6

Counting Weighted Objects

In earlier chapters, we have spent a lot of time studying the *counting problem*: given a finite set S, how many elements does S have? This chapter generalizes the counting problem to the following situation. Given a finite set S of objects, where each object is assigned an integer-valued *weight*, how many objects in S are there of each given weight? A convenient way to present the answer to this question is via a *generating function*, which is a polynomial $a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ such that a_k (the coefficient of x^k) is the number of objects in S of weight k. After giving the basic definitions, we will develop rules for manipulating generating functions that are analogous to the sum rule and product rule from Chapter 1. We will also derive formulas for certain generating functions that generalize factorials, binomial coefficients, and multinomial coefficients. In later chapters, we extend all of these ideas to the more general situation where S is an *infinite* set of weighted objects.

6.1 Weighted Sets

This section presents the basic definitions needed to discuss sets of weighted objects, together with many examples.

6.1. Definition: Weighted Sets. A *weighted set* is a pair (S, wt), where S is a set and $\mathrm{wt} : S \to \mathbb{N}$ is a function from S to the nonnegative integers. For each $z \in S$, the integer $\mathrm{wt}(z)$ is called the *weight* of z. In this definition, S is not required to be finite, although we shall always make that assumption in this chapter. The weight function wt is also sometimes referred to as a *statistic* on S. If the weight function is understood from the context, we may sometimes refer to "the weighted set S."

6.2. Definition: Generating Function for a Weighted Set. Given a finite weighted set (S, wt), the *generating function for S* is the polynomial

$$G_{S,\mathrm{wt}}(x) = \sum_{z \in S} x^{\mathrm{wt}(z)}.$$

We also write $G_S(x)$ or $G(x)$ if the weight function and set are understood from context. Note that the sum on the right side is well-defined, since S is finite and addition of polynomials is an associative and commutative operation (see the discussion following 2.2 and 2.149).

6.3. Example. Suppose $S = \{a, b, c, d, e, f\}$, and $\mathrm{wt} : S \to \mathbb{N}$ is given by

$$\mathrm{wt}(a) = 4,\ \mathrm{wt}(b) = 1,\ \mathrm{wt}(c) = 0,\ \mathrm{wt}(d) = 4,\ \mathrm{wt}(e) = 4,\ \mathrm{wt}(f) = 1.$$

The generating function for (S, wt) is

$$G_{S,\mathrm{wt}}(x) = x^{\mathrm{wt}(a)} + x^{\mathrm{wt}(b)} + \cdots + x^{\mathrm{wt}(f)} = x^4 + x^1 + x^0 + x^4 + x^4 + x^1 = 1 + 2x + 3x^4.$$

Consider another weight function $w : S \to \mathbb{N}$ given by $w(a) = 0$, $w(b) = 1$, $w(c) = 2$, $w(d) = 3$, $w(e) = 4$, and $w(f) = 5$. Using this weight function, we obtain a different generating function, namely

$$G_{S,w}(x) = 1 + x + x^2 + x^3 + x^4 + x^5.$$

6.4. Example. Suppose S is the set of all subsets of $\{1, 2, 3\}$, so

$$S = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}.$$

Consider three different weight functions $w_i : S \to \mathbb{N}$, given by

$$w_1(A) = |A|; \qquad w_2(A) = \sum_{i \in A} i; \qquad w_3(A) = \min_{i \in A} i; \qquad \text{(for all } A \in S).$$

(By convention, define $w_3(\emptyset) = 0$.) Each of these statistics leads to a different generating function:

$$\begin{aligned} G_{S,w_1}(x) &= x^0 + x^1 + x^1 + x^1 + x^2 + x^2 + x^2 + x^3 = 1 + 3x + 3x^2 + x^3 = (1 + x)^3; \\ G_{S,w_2}(x) &= x^0 + x^1 + x^2 + x^3 + x^3 + x^4 + x^5 + x^6 = 1 + x + x^2 + 2x^3 + x^4 + x^5 + x^6; \\ G_{S,w_3}(x) &= x^0 + x^1 + x^2 + x^3 + x^1 + x^1 + x^2 + x^1 = 1 + 4x + 2x^2 + x^3. \end{aligned}$$

6.5. Example. For each integer $n \geq 0$, consider the set $\underline{\mathbf{n}} = \{0, 1, 2, \ldots, n - 1\}$. Define a weight function on this set by letting $\mathrm{wt}(i) = i$ for all $i \in \underline{\mathbf{n}}$. The associated generating function is

$$G_{\underline{\mathbf{n}},\mathrm{wt}}(x) = x^0 + x^1 + x^2 + \cdots + x^{n-1} = \frac{x^n - 1}{x - 1}.$$

The last equality can be verified by using the distributive law to calculate

$$(x - 1)(1 + x + x^2 + \cdots + x^{n-1}) = x^n - 1.$$

The generating function in this example will be a recurring building block in our later work, so we give it a special name.

6.6. Definition: Quantum Integers. If n is a positive integer and x is any variable, we define

$$[n]_x = 1 + x + x^2 + \cdots + x^{n-1} = \frac{x^n - 1}{x - 1}.$$

We also define $[0]_x = 0$. The polynomial $[n]_x$ is called the *quantum integer* n (relative to the variable x).

6.7. Example. Let S be the set of all lattice paths from $(0, 0)$ to $(2, 3)$. For $P \in S$, let $w(P)$ be the number of unit squares in the region bounded by P, the x-axis, and the line $x = 2$. Let $w'(P)$ be the number of unit squares in the region bounded by P, the y-axis, and the line $y = 3$. By examining the paths in Figure 1.7, we compute

$$\begin{aligned} G_{S,w}(x) &= x^6 + x^5 + x^4 + x^4 + x^3 + x^2 + x^3 + x^2 + x^1 + x^0 \\ &= 1 + x + 2x^2 + 2x^3 + 2x^4 + x^5 + x^6; \\ G_{S,w'}(x) &= x^0 + x^1 + x^2 + x^2 + x^3 + x^4 + x^3 + x^4 + x^5 + x^6 \\ &= 1 + x + 2x^2 + 2x^3 + 2x^4 + x^5 + x^6. \end{aligned}$$

Although the two weight functions are not equal (since there are paths P with $w(P) \neq w'(P)$), it happens that $G_{S,w} = G_{S,w'}$ in this example.

Now, consider the set T of Dyck paths from $(0,0)$ to $(3,3)$. For $P \in T$, let $\mathrm{wt}(P)$ be the number of complete unit squares located between P and the diagonal line $y = x$. Using Figure 1.8, we find that

$$G_{T,\mathrm{wt}}(x) = x^3 + x^2 + x^1 + x^1 + x^0 = 1 + 2x + x^2 + x^3.$$

6.8. Remark. Let (S, wt) be a finite set of weighted objects. We know $G_S(x) = \sum_{z \in S} x^{\mathrm{wt}(z)}$. By collecting together equal powers of x (as done in the calculations above), we can write G_S in the standard form

$$G_S(x) = a_0x^0 + a_1x^1 + a_2x^2 + \cdots + a_mx^m \qquad (a_i \in \mathbb{N}).$$

Comparing the two formulas for G_S, we see that *the coefficient a_i of x^i in $G_{S,\mathrm{wt}}(x)$ is the number of objects z in S such that* $\mathrm{wt}(z) = i$. We now illustrate this observation with several examples.

6.9. Example. Suppose T is the set of all set partitions of an n-element set, and the weight of a partition is the number of blocks in the partition. By definition of the Stirling number of the second kind (see 2.51), we have

$$G_T(x) = \sum_{k=0}^{n} S(n,k)x^k.$$

Similarly, if U is the set of all permutations of n elements, weighted by the number of cycles in the disjoint cycle decomposition, then

$$G_U(x) = \sum_{k=0}^{n} s'(n,k)x^k$$

where $s'(n,k)$ is a signless Stirling number of the first kind (see §3.6). Finally, if V is the set of all integer partitions of n, weighted by number of parts, then

$$G_V(x) = \sum_{k=0}^{n} p(n,k)x^k.$$

This is also the generating function for V if we weight a partition by the length of its largest part (§2.8).

6.10. Remark. Suppose we replace the variable x in $G_S(x)$ by the value 1. We obtain $G_S(1) = \sum_{z \in S} 1^{\mathrm{wt}(z)} = \sum_{z \in S} 1 = |S|$. For example, in 6.7, $G_{T,\mathrm{wt}}(1) = 5 = C_3$; in 6.9, $G_T(1) = B(n)$ (the Bell number), $G_U(1) = n!$, and $G_V(1) = p(n)$. Thus, the generating function $G_S(x)$ can be viewed as a weighted analogue of "the number of elements in S." On the other hand, using the convention that $0^0 = 1$, $G_S(0)$ is the number of objects in S having weight zero.

We also note that the polynomial $G_S(x)$ can sometimes be factored or otherwise simplified, as illustrated by the first weight function in 6.4. Different statistics on S usually lead to different generating functions, but this is not always true (see 6.4 and 6.7).

Our goal in this chapter is to develop techniques for finding and manipulating generating functions that avoid listing all the objects in S, as we did in the examples above.

6.2 Inversions

Before presenting the sum and product rules for generating functions, we introduce an example of a weight function that arises frequently in algebraic combinatorics.

6.11. Definition: Inversions. Suppose $w = w_1 w_2 \cdots w_n$ is a word, where each letter w_i is an integer. An *inversion* of w is a pair of indices $i < j$ such that $w_i > w_j$. We write $\mathrm{inv}(w)$ for the number of inversions of w; in other words,

$$\mathrm{inv}(w_1 w_2 \cdots w_n) = \sum_{1 \leq i < j \leq n} \chi(w_i > w_j).$$

Thus $\mathrm{inv}(w)$ counts pairs of letters in w (not necessarily adjacent) that are out of numerical order. We also define $\mathrm{Inv}(w)$ to be the *set* of all inversion pairs (i, j), so $|\mathrm{Inv}(w)| = \mathrm{inv}(w)$. If S is any finite set of words over the alphabet $\mathbb{Z}$, then

$$G_{S,\mathrm{inv}}(x) = \sum_{w \in S} x^{\mathrm{inv}(w)}$$

is the *inversion generating function for S*. These definitions extend to words over any *totally ordered* alphabet.

6.12. Example. Consider the word $w = 414253$; here $w_1 = 4$, $w_2 = 1$, $w_3 = 4$, etc. The pair $(1, 2)$ is an inversion of w since $w_1 = 4 > 1 = w_2$. The pair $(2, 3)$ is not an inversion, since $w_2 = 1 \leq 4 = w_3$. Similarly, $(1, 3)$ is not an inversion. Continuing in this way, we find that

$$\mathrm{Inv}(w) = \{(1, 2), (1, 4), (1, 6), (3, 4), (3, 6), (5, 6)\},$$

so $\mathrm{inv}(w) = 6$.

6.13. Example. Let S be the set of all permutations of $\{1, 2, 3\}$. We know that

$$S = \{123,\ 132,\ 213,\ 231,\ 312,\ 321\}.$$

Counting inversions, we conclude that

$$G_{S,\mathrm{inv}}(x) = x^0 + x^1 + x^1 + x^2 + x^2 + x^3 = 1 + 2x + 2x^2 + x^3 = 1(1 + x)(1 + x + x^2).$$

Note that $G_S(1) = 6 = 3! = |S|$. Similarly, if T is the set of all permutations of $\{1, 2, 3, 4\}$, a longer calculation (see 6.50) leads to

$$G_{T,\mathrm{inv}}(x) = 1 + 3x + 5x^2 + 6x^3 + 5x^4 + 3x^5 + x^6 = 1(1 + x)(1 + x + x^2)(1 + x + x^2 + x^3).$$

The factorization patterns in these examples will be explained and generalized below.

6.14. Example. Let $S = \mathcal{R}(0^2 1^3)$ be the set of all rearrangements of two zeroes and three ones. We know that

$$S = \{00111,\ 01011,\ 01101,\ 01110,\ 10011,\ 10101,\ 10110,\ 11001,\ 11010,\ 11100\}.$$

Counting inversions, we conclude that

$$G_{S,\mathrm{inv}}(x) = x^0 + x^1 + x^2 + x^3 + x^2 + x^3 + x^4 + x^4 + x^5 + x^6 = 1 + x + 2x^2 + 2x^3 + 2x^4 + x^5 + x^6.$$

The reader may notice that this is the same generating function that appeared in 6.7. This is not a coincidence; we explain why this happens in the next section.

6.15. Example. Let $S = \mathcal{R}(a^1b^1c^2)$, where we use $a < b < c$ as the ordering of the alphabet. We know that

$$S = \{\text{abcc, acbc, accb, bacc, bcac, bcca, cabc, cacb, cbac, cbca, ccab, ccba}\}.$$

Counting inversions leads to

$$G_{S,\text{inv}}(x) = 1 + 2x + 3x^2 + 3x^3 + 2x^4 + x^5.$$

Now let $T = \mathcal{R}(a^1b^2c^1)$ and $U = \mathcal{R}(a^2b^1c^1)$ with the same ordering of the alphabet. The reader is invited to confirm that

$$G_{S,\text{inv}}(x) = G_{T,\text{inv}}(x) = G_{U,\text{inv}}(x),$$

although the sets of words in question are all different. This phenomenon will also be explained in the coming sections.

6.16. Remark. It can be shown that for any word w, $\text{inv}(w)$ is the minimum number of transpositions of adjacent letters required to sort the letters of w into weakly increasing order (see 9.29 and 9.179).

6.3 Weight-Preserving Bijections

In the next few sections, we introduce three fundamental rules that we will use to give combinatorial derivations of many generating function formulas. These rules are weighted analogues of the counting rules studied in Chapter 1. The first rule generalizes 1.30. We need one new definition to state this rule.

6.17. Definition: Weight-Preserving Bijections. Let (S, w_1) and (T, w_2) be two weighted sets. A *weight-preserving bijection* from (S, w_1) to (T, w_2) is a bijection $f : S \to T$ such that

$$w_2(f(z)) = w_1(z) \qquad \text{for all } z \in S.$$

6.18. Theorem: Bijection Rule for Generating Functions. Suppose (S, w_1) and (T, w_2) are two finite weighted sets such that there exists a weight-preserving bijection $f : S \to T$. Then

$$G_{S,w_1}(x) = G_{T,w_2}(x).$$

Proof. Let $g : T \to S$ be the inverse of f. One verifies that g is a weight-preserving bijection, since f is. For each $k \geq 0$, let $S_k = \{z \in S : w_1(z) = k\}$ and $T_k = \{u \in T : w_2(u) = k\}$. Since f and g preserve weights, they restrict to give maps $f_k : S_k \to T_k$ and $g_k : T_k \to S_k$ that are mutual inverses. Therefore, $|S_k| = |T_k|$ for all $k \geq 0$. It follows that

$$G_{S,w_1}(x) = \sum_{k\geq 0} |S_k|x^k = \sum_{k\geq 0} |T_k|x^k = G_{T,w_2}(x). \qquad \square$$

6.19. Example. Let S be the set of all lattice paths P from $(0, 0)$ to (a, b), and let $\text{area}(P)$ be the area below the path and above the x-axis (cf. 6.7). Let $T = \mathcal{R}(0^a1^b)$ be the set of all words consisting of a zeroes and b ones, weighted by inversions. There is a bijection $g : T \to S$ obtained by converting zeroes to east steps and ones to north steps. By examining a picture, one sees that $\text{inv}(w) = \text{area}(g(w))$ for all $w \in T$. For example, if $w = 1001010$,

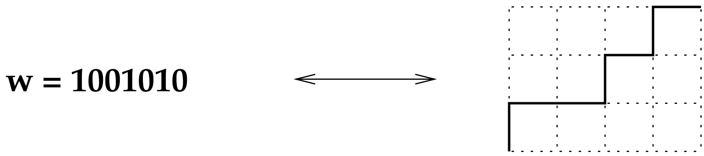

FIGURE 6.1
Inversions of a word vs. area under a lattice path.

then $g(w)$ is the lattice path shown in Figure 6.1. The four area cells in the lowest row correspond to the inversions between the first 1 in w and the four zeroes occurring later. Similarly, the two area cells in the next lowest row come from the inversions between the second 1 in w and the two zeroes occurring later. Since g is a weight-preserving bijection, we conclude that

$$G_{T,\mathrm{inv}}(x) = G_{S,\mathrm{area}}(x).$$

When $a = 2$ and $b = 3$, this explains the equality of generating functions observed in 6.7 and 6.14. In 6.7, we also considered another weight on paths $P \in S$, namely the number of area squares between P and the y-axis. Denoting this weight by area$'$, we have

$$G_{S,\mathrm{area}}(x) = G_{S,\mathrm{area}'}(x)$$

(for arbitrary a and b). This follows from the weight-preserving bijection rule, since rotating a path 180° about $(a/2, b/2)$ defines a bijection $r : S \to S$ that sends area to area$'$. Similarly, letting S' be the set of paths from $(0,0)$ to (b,a), we have

$$G_{S,\mathrm{area}}(x) = G_{S',\mathrm{area}'}(x) \quad (= G_{S',\mathrm{area}}(x))$$

since reflection in the diagonal line $y = x$ defines a weight-preserving bijection from (S, area) to (S', area'). We will soon derive an explicit formula for the generating functions occurring in this example (§6.7).

6.4 Sum and Product Rules for Weighted Sets

Now we discuss the weighted analogues of the sum and product rules.

6.20. Theorem: Sum Rule for Weighted Sets. Suppose (S_1, wt_1), (S_2, wt_2), $\ldots$, (S_k, wt_k) are finite weighted sets such that $S_1, \ldots, S_k$ are pairwise disjoint sets. Let $S = S_1 \cup \cdots \cup S_k$, and define $\mathrm{wt} : S \to \mathbb{N}$ by setting $\mathrm{wt}(z) = \mathrm{wt}_i(z)$ for all $z \in S_i$. Then

$$G_{S,\mathrm{wt}}(x) = G_{S_1,\mathrm{wt}_1}(x) + G_{S_2,\mathrm{wt}_2}(x) + \cdots + G_{S_k,\mathrm{wt}_k}(x).$$

Proof. By definition, $G_{S,\mathrm{wt}}(x) = \sum_{z \in S} x^{\mathrm{wt}(z)}$. Because addition of polynomials is commutative and associative, we can order the terms of this sum so that all objects in S_1 come first, followed by objects in S_2, and so on, ending with all objects in S_k. We obtain

$$\begin{aligned} G_{S,\mathrm{wt}}(x) &= \sum_{z \in S_1} x^{\mathrm{wt}(z)} + \sum_{z \in S_2} x^{\mathrm{wt}(z)} + \cdots + \sum_{z \in S_k} x^{\mathrm{wt}(z)} \\ &= \sum_{z \in S_1} x^{\mathrm{wt}_1(z)} + \sum_{z \in S_2} x^{\mathrm{wt}_2(z)} + \cdots + \sum_{z \in S_k} x^{\mathrm{wt}_k(z)} \\ &= G_{S_1,\mathrm{wt}_1}(x) + G_{S_2,\mathrm{wt}_2}(x) + \cdots + G_{S_k,\mathrm{wt}_k}(x). \quad \square \end{aligned}$$

6.21. Theorem: Product Rule for Two Weighted Sets. Suppose (T, w_1) and (U, w_2) are two finite weighted sets. On the product set $S = T \times U$, define a weight w by setting $w((t,u)) = w_1(t) + w_2(u)$ for $t \in T$ and $u \in U$. Then

$$G_{S,w}(x) = G_{T,w_1}(x) \cdot G_{U,w_2}(x).$$

Proof. The proof consists of the following calculation, which requires the generalized distributive law (in the form given by 2.5):

$$\begin{aligned} G_{S,w}(x) &= \sum_{(t,u)\in S} x^{w((t,u))} = \sum_{(t,u)\in T\times U} x^{w_1(t)+w_2(u)} \\ &= \sum_{(t,u)\in T\times U} x^{w_1(t)} \cdot x^{w_2(u)} \\ &= \left(\sum_{t\in T} x^{w_1(t)}\right) \cdot \left(\sum_{u\in U} x^{w_2(u)}\right) \\ &= G_{T,w_1}(x) \cdot G_{U,w_2}(x). \quad \square \end{aligned}$$

6.22. Theorem: Product Rule for k Weighted Sets. Suppose that (S_i, w_i) is a finite weighted set for $1 \leq i \leq k$. On the product set $S = S_1 \times S_2 \times \cdots \times S_k$, define a weight w by $w((z_1, \ldots, z_k)) = \sum_{i=1}^{k} w_i(z_i)$ for all $z_i \in S_i$. Then

$$G_{S,w}(x) = \prod_{i=1}^{k} G_{S_i,w_i}(x).$$

Proof. This formula follows from the product rule for two weighted sets by induction on k. Alternatively, one can mimic the proof given above for two sets, this time using the full-blown version of the generalized distributive law (see 2.6). $\square$

When discussing a Cartesian product of weighted sets, we always use the weight function on the product set given in the statement of the product rule (i.e., the weight is the sum of the weights of the component objects) unless otherwise specified.

6.23. Example. Given a positive integer n, consider the n weighted sets $S_i = \underline{\mathbf{i}} = \{0, 1, \ldots, i-1\}$ (for $1 \leq i \leq n$), where $\mathrm{wt}(z) = z$ for each $z \in \underline{\mathbf{i}}$. We have seen in 6.5 that $G_{S_i}(x) = [i]_x$, the quantum integer i. Now consider the product set $S = S_1 \times S_2 \times \cdots \times S_n = \underline{\mathbf{1}} \times \underline{\mathbf{2}} \times \cdots \times \underline{\mathbf{n}}$. By the product rule, the generating function for S is

$$G_S(x) = \prod_{i=1}^{n} [i]_x.$$

Since $G_S(1) = |S| = n!$, the polynomial $G_S(x)$ is a weighted analogue of a factorial. This polynomial will arise frequently, so we give it a special name.

6.24. Definition: Quantum Factorials. For each $n \geq 1$ and every variable x, define the *quantum factorial of n relative to x* to be the polynomial

$$[n]!_x = \prod_{i=1}^{n} [i]_x = \prod_{i=1}^{n} (1 + x + x^2 + \cdots + x^{i-1}) = \prod_{i=1}^{n} \frac{x^i - 1}{x - 1}.$$

Also define $[0]!_x = 1$.

Observe that $[n]!_x = [n-1]!_x [n]_x$ for all $n \geq 1$.

6.25. Example. We have $[0]!_x = 1 = [1]!_x$; $[2]!_x = [2]_x = 1 + x$;

$$[3]!_x = (1 + x)(1 + x + x^2) = 1 + 2x + 2x^2 + x^3;$$

$$[4]!_x = (1 + x)(1 + x + x^2)(1 + x + x^2 + x^3) = 1 + 3x + 5x^2 + 6x^3 + 5x^4 + 3x^5 + x^6;$$

$$[5]!_x = 1 + 4x + 9x^2 + 15x^3 + 20x^4 + 22x^5 + 20x^6 + 15x^7 + 9x^8 + 4x^9 + x^{10}.$$

We can use other variables besides x; for instance, $[3]!_q = 1 + 2q + 2q^2 + q^3$. Sometimes we will replace the variable here by a specific integer or real number; then the quantum factorial will evaluate to some specific number. For example, when $q = 4$, $[3]!_q = 1+8+32+64 = 105$. As another example, when $x = 1$, $[n]!_x = n!$.

6.5 Inversions and Quantum Factorials

The reader may have recognized some of the quantum factorial polynomials above as matching the inversion generating functions for permutations in 6.13. The next theorem proves that this pattern holds in general.

6.26. Theorem: Quantum Factorials and Inversions. For every $n \geq 0$, let S_n be the set of all permutations of $\{1, 2, \ldots, n\}$, weighted by inversions. Then

$$G_{S_n,\mathrm{inv}}(x) = \sum_{w \in S_n} x^{\mathrm{inv}(w)} = [n]!_x.$$

Proof. Let $T_n = \underline{\mathbf{1}} \times \underline{\mathbf{2}} \times \cdots \times \underline{\mathbf{n}}$ with the usual product weight; we saw in 6.23 that $G_{T_n,\mathrm{wt}}(x) = [n]!_x$. Therefore, to prove the theorem, it suffices to define a weight-preserving bijection $f_n : S_n \to T_n$.

Let $w = w_1 w_2 \cdots w_n \in S_n$ be a permutation of $\{1, 2, \ldots, n\}$. For each k between 1 and n, define t_k to be the number of pairs $i < j$ such that $w_i > w_j$ and $w_i = k$; then define $f_n(w) = (t_1, t_2, \ldots, t_n)$. In other words, $t_k = |\{(i, j) \in \mathrm{Inv}(w) : w_i = k\}|$ is the number of inversions that the symbol k has with smaller symbols to its right. There are $k - 1$ possible symbols less than k, so $t_k \in \{0, 1, 2, \ldots, k - 1\} = \underline{\mathbf{k}}$ for every k. Thus, $f_n(w)$ does lie in the set T_n. For example, if $w = 4, 2, 8, 5, 1, 6, 7, 3$, then $f_n(w) = (0, 1, 0, 3, 2, 1, 1, 5)$. We have $t_5 = 2$, for instance, because of the two inversions $(4, 5)$ and $(4, 8)$ caused by the entries $5 > 1$ and $5 > 3$ in w. Every inversion of w is counted by exactly one of the numbers t_k, so that $\mathrm{inv}(w) = \sum_{k=1}^{n} t_k = \mathrm{wt}(f_n(w))$ for all $w \in S_n$. This shows that f_n is a weight-preserving map.

To show that f_n is a bijection, we display a two-sided inverse map $g_n : T_n \to S_n$. We define g_n by means of a recursive *insertion procedure*. The cases $n \leq 1$ are immediate since the sets involved have only one element. Assume $n > 1$ and g_{n-1} has already been defined. Given $(t_1, \ldots, t_n) \in T_n = T_{n-1} \times \underline{\mathbf{n}}$, begin by computing $v = g_{n-1}(t_1, \ldots, t_{n-1})$, which is a permutation of $\{1, 2, \ldots, n - 1\}$. To find $g_n(t_1, \ldots, t_n)$, we insert the symbol n into the permutation v in such a way that n will cause t_n new inversions. This can always be done in a unique way. For, there are n possible positions in v where the symbol n could be inserted (after the last letter, or immediately before one of the $n - 1$ existing letters). If we insert n after the last letter, it will create no new inversions. Scanning to the left, if we insert n immediately before the kth letter from the far right, then this insertion will cause exactly k new inversions, since n exceeds all letters to its right, and no letter to the left of n exceeds n. Thus, the different insertion positions for n lead to $0, 1, \ldots,$ or $n - 1$ new inversions, and this is exactly the range of values for t_n. The recursive construction ensures that, for all $k \leq n$, t_k is the number of inversion pairs involving the symbol k on the left and some smaller symbol on the right. One may check that g_n is the two-sided inverse of f_n.

Here is an iterative description of the computation of $g_n(t_1, \ldots, t_n)$. Beginning with an empty word, successively insert $1, 2, \ldots, n$. At stage k, insert k in the unique position that will increase the total inversion count by t_k. For example, let us compute $g_8(0, 0, 1, 3, 2, 1, 5, 5)$. In the first two steps, we generate $1, 2$, which has zero inversions. Then we place the 3 one position left of the far right slot, obtaining the permutation $1, 3, 2$ with one inversion. Next we count three positions from the far right (arriving at the far left), and insert 4 to obtain the permutation $4, 1, 3, 2$ with three new inversions and four total inversions. The process continues, leading to $4, 1, 5, 3, 2$; then $4, 1, 5, 3, 6, 2$; then $4, 7, 1, 5, 3, 6, 2$; and finally to $w = 4, 7, 8, 1, 5, 3, 6, 2$. The reader may check that $f_8(w) = (0, 0, 1, 3, 2, 1, 5, 5)$. $\square$

Because of the previous proof, we sometimes call the elements of T_n *inversion tables.* Other types of inversion tables for permutations can be constructed by classifying the inversions of w in different ways. For example, our proof classified inversions by the value of the leftmost symbol in the inversion pair. One can also classify inversions using the value of the rightmost symbol, the position of the leftmost symbol, or the position of the rightmost symbol. These possibilities are explored in the exercises.

6.6 Descents and Major Index

This section introduces more statistics on words, which will lead to another combinatorial interpretation for the quantum factorial $[n]!_x$.

6.27. Definition: Descents and Major Index. Let $w = w_1 w_2 \cdots w_n$ be a word over a totally ordered alphabet A. The *descent set* of w, denoted $\mathrm{Des}(w)$, is the set of all $i < n$ such that $w_i > w_{i+1}$. This is the set of *positions* in w where a letter is immediately followed by a smaller letter. Define the *descent count* for w by $\mathrm{des}(w) = |\,\mathrm{Des}(w)|$. Define the *major index* of w, denoted $\mathrm{maj}(w)$, to be the sum of the elements of the set $\mathrm{Des}(w)$. In symbols, we can write

$$\mathrm{des}(w) = \sum_{i=1}^{n-1} \chi(w_i > w_{i+1}); \qquad \mathrm{maj}(w) = \sum_{i=1}^{n-1} i\chi(w_i > w_{i+1}).$$

6.28. Example. If $w = 47815362$, then $\mathrm{Des}(w) = \{3, 5, 7\}$, $\mathrm{des}(w) = 3$, and $\mathrm{maj}(w) = 3 + 5 + 7 = 15$. If $w = 101100101$, then $\mathrm{Des}(w) = \{1, 4, 7\}$, $\mathrm{des}(w) = 3$, and $\mathrm{maj}(w) = 12$. If $w = 33555789$, then $\mathrm{Des}(w) = \emptyset$, $\mathrm{des}(w) = 0$, and $\mathrm{maj}(w) = 0$.

6.29. Theorem: Quantum Factorials and Major Index. For every $n \geq 0$, let S_n be the set of all permutations of $\{1, 2, \ldots, n\}$, weighted by major index. Then $G_{S_n,\mathrm{maj}}(x) = \sum_{w \in S_n} x^{\mathrm{maj}(w)} = [n]!_x$.

Proof. As in the case of inversions, it suffices to define a weight-preserving bijection $f_n : S_n \to T_n = \underline{\mathbf{1}} \times \underline{\mathbf{2}} \times \cdots \times \underline{\mathbf{n}}$. We use a variation of the "inversion-table" idea, adapted to the major index statistic. Given $w \in S_n$ and $0 \leq k \leq n$, let $w^{(k)}$ be the word obtained from w by erasing all letters larger than k. Then define $f_n(w) = (t_1, t_2, \ldots, t_n)$, where $t_k = \mathrm{maj}(w^{(k)}) - \mathrm{maj}(w^{(k-1)})$. Intuitively, if we imagine building up w from the empty word by inserting $1, 2, \ldots, n$ in this order, then t_k records the extra contribution to maj caused by the insertion of the new symbol k. For example, given $w = 42851673$, we compute $\mathrm{maj}(\epsilon) = 0$, $\mathrm{maj}(1) = 0$, $\mathrm{maj}(21) = 1$, $\mathrm{maj}(213) = 1$, $\mathrm{maj}(4213) = 3$, $\mathrm{maj}(42513) = 4$, $\mathrm{maj}(425163) = 9$, $\mathrm{maj}(4251673) = 10$ and finally $\mathrm{maj}(w) = 15$. It follows that $f_8(w) = (0, 1, 0, 2, 1, 5, 1, 5)$. Observe that $\sum_{k=1}^{n} t_k = \mathrm{maj}(w^{(n)}) - \mathrm{maj}(w^{(0)}) = \mathrm{maj}(w)$, so the map f_n is weight-preserving.

We see from the definition that $f_n(w) = (f_{n-1}(w^{(n-1)}), t_n = \mathrm{maj}(w) - \mathrm{maj}(w^{(n-1)}))$. To show that $f_n(w)$ does lie in T_n, it suffices by induction to show that $t_n \in \underline{\mathbf{n}} = \{0, 1, 2, \ldots, n-1\}$. Let us first consider an example. Suppose we wish to insert the new symbol 8 into the permutation $w' = 4 > 2, 5 > 1, 6, 7 > 3$, which satisfies $\mathrm{maj}(w') = 1 + 3 + 6 = 10$. There are eight gaps into which the symbol 8 might be placed. Let us compute the major index of each of the resulting permutations:

$$\begin{array}{llll}
\mathrm{maj}(8 > 4 > 2, 5 > 1, 6, 7 > 3) & = 1+2+4+7 & = 14 = & \mathrm{maj}(w') + 4; \\
\mathrm{maj}(4, 8 > 2, 5 > 1, 6, 7 > 3) & = 2+4+7 & = 13 = & \mathrm{maj}(w') + 3; \\
\mathrm{maj}(4 > 2, 8 > 5 > 1, 6, 7 > 3) & = 1+3+4+7 & = 15 = & \mathrm{maj}(w') + 5; \\
\mathrm{maj}(4 > 2, 5, 8 > 1, 6, 7 > 3) & = 1+4+7 & = 12 = & \mathrm{maj}(w') + 2; \\
\mathrm{maj}(4 > 2, 5 > 1, 8 > 6, 7 > 3) & = 1+3+5+7 & = 16 = & \mathrm{maj}(w') + 6; \\
\mathrm{maj}(4 > 2, 5 > 1, 6, 8 > 7 > 3) & = 1+3+6+7 & = 17 = & \mathrm{maj}(w') + 7; \\
\mathrm{maj}(4 > 2, 5 > 1, 6, 7, 8 > 3) & = 1+3+7 & = 11 = & \mathrm{maj}(w') + 1; \\
\mathrm{maj}(4 > 2, 5 > 1, 6, 7 > 3, 8) & = 1+3+6 & = 10 = & \mathrm{maj}(w') + 0.
\end{array}$$

Observe that the possible values of t_n are precisely $0, 1, 2, \ldots, 7$ (in some order).

To see that this always happens, suppose $w^{(n-1)}$ has descents at positions $i_1 > i_2 > \cdots > i_d$, where $1 \le i_j \le n-2$ for all i_j. There are n *gaps* between the $n-1$ letters in $w^{(n-1)}$, including the positions at the far left and far right ends. Let us number these gaps $0, 1, 2, \ldots, n-1$ as follows. The gap at the far right end is numbered zero. Next, the gaps immediately to the right of the descents are numbered $1, 2, \ldots, d$ *from right to left.* (We have chosen the indexing of the descent positions so that the gap between positions i_j and $i_j + 1$ receives the number j.) Then, the remaining gaps are numbered $d+1, \ldots, n-1$ starting at the far left end and working to the right. In the example considered above, the gaps would be numbered as follows:

$$\begin{array}{ccccccccccccccc}
. & 4 & > & 2 & , & 5 & > & 1 & , & 6 & , & 7 & > & 3 & . \\
4 & & 3 & & 5 & & 2 & & 6 & & 7 & & 1 & & 0
\end{array}$$

Note that inserting the symbol 8 into the gap labeled j causes the major index to increase by exactly j.

Let us prove that this happens in the general case. If we insert n into the far right gap of $w^{(n-1)}$ (which is labeled zero), there will be no new descents, so $\mathrm{maj}(w^{(n)}) = \mathrm{maj}(w^{(n-1)}) + 0$ as desired. Suppose we insert n into the gap labeled j, where $1 \le j \le d$. In $w^{(n-1)}$, we had $w_{i_j} > w_{i_j+1}$, but the insertion of n changes this configuration to $w_{i_j} < n > w_{i_j+1}$. This pushes the descent at position i_j one position to the right. Furthermore, the descents that formerly occurred at positions $i_{j-1}, \ldots, i_1$ (which are to the right of i_j) also get pushed one position to the right because of the new symbol n. It follows that the major index increases by exactly j, as desired. Finally, suppose $d < j \le n-1$. Let the gap labeled j occur at position u in the new word, and let t be the number of descents in the old word preceding this gap. By definition of the gap labeling, we must have $j = (u - t) + d$. On the other hand, inserting n in this gap produces a new descent at position u, and pushes the $(d - t)$ descents located to the right of position u one position further right. The net change to the major index is therefore $u + (d - t) = j$, as desired.

We now know that $t_k \in \{0, 1, 2, \ldots, k-1\}$ for all k, so that f_n does map S_n into T_n. The preceding discussion also tells us how to invert the action of f_n. Given $(t_1, \ldots, t_n) \in T_n$, we use the t_k's to insert the numbers $1, \ldots, n$ into an initially empty permutation. After $1, \ldots, k-1$ have been inserted, we number the gaps according to the rules above and then insert k in the unique gap labeled t_k. We just proved that this insertion will increase the major index by t_k. It follows that the resulting permutation $w \in S_n$ is the unique object satisfying $f_n(w) = (t_1, \ldots, t_n)$. Therefore, f_n is a bijection. $\square$

6.30. Example. Let us compute $f_6^{-1}(0, 1, 1, 0, 4, 3)$. Using the insertion algorithm from the preceding proof, we generate the following sequence of permutations: 1; then 2, 1; then 2, 3, 1; then 2, 3, 1, 4; then 2, 3, 1, 5, 4; and finally 6, 2, 3, 1, 5, 4.

6.7 Quantum Binomial Coefficients

The formula $[n]!_x = \prod_{i=1}^{n} [i]_x$ for the quantum factorial is analogous to the formula $n! = \prod_{i=1}^{n} i$ for the ordinary factorial. We can extend this analogy to binomial coefficients and multinomial coefficients. This leads to the following definitions.

6.31. Definition: Quantum Binomial Coefficients. Suppose $n \geq 0$, $0 \leq k \leq n$, and x is any variable. We define the *quantum binomial coefficients* by the formula

$$\begin{bmatrix} n \\ k \end{bmatrix}_x = \frac{[n]!_x}{[k]!_x [n-k]!_x} = \frac{(x^n - 1)(x^{n-1} - 1) \cdots (x - 1)}{(x^k - 1)(x^{k-1} - 1) \cdots (x - 1)(x^{n-k} - 1)(x^{n-k-1} - 1) \cdots (x - 1)}.$$

6.32. Definition: Quantum Multinomial Coefficients. Suppose $n_1, \ldots, n_k \geq 0$ and x is any variable. We define the *quantum multinomial coefficients* by the formula

$$\begin{bmatrix} n_1 + \cdots + n_k \\ n_1, \ldots, n_k \end{bmatrix}_x = \frac{[n_1 + \cdots + n_k]!_x}{[n_1]!_x [n_2]!_x \cdots [n_k]!_x} = \frac{(x^{n_1 + \cdots + n_k} - 1) \cdots (x^2 - 1)(x - 1)}{\prod_{i=1}^{k} [(x^{n_i} - 1)(x^{n_i - 1} - 1) \cdots (x - 1)]}.$$

One cannot immediately see from the defining formulas that the quantum binomial and multinomial coefficients are actually polynomials in x (as opposed to quotients of polynomials). However, we will soon prove that these entities are polynomials with nonnegative integer coefficients. We will also give several combinatorial interpretations for quantum binomial and multinomial coefficients in terms of suitable weighted sets of objects. Before doing so, we need to develop a few more tools.

It is immediate from the definitions that $\begin{bmatrix} n \\ k \end{bmatrix}_x = \begin{bmatrix} n \\ n-k \end{bmatrix}_x = \begin{bmatrix} n \\ k, n-k \end{bmatrix}_x$; in particular, quantum binomial coefficients are special cases of quantum multinomial coefficients. We will usually prefer to use multinomial coefficients, writing $\begin{bmatrix} a+b \\ a,b \end{bmatrix}_x$ in preference to $\begin{bmatrix} a+b \\ a \end{bmatrix}_x$ or $\begin{bmatrix} a+b \\ b \end{bmatrix}_x$, because in most combinatorial applications the parameters a and b are more natural than a and $a + b$.

Before entering into further combinatorial discussions, we pause to give an algebraic proof of two fundamental recursions satisfied by the quantum binomial coefficients. These recursions are both "quantum analogues" of the binomial coefficient recursion 2.25.

6.33. Theorem: Recursions for Quantum Binomial Coefficients. For all $a, b > 0$, we have the recursion

$$\begin{aligned} \begin{bmatrix} a+b \\ a,b \end{bmatrix}_x &= x^b \begin{bmatrix} a+b-1 \\ a-1,b \end{bmatrix}_x + \begin{bmatrix} a+b-1 \\ a,b-1 \end{bmatrix}_x \\ &= \begin{bmatrix} a+b-1 \\ a-1,b \end{bmatrix}_x + x^a \begin{bmatrix} a+b-1 \\ a,b-1 \end{bmatrix}_x. \end{aligned}$$

The initial conditions are $\begin{bmatrix} a \\ a,0 \end{bmatrix}_x = \begin{bmatrix} b \\ 0,b \end{bmatrix}_x = 1$.

Proof. We prove the first equality, leaving the second one as an exercise. Writing out the definitions, the right side of the first recursion is

$$x^b \begin{bmatrix} a+b-1 \\ a-1, b \end{bmatrix}_x + \begin{bmatrix} a+b-1 \\ a, b-1 \end{bmatrix}_x = \frac{x^b[a+b-1]!_x}{[a-1]!_x[b]!_x} + \frac{[a+b-1]!_x}{[a]!_x[b-1]!_x}.$$

Multiply the first fraction by $[a]_x/[a]_x$ and the second fraction by $[b]_x/[b]_x$ to create a common denominator. Bringing out common factors, we obtain

$$\left(\frac{[a+b-1]!_x}{[a]!_x[b]!_x}\right) \cdot (x^b[a]_x + [b]_x).$$

By definition of quantum integers,

$$\begin{aligned} [b]_x + x^b[a]_x &= (1+x+x^2+\cdots+x^{b-1}) + x^b(1+x+\cdots+x^{a-1}) \\ &= (1+\cdots+x^{b-1}) + (x^b+\cdots+x^{a+b-1}) = [a+b]_x. \end{aligned}$$

Putting this into the previous formula, we get

$$\frac{[a+b-1]!_x[a+b]_x}{[a]!_x[b]!_x} = \begin{bmatrix} a+b \\ a, b \end{bmatrix}_x.$$

The initial conditions follow immediately from the definitions. □

6.34. Corollary: Polynomiality of Quantum Binomial Coefficients. For all $n \geq 0$ and $0 \leq k \leq n$, $\left[{n \atop k}\right]_x$ is a polynomial in x with nonnegative integer coefficients.

Proof. Use induction on $n \geq 0$, the base case $n = 0$ being evident. Assume $n > 0$ and that $\left[{n-1 \atop j}\right]_x$ is already known to be a polynomial with coefficients in $\mathbb{N}$ for $0 \leq j \leq n-1$. Then, by the recursion derived above (taking $n = a+b$ and $k = a$, so $b = n-k$),

$$\begin{bmatrix} n \\ k \end{bmatrix}_x = x^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_x + \begin{bmatrix} n-1 \\ k \end{bmatrix}_x.$$

Thanks to the induction hypothesis, we know that the right side is a polynomial with coefficients in $\mathbb{N}$. This completes the induction argument. □

We need one more result before describing the combinatorial interpretations for the quantum binomial coefficients. This result can be viewed as a generalization of the weighted bijection rule.

6.35. Theorem: Weight-Shifting Rule. Suppose (S, w_1) and (T, w_2) are finite weighted sets, and $f : S \to T$ is a bijection such that, for some constant b, $w_1(z) = w_2(f(z)) + b$ holds for all $z \in S$. Then

$$G_{S,w_1}(x) = x^b G_{T,w_2}(x).$$

Proof. Let $\{\star\}$ be a one-point set with $\mathrm{wt}(\star) = b$. The generating function for this weighted set is x^b. There is a bijection $i : T \to T \times \{\star\}$ given by $t \mapsto (t, \star)$ for $t \in T$. Observe that $i \circ f : S \to T \times \{\star\}$ is a *weight-preserving* bijection, since

$$w_1(z) = w_2(f(z)) + b = w_2(f(z)) + \mathrm{wt}(\star) = \mathrm{wt}((f(z), \star)) = \mathrm{wt}(i(f(z))) \qquad (z \in S).$$

Therefore, by the bijection rule and product rule for weighted sets,

$$G_S(x) = G_{T\times\{\star\}}(x) = G_T(x)G_{\{\star\}}(x) = x^b G_T(x).$$ □

6.36. Theorem: Combinatorial Interpretations of Quantum Binomial Coefficients. Fix integers $a, b \geq 0$. Let $L(a, b)$ be the set of all lattice paths from $(0, 0)$ to (a, b). Let $P(a, b)$ be the set of integer partitions μ with largest part at most a and with at most b parts. Then

$$\begin{bmatrix} a+b \\ a, b \end{bmatrix}_x = \sum_{w \in \mathcal{R}(0^a 1^b)} x^{\mathrm{inv}(w)} = \sum_{\pi \in L(a,b)} x^{\mathrm{area}(\pi)} = \sum_{\pi \in L(a,b)} x^{\mathrm{area}'(\pi)} = \sum_{\mu \in P(a,b)} x^{|\mu|}.$$

Proof. In 6.19, we constructed weight-preserving bijections between the three weighted sets $(\mathcal{R}(0^a 1^b), \mathrm{inv})$, $(L(a, b), \mathrm{area})$, and $(L(a, b), \mathrm{area}')$. Furthermore, Figure 2.18 shows that the weighted set $(P(a, b), |\cdot|)$ is essentially identical to the weighted set $(L(a, b), \mathrm{area}')$. So all of the combinatorial summations in the theorem are equal by the weighted bijection rule. We must prove that these all equal $\left[\begin{smallmatrix} a+b \\ a, b \end{smallmatrix}\right]_x$. We give two proofs illustrating different techniques.

First Proof. For each $a, b \geq 0$, let $g(a, b) = \sum_{\pi \in L(a,b)} x^{\mathrm{area}(\pi)} = G_{L(a,b),\mathrm{area}}(x)$. Suppose we can show that this function satisfies the same recursion and initial conditions as the quantum binomial coefficients do, namely

$$g(a, b) = x^b g(a - 1, b) + g(a, b - 1); \quad g(a, 0) = g(0, b) = 1.$$

Then a routine induction on $a + b$ will prove that $\left[\begin{smallmatrix} a+b \\ a, b \end{smallmatrix}\right]_x = g(a, b)$ for all $a, b \geq 0$.

To check the initial conditions, note that there is only one lattice path from $(0, 0)$ to $(a, 0)$, and the area underneath this path is zero. So $g(a, 0) = x^0 = 1$. Similarly, $g(0, b) = 1$. Now let us prove the recursion for $g(a, b)$, assuming $a, b > 0$. The set $L(a, b)$ is the disjoint union of sets L_1 and L_2, where L_1 consists of all paths from $(0, 0)$ to (a, b) ending in an east step, and L_2 consists of all paths from $(0, 0)$ to (a, b) ending in a north step. See Figure 6.2. Deleting the final north step from a path in L_2 defines a bijection from L_2 to $L(a, b-1)$, which is weight-preserving since the area below the path is not affected by the deletion of the north step. It follows that $G_{L_2,\mathrm{area}}(x) = G_{L(a,b-1),\mathrm{area}}(x) = g(a, b-1)$. On the other hand, deleting the final east step from a path in L_1 defines a bijection from L_1 to $L(a - 1, b)$ that is *not* weight-preserving. The reason is that the b area cells below the final east step in a path in L_1 no longer contribute to the area of the path in $L(a-1, b)$. However, since the area drops by b for all objects in L_1, we can conclude that $G_{L_1,\mathrm{area}}(x) = x^b G_{L(a-1,b),\mathrm{area}}(x) = x^b g(a-1, b)$. Now, by the sum rule for weighted sets,

$$g(a, b) = G_{L(a,b),\mathrm{area}}(x) = G_{L_1}(x) + G_{L_2}(x) = x^b g(a - 1, b) + g(a, b - 1).$$

We remark that a similar argument involving deletion of the initial step of a path in $L(a, b)$ establishes the dual recursion $g(a, b) = g(a - 1, b) + x^a g(a, b - 1)$.

Second Proof. Here we will prove that

$$[a + b]!_x = [a]!_x [b]!_x \sum_{w \in \mathcal{R}(0^a 1^b)} x^{\mathrm{inv}(w)},$$

which is equivalent to the first equality in the theorem statement. We know from 6.26 that the left side here is the generating function for the set S_{a+b} of permutations of $\{1, 2, \ldots, a+b\}$, weighted by inversions. By the product rule, the right side is the generating function for the product set $S_a \times S_b \times \mathcal{R}(0^a 1^b)$, with the usual product weight ($\mathrm{wt}(u, v, w) = \mathrm{inv}(u) + \mathrm{inv}(v) + \mathrm{inv}(w)$). Therefore, it suffices to define a bijection

$$f : S_a \times S_b \times \mathcal{R}(0^a 1^b) \to S_{a+b}$$

such that $\mathrm{inv}(f(u, v, w)) = \mathrm{inv}(u) + \mathrm{inv}(v) + \mathrm{inv}(w)$ for $u \in S_a$, $v \in S_b$, and $w \in \mathcal{R}(0^a 1^b)$.

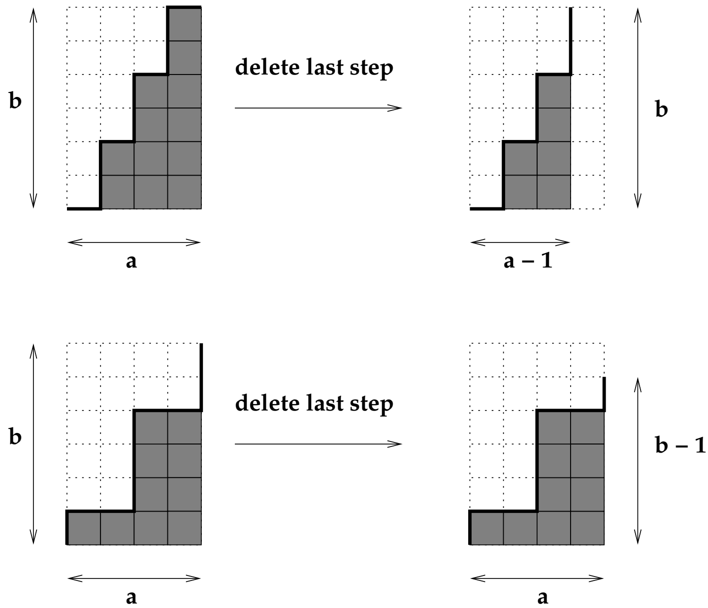

FIGURE 6.2
Deleting the final step of a lattice path.

Given (u, v, w) in the domain of f, note that u is a permutation of the a letters $1, 2, \ldots, a$. Replace the a zeroes in w with these a letters, in the same order that they occur in u. Next, add a to each of the letters in the permutation v, and then replace the b ones in w by these new letters in the same order that they occur in v. The resulting object z is evidently a permutation of $\{1, 2, \ldots, a + b\}$. For example, if $a = 3$ and $b = 5$, then

$$f(132, 24531, 01100111) = 15732864.$$

Since a and b are fixed and known, we can reverse the action of f. Starting with a permutation z of $a + b$ elements, we first recover the word $w \in \mathcal{R}(0^a 1^b)$ by replacing the numbers $1, 2, \ldots, a$ in z by zeroes and replacing the numbers $a + 1, \ldots, a + b$ in z by ones. Next, we take the subword of z consisting of the numbers $1, 2, \ldots, a$ to recover u. Similarly, let v' be the subword of z consisting of the numbers $a + 1, \ldots, a + b$. We recover v by subtracting a from each of these numbers. This algorithm defines a two-sided inverse map to f. For example, still taking $a = 3$ and $b = 5$, we have

$$f^{-1}(35162847) = (312, 23514, 01010111).$$

All that remains is to check that f is weight-preserving. Fix u, v, w, z with $z = f(u, v, w)$. Let A be the set of positions in z occupied by letters in u, and let B be the remaining positions (occupied by shifted letters of v). Equivalently, by definition of f, $A = \{i : w_i = 0\}$ and $B = \{i : w_i = 1\}$. The inversions of z can be classified into three kinds. First, there are inversions (i, j) such that $i, j \in A$. These inversions correspond bijectively to the inversions of u. Second, there are inversions (i, j) such that $i, j \in B$. These inversions correspond bijectively to the inversions of v. Third, there are inversions (i, j) such that $i \in A$ and $j \in B$, or $i \in B$ and $j \in A$. The first case $(i \in A, j \in B)$ cannot occur, because every position in A is filled with a lower number than every position in B. The second case $(i \in B, j \in A)$ occurs iff $i < j$ and $w_i = 1$ and $w_j = 0$. This means that the inversions of the third kind in z correspond bijectively to the inversions of the binary word w. Conclusion: $\mathrm{inv}(z) = \mathrm{inv}(u) + \mathrm{inv}(v) + \mathrm{inv}(w)$, as desired. □

Like ordinary binomial coefficients, the quantum binomial coefficients appear in a plethora of identities, many of which have combinatorial proofs. Here is a typical example, which is a weighted analogue of the Chu-Vandermonde identity in 2.21.

6.37. Theorem: Quantum Chu-Vandermonde Identity. For all integers $a, b, c \geq 0$,

$$\begin{bmatrix} a+b+c+1 \\ a, b+c+1 \end{bmatrix}_x = \sum_{k=0}^{a} x^{(b+1)(a-k)} \begin{bmatrix} k+b \\ k, b \end{bmatrix}_x \begin{bmatrix} a-k+c \\ a-k, c \end{bmatrix}_x .$$

Proof. Recall the picture we used to prove the original version of the identity (Figure 2.3), which is reprinted here as Figure 6.3. The path dissection in this picture defines a bijection

$$f : L(a, b+c+1) \to \bigcup_{k=0}^{a} L(k, b) \times L(a-k, c).$$

Here, k is the x-coordinate where the given path in $L(a, b+c+1)$ crosses the line $y = b+(1/2)$. The bijection f is not weight-preserving. However, if a path $P \in L(a, b+c+1)$ maps to $(Q, R) \in L(k, b) \times L(a-k, c)$ under f, then it is evident from the picture that

$$\mathrm{area}(P) = \mathrm{area}(Q) + \mathrm{area}(R) + (b+1)(a-k).$$

(The extra factor comes from the lower-right rectangle of width $a-k$ and height $b+1$.) It now follows from the weight-shifting rule, the sum rule, and the product rule that

$$G_{L(a,b+c+1),\mathrm{area}}(x) = \sum_{k=0}^{a} x^{(b+1)(a-k)} G_{L(k,b),\mathrm{area}}(x) \cdot G_{L(a-k,c),\mathrm{area}}(x).$$

We complete the proof by using 6.36 to replace each area generating function here by a suitable quantum binomial coefficient. □

6.38. Remark. There are also linear-algebraic interpretations of the quantum binomial coefficients. Specifically, let F be a finite field with q elements, where q is necessarily a prime power. Then the integer $\begin{bmatrix} n \\ k \end{bmatrix}_q$ (which is $\begin{bmatrix} n \\ k \end{bmatrix}_x$ evaluated at $x = q$) is the number of k-dimensional subspaces of the n-dimensional vector space F^n. We prove this fact in §12.7.

6.8 Quantum Multinomial Coefficients

Recall from 2.27 that the ordinary multinomial coefficients $C(n; n_1, \ldots, n_s)$ (where $n = \sum_k n_k$) satisfy the recursion

$$C(n; n_1, \ldots, n_k) = \sum_{k=1}^{s} C(n-1; n_1, \ldots, n_k - 1, \ldots, n_s).$$

The quantum multinomial coefficients satisfy the following analogous recursion.

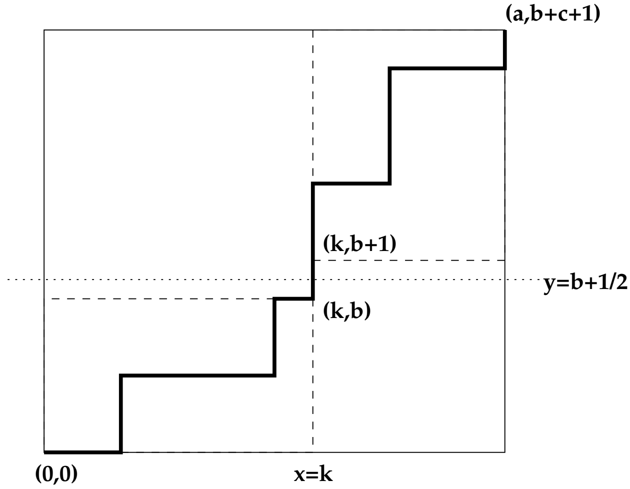

FIGURE 6.3
Picture used to prove the q-Chu-Vandermonde identity.

6.39. Theorem: Recursions for Quantum Multinomial Coefficients. Let $n_1, \ldots, n_s$ be nonnegative integers, and set $n = \sum_{k=1}^{s} n_k$. Then

$$\left[\begin{matrix} n \\ n_1, \ldots, n_s \end{matrix}\right]_x = \sum_{k=1}^{s} x^{n_1+n_2+\cdots+n_{k-1}} \left[\begin{matrix} n-1 \\ n_1, \ldots, n_k - 1, \ldots, n_s \end{matrix}\right]_x .$$

(If $n_k = 0$, the kth summand on the right side is zero.) The initial condition is $\left[\begin{smallmatrix} 0 \\ 0,\ldots,0 \end{smallmatrix}\right]_x = 1$. Moreover, $\left[\begin{smallmatrix} n \\ n_1,\ldots,n_s \end{smallmatrix}\right]_x$ is a polynomial in x with coefficients in $\mathbb{N}$.

Proof. Neither side of the claimed recursion changes if we drop all n_i's that are equal to zero; so, without loss of generality, assume every n_i is positive. We can create a common factor of $[n-1]!_x / \prod_{j=1}^{s} [n_j]!_x$ on the right side by multiplying the kth summand by $[n_k]_x/[n_k]_x$, for $1 \leq k \leq s$. Pulling out this common factor, we are left with

$$\sum_{k=1}^{s} x^{n_1+n_2+\cdots+n_{k-1}} [n_k]_x = \sum_{k=1}^{s} x^{n_1+\cdots+n_{k-1}} (1 + x + x^2 + \cdots + x^{n_k - 1}).$$

The kth summand consists of the sum of consecutive powers of x starting at $x^{n_1+\cdots+n_{k-1}}$ and ending at $x^{n_1+\cdots+n_k-1}$. Chaining these together, we see that the sum evaluates to $x^0 + x^1 + \cdots + x^{n-1} = [n]_x$. Multiplying by the common factor mentioned above, we obtain

$$\frac{[n]!_x}{\prod_{j=1}^{s} [n_j]!_x} = \left[\begin{matrix} n \\ n_1, \ldots, n_s \end{matrix}\right]_x ,$$

as desired. The initial condition is immediate. Finally, we deduce polynomiality of the quantum multinomial coefficients using induction on n and the recursion just proved, as in the proof of 6.34. □

6.40. Theorem: Quantum Multinomial Coefficients and Inversions of Words. Suppose $A = \{a_1 < a_2 < \cdots < a_s\}$ is a totally ordered alphabet. For all integers

$n_1, \ldots, n_s \geq 0$,

$$\begin{bmatrix} n_1 + \cdots + n_s \\ n_1, n_2, \ldots, n_s \end{bmatrix}_x = \sum_{w \in \mathcal{R}(a_1^{n_1} \cdots a_s^{n_s})} x^{\mathrm{inv}(w)}.$$

Proof. For all integers $n_1, \ldots, n_s$, define

$$g(n_1, \ldots, n_s) = \sum_{w \in \mathcal{R}(a_1^{n_1} \cdots a_s^{n_s})} x^{\mathrm{inv}(w)}.$$

(This is zero by convention if any n_i is negative.) By induction on $\sum_k n_k$, it suffices to show that g satisfies the recursion in 6.39. Now $g(0, 0, \ldots, 0) = x^0 = 1$, so the initial condition is correct. Next, fix $n_1, \ldots, n_s \geq 0$, and let W be the set of words appearing in the definition of $g(n_1, \ldots, n_s)$. Write W as the disjoint union of sets $W_1, \ldots, W_s$, where W_k consists of the words in W with first letter a_k. By the sum rule,

$$g(n_1, \ldots, n_s) = G_W(x) = \sum_{k=1}^{s} G_{W_k}(x).$$

Fix a value of k in the range $1 \leq k \leq s$ such that W_k is nonempty. Erasing the first letter of a word w in W_k defines a bijection from W_k to the set $\mathcal{R}(a_1^{n_1} \cdots a_k^{n_k - 1} \cdots a_s^{n_s})$. The generating function for the latter set is $g(n_1, \ldots, n_k - 1, \ldots, n_s)$. The bijection in question does not preserve weights, because inversions involving the first letter of $w \in W_k$ disappear when this letter is erased. However, no matter what word w we pick in W_k, the number of inversions that involve the first letter in w will always be the same. Specifically, this first letter (namely a_k) will cause inversions with all of the a_1's, a_2's, ..., and a_{k-1}'s that follow it in w. The number of such letters is $n_1 + \cdots + n_{k-1}$. Therefore, by the weight-shifting rule,

$$G_{W_k, \mathrm{inv}}(x) = x^{n_1 + \cdots + n_{k-1}} g(n_1, \ldots, n_k - 1, \ldots, n_s).$$

This equation is also correct if $W_k = \emptyset$ (which occurs iff $n_k = 0$). Using these results in the formula above, we conclude that

$$g(n_1, \ldots, n_s) = \sum_{k=1}^{s} x^{n_1 + \cdots + n_{k-1}} g(n_1, \ldots, n_k - 1, \ldots, n_s),$$

which is precisely the recursion occurring in 6.39. $\square$

6.41. Remark. This theorem can also be proved by generalizing the second proof of 6.36. Specifically, one can prove that

$$[n_1 + \cdots + n_s]!_x = [n_1]!_x \cdots [n_s]!_x \sum_{w \in \mathcal{R}(1^{n_1} \cdots s^{n_s})} x^{\mathrm{inv}(w)}$$

by defining a weight-preserving bijection

$$f : S_{n_1 + \cdots + n_s} \to S_{n_1} \times \cdots \times S_{n_s} \times \mathcal{R}(1^{n_1} \cdots s^{n_s})$$

(where S_{n_i} is the set of all permutations of $\{1, 2, \ldots, n_i\}$, and all sets in the Cartesian product are weighted by inversions). We leave the details as an exercise.

6.9 Foata's Map

We know from 6.26 and 6.29 that $\sum_{w\in S_n} x^{\operatorname{inv}(w)} = [n]!_x = \sum_{w\in S_n} x^{\operatorname{maj}(w)}$, where S_n is the set of permutations of $\{1,2,\ldots,n\}$. We can express this result by saying that the statistics inv and maj are *equidistributed* on S_n. We have just derived a formula for the distribution of inv on more general sets of words, namely

$$\sum_{w\in\mathcal{R}(1^{n_1}\cdots s^{n_s})} x^{\operatorname{inv}(w)} = \begin{bmatrix} n_1+\cdots+n_s \\ n_1,\ldots,n_s \end{bmatrix}_x .$$

Could it be true that inv and maj are still equidistributed on these more general sets? MacMahon [90] proved that this is indeed the case. We present a combinatorial proof of this result based on a bijection due to Dominique Foata. For each set $S = \mathcal{R}(1^{n_1}\cdots s^{n_s})$, our goal is to define a weight-preserving bijection $f : (S, \operatorname{maj}) \to (S, \operatorname{inv})$.

To achieve our goal, let W be the set of *all* words in the alphabet $\{1,2,\ldots,s\}$. We shall define a function $g : W \to W$ with the following properties: (a) g is a bijection; (b) for all $w \in W$, w and $g(w)$ are anagrams (§1.9); (c) if w is not the empty word, then w and $g(w)$ have the same last letter; (d) for all $w \in W$, $\operatorname{inv}(g(w)) = \operatorname{maj}(w)$. We can then obtain the desired weight-preserving bijections f by restricting g to the various anagram classes $\mathcal{R}(1^{n_1}\cdots s^{n_s})$.

We will define g by recursion on the length of $w \in W$. If this length is 0 or 1, set $g(w) = w$. Then conditions (b), (c), and (d) hold in this case. Now suppose w has length $n \geq 2$. Write $w = w'yz$, where $w' \in W$ and y, z are the last two letters of w. We can assume by induction that $u = g(w'y)$ has already been defined, and that u is an anagram of $w'y$ ending in y such that $\operatorname{inv}(u) = \operatorname{maj}(w'y)$. We will define $g(w) = h_z(u)z$, where $h_z : W \to W$ is a certain map (to be described momentarily) that satisfies conditions (a) and (b) above. No matter what the details of the definition of h_z, it is already evident that g will satisfy conditions (b) and (c) for words of length n.

To motivate the definition of h_z, we first give a lemma that analyzes the effect on inv and maj of appending a letter to the end of a word. The lemma will use the following convenient notation. If u is any word and z is any letter, let $n_{\leq z}(u)$ be the number of letters in u (counting repetitions) that are $\leq z$; define $n_{<z}(u)$, $n_{>z}(u)$, and $n_{\geq z}(u)$ similarly.

6.42. Lemma. Suppose u is a word of length m with last letter y, and z is any letter.
(a) If $y \leq z$, then $\operatorname{maj}(uz) = \operatorname{maj}(u)$. (b) If $y > z$, then $\operatorname{maj}(uz) = \operatorname{maj}(u) + m$.
(c) $\operatorname{inv}(uz) = \operatorname{inv}(u) + n_{>z}(u)$.

Proof. All statements follow routinely from the definitions of inv and maj. □

Let us now describe the map $h_z : W \to W$. First, h_z sends the empty word to itself. Now suppose u is a nonempty word ending in y. There are two cases. **Case 1:** $y \leq z$. In this case, we break the word u into *runs* of consecutive letters such that the last letter in each run is $\leq z$, while all preceding letters in the run are $> z$. For example, if $u = 1342434453552$ and $z = 3$, then the decomposition of u into runs is

$$u = 1/3/4,2/4,3/4,4,5,3/5,5,2/$$

where we use slashes to delimit consecutive runs. Now, h_z operates on u by cyclically shifting the letters in each run one step to the right. Continuing the preceding example,

$$h_3(u) = 1/3/2,4/3,4/3,4,4,5/2,5,5/.$$

What effect does this process have on $\mathrm{inv}(u)$? The last element in each run (which is $\leq z$) is strictly less than all elements before it in its run (which are $> z$). So, moving the last element to the front of its run causes the inversion number to drop by the number of elements $> z$ in the run. Adding up these changes over all the runs, we see that

$$\mathrm{inv}(h_z(u)) = \mathrm{inv}(u) - n_{>z}(u) \tag{6.1}$$

in case 1. Furthermore, note that the first letter of $h_z(u)$ is always $\leq z$ in this case.

Case 2: $y > z$. Again we break the word u into runs, but here the last letter of each run must be $> z$, while all preceding letters in the run are $\leq z$. For example, if $z = 3$ and $u = 134243445355$, we decompose u as

$$u = 1, 3, 4/2, 4/3, 4/4/5/3, 5/5/.$$

As before, we cyclically shift the letters in each run one step right, which gives

$$h_3(u) = 4, 1, 3/4, 2/4, 3/4/5/5, 3/5/$$

in our example. This time, the last element in each run is $> z$ and is strictly greater than the elements $\leq z$ that precede it in its run. So, the cyclic shift of each run will increase the inversion count by the number of elements $\leq z$ in the run. Adding over all runs, we see that

$$\mathrm{inv}(h_z(u)) = \mathrm{inv}(u) + n_{\leq z}(u) \tag{6.2}$$

in case 2. Furthermore, note that the first letter of $h_z(u)$ is always $> z$ in this case.

In both cases, $h_z(u)$ is an anagram of u. Moreover, we can invert the action of h_z as follows. Examination of the first letter of $h_z(u)$ tells us whether we were in case 1 or case 2 above. To invert in case 1, break the word into runs whose first letter is $\leq z$ and whose other letters are $> z$, and cyclically shift each run one step left. To invert in case 2, break the word into runs whose first letter is $> z$ and whose other letters are $\leq z$, and cyclically shift each run one step left. We now see that h_z is a bijection. For example, to compute $h_3^{-1}(1342434453552)$, first write

$$1/3, 4/2, 4/3, 4, 4, 5/3, 5, 5/2/$$

and then cyclically shift to get the answer $1/4, 3/4, 2/4, 4, 5, 3/5, 5, 3/2/$.

Now we can return to the discussion of g. Recall that we have set $g(w) = g(w'yz) = h_z(u)z$, where $u = g(w'y)$ is an anagram of $w'y$ ending in y and satisfying $\mathrm{inv}(u) = \mathrm{maj}(w'y)$. To check condition (d) for this w, we must show that $\mathrm{inv}(h_z(u)z) = \mathrm{maj}(w)$. Again consider two cases. If $y \leq z$, then

$$\mathrm{maj}(w) = \mathrm{maj}(w'yz) = \mathrm{maj}(w'y) = \mathrm{inv}(u).$$

On the other hand, by the lemma and (6.1), we have

$$\mathrm{inv}(h_z(u)z) = \mathrm{inv}(h_z(u)) + n_{>z}(h_z(u)) = \mathrm{inv}(u) - n_{>z}(u) + n_{>z}(u) = \mathrm{inv}(u)$$

(observe that $n_{>z}(h_z(u)) = n_{>z}(u)$ since $h_z(u)$ and u are anagrams). In the second case, where $y > z$, we have

$$\mathrm{maj}(w) = \mathrm{maj}(w'yz) = \mathrm{maj}(w'y) + n - 1 = \mathrm{inv}(u) + n - 1.$$

On the other hand, the lemma and (6.2) give

$$\mathrm{inv}(h_z(u)z) = \mathrm{inv}(h_z(u)) + n_{>z}(h_z(u)) = \mathrm{inv}(u) + n_{\leq z}(u) + n_{>z}(u) = \mathrm{inv}(u) + n - 1,$$

$$\begin{array}{rcll} & & & \text{current word:} \\ & & & 2,1,3,3,1,3,2,2 \\ h_1(2) & = & 2; & 2,1,3,3,1,3,2,2 \\ h_3(2,1) & = & 2,1; & 2,1,3,3,1,3,2,2 \\ h_3(2,1,3) & = & 2,1,3; & 2,1,3,3,1,3,2,2 \\ h_1(2,1,3,3) & = & 2,3,1,3; & 2,3,1,3,1,3,2,2 \\ h_3(2,3,1,3,1) & = & 2,3,1,3,1; & 2,3,1,3,1,3,2,2 \\ h_2(2,3,1,3,1,3) & = & 3,2,3,1,3,1; & 3,2,3,1,3,1,2,2 \\ h_2(3,2,3,1,3,1,2) & = & 2,3,1,3,1,3,2; & 2,3,1,3,1,3,2,2 \end{array}$$

FIGURE 6.4
Computation of $g(w)$.

$$\begin{array}{rcll} & & & \text{current word:} \\ & & & 2,1,3,3,1,3,2,2 \\ h_2^{-1}(2,1,3,3,1,3,2) & = & 2,3,3,1,3,1,2; & 2,3,3,1,3,1,2,2 \\ h_2^{-1}(2,3,3,1,3,1) & = & 3,3,2,3,1,1; & 3,3,2,3,1,1,2,2 \\ h_1^{-1}(3,3,2,3,1) & = & 3,3,2,1,3; & 3,3,2,1,3,1,2,2 \\ h_3^{-1}(3,3,2,1) & = & 3,3,2,1; & 3,3,2,1,3,1,2,2 \\ h_1^{-1}(3,3,2) & = & 3,3,2; & 3,3,2,1,3,1,2,2 \\ h_2^{-1}(3,3) & = & 3,3; & 3,3,2,1,3,1,2,2 \\ h_3^{-1}(3) & = & 3; & 3,3,2,1,3,1,2,2 \end{array}$$

FIGURE 6.5
Computation of $g^{-1}(w)$.

since u has $n-1$ letters, each of which is either $\leq z$ or $> z$.

It remains to prove that g is a bijection, by describing the two-sided inverse map g^{-1}. This is the identity map on words of length at most 1. To compute $g^{-1}(uz)$, first compute $u' = h_z^{-1}(u)$. Then return the answer $g^{-1}(uz) = (g^{-1}(u'))z$. Here is a nonrecursive description of the maps g and g^{-1}, obtained by "unrolling" the recursive applications of g and g^{-1} in the preceding definitions.

> *To compute* $g(w_1w_2\cdots w_n)$*:* for $i = 2, \ldots, n$ in this order, apply h_{w_i} to the first $i-1$ letters of the current word.
>
> *To compute* $g^{-1}(z_1z_2\cdots z_n)$*:* for $i = n, n-1, \ldots, 2$ in this order, let z_i' be the ith letter of the current word, and apply $h_{z_i'}^{-1}$ to the first $i-1$ letters of the current word.

6.43. Example. Figure 6.4 illustrates the computation of $g(w)$ for $w = 21331322$. We find that $g(w) = 23131322$. Observe that $\text{maj}(w) = 1+4+6 = 11 = \text{inv}(g(w))$. Next, Figure 6.5 illustrates the calculation of $g^{-1}(w)$. We have $g^{-1}(w) = 33213122$ and $\text{inv}(w) = 10 = \text{maj}(g^{-1}(w))$.

We summarize the results of this section in the following theorem.

6.44. Theorem. For all $n_1, \ldots, n_s \geq 0$,

$$\sum_{w \in \mathcal{R}(1^{n_1}\cdots s^{n_s})} x^{\text{maj}(w)} = \sum_{w \in \mathcal{R}(1^{n_1}\cdots s^{n_s})} x^{\text{inv}(w)} = \begin{bmatrix} n_1+\cdots+n_s \\ n_1,\ldots,n_s \end{bmatrix}_x.$$

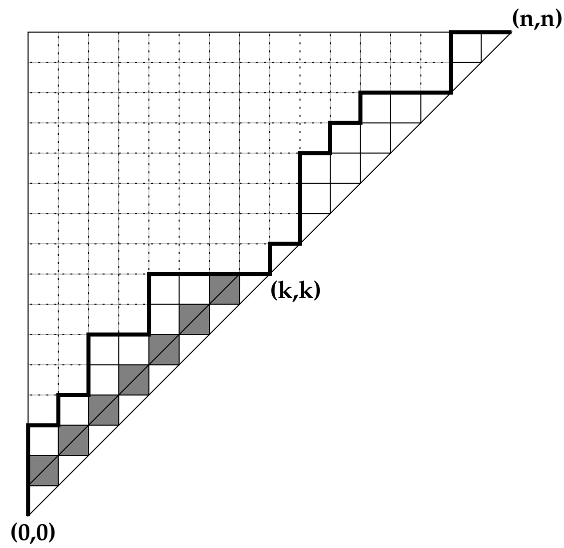

FIGURE 6.6
First-return analysis for weighted Dyck paths.

More precisely, there is a bijection on $\mathcal{R}(1^{n_1}\cdots s^{n_s})$ sending maj to inv and preserving the last letter of each word.

6.10 Quantum Catalan Numbers

In this section, we investigate two weighted analogues of the Catalan numbers. Recall that the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n} = \binom{2n}{n,n} - \binom{2n}{n-1,n+1}$ counts the collection of all lattice paths from $(0,0)$ to (n,n) that never go below the line $y = x$ (§1.10). Let D_n be the collection of these *Dyck paths.* Also, let W_n be the set of words that encode the Dyck paths, where we use 0 to encode a north step and 1 to encode an east step. Elements of W_n will be called *Dyck words.*

6.45. Definition: Statistics on Dyck Paths. For every Dyck path $P \in D_n$, let area(P) be the number of complete unit squares located between P and the line $y = x$. If P is encoded by the Dyck word $w \in W_n$, let $\text{inv}(P) = \text{inv}(w)$ and $\text{maj}(P) = \text{maj}(w)$.

For example, the path P shown in Figure 6.6 has $\text{area}(P) = 23$. One sees that $\text{inv}(P)$ is the number of unit squares in the region bounded by P, the y-axis, and the line $y = n$. We also have $\text{inv}(P) + \text{area}(P) = \binom{n}{2}$ since $\binom{n}{2}$ is the total number of area squares in the bounding triangle. The statistic $\text{maj}(P)$ is the sum of the number of steps in the path that precede each "left-turn" where an east step (1) is immediately followed by a north step (0). For the path in Figure 6.6, we have $\text{inv}(P) = 97$ and $\text{maj}(P) = 4 + 6 + 10 + 16 + 18 + 22 + 24 + 28 = 128$.

6.46. Example. When $n = 3$, examination of Figure 1.8 shows that

$$\begin{aligned} G_{D_3,\text{area}}(x) &= 1 + 2x + x^2 + x^3; \\ G_{D_3,\text{inv}}(x) &= 1 + x + 2x^2 + x^3; \\ G_{D_3,\text{maj}}(x) &= 1 + x^2 + x^3 + x^4 + x^6. \end{aligned}$$

When $n = 4$, a longer calculation gives

$$\begin{aligned} G_{D_4,\text{area}}(x) &= 1 + 3x + 3x^2 + 3x^3 + 2x^4 + x^5 + x^6; \\ G_{D_4,\text{maj}}(x) &= 1 + x^2 + x^3 + 2x^4 + x^5 + 2x^6 + x^7 + 2x^8 + x^9 + x^{10} + x^{12}. \end{aligned}$$

There is no particularly nice closed formula for $G_{D_n,\text{area}}(x)$ (although determinant formulas do exist for this polynomial). However, these generating functions do satisfy a recursion, which is the analogue of the "first-return" recursion used in the unweighted case (§2.7).

6.47. Theorem: Recursion for Dyck Paths Weighted by Area. For all $n \geq 0$, set $C_n(x) = G_{D_n,\text{area}}(x)$. Then $C_0(x) = 1$ and, for all $n \geq 1$,

$$C_n(x) = \sum_{k=1}^{n} x^{k-1} C_{k-1}(x) C_{n-k}(x).$$

Proof. We imitate the proof of 2.33, but now we must take weights into account. For $1 \leq k \leq n$, write $D_{n,k}$ for the set of Dyck paths of order n whose first return to the line $y = x$ occurs at (k, k). Evidently, D_n is the disjoint union of the $D_{n,k}$'s, so the sum rule gives

$$C_n(x) = \sum_{k=1}^{n} G_{D_{n,k},\text{area}}(x).$$

For fixed k, we have a bijection from $D_{n,k}$ to $D_{k-1} \times D_{n-k}$ defined by sending $P = 0, P_1, 1, P_2$ to (P_1, P_2), where the displayed 1 encodes the east step that arrives at (k, k). See Figure 6.6. Examination of the figure shows that

$$\text{area}(P) = \text{area}(P_1) + \text{area}(P_2) + (k - 1),$$

where the $k-1$ counts the shaded cells in the figure which are not included in the calculation of $\text{area}(P_1)$. By the product rule and weight-shifting rule, we see that

$$G_{D_{n,k},\text{area}}(x) = C_{k-1}(x) C_{n-k}(x) x^{k-1}.$$

Inserting this into the previous formula gives the recursion. □

Now let us consider the generating function $G_{D_n,\text{maj}}(x)$. This polynomial does have a nice closed formula, as we see in the next theorem.

6.48. Theorem: Dyck Paths Weighted by Major Index. For all $n \geq 0$,

$$G_{D_n,\text{maj}}(x) = \begin{bmatrix} 2n \\ n, n \end{bmatrix}_x - x \begin{bmatrix} 2n \\ n-1, n+1 \end{bmatrix}_x = \frac{1}{[n+1]_x} \begin{bmatrix} 2n \\ n, n \end{bmatrix}_x.$$

Proof. The second equality follows from the manipulation

$$\begin{aligned} &\begin{bmatrix} 2n \\ n, n \end{bmatrix}_x - x \begin{bmatrix} 2n \\ n-1, n+1 \end{bmatrix}_x = \begin{bmatrix} 2n \\ n, n \end{bmatrix}_x \cdot \left(1 - \frac{x[n]_x}{[n+1]_x}\right) \\ &= \begin{bmatrix} 2n \\ n, n \end{bmatrix}_x \cdot \left(\frac{(1 + x + x^2 + \cdots + x^n) - x(1 + x + \cdots + x^{n-1})}{[n+1]_x}\right) = \begin{bmatrix} 2n \\ n, n \end{bmatrix}_x \cdot \frac{1}{[n+1]_x}. \end{aligned}$$

The other equality in the theorem statement can be rewritten

$$x \begin{bmatrix} 2n \\ n-1, n+1 \end{bmatrix}_x + G_{D_n,\mathrm{maj}}(x) = \begin{bmatrix} 2n \\ n, n \end{bmatrix}_x .$$

We will give a bijective proof of this result reminiscent of André's reflection principle (see 1.56). Consider the set of words $S = \mathcal{R}(0^n 1^n)$, weighted by major index. By 6.44, the generating function for this set is $\left[\begin{smallmatrix} 2n \\ n,n \end{smallmatrix}\right]_x$. On the other hand, we can write S as the disjoint union of W_n and T, where W_n is the set of Dyck words and T consists of all other words in $\mathcal{R}(0^n 1^n)$. We will define a bijection $g : T \to \mathcal{R}(0^{n+1}1^{n-1})$ such that $\mathrm{maj}(w) = 1+\mathrm{maj}(g(w))$ for all $w \in T$. This will give

$$\begin{bmatrix} 2n \\ n, n \end{bmatrix}_x = G_{S,\mathrm{maj}}(x) = G_{T,\mathrm{maj}}(x) + G_{W_n,\mathrm{maj}}(x) = x \begin{bmatrix} 2n \\ n-1, n+1 \end{bmatrix}_x + G_{D_n,\mathrm{maj}}(x),$$

as desired. To define $g(w)$ for $w \in T$, regard w as a lattice path in an $n \times n$ rectangle by interpreting 0's as north steps and 1's as east steps. Find the largest $k > 0$ such that the path w touches the line $y = x - k$. Such a k must exist, because $w \in T$ is not a Dyck path. Consider the first vertex v on w that touches the line in question. (See Figure 6.7.) The path w must arrive at v by taking an east step, and w must leave v by taking a north step. These steps correspond to certain adjacent letters $w_i = 1$ and $w_{i+1} = 0$ in the word w. Furthermore, since v is the first visitation to this line, we must have either $i = 1$ or $w_{i-1} = 1$ (i.e., the step before w_i must be an east step if it exists). Let $g(w)$ be the word obtained by changing w_i from 1 to 0. Pictorially, we "tip" the east step arriving at v upwards, changing it to a north step (which causes the following steps to shift to the northwest). The word $w = \cdots 1, 1, 0 \cdots$ turns into $g(w) = \cdots 1, 0, 0 \cdots$, so the major index drops by exactly 1 when we pass from w to $g(w)$. This result also holds if $i = 1$. The new word $g(w)$ has $n-1$ east steps and $n+1$ north steps, so $g(w) \in \mathcal{R}(0^{n+1}1^{n-1})$. Finally, g is invertible. Given a path/word $P \in \mathcal{R}(0^{n+1}1^{n-1})$, again take the largest $k \geq 0$ such that P touches the line $y = x - k$, and let v be the *last* time P touches this line. Here, v is preceded by an east step and followed by two north steps (or v is the origin and is followed by a north step). Changing the first north step following v into an east step produces a path $g'(P) \in \mathcal{R}(0^n 1^n)$. One routinely checks that $g'(P)$ cannot be a Dyck path (so g' maps into T), and that g' is the two-sided inverse of g. The key is that the selection rules for v ensure that the same step is "tipped" when we apply g followed by g', and similarly in the other order.

□

Summary

- *Generating Functions for Weighted Sets.* A weighted set is a pair (S, wt) where S is a set and $\mathrm{wt} : S \to \mathbb{N}$ is a function (called a *statistic* on S). The generating function for this weighted set is $G_{S,\mathrm{wt}}(x) = G_S(x) = \sum_{z \in S} x^{\mathrm{wt}(z)}$. Writing $G_S(x) = \sum_{k \geq 0} a_k x^k$, a_k is the number of objects in S having weight k.

- *Weight-Preserving Bijections.* A weight-preserving bijection from (S, wt) to (T, wt') is a bijection $f : S \to T$ with $\mathrm{wt}'(f(z)) = \mathrm{wt}(z)$ for all $z \in S$. When such an f exists, $G_{S,\mathrm{wt}}(x) = G_{T,\mathrm{wt}'}(x)$. More generally, if there is $b \in \mathbb{Z}$ with $\mathrm{wt}'(f(z)) = b + \mathrm{wt}(z)$ for all $z \in S$, then $G_{T,\mathrm{wt}'}(x) = x^b G_{S,\mathrm{wt}}(x)$.

- *Sum Rule for Weighted Sets.* Suppose (S_i, w_i) are weighted sets for $1 \leq i \leq k$, S is the

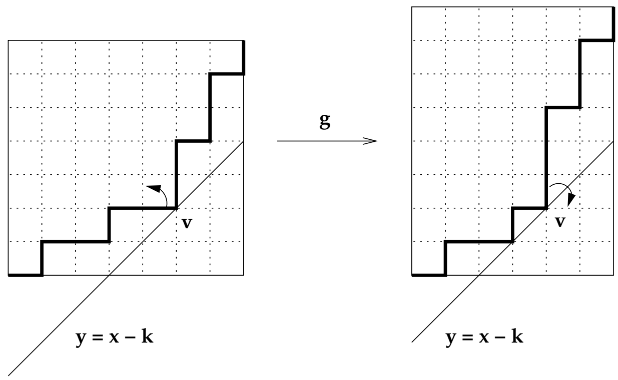

FIGURE 6.7
The tipping bijection.

disjoint union of the S_i, and we define $w : S \to \mathbb{N}$ by $w(z) = w_i(z)$ for $z \in S_i$. Then $G_S(x) = \sum_{i=1}^{k} G_{S_i}(x)$.

- *Product Rule for Weighted Sets.* Suppose (S_i, w_i) are weighted sets for $1 \le i \le k$, $S = S_1 \times S_2 \times \cdots \times S_k$ is the product of the S_i, and we define $w : S \to \mathbb{N}$ by $w(z_1, \ldots, z_k) = \sum_{i=1}^{k} w_i(z_i)$ for $z_i \in S_i$. Then $G_S(x) = \prod_{i=1}^{k} G_{S_i}(x)$.

- *Quantum Integers, Factorials, Binomial Coefficients, and Multinomial Coefficients.* Suppose x is a variable, $n, k, n_i \in \mathbb{N}$, $0 \le k \le n$, and $\sum_i n_i = n$. We define $[n]_x = \sum_{i=0}^{n-1} x^i = (x^n - 1)/(x - 1)$, $[n]!_x = \prod_{i=1}^{n} [i]_x$, $\left[{n \atop k}\right]_x = \frac{[n]!_x}{[k]!_x [n-k]!_x}$, $\left[{n \atop n_1, \ldots, n_s}\right]_x = \frac{[n]!_x}{\prod_{i=1}^{s} [n_i]!_x}$. These are all polynomials in x with coefficients in $\mathbb{N}$.

- *Recursions for Quantum Binomial Coefficients, etc.* The following recursions hold:

$$\begin{aligned}
[n]!_x &= [n-1]!_x \cdot [n]_x \\
\begin{bmatrix} a+b \\ a, b \end{bmatrix}_x &= x^b \begin{bmatrix} a+b-1 \\ a-1, b \end{bmatrix}_x + \begin{bmatrix} a+b-1 \\ a, b-1 \end{bmatrix}_x = \begin{bmatrix} a+b-1 \\ a-1, b \end{bmatrix}_x + x^a \begin{bmatrix} a+b-1 \\ a, b-1 \end{bmatrix}_x \\
\begin{bmatrix} n_1 + \cdots + n_s \\ n_1, \ldots, n_s \end{bmatrix}_x &= \sum_{k=1}^{s} x^{n_1 + \cdots + n_{k-1}} \begin{bmatrix} n_1 + \cdots + n_s - 1 \\ n_1, \ldots, n_k - 1, \ldots, n_s \end{bmatrix}_x
\end{aligned}$$

- *Statistics on Words.* Given a word $w = w_1 w_2 \cdots w_n$ over a totally ordered alphabet, $\operatorname{Inv}(w) = \{(i, j) : i < j \text{ and } w_i > w_j\}$, $\operatorname{inv}(w) = |\operatorname{Inv}(w)|$, $\operatorname{Des}(w) = \{i < n : w_i > w_{i+1}\}$, $\operatorname{des}(w) = |\operatorname{Des}(w)|$, and $\operatorname{maj}(w) = \sum_{i \in \operatorname{Des}(w)} i$. We have

$$\begin{bmatrix} n_1 + \cdots + n_s \\ n_1, \ldots, n_s \end{bmatrix}_x = \sum_{w \in \mathcal{R}(a_1^{n_1} \cdots a_s^{n_s})} x^{\operatorname{inv}(w)} = \sum_{w \in \mathcal{R}(a_1^{n_1} \cdots a_s^{n_s})} x^{\operatorname{maj}(w)}.$$

The second equality follows from a subtle bijection due to Foata, which maps maj to inv while preserving the last letter of the word. In particular, letting $S_n = \mathcal{R}(1^1 2^1 \cdots n^1)$,

$$[n]!_x = \sum_{w \in S_n} x^{\operatorname{inv}(w)} = \sum_{w \in S_n} x^{\operatorname{maj}(w)}.$$

These two formulas can be proved bijectively by mapping $w \in S_n$ to its "inversion table" $(t_1, \ldots, t_n)$, where t_i records the change in inversions (resp. major index) caused by inserting the symbol i into the subword of w consisting of $1, 2, \ldots, i-1$.

- *Weighted Lattice Paths.* The quantum binomial coefficient $\begin{bmatrix} a+b \\ a,b \end{bmatrix}_x = \begin{bmatrix} b+a \\ b,a \end{bmatrix}_x$ counts lattice paths in an $a \times b$ (or $b \times a$) rectangle, weighted either by area above the path or area below the path. This coefficient also counts integer partitions with first part $\leq a$ and length $\leq b$, weighted by area.

- *Weighted Dyck Paths.* Let $C_n(x)$ be the generating function for Dyck paths of order n, weighted by area between the path and $y = x$. Then $C_0(x) = 1$ and $C_n(x) = \sum_{k=1}^{n} x^{k-1} C_{k-1}(x) C_{n-k}(x)$. The generating function for Dyck paths (viewed as words in $\mathcal{R}(0^n 1^n)$) weighted by major index is $\frac{1}{[n+1]_x} \begin{bmatrix} 2n \\ n,n \end{bmatrix}_x = \begin{bmatrix} 2n \\ n,n \end{bmatrix}_x - x \begin{bmatrix} 2n \\ n-1,n+1 \end{bmatrix}_x$.

Exercises

In the exercises below, S_n denotes the set $\mathcal{R}(12 \cdots n)$ of permutations of $\{1, 2, \ldots, n\}$, unless otherwise specified.

6.49. Let $S = \mathcal{R}(a^1 b^1 c^2)$, $T = \mathcal{R}(a^1 b^2 c^1)$, and $U = \mathcal{R}(a^2 b^1 c^1)$. Confirm that $G_{S,\mathrm{inv}}(x) = G_{T,\mathrm{inv}}(x) = G_{U,\mathrm{inv}}(x)$ (as asserted in 6.15) by listing all weighted objects in T and U.

6.50. (a) Compute $\mathrm{inv}(w)$, $\mathrm{des}(w)$, and $\mathrm{maj}(w)$ for each $w \in S_4$. (b) Use (a) to find the generating functions $G_{S_4,\mathrm{inv}}(x)$, $G_{S_4,\mathrm{des}}(x)$, and $G_{S_4,\mathrm{maj}}(x)$. (c) Compute $[4]!_x$ by polynomial multiplication, and compare to the answers in (b).

6.51. (a) Compute $\mathrm{inv}(w)$ for the following words w: 4251673, 101101110001, 314423313, 55233514425331. (b) Compute $\mathrm{Des}(w)$, $\mathrm{des}(w)$, and $\mathrm{maj}(w)$ for each word w in (a).

6.52. Confirm the formulas for $G_{D_4,\mathrm{area}}(x)$ and $G_{D_4,\mathrm{maj}}(x)$ stated in 6.46 by listing weighted Dyck paths of order 4.

6.53. (a) Find the maximum value of $\mathrm{inv}(w)$, $\mathrm{des}(w)$, and $\mathrm{maj}(w)$ as w ranges over S_n. (b) Repeat (a) for w ranging over $\mathcal{R}(1^{n_1} 2^{n_2} \cdots s^{n_s})$.

6.54. Let S be the set of k-letter words over the alphabet $\underline{\mathbf{n}}$. For $w \in S$, let $\mathrm{wt}(w)$ be the sum of all letters in w. Compute $G_{S,\mathrm{wt}}(x)$.

6.55. Let S be the set of 5-letter words using the 26-letter English alphabet. For $w \in S$, let $\mathrm{wt}(w)$ be the number of vowels in w. Compute $G_{S,\mathrm{wt}}(x)$.

6.56. Let S be the set of all subsets of $\{1, 2, \ldots, n\}$. For $A \in S$, let $\mathrm{wt}(A) = |A|$. Use the product rule for weighted sets to compute $G_{S,\mathrm{wt}}(x)$ (cf. 1.20).

6.57. Let S be the set of all k-element multisets using the alphabet $\underline{\mathbf{n}}$. For $M \in S$, let $\mathrm{wt}(M)$ be the sum of the elements in M, counting multiplicities. Express $G_{S,\mathrm{wt}}(x)$ in terms of quantum binomial coefficients.

6.58. (a) How many permutations of $\{1, 2, \ldots, 8\}$ have exactly 17 inversions? (b) How many permutations of $\{1, 2, \ldots, 9\}$ have major index 29?

6.59. (a) How many lattice paths from $(0, 0)$ to $(8, 6)$ have area 21? (b) How many words in $\mathcal{R}(0^5 1^6 2^7)$ have ten inversions? (c) How many Dyck paths of order 7 have major index 30?

6.60. Quantum Binomial Theorem. Let S be the set of all subsets of $\{1,2,\ldots,n\}$. For $A \in S$, let $\mathrm{wt}(A)$ be the sum of the elements in A. Show that

$$\prod_{i=1}^{n}(1+x^i) = G_{S,\mathrm{wt}}(x) = \sum_{k=0}^{n} x^{k(k+1)/2} \begin{bmatrix} n \\ k \end{bmatrix}_x .$$

6.61. Use an involution to prove $\sum_{k=0}^{n}(-1)^k x^{k(k-1)/2} \begin{bmatrix} n \\ k \end{bmatrix}_x = 0$ for all $n > 0$.

6.62. Compute each of the following polynomials by any method, expressing the answer in the form $\sum_{k\ge 0} a_k x^k$: (a) $[7]_x$; (b) $[6]!_x$; (c) $\begin{bmatrix} 8 \\ 5 \end{bmatrix}_x$; (d) $\begin{bmatrix} 7 \\ 2,3,2 \end{bmatrix}_x$; (e) $\frac{1}{[8]_x}\begin{bmatrix} 8 \\ 5,3 \end{bmatrix}_x$; (f) $\frac{1}{[11]_x}\begin{bmatrix} 11 \\ 5,6 \end{bmatrix}_x$.

6.63. (a) Factor the polynomials $[4]_x$, $[5]_x$, $[6]_x$, and $[12]_x$ in $\mathbb{Z}[x]$. (b) How do these polynomials factor in $\mathbb{C}[x]$?

6.64. Compute $\begin{bmatrix} 4 \\ 2 \end{bmatrix}_x$ in six ways, by: (a) simplifying the defining formula in 6.31; (b) using the first recursion in 6.33; (c) using the second recursion in 6.33; (d) enumerating words in $\mathcal{R}(0011)$ by inversions; (e) enumerating words in $\mathcal{R}(0011)$ by major index; (f) enumerating partitions contained in a 2×2 box by area.

6.65. (a) Prove the identity $\begin{bmatrix} n_1+\cdots+n_k \\ n_1,\ldots,n_k \end{bmatrix}_x = \prod_{i=1}^{k} \begin{bmatrix} n_i+\cdots+n_k \\ n_i, n_{i+1}+\cdots+n_k \end{bmatrix}_x$ algebraically. (b) Give a combinatorial proof of the identity in (a).

6.66. For $1 \le i \le 3$, let (T_i, w_i) be a set of weighted objects. (a) Prove that $\mathrm{id}_{T_1} : T_1 \to T_1$ is a weight-preserving bijection. (b) Prove that if $f : T_1 \to T_2$ is a weight-preserving bijection, then $f^{-1} : T_2 \to T_1$ is weight-preserving. (c) Prove that if $f : T_1 \to T_2$ and $g : T_2 \to T_3$ are weight-preserving bijections, so is $g \circ f$.

6.67. Prove the second recursion in 6.33: (a) by an algebraic manipulation; (b) by removing the first step from a lattice path in an $a \times b$ rectangle.

6.68. Let f be the map in the second proof of 6.36, with $a = b = 4$. Compute: (a) $f(2413, 1423, 10011010)$; (b) $f(4321, 4321, 11110000)$; (c) $f(2134, 3214, 01010101)$. Verify that weights are preserved in each case.

6.69. Let f be the map in the second proof of 6.36, with $a = 5$ and $b = 4$. For each w given here, compute $f^{-1}(w)$ and verify that weights are preserved: (a) $w = 123456789$; (b) $w = 371945826$; (c) $w = 987456321$.

6.70. Repeat 6.69 assuming $a = 2$ and $b = 7$.

6.71. Prove that $[n_1+\cdots+n_s]!_x = [n_1]!_x \cdots [n_s]!_x \sum_{w\in\mathcal{R}(1^{n_1}\cdots s^{n_s})} x^{\mathrm{inv}(w)}$ by defining a weight-preserving bijection $f : S_{n_1+\cdots+n_s} \to S_{n_1} \times \cdots \times S_{n_s} \times \mathcal{R}(1^{n_1}\cdots s^{n_s})$.

6.72. (a) Find and prove an analogue of the identity $\sum_{k=0}^{n}\binom{n}{k}^2 = \binom{2n}{n}$ involving quantum binomial coefficients (cf. 2.19 and Figure 2.1). (b) Similarly, derive a quantum analogue of the identity $\sum_{k=0}^{a}\binom{k+b-1}{k,b-1} = \binom{a+b}{a,b}$.

6.73. Let S be the set of two-element subsets of Deck. For $H \in S$, let $\mathrm{wt}(H)$ be the sum of the values of the two cards in H, where aces count as 11 and jacks, queens, and kings count as 10. Find $G_{S,\mathrm{wt}}(x)$.

6.74. Define the weight of a five-card poker hand to be the number of face cards in the hand (the face cards are aces, jacks, queens, and kings). Compute the generating functions for the following sets of poker hands relative to this weight: (a) full house hands; (b) three-of-a-kind hands; (c) flush hands; (d) straight hands.

6.75. Define the weight of a five-card poker hand to be the number of diamond cards in the hand. Compute the generating functions for the following sets of poker hands relative to this weight: (a) full house hands; (b) three-of-a-kind hands; (c) flush hands; (d) straight hands.

6.76. Let T_n be the set of connected simple graphs with vertex set $\{1, 2, \ldots, n\}$. Let the weight of a graph in T_n be the number of edges. Compute $G_{T_n}(x)$ for $1 \leq n \leq 5$.

6.77. Let f_n and g_n be the maps in 6.26. Compute $f_6(341265)$ and $g_6(0, 0, 1, 3, 2, 3)$, and verify that weights are preserved for these two objects.

6.78. Let f_n and g_n be the maps in 6.26. Compute $f_8(35261784)$ and $g_8(0, 1, 0, 3, 2, 4, 6, 5)$, and verify that weights are preserved for these two objects.

6.79. In 6.26, we constructed an inversion table for $w \in S_n$ by classifying inversions $(i, j) \in \mathrm{Inv}(w)$ based on the left-hand value w_i. Define a new map $f : S_n \to \underline{\mathbf{1}} \times \underline{\mathbf{2}} \times \cdots \times \underline{\mathbf{n}}$ by classifying inversions $(i, j) \in \mathrm{Inv}(w)$ based on the right-hand value w_j. Show that f is a bijection, and compute $f(35261784)$ and $f^{-1}(0, 1, 0, 3, 2, 4, 6, 5)$.

6.80. Define a map $f : S_n \to \underline{\mathbf{1}} \times \underline{\mathbf{2}} \times \cdots \times \underline{\mathbf{n}}$ by setting $f(w) = (t_n, \ldots, t_1)$, where $t_i = |\{j : (i, j) \in \mathrm{Inv}(w)\}|$. Show that f is a bijection. (Informally, f classifies inversions of w based on the left position of the inversion pair.) Compute $f(35261784)$ and $f^{-1}(0, 1, 0, 3, 2, 4, 6, 5)$.

6.81. Define a map $f : S_n \to \underline{\mathbf{1}} \times \underline{\mathbf{2}} \times \cdots \times \underline{\mathbf{n}}$ that classifies inversions of w based on the right position of the inversion pair (cf. 6.80). Show that f is a bijection, and compute $f(35261784)$ and $f^{-1}(0, 1, 0, 3, 2, 4, 6, 5)$.

6.82. Let f_n be the map in 6.29. Compute $f_6(341265)$ and $f_6^{-1}(0, 0, 1, 3, 2, 3)$, and verify that weights are preserved for these two objects.

6.83. Let f_n be the map in 6.29. Compute $f_8(35261784)$ and $f_8^{-1}(0, 1, 0, 3, 2, 4, 6, 5)$, and verify that weights are preserved for these two objects.

6.84. Coinversions. Define the *coinversions* of a word $w = w_1 w_2 \cdots w_n$ by $\mathrm{coinv}(w) = \sum_{i<j} \chi(w_i < w_j)$. Prove that $\sum_{w \in \mathcal{R}(1^{n_1} 2^{n_2} \cdots s^{n_s})} x^{\mathrm{coinv}(w)} = \begin{bmatrix} n_1+\cdots+n_s \\ n_1,\ldots,n_s \end{bmatrix}_x$ (a) by using a bijection to reduce to the corresponding result for inv; (b) by verifying a suitable recursion.

6.85. Given a word $w = w_1 \cdots w_n$, define $\mathrm{comaj}(w) = \sum_{i<n} i\chi(w_i < w_{i+1})$ and $\mathrm{rlmaj}(w) = \sum_{i<n} (n-i)\chi(w_i > w_{i+1})$. Calculate $\sum_{w \in S_n} x^{\mathrm{comaj}(w)}$ and $\sum_{w \in S_n} x^{\mathrm{rlmaj}(w)}$.

6.86. For $w \in S_n$, let $\mathrm{wt}(w)$ be the sum of all $i < n$ such that $i + 1$ appears to the left of i in w. Compute $G_{S_n,\mathrm{wt}}(x)$.

6.87. (a) Suppose $w = w_1 w_2 \cdots w_{n-1}$ is a fixed permutation of $n - 1$ distinct letters. Let a be a new letter less than all letters appearing in w. Let S be the set of n words that can be obtained from w by inserting a in some gap. Prove that $\sum_{z \in S} x^{\mathrm{maj}(z)} = x^{\mathrm{maj}(w)}[n]_x$. (b) Use (a) to obtain another proof that $\sum_{w \in S_n} x^{\mathrm{maj}(w)} = [n]!_x$.

6.88. Suppose k is fixed in $\{1, 2, \ldots, n\}$, and $w = w_1 w_2 \cdots w_{n-1}$ is a fixed permutation of $\{1, 2, \ldots, k - 1, k + 1, \ldots, n\}$. Let S be the set of n words that can be obtained from w by inserting k in some gap. Prove or disprove: $\sum_{z \in S} x^{\mathrm{maj}(z)} = x^{\mathrm{maj}(w)}[n]_x$.

6.89. Define a *cyclic shift* function $c : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ by $c(i) = i+1$ for $i < n$, and $c(n) = 1$. Define a map $C : S_n \to S_n$ by setting $C(w_1 w_2 \cdots w_n) = c(w_1)c(w_2) \cdots c(w_n)$. (a) Prove: for all $w \in S_n$, $\mathrm{maj}(C(w)) = \mathrm{maj}(w) - 1$ if $w_n \neq n$, and $\mathrm{maj}(C(w)) = \mathrm{maj}(w) + n - 1$ if $w_n = n$. (b) Use (a) to show combinatorially that, for $1 \leq k \leq n$, $\sum_{w \in S_n:\ w_n = k} x^{\mathrm{maj}(w)} = x^{n-k} \sum_{v \in S_{n-1}} x^{\mathrm{maj}(v)}$. (c) Use (b), the sum rule, and induction to obtain another proof of 6.29.

6.90. For all $n \geq 1$, all $T \subseteq \{1, 2, \ldots, n-1\}$, and $1 \leq k \leq n$, let $G(n, T, k)$ be the number of permutations w of $\{1, 2, \ldots, n\}$ with $\mathrm{Des}(w) = T$ and $w_n = k$. (a) Find a recursion for the quantities $G(n, T, k)$. (b) Count the number of permutations of 10 objects with descent set $\{2, 3, 5, 7\}$.

6.91. Let w be the word 4523351452511332, and let h_z be the map from §6.9. Compute $h_z(w)$ for $z = 1, 2, 3, 4, 5, 6$. Verify that (6.1) or (6.2) holds in each case.

6.92. Let w be the word 4523351452511332, and let h_z be the map from §6.9. Compute $h_z^{-1}(w)$ for $z = 1, 2, 3, 4, 5, 6$.

6.93. Compute the image of each $w \in S_4$ under the map g from §6.9.

6.94. Let g be the map in §6.9. Compute $g(w)$ for each of these words: (a) 4251673; (b) 27418563; (c) 101101110001; (d) 314423313. Verify that $\mathrm{inv}(g(w)) = \mathrm{maj}(w)$ in each case.

6.95. Let g be the map in §6.9. Compute $g^{-1}(w)$ for each word w in 6.94.

6.96. Let g be the bijection in the proof of 6.48. Compute $g(w)$ for each non-Dyck word $w \in \mathcal{R}(0^3 1^3)$.

6.97. Quantum Fibonacci Numbers. (a) Let W_n be the set of words in $\{0, 1\}^n$ with no two consecutive zeroes, and let the weight of a word be the number of zeroes in it. Find a recursion for the generating functions $G_{W_n}(x)$, and use this to compute $G_{W_6}(x)$. (b) Repeat part (a), taking the weight to be the number of ones in the word.

6.98. Let $S_{n,k}$ be the set of non-attacking placements of $n - k$ rooks on the board Δ_n (see 2.63). Define the weight of such a placement as follows. Each rook in the placement "cancels" all squares above it in its column. The weight of the placement is the total number of uncanceled squares located due west of rooks in the placement. Find a recursion for the generating functions $G_{n,k} = G_{S_{n,k},\mathrm{wt}}(x)$, which are quantum analogues of the Stirling numbers of the second kind. Compute these generating functions for $0 \leq k \leq n \leq 5$.

6.99. Let $C_{n,k}$ be the set of permutations of $\underline{\mathbf{n}}$ consisting of k disjoint cycles. Define a statistic on permutations $w \in C_{n,k}$ so that the associated generating functions satisfy the recursion

$$G_{C_{n,k}}(x) = G_{C_{n-1,k-1}}(x) + [n-1]_x G_{C_{n-1,k}}(x).$$

6.100. Let T_n be the set of trees with vertex set $\underline{\mathbf{n}}$. Can you find a statistic on trees such that the associated generating function satisfies $G_{T_n}(x) = [n]_x^{n-2}$?

6.101. Multivariable Generating Functions. Suppose S is a finite set, and $w_1, \ldots, w_n : S \to \mathbb{N}$ are n statistics on S. The *generating function for S relative to the n weights $w_1, \ldots, w_n$* is the polynomial $G_{S,w_1,\ldots,w_n}(x_1, \ldots, x_n) = \sum_{z \in S} \prod_{i=1}^{n} x_i^{w_i(z)}$. Formulate and prove versions of the sum rule, product rule, bijection rule, and weight-shifting rule for such generating functions.

6.102. Extend 6.60 to a formula for $\prod_{i=1}^{n}(1 + tx^i)$ by weighting subsets of $\{1, 2, \ldots, n\}$ by the number of elements in the subset and by the sum of the elements in the subset.

6.103. Recall from 1.29 that we can view permutations $w \in S_n$ as bijective maps of $\{1, 2, \ldots, n\}$ into itself. Define $I : S_n \to S_n$ by $I(w) = w^{-1}$ for $w \in S_n$. (a) Show that $I \circ I = \mathrm{id}_{S_n}$. (b) Show that $\mathrm{inv}(I(w)) = \mathrm{inv}(w)$ for all $w \in S_n$. (c) Define $\mathrm{imaj}(w) = \mathrm{maj}(I(w))$ for all $w \in S_n$. Compute the two-variable generating function $G_n(x, y) = \sum_{w \in S_n} x^{\mathrm{maj}(w)} y^{\mathrm{imaj}(w)}$ for $1 \leq n \leq 4$. Prove that $G_n(x, y) = G_n(y, x)$.

6.104. Let g be the map in §6.9, and let $\mathrm{IDes}(w) = \mathrm{Des}(w^{-1})$ for $w \in S_n$. (a) Show that for all $w \in S_n$, $\mathrm{IDes}(g(w)) = \mathrm{IDes}(w)$. (b) Construct a bijection $h : S_n \to S_n$ such that, for all $w \in S_n$, $\mathrm{inv}(h(w)) = \mathrm{maj}(w)$ *and* $\mathrm{maj}(h(w)) = \mathrm{inv}(w)$.

6.105. Let P_n be the set of integer partitions whose diagrams fit in the diagram of $(n-1, n-2, \ldots, 2, 1, 0)$, i.e., $\mu \in P_n$ iff $\ell(\mu) < n$ and $\mu_i \leq n-i$ for $1 \leq i < n$. Let $G_n(x) = \sum_{\mu \in P_n} x^{|\mu|}$. Find a recursion satisfied by $G_n(x)$ and use this to calculate $G_5(x)$. What is the relation between $G_n(x)$ and the quantum Catalan number $C_n(x)$ from §6.10?

6.106. Bounce Statistic on Dyck Paths. Given a Dyck path $P \in D_n$, define a new weight $\mathrm{bounce}(P)$ as follows. A ball starts at $(0,0)$ and moves north and east to (n,n) according to the following rules. The ball moves north v_0 steps until blocked by an east step of P, then moves east v_0 steps to the line $y = x$. The ball then moves north v_1 steps until blocked by the east step of P starting on the line $x = v_0$, then moves east v_1 steps to the line $y = x$. This bouncing process continues, generating a sequence $(v_0, v_1, \ldots, v_s)$ of vertical moves adding to n. We define $\mathrm{bounce}(P) = \sum_{i=0}^{s} i v_i$ and $C_n(q,t) = \sum_{P \in D_n} q^{\mathrm{area}(P)} t^{\mathrm{bounce}(P)}$. (a) Calculate $C_n(q,t)$ for $n \leq 4$ by enumerating Dyck paths. (b) Let $C_{n,k}(q,t) = \sum_{P \in D_n :\, v_0(P)=k} q^{\mathrm{area}(P)} t^{\mathrm{bounce}(P)}$ be the generating function for Dyck paths that start with exactly k north steps. Establish the recursion

$$C_{n,k}(q,t) = \sum_{r=1}^{n-k} t^{n-k} q^{k(k-1)/2} \begin{bmatrix} r+k-1 \\ r, k-1 \end{bmatrix}_q C_{n-k,r}(q,t)$$

by "removing the first bounce." Show also that $C_n(q,t) = t^{-n} C_{n+1,1}(q,t)$. (c) Use the recursion in (b) to calculate $C_n(q,t)$ for $n = 5, 6$. (d) Prove that $q^{n(n-1)/2} C_n(q, 1/q) = \sum_{P \in D_n} q^{\mathrm{maj}(P)}$. (e) Can you prove $C_n(q,t) = C_n(t,q)$ for all $n \geq 1$?

6.107. Let G_n be the set of sequences $g = (g_0, g_1, \ldots, g_{n-1})$ of nonnegative integers with $g_0 = 0$ and $g_{i+1} \leq g_i + 1$ for all $i < n-1$ (cf. 2.120). For $g \in G_n$, define $\mathrm{area}(g) = \sum_{i=0}^{n-1} g_i$ and $\mathrm{dinv}(g) = \sum_{i<j} \chi(g_i - g_j \in \{0,1\})$. (a) Find a bijection $k : G_n \to D_n$ such that $\mathrm{area}(k(g)) = \mathrm{area}(g)$ for all $g \in G_n$. (b) Find a bijection $h : G_n \to D_n$ such that $\mathrm{area}(h(g)) = \mathrm{dinv}(g)$ and $\mathrm{bounce}(h(g)) = \mathrm{area}(g)$ for all $g \in G_n$ (see 6.106). Conclude that the statistics dinv, bounce, area (on G_n), and area (on D_n) all have the same distribution.

Notes

The idea used to prove 6.29 seems to have first appeared in Gupta [63]. The bijection in §6.9 is due to Foata [38]. For related material, see Foata and Schützenberger [40]. Much of the early work on permutation statistics, including proofs of 6.44 and 6.48, is due to Major Percy MacMahon [90]. The bijective proof of 6.48, along with other material on quantum Catalan numbers, may be found in Fürlinger and Hofbauer [47]. The bounce statistic in 6.106 was introduced by Haglund [64]; for more on this topic, see Haglund [65].