, x_2, ... is defined by the equalities $x_1 = x_2 = 1$ and

$$x_{n+2} = 14x_{n+1} - x_n - 4, \quad n \geq 1.$$

Prove that each number of the given sequence is a perfect square. (Bulgarian MO, 1987)

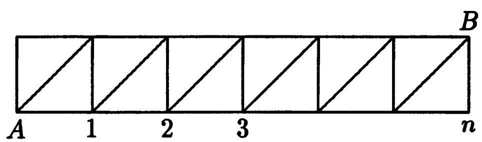
86. How many words with n digits can be formed from the alphabet $\{0, 1, 2, 3, 4\}$, if adjacent digits must differ by exactly one? (West Germany, 1987)

87. The sequence (a_n) of integers is defined by

$$-\frac{1}{2} < a_{n+1} - \frac{a_n^2}{a_{n-1}} \leq \frac{1}{2}$$

with $a_1 = 2$ and $a_2 = 7$. Show that a_n is odd for all values of $n \geq 2$. (British MO, 1988)

88. In the network illustrated by the figure below, where there are n adjacent squares, what is the number of paths (not necessarily shortest) from A to B which do not pass through any intersection twice?



(Proposed by P. Andrews and E. T. H. Wang, see *CRUX Mathematicorum*, **14** (1988), 62-64.)

89. Let $a_1 = 1$ and $a_{n+1} = a_n + \lfloor\sqrt{a_n}\rfloor$ for $n \in \mathbb{N}$. Show that a_n is a square iff $n = 2^k + k - 2$ for some $k \in \mathbb{N}$. (Proposed by T. C. Brown, see *Amer. Math. Monthly*, **85** (1978), 52-53.)

90. Determine all pairs (h, s) of positive integers with the following property: If one draws h horizontal lines and another s lines which satisfy

(i) they are not horizontal,

(ii) no two of them are parallel,

(iii) no three of the $h + s$ lines are concurrent,

then the number of regions formed by these $h+s$ lines is 1992. (APMO, 1992)

91. Show that

$$S(r,n) = \sum_{k=n-1}^{r-1} \binom{r-1}{k} S(k,n-1),$$

where $r \geq n \geq 2$.

92. Let $B_0 = 1$ and for $r \in \mathbf{N}$, let $B_r = \sum_{n=1}^{r} S(r,n)$ denote the rth Bell number (see Section 1.7). Show that

$$B_r = \sum_{k=0}^{r-1} \binom{r-1}{k} B_k,$$

where $r \geq 1$.

93. Two sequences $P(m,n)$ and $Q(m,n)$ are defined as follows (m, n are integers). $P(m,0) = 1$ for $m \geq 0$, $P(0,n) = 0$ for $n \geq 1$, $P(m,n) = 0$ for $m,n < 0$. $P(m,n) = \sum_{j=0}^{n} P(m-1,j)$ for $m \geq 1$. $Q(m,n) = P(m-1,n) + P(m-1,n-1) + P(m-1,n-2)$ for $m \geq 1$. Express $Q(m,n)$ in terms of m and n for $m \geq 1$. (Proposed by L. Kuipers, see *Amer. Math. Monthly*, **76** (1969), 97-98.)

94. For $n, k \in \mathbf{N}$, let $S_k(n) = \sum_{j=1}^{n} j^k$ (see Problem 2.85). Show that

(i) $S_k(n) = n^{k+1} - \sum_{r=0}^{k-1} \binom{k}{r} S_{r+1}(n-1)$ for $n \geq 2$,

(ii) $(k+1)S_k(n) = (n+1)^{k+1} - (n+1)^k - \sum_{r=0}^{k-2} \binom{k}{r} S_{r+1}(n).$

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