Chapter 3

Convex functions

3.1 Basic properties and examples

3.1.1 Definition

A function $f : \mathbf{R}^n \to \mathbf{R}$ is *convex* if $\mathbf{dom}\, f$ is a convex set and if for all x, $y \in \mathbf{dom}\, f$, and θ with $0 \leq \theta \leq 1$, we have

$$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta)f(y). \tag{3.1}$$

Geometrically, this inequality means that the line segment between $(x, f(x))$ and $(y, f(y))$, which is the *chord* from x to y, lies above the graph of f (figure 3.1). A function f is *strictly convex* if strict inequality holds in (3.1) whenever $x \neq y$ and $0 < \theta < 1$. We say f is *concave* if $-f$ is convex, and *strictly concave* if $-f$ is strictly convex.

For an affine function we always have equality in (3.1), so all affine (and therefore also linear) functions are both convex and concave. Conversely, any function that is convex and concave is affine.

A function is convex if and only if it is convex when restricted to any line that intersects its domain. In other words f is convex if and only if for all $x \in \mathbf{dom}\, f$ and

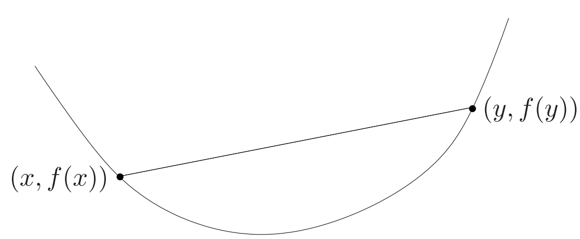


Figure 3.1 Graph of a convex function. The chord (*i.e.*, line segment) between any two points on the graph lies above the graph.

all v, the function $g(t) = f(x+tv)$ is convex (on its domain, $\{t \mid x+tv \in \mathbf{dom}\, f\}$). This property is very useful, since it allows us to check whether a function is convex by restricting it to a line.

The *analysis* of convex functions is a well developed field, which we will not pursue in any depth. One simple result, for example, is that a convex function is continuous on the relative interior of its domain; it can have discontinuities only on its relative boundary.

3.1.2 Extended-value extensions

It is often convenient to extend a convex function to all of $\mathbf{R}^n$ by defining its value to be ∞ outside its domain. If f is convex we define its *extended-value extension* $\tilde{f} : \mathbf{R}^n \to \mathbf{R} \cup \{\infty\}$ by

$$\tilde{f}(x) = \begin{cases} f(x) & x \in \mathbf{dom}\, f \\ \infty & x \notin \mathbf{dom}\, f. \end{cases}$$

The extension $\tilde{f}$ is defined on all $\mathbf{R}^n$, and takes values in $\mathbf{R} \cup \{\infty\}$. We can recover the domain of the original function f from the extension $\tilde{f}$ as $\mathbf{dom}\, f = \{x \mid \tilde{f}(x) < \infty\}$.

The extension can simplify notation, since we do not need to explicitly describe the domain, or add the qualifier 'for all $x \in \mathbf{dom}\, f$' every time we refer to $f(x)$. Consider, for example, the basic defining inequality (3.1). In terms of the extension $\tilde{f}$, we can express it as: for $0 < \theta < 1$,

$$\tilde{f}(\theta x + (1-\theta)y) \leq \theta \tilde{f}(x) + (1-\theta)\tilde{f}(y)$$

for *any* x and y. (For $\theta = 0$ or $\theta = 1$ the inequality always holds.) Of course here we must interpret the inequality using extended arithmetic and ordering. For x and y both in $\mathbf{dom}\, f$, this inequality coincides with (3.1); if either is outside $\mathbf{dom}\, f$, then the righthand side is ∞, and the inequality therefore holds. As another example of this notational device, suppose f_1 and f_2 are two convex functions on $\mathbf{R}^n$. The pointwise sum $f = f_1 + f_2$ is the function with domain $\mathbf{dom}\, f = \mathbf{dom}\, f_1 \cap \mathbf{dom}\, f_2$, with $f(x) = f_1(x) + f_2(x)$ for any $x \in \mathbf{dom}\, f$. Using extended-value extensions we can simply say that for any x, $\tilde{f}(x) = \tilde{f}_1(x) + \tilde{f}_2(x)$. In this equation the domain of f has been automatically defined as $\mathbf{dom}\, f = \mathbf{dom}\, f_1 \cap \mathbf{dom}\, f_2$, since $\tilde{f}(x) = \infty$ whenever $x \notin \mathbf{dom}\, f_1$ or $x \notin \mathbf{dom}\, f_2$. In this example we are relying on extended arithmetic to automatically define the domain.

In this book we will use the same symbol to denote a convex function and its extension, whenever there is no harm from the ambiguity. This is the same as assuming that all convex functions are implicitly extended, *i.e.*, are defined as ∞ outside their domains.

Example 3.1 *Indicator function of a convex set.* Let $C \subseteq \mathbf{R}^n$ be a convex set, and consider the (convex) function I_C with domain C and $I_C(x) = 0$ for all $x \in C$. In other words, the function is identically zero on the set C. Its extended-value extension

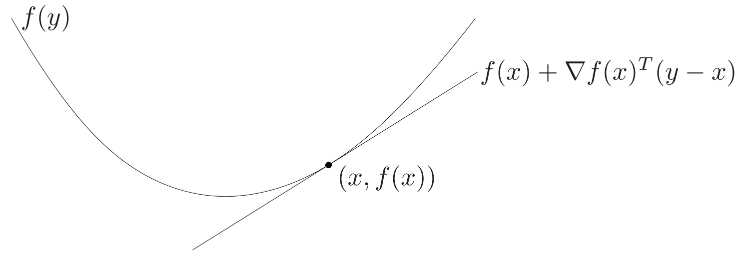


Figure 3.2 If f is convex and differentiable, then $f(x)+\nabla f(x)^T(y-x) \le f(y)$ for all $x,\ y \in \mathbf{dom}\, f$.

is given by

$$\tilde{I}_C(x) = \begin{cases} 0 & x \in C \\ \infty & x \notin C. \end{cases}$$

The convex function $\tilde{I}_C$ is called the *indicator function* of the set C.

We can play several notational tricks with the indicator function $\tilde{I}_C$. For example the problem of minimizing a function f (defined on all of $\mathbf{R}^n$, say) on the set C is the same as minimizing the function $f + \tilde{I}_C$ over all of $\mathbf{R}^n$. Indeed, the function $f + \tilde{I}_C$ is (by our convention) f restricted to the set C.

In a similar way we can extend a concave function by defining it to be $-\infty$ outside its domain.

3.1.3 First-order conditions

Suppose f is differentiable (*i.e.*, its gradient ∇f exists at each point in $\mathbf{dom}\, f$, which is open). Then f is convex if and only if $\mathbf{dom}\, f$ is convex and

$$f(y) \ge f(x) + \nabla f(x)^T(y - x) \tag{3.2}$$

holds for all $x,\ y \in \mathbf{dom}\, f$. This inequality is illustrated in figure 3.2.

The affine function of y given by $f(x)+\nabla f(x)^T(y-x)$ is, of course, the first-order Taylor approximation of f near x. The inequality (3.2) states that for a convex function, the first-order Taylor approximation is in fact a *global underestimator* of the function. Conversely, if the first-order Taylor approximation of a function is always a global underestimator of the function, then the function is convex.

The inequality (3.2) shows that from *local information* about a convex function (*i.e.*, its value and derivative at a point) we can derive *global information* (*i.e.*, a global underestimator of it). This is perhaps the most important property of convex functions, and explains some of the remarkable properties of convex functions and convex optimization problems. As one simple example, the inequality (3.2) shows that if $\nabla f(x) = 0$, then for all $y \in \mathbf{dom}\, f$, $f(y) \ge f(x)$, *i.e.*, x is a global minimizer of the function f.

Strict convexity can also be characterized by a first-order condition: f is strictly convex if and only if $\mathbf{dom}\, f$ is convex and for $x,\ y \in \mathbf{dom}\, f$, $x \neq y$, we have

$$f(y) > f(x) + \nabla f(x)^T (y - x). \tag{3.3}$$

For concave functions we have the corresponding characterization: f is concave if and only if $\mathbf{dom}\, f$ is convex and

$$f(y) \leq f(x) + \nabla f(x)^T (y - x)$$

for all $x,\ y \in \mathbf{dom}\, f$.

Proof of first-order convexity condition

To prove (3.2), we first consider the case $n = 1$: We show that a differentiable function $f : \mathbf{R} \to \mathbf{R}$ is convex if and only if

$$f(y) \geq f(x) + f'(x)(y - x) \tag{3.4}$$

for all x and y in $\mathbf{dom}\, f$.

Assume first that f is convex and $x,\ y \in \mathbf{dom}\, f$. Since $\mathbf{dom}\, f$ is convex (*i.e.*, an interval), we conclude that for all $0 < t \leq 1$, $x + t(y - x) \in \mathbf{dom}\, f$, and by convexity of f,

$$f(x + t(y - x)) \leq (1 - t)f(x) + tf(y).$$

If we divide both sides by t, we obtain

$$f(y) \geq f(x) + \frac{f(x + t(y - x)) - f(x)}{t},$$

and taking the limit as $t \to 0$ yields (3.4).

To show sufficiency, assume the function satisfies (3.4) for all x and y in $\mathbf{dom}\, f$ (which is an interval). Choose any $x \neq y$, and $0 \leq \theta \leq 1$, and let $z = \theta x + (1 - \theta)y$. Applying (3.4) twice yields

$$f(x) \geq f(z) + f'(z)(x - z), \qquad f(y) \geq f(z) + f'(z)(y - z).$$

Multiplying the first inequality by θ, the second by $1 - \theta$, and adding them yields

$$\theta f(x) + (1 - \theta)f(y) \geq f(z),$$

which proves that f is convex.

Now we can prove the general case, with $f : \mathbf{R}^n \to \mathbf{R}$. Let $x,\ y \in \mathbf{R}^n$ and consider f restricted to the line passing through them, *i.e.*, the function defined by $g(t) = f(ty + (1 - t)x)$, so $g'(t) = \nabla f(ty + (1 - t)x)^T (y - x)$.

First assume f is convex, which implies g is convex, so by the argument above we have $g(1) \geq g(0) + g'(0)$, which means

$$f(y) \geq f(x) + \nabla f(x)^T (y - x).$$

Now assume that this inequality holds for any x and y, so if $ty + (1 - t)x \in \mathbf{dom}\, f$ and $\tilde{t}y + (1 - \tilde{t})x \in \mathbf{dom}\, f$, we have

$$f(ty + (1 - t)x) \geq f(\tilde{t}y + (1 - \tilde{t})x) + \nabla f(\tilde{t}y + (1 - \tilde{t})x)^T (y - x)(t - \tilde{t}),$$

i.e., $g(t) \geq g(\tilde{t}) + g'(\tilde{t})(t - \tilde{t})$. We have seen that this implies that g is convex.

3.1.4 Second-order conditions

We now assume that f is twice differentiable, that is, its *Hessian* or second derivative $\nabla^2 f$ exists at each point in $\mathbf{dom}\, f$, which is open. Then f is convex if and only if $\mathbf{dom}\, f$ is convex and its Hessian is positive semidefinite: for all $x \in \mathbf{dom}\, f$,

$$\nabla^2 f(x) \succeq 0.$$

For a function on $\mathbf{R}$, this reduces to the simple condition $f''(x) \geq 0$ (and $\mathbf{dom}\, f$ convex, *i.e.*, an interval), which means that the derivative is nondecreasing. The condition $\nabla^2 f(x) \succeq 0$ can be interpreted geometrically as the requirement that the graph of the function have positive (upward) curvature at x. We leave the proof of the second-order condition as an exercise (exercise 3.8).

Similarly, f is concave if and only if $\mathbf{dom}\, f$ is convex and $\nabla^2 f(x) \preceq 0$ for all $x \in \mathbf{dom}\, f$. Strict convexity can be partially characterized by second-order conditions. If $\nabla^2 f(x) \succ 0$ for all $x \in \mathbf{dom}\, f$, then f is strictly convex. The converse, however, is not true: for example, the function $f : \mathbf{R} \to \mathbf{R}$ given by $f(x) = x^4$ is strictly convex but has zero second derivative at $x = 0$.

Example 3.2 *Quadratic functions.* Consider the quadratic function $f : \mathbf{R}^n \to \mathbf{R}$, with $\mathbf{dom}\, f = \mathbf{R}^n$, given by

$$f(x) = (1/2)x^T P x + q^T x + r,$$

with $P \in \mathbf{S}^n$, $q \in \mathbf{R}^n$, and $r \in \mathbf{R}$. Since $\nabla^2 f(x) = P$ for all x, f is convex if and only if $P \succeq 0$ (and concave if and only if $P \preceq 0$).

For quadratic functions, strict convexity is easily characterized: f is strictly convex if and only if $P \succ 0$ (and strictly concave if and only if $P \prec 0$).

Remark 3.1 The separate requirement that $\mathbf{dom}\, f$ be convex cannot be dropped from the first- or second-order characterizations of convexity and concavity. For example, the function $f(x) = 1/x^2$, with $\mathbf{dom}\, f = \{x \in \mathbf{R} \mid x \neq 0\}$, satisfies $f''(x) > 0$ for all $x \in \mathbf{dom}\, f$, but is not a convex function.

3.1.5 Examples

We have already mentioned that all linear and affine functions are convex (and concave), and have described the convex and concave quadratic functions. In this section we give a few more examples of convex and concave functions. We start with some functions on $\mathbf{R}$, with variable x.

- *Exponential.* e^{ax} is convex on $\mathbf{R}$, for any $a \in \mathbf{R}$.
- *Powers.* x^a is convex on $\mathbf{R}_{++}$ when $a \geq 1$ or $a \leq 0$, and concave for $0 \leq a \leq 1$.
- *Powers of absolute value.* $|x|^p$, for $p \geq 1$, is convex on $\mathbf{R}$.
- *Logarithm.* $\log x$ is concave on $\mathbf{R}_{++}$.

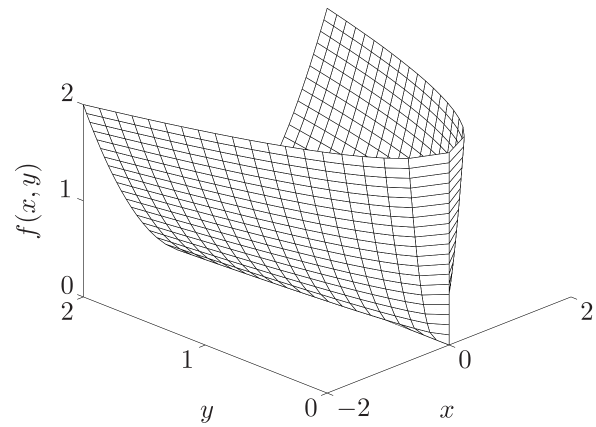


Figure 3.3 Graph of $f(x, y) = x^2/y$.

- *Negative entropy.* $x \log x$ (either on $\mathbf{R}_{++}$, or on $\mathbf{R}_+$, defined as 0 for $x = 0$) is convex.

Convexity or concavity of these examples can be shown by verifying the basic inequality (3.1), or by checking that the second derivative is nonnegative or nonpositive. For example, with $f(x) = x \log x$ we have

$$f'(x) = \log x + 1, \qquad f''(x) = 1/x,$$

so that $f''(x) > 0$ for $x > 0$. This shows that the negative entropy function is (strictly) convex.

We now give a few interesting examples of functions on $\mathbf{R}^n$.

- *Norms.* Every norm on $\mathbf{R}^n$ is convex.
- *Max function.* $f(x) = \max\{x_1, \dots, x_n\}$ is convex on $\mathbf{R}^n$.
- *Quadratic-over-linear function.* The function $f(x, y) = x^2/y$, with

  $$\mathbf{dom}\, f = \mathbf{R} \times \mathbf{R}_{++} = \{(x, y) \in \mathbf{R}^2 \mid y > 0\},$$

  is convex (figure 3.3).
- *Log-sum-exp.* The function $f(x) = \log(e^{x_1} + \cdots + e^{x_n})$ is convex on $\mathbf{R}^n$. This function can be interpreted as a differentiable (in fact, analytic) approximation of the max function, since

  $$\max\{x_1, \dots, x_n\} \le f(x) \le \max\{x_1, \dots, x_n\} + \log n$$

  for all x. (The second inequality is tight when all components of x are equal.) Figure 3.4 shows f for $n = 2$.

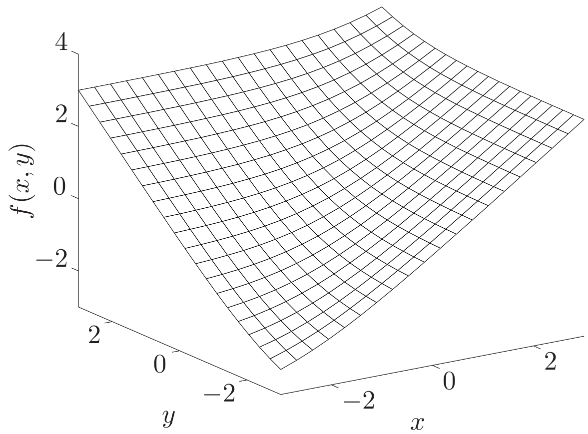


Figure 3.4 Graph of $f(x,y) = \log(e^x + e^y)$.

- *Geometric mean.* The geometric mean $f(x) = \left(\prod_{i=1}^n x_i\right)^{1/n}$ is concave on $\mathbf{dom}\, f = \mathbf{R}^n_{++}$.
- *Log-determinant.* The function $f(X) = \log\det X$ is concave on $\mathbf{dom}\, f = \mathbf{S}^n_{++}$.

Convexity (or concavity) of these examples can be verified in several ways, such as directly verifying the inequality (3.1), verifying that the Hessian is positive semidefinite, or restricting the function to an arbitrary line and verifying convexity of the resulting function of one variable.

Norms. If $f : \mathbf{R}^n \to \mathbf{R}$ is a norm, and $0 \leq \theta \leq 1$, then

$$f(\theta x + (1-\theta)y) \leq f(\theta x) + f((1-\theta)y) = \theta f(x) + (1-\theta) f(y).$$

The inequality follows from the triangle inequality, and the equality follows from homogeneity of a norm.

Max function. The function $f(x) = \max_i x_i$ satisfies, for $0 \leq \theta \leq 1$,

$$\begin{array}{rcl} f(\theta x + (1-\theta)y) & = & \max\limits_i (\theta x_i + (1-\theta) y_i) \\ & \leq & \theta \max\limits_i x_i + (1-\theta) \max\limits_i y_i \\ & = & \theta f(x) + (1-\theta) f(y). \end{array}$$

Quadratic-over-linear function. To show that the quadratic-over-linear function $f(x,y) = x^2/y$ is convex, we note that (for $y > 0$),

$$\nabla^2 f(x,y) = \frac{2}{y^3} \left[\begin{array}{cc} y^2 & -xy \\ -xy & x^2 \end{array} \right] = \frac{2}{y^3} \left[\begin{array}{c} y \\ -x \end{array} \right] \left[\begin{array}{c} y \\ -x \end{array} \right]^T \succeq 0.$$

Log-sum-exp. The Hessian of the log-sum-exp function is

$$\nabla^2 f(x) = \frac{1}{(\mathbf{1}^T z)^2}\left((\mathbf{1}^T z)\,\mathbf{diag}(z) - zz^T\right),$$

where $z = (e^{x_1}, \ldots, e^{x_n})$. To verify that $\nabla^2 f(x) \succeq 0$ we must show that for all v, $v^T \nabla^2 f(x) v \geq 0$, *i.e.*,

$$v^T \nabla^2 f(x) v = \frac{1}{(\mathbf{1}^T z)^2}\left(\left(\sum_{i=1}^n z_i\right)\left(\sum_{i=1}^n v_i^2 z_i\right) - \left(\sum_{i=1}^n v_i z_i\right)^2\right) \geq 0.$$

But this follows from the Cauchy-Schwarz inequality $(a^T a)(b^T b) \geq (a^T b)^2$ applied to the vectors with components $a_i = v_i\sqrt{z_i}$, $b_i = \sqrt{z_i}$.

Geometric mean. In a similar way we can show that the geometric mean $f(x) = \left(\prod_{i=1}^n x_i\right)^{1/n}$ is concave on $\mathbf{dom}\, f = \mathbf{R}^n_{++}$. Its Hessian $\nabla^2 f(x)$ is given by

$$\frac{\partial^2 f(x)}{\partial x_k^2} = -(n-1)\frac{\left(\prod_{i=1}^n x_i\right)^{1/n}}{n^2 x_k^2}, \qquad \frac{\partial^2 f(x)}{\partial x_k \partial x_l} = \frac{\left(\prod_{i=1}^n x_i\right)^{1/n}}{n^2 x_k x_l} \quad \text{for } k \neq l,$$

and can be expressed as

$$\nabla^2 f(x) = -\frac{\prod_{i=1}^n x_i^{1/n}}{n^2}\left(n\,\mathbf{diag}(1/x_1^2, \ldots, 1/x_n^2) - qq^T\right)$$

where $q_i = 1/x_i$. We must show that $\nabla^2 f(x) \preceq 0$, *i.e.*, that

$$v^T \nabla^2 f(x) v = -\frac{\prod_{i=1}^n x_i^{1/n}}{n^2}\left(n\sum_{i=1}^n v_i^2/x_i^2 - \left(\sum_{i=1}^n v_i/x_i\right)^2\right) \leq 0$$

for all v. Again this follows from the Cauchy-Schwarz inequality $(a^T a)(b^T b) \geq (a^T b)^2$, applied to the vectors $a = \mathbf{1}$ and $b_i = v_i/x_i$.

Log-determinant. For the function $f(X) = \log\det X$, we can verify concavity by considering an arbitrary line, given by $X = Z + tV$, where $Z,\ V \in \mathbf{S}^n$. We define $g(t) = f(Z + tV)$, and restrict g to the interval of values of t for which $Z + tV \succ 0$. Without loss of generality, we can assume that $t = 0$ is inside this interval, *i.e.*, $Z \succ 0$. We have

$$\begin{aligned} g(t) &= \log\det(Z + tV) \\ &= \log\det(Z^{1/2}(I + tZ^{-1/2}VZ^{-1/2})Z^{1/2}) \\ &= \sum_{i=1}^n \log(1 + t\lambda_i) + \log\det Z \end{aligned}$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $Z^{-1/2}VZ^{-1/2}$. Therefore we have

$$g'(t) = \sum_{i=1}^n \frac{\lambda_i}{1 + t\lambda_i}, \qquad g''(t) = -\sum_{i=1}^n \frac{\lambda_i^2}{(1 + t\lambda_i)^2}.$$

Since $g''(t) \leq 0$, we conclude that f is concave.

3.1.6 Sublevel sets

The α-*sublevel set* of a function $f : \mathbf{R}^n \to \mathbf{R}$ is defined as

$$C_\alpha = \{x \in \mathbf{dom}\, f \mid f(x) \leq \alpha\}.$$

Sublevel sets of a convex function are convex, for any value of α. The proof is immediate from the definition of convexity: if $x,\ y \in C_\alpha$, then $f(x) \leq \alpha$ and $f(y) \leq \alpha$, and so $f(\theta x + (1-\theta)y) \leq \alpha$ for $0 \leq \theta \leq 1$, and hence $\theta x + (1-\theta)y \in C_\alpha$.

The converse is not true: a function can have all its sublevel sets convex, but not be a convex function. For example, $f(x) = -e^x$ is not convex on $\mathbf{R}$ (indeed, it is strictly concave) but all its sublevel sets are convex.

If f is concave, then its α-*superlevel set*, given by $\{x \in \mathbf{dom}\, f \mid f(x) \geq \alpha\}$, is a convex set. The sublevel set property is often a good way to establish convexity of a set, by expressing it as a sublevel set of a convex function, or as the superlevel set of a concave function.

Example 3.3 The geometric and arithmetic means of $x \in \mathbf{R}^n_+$ are, respectively,

$$G(x) = \left(\prod_{i=1}^n x_i\right)^{1/n}, \qquad A(x) = \frac{1}{n}\sum_{i=1}^n x_i,$$

(where we take $0^{1/n} = 0$ in our definition of G). The arithmetic-geometric mean inequality states that $G(x) \leq A(x)$.

Suppose $0 \leq \alpha \leq 1$, and consider the set

$$\{x \in \mathbf{R}^n_+ \mid G(x) \geq \alpha A(x)\},$$

i.e., the set of vectors with geometric mean at least as large as a factor α times the arithmetic mean. This set is convex, since it is the 0-superlevel set of the function $G(x) - \alpha A(x)$, which is concave. In fact, the set is positively homogeneous, so it is a convex cone.

3.1.7 Epigraph

The graph of a function $f : \mathbf{R}^n \to \mathbf{R}$ is defined as

$$\{(x, f(x)) \mid x \in \mathbf{dom}\, f\},$$

which is a subset of $\mathbf{R}^{n+1}$. The *epigraph* of a function $f : \mathbf{R}^n \to \mathbf{R}$ is defined as

$$\mathbf{epi}\, f = \{(x, t) \mid x \in \mathbf{dom}\, f,\ f(x) \leq t\},$$

which is a subset of $\mathbf{R}^{n+1}$. ('Epi' means 'above' so epigraph means 'above the graph'.) The definition is illustrated in figure 3.5.

The link between convex sets and convex functions is via the epigraph: A function is convex if and only if its epigraph is a convex set. A function is concave if and only if its *hypograph*, defined as

$$\mathbf{hypo}\, f = \{(x, t) \mid t \leq f(x)\},$$

is a convex set.

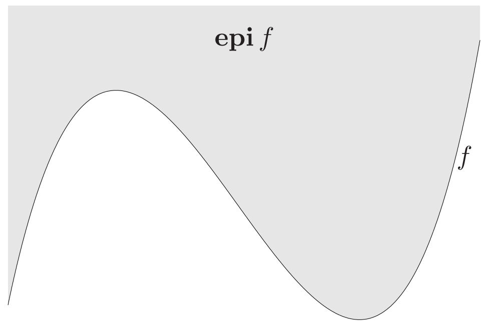


Figure 3.5 Epigraph of a function f, shown shaded. The lower boundary, shown darker, is the graph of f.

Example 3.4 *Matrix fractional function.* The function $f : \mathbf{R}^n \times \mathbf{S}^n \to \mathbf{R}$, defined as

$$f(x, Y) = x^T Y^{-1} x$$

is convex on $\mathbf{dom}\, f = \mathbf{R}^n \times \mathbf{S}^n_{++}$. (This generalizes the quadratic-over-linear function $f(x, y) = x^2/y$, with $\mathbf{dom}\, f = \mathbf{R} \times \mathbf{R}_{++}$.)

One easy way to establish convexity of f is via its epigraph:

$$\begin{aligned} \mathbf{epi}\, f &= \{(x, Y, t) \mid Y \succ 0,\ x^T Y^{-1} x \le t\} \\ &= \left\{ (x, Y, t) \;\middle|\; \begin{bmatrix} Y & x \\ x^T & t \end{bmatrix} \succeq 0,\ Y \succ 0 \right\}, \end{aligned}$$

using the Schur complement condition for positive semidefiniteness of a block matrix (see §A.5.5). The last condition is a linear matrix inequality in (x, Y, t), and therefore $\mathbf{epi}\, f$ is convex.

For the special case $n = 1$, the matrix fractional function reduces to the quadratic-over-linear function x^2/y, and the associated LMI representation is

$$\begin{bmatrix} y & x \\ x & t \end{bmatrix} \succeq 0, \qquad y > 0$$

(the graph of which is shown in figure 3.3).

Many results for convex functions can be proved (or interpreted) geometrically using epigraphs, and applying results for convex sets. As an example, consider the first-order condition for convexity:

$$f(y) \ge f(x) + \nabla f(x)^T (y - x),$$

where f is convex and $x,\ y \in \mathbf{dom}\, f$. We can interpret this basic inequality geometrically in terms of $\mathbf{epi}\, f$. If $(y, t) \in \mathbf{epi}\, f$, then

$$t \ge f(y) \ge f(x) + \nabla f(x)^T (y - x).$$

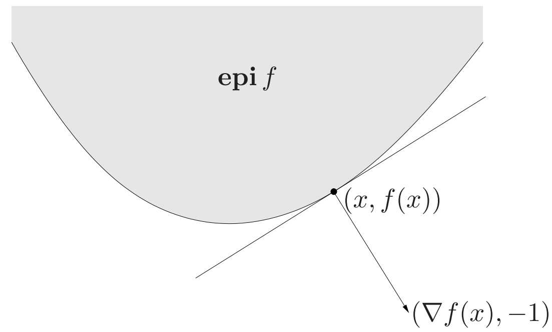


Figure 3.6 For a differentiable convex function f, the vector $(\nabla f(x), -1)$ defines a supporting hyperplane to the epigraph of f at x.

We can express this as:

$$(y,t) \in \mathbf{epi}\, f \implies \left[\begin{array}{c} \nabla f(x) \\ -1 \end{array}\right]^T \left(\left[\begin{array}{c} y \\ t \end{array}\right] - \left[\begin{array}{c} x \\ f(x) \end{array}\right] \right) \leq 0.$$

This means that the hyperplane defined by $(\nabla f(x), -1)$ supports $\mathbf{epi}\, f$ at the boundary point $(x, f(x))$; see figure 3.6.

3.1.8 Jensen's inequality and extensions

The basic inequality (3.1), *i.e.*,

$$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta) f(y),$$

is sometimes called *Jensen's inequality*. It is easily extended to convex combinations of more than two points: If f is convex, $x_1, \ldots, x_k \in \mathbf{dom}\, f$, and $\theta_1, \ldots, \theta_k \geq 0$ with $\theta_1 + \cdots + \theta_k = 1$, then

$$f(\theta_1 x_1 + \cdots + \theta_k x_k) \leq \theta_1 f(x_1) + \cdots + \theta_k f(x_k).$$

As in the case of convex sets, the inequality extends to infinite sums, integrals, and expected values. For example, if $p(x) \geq 0$ on $S \subseteq \mathbf{dom}\, f$, $\int_S p(x)\, dx = 1$, then

$$f\left(\int_S p(x) x\, dx\right) \leq \int_S f(x) p(x)\, dx,$$

provided the integrals exist. In the most general case we can take any probability measure with support in $\mathbf{dom}\, f$. If x is a random variable such that $x \in \mathbf{dom}\, f$ with probability one, and f is convex, then we have

$$f(\mathbf{E}\, x) \leq \mathbf{E}\, f(x), \tag{3.5}$$

provided the expectations exist. We can recover the basic inequality (3.1) from this general form, by taking the random variable x to have support $\{x_1, x_2\}$, with

$\mathbf{prob}(x = x_1) = \theta$, $\mathbf{prob}(x = x_2) = 1 - \theta$. Thus the inequality (3.5) characterizes convexity: If f is not convex, there is a random variable x, with $x \in \mathbf{dom}\, f$ with probability one, such that $f(\mathbf{E}\, x) > \mathbf{E}\, f(x)$.

All of these inequalities are now called *Jensen's inequality*, even though the inequality studied by Jensen was the very simple one

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2}.$$

Remark 3.2 We can interpret (3.5) as follows. Suppose $x \in \mathbf{dom}\, f \subseteq \mathbf{R}^n$ and z is any zero mean random vector in $\mathbf{R}^n$. Then we have

$$\mathbf{E}\, f(x+z) \geq f(x).$$

Thus, randomization or dithering (*i.e.*, adding a zero mean random vector to the argument) cannot decrease the value of a convex function on average.

3.1.9 Inequalities

Many famous inequalities can be derived by applying Jensen's inequality to some appropriate convex function. (Indeed, convexity and Jensen's inequality can be made the foundation of a theory of inequalities.) As a simple example, consider the arithmetic-geometric mean inequality:

$$\sqrt{ab} \leq (a+b)/2 \tag{3.6}$$

for $a, b \geq 0$. The function $-\log x$ is convex; Jensen's inequality with $\theta = 1/2$ yields

$$-\log\left(\frac{a+b}{2}\right) \leq \frac{-\log a - \log b}{2}.$$

Taking the exponential of both sides yields (3.6).

As a less trivial example we prove Hölder's inequality: for $p > 1$, $1/p + 1/q = 1$, and $x,\ y \in \mathbf{R}^n$,

$$\sum_{i=1}^n x_i y_i \leq \left(\sum_{i=1}^n |x_i|^p\right)^{1/p} \left(\sum_{i=1}^n |y_i|^q\right)^{1/q}.$$

By convexity of $-\log x$, and Jensen's inequality with general θ, we obtain the more general arithmetic-geometric mean inequality

$$a^\theta b^{1-\theta} \leq \theta a + (1-\theta) b,$$

valid for $a,\ b \geq 0$ and $0 \leq \theta \leq 1$. Applying this with

$$a = \frac{|x_i|^p}{\sum_{j=1}^n |x_j|^p}, \qquad b = \frac{|y_i|^q}{\sum_{j=1}^n |y_j|^q}, \qquad \theta = 1/p,$$

yields

$$\left(\frac{|x_i|^p}{\sum_{j=1}^n |x_j|^p}\right)^{1/p} \left(\frac{|y_i|^q}{\sum_{j=1}^n |y_j|^q}\right)^{1/q} \leq \frac{|x_i|^p}{p \sum_{j=1}^n |x_j|^p} + \frac{|y_i|^q}{q \sum_{j=1}^n |y_j|^q}.$$

Summing over i then yields Hölder's inequality.

3.2 Operations that preserve convexity

In this section we describe some operations that preserve convexity or concavity of functions, or allow us to construct new convex and concave functions. We start with some simple operations such as addition, scaling, and pointwise supremum, and then describe some more sophisticated operations (some of which include the simple operations as special cases).

3.2.1 Nonnegative weighted sums

Evidently if f is a convex function and $\alpha \geq 0$, then the function αf is convex. If f_1 and f_2 are both convex functions, then so is their sum $f_1 + f_2$. Combining nonnegative scaling and addition, we see that the set of convex functions is itself a convex cone: a nonnegative weighted sum of convex functions,

$$f = w_1 f_1 + \cdots + w_m f_m,$$

is convex. Similarly, a nonnegative weighted sum of concave functions is concave. A nonnegative, nonzero weighted sum of strictly convex (concave) functions is strictly convex (concave).

These properties extend to infinite sums and integrals. For example if $f(x, y)$ is convex in x for each $y \in \mathcal{A}$, and $w(y) \geq 0$ for each $y \in \mathcal{A}$, then the function g defined as

$$g(x) = \int_{\mathcal{A}} w(y) f(x, y) \, dy$$

is convex in x (provided the integral exists).

The fact that convexity is preserved under nonnegative scaling and addition is easily verified directly, or can be seen in terms of the associated epigraphs. For example, if $w \geq 0$ and f is convex, we have

$$\mathbf{epi}(wf) = \begin{bmatrix} I & 0 \\ 0 & w \end{bmatrix} \mathbf{epi}\, f,$$

which is convex because the image of a convex set under a linear mapping is convex.

3.2.2 Composition with an affine mapping

Suppose $f : \mathbf{R}^n \to \mathbf{R}$, $A \in \mathbf{R}^{n \times m}$, and $b \in \mathbf{R}^n$. Define $g : \mathbf{R}^m \to \mathbf{R}$ by

$$g(x) = f(Ax + b),$$

with $\mathbf{dom}\, g = \{x \mid Ax + b \in \mathbf{dom}\, f\}$. Then if f is convex, so is g; if f is concave, so is g.

3.2.3 Pointwise maximum and supremum

If f_1 and f_2 are convex functions then their *pointwise maximum* f, defined by

$$f(x) = \max\{f_1(x), f_2(x)\},$$

with $\mathbf{dom}\, f = \mathbf{dom}\, f_1 \cap \mathbf{dom}\, f_2$, is also convex. This property is easily verified: if $0 \le \theta \le 1$ and $x,\ y \in \mathbf{dom}\, f$, then

$$\begin{array}{rcl} f(\theta x + (1-\theta)y) & = & \max\{f_1(\theta x + (1-\theta)y), f_2(\theta x + (1-\theta)y)\} \\ & \le & \max\{\theta f_1(x) + (1-\theta) f_1(y), \theta f_2(x) + (1-\theta) f_2(y)\} \\ & \le & \theta \max\{f_1(x), f_2(x)\} + (1-\theta)\max\{f_1(y), f_2(y)\} \\ & = & \theta f(x) + (1-\theta) f(y), \end{array}$$

which establishes convexity of f. It is easily shown that if $f_1, \ldots, f_m$ are convex, then their pointwise maximum

$$f(x) = \max\{f_1(x), \ldots, f_m(x)\}$$

is also convex.

Example 3.5 *Piecewise-linear functions.* The function

$$f(x) = \max\{a_1^T x + b_1, \ldots, a_L^T x + b_L\}$$

defines a piecewise-linear (or really, affine) function (with L or fewer regions). It is convex since it is the pointwise maximum of affine functions.

The converse can also be shown: any piecewise-linear convex function with L or fewer regions can be expressed in this form. (See exercise 3.29.)

Example 3.6 *Sum of r largest components.* For $x \in \mathbf{R}^n$ we denote by $x_{[i]}$ the ith largest component of x, *i.e.*,

$$x_{[1]} \ge x_{[2]} \ge \cdots \ge x_{[n]}$$

are the components of x sorted in nonincreasing order. Then the function

$$f(x) = \sum_{i=1}^{r} x_{[i]},$$

i.e., the sum of the r largest elements of x, is a convex function. This can be seen by writing it as

$$f(x) = \sum_{i=1}^{r} x_{[i]} = \max\{x_{i_1} + \cdots + x_{i_r} \mid 1 \le i_1 < i_2 < \cdots < i_r \le n\},$$

i.e., the maximum of all possible sums of r different components of x. Since it is the pointwise maximum of $n!/(r!(n-r)!)$ linear functions, it is convex.

As an extension it can be shown that the function $\sum_{i=1}^{r} w_i x_{[i]}$ is convex, provided $w_1 \ge w_2 \ge \cdots \ge w_r \ge 0$. (See exercise 3.19.)

The pointwise maximum property extends to the pointwise supremum over an infinite set of convex functions. If for each $y \in \mathcal{A}$, $f(x, y)$ is convex in x, then the function g, defined as

$$g(x) = \sup_{y \in \mathcal{A}} f(x, y) \tag{3.7}$$

is convex in x. Here the domain of g is

$$\mathbf{dom}\, g = \{x \mid (x, y) \in \mathbf{dom}\, f \text{ for all } y \in \mathcal{A},\ \sup_{y \in \mathcal{A}} f(x, y) < \infty\}.$$

Similarly, the pointwise infimum of a set of concave functions is a concave function.

In terms of epigraphs, the pointwise supremum of functions corresponds to the intersection of epigraphs: with f, g, and $\mathcal{A}$ as defined in (3.7), we have

$$\mathbf{epi}\, g = \bigcap_{y \in \mathcal{A}} \mathbf{epi}\, f(\cdot, y).$$

Thus, the result follows from the fact that the intersection of a family of convex sets is convex.

Example 3.7 *Support function of a set.* Let $C \subseteq \mathbf{R}^n$, with $C \neq \emptyset$. The *support function* S_C associated with the set C is defined as

$$S_C(x) = \sup\{x^T y \mid y \in C\}$$

(and, naturally, $\mathbf{dom}\, S_C = \{x \mid \sup_{y \in C} x^T y < \infty\}$).

For each $y \in C$, $x^T y$ is a linear function of x, so S_C is the pointwise supremum of a family of linear functions, hence convex.

Example 3.8 *Distance to farthest point of a set.* Let $C \subseteq \mathbf{R}^n$. The distance (in any norm) to the farthest point of C,

$$f(x) = \sup_{y \in C} \|x - y\|,$$

is convex. To see this, note that for any y, the function $\|x - y\|$ is convex in x. Since f is the pointwise supremum of a family of convex functions (indexed by $y \in C$), it is a convex function of x.

Example 3.9 *Least-squares cost as a function of weights.* Let $a_1, \ldots, a_n \in \mathbf{R}^m$. In a weighted least-squares problem we minimize the objective function $\sum_{i=1}^n w_i(a_i^T x - b_i)^2$ over $x \in \mathbf{R}^m$. We refer to w_i as *weights*, and allow negative w_i (which opens the possibility that the objective function is unbounded below).

We define the (optimal) *weighted least-squares cost* as

$$g(w) = \inf_x \sum_{i=1}^n w_i(a_i^T x - b_i)^2,$$

with domain

$$\mathbf{dom}\, g = \left\{ w \;\middle|\; \inf_x \sum_{i=1}^n w_i(a_i^T x - b_i)^2 > -\infty \right\}.$$

Since g is the infimum of a family of linear functions of w (indexed by $x \in \mathbf{R}^m$), it is a concave function of w.

We can derive an explicit expression for g, at least on part of its domain. Let $W = \mathbf{diag}(w)$, the diagonal matrix with elements $w_1, \ldots, w_n$, and let $A \in \mathbf{R}^{n\times m}$ have rows a_i^T, so we have

$$g(w) = \inf_x (Ax-b)^T W(Ax-b) = \inf_x (x^T A^T W Ax - 2b^T W Ax + b^T W b).$$

From this we see that if $A^T W A \not\succeq 0$, the quadratic function is unbounded below in x, so $g(w) = -\infty$, *i.e.*, $w \notin \mathbf{dom}\, g$. We can give a simple expression for g when $A^T W A \succ 0$ (which defines a strict linear matrix inequality), by analytically minimizing the quadratic function:

$$\begin{aligned} g(w) &= b^T W b - b^T W A(A^T W A)^{-1} A^T W b \\ &= \sum_{i=1}^n w_i b_i^2 - \sum_{i=1}^n w_i^2 b_i^2 a_i^T \left(\sum_{j=1}^n w_j a_j a_j^T\right)^{-1} a_i. \end{aligned}$$

Concavity of g from this expression is not immediately obvious (but does follow, for example, from convexity of the matrix fractional function; see example 3.4).

Example 3.10 *Maximum eigenvalue of a symmetric matrix.* The function $f(X) = \lambda_{\max}(X)$, with $\mathbf{dom}\, f = \mathbf{S}^m$, is convex. To see this, we express f as

$$f(X) = \sup\{y^T X y \mid \|y\|_2 = 1\},$$

i.e., as the pointwise supremum of a family of linear functions of X (*i.e.*, $y^T X y$) indexed by $y \in \mathbf{R}^m$.

Example 3.11 *Norm of a matrix.* Consider $f(X) = \|X\|_2$ with $\mathbf{dom}\, f = \mathbf{R}^{p\times q}$, where $\|\cdot\|_2$ denotes the spectral norm or maximum singular value. Convexity of f follows from

$$f(X) = \sup\{u^T X v \mid \|u\|_2 = 1,\ \|v\|_2 = 1\},$$

which shows it is the pointwise supremum of a family of linear functions of X.

As a generalization suppose $\|\cdot\|_a$ and $\|\cdot\|_b$ are norms on $\mathbf{R}^p$ and $\mathbf{R}^q$, respectively. The induced norm of a matrix $X \in \mathbf{R}^{p\times q}$ is defined as

$$\|X\|_{a,b} = \sup_{v\neq 0} \frac{\|Xv\|_a}{\|v\|_b}.$$

(This reduces to the spectral norm when both norms are Euclidean.) The induced norm can be expressed as

$$\begin{aligned} \|X\|_{a,b} &= \sup\{\|Xv\|_a \mid \|v\|_b = 1\} \\ &= \sup\{u^T X v \mid \|u\|_{a*} = 1,\ \|v\|_b = 1\}, \end{aligned}$$

where $\|\cdot\|_{a*}$ is the dual norm of $\|\cdot\|_a$, and we use the fact that

$$\|z\|_a = \sup\{u^T z \mid \|u\|_{a*} = 1\}.$$

Since we have expressed $\|X\|_{a,b}$ as a supremum of linear functions of X, it is a convex function.

Representation as pointwise supremum of affine functions

The examples above illustrate a good method for establishing convexity of a function: by expressing it as the pointwise supremum of a family of affine functions. Except for a technical condition, a converse holds: almost every convex function can be expressed as the pointwise supremum of a family of affine functions. For example, if $f : \mathbf{R}^n \to \mathbf{R}$ is convex, with $\mathbf{dom}\, f = \mathbf{R}^n$, then we have

$$f(x) = \sup\{g(x) \mid g \text{ affine}, \ g(z) \le f(z) \text{ for all } z\}.$$

In other words, f is the pointwise supremum of the set of all affine global underestimators of it. We give the proof of this result below, and leave the case where $\mathbf{dom}\, f \neq \mathbf{R}^n$ as an exercise (exercise 3.28).

Suppose f is convex with $\mathbf{dom}\, f = \mathbf{R}^n$. The inequality

$$f(x) \ge \sup\{g(x) \mid g \text{ affine}, \ g(z) \le f(z) \text{ for all } z\}$$

is clear, since if g is any affine underestimator of f, we have $g(x) \le f(x)$. To establish equality, we will show that for each $x \in \mathbf{R}^n$, there is an affine function g, which is a global underestimator of f, and satisfies $g(x) = f(x)$.

The epigraph of f is, of course, a convex set. Hence we can find a supporting hyperplane to it at $(x, f(x))$, *i.e.*, $a \in \mathbf{R}^n$ and $b \in \mathbf{R}$ with $(a, b) \neq 0$ and

$$\begin{bmatrix} a \\ b \end{bmatrix}^T \begin{bmatrix} x - z \\ f(x) - t \end{bmatrix} \le 0$$

for all $(z, t) \in \mathbf{epi}\, f$. This means that

$$a^T(x - z) + b(f(x) - f(z) - s) \le 0 \qquad (3.8)$$

for all $z \in \mathbf{dom}\, f = \mathbf{R}^n$ and all $s \ge 0$ (since $(z, t) \in \mathbf{epi}\, f$ means $t = f(z) + s$ for some $s \ge 0$). For the inequality (3.8) to hold for all $s \ge 0$, we must have $b \ge 0$. If $b = 0$, then the inequality (3.8) reduces to $a^T(x - z) \le 0$ for all $z \in \mathbf{R}^n$, which implies $a = 0$ and contradicts $(a, b) \neq 0$. We conclude that $b > 0$, *i.e.*, that the supporting hyperplane is not vertical.

Using the fact that $b > 0$ we rewrite (3.8) for $s = 0$ as

$$g(z) = f(x) + (a/b)^T(x - z) \le f(z)$$

for all z. The function g is an affine underestimator of f, and satisfies $g(x) = f(x)$.

3.2.4 Composition

In this section we examine conditions on $h : \mathbf{R}^k \to \mathbf{R}$ and $g : \mathbf{R}^n \to \mathbf{R}^k$ that guarantee convexity or concavity of their composition $f = h \circ g : \mathbf{R}^n \to \mathbf{R}$, defined by

$$f(x) = h(g(x)), \qquad \mathbf{dom}\, f = \{x \in \mathbf{dom}\, g \mid g(x) \in \mathbf{dom}\, h\}.$$

Scalar composition

We first consider the case $k = 1$, so $h : \mathbf{R} \to \mathbf{R}$ and $g : \mathbf{R}^n \to \mathbf{R}$. We can restrict ourselves to the case $n = 1$ (since convexity is determined by the behavior of a function on arbitrary lines that intersect its domain).

To discover the composition rules, we start by assuming that h and g are twice differentiable, with $\mathbf{dom}\, g = \mathbf{dom}\, h = \mathbf{R}$. In this case, convexity of f reduces to $f'' \geq 0$ (meaning, $f''(x) \geq 0$ for all $x \in \mathbf{R}$).

The second derivative of the composition function $f = h \circ g$ is given by

$$f''(x) = h''(g(x))g'(x)^2 + h'(g(x))g''(x). \tag{3.9}$$

Now suppose, for example, that g is convex (so $g'' \geq 0$) and h is convex and nondecreasing (so $h'' \geq 0$ and $h' \geq 0$). It follows from (3.9) that $f'' \geq 0$, *i.e.*, f is convex. In a similar way, the expression (3.9) gives the results:

$$\begin{array}{l} f \text{ is convex if } h \text{ is convex and nondecreasing, and } g \text{ is convex,} \\ f \text{ is convex if } h \text{ is convex and nonincreasing, and } g \text{ is concave,} \\ f \text{ is concave if } h \text{ is concave and nondecreasing, and } g \text{ is concave,} \\ f \text{ is concave if } h \text{ is concave and nonincreasing, and } g \text{ is convex.} \end{array} \tag{3.10}$$

These statements are valid when the functions g and h are twice differentiable and have domains that are all of $\mathbf{R}$. It turns out that very similar composition rules hold in the general case $n > 1$, without assuming differentiability of h and g, or that $\mathbf{dom}\, g = \mathbf{R}^n$ and $\mathbf{dom}\, h = \mathbf{R}$:

$$\begin{array}{l} f \text{ is convex if } h \text{ is convex, } \tilde{h} \text{ is nondecreasing, and } g \text{ is convex,} \\ f \text{ is convex if } h \text{ is convex, } \tilde{h} \text{ is nonincreasing, and } g \text{ is concave,} \\ f \text{ is concave if } h \text{ is concave, } \tilde{h} \text{ is nondecreasing, and } g \text{ is concave,} \\ f \text{ is concave if } h \text{ is concave, } \tilde{h} \text{ is nonincreasing, and } g \text{ is convex.} \end{array} \tag{3.11}$$

Here $\tilde{h}$ denotes the extended-value extension of the function h, which assigns the value ∞ ($-\infty$) to points not in $\mathbf{dom}\, h$ for h convex (concave). The only difference between these results, and the results in (3.10), is that we require that the *extended-value extension* function $\tilde{h}$ be nonincreasing or nondecreasing, on all of $\mathbf{R}$.

To understand what this means, suppose h is convex, so $\tilde{h}$ takes on the value ∞ outside $\mathbf{dom}\, h$. To say that $\tilde{h}$ is nondecreasing means that for *any* x, $y \in \mathbf{R}$, with $x < y$, we have $\tilde{h}(x) \leq \tilde{h}(y)$. In particular, this means that if $y \in \mathbf{dom}\, h$, then $x \in \mathbf{dom}\, h$. In other words, the domain of h extends infinitely in the negative direction; it is either $\mathbf{R}$, or an interval of the form $(-\infty, a)$ or $(-\infty, a]$. In a similar way, to say that h is convex and $\tilde{h}$ is nonincreasing means that h is nonincreasing and $\mathbf{dom}\, h$ extends infinitely in the positive direction. This is illustrated in figure 3.7.

Example 3.12 Some simple examples will illustrate the conditions on h that appear in the composition theorems.

- The function $h(x) = \log x$, with $\mathbf{dom}\, h = \mathbf{R}_{++}$, is concave and satisfies $\tilde{h}$ nondecreasing.

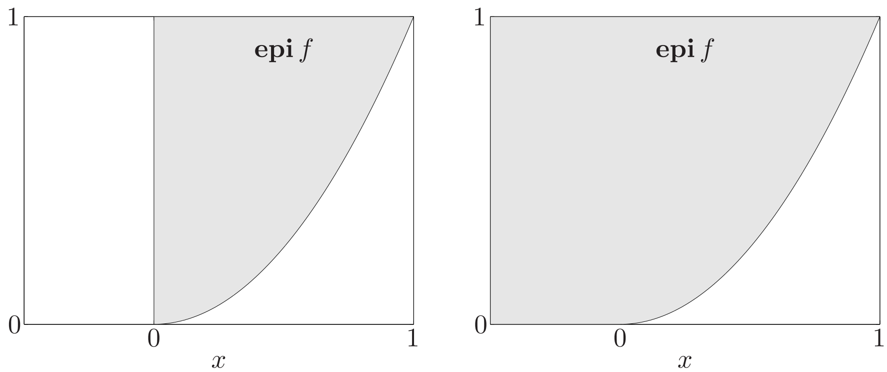


Figure 3.7 *Left.* The function x^2, with domain $\mathbf{R}_+$, is convex and nondecreasing on its domain, but its extended-value extension is *not* nondecreasing. *Right.* The function $\max\{x, 0\}^2$, with domain $\mathbf{R}$, is convex, and its extended-value extension is nondecreasing.

- The function $h(x) = x^{1/2}$, with $\mathbf{dom}\, h = \mathbf{R}_+$, is concave and satisfies the condition $\tilde{h}$ nondecreasing.
- The function $h(x) = x^{3/2}$, with $\mathbf{dom}\, h = \mathbf{R}_+$, is convex but *does not* satisfy the condition $\tilde{h}$ nondecreasing. For example, we have $\tilde{h}(-1) = \infty$, but $\tilde{h}(1) = 1$.
- The function $h(x) = x^{3/2}$ for $x \geq 0$, and $h(x) = 0$ for $x < 0$, with $\mathbf{dom}\, h = \mathbf{R}$, is convex and *does* satisfy the condition $\tilde{h}$ nondecreasing.

The composition results (3.11) can be proved directly, without assuming differentiability, or using the formula (3.9). As an example, we will prove the following composition theorem: if g is convex, h is convex, and $\tilde{h}$ is nondecreasing, then $f = h \circ g$ is convex. Assume that $x, y \in \mathbf{dom}\, f$, and $0 \leq \theta \leq 1$. Since $x, y \in \mathbf{dom}\, f$, we have that $x, y \in \mathbf{dom}\, g$ and $g(x), g(y) \in \mathbf{dom}\, h$. Since $\mathbf{dom}\, g$ is convex, we conclude that $\theta x + (1-\theta)y \in \mathbf{dom}\, g$, and from convexity of g, we have

$$g(\theta x + (1-\theta)y) \leq \theta g(x) + (1-\theta)g(y). \tag{3.12}$$

Since $g(x), g(y) \in \mathbf{dom}\, h$, we conclude that $\theta g(x) + (1-\theta)g(y) \in \mathbf{dom}\, h$, *i.e.*, the righthand side of (3.12) is in $\mathbf{dom}\, h$. Now we use the assumption that $\tilde{h}$ is nondecreasing, which means that its domain extends infinitely in the negative direction. Since the righthand side of (3.12) is in $\mathbf{dom}\, h$, we conclude that the lefthand side, *i.e.*, $g(\theta x+(1-\theta)y) \in \mathbf{dom}\, h$. This means that $\theta x+(1-\theta)y \in \mathbf{dom}\, f$. At this point, we have shown that $\mathbf{dom}\, f$ is convex.

Now using the fact that $\tilde{h}$ is nondecreasing and the inequality (3.12), we get

$$h(g(\theta x + (1-\theta)y)) \leq h(\theta g(x) + (1-\theta)g(y)). \tag{3.13}$$

From convexity of h, we have

$$h(\theta g(x) + (1-\theta)g(y)) \leq \theta h(g(x)) + (1-\theta)h(g(y)). \tag{3.14}$$

Putting (3.13) and (3.14) together, we have

$$h(g(\theta x + (1-\theta)y)) \leq \theta h(g(x)) + (1-\theta)h(g(y)).$$

which proves the composition theorem.

Example 3.13 *Simple composition results.*

- If g is convex then $\exp g(x)$ is convex.
- If g is concave and positive, then $\log g(x)$ is concave.
- If g is concave and positive, then $1/g(x)$ is convex.
- If g is convex and nonnegative and $p \geq 1$, then $g(x)^p$ is convex.
- If g is convex then $-\log(-g(x))$ is convex on $\{x \mid g(x) < 0\}$.

Remark 3.3 The requirement that monotonicity hold for the extended-value extension $\tilde{h}$, and not just the function h, cannot be removed. For example, consider the function $g(x) = x^2$, with $\mathbf{dom}\, g = \mathbf{R}$, and $h(x) = 0$, with $\mathbf{dom}\, h = [1, 2]$. Here g is convex, and h is convex and nondecreasing. But the function $f = h \circ g$, given by

$$f(x) = 0, \qquad \mathbf{dom}\, f = [-\sqrt{2}, -1] \cup [1, \sqrt{2}],$$

is not convex, since its domain is not convex. Here, of course, the function $\tilde{h}$ is *not* nondecreasing.

Vector composition

We now turn to the more complicated case when $k \geq 1$. Suppose

$$f(x) = h(g(x)) = h(g_1(x), \ldots, g_k(x)),$$

with $h : \mathbf{R}^k \to \mathbf{R}$, $g_i : \mathbf{R}^n \to \mathbf{R}$. Again without loss of generality we can assume $n = 1$. As in the case $k = 1$, we start by assuming the functions are twice differentiable, with $\mathbf{dom}\, g = \mathbf{R}$ and $\mathbf{dom}\, h = \mathbf{R}^k$, in order to discover the composition rules. We have

$$f''(x) = g'(x)^T \nabla^2 h(g(x)) g'(x) + \nabla h(g(x))^T g''(x), \tag{3.15}$$

which is the vector analog of (3.9). Again the issue is to determine conditions under which $f''(x) \geq 0$ for all x (or $f''(x) \leq 0$ for all x for concavity). From (3.15) we can derive many rules, for example:

f is convex if h is convex, h is nondecreasing in each argument, and g_i are convex,

f is convex if h is convex, h is nonincreasing in each argument, and g_i are concave,

f is concave if h is concave, h is nondecreasing in each argument, and g_i are concave.

As in the scalar case, similar composition results hold in general, with $n > 1$, no assumption of differentiability of h or g, and general domains. For the general results, the monotonicity condition on h must hold for the extended-value extension $\tilde{h}$.

To understand the meaning of the condition that the extended-value extension $\tilde{h}$ be monotonic, we consider the case where $h : \mathbf{R}^k \to \mathbf{R}$ is convex, and $\tilde{h}$ nondecreasing, *i.e.*, whenever $u \preceq v$, we have $\tilde{h}(u) \leq \tilde{h}(v)$. This implies that if $v \in \mathbf{dom}\, h$, then so is u: the domain of h must extend infinitely in the $-\mathbf{R}^k_+$ directions. We can express this compactly as $\mathbf{dom}\, h - \mathbf{R}^k_+ = \mathbf{dom}\, h$.

Example 3.14 *Vector composition examples.*

- Let $h(z) = z_{[1]} + \cdots + z_{[r]}$, the sum of the r largest components of $z \in \mathbf{R}^k$. Then h is convex and nondecreasing in each argument. Suppose $g_1, \ldots, g_k$ are convex functions on $\mathbf{R}^n$. Then the composition function $f = h \circ g$, *i.e.*, the pointwise sum of the r largest g_i's, is convex.
- The function $h(z) = \log(\sum_{i=1}^k e^{z_i})$ is convex and nondecreasing in each argument, so $\log(\sum_{i=1}^k e^{g_i})$ is convex whenever g_i are.
- For $0 < p \leq 1$, the function $h(z) = (\sum_{i=1}^k z_i^p)^{1/p}$ on $\mathbf{R}^k_+$ is concave, and its extension (which has the value $-\infty$ for $z \not\succeq 0$) is nondecreasing in each component. So if g_i are concave and nonnegative, we conclude that $f(x) = (\sum_{i=1}^k g_i(x)^p)^{1/p}$ is concave.
- Suppose $p \geq 1$, and $g_1, \ldots, g_k$ are convex and nonnegative. Then the function $(\sum_{i=1}^k g_i(x)^p)^{1/p}$ is convex.

  To show this, we consider the function $h : \mathbf{R}^k \to \mathbf{R}$ defined as
  $$h(z) = \left(\sum_{i=1}^k \max\{z_i, 0\}^p \right)^{1/p},$$
  with $\mathbf{dom}\, h = \mathbf{R}^k$, so $h = \tilde{h}$. This function is convex, and nondecreasing, so we conclude $h(g(x))$ is a convex function of x. For $z \succeq 0$, we have $h(z) = (\sum_{i=1}^k z_i^p)^{1/p}$, so our conclusion is that $(\sum_{i=1}^k g_i(x)^p)^{1/p}$ is convex.
- The geometric mean $h(z) = (\prod_{i=1}^k z_i)^{1/k}$ on $\mathbf{R}^k_+$ is concave and its extension is nondecreasing in each argument. It follows that if $g_1, \ldots, g_k$ are nonnegative concave functions, then so is their geometric mean, $(\prod_{i=1}^k g_i)^{1/k}$.

3.2.5 Minimization

We have seen that the maximum or supremum of an arbitrary family of convex functions is convex. It turns out that some special forms of minimization also yield convex functions. If f is convex in (x, y), and C is a convex nonempty set, then the function
$$g(x) = \inf_{y \in C} f(x, y) \tag{3.16}$$

is convex in x, provided $g(x) > -\infty$ for all x. The domain of g is the projection of $\mathbf{dom}\, f$ on its x-coordinates, *i.e.*,

$$\mathbf{dom}\, g = \{x \mid (x, y) \in \mathbf{dom}\, f \text{ for some } y \in C\}.$$

We prove this by verifying Jensen's inequality for $x_1,\ x_2 \in \mathbf{dom}\, g$. Let $\epsilon > 0$. Then there are $y_1,\ y_2 \in C$ such that $f(x_i, y_i) \leq g(x_i) + \epsilon$ for $i = 1,\ 2$. Now let $\theta \in [0, 1]$. We have

$$\begin{aligned} g(\theta x_1 + (1-\theta)x_2) &= \inf_{y \in C} f(\theta x_1 + (1-\theta)x_2, y) \\ &\leq f(\theta x_1 + (1-\theta)x_2, \theta y_1 + (1-\theta)y_2) \\ &\leq \theta f(x_1, y_1) + (1-\theta) f(x_2, y_2) \\ &\leq \theta g(x_1) + (1-\theta) g(x_2) + \epsilon. \end{aligned}$$

Since this holds for any $\epsilon > 0$, we have

$$g(\theta x_1 + (1-\theta)x_2) \leq \theta g(x_1) + (1-\theta) g(x_2).$$

The result can also be seen in terms of epigraphs. With f, g, and C defined as in (3.16), and assuming the infimum over $y \in C$ is attained for each x, we have

$$\mathbf{epi}\, g = \{(x, t) \mid (x, y, t) \in \mathbf{epi}\, f \text{ for some } y \in C\}.$$

Thus $\mathbf{epi}\, g$ is convex, since it is the projection of a convex set on some of its components.

Example 3.15 *Schur complement.* Suppose the quadratic function

$$f(x, y) = x^T A x + 2x^T B y + y^T C y,$$

(where A and C are symmetric) is convex in (x, y), which means

$$\begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \succeq 0.$$

We can express $g(x) = \inf_y f(x, y)$ as

$$g(x) = x^T (A - BC^\dagger B^T) x,$$

where $C^\dagger$ is the pseudo-inverse of C (see §A.5.4). By the minimization rule, g is convex, so we conclude that $A - BC^\dagger B^T \succeq 0$.

If C is invertible, *i.e.*, $C \succ 0$, then the matrix $A - BC^{-1}B^T$ is called the *Schur complement* of C in the matrix

$$\begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$$

(see §A.5.5).

Example 3.16 *Distance to a set.* The distance of a point x to a set $S \subseteq \mathbf{R}^n$, in the norm $\|\cdot\|$, is defined as

$$\mathbf{dist}(x, S) = \inf_{y \in S} \|x - y\|.$$

The function $\|x - y\|$ is convex in (x, y), so if the set S is convex, the distance function $\mathbf{dist}(x, S)$ is a convex function of x.

Example 3.17 Suppose h is convex. Then the function g defined as

$$g(x) = \inf\{h(y) \mid Ay = x\}$$

is convex. To see this, we define f by

$$f(x, y) = \begin{cases} h(y) & \text{if } Ay = x \\ \infty & \text{otherwise,} \end{cases}$$

which is convex in (x, y). Then g is the minimum of f over y, and hence is convex. (It is not hard to show directly that g is convex.)

3.2.6 Perspective of a function

If $f : \mathbf{R}^n \to \mathbf{R}$, then the *perspective* of f is the function $g : \mathbf{R}^{n+1} \to \mathbf{R}$ defined by

$$g(x, t) = tf(x/t),$$

with domain

$$\mathbf{dom}\, g = \{(x, t) \mid x/t \in \mathbf{dom}\, f,\ t > 0\}.$$

The perspective operation preserves convexity: If f is a convex function, then so is its perspective function g. Similarly, if f is concave, then so is g.

This can be proved several ways, for example, direct verification of the defining inequality (see exercise 3.33). We give a short proof here using epigraphs and the perspective mapping on $\mathbf{R}^{n+1}$ described in §2.3.3 (which will also explain the name 'perspective'). For $t > 0$ we have

$$\begin{aligned} (x, t, s) \in \mathbf{epi}\, g \quad &\Longleftrightarrow \quad tf(x/t) \le s \\ &\Longleftrightarrow \quad f(x/t) \le s/t \\ &\Longleftrightarrow \quad (x/t, s/t) \in \mathbf{epi}\, f. \end{aligned}$$

Therefore $\mathbf{epi}\, g$ is the inverse image of $\mathbf{epi}\, f$ under the perspective mapping that takes (u, v, w) to $(u, w)/v$. It follows (see §2.3.3) that $\mathbf{epi}\, g$ is convex, so the function g is convex.

Example 3.18 *Euclidean norm squared.* The perspective of the convex function $f(x) = x^T x$ on $\mathbf{R}^n$ is

$$g(x, t) = t(x/t)^T(x/t) = \frac{x^T x}{t},$$

which is convex in (x, t) for $t > 0$.

We can deduce convexity of g using several other methods. First, we can express g as the sum of the quadratic-over-linear functions x_i^2/t, which were shown to be convex in §3.1.5. We can also express g as a special case of the matrix fractional function $x^T(tI)^{-1}x$ (see example 3.4).

Example 3.19 *Negative logarithm.* Consider the convex function $f(x) = -\log x$ on $\mathbf{R}_{++}$. Its perspective is

$$g(x,t) = -t\log(x/t) = t\log(t/x) = t\log t - t\log x,$$

and is convex on $\mathbf{R}^2_{++}$. The function g is called the *relative entropy* of t and x. For $x = 1$, g reduces to the negative entropy function.

From convexity of g we can establish convexity or concavity of several interesting related functions. First, the relative entropy of two vectors $u,\ v \in \mathbf{R}^n_{++}$, defined as

$$\sum_{i=1}^n u_i \log(u_i/v_i),$$

is convex in (u,v), since it is a sum of relative entropies of $u_i,\ v_i$.

A closely related function is the *Kullback-Leibler divergence* between $u,\ v \in \mathbf{R}^n_{++}$, given by

$$D_{\mathrm{kl}}(u,v) = \sum_{i=1}^n \left(u_i \log(u_i/v_i) - u_i + v_i\right), \tag{3.17}$$

which is convex, since it is the relative entropy plus a linear function of (u,v). The Kullback-Leibler divergence satisfies $D_{\mathrm{kl}}(u,v) \geq 0$, and $D_{\mathrm{kl}}(u,v) = 0$ if and only if $u = v$, and so can be used as a measure of deviation between two positive vectors; see exercise 3.13. (Note that the relative entropy and the Kullback-Leibler divergence are the same when u and v are probability vectors, *i.e.*, satisfy $\mathbf{1}^T u = \mathbf{1}^T v = 1$.)

If we take $v_i = \mathbf{1}^T u$ in the relative entropy function, we obtain the concave (and homogeneous) function of $u \in \mathbf{R}^n_{++}$ given by

$$\sum_{i=1}^n u_i \log(\mathbf{1}^T u/u_i) = (\mathbf{1}^T u)\sum_{i=1}^n z_i \log(1/z_i),$$

where $z = u/(\mathbf{1}^T u)$, which is called the *normalized entropy* function. The vector $z = u/\mathbf{1}^T u$ is a normalized vector or probability distribution, since its components sum to one; the normalized entropy of u is $\mathbf{1}^T u$ times the entropy of this normalized distribution.

Example 3.20 Suppose $f : \mathbf{R}^m \to \mathbf{R}$ is convex, and $A \in \mathbf{R}^{m\times n}$, $b \in \mathbf{R}^m$, $c \in \mathbf{R}^n$, and $d \in \mathbf{R}$. We define

$$g(x) = (c^T x + d) f\left((Ax+b)/(c^T x + d)\right),$$

with

$$\mathbf{dom}\, g = \{x \mid c^T x + d > 0,\ (Ax+b)/(c^T x+d) \in \mathbf{dom}\, f\}.$$

Then g is convex.

3.3 The conjugate function

In this section we introduce an operation that will play an important role in later chapters.

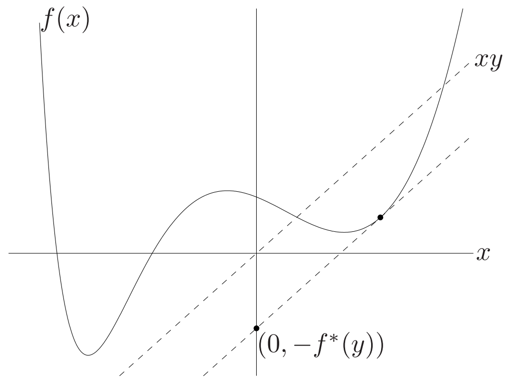


Figure 3.8 A function $f : \mathbf{R} \to \mathbf{R}$, and a value $y \in \mathbf{R}$. The conjugate function $f^*(y)$ is the maximum gap between the linear function yx and $f(x)$, as shown by the dashed line in the figure. If f is differentiable, this occurs at a point x where $f'(x) = y$.

3.3.1 Definition and examples

Let $f : \mathbf{R}^n \to \mathbf{R}$. The function $f^* : \mathbf{R}^n \to \mathbf{R}$, defined as

$$f^*(y) = \sup_{x \in \mathbf{dom}\, f} \left(y^T x - f(x) \right), \tag{3.18}$$

is called the *conjugate* of the function f. The domain of the conjugate function consists of $y \in \mathbf{R}^n$ for which the supremum is finite, *i.e.*, for which the difference $y^T x - f(x)$ is bounded above on $\mathbf{dom}\, f$. This definition is illustrated in figure 3.8.

We see immediately that f^* is a convex function, since it is the pointwise supremum of a family of convex (indeed, affine) functions of y. This is true whether or not f is convex. (Note that when f is convex, the subscript $x \in \mathbf{dom}\, f$ is not necessary since, by convention, $y^T x - f(x) = -\infty$ for $x \notin \mathbf{dom}\, f$.)

We start with some simple examples, and then describe some rules for conjugating functions. This allows us to derive an analytical expression for the conjugate of many common convex functions.

Example 3.21 We derive the conjugates of some convex functions on $\mathbf{R}$.

- *Affine function.* $f(x) = ax + b$. As a function of x, $yx - ax - b$ is bounded if and only if $y = a$, in which case it is constant. Therefore the domain of the conjugate function f^* is the singleton $\{a\}$, and $f^*(a) = -b$.
- *Negative logarithm.* $f(x) = -\log x$, with $\mathbf{dom}\, f = \mathbf{R}_{++}$. The function $xy + \log x$ is unbounded above if $y \geq 0$ and reaches its maximum at $x = -1/y$ otherwise. Therefore, $\mathbf{dom}\, f^* = \{y \mid y < 0\} = -\mathbf{R}_{++}$ and $f^*(y) = -\log(-y) - 1$ for $y < 0$.
- *Exponential.* $f(x) = e^x$. $xy - e^x$ is unbounded if $y < 0$. For $y > 0$, $xy - e^x$ reaches its maximum at $x = \log y$, so we have $f^*(y) = y \log y - y$. For $y = 0$,

$f^*(y) = \sup_x -e^x = 0$. In summary, $\mathbf{dom}\, f^* = \mathbf{R}_+$ and $f^*(y) = y\log y - y$ (with the interpretation $0 \log 0 = 0$).

- *Negative entropy.* $f(x) = x \log x$, with $\mathbf{dom}\, f = \mathbf{R}_+$ (and $f(0) = 0$). The function $xy - x\log x$ is bounded above on $\mathbf{R}_+$ for all y, hence $\mathbf{dom}\, f^* = \mathbf{R}$. It attains its maximum at $x = e^{y-1}$, and substituting we find $f^*(y) = e^{y-1}$.
- *Inverse.* $f(x) = 1/x$ on $\mathbf{R}_{++}$. For $y > 0$, $yx - 1/x$ is unbounded above. For $y = 0$ this function has supremum 0; for $y < 0$ the supremum is attained at $x = (-y)^{-1/2}$. Therefore we have $f^*(y) = -2(-y)^{1/2}$, with $\mathbf{dom}\, f^* = -\mathbf{R}_+$.

Example 3.22 *Strictly convex quadratic function.* Consider $f(x) = \frac{1}{2}x^TQx$, with $Q \in \mathbf{S}^n_{++}$. The function $y^Tx - \frac{1}{2}x^TQx$ is bounded above as a function of x for all y. It attains its maximum at $x = Q^{-1}y$, so

$$f^*(y) = \frac{1}{2}y^TQ^{-1}y.$$

Example 3.23 *Log-determinant.* We consider $f(X) = \log\det X^{-1}$ on $\mathbf{S}^n_{++}$. The conjugate function is defined as

$$f^*(Y) = \sup_{X \succ 0} \left(\mathbf{tr}(YX) + \log\det X\right),$$

since $\mathbf{tr}(YX)$ is the standard inner product on $\mathbf{S}^n$. We first show that $\mathbf{tr}(YX) + \log\det X$ is unbounded above unless $Y \prec 0$. If $Y \not\prec 0$, then Y has an eigenvector v, with $\|v\|_2 = 1$, and eigenvalue $\lambda \geq 0$. Taking $X = I + tvv^T$ we find that

$$\mathbf{tr}(YX) + \log\det X = \mathbf{tr}\, Y + t\lambda + \log\det(I + tvv^T) = \mathbf{tr}\, Y + t\lambda + \log(1+t),$$

which is unbounded above as $t \to \infty$.

Now consider the case $Y \prec 0$. We can find the maximizing X by setting the gradient with respect to X equal to zero:

$$\nabla_X \left(\mathbf{tr}(YX) + \log\det X\right) = Y + X^{-1} = 0$$

(see §A.4.1), which yields $X = -Y^{-1}$ (which is, indeed, positive definite). Therefore we have

$$f^*(Y) = \log\det(-Y)^{-1} - n,$$

with $\mathbf{dom}\, f^* = -\mathbf{S}^n_{++}$.

Example 3.24 *Indicator function.* Let I_S be the indicator function of a (not necessarily convex) set $S \subseteq \mathbf{R}^n$, *i.e.*, $I_S(x) = 0$ on $\mathbf{dom}\, I_S = S$. Its conjugate is

$$I_S^*(y) = \sup_{x \in S} y^Tx,$$

which is the support function of the set S.

Example 3.25 *Log-sum-exp function.* To derive the conjugate of the log-sum-exp function $f(x) = \log(\sum_{i=1}^n e^{x_i})$, we first determine the values of y for which the maximum over x of $y^T x - f(x)$ is attained. By setting the gradient with respect to x equal to zero, we obtain the condition

$$y_i = \frac{e^{x_i}}{\sum_{j=1}^n e^{x_j}}, \quad i = 1, \ldots, n.$$

These equations are solvable for x if and only if $y \succ 0$ and $\mathbf{1}^T y = 1$. By substituting the expression for y_i into $y^T x - f(x)$ we obtain $f^*(y) = \sum_{i=1}^n y_i \log y_i$. This expression for f^* is still correct if some components of y are zero, as long as $y \succeq 0$ and $\mathbf{1}^T y = 1$, and we interpret $0 \log 0$ as 0.

In fact the domain of f^* is exactly given by $\mathbf{1}^T y = 1$, $y \succeq 0$. To show this, suppose that a component of y is negative, say, $y_k < 0$. Then we can show that $y^T x - f(x)$ is unbounded above by choosing $x_k = -t$, and $x_i = 0$, $i \neq k$, and letting t go to infinity.

If $y \succeq 0$ but $\mathbf{1}^T y \neq 1$, we choose $x = t\mathbf{1}$, so that

$$y^T x - f(x) = t\mathbf{1}^T y - t - \log n.$$

If $\mathbf{1}^T y > 1$, this grows unboundedly as $t \to \infty$; if $\mathbf{1}^T y < 1$, it grows unboundedly as $t \to -\infty$.

In summary,

$$f^*(y) = \begin{cases} \sum_{i=1}^n y_i \log y_i & \text{if } y \succeq 0 \text{ and } \mathbf{1}^T y = 1 \\ \infty & \text{otherwise.} \end{cases}$$

In other words, the conjugate of the log-sum-exp function is the negative entropy function, restricted to the probability simplex.

Example 3.26 *Norm.* Let $\|\cdot\|$ be a norm on $\mathbf{R}^n$, with dual norm $\|\cdot\|_*$. We will show that the conjugate of $f(x) = \|x\|$ is

$$f^*(y) = \begin{cases} 0 & \|y\|_* \leq 1 \\ \infty & \text{otherwise,} \end{cases}$$

i.e., the conjugate of a norm is the indicator function of the dual norm unit ball.

If $\|y\|_* > 1$, then by definition of the dual norm, there is a $z \in \mathbf{R}^n$ with $\|z\| \leq 1$ and $y^T z > 1$. Taking $x = tz$ and letting $t \to \infty$, we have

$$y^T x - \|x\| = t(y^T z - \|z\|) \to \infty,$$

which shows that $f^*(y) = \infty$. Conversely, if $\|y\|_* \leq 1$, then we have $y^T x \leq \|x\| \|y\|_*$ for all x, which implies for all x, $y^T x - \|x\| \leq 0$. Therefore $x = 0$ is the value that maximizes $y^T x - \|x\|$, with maximum value 0.

Example 3.27 *Norm squared.* Now consider the function $f(x) = (1/2)\|x\|^2$, where $\|\cdot\|$ is a norm, with dual norm $\|\cdot\|_*$. We will show that its conjugate is $f^*(y) = (1/2)\|y\|_*^2$. From $y^T x \leq \|y\|_* \|x\|$, we conclude

$$y^T x - (1/2)\|x\|^2 \leq \|y\|_* \|x\| - (1/2)\|x\|^2$$

for all x. The righthand side is a quadratic function of $\|x\|$, which has maximum value $(1/2)\|y\|_*^2$. Therefore for all x, we have

$$y^T x - (1/2)\|x\|^2 \leq (1/2)\|y\|_*^2,$$

which shows that $f^*(y) \leq (1/2)\|y\|_*^2$.

To show the other inequality, let x be any vector with $y^T x = \|y\|_* \|x\|$, scaled so that $\|x\| = \|y\|_*$. Then we have, for this x,

$$y^T x - (1/2)\|x\|^2 = (1/2)\|y\|_*^2,$$

which shows that $f^*(y) \geq (1/2)\|y\|_*^2$.

Example 3.28 *Revenue and profit functions.* We consider a business or enterprise that consumes n resources and produces a product that can be sold. We let $r = (r_1, \ldots, r_n)$ denote the vector of resource quantities consumed, and $S(r)$ denote the sales revenue derived from the product produced (as a function of the resources consumed). Now let p_i denote the price (per unit) of resource i, so the total amount paid for resources by the enterprise is $p^T r$. The profit derived by the firm is then $S(r) - p^T r$. Let us fix the prices of the resources, and ask what is the maximum profit that can be made, by wisely choosing the quantities of resources consumed. This maximum profit is given by

$$M(p) = \sup_r \left(S(r) - p^T r \right).$$

The function $M(p)$ gives the maximum profit attainable, as a function of the resource prices. In terms of conjugate functions, we can express M as

$$M(p) = (-S)^*(-p).$$

Thus the maximum profit (as a function of resource prices) is closely related to the conjugate of gross sales (as a function of resources consumed).

3.3.2 Basic properties

Fenchel's inequality

From the definition of conjugate function, we immediately obtain the inequality

$$f(x) + f^*(y) \geq x^T y$$

for all x, y. This is called *Fenchel's inequality* (or *Young's inequality* when f is differentiable).

For example with $f(x) = (1/2)x^T Q x$, where $Q \in \mathbf{S}_{++}^n$, we obtain the inequality

$$x^T y \leq (1/2)x^T Q x + (1/2)y^T Q^{-1} y.$$

Conjugate of the conjugate

The examples above, and the name 'conjugate', suggest that the conjugate of the conjugate of a convex function is the original function. This is the case provided a technical condition holds: if f is convex, and f is closed (*i.e.*, $\mathbf{epi}\, f$ is a closed set; see §A.3.3), then $f^{**} = f$. For example, if $\mathbf{dom}\, f = \mathbf{R}^n$, then we have $f^{**} = f$, *i.e.*, the conjugate of the conjugate of f is f again (see exercise 3.39).

Differentiable functions

The conjugate of a differentiable function f is also called the *Legendre transform* of f. (To distinguish the general definition from the differentiable case, the term *Fenchel conjugate* is sometimes used instead of conjugate.)

Suppose f is convex and differentiable, with $\mathbf{dom}\, f = \mathbf{R}^n$. Any maximizer x^* of $y^T x - f(x)$ satisfies $y = \nabla f(x^*)$, and conversely, if x^* satisfies $y = \nabla f(x^*)$, then x^* maximizes $y^T x - f(x)$. Therefore, if $y = \nabla f(x^*)$, we have

$$f^*(y) = x^{*T} \nabla f(x^*) - f(x^*).$$

This allows us to determine $f^*(y)$ for any y for which we can solve the gradient equation $y = \nabla f(z)$ for z.

We can express this another way. Let $z \in \mathbf{R}^n$ be arbitrary and define $y = \nabla f(z)$. Then we have

$$f^*(y) = z^T \nabla f(z) - f(z).$$

Scaling and composition with affine transformation

For $a > 0$ and $b \in \mathbf{R}$, the conjugate of $g(x) = af(x) + b$ is $g^*(y) = af^*(y/a) - b$.

Suppose $A \in \mathbf{R}^{n \times n}$ is nonsingular and $b \in \mathbf{R}^n$. Then the conjugate of $g(x) = f(Ax + b)$ is

$$g^*(y) = f^*(A^{-T} y) - b^T A^{-T} y,$$

with $\mathbf{dom}\, g^* = A^T\, \mathbf{dom}\, f^*$.

Sums of independent functions

If $f(u, v) = f_1(u) + f_2(v)$, where f_1 and f_2 are convex functions with conjugates f_1^* and f_2^*, respectively, then

$$f^*(w, z) = f_1^*(w) + f_2^*(z).$$

In other words, the conjugate of the sum of *independent* convex functions is the sum of the conjugates. ('Independent' means they are functions of different variables.)

3.4 Quasiconvex functions

3.4.1 Definition and examples

A function $f : \mathbf{R}^n \to \mathbf{R}$ is called *quasiconvex* (or *unimodal*) if its domain and all its sublevel sets

$$S_\alpha = \{x \in \mathbf{dom}\, f \mid f(x) \le \alpha\},$$

for $\alpha \in \mathbf{R}$, are convex. A function is *quasiconcave* if $-f$ is quasiconvex, *i.e.*, every superlevel set $\{x \mid f(x) \ge \alpha\}$ is convex. A function that is both quasiconvex and quasiconcave is called *quasilinear*. If a function f is quasilinear, then its domain, and every level set $\{x \mid f(x) = \alpha\}$ is convex.

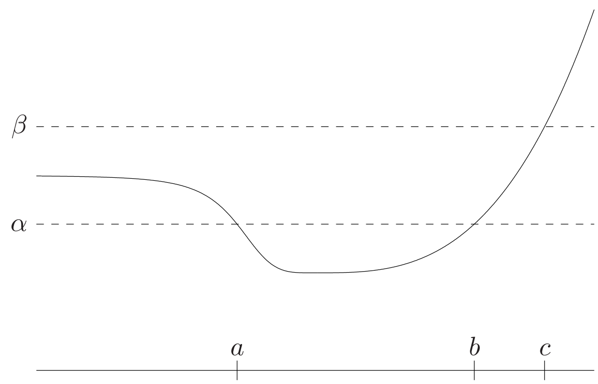


Figure 3.9 A quasiconvex function on $\mathbf{R}$. For each α, the α-sublevel set S_α is convex, *i.e.*, an interval. The sublevel set S_α is the interval $[a, b]$. The sublevel set S_β is the interval $(-\infty, c]$.

For a function on $\mathbf{R}$, quasiconvexity requires that each sublevel set be an interval (including, possibly, an infinite interval). An example of a quasiconvex function on $\mathbf{R}$ is shown in figure 3.9.

Convex functions have convex sublevel sets, and so are quasiconvex. But simple examples, such as the one shown in figure 3.9, show that the converse is not true.

Example 3.29 Some examples on $\mathbf{R}$:

- *Logarithm.* $\log x$ on $\mathbf{R}_{++}$ is quasiconvex (and quasiconcave, hence quasilinear).
- *Ceiling function.* $\mathrm{ceil}(x) = \inf\{z \in \mathbf{Z} \mid z \geq x\}$ is quasiconvex (and quasiconcave).

These examples show that quasiconvex functions can be concave, or discontinuous. We now give some examples on $\mathbf{R}^n$.

Example 3.30 *Length of a vector.* We define the *length* of $x \in \mathbf{R}^n$ as the largest index of a nonzero component, *i.e.*,

$$f(x) = \max\{i \mid x_i \neq 0\}.$$

(We define the length of the zero vector to be zero.) This function is quasiconvex on $\mathbf{R}^n$, since its sublevel sets are subspaces:

$$f(x) \leq \alpha \iff x_i = 0 \text{ for } i = \lfloor \alpha \rfloor + 1, \ldots, n.$$

Example 3.31 Consider $f : \mathbf{R}^2 \to \mathbf{R}$, with $\mathbf{dom}\, f = \mathbf{R}^2_+$ and $f(x_1, x_2) = x_1 x_2$. This function is neither convex nor concave since its Hessian

$$\nabla^2 f(x) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

is indefinite; it has one positive and one negative eigenvalue. The function f is quasiconcave, however, since the superlevel sets

$$\{x \in \mathbf{R}_+^2 \mid x_1 x_2 \geq \alpha\}$$

are convex sets for all α. (Note, however, that f is *not* quasiconcave on $\mathbf{R}^2$.)

Example 3.32 *Linear-fractional function.* The function

$$f(x) = \frac{a^T x + b}{c^T x + d},$$

with $\mathbf{dom}\, f = \{x \mid c^T x + d > 0\}$, is quasiconvex, and quasiconcave, *i.e.*, quasilinear. Its α-sublevel set is

$$\begin{aligned} S_\alpha &= \{x \mid c^T x + d > 0,\ (a^T x + b)/(c^T x + d) \leq \alpha\} \\ &= \{x \mid c^T x + d > 0,\ a^T x + b \leq \alpha(c^T x + d)\}, \end{aligned}$$

which is convex, since it is the intersection of an open halfspace and a closed halfspace. (The same method can be used to show its superlevel sets are convex.)

Example 3.33 *Distance ratio function.* Suppose $a, b \in \mathbf{R}^n$, and define

$$f(x) = \frac{\|x - a\|_2}{\|x - b\|_2},$$

i.e., the ratio of the Euclidean distance to a to the distance to b. Then f is quasiconvex on the halfspace $\{x \mid \|x - a\|_2 \leq \|x - b\|_2\}$. To see this, we consider the α-sublevel set of f, with $\alpha \leq 1$ since $f(x) \leq 1$ on the halfspace $\{x \mid \|x - a\|_2 \leq \|x - b\|_2\}$. This sublevel set is the set of points satisfying

$$\|x - a\|_2 \leq \alpha \|x - b\|_2.$$

Squaring both sides, and rearranging terms, we see that this is equivalent to

$$(1 - \alpha^2) x^T x - 2(a - \alpha^2 b)^T x + a^T a - \alpha^2 b^T b \leq 0.$$

This describes a convex set (in fact a Euclidean ball) if $\alpha \leq 1$.

Example 3.34 *Internal rate of return.* Let $x = (x_0, x_1, \ldots, x_n)$ denote a cash flow sequence over n periods, where $x_i > 0$ means a payment to us in period i, and $x_i < 0$ means a payment by us in period i. We define the *present value* of a cash flow, with interest rate $r \geq 0$, to be

$$\mathrm{PV}(x, r) = \sum_{i=0}^{n} (1 + r)^{-i} x_i.$$

(The factor $(1 + r)^{-i}$ is a *discount factor* for a payment by or to us in period i.)

Now we consider cash flows for which $x_0 < 0$ and $x_0 + x_1 + \cdots + x_n > 0$. This means that we start with an investment of $|x_0|$ in period 0, and that the total of the remaining cash flow, $x_1 + \cdots + x_n$, (not taking any discount factors into account) exceeds our initial investment.

For such a cash flow, $\mathrm{PV}(x, 0) > 0$ and $\mathrm{PV}(x, r) \to x_0 < 0$ as $r \to \infty$, so it follows that for at least one $r \geq 0$, we have $\mathrm{PV}(x, r) = 0$. We define the *internal rate of return* of the cash flow as the smallest interest rate $r \geq 0$ for which the present value is zero:

$$\mathrm{IRR}(x) = \inf\{r \geq 0 \mid \mathrm{PV}(x, r) = 0\}.$$

Internal rate of return is a quasiconcave function of x (restricted to $x_0 < 0$, $x_1 + \cdots + x_n > 0$). To see this, we note that

$$\mathrm{IRR}(x) \geq R \iff \mathrm{PV}(x, r) > 0 \text{ for } 0 \leq r < R.$$

The lefthand side defines the R-superlevel set of IRR. The righthand side is the intersection of the sets $\{x \mid \mathrm{PV}(x, r) > 0\}$, indexed by r, over the range $0 \leq r < R$. For each r, $\mathrm{PV}(x, r) > 0$ defines an open halfspace, so the righthand side defines a convex set.

3.4.2 Basic properties

The examples above show that quasiconvexity is a considerable generalization of convexity. Still, many of the properties of convex functions hold, or have analogs, for quasiconvex functions. For example, there is a variation on Jensen's inequality that characterizes quasiconvexity: A function f is quasiconvex if and only if $\mathbf{dom}\, f$ is convex and for any $x,\ y \in \mathbf{dom}\, f$ and $0 \leq \theta \leq 1$,

$$f(\theta x + (1 - \theta) y) \leq \max\{f(x), f(y)\}, \tag{3.19}$$

i.e., the value of the function on a segment does not exceed the maximum of its values at the endpoints. The inequality (3.19) is sometimes called Jensen's inequality for quasiconvex functions, and is illustrated in figure 3.10.

Example 3.35 *Cardinality of a nonnegative vector.* The *cardinality* or *size* of a vector $x \in \mathbf{R}^n$ is the number of nonzero components, and denoted $\mathbf{card}(x)$. The function $\mathbf{card}$ is quasiconcave on $\mathbf{R}^n_+$ (but not $\mathbf{R}^n$). This follows immediately from the modified Jensen inequality

$$\mathbf{card}(x + y) \geq \min\{\mathbf{card}(x), \mathbf{card}(y)\},$$

which holds for $x,\ y \succeq 0$.

Example 3.36 *Rank of positive semidefinite matrix.* The function $\mathbf{rank}\, X$ is quasiconcave on $\mathbf{S}^n_+$. This follows from the modified Jensen inequality (3.19),

$$\mathbf{rank}(X + Y) \geq \min\{\mathbf{rank}\, X, \mathbf{rank}\, Y\}$$

which holds for $X,\ Y \in \mathbf{S}^n_+$. (This can be considered an extension of the previous example, since $\mathbf{rank}(\mathbf{diag}(x)) = \mathbf{card}(x)$ for $x \succeq 0$.)

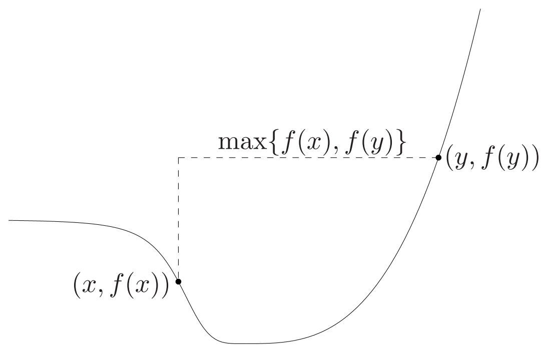


Figure 3.10 A quasiconvex function on $\mathbf{R}$. The value of f between x and y is no more than $\max\{f(x), f(y)\}$.

Like convexity, quasiconvexity is characterized by the behavior of a function f on lines: f is quasiconvex if and only if its restriction to any line intersecting its domain is quasiconvex. In particular, quasiconvexity of a function can be verified by restricting it to an arbitrary line, and then checking quasiconvexity of the resulting function on $\mathbf{R}$.

Quasiconvex functions on R

We can give a simple characterization of quasiconvex functions on $\mathbf{R}$. We consider continuous functions, since stating the conditions in the general case is cumbersome. A continuous function $f : \mathbf{R} \to \mathbf{R}$ is quasiconvex if and only if at least one of the following conditions holds:

- f is nondecreasing
- f is nonincreasing
- there is a point $c \in \mathbf{dom}\, f$ such that for $t \leq c$ (and $t \in \mathbf{dom}\, f$), f is nonincreasing, and for $t \geq c$ (and $t \in \mathbf{dom}\, f$), f is nondecreasing.

The point c can be chosen as any point which is a global minimizer of f. Figure 3.11 illustrates this.

3.4.3 Differentiable quasiconvex functions

First-order conditions

Suppose $f : \mathbf{R}^n \to \mathbf{R}$ is differentiable. Then f is quasiconvex if and only if $\mathbf{dom}\, f$ is convex and for all x, $y \in \mathbf{dom}\, f$

$$f(y) \leq f(x) \Longrightarrow \nabla f(x)^T (y - x) \leq 0. \tag{3.20}$$

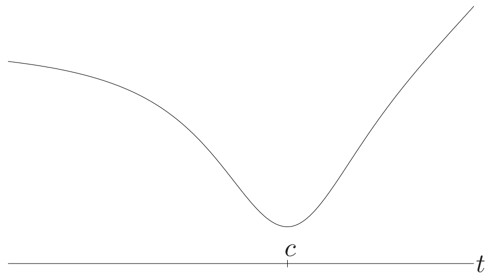


Figure 3.11 A quasiconvex function on $\mathbf{R}$. The function is nonincreasing for $t \leq c$ and nondecreasing for $t \geq c$.

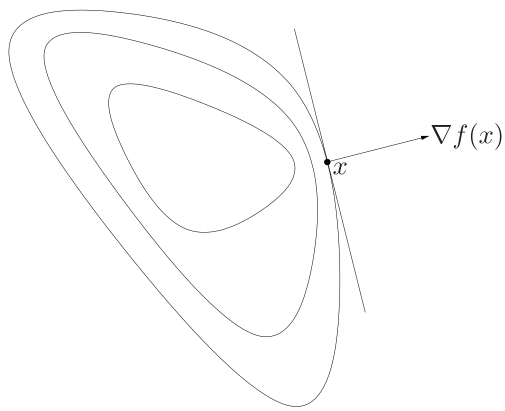


Figure 3.12 Three level curves of a quasiconvex function f are shown. The vector $\nabla f(x)$ defines a supporting hyperplane to the sublevel set $\{z \mid f(z) \leq f(x)\}$ at x.

This is the analog of inequality (3.2), for quasiconvex functions. We leave the proof as an exercise (exercise 3.43).

The condition (3.20) has a simple geometric interpretation when $\nabla f(x) \neq 0$. It states that $\nabla f(x)$ defines a supporting hyperplane to the sublevel set $\{y \mid f(y) \leq f(x)\}$, at the point x, as illustrated in figure 3.12.

While the first-order condition for convexity (3.2), and the first-order condition for quasiconvexity (3.20) are similar, there are some important differences. For example, if f is convex and $\nabla f(x) = 0$, then x is a global minimizer of f. But this statement is *false* for quasiconvex functions: it is possible that $\nabla f(x) = 0$, but x is not a global minimizer of f.

Second-order conditions

Now suppose f is twice differentiable. If f is quasiconvex, then for all $x \in \mathbf{dom}\, f$, and all $y \in \mathbf{R}^n$, we have

$$y^T \nabla f(x) = 0 \Longrightarrow y^T \nabla^2 f(x) y \geq 0. \tag{3.21}$$

For a quasiconvex function on $\mathbf{R}$, this reduces to the simple condition

$$f'(x) = 0 \Longrightarrow f''(x) \geq 0,$$

i.e., at any point with zero slope, the second derivative is nonnegative. For a quasiconvex function on $\mathbf{R}^n$, the interpretation of the condition (3.21) is a bit more complicated. As in the case $n = 1$, we conclude that whenever $\nabla f(x) = 0$, we must have $\nabla^2 f(x) \succeq 0$. When $\nabla f(x) \neq 0$, the condition (3.21) means that $\nabla^2 f(x)$ is positive semidefinite on the $(n-1)$-dimensional subspace $\nabla f(x)^\perp$. This implies that $\nabla^2 f(x)$ can have at most one negative eigenvalue.

As a (partial) converse, if f satisfies

$$y^T \nabla f(x) = 0 \Longrightarrow y^T \nabla^2 f(x) y > 0 \tag{3.22}$$

for all $x \in \mathbf{dom}\, f$ and all $y \in \mathbf{R}^n$, $y \neq 0$, then f is quasiconvex. This condition is the same as requiring $\nabla^2 f(x)$ to be positive definite for any point with $\nabla f(x) = 0$, and for all other points, requiring $\nabla^2 f(x)$ to be positive definite on the $(n-1)$-dimensional subspace $\nabla f(x)^\perp$.

Proof of second-order conditions for quasiconvexity

By restricting the function to an arbitrary line, it suffices to consider the case in which $f : \mathbf{R} \to \mathbf{R}$.

We first show that if $f : \mathbf{R} \to \mathbf{R}$ is quasiconvex on an interval (a, b), then it must satisfy (3.21), *i.e.*, if $f'(c) = 0$ with $c \in (a, b)$, then we must have $f''(c) \geq 0$. If $f'(c) = 0$ with $c \in (a, b)$, $f''(c) < 0$, then for small positive ϵ we have $f(c-\epsilon) < f(c)$ and $f(c+\epsilon) < f(c)$. It follows that the sublevel set $\{x \mid f(x) \leq f(c) - \epsilon\}$ is disconnected for small positive ϵ, and therefore not convex, which contradicts our assumption that f is quasiconvex.

Now we show that if the condition (3.22) holds, then f is quasiconvex. Assume that (3.22) holds, *i.e.*, for each $c \in (a, b)$ with $f'(c) = 0$, we have $f''(c) > 0$. This means that whenever the function f' crosses the value 0, it is strictly increasing. Therefore it can cross the value 0 at most once. If f' does not cross the value 0 at all, then f is either nonincreasing or nondecreasing on (a, b), and therefore quasiconvex. Otherwise it must cross the value 0 exactly once, say at $c \in (a, b)$. Since $f''(c) > 0$, it follows that $f'(t) \leq 0$ for $a < t \leq c$, and $f'(t) \geq 0$ for $c \leq t < b$. This shows that f is quasiconvex.

3.4.4 Operations that preserve quasiconvexity

Nonnegative weighted maximum

A nonnegative weighted maximum of quasiconvex functions, *i.e.*,

$$f = \max\{w_1 f_1, \ldots, w_m f_m\},$$

with $w_i \geq 0$ and f_i quasiconvex, is quasiconvex. The property extends to the general pointwise supremum

$$f(x) = \sup_{y \in C}(w(y)g(x,y))$$

where $w(y) \geq 0$ and $g(x,y)$ is quasiconvex in x for each y. This fact can be easily verified: $f(x) \leq \alpha$ if and only if

$$w(y)g(x,y) \leq \alpha \text{ for all } y \in C,$$

i.e., the α-sublevel set of f is the intersection of the α-sublevel sets of the functions $w(y)g(x,y)$ in the variable x.

Example 3.37 *Generalized eigenvalue.* The *maximum generalized eigenvalue* of a pair of symmetric matrices (X,Y), with $Y \succ 0$, is defined as

$$\lambda_{\max}(X,Y) = \sup_{u \neq 0} \frac{u^T X u}{u^T Y u} = \sup\{\lambda \mid \det(\lambda Y - X) = 0\}.$$

(See §A.5.3). This function is quasiconvex on $\mathbf{dom}\, f = \mathbf{S}^n \times \mathbf{S}^n_{++}$.

To see this we consider the expression

$$\lambda_{\max}(X,Y) = \sup_{u \neq 0} \frac{u^T X u}{u^T Y u}.$$

For each $u \neq 0$, the function $u^T X u / u^T Y u$ is linear-fractional in (X,Y), hence a quasiconvex function of (X,Y). We conclude that $\lambda_{\max}$ is quasiconvex, since it is the supremum of a family of quasiconvex functions.

Composition

If $g : \mathbf{R}^n \to \mathbf{R}$ is quasiconvex and $h : \mathbf{R} \to \mathbf{R}$ is nondecreasing, then $f = h \circ g$ is quasiconvex.

The composition of a quasiconvex function with an affine or linear-fractional transformation yields a quasiconvex function. If f is quasiconvex, then $g(x) = f(Ax+b)$ is quasiconvex, and $\tilde{g}(x) = f((Ax+b)/(c^T x + d))$ is quasiconvex on the set

$$\{x \mid c^T x + d > 0,\ (Ax+b)/(c^T x + d) \in \mathbf{dom}\, f\}.$$

Minimization

If $f(x,y)$ is quasiconvex jointly in x and y and C is a convex set, then the function

$$g(x) = \inf_{y \in C} f(x,y)$$

is quasiconvex.

To show this, we need to show that $\{x \mid g(x) \leq \alpha\}$ is convex, where $\alpha \in \mathbf{R}$ is arbitrary. From the definition of g, $g(x) \leq \alpha$ if and only if for any $\epsilon > 0$ there exists a $y \in C$ with $f(x,y) \leq \alpha + \epsilon$. Now let x_1 and x_2 be two points in the α-sublevel set of g. Then for any $\epsilon > 0$, there exists $y_1, y_2 \in C$ with

$$f(x_1, y_1) \leq \alpha + \epsilon, \qquad f(x_2, y_2) \leq \alpha + \epsilon,$$

and since f is quasiconvex in x and y, we also have

$$f(\theta x_1 + (1-\theta)x_2, \theta y_1 + (1-\theta)y_2) \leq \alpha + \epsilon,$$

for $0 \leq \theta \leq 1$. Hence $g(\theta x_1 + (1-\theta)x_2) \leq \alpha$, which proves that $\{x \mid g(x) \leq \alpha\}$ is convex.

3.4.5 Representation via family of convex functions

In the sequel, it will be convenient to represent the sublevel sets of a quasiconvex function f (which are convex) via inequalities of convex functions. We seek a family of convex functions $\phi_t : \mathbf{R}^n \to \mathbf{R}$, indexed by $t \in \mathbf{R}$, with

$$f(x) \leq t \iff \phi_t(x) \leq 0, \tag{3.23}$$

i.e., the t-sublevel set of the quasiconvex function f is the 0-sublevel set of the convex function ϕ_t. Evidently ϕ_t must satisfy the property that for all $x \in \mathbf{R}^n$, $\phi_t(x) \leq 0 \implies \phi_s(x) \leq 0$ for $s \geq t$. This is satisfied if for each x, $\phi_t(x)$ is a nonincreasing function of t, *i.e.*, $\phi_s(x) \leq \phi_t(x)$ whenever $s \geq t$.

To see that such a representation always exists, we can take

$$\phi_t(x) = \begin{cases} 0 & f(x) \leq t \\ \infty & \text{otherwise,} \end{cases}$$

i.e., ϕ_t is the indicator function of the t-sublevel of f. Obviously this representation is not unique; for example if the sublevel sets of f are closed, we can take

$$\phi_t(x) = \mathbf{dist}\left(x, \{z \mid f(z) \leq t\}\right).$$

We are usually interested in a family ϕ_t with nice properties, such as differentiability.

Example 3.38 *Convex over concave function.* Suppose p is a convex function, q is a concave function, with $p(x) \geq 0$ and $q(x) > 0$ on a convex set C. Then the function f defined by $f(x) = p(x)/q(x)$, on C, is quasiconvex.

Here we have

$$f(x) \leq t \iff p(x) - tq(x) \leq 0,$$

so we can take $\phi_t(x) = p(x) - tq(x)$ for $t \geq 0$. For each t, ϕ_t is convex and for each x, $\phi_t(x)$ is decreasing in t.

3.5 Log-concave and log-convex functions

3.5.1 Definition

A function $f : \mathbf{R}^n \to \mathbf{R}$ is *logarithmically concave* or *log-concave* if $f(x) > 0$ for all $x \in \mathbf{dom}\, f$ and $\log f$ is concave. It is said to be *logarithmically convex* or *log-convex* if $\log f$ is convex. Thus f is log-convex if and only if $1/f$ is log-concave. It is convenient to allow f to take on the value zero, in which case we take $\log f(x) = -\infty$. In this case we say f is log-concave if the extended-value function $\log f$ is concave.

We can express log-concavity directly, without logarithms: a function $f : \mathbf{R}^n \to \mathbf{R}$, with convex domain and $f(x) > 0$ for all $x \in \mathbf{dom}\, f$, is log-concave if and only if for all $x,\ y \in \mathbf{dom}\, f$ and $0 \leq \theta \leq 1$, we have

$$f(\theta x + (1-\theta)y) \geq f(x)^\theta f(y)^{1-\theta}.$$

In particular, the value of a log-concave function at the average of two points is at least the *geometric mean* of the values at the two points.

From the composition rules we know that e^h is convex if h is convex, so a log-convex function is convex. Similarly, a nonnegative concave function is log-concave. It is also clear that a log-convex function is quasiconvex and a log-concave function is quasiconcave, since the logarithm is monotone increasing.

Example 3.39 *Some simple examples of log-concave and log-convex functions.*

- *Affine function.* $f(x) = a^T x + b$ is log-concave on $\{x \mid a^T x + b > 0\}$.
- *Powers.* $f(x) = x^a$, on $\mathbf{R}_{++}$, is log-convex for $a \leq 0$, and log-concave for $a \geq 0$.
- *Exponentials.* $f(x) = e^{ax}$ is log-convex and log-concave.
- The cumulative distribution function of a Gaussian density,

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2}\, du,$$

  is log-concave (see exercise 3.54).
- *Gamma function.* The Gamma function,

$$\Gamma(x) = \int_0^\infty u^{x-1} e^{-u}\, du,$$

  is log-convex for $x \geq 1$ (see exercise 3.52).
- *Determinant.* $\det X$ is log concave on $\mathbf{S}^n_{++}$.
- *Determinant over trace.* $\det X / \mathbf{tr}\, X$ is log concave on $\mathbf{S}^n_{++}$ (see exercise 3.49).

Example 3.40 *Log-concave density functions.* Many common probability density functions are log-concave. Two examples are the multivariate normal distribution,

$$f(x) = \frac{1}{\sqrt{(2\pi)^n \det \Sigma}} e^{-\frac{1}{2}(x-\bar{x})^T \Sigma^{-1} (x-\bar{x})}$$

(where $\bar{x} \in \mathbf{R}^n$ and $\Sigma \in \mathbf{S}^n_{++}$), and the exponential distribution on $\mathbf{R}^n_+$,

$$f(x) = \left(\prod_{i=1}^{n} \lambda_i \right) e^{-\lambda^T x}$$

(where $\lambda \succ 0$). Another example is the uniform distribution over a convex set C,

$$f(x) = \begin{cases} 1/\alpha & x \in C \\ 0 & x \notin C \end{cases}$$

where $\alpha = \mathbf{vol}(C)$ is the volume (Lebesgue measure) of C. In this case $\log f$ takes on the value $-\infty$ outside C, and $-\log \alpha$ on C, hence is concave.

As a more exotic example consider the Wishart distribution, defined as follows. Let $x_1, \ldots, x_p \in \mathbf{R}^n$ be independent Gaussian random vectors with zero mean and covariance $\Sigma \in \mathbf{S}^n$, with $p > n$. The random matrix $X = \sum_{i=1}^p x_i x_i^T$ has the Wishart density

$$f(X) = a\,(\det X)^{(p-n-1)/2}\, e^{-\frac{1}{2}\mathbf{tr}(\Sigma^{-1}X)},$$

with $\mathbf{dom}\, f = \mathbf{S}^n_{++}$, and a is a positive constant. The Wishart density is log-concave, since

$$\log f(X) = \log a + \frac{p-n-1}{2} \log\det X - \frac{1}{2}\mathbf{tr}(\Sigma^{-1}X),$$

which is a concave function of X.

3.5.2 Properties

Twice differentiable log-convex/concave functions

Suppose f is twice differentiable, with $\mathbf{dom}\, f$ convex, so

$$\nabla^2 \log f(x) = \frac{1}{f(x)} \nabla^2 f(x) - \frac{1}{f(x)^2} \nabla f(x) \nabla f(x)^T.$$

We conclude that f is log-convex if and only if for all $x \in \mathbf{dom}\, f$,

$$f(x)\nabla^2 f(x) \succeq \nabla f(x) \nabla f(x)^T,$$

and log-concave if and only if for all $x \in \mathbf{dom}\, f$,

$$f(x)\nabla^2 f(x) \preceq \nabla f(x) \nabla f(x)^T.$$

Multiplication, addition, and integration

Log-convexity and log-concavity are closed under multiplication and positive scaling. For example, if f and g are log-concave, then so is the pointwise product $h(x) = f(x)g(x)$, since $\log h(x) = \log f(x) + \log g(x)$, and $\log f(x)$ and $\log g(x)$ are concave functions of x.

Simple examples show that the sum of log-concave functions is not, in general, log-concave. Log-convexity, however, is preserved under sums. Let f and g be log-convex functions, *i.e.*, $F = \log f$ and $G = \log g$ are convex. From the composition rules for convex functions, it follows that

$$\log(\exp F + \exp G) = \log(f+g)$$

is convex. Therefore the sum of two log-convex functions is log-convex.

More generally, if $f(x, y)$ is log-convex in x for each $y \in C$ then

$$g(x) = \int_C f(x, y)\, dy$$

is log-convex.

Example 3.41 *Laplace transform of a nonnegative function and the moment and cumulant generating functions.* Suppose $p : \mathbf{R}^n \to \mathbf{R}$ satisfies $p(x) \geq 0$ for all x. The Laplace transform of p,

$$P(z) = \int p(x) e^{-z^T x}\, dx,$$

is log-convex on $\mathbf{R}^n$. (Here $\mathbf{dom}\, P$ is, naturally, $\{z \mid P(z) < \infty\}$.)

Now suppose p is a density, *i.e.*, satisfies $\int p(x)\, dx = 1$. The function $M(z) = P(-z)$ is called the *moment generating function* of the density. It gets its name from the fact that the moments of the density can be found from the derivatives of the moment generating function, evaluated at $z = 0$, *e.g.*,

$$\nabla M(0) = \mathbf{E}\, v, \qquad \nabla^2 M(0) = \mathbf{E}\, v v^T,$$

where v is a random variable with density p.

The function $\log M(z)$, which is convex, is called the *cumulant generating function* for p, since its derivatives give the cumulants of the density. For example, the first and second derivatives of the cumulant generating function, evaluated at zero, are the mean and covariance of the associated random variable:

$$\nabla \log M(0) = \mathbf{E}\, v, \qquad \nabla^2 \log M(0) = \mathbf{E}(v - \mathbf{E}\, v)(v - \mathbf{E}\, v)^T.$$

Integration of log-concave functions

In some special cases log-concavity is preserved by integration. If $f : \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}$ is log-concave, then

$$g(x) = \int f(x, y)\, dy$$

is a log-concave function of x (on $\mathbf{R}^n$). (The integration here is over $\mathbf{R}^m$.) A proof of this result is not simple; see the references.

This result has many important consequences, some of which we describe in the rest of this section. It implies, for example, that marginal distributions of log-concave probability densities are log-concave. It also implies that log-concavity is closed under convolution, *i.e.*, if f and g are log-concave on $\mathbf{R}^n$, then so is the convolution

$$(f * g)(x) = \int f(x - y) g(y)\, dy.$$

(To see this, note that $g(y)$ and $f(x-y)$ are log-concave in (x, y), hence the product $f(x - y)g(y)$ is; then the integration result applies.)

Suppose $C \subseteq \mathbf{R}^n$ is a convex set and w is a random vector in $\mathbf{R}^n$ with log-concave probability density p. Then the function

$$f(x) = \mathbf{prob}(x + w \in C)$$

is log-concave in x. To see this, express f as

$$f(x) = \int g(x + w)p(w)\, dw,$$

where g is defined as

$$g(u) = \begin{cases} 1 & u \in C \\ 0 & u \notin C, \end{cases}$$

(which is log-concave) and apply the integration result.

Example 3.42 The *cumulative distribution function* of a probability density function $f : \mathbf{R}^n \to \mathbf{R}$ is defined as

$$F(x) = \mathbf{prob}(w \preceq x) = \int_{-\infty}^{x_n} \cdots \int_{-\infty}^{x_1} f(z)\, dz_1 \cdots dz_n,$$

where w is a random variable with density f. If f is log-concave, then F is log-concave. We have already encountered a special case: the cumulative distribution function of a Gaussian random variable,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dt,$$

is log-concave. (See example 3.39 and exercise 3.54.)

Example 3.43 *Yield function.* Let $x \in \mathbf{R}^n$ denote the nominal or target value of a set of parameters of a product that is manufactured. Variation in the manufacturing process causes the parameters of the product, when manufactured, to have the value $x + w$, where $w \in \mathbf{R}^n$ is a random vector that represents manufacturing variation, and is usually assumed to have zero mean. The *yield* of the manufacturing process, as a function of the nominal parameter values, is given by

$$Y(x) = \mathbf{prob}(x + w \in S),$$

where $S \subseteq \mathbf{R}^n$ denotes the set of acceptable parameter values for the product, *i.e.*, the product *specifications*.

If the density of the manufacturing error w is log-concave (for example, Gaussian) and the set S of product specifications is convex, then the yield function Y is log-concave. This implies that the α-*yield region*, defined as the set of nominal parameters for which the yield exceeds α, is convex. For example, the 95% yield region

$$\{x \mid Y(x) \geq 0.95\} = \{x \mid \log Y(x) \geq \log 0.95\}$$

is convex, since it is a superlevel set of the concave function $\log Y$.

Example 3.44 *Volume of polyhedron.* Let $A \in \mathbf{R}^{m\times n}$. Define

$$P_u = \{x \in \mathbf{R}^n \mid Ax \preceq u\}.$$

Then its volume $\mathbf{vol}\, P_u$ is a log-concave function of u.

To prove this, note that the function

$$\Psi(x,u) = \begin{cases} 1 & Ax \preceq u \\ 0 & \text{otherwise,} \end{cases}$$

is log-concave. By the integration result, we conclude that

$$\int \Psi(x,u)\, dx = \mathbf{vol}\, P_u$$

is log-concave.

3.6 Convexity with respect to generalized inequalities

We now consider generalizations of the notions of monotonicity and convexity, using generalized inequalities instead of the usual ordering on $\mathbf{R}$.

3.6.1 Monotonicity with respect to a generalized inequality

Suppose $K \subseteq \mathbf{R}^n$ is a proper cone with associated generalized inequality $\preceq_K$. A function $f : \mathbf{R}^n \to \mathbf{R}$ is called K-*nondecreasing* if

$$x \preceq_K y \Longrightarrow f(x) \le f(y),$$

and K-*increasing* if

$$x \preceq_K y,\ x \ne y \Longrightarrow f(x) < f(y).$$

We define K-*nonincreasing* and K-*decreasing* functions in a similar way.

Example 3.45 *Monotone vector functions.* A function $f : \mathbf{R}^n \to \mathbf{R}$ is nondecreasing with respect to $\mathbf{R}^n_+$ if and only if

$$x_1 \le y_1, \ldots, x_n \le y_n \implies f(x) \le f(y)$$

for all x, y. This is the same as saying that f, when restricted to any component x_i (*i.e.*, x_i is considered the variable while x_j for $j \ne i$ are fixed), is nondecreasing.

Example 3.46 *Matrix monotone functions.* A function $f : \mathbf{S}^n \to \mathbf{R}$ is called *matrix monotone* (increasing, decreasing) if it is monotone with respect to the positive semidefinite cone. Some examples of matrix monotone functions of the variable $X \in \mathbf{S}^n$:

- $\mathbf{tr}(WX)$, where $W \in \mathbf{S}^n$, is matrix nondecreasing if $W \succeq 0$, and matrix increasing if $W \succ 0$ (it is matrix nonincreasing if $W \preceq 0$, and matrix decreasing if $W \prec 0$).
- $\mathbf{tr}(X^{-1})$ is matrix decreasing on $\mathbf{S}^n_{++}$.
- $\det X$ is matrix increasing on $\mathbf{S}^n_{++}$, and matrix nondecreasing on $\mathbf{S}^n_{+}$.

Gradient conditions for monotonicity

Recall that a differentiable function $f : \mathbf{R} \to \mathbf{R}$, with convex (*i.e.*, interval) domain, is nondecreasing if and only if $f'(x) \geq 0$ for all $x \in \mathbf{dom}\, f$, and increasing if $f'(x) > 0$ for all $x \in \mathbf{dom}\, f$ (but the converse is not true). These conditions are readily extended to the case of monotonicity with respect to a generalized inequality. A differentiable function f, with convex domain, is K-nondecreasing if and only if

$$\nabla f(x) \succeq_{K^*} 0 \tag{3.24}$$

for all $x \in \mathbf{dom}\, f$. Note the difference with the simple scalar case: the gradient must be nonnegative in the *dual* inequality. For the strict case, we have the following: If

$$\nabla f(x) \succ_{K^*} 0 \tag{3.25}$$

for all $x \in \mathbf{dom}\, f$, then f is K-increasing. As in the scalar case, the converse is not true.

Let us prove these first-order conditions for monotonicity. First, assume that f satisfies (3.24) for all x, but is not K-nondecreasing, *i.e.*, there exist x, y with $x \preceq_K y$ and $f(y) < f(x)$. By differentiability of f there exists a $t \in [0, 1]$ with

$$\frac{d}{dt} f(x + t(y - x)) = \nabla f(x + t(y - x))^T (y - x) < 0.$$

Since $y - x \in K$ this means

$$\nabla f(x + t(y - x)) \notin K^*,$$

which contradicts our assumption that (3.24) is satisfied everywhere. In a similar way it can be shown that (3.25) implies f is K-increasing.

It is also straightforward to see that it is necessary that (3.24) hold everywhere. Assume (3.24) does not hold for $x = z$. By the definition of dual cone this means there exists a $v \in K$ with

$$\nabla f(z)^T v < 0.$$

Now consider $h(t) = f(z + tv)$ as a function of t. We have $h'(0) = \nabla f(z)^T v < 0$, and therefore there exists $t > 0$ with $h(t) = f(z + tv) < h(0) = f(z)$, which means f is not K-nondecreasing.

3.6.2 Convexity with respect to a generalized inequality

Suppose $K \subseteq \mathbf{R}^m$ is a proper cone with associated generalized inequality $\preceq_K$. We say $f : \mathbf{R}^n \to \mathbf{R}^m$ is *K-convex* if for all x, y, and $0 \leq \theta \leq 1$,

$$f(\theta x + (1 - \theta) y) \preceq_K \theta f(x) + (1 - \theta) f(y).$$

The function is *strictly K-convex* if

$$f(\theta x + (1-\theta)y) \prec_K \theta f(x) + (1-\theta)f(y)$$

for all $x \neq y$ and $0 < \theta < 1$. These definitions reduce to ordinary convexity and strict convexity when $m = 1$ (and $K = \mathbf{R}_+$).

Example 3.47 *Convexity with respect to componentwise inequality.* A function $f : \mathbf{R}^n \to \mathbf{R}^m$ is convex with respect to componentwise inequality (*i.e.*, the generalized inequality induced by $\mathbf{R}^m_+$) if and only if for all x, y and $0 \leq \theta \leq 1$,

$$f(\theta x + (1-\theta)y) \preceq \theta f(x) + (1-\theta)f(y),$$

i.e., each component f_i is a convex function. The function f is strictly convex with respect to componentwise inequality if and only if each component f_i is strictly convex.

Example 3.48 *Matrix convexity.* Suppose f is a symmetric matrix valued function, *i.e.*, $f : \mathbf{R}^n \to \mathbf{S}^m$. The function f is convex with respect to matrix inequality if

$$f(\theta x + (1-\theta)y) \preceq \theta f(x) + (1-\theta)f(y)$$

for any x and y, and for $\theta \in [0, 1]$. This is sometimes called *matrix convexity.* An equivalent definition is that the scalar function $z^T f(x) z$ is convex for all vectors z. (This is often a good way to prove matrix convexity). A matrix function is strictly matrix convex if

$$f(\theta x + (1-\theta)y) \prec \theta f(x) + (1-\theta)f(y)$$

when $x \neq y$ and $0 < \theta < 1$, or, equivalently, if $z^T f z$ is strictly convex for every $z \neq 0$.

Some examples:

- The function $f(X) = XX^T$ where $X \in \mathbf{R}^{n \times m}$ is matrix convex, since for fixed z the function $z^T XX^T z = \|X^T z\|_2^2$ is a convex quadratic function of (the components of) X. For the same reason, $f(X) = X^2$ is matrix convex on $\mathbf{S}^n$.
- The function X^p is matrix convex on $\mathbf{S}^n_{++}$ for $1 \leq p \leq 2$ or $-1 \leq p \leq 0$, and matrix concave for $0 \leq p \leq 1$.
- The function $f(X) = e^X$ is *not* matrix convex on $\mathbf{S}^n$, for $n \geq 2$.

Many of the results for convex functions have extensions to K-convex functions. As a simple example, a function is K-convex if and only if its restriction to any line in its domain is K-convex. In the rest of this section we list a few results for K-convexity that we will use later; more results are explored in the exercises.

Dual characterization of K-convexity

A function f is K-convex if and only if for every $w \succeq_{K^*} 0$, the (real-valued) function $w^T f$ is convex (in the ordinary sense); f is strictly K-convex if and only if for every nonzero $w \succeq_{K^*} 0$ the function $w^T f$ is strictly convex. (These follow directly from the definitions and properties of dual inequality.)

Differentiable K-convex functions

A differentiable function f is K-convex if and only if its domain is convex, and for all $x,\ y \in \mathbf{dom}\, f$,

$$f(y) \succeq_K f(x) + Df(x)(y - x).$$

(Here $Df(x) \in \mathbf{R}^{m \times n}$ is the derivative or Jacobian matrix of f at x; see §A.4.1.) The function f is strictly K-convex if and only if for all $x,\ y \in \mathbf{dom}\, f$ with $x \neq y$,

$$f(y) \succ_K f(x) + Df(x)(y - x).$$

Composition theorem

Many of the results on composition can be generalized to K-convexity. For example, if $g : \mathbf{R}^n \to \mathbf{R}^p$ is K-convex, $h : \mathbf{R}^p \to \mathbf{R}$ is convex, and $\tilde{h}$ (the extended-value extension of h) is K-nondecreasing, then $h \circ g$ is convex. This generalizes the fact that a nondecreasing convex function of a convex function is convex. The condition that $\tilde{h}$ be K-nondecreasing implies that $\mathbf{dom}\, h - K = \mathbf{dom}\, h$.

Example 3.49 The quadratic matrix function $g : \mathbf{R}^{m \times n} \to \mathbf{S}^n$ defined by

$$g(X) = X^T A X + B^T X + X^T B + C,$$

where $A \in \mathbf{S}^m$, $B \in \mathbf{R}^{m \times n}$, and $C \in \mathbf{S}^n$, is convex when $A \succeq 0$.

The function $h : \mathbf{S}^n \to \mathbf{R}$ defined by $h(Y) = -\log\det(-Y)$ is convex and increasing on $\mathbf{dom}\, h = -\mathbf{S}^n_{++}$.

By the composition theorem, we conclude that

$$f(X) = -\log\det(-(X^T A X + B^T X + X^T B + C))$$

is convex on

$$\mathbf{dom}\, f = \{X \in \mathbf{R}^{m \times n} \mid X^T A X + B^T X + X^T B + C \prec 0\}.$$

This generalizes the fact that

$$-\log(-(ax^2 + bx + c))$$

is convex on

$$\{x \in \mathbf{R} \mid ax^2 + bx + c < 0\},$$

provided $a \geq 0$.

Bibliography

The standard reference on convex analysis is Rockafellar [Roc70]. Other books on convex functions are Stoer and Witzgall [SW70], Roberts and Varberg [RV73], Van Tiel [vT84], Hiriart-Urruty and Lemaréchal [HUL93], Ekeland and Témam [ET99], Borwein and Lewis [BL00], Florenzano and Le Van [FL01], Barvinok [Bar02], and Bertsekas, Nedić, and Ozdaglar [Ber03]. Most nonlinear programming texts also include chapters on convex functions (see, for example, Mangasarian [Man94], Bazaraa, Sherali, and Shetty [BSS93], Bertsekas [Ber99], Polyak [Pol87], and Peressini, Sullivan, and Uhl [PSU88]).

Jensen's inequality appears in [Jen06]. A general study of inequalities, in which Jensen's inequality plays a central role, is presented by Hardy, Littlewood, and Pólya [HLP52], and Beckenbach and Bellman [BB65].

The term *perspective function* is from Hiriart-Urruty and Lemaréchal [HUL93, volume 1, page 100]. For the definitions in example 3.19 (relative entropy and Kullback-Leibler divergence), and the related exercise 3.13, see Cover and Thomas [CT91].

Some important early references on quasiconvex functions (as well as other extensions of convexity) are Nikaidô [Nik54], Mangasarian [Man94, chapter 9], Arrow and Enthoven [AE61], Ponstein [Pon67], and Luenberger [Lue68]. For a more comprehensive reference list, we refer to Bazaraa, Sherali, and Shetty [BSS93, page 126].

Prékopa [Pré80] gives a survey of log-concave functions. Log-convexity of the Laplace transform is mentioned in Barndorff-Nielsen [BN78, §7]. For a proof of the integration result of log-concave functions, see Prékopa [Pré71, Pré73].

Generalized inequalities are used extensively in the recent literature on cone programming, starting with Nesterov and Nemirovski [NN94, page 156]; see also Ben-Tal and Nemirovski [BTN01] and the references at the end of chapter 4. Convexity with respect to generalized inequalities also appears in the work of Luenberger [Lue69, §8.2] and Isii [Isi64]. Matrix monotonicity and matrix convexity are attributed to Löwner [Löw34], and are discussed in detail by Davis [Dav63], Roberts and Varberg [RV73, page 216] and Marshall and Olkin [MO79, §16E]. For the result on convexity and concavity of the function X^p in example 3.48, see Bondar [Bon94, theorem 16.1]. For a simple example that demonstrates that e^X is not matrix convex, see Marshall and Olkin [MO79, page 474].

Exercises

Definition of convexity

3.1 Suppose $f : \mathbf{R} \to \mathbf{R}$ is convex, and $a,\ b \in \mathbf{dom}\, f$ with $a < b$.

(a) Show that

$$f(x) \leq \frac{b-x}{b-a} f(a) + \frac{x-a}{b-a} f(b)$$

for all $x \in [a, b]$.

(b) Show that

$$\frac{f(x) - f(a)}{x - a} \leq \frac{f(b) - f(a)}{b - a} \leq \frac{f(b) - f(x)}{b - x}$$

for all $x \in (a, b)$. Draw a sketch that illustrates this inequality.

(c) Suppose f is differentiable. Use the result in (b) to show that

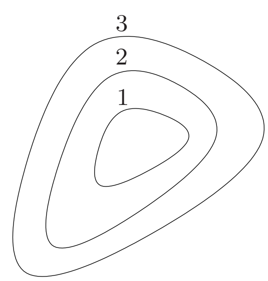
$$f'(a) \leq \frac{f(b) - f(a)}{b - a} \leq f'(b).$$

Note that these inequalities also follow from (3.2):

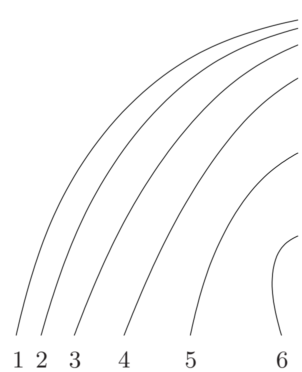
$$f(b) \geq f(a) + f'(a)(b - a), \qquad f(a) \geq f(b) + f'(b)(a - b).$$

(d) Suppose f is twice differentiable. Use the result in (c) to show that $f''(a) \geq 0$ and $f''(b) \geq 0$.

3.2 *Level sets of convex, concave, quasiconvex, and quasiconcave functions.* Some level sets of a function f are shown below. The curve labeled 1 shows $\{x \mid f(x) = 1\}$, etc.



Could f be convex (concave, quasiconvex, quasiconcave)? Explain your answer. Repeat for the level curves shown below.



3.3 *Inverse of an increasing convex function.* Suppose $f : \mathbf{R} \to \mathbf{R}$ is increasing and convex on its domain (a, b). Let g denote its inverse, *i.e.*, the function with domain $(f(a), f(b))$ and $g(f(x)) = x$ for $a < x < b$. What can you say about convexity or concavity of g?

3.4 [RV73, page 15] Show that a continuous function $f : \mathbf{R}^n \to \mathbf{R}$ is convex if and only if for every line segment, its average value on the segment is less than or equal to the average of its values at the endpoints of the segment: For every $x,\ y \in \mathbf{R}^n$,

$$\int_0^1 f(x + \lambda(y - x))\, d\lambda \le \frac{f(x) + f(y)}{2}.$$

3.5 [RV73, page 22] *Running average of a convex function.* Suppose $f : \mathbf{R} \to \mathbf{R}$ is convex, with $\mathbf{R}_+ \subseteq \mathbf{dom}\, f$. Show that its *running average* F, defined as

$$F(x) = \frac{1}{x}\int_0^x f(t)\, dt, \qquad \mathbf{dom}\, F = \mathbf{R}_{++},$$

is convex. *Hint.* For each s, $f(sx)$ is convex in x, so $\int_0^1 f(sx)\, ds$ is convex.

3.6 *Functions and epigraphs.* When is the epigraph of a function a halfspace? When is the epigraph of a function a convex cone? When is the epigraph of a function a polyhedron?

3.7 Suppose $f : \mathbf{R}^n \to \mathbf{R}$ is convex with $\mathbf{dom}\, f = \mathbf{R}^n$, and bounded above on $\mathbf{R}^n$. Show that f is constant.

3.8 *Second-order condition for convexity.* Prove that a twice differentiable function f is convex if and only if its domain is convex and $\nabla^2 f(x) \succeq 0$ for all $x \in \mathbf{dom}\, f$. *Hint.* First consider the case $f : \mathbf{R} \to \mathbf{R}$. You can use the first-order condition for convexity (which was proved on page 70).

3.9 *Second-order conditions for convexity on an affine set.* Let $F \in \mathbf{R}^{n\times m}$, $\hat{x} \in \mathbf{R}^n$. The *restriction* of $f : \mathbf{R}^n \to \mathbf{R}$ to the affine set $\{Fz + \hat{x} \mid z \in \mathbf{R}^m\}$ is defined as the function $\tilde{f} : \mathbf{R}^m \to \mathbf{R}$ with

$$\tilde{f}(z) = f(Fz + \hat{x}), \qquad \mathbf{dom}\, \tilde{f} = \{z \mid Fz + \hat{x} \in \mathbf{dom}\, f\}.$$

Suppose f is twice differentiable with a convex domain.

(a) Show that $\tilde{f}$ is convex if and only if for all $z \in \mathbf{dom}\, \tilde{f}$

$$F^T \nabla^2 f(Fz + \hat{x}) F \succeq 0.$$

(b) Suppose $A \in \mathbf{R}^{p\times n}$ is a matrix whose nullspace is equal to the range of F, *i.e.*, $AF = 0$ and $\mathbf{rank}\, A = n - \mathbf{rank}\, F$. Show that $\tilde{f}$ is convex if for all $z \in \mathbf{dom}\, \tilde{f}$ there exists a $\lambda \in \mathbf{R}$ such that

$$\nabla^2 f(Fz + \hat{x}) + \lambda A^T A \succeq 0.$$

Hint. Use the following result: If $B \in \mathbf{S}^n$ and $A \in \mathbf{R}^{p\times n}$, then $x^T B x \ge 0$ for all $x \in \mathcal{N}(A)$ if there exists a λ such that $B + \lambda A^T A \succeq 0$.

3.10 *An extension of Jensen's inequality.* One interpretation of Jensen's inequality is that randomization or dithering hurts, *i.e.*, raises the average value of a convex function: For f convex and v a zero mean random variable, we have $\mathbf{E}\, f(x_0 + v) \ge f(x_0)$. This leads to the following conjecture. If f is convex, then the larger the variance of v, the larger $\mathbf{E}\, f(x_0 + v)$.

(a) Give a counterexample that shows that this conjecture is false. Find zero mean random variables v and w, with $\mathbf{var}(v) > \mathbf{var}(w)$, a convex function f, and a point x_0, such that $\mathbf{E}\, f(x_0 + v) < \mathbf{E}\, f(x_0 + w)$.

(b) The conjecture is true when v and w are scaled versions of each other. Show that $\mathbf{E} f(x_0 + tv)$ is monotone increasing in $t \geq 0$, when f is convex and v is zero mean.

3.11 *Monotone mappings.* A function $\psi : \mathbf{R}^n \to \mathbf{R}^n$ is called *monotone* if for all $x,\ y \in \mathbf{dom}\,\psi$,

$$(\psi(x) - \psi(y))^T (x - y) \geq 0.$$

(Note that 'monotone' as defined here is not the same as the definition given in §3.6.1. Both definitions are widely used.) Suppose $f : \mathbf{R}^n \to \mathbf{R}$ is a differentiable convex function. Show that its gradient ∇f is monotone. Is the converse true, *i.e.*, is every monotone mapping the gradient of a convex function?

3.12 Suppose $f : \mathbf{R}^n \to \mathbf{R}$ is convex, $g : \mathbf{R}^n \to \mathbf{R}$ is concave, $\mathbf{dom}\,f = \mathbf{dom}\,g = \mathbf{R}^n$, and for all x, $g(x) \leq f(x)$. Show that there exists an affine function h such that for all x, $g(x) \leq h(x) \leq f(x)$. In other words, if a concave function g is an underestimator of a convex function f, then we can fit an affine function between f and g.

3.13 *Kullback-Leibler divergence and the information inequality.* Let D_{kl} be the Kullback-Leibler divergence, as defined in (3.17). Prove the *information inequality*: $D_{\mathrm{kl}}(u,v) \geq 0$ for all $u,\ v \in \mathbf{R}^n_{++}$. Also show that $D_{\mathrm{kl}}(u,v) = 0$ if and only if $u = v$.

Hint. The Kullback-Leibler divergence can be expressed as

$$D_{\mathrm{kl}}(u,v) = f(u) - f(v) - \nabla f(v)^T (u - v),$$

where $f(v) = \sum_{i=1}^n v_i \log v_i$ is the negative entropy of v.

3.14 *Convex-concave functions and saddle-points.* We say the function $f : \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}$ is *convex-concave* if $f(x,z)$ is a concave function of z, for each fixed x, and a convex function of x, for each fixed z. We also require its domain to have the product form $\mathbf{dom}\,f = A \times B$, where $A \subseteq \mathbf{R}^n$ and $B \subseteq \mathbf{R}^m$ are convex.

(a) Give a second-order condition for a twice differentiable function $f : \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}$ to be convex-concave, in terms of its Hessian $\nabla^2 f(x,z)$.

(b) Suppose that $f : \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}$ is convex-concave and differentiable, with $\nabla f(\tilde{x}, \tilde{z}) = 0$. Show that the *saddle-point property* holds: for all x, z, we have

$$f(\tilde{x}, z) \leq f(\tilde{x}, \tilde{z}) \leq f(x, \tilde{z}).$$

Show that this implies that f satisfies the *strong max-min property*:

$$\sup_z \inf_x f(x,z) = \inf_x \sup_z f(x,z)$$

(and their common value is $f(\tilde{x}, \tilde{z})$).

(c) Now suppose that $f : \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}$ is differentiable, but not necessarily convex-concave, and the saddle-point property holds at $\tilde{x}, \tilde{z}$:

$$f(\tilde{x}, z) \leq f(\tilde{x}, \tilde{z}) \leq f(x, \tilde{z})$$

for all x, z. Show that $\nabla f(\tilde{x}, \tilde{z}) = 0$.

Examples

3.15 *A family of concave utility functions.* For $0 < \alpha \leq 1$ let

$$u_\alpha(x) = \frac{x^\alpha - 1}{\alpha},$$

with $\mathbf{dom}\,u_\alpha = \mathbf{R}_+$. We also define $u_0(x) = \log x$ (with $\mathbf{dom}\,u_0 = \mathbf{R}_{++}$).

(a) Show that for $x > 0$, $u_0(x) = \lim_{\alpha \to 0} u_\alpha(x)$.

(b) Show that u_α are concave, monotone increasing, and all satisfy $u_\alpha(1) = 0$.

These functions are often used in economics to model the benefit or utility of some quantity of goods or money. Concavity of u_α means that the marginal utility (*i.e.*, the increase in utility obtained for a fixed increase in the goods) decreases as the amount of goods increases. In other words, concavity models the effect of *satiation*.

3.16 For each of the following functions determine whether it is convex, concave, quasiconvex, or quasiconcave.

(a) $f(x) = e^x - 1$ on $\mathbf{R}$.

(b) $f(x_1, x_2) = x_1 x_2$ on $\mathbf{R}^2_{++}$.

(c) $f(x_1, x_2) = 1/(x_1 x_2)$ on $\mathbf{R}^2_{++}$.

(d) $f(x_1, x_2) = x_1/x_2$ on $\mathbf{R}^2_{++}$.

(e) $f(x_1, x_2) = x_1^2/x_2$ on $\mathbf{R} \times \mathbf{R}_{++}$.

(f) $f(x_1, x_2) = x_1^\alpha x_2^{1-\alpha}$, where $0 \le \alpha \le 1$, on $\mathbf{R}^2_{++}$.

3.17 Suppose $p < 1$, $p \neq 0$. Show that the function

$$f(x) = \left(\sum_{i=1}^n x_i^p \right)^{1/p}$$

with $\mathbf{dom}\, f = \mathbf{R}^n_{++}$ is concave. This includes as special cases $f(x) = (\sum_{i=1}^n x_i^{1/2})^2$ and the *harmonic mean* $f(x) = (\sum_{i=1}^n 1/x_i)^{-1}$. *Hint.* Adapt the proofs for the log-sum-exp function and the geometric mean in §3.1.5.

3.18 Adapt the proof of concavity of the log-determinant function in §3.1.5 to show the following.

(a) $f(X) = \mathbf{tr}\left(X^{-1}\right)$ is convex on $\mathbf{dom}\, f = \mathbf{S}^n_{++}$.

(b) $f(X) = (\det X)^{1/n}$ is concave on $\mathbf{dom}\, f = \mathbf{S}^n_{++}$.

3.19 *Nonnegative weighted sums and integrals.*

(a) Show that $f(x) = \sum_{i=1}^r \alpha_i x_{[i]}$ is a convex function of x, where $\alpha_1 \ge \alpha_2 \ge \cdots \ge \alpha_r \ge 0$, and $x_{[i]}$ denotes the ith largest component of x. (You can use the fact that $f(x) = \sum_{i=1}^k x_{[i]}$ is convex on $\mathbf{R}^n$.)

(b) Let $T(x, \omega)$ denote the trigonometric polynomial

$$T(x, \omega) = x_1 + x_2 \cos\omega + x_3 \cos 2\omega + \cdots + x_n \cos(n-1)\omega.$$

Show that the function

$$f(x) = -\int_0^{2\pi} \log T(x, \omega)\, d\omega$$

is convex on $\{x \in \mathbf{R}^n \mid T(x, \omega) > 0,\ 0 \le \omega \le 2\pi\}$.

3.20 *Composition with an affine function.* Show that the following functions $f : \mathbf{R}^n \to \mathbf{R}$ are convex.

(a) $f(x) = \|Ax - b\|$, where $A \in \mathbf{R}^{m \times n}$, $b \in \mathbf{R}^m$, and $\|\cdot\|$ is a norm on $\mathbf{R}^m$.

(b) $f(x) = -\left(\det(A_0 + x_1 A_1 + \cdots + x_n A_n)\right)^{1/m}$, on $\{x \mid A_0 + x_1 A_1 + \cdots + x_n A_n \succ 0\}$, where $A_i \in \mathbf{S}^m$.

(c) $f(X) = \mathbf{tr}\left(A_0 + x_1 A_1 + \cdots + x_n A_n\right)^{-1}$, on $\{x \mid A_0 + x_1 A_1 + \cdots + x_n A_n \succ 0\}$, where $A_i \in \mathbf{S}^m$. (Use the fact that $\mathbf{tr}(X^{-1})$ is convex on $\mathbf{S}^m_{++}$; see exercise 3.18.)

3.21 *Pointwise maximum and supremum.* Show that the following functions $f : \mathbf{R}^n \to \mathbf{R}$ are convex.

(a) $f(x) = \max_{i=1,\ldots,k} \|A^{(i)}x - b^{(i)}\|$, where $A^{(i)} \in \mathbf{R}^{m\times n}$, $b^{(i)} \in \mathbf{R}^m$ and $\|\cdot\|$ is a norm on $\mathbf{R}^m$.

(b) $f(x) = \sum_{i=1}^r |x|_{[i]}$ on $\mathbf{R}^n$, where $|x|$ denotes the vector with $|x|_i = |x_i|$ (*i.e.*, $|x|$ is the absolute value of x, componentwise), and $|x|_{[i]}$ is the ith largest component of $|x|$. In other words, $|x|_{[1]}, |x|_{[2]}, \ldots, |x|_{[n]}$ are the absolute values of the components of x, sorted in nonincreasing order.

3.22 *Composition rules.* Show that the following functions are convex.

(a) $f(x) = -\log(-\log(\sum_{i=1}^m e^{a_i^T x+b_i}))$ on $\mathbf{dom}\, f = \{x \mid \sum_{i=1}^m e^{a_i^T x+b_i} < 1\}$. You can use the fact that $\log(\sum_{i=1}^n e^{y_i})$ is convex.

(b) $f(x,u,v) = -\sqrt{uv - x^T x}$ on $\mathbf{dom}\, f = \{(x,u,v) \mid uv > x^T x,\ u,\ v > 0\}$. Use the fact that $x^T x/u$ is convex in (x,u) for $u > 0$, and that $-\sqrt{x_1 x_2}$ is convex on $\mathbf{R}^2_{++}$.

(c) $f(x,u,v) = -\log(uv - x^T x)$ on $\mathbf{dom}\, f = \{(x,u,v) \mid uv > x^T x,\ u,\ v > 0\}$.

(d) $f(x,t) = -(t^p - \|x\|_p^p)^{1/p}$ where $p > 1$ and $\mathbf{dom}\, f = \{(x,t) \mid t \geq \|x\|_p\}$. You can use the fact that $\|x\|_p^p/u^{p-1}$ is convex in (x,u) for $u > 0$ (see exercise 3.23), and that $-x^{1/p}y^{1-1/p}$ is convex on $\mathbf{R}^2_+$ (see exercise 3.16).

(e) $f(x,t) = -\log(t^p - \|x\|_p^p)$ where $p > 1$ and $\mathbf{dom}\, f = \{(x,t) \mid t > \|x\|_p\}$. You can use the fact that $\|x\|_p^p/u^{p-1}$ is convex in (x,u) for $u > 0$ (see exercise 3.23).

3.23 *Perspective of a function.*

(a) Show that for $p > 1$,

$$f(x,t) = \frac{|x_1|^p + \cdots + |x_n|^p}{t^{p-1}} = \frac{\|x\|_p^p}{t^{p-1}}$$

is convex on $\{(x,t) \mid t > 0\}$.

(b) Show that

$$f(x) = \frac{\|Ax+b\|_2^2}{c^T x + d}$$

is convex on $\{x \mid c^T x + d > 0\}$, where $A \in \mathbf{R}^{m\times n}$, $b \in \mathbf{R}^m$, $c \in \mathbf{R}^n$ and $d \in \mathbf{R}$.

3.24 *Some functions on the probability simplex.* Let x be a real-valued random variable which takes values in $\{a_1, \ldots, a_n\}$ where $a_1 < a_2 < \cdots < a_n$, with $\mathbf{prob}(x = a_i) = p_i$, $i = 1, \ldots, n$. For each of the following functions of p (on the probability simplex $\{p \in \mathbf{R}^n_+ \mid \mathbf{1}^T p = 1\}$), determine if the function is convex, concave, quasiconvex, or quasiconcave.

(a) $\mathbf{E}\, x$.

(b) $\mathbf{prob}(x \geq \alpha)$.

(c) $\mathbf{prob}(\alpha \leq x \leq \beta)$.

(d) $\sum_{i=1}^n p_i \log p_i$, the negative entropy of the distribution.

(e) $\mathbf{var}\, x = \mathbf{E}(x - \mathbf{E}\, x)^2$.

(f) $\mathbf{quartile}(x) = \inf\{\beta \mid \mathbf{prob}(x \leq \beta) \geq 0.25\}$.

(g) The cardinality of the smallest set $\mathcal{A} \subseteq \{a_1, \ldots, a_n\}$ with probability $\geq 90\%$. (By cardinality we mean the number of elements in $\mathcal{A}$.)

(h) The minimum width interval that contains 90% of the probability, *i.e.*,

$$\inf\{\beta - \alpha \mid \mathbf{prob}(\alpha \leq x \leq \beta) \geq 0.9\}.$$

3.25 *Maximum probability distance between distributions.* Let $p,\ q \in \mathbf{R}^n$ represent two probability distributions on $\{1,\ldots,n\}$ (so $p,\ q \succeq 0$, $\mathbf{1}^T p = \mathbf{1}^T q = 1$). We define the *maximum probability distance* $d_{\rm mp}(p,q)$ between p and q as the maximum difference in probability assigned by p and q, over all events:

$$d_{\rm mp}(p,q) = \max\{|\,\mathbf{prob}(p,C) - \mathbf{prob}(q,C)| \mid C \subseteq \{1,\ldots,n\}\}.$$

Here $\mathbf{prob}(p,C)$ is the probability of C, under the distribution p, *i.e.*, $\mathbf{prob}(p,C) = \sum_{i\in C} p_i$.

Find a simple expression for $d_{\rm mp}$, involving $\|p-q\|_1 = \sum_{i=1}^n |p_i - q_i|$, and show that $d_{\rm mp}$ is a convex function on $\mathbf{R}^n \times \mathbf{R}^n$. (Its domain is $\{(p,q) \mid p,\ q \succeq 0,\ \mathbf{1}^T p = \mathbf{1}^T q = 1\}$, but it has a natural extension to all of $\mathbf{R}^n \times \mathbf{R}^n$.)

3.26 *More functions of eigenvalues.* Let $\lambda_1(X) \geq \lambda_2(X) \geq \cdots \geq \lambda_n(X)$ denote the eigenvalues of a matrix $X \in \mathbf{S}^n$. We have already seen several functions of the eigenvalues that are convex or concave functions of X.

- The maximum eigenvalue $\lambda_1(X)$ is convex (example 3.10). The minimum eigenvalue $\lambda_n(X)$ is concave.
- The sum of the eigenvalues (or trace), $\mathbf{tr}\, X = \lambda_1(X) + \cdots + \lambda_n(X)$, is linear.
- The sum of the inverses of the eigenvalues (or trace of the inverse), $\mathbf{tr}(X^{-1}) = \sum_{i=1}^n 1/\lambda_i(X)$, is convex on $\mathbf{S}^n_{++}$ (exercise 3.18).
- The geometric mean of the eigenvalues, $(\det X)^{1/n} = (\prod_{i=1}^n \lambda_i(X))^{1/n}$, and the logarithm of the product of the eigenvalues, $\log\det X = \sum_{i=1}^n \log \lambda_i(X)$, are concave on $X \in \mathbf{S}^n_{++}$ (exercise 3.18 and page 74).

In this problem we explore some more functions of eigenvalues, by exploiting variational characterizations.

(a) *Sum of k largest eigenvalues.* Show that $\sum_{i=1}^k \lambda_i(X)$ is convex on $\mathbf{S}^n$. *Hint.* [HJ85, page 191] Use the variational characterization

$$\sum_{i=1}^k \lambda_i(X) = \sup\{\mathbf{tr}(V^T X V) \mid V \in \mathbf{R}^{n\times k},\ V^T V = I\}.$$

(b) *Geometric mean of k smallest eigenvalues.* Show that $(\prod_{i=n-k+1}^n \lambda_i(X))^{1/k}$ is concave on $\mathbf{S}^n_{++}$. *Hint.* [MO79, page 513] For $X \succ 0$, we have

$$\left(\prod_{i=n-k+1}^n \lambda_i(X)\right)^{1/k} = \frac{1}{k}\inf\{\mathbf{tr}(V^T X V) \mid V \in \mathbf{R}^{n\times k},\ \det V^T V = 1\}.$$

(c) *Log of product of k smallest eigenvalues.* Show that $\sum_{i=n-k+1}^n \log \lambda_i(X)$ is concave on $\mathbf{S}^n_{++}$. *Hint.* [MO79, page 513] For $X \succ 0$,

$$\prod_{i=n-k+1}^n \lambda_i(X) = \inf\left\{\prod_{i=1}^k (V^T X V)_{ii} \;\middle|\; V \in \mathbf{R}^{n\times k},\ V^T V = I\right\}.$$

3.27 *Diagonal elements of Cholesky factor.* Each $X \in \mathbf{S}^n_{++}$ has a unique *Cholesky factorization* $X = LL^T$, where L is lower triangular, with $L_{ii} > 0$. Show that L_{ii} is a concave function of X (with domain $\mathbf{S}^n_{++}$).

Hint. L_{ii} can be expressed as $L_{ii} = (w - z^T Y^{-1} z)^{1/2}$, where

$$\begin{bmatrix} Y & z \\ z^T & w \end{bmatrix}$$

is the leading $i \times i$ submatrix of X.

Operations that preserve convexity

3.28 *Expressing a convex function as the pointwise supremum of a family of affine functions.* In this problem we extend the result proved on page 83 to the case where $\mathbf{dom}\, f \neq \mathbf{R}^n$. Let $f : \mathbf{R}^n \to \mathbf{R}$ be a convex function. Define $\tilde{f} : \mathbf{R}^n \to \mathbf{R}$ as the pointwise supremum of all affine functions that are global underestimators of f:

$$\tilde{f}(x) = \sup\{g(x) \mid g \text{ affine, } g(z) \leq f(z) \text{ for all } z\}.$$

(a) Show that $f(x) = \tilde{f}(x)$ for $x \in \mathbf{int\, dom}\, f$.

(b) Show that $f = \tilde{f}$ if f is closed (*i.e.*, $\mathbf{epi}\, f$ is a closed set; see §A.3.3).

3.29 *Representation of piecewise-linear convex functions.* A convex function $f : \mathbf{R}^n \to \mathbf{R}$, with $\mathbf{dom}\, f = \mathbf{R}^n$, is called *piecewise-linear* if there exists a partition of $\mathbf{R}^n$ as

$$\mathbf{R}^n = X_1 \cup X_2 \cup \cdots \cup X_L,$$

where $\mathbf{int}\, X_i \neq \emptyset$ and $\mathbf{int}\, X_i \cap \mathbf{int}\, X_j = \emptyset$ for $i \neq j$, and a family of affine functions $a_1^T x + b_1, \ldots, a_L^T x + b_L$ such that $f(x) = a_i^T x + b_i$ for $x \in X_i$.

Show that such a function has the form $f(x) = \max\{a_1^T x + b_1, \ldots, a_L^T x + b_L\}$.

3.30 *Convex hull or envelope of a function.* The *convex hull* or *convex envelope* of a function $f : \mathbf{R}^n \to \mathbf{R}$ is defined as

$$g(x) = \inf\{t \mid (x, t) \in \mathbf{conv\, epi}\, f\}.$$

Geometrically, the epigraph of g is the convex hull of the epigraph of f.

Show that g is the largest convex underestimator of f. In other words, show that if h is convex and satisfies $h(x) \leq f(x)$ for all x, then $h(x) \leq g(x)$ for all x.

3.31 [Roc70, page 35] *Largest homogeneous underestimator.* Let f be a convex function. Define the function g as

$$g(x) = \inf_{\alpha > 0} \frac{f(\alpha x)}{\alpha}.$$

(a) Show that g is homogeneous ($g(tx) = tg(x)$ for all $t \geq 0$).

(b) Show that g is the largest homogeneous underestimator of f: If h is homogeneous and $h(x) \leq f(x)$ for all x, then we have $h(x) \leq g(x)$ for all x.

(c) Show that g is convex.

3.32 *Products and ratios of convex functions.* In general the product or ratio of two convex functions is not convex. However, there are some results that apply to functions on $\mathbf{R}$. Prove the following.

(a) If f and g are convex, both nondecreasing (or nonincreasing), and positive functions on an interval, then fg is convex.

(b) If f, g are concave, positive, with one nondecreasing and the other nonincreasing, then fg is concave.

(c) If f is convex, nondecreasing, and positive, and g is concave, nonincreasing, and positive, then f/g is convex.

3.33 *Direct proof of perspective theorem.* Give a direct proof that the perspective function g, as defined in §3.2.6, of a convex function f is convex: Show that $\mathbf{dom}\, g$ is a convex set, and that for (x, t), $(y, s) \in \mathbf{dom}\, g$, and $0 \leq \theta \leq 1$, we have

$$g(\theta x + (1 - \theta)y, \theta t + (1 - \theta)s) \leq \theta g(x, t) + (1 - \theta)g(y, s).$$

3.34 *The Minkowski function.* The *Minkowski function* of a convex set C is defined as

$$M_C(x) = \inf\{t > 0 \mid t^{-1}x \in C\}.$$

(a) Draw a picture giving a geometric interpretation of how to find $M_C(x)$.

(b) Show that M_C is homogeneous, *i.e.*, $M_C(\alpha x) = \alpha M_C(x)$ for $\alpha \geq 0$.

(c) What is $\mathbf{dom}\, M_C$?

(d) Show that M_C is a convex function.

(e) Suppose C is also closed, bounded, symmetric (if $x \in C$ then $-x \in C$), and has nonempty interior. Show that M_C is a norm. What is the corresponding unit ball?

3.35 *Support function calculus.* Recall that the support function of a set $C \subseteq \mathbf{R}^n$ is defined as $S_C(y) = \sup\{y^T x \mid x \in C\}$. On page 81 we showed that S_C is a convex function.

(a) Show that $S_B = S_{\mathbf{conv}\, B}$.

(b) Show that $S_{A+B} = S_A + S_B$.

(c) Show that $S_{A\cup B} = \max\{S_A, S_B\}$.

(d) Let B be closed and convex. Show that $A \subseteq B$ if and only if $S_A(y) \leq S_B(y)$ for all y.

Conjugate functions

3.36 Derive the conjugates of the following functions.

(a) *Max function.* $f(x) = \max_{i=1,\ldots,n} x_i$ on $\mathbf{R}^n$.

(b) *Sum of largest elements.* $f(x) = \sum_{i=1}^r x_{[i]}$ on $\mathbf{R}^n$.

(c) *Piecewise-linear function on* $\mathbf{R}$. $f(x) = \max_{i=1,\ldots,m}(a_i x + b_i)$ on $\mathbf{R}$. You can assume that the a_i are sorted in increasing order, *i.e.*, $a_1 \leq \cdots \leq a_m$, and that none of the functions $a_i x + b_i$ is redundant, *i.e.*, for each k there is at least one x with $f(x) = a_k x + b_k$.

(d) *Power function.* $f(x) = x^p$ on $\mathbf{R}_{++}$, where $p > 1$. Repeat for $p < 0$.

(e) *Negative geometric mean.* $f(x) = -(\prod x_i)^{1/n}$ on $\mathbf{R}^n_{++}$.

(f) *Negative generalized logarithm for second-order cone.* $f(x,t) = -\log(t^2 - x^T x)$ on $\{(x,t) \in \mathbf{R}^n \times \mathbf{R} \mid \|x\|_2 < t\}$.

3.37 Show that the conjugate of $f(X) = \mathbf{tr}(X^{-1})$ with $\mathbf{dom}\, f = \mathbf{S}^n_{++}$ is given by

$$f^*(Y) = -2\,\mathbf{tr}(-Y)^{1/2}, \qquad \mathbf{dom}\, f^* = -\mathbf{S}^n_+.$$

Hint. The gradient of f is $\nabla f(X) = -X^{-2}$.

3.38 *Young's inequality.* Let $f : \mathbf{R} \to \mathbf{R}$ be an increasing function, with $f(0) = 0$, and let g be its inverse. Define F and G as

$$F(x) = \int_0^x f(a)\, da, \qquad G(y) = \int_0^y g(a)\, da.$$

Show that F and G are conjugates. Give a simple graphical interpretation of Young's inequality,

$$xy \leq F(x) + G(y).$$

3.39 *Properties of conjugate functions.*

(a) *Conjugate of convex plus affine function.* Define $g(x) = f(x) + c^T x + d$, where f is convex. Express g^* in terms of f^* (and c, d).

(b) *Conjugate of perspective.* Express the conjugate of the perspective of a convex function f in terms of f^*.

(c) *Conjugate and minimization.* Let $f(x,z)$ be convex in (x,z) and define $g(x) = \inf_z f(x,z)$. Express the conjugate g^* in terms of f^*.

As an application, express the conjugate of $g(x) = \inf_z\{h(z) \mid Az + b = x\}$, where h is convex, in terms of h^*, A, and b.

(d) *Conjugate of conjugate.* Show that the conjugate of the conjugate of a closed convex function is itself: $f = f^{**}$ if f is closed and convex. (A function is closed if its epigraph is closed; see §A.3.3.) *Hint.* Show that f^{**} is the pointwise supremum of all affine global underestimators of f. Then apply the result of exercise 3.28.

3.40 *Gradient and Hessian of conjugate function.* Suppose $f : \mathbf{R}^n \to \mathbf{R}$ is convex and twice continuously differentiable. Suppose $\bar{y}$ and $\bar{x}$ are related by $\bar{y} = \nabla f(\bar{x})$, and that $\nabla^2 f(\bar{x}) \succ 0$.

(a) Show that $\nabla f^*(\bar{y}) = \bar{x}$.

(b) Show that $\nabla^2 f^*(\bar{y}) = \nabla^2 f(\bar{x})^{-1}$.

3.41 *Conjugate of negative normalized entropy.* Show that the conjugate of the negative normalized entropy

$$f(x) = \sum_{i=1}^n x_i \log(x_i/\mathbf{1}^T x),$$

with $\mathbf{dom}\, f = \mathbf{R}^n_{++}$, is given by

$$f^*(y) = \begin{cases} 0 & \sum_{i=1}^n e^{y_i} \le 1 \\ +\infty & \text{otherwise.} \end{cases}$$

Quasiconvex functions

3.42 *Approximation width.* Let $f_0, \ldots, f_n : \mathbf{R} \to \mathbf{R}$ be given continuous functions. We consider the problem of approximating f_0 as a linear combination of $f_1, \ldots, f_n$. For $x \in \mathbf{R}^n$, we say that $f = x_1 f_1 + \cdots + x_n f_n$ approximates f_0 with tolerance $\epsilon > 0$ over the interval $[0,T]$ if $|f(t) - f_0(t)| \le \epsilon$ for $0 \le t \le T$. Now we choose a fixed tolerance $\epsilon > 0$ and define the *approximation width* as the largest T such that f approximates f_0 over the interval $[0,T]$:

$$W(x) = \sup\{T \mid |x_1 f_1(t) + \cdots + x_n f_n(t) - f_0(t)| \le \epsilon \text{ for } 0 \le t \le T\}.$$

Show that W is quasiconcave.

3.43 *First-order condition for quasiconvexity.* Prove the first-order condition for quasiconvexity given in §3.4.3: A differentiable function $f : \mathbf{R}^n \to \mathbf{R}$, with $\mathbf{dom}\, f$ convex, is quasiconvex if and only if for all $x, y \in \mathbf{dom}\, f$,

$$f(y) \le f(x) \Longrightarrow \nabla f(x)^T (y - x) \le 0.$$

Hint. It suffices to prove the result for a function on $\mathbf{R}$; the general result follows by restriction to an arbitrary line.

3.44 *Second-order conditions for quasiconvexity.* In this problem we derive alternate representations of the second-order conditions for quasiconvexity given in §3.4.3. Prove the following.

(a) A point $x \in \mathbf{dom}\, f$ satisfies (3.21) if there exists a σ such that

$$\nabla^2 f(x) + \sigma \nabla f(x) \nabla f(x)^T \succeq 0. \tag{3.26}$$

It satisfies (3.22) for all $y \neq 0$ if and only if there exists a σ such

$$\nabla^2 f(x) + \sigma \nabla f(x) \nabla f(x)^T \succ 0. \tag{3.27}$$

Hint. We can assume without loss of generality that $\nabla^2 f(x)$ is diagonal.

(b) A point $x \in \mathbf{dom}\, f$ satisfies (3.21) if and only if either $\nabla f(x) = 0$ and $\nabla^2 f(x) \succeq 0$, or $\nabla f(x) \neq 0$ and the matrix

$$H(x) = \begin{bmatrix} \nabla^2 f(x) & \nabla f(x) \\ \nabla f(x)^T & 0 \end{bmatrix}$$

has exactly one negative eigenvalue. It satisfies (3.22) for all $y \neq 0$ if and only if $H(x)$ has exactly one nonpositive eigenvalue.

Hint. You can use the result of part (a). The following result, which follows from the eigenvalue interlacing theorem in linear algebra, may also be useful: If $B \in \mathbf{S}^n$ and $a \in \mathbf{R}^n$, then

$$\lambda_n \left(\begin{bmatrix} B & a \\ a^T & 0 \end{bmatrix} \right) \geq \lambda_n(B).$$

3.45 Use the first and second-order conditions for quasiconvexity given in §3.4.3 to verify quasiconvexity of the function $f(x) = -x_1 x_2$, with $\mathbf{dom}\, f = \mathbf{R}^2_{++}$.

3.46 *Quasilinear functions with domain* $\mathbf{R}^n$. A function on $\mathbf{R}$ that is quasilinear (*i.e.*, quasiconvex and quasiconcave) is monotone, *i.e.*, either nondecreasing or nonincreasing. In this problem we consider a generalization of this result to functions on $\mathbf{R}^n$.

Suppose the function $f : \mathbf{R}^n \to \mathbf{R}$ is quasilinear and continuous with $\mathbf{dom}\, f = \mathbf{R}^n$. Show that it can be expressed as $f(x) = g(a^T x)$, where $g : \mathbf{R} \to \mathbf{R}$ is monotone and $a \in \mathbf{R}^n$. In other words, a quasilinear function with domain $\mathbf{R}^n$ must be a monotone function of a linear function. (The converse is also true.)

Log-concave and log-convex functions

3.47 Suppose $f : \mathbf{R}^n \to \mathbf{R}$ is differentiable, $\mathbf{dom}\, f$ is convex, and $f(x) > 0$ for all $x \in \mathbf{dom}\, f$. Show that f is log-concave if and only if for all $x, y \in \mathbf{dom}\, f$,

$$\frac{f(y)}{f(x)} \leq \exp\left(\frac{\nabla f(x)^T (y - x)}{f(x)} \right).$$

3.48 Show that if $f : \mathbf{R}^n \to \mathbf{R}$ is log-concave and $a \geq 0$, then the function $g = f - a$ is log-concave, where $\mathbf{dom}\, g = \{x \in \mathbf{dom}\, f \mid f(x) > a\}$.

3.49 Show that the following functions are log-concave.

(a) *Logistic function:* $f(x) = e^x/(1 + e^x)$ with $\mathbf{dom}\, f = \mathbf{R}$.

(b) *Harmonic mean:*

$$f(x) = \frac{1}{1/x_1 + \cdots + 1/x_n}, \qquad \mathbf{dom}\, f = \mathbf{R}^n_{++}.$$

(c) *Product over sum:*

$$f(x) = \frac{\prod_{i=1}^n x_i}{\sum_{i=1}^n x_i}, \qquad \mathbf{dom}\, f = \mathbf{R}^n_{++}.$$

(d) *Determinant over trace:*

$$f(X) = \frac{\det X}{\mathbf{tr}\, X}, \qquad \mathbf{dom}\, f = \mathbf{S}^n_{++}.$$

3.50 *Coefficients of a polynomial as a function of the roots.* Show that the coefficients of a polynomial with real negative roots are log-concave functions of the roots. In other words, the functions $a_i : \mathbf{R}^n \to \mathbf{R}$, defined by the identity

$$s^n + a_1(\lambda)s^{n-1} + \cdots + a_{n-1}(\lambda)s + a_n(\lambda) = (s - \lambda_1)(s - \lambda_2) \cdots (s - \lambda_n),$$

are log-concave on $-\mathbf{R}^n_{++}$.
Hint. The function

$$S_k(x) = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} x_{i_1} x_{i_2} \cdots x_{i_k},$$

with $\mathbf{dom}\, S_k \in \mathbf{R}^n_+$ and $1 \le k \le n$, is called the kth elementary symmetric function on $\mathbf{R}^n$. It can be shown that $S_k^{1/k}$ is concave (see [ML57]).

3.51 [BL00, page 41] Let p be a polynomial on $\mathbf{R}$, with all its roots real. Show that it is log-concave on any interval on which it is positive.

3.52 [MO79, §3.E.2] *Log-convexity of moment functions.* Suppose $f : \mathbf{R} \to \mathbf{R}$ is nonnegative with $\mathbf{R}_+ \subseteq \mathbf{dom}\, f$. For $x \ge 0$ define

$$\phi(x) = \int_0^\infty u^x f(u)\, du.$$

Show that ϕ is a log-convex function. (If x is a positive integer, and f is a probability density function, then $\phi(x)$ is the xth moment of the distribution.)
Use this to show that the Gamma function,

$$\Gamma(x) = \int_0^\infty u^{x-1} e^{-u}\, du,$$

is log-convex for $x \ge 1$.

3.53 Suppose x and y are independent random vectors in $\mathbf{R}^n$, with log-concave probability density functions f and g, respectively. Show that the probability density function of the sum $z = x + y$ is log-concave.

3.54 *Log-concavity of Gaussian cumulative distribution function.* The cumulative distribution function of a Gaussian random variable,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-t^2/2}\, dt,$$

is log-concave. This follows from the general result that the convolution of two log-concave functions is log-concave. In this problem we guide you through a simple self-contained proof that f is log-concave. Recall that f is log-concave if and only if $f''(x)f(x) \le f'(x)^2$ for all x.

(a) Verify that $f''(x)f(x) \le f'(x)^2$ for $x \ge 0$. That leaves us the hard part, which is to show the inequality for $x < 0$.

(b) Verify that for any t and x we have $t^2/2 \ge -x^2/2 + xt$.

(c) Using part (b) show that $e^{-t^2/2} \le e^{x^2/2 - xt}$. Conclude that, for $x < 0$,

$$\int_{-\infty}^x e^{-t^2/2}\, dt \le e^{x^2/2} \int_{-\infty}^x e^{-xt}\, dt.$$

(d) Use part (c) to verify that $f''(x)f(x) \le f'(x)^2$ for $x \le 0$.

3.55 *Log-concavity of the cumulative distribution function of a log-concave probability density.* In this problem we extend the result of exercise 3.54. Let $g(t) = \exp(-h(t))$ be a differentiable log-concave probability density function, and let

$$f(x) = \int_{-\infty}^{x} g(t)\,dt = \int_{-\infty}^{x} e^{-h(t)}\,dt$$

be its cumulative distribution. We will show that f is log-concave, *i.e.*, it satisfies $f''(x)f(x) \le (f'(x))^2$ for all x.

(a) Express the derivatives of f in terms of the function h. Verify that $f''(x)f(x) \le (f'(x))^2$ if $h'(x) \ge 0$.

(b) Assume that $h'(x) < 0$. Use the inequality

$$h(t) \ge h(x) + h'(x)(t - x)$$

(which follows from convexity of h), to show that

$$\int_{-\infty}^{x} e^{-h(t)}\,dt \le \frac{e^{-h(x)}}{-h'(x)}.$$

Use this inequality to verify that $f''(x)f(x) \le (f'(x))^2$ if $h'(x) < 0$.

3.56 *More log-concave densities.* Show that the following densities are log-concave.

(a) [MO79, page 493] The *gamma density*, defined by

$$f(x) = \frac{\alpha^\lambda}{\Gamma(\lambda)} x^{\lambda-1} e^{-\alpha x},$$

with $\mathbf{dom}\, f = \mathbf{R}_+$. The parameters λ and α satisfy $\lambda \ge 1$, $\alpha > 0$.

(b) [MO79, page 306] The *Dirichlet density*

$$f(x) = \frac{\Gamma(\mathbf{1}^T\lambda)}{\Gamma(\lambda_1)\cdots\Gamma(\lambda_{n+1})} x_1^{\lambda_1-1} \cdots x_n^{\lambda_n-1} \left(1 - \sum_{i=1}^n x_i\right)^{\lambda_{n+1}-1}$$

with $\mathbf{dom}\, f = \{x \in \mathbf{R}^n_{++} \mid \mathbf{1}^T x < 1\}$. The parameter λ satisfies $\lambda \succeq \mathbf{1}$.

Convexity with respect to a generalized inequality

3.57 Show that the function $f(X) = X^{-1}$ is matrix convex on $\mathbf{S}^n_{++}$.

3.58 *Schur complement.* Suppose $X \in \mathbf{S}^n$ partitioned as

$$X = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix},$$

where $A \in \mathbf{S}^k$. The *Schur complement* of X (with respect to A) is $S = C - B^T A^{-1} B$ (see §A.5.5). Show that the Schur complement, viewed as a function from $\mathbf{S}^n$ into $\mathbf{S}^{n-k}$, is matrix concave on $\mathbf{S}^n_{++}$.

3.59 *Second-order conditions for K-convexity.* Let $K \subseteq \mathbf{R}^m$ be a proper convex cone, with associated generalized inequality $\preceq_K$. Show that a twice differentiable function $f : \mathbf{R}^n \to \mathbf{R}^m$, with convex domain, is K-convex if and only if for all $x \in \mathbf{dom}\, f$ and all $y \in \mathbf{R}^n$,

$$\sum_{i,j=1}^n \frac{\partial^2 f(x)}{\partial x_i \partial x_j} y_i y_j \succeq_K 0,$$

i.e., the second derivative is a K-nonnegative bilinear form. (Here $\partial^2 f/\partial x_i \partial x_j \in \mathbf{R}^m$, with components $\partial^2 f_k/\partial x_i \partial x_j$, for $k = 1, \ldots, m$; see §A.4.1.)

3.60 *Sublevel sets and epigraph of K-convex functions.* Let $K \subseteq \mathbf{R}^m$ be a proper convex cone with associated generalized inequality $\preceq_K$, and let $f : \mathbf{R}^n \to \mathbf{R}^m$. For $\alpha \in \mathbf{R}^m$, the α-sublevel set of f (with respect to $\preceq_K$) is defined as

$$C_\alpha = \{x \in \mathbf{R}^n \mid f(x) \preceq_K \alpha\}.$$

The epigraph of f, with respect to $\preceq_K$, is defined as the set

$$\mathbf{epi}_K f = \{(x,t) \in \mathbf{R}^{n+m} \mid f(x) \preceq_K t\}.$$

Show the following:

(a) If f is K-convex, then its sublevel sets C_α are convex for all α.

(b) f is K-convex if and only if $\mathbf{epi}_K f$ is a convex set.