CHAPTER 1

Naive Set Theory

In this chapter we introduce the basic notions and notations needed for dealing with **sets** (collections of objects of various kinds, such as numbers or pairs of numbers, called **points**). Although you may have dealt with these concepts since grade school, it is important that this chapter be utilized (at least by independent study) to assure a sound base for the applications in probability that follow in Chapter 2. **Most courses will not spend too much time on this chapter.**

"Set" is an undefined notion (like "point" or "line" in high-school geometry); it is assumed that at least one exists, and attention is restricted to a **universe** or **universal set** Ω. All operations are with respect to Ω in order to avoid paradoxical situations. For further details, see the Problems at the end of this chapter or Feferman (1964). ("Naive" set theory is to be contrasted with "axiomatic" set theory, which is an advanced treatment arranged to obviate the possibility of paradoxes.)

How sets arise in probability and statistics is illustrated by examples and discussed in Section 1.4. At this point, the examples are simple and convenient probability-related illustrations of set concepts and operations. Realistic probabilistic problems using these concepts, such as the post office operations problem discussed following Definition 2.2.20 and Problem 2.8H, are deferred since their proper discussion entails probabilistic concepts introduced in Chapter 2.

On reaching Section 1.4 (with the resulting universe Ω given in Example 1.4.1), you should return to Sections 1.1 through 1.3 and draw illustrations of the definitions and theorems with the universe Ω of ordered pairs.

1.1. BASIC DEFINITIONS

Let A and B be sets.[1] We now define four set operations and illustrate their meanings with diagrams called **Venn diagrams**. These diagrams are slightly unreliable, as are arguments by diagram in geometry (e.g., the "typical" cases

[1] We generally denote *points* by lowercase Greek letters, *sets* by capital italic letters, collections of sets by capital script letters, and the *universe* by capital omega.

drawn often do not show any relations of inclusion or disjointness; a specific instance is given in Example 1.2.9).

Definition 1.1.1.[2] **Union** (or **join,**[3] or **sum**[3]) of sets A and B (see Figure 1.1-1):

$$A \cup B \equiv A + B \equiv \{\omega\colon \omega \in A \text{ or } \omega \in B\}. \tag{1.1.2}$$

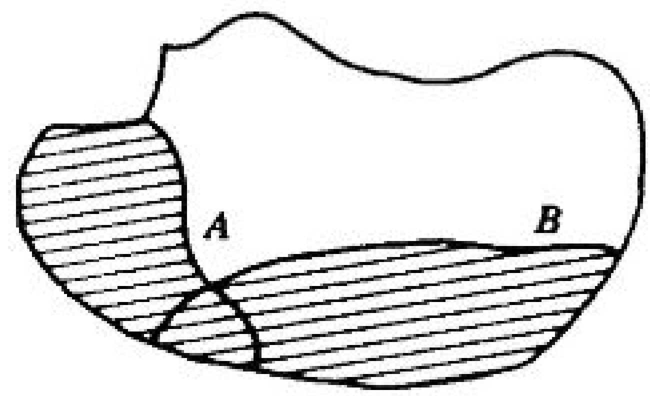

Figure 1.1-1. The shaded region is $A \cup B$.

Definition 1.1.3. **Intersection** (or **meet,**[3] or **product**[3]) of sets A and B (see Figure 1.1-2):

$$A \cap B \equiv AB \equiv A \cdot B \equiv \{\omega\colon \omega \in A \text{ and } \omega \in B\}. \tag{1.1.4}$$

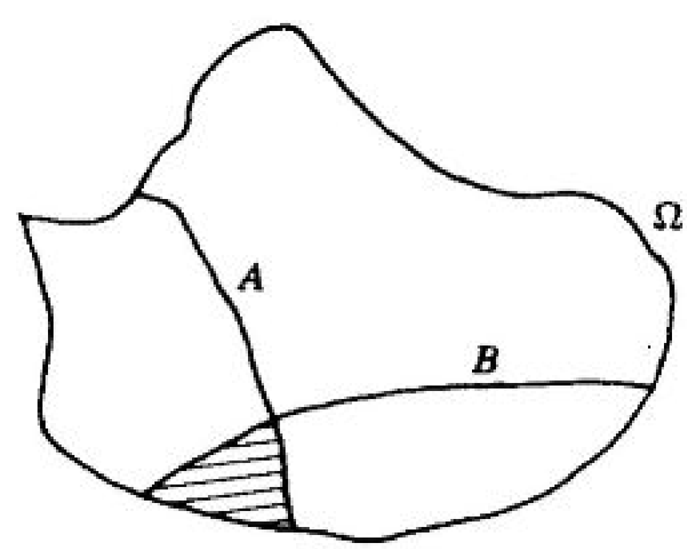

Figure 1.1-2. The shaded region is AB.

Definition 1.1.5. **Complement** of a set A (see Figure 1.1-3):

$$A^c \equiv \bar{A} \equiv A' \equiv \{\omega\colon \omega \in \Omega \text{ and } \omega \notin A\}. \tag{1.1.6}$$

Note that we have two notations for set unions ($A \cup B$ and $A + B$), three for set intersections ($A \cap B$, AB, and $A \cdot B$), and three for set complements (A^c, $\bar{A}$, and A'). Each notation is widely used in the fields of statistics and probability, and each seems more convenient in certain contexts. We try to use the first or

[2] In this text "or" is used in the usual mathematical sense; that is, "$\omega \in A$ or $\omega \in B$" means "either $\omega \in A$, or $\omega \in B$, *or both*." As used in equation (1.1.2), $\equiv$ defines the symbols $A \cup B$ and $A + B$; they are by definition shorthand for $\{\omega\colon \omega \in A \text{ or } \omega \in B\}$.

[3] Although this terminology (and some of the notation that follows) is not now in widespread use, it is important to be exposed to it in order to be able to read other (and older) books and writings.

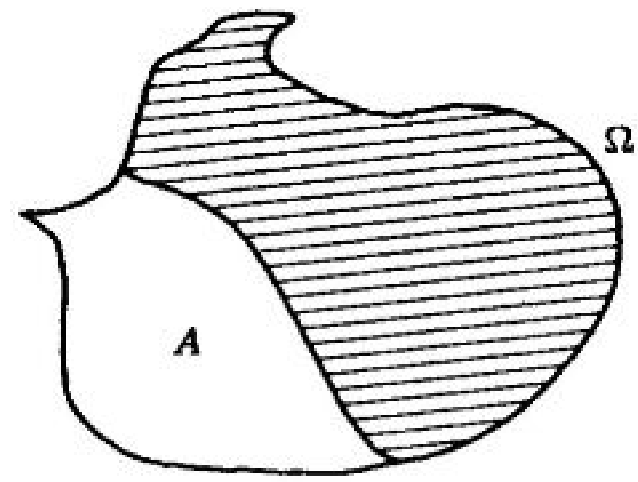

Figure 1.1-3. The shaded region is $\bar{A}$.

second notations given most of the time. However, as all these notations are in widespread use, it is important to become familiar with them. Since various ones are easier to use in various situations, we have not settled on one to use exclusively in this text.

Definition 1.1.7. **Difference** of sets A and B (see Figure 1.1-4):

$$A - B \equiv A \cap B^c. \tag{1.1.8}$$

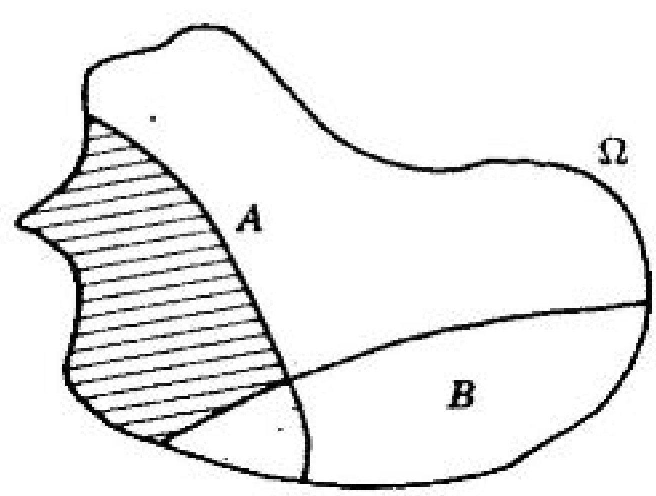

Figure 1.1-4. The shaded region is $A - B$.

Example 1.1.9. As an illustration, suppose that Ω is the set of all positive integers, that A is the set of even positive integers, that B is the set of odd positive integers, and that C is the set of positive integers which are powers of 2. Then

$$\begin{cases} A = \{2,4,6,8,10,12,\dots\}, \\ B = \{1,3,5,7,9,11,\dots\}, \\ C = \{1,2,4,8,16,\dots\}, \end{cases} \tag{1.1.10}$$

are each subsets of $\Omega = \{1,2,3,4,5,6,7,8,9,10,11,12,\dots\}$. Simple calculations from Definitions 1.1.1, 1.1.3, 1.1.5, and 1.1.7 yield

$$\begin{cases} A \cup B = \Omega, \\ A \cap B \quad \text{contains no elements,} \\ A \cap C = \{2,4,8,16,\dots\}, \\ B \cap C = \{1\}, \\ \quad \overline{A} = B, \\ \quad \overline{B} = A, \\ \quad \overline{C} = \{3,5,6,7,9,10,11,12,13,14,15,17,\dots\} = \Omega - C, \text{ and} \\ A - C = \{6,10,12,14,18,\dots\}. \end{cases} \tag{1.1.11}$$

PROBLEMS FOR SECTION 1.1

1.1.1 Draw a Venn diagram similar to Figure 1.1-1 or Figure 1.1-4, and on it shade the area corresponding to the set described by: **(a)** $\overline{A}B$; **(b)** $\overline{AB}$; **(c)** $\overline{A} + \overline{B}$; **(d)** $\overline{A} - \overline{B}$. Use a different diagram for each part of this problem.

1.1.2 Let $\Omega = \{1,2,3,4,5,6,7,8,9,10\}$, $A = \{1,2,3,4,5\}$, and $B = \{3,4,5,6,7,8\}$. For each of the sets in Problem 1.1.1, find which numbers are members of that set.

1.1.3 Let $\Omega = \{(x, y)\colon\ x \ge 0,\ y \ge 0\}$, $A = \{(x, y)\colon\ x \ge 0,\ 0 \le y \le 10\}$, $B = \{(x, y)\colon\ 0 \le x \le 10,\ y \ge 0\}$. On a set of coordinate axes, shade the area corresponding to each of the sets in Problem 1.1.1. Write each of the resulting sets in the simplest possible form of the type used to describe A and B in this problem. Use a different diagram for each shaded area.

1.2. ADDITIONAL DEFINITIONS

Definition 1.2.1. **Identity** of sets A and B:

$$A = B \text{ iff } (\text{for each } \omega \in \Omega)\, \omega \in A \text{ iff } \omega \in B. \tag{1.2.2}$$

Definition 1.2.3. A is a **subset** of B, written $A \subseteq B$, if $\omega \in A \Rightarrow \omega \in B$.

Definition 1.2.4. A is a **proper subset** of B, written $A \subsetneqq B$, if $A \subseteq B$ and $A \neq B$ [where, of course, "$A \neq B$" means "not $A = B$"; i.e., the right side of equivalence (1.2.2) is false].

[*Note*: $A \subseteq B$ will sometimes be shortened to $A \subset B$; some authors, however, use the latter notation for $A \subsetneqq B$.]

Definition 1.2.5. The **empty set** (or **null set**), usually written $\varnothing$ or 0, is the unique set such that for all $\omega \in \Omega$, $\omega \notin \varnothing$. (In other words, the empty set is the set containing no points at all. Such a set, $A \cap B$, was encountered in Example 1.1.9.)

Definition 1.2.6. $A \supseteq B$, $A \supset B$, and $A \supsetneqq B$ mean, respectively, $B \subseteq A$, $B \subset A$, and $B \subsetneqq A$.

Definition 1.2.7. Sets A and B are said to be **mutually exclusive** (or **disjoint**) if $A \cap B = \varnothing$.

Definition 1.2.8. Three (or more) sets A, B, C are said to be **pairwise disjoint** (or **mutually exclusive**) if every two of them are disjoint (as in Definition 1.2.7).

Example 1.2.9. At the beginning of Section 1.1 we noted that Venn diagrams are not considered sufficient proof of a theorem of set theory since their "typical" cases may be misleading. Figure 1.2-1 could, for example, be used as a proof of the statement "$A \cup B$ is *not* a subset of B" *if* Venn diagrams were considered to be sufficient proof. As we see in Figure 1.2-2, the statement can be false. We therefore employ Venn diagrams solely as convenient aids to our intuition, recognizing that proofs of theorems about sets cannot rely solely on Venn diagrams.

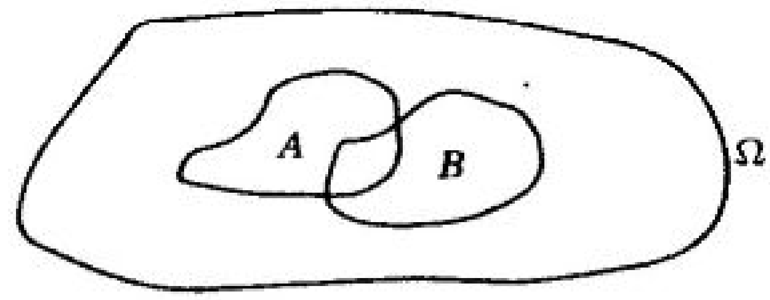

Figure 1.2-1. Is $A \cup B$ a subset of B?

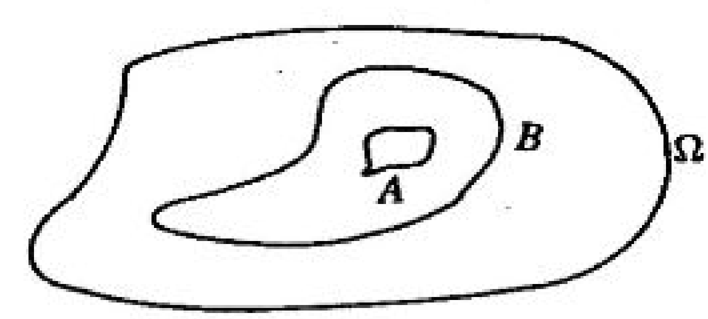

Figure 1.2-2. $A \cup B$ may be a subset of B.

PROBLEMS FOR SECTION 1.2

1.2.1 For the sets A and B of Example 1.1.9, determine whether A and B are disjoint. Similarly for sets B and C. Similarly for sets A and C. Are sets A, B, C pairwise disjoint?

1.2.2 For sets A and B of Example 1.1.9, determine for each of the following whether or not it is a true statement. **(a)** $A = B$. **(b)** $A \subseteq B$. **(c)** $B \subseteq A$. **(d)** $A \neq B$. **(e)** A and B are mutually exclusive.

1.2.3 As in Problem 1.2.2, but for the sets A and B of Problem 1.1.2.

1.2.4 As in Problem 1.2.2, but for the sets A and B of Problem 1.1.3.

1.3. SIMPLE LAWS AND SOME EXTENSIONS

In the following theorem, we state some simple laws dealing with operations (such as union and intersection) on sets and prove one in order to show how a formal proof (as contrasted with an intuitive Venn diagram proof) proceeds. It is known that the set of statements 1 through 9 is **complete** in the sense that any equation formed using $\varnothing$, Ω, $\cap$, $\cup$, and $\subset$ and any number of variables $A, B, C, \ldots$ which is true for every set Ω and subsets $A, B, C, \ldots$ of Ω can be deduced from statements 1 through 9.

Theorem 1.3.1. Let Ω be a set and let A, B, C be subsets of Ω. Then

1. **Idempotency laws**

$$A \cap A = A,$$

$$A \cup A = A.$$

2. **Commutative laws**

$$A \cap B = B \cap A,$$

$$A \cup B = B \cup A.$$

3. **Associative laws**

$$A \cap (B \cap C) = (A \cap B) \cap C,$$

$$A \cup (B \cup C) = (A \cup B) \cup C.$$

4. **Distributive laws**

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C),$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

5. $A \cap \varnothing = \varnothing$, and $A \cup \varnothing = A$.
6. Ω and $\varnothing$ act as **identity elements** for $\cap$ and $\cup$:

$$A \cap \Omega = A,$$

$$A \cup \varnothing = A.$$

7.

$$A \cap A^c = \varnothing,$$

$$A \cup A^c = \Omega.$$

8. De Morgan's laws

$$(A \cap B)^c = A^c \cup B^c,$$

$$(A \cup B)^c = A^c \cap B^c.$$

9.
$$(A^c)^c = A.$$

10.
$$B \subseteq A \text{ iff } A \cap B = B.$$

11.
$$B \subseteq A \text{ iff } A \cup B = A.$$

Proof of the commutative law $A \cap B = B \cap A$. We will prove that $A \cap B \subseteq B \cap A$. Since the proof that $B \cap A \subseteq A \cap B$ is similar, it then follows from Definitions 1.2.1 and 1.2.3 that $A \cap B = B \cap A$. Proof that $A \cap B \subseteq B \cap A$:

$$\begin{aligned} \omega \in A \cap B &\Rightarrow \omega \in A \text{ and } \omega \in B \\ &\Rightarrow \omega \in B \text{ and } \omega \in A \\ &\Rightarrow \omega \in B \cap A. \end{aligned}$$ ■

Extended intersections and unions. Let $\mathscr{M}$ be any nonempty collection of subsets of Ω. Define

$$\bigcap_{A \in \mathscr{M}} A = \{\omega \colon \omega \in A \text{ for every } A \in \mathscr{M}\}, \tag{1.3.2}$$

$$\bigcup_{A \in \mathscr{M}} A = \{\omega \colon \omega \in A \text{ for some } A \in \mathscr{M}\}. \tag{1.3.3}$$

[Thus, (1.3.2) is the collection of those points ω that are in every set $A \in \mathscr{M}$, while (1.3.3) is the collection of those points ω that are in at least one set $A \in \mathscr{M}$. This notation allows us to talk with ease of intersections and unions of infinitely many sets.]

Many of the laws of Theorem 1.3.1 can be generalized to arbitrary unions and intersections. For example, De Morgan's laws become

$$\left(\bigcap_{A \in \mathscr{M}} A\right)^c = \bigcup_{A \in \mathscr{M}} A^c, \tag{1.3.4}$$

$$\left(\bigcup_{A \in \mathscr{M}} A\right)^c = \bigcap_{A \in \mathscr{M}} A^c. \tag{1.3.5}$$

Example 1.3.6. Let Ω be a set and let A, B, C be subsets of Ω. Prove that

$$A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C).$$

We will in fact prove the two sets are equal, as follows:

$$\begin{aligned}\omega \in A \cap (B \cup C) &\Leftrightarrow \omega \in A \text{ and } \omega \in B \cup C\\ &\Leftrightarrow (\omega \in A) \text{ and } (\omega \in B \text{ or } \omega \in C)\\ &\Leftrightarrow (\omega \in A \text{ and } \omega \in B) \text{ or } (\omega \in A \text{ and } \omega \in C)\\ &\Leftrightarrow (\omega \in A \cap B) \text{ or } (\omega \in A \cap C)\\ &\Leftrightarrow \omega \in (A \cap B) \cup (A \cap C).\end{aligned}$$

Example 1.3.7. For the Ω, A, B specified in Example 1.1.9,

$$\begin{cases} A \cap B = \varnothing, \\ A \cup B = \Omega, \\ \qquad \bar{A} = B, \\ \qquad \bar{B} = A. \end{cases} \tag{1.3.8}$$

Hence,

$$\begin{cases} (A \cap B)^c = \Omega = A \cup B = A^c \cup B^c, \\ (A \cup B)^c = \varnothing = A \cap B = A^c \cap B^c, \end{cases} \tag{1.3.9}$$

which verifies some special cases of De Morgan's laws (statement 8 of Theorem 1.3.1).

PROBLEMS FOR SECTION 1.3

1.3.1 For the Ω, A, B, and C specified in Example 1.1.9, directly verify the validity of the following parts of Theorem 1.3.1: **(1)** Idempotency laws; **(2)** Commutative laws; **(3)** Associative laws; **(4)** Distributive laws; **(5)**; **(6)** Identity elements; **(7)**; **(8)** De Morgan's laws; **(9)**; **(10)**; **(11)**.

1.3.2 Directly verify De Morgan's laws for the Ω, A, and B specified in Problem 1.1.2.

1.3.3 Directly verify De Morgan's laws for the Ω, A, and B specified in Problem 1.1.3.

1.4. SETS IN PROBABILITY AND STATISTICS

In probability and statistics we usually study phenomena that are **random** (as opposed to phenomena that are **deterministic**); that is, we wish to consider some **experiment's results** and the experiment does not always produce the same results. We then become involved with the collection of **all possible outcomes** of the experiment, and this functions as our universal set Ω. Subsets $A, B, C, \ldots$ of Ω will then represent possible occurrences, and we will be interested in the "chances" (or probability) of their occurrence; in calculating those "chances," it will often be desirable (and sometimes necessary) to use the manipulations of set theory covered in the preceding sections. Example 1.4.1 details some sets of interest when a pair of dice is rolled. (This example is simply a convenient illustration of the concepts involved; more realistic examples will be covered in due course, as in Problem 1.8.)

Example 1.4.1. Suppose that two six-sided dice are rolled. If each has its sides marked with one through six dots in the usual manner, one way of looking at the experiment is as resulting in a pair of integers (i, j) with $1 \le i, j \le 6$, where i is the number that "shows" on die 1 and j is the number that "shows" on die 2 (see Figure 1.4-1). We may then take our universal set Ω as

$$\Omega = \left\{ \begin{matrix} (1,1),(2,1),(3,1),(4,1),(5,1),(6,1), \\ (1,2),(2,2),(3,2),(4,2),(5,2),(6,2), \\ (1,3),(2,3),(3,3),(4,3),(5,3),(6,3), \\ (1,4),(2,4),(3,4),(4,4),(5,4),(6,4), \\ (1,5),(2,5),(3,5),(4,5),(5,5),(6,5), \\ (1,6),(2,6),(3,6),(4,6),(5,6),(6,6) \end{matrix} \right\}.$$

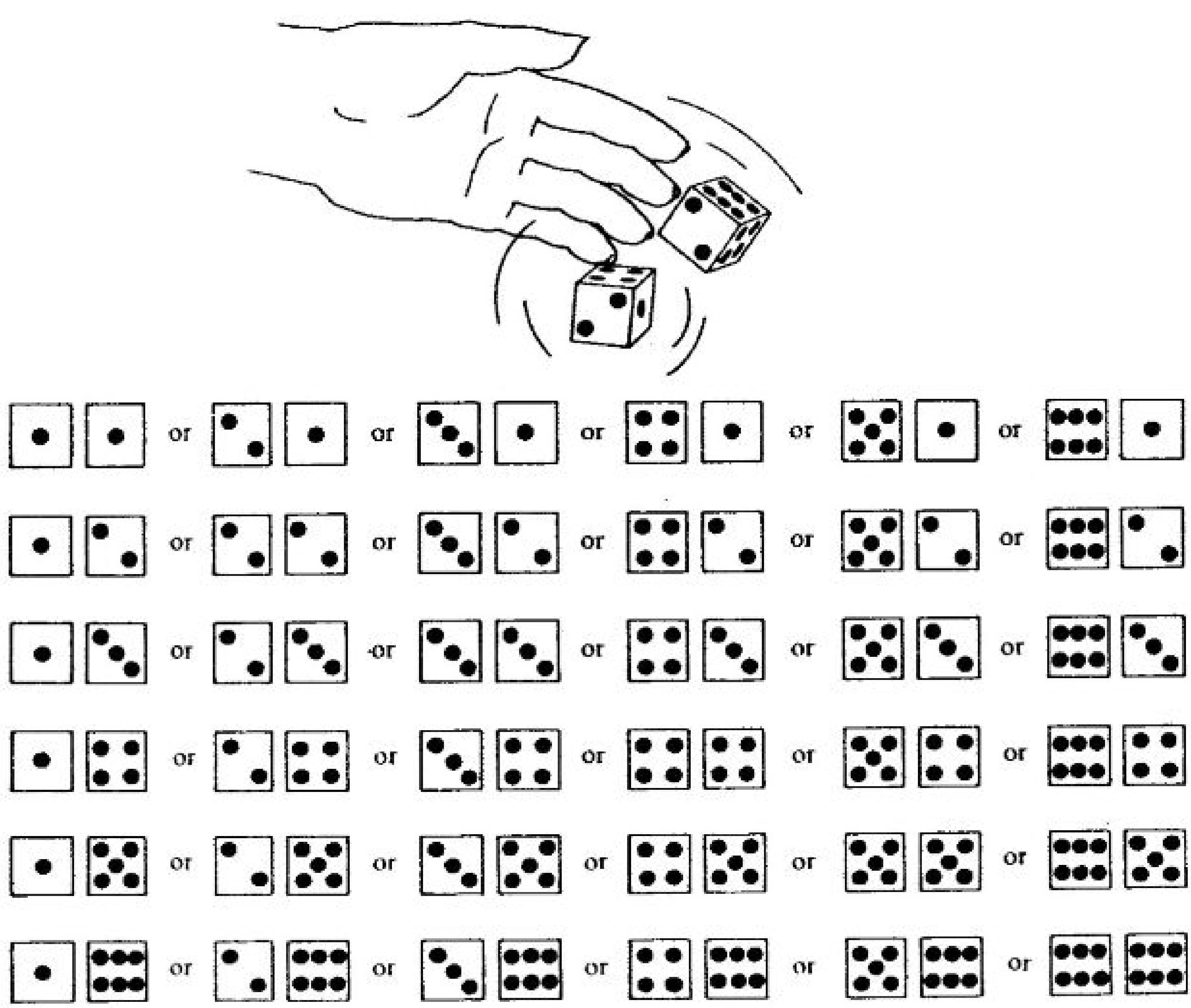

Figure 1.4-1. Results of dice-rolling experiment (with only tops of dice shown in results).

The "sum is 7" when and only when the result of the experiment is a point in

$$A = \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\},$$

which is a subset of Ω. "Neither die shows a 1" when and only when the result of the experiment is a point in

$$B = \left\{\begin{matrix} (2,2),(3,2),(4,2),(5,2),(6,2), \\ (2,3),(3,3),(4,3),(5,3),(6,3), \\ (2,4),(3,4),(4,4),(5,4),(6,4), \\ (2,5),(3,5),(4,5),(5,5),(6,5), \\ (2,6),(3,6),(4,6),(5,6),(6,6) \end{matrix}\right\}.$$

The "sum is 7, and neither die shows a 1" when and only when the result of the experiment is a point in $A \cap B = \{(2,5),(3,4),(4,3),(5,2)\}$. (See Figure 1.4-2.)

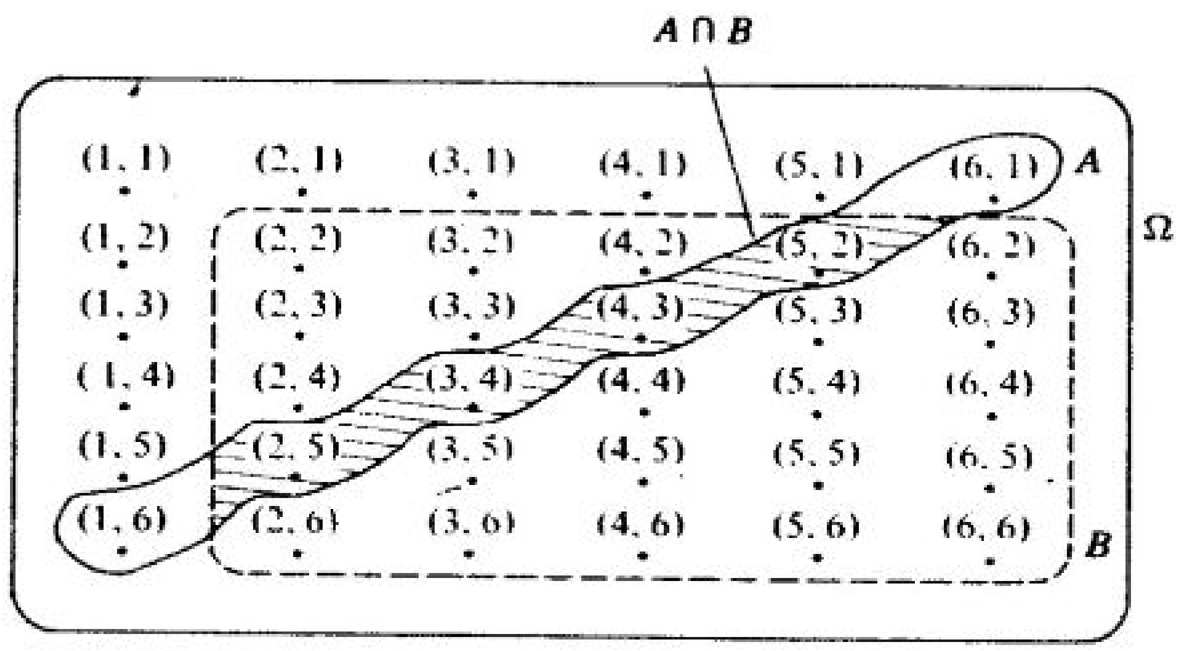

Figure 1.4-2. Ω, A, B, $A \cap B$ for Example 1.4.1.

Example 1.4.2. U.S. Department of Agriculture guidelines suggest that teenagers drink four 8-ounce glasses of milk a day. For skim milk, this will provide 80% of the riboflavin (vitamin B_2) in the U.S. recommended daily allowance (RDA) for adults and children older than four. (From whole milk, 100% of the U.S. RDA will be obtained.)

However, light destroys vitamins, and milk containers are usually stored under fluorescent lights in store display cases. In recent studies (Levey (1982)), milk in paper cartons for 2 days had 2% of its riboflavin destroyed, whereas skim milk in plastic containers for 2 days lost 20% of its riboflavin. (Studies show 37% of milk remains in a store more than 1 day, and, of course, it is exposed to additional light after sale.)

Suppose an experiment is to be run to study nutrition of teenagers, in particular their intake of riboflavin and its relation to the type of milk container bought. If for each teenager the information collected is (1) number of glasses of milk consumed in a given day (0, 1, 2, 3, 4, or 5-or-more), (2) how long that milk was exposed to light in its containers (assume this is 0, 1, or ≥ 2 days), and (3) type of milk container used (plastic or paper), then for each teenager studied the sample space Ω will have 36 possible points (i, j, k), where i denotes the number of glasses of milk consumed, j denotes days of light exposure, and k denotes type of container (0 for plastic, 1 for paper). This is shown in Figure 1.4-3, on which are marked the sets A (4 or 5-or-more glasses a day) and B (paper cartons).

$$\Omega = \left\{ \begin{array}{l} (0,0,0),\ (0,1,0),\ (0,2,0),\ (0,0,1),\ (0,1,1),\ (0,2,1) \\ (1,0,0),\ (1,1,0),\ (1,2,0),\ (1,0,1),\ (1,1,1),\ (1,2,1) \\ (2,0,0),\ (2,1,0),\ (2,2,0),\ (2,0,1),\ (2,1,1),\ (2,2,1) \\ (3,0,0),\ (3,1,0),\ (3,2,0),\ (3,0,1),\ (3,1,1),\ (3,2,1) \\ (4,0,0),\ (4,1,0),\ (4,2,0),\ (4,0,1),\ (4,1,1),\ (4,2,1) \\ (5,0,0),\ (5,1,0),\ (5,2,0),\ (5,0,1),\ (5,1,1),\ (5,2,1) \end{array} \right\}$$

A (4 or 5-or-more glasses a day)

B (paper cartons)

Figure 1.4-3. Sample space Ω and events A, B for nutrition study of Example 1.4.2.

PROBLEMS FOR SECTION 1.4

1.4.1 In Example 1.4.1, another possible choice of the set of all possible outcomes is $\Omega_1 = \{S_2, S_3, S_4, S_5, S_6, S_7, S_8, S_9, S_{10}, S_{11}, S_{12}\}$, where S_i denotes that "the sum of the numbers showing on die 1 and die 2 is i." Is A of Example 1.4.1 ("sum is 7") a subset of Ω_1 (and if so, list its members)? Is B of Example 1.4.1 ("neither die shows a 1") a subset of Ω_1 (and if so, list its members)?

1.4.2 Specify a universal set Ω for all possible outcomes of the experiment of tossing four coins. Be sure that the outcome set "at least two heads, and no tails before the first head" is a subset of the Ω you choose. (As we saw in Problem 1.4.1, in general there are many ways to specify the possible outcomes of an experiment, some containing more detail than others. Here, we wish to have enough detail retained to study outcome sets such as the one specified.)

1.4.3 Specify a universal set Ω for all possible outcomes of the experiment of interviewing families and determining the sexes of their children. Choose the universal set as small as possible, but such that "the family has i boys and j girls" is a subset of the Ω you choose $(i, j = 0, 1, 2, \ldots)$.

1.4.4 List the universal set Ω of Problem 1.4.3 that you would use when it is known that no family has more than 25 children. On that list, mark (as done in Figure 1.4-2, for example) the sets

A: the family has more boys than girls

and

B: the family has enough children to field a baseball team.

[*Note*: A baseball team has nine players.]

PROBLEMS FOR CHAPTER 1[4]

1.1 Sets A, B, C, D are defined by $A = \{w_1, w_2, w_3\}$, $B = \{w_3, w_4, w_5\}$, $C = \{w_6\}$, $D = \{w_1, w_2, w_4, w_5\}$. Find $A \cup B$, $A \cap B$, $S(A, B)$ (see Problem 1.6 for the definition of this set), $A \cap B \cap C \cap D$, and $(B \cap C)^c$.

1.2 Suppose that the Librarian of Congress compiles, for inclusion in the Library of Congress, a bibliography of those bibliographies in the Library of Congress that do not list themselves. Should this bibliography list itself? How does this show that "not every property determines a set?" (Be explicit; e.g., what property?)

1.3 B. Russell discovered that unrestricted formation of sets can lead to a contradiction that, in consequence, has come to be known as **Russell's paradox**. Namely, form the set $A = \{B\colon B \text{ is a set, and } B \notin B\}$. Show that
(a) $A \in A \Rightarrow A \notin A$ (a contradiction).
(b) $A \notin A \Rightarrow A \in A$ (a contradiction).
How was such a contradiction avoided in the formulation of set theory in Chapter 1? (Prove that it was avoided.) [In "axiomatic set theory," there exists a systematization in which one explicitly lists all assumptions needed for a development consistently extending our conception of sets to the realm of infinite sets.]

1.4 **The Barber of Seville** shaves all men who don't shave themselves. Who shaves the Barber of Seville?

1.5 **(Poretsky's law)** Let X and T be given sets. Prove that $X = \varnothing$ iff

$$T = (X \cap T^c) \cup (X^c \cap T).$$

1.6 Let A and B be given sets. Define a new set $S(A, B) = (A - B) \cup (B - A)$, called the **symmetric difference** of A and B. Prove that

$$S(A_1 \cup A_2, B_1 \cup B_2) \subseteq S(A_1, B_1) \cup S(A_2, B_2).$$

[*Note*: The result of Problem 1.5 says $X = \varnothing$ iff $T = S(T, X)$. Thus, introduction of the concept of the symmetric difference of two sets allows for a simplified notation.]

1.7 (*Suggested by P. Zuckerman*) In the definitions of extended intersections and unions, (1.3.2) and (1.3.3), we required $\mathscr{M} \neq \varnothing$. If we make the same definitions but allow $\mathscr{M} = \varnothing$, what sets will $\bigcap_{A \in \mathscr{M}} A$ and $\bigcup_{A \in \mathscr{M}} A$ be identical to when in fact $\mathscr{M} = \varnothing$? Check to see that your answer satisfies De Morgan's laws, (1.3.4) and (1.3.5). (A reference is Halmos, (1950), pp. 13–14.)

1.8 Suppose that one part of an experiment is to consist of observing the random phenomena (such as time of birth) that arise in a human birth. We may regard these phenomena in many different ways, depending on our experiment's purpose. (For example, we could regard the random phenomena as simply "year of birth," or more complexly as "year of birth, month of birth, day of birth, hour of birth, place of birth, weight at birth, sex at birth.")
(a) If the purpose of the experiment is to answer the question "Do more births occur from 1 A.M. to 6 A.M. than from 6 A.M. to 1 A.M.?" what minimal information

[4] Problems 1.2 through 1.4 relate to the paradoxes of set theory mentioned in the introduction to this chapter. For details see Nakano (1969), p. 764, and Takeuti and Zaring (1971), pp. 9–13.

should the statement of results contain? In that case, what is the universal set Ω? How many points does Ω contain?

(b) If the purpose of the experiment is as stated in (a), it may still be desirable to have a more detailed statement of results. Why? Give an example.

1.9 Suppose that three six-sided dice are rolled. If each has its sides marked with one through six dots in the usual manner, what universal set Ω would you, in analogy with Example 1.4.1, choose to represent the outcomes of the experiment? For what subset of Ω is the sum equal to 10? 11?

HONORS PROBLEMS FOR CHAPTER 1

1.1H Let $\Omega = \{1,2,3,4,5,\dots\}$ be the set of all positive integers. Let $\mathscr{F}$ be the set whose members are Ω; $\varnothing$; all subsets of Ω that contain only a finite number of positive integers; and all subsets of Ω whose complements contain only a finite number of positive integers. Prove that $A \in \mathscr{F} \Rightarrow \bar{A} \in \mathscr{F}$, and that $A, B \in \mathscr{F} \Rightarrow A \cup B \in \mathscr{F}$. Prove that if $A_n \in \mathscr{F}$ $(n = 1, 2, \dots)$, it is not necessarily the case that

$$\bigcup_{n=1}^{\infty} A_n \in \mathscr{F}.$$

[*Hint*: Consider the set $C = \{1,3,5,7,\dots\}$ of odd positive integers.] (In terms of the terminology of Problem 2.2, $\mathscr{F}$ is an algebra of sets, but is not a σ-algebra of sets.)

1.2H Let $A_1, \dots, A_n$ be sets each of which contains a finite number of elements. Let

$$\Omega = A_1 \cup A_2 \cup \cdots \cup A_n.$$

Suppose that for some fixed integer k $(1 \le k \le n)$,

$$A_{\lambda_1} \cup \cdots \cup A_{\lambda_k} = \Omega$$

for every specification $(\lambda_1, \dots, \lambda_k)$ of k of the integers $(1, 2, 3, \dots, k, k+1, \dots, n)$, *but* that for $j = 1, 2, \dots, k-1$,

$$A_{\lambda_1} \cup \cdots \cup A_{\lambda_j} \subsetneqq \Omega$$

for every specification $(\lambda_1, \dots, \lambda_j)$ of j of the integers $(1, 2, 3, \dots, k, k+1, \dots, n)$. *Find* formulas (involving k and n) for **(a)** the smallest possible number of elements in Ω; **(b)** the number of elements in each of $A_1, \dots, A_n$ when Ω contains the smallest possible number of elements; and **(c)** the number of elements in $A_{\lambda_1} \cap \cdots \cap A_{\lambda_j}$ when Ω contains the smallest possible number of elements. Illustrate your answers for the case $n = 5$, $k = 4$. [*Hint*: See the problem by H. Lass and P. Gottlieb that is solved in Bloom (1972).]