4

Determinants

The determinant, which has played a prominent role in the theory of linear algebra, is a special scalar-valued function defined on the set of square matrices. Although it still has a place in the study of linear algebra and its applications, its role is less central than in former times. Yet no linear algebra book would be complete without a systematic treatment of the determinant, and we present one here. However, the main use of determinants in this book is to compute and establish the properties of eigenvalues, which we discuss in Chapter 5.

Although the determinant is not a linear transformation on $\mathsf{M}_{n\times n}(F)$ for $n > 1$, it does possess a kind of linearity (called *n-linearity*) as well as other properties that are examined in this chapter. In Section 4.1, we consider the determinant on the set of 2×2 matrices and derive its important properties and develop an efficient computational procedure. To illustrate the important role that determinants play in geometry, we also include optional material that explores the applications of the determinant to the study of area and orientation. In Sections 4.2 and 4.3, we extend the definition of the determinant to all square matrices and derive its important properties and develop an efficient computational procedure. For the reader who prefers to treat determinants lightly, Section 4.4 contains the essential properties that are needed in later chapters. Finally, Section 4.5, which is optional, offers an axiomatic approach to determinants by showing how to characterize the determinant in terms of three key properties.

4.1 DETERMINANTS OF ORDER 2

In this section, we define the determinant of a 2×2 matrix and investigate its geometric significance in terms of area and orientation.

Definition. *If*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is a 2×2 *matrix with entries from a field* F, *then we define the* **determinant** *of* A, *denoted* $\det(A)$ *or* $|A|$, *to be the scalar* $ad - bc$.

Example 1

For the matrices

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 3 & 2 \\ 6 & 4 \end{pmatrix}$$

in $\mathsf{M}_{2\times 2}(R)$, we have

$$\det(A) = 1\cdot 4 - 2\cdot 3 = -2 \text{ and } \det(B) = 3\cdot 4 - 2\cdot 6 = 0. \quad \blacklozenge$$

For the matrices A and B in Example 1, we have

$$A + B = \begin{pmatrix} 4 & 4 \\ 9 & 8 \end{pmatrix},$$

and so

$$\det(A+B) = 4\cdot 8 - 4\cdot 9 = -4.$$

Since $\det(A + B) \neq \det(A) + \det(B)$, the function $\det\colon \mathsf{M}_{2\times 2}(R) \to R$ is *not* a linear transformation. Nevertheless, the determinant does possess an important linearity property, which is explained in the following theorem.

Theorem 4.1. *The function* $\det\colon \mathsf{M}_{2\times 2}(F) \to F$ *is a linear function of each row of a* 2×2 *matrix when the other row is held fixed. That is, if* u, v, *and* w *are in* F^2 *and* k *is a scalar, then*

$$\det\begin{pmatrix} u + kv \\ w \end{pmatrix} = \det\begin{pmatrix} u \\ w \end{pmatrix} + k\det\begin{pmatrix} v \\ w \end{pmatrix}$$

and

$$\det\begin{pmatrix} w \\ u + kv \end{pmatrix} = \det\begin{pmatrix} w \\ u \end{pmatrix} + k\det\begin{pmatrix} w \\ v \end{pmatrix}.$$

Proof. Let $u = (a_1, a_2)$, $v = (b_1, b_2)$, and $w = (c_1, c_2)$ be in F^2 and k be a scalar. Then

$$\det\begin{pmatrix} u \\ w \end{pmatrix} + k\det\begin{pmatrix} v \\ w \end{pmatrix} = \det\begin{pmatrix} a_1 & a_2 \\ c_1 & c_2 \end{pmatrix} + k\det\begin{pmatrix} b_1 & b_2 \\ c_1 & c_2 \end{pmatrix}$$

$$\begin{aligned}
&= (a_1c_2 - a_2c_1) + k(b_1c_2 - b_2c_1)\\
&= (a_1 + kb_1)c_2 - (a_2 + kb_2)c_1\\
&= \det\begin{pmatrix} a_1 + kb_1 & a_2 + kb_2 \\ c_1 & c_2 \end{pmatrix}\\
&= \det\begin{pmatrix} u + kv \\ w \end{pmatrix}.
\end{aligned}$$

A similar calculation shows that

$$\det\begin{pmatrix} w \\ u \end{pmatrix} + k\det\begin{pmatrix} w \\ v \end{pmatrix} = \det\begin{pmatrix} w \\ u + kv \end{pmatrix}. \qquad \blacksquare$$

For the 2×2 matrices A and B in Example 1, it is easily checked that A is invertible but B is not. Note that $\det(A) \neq 0$ but $\det(B) = 0$. We now show that this property is true in general.

Theorem 4.2. *Let $A \in \mathsf{M}_{2\times2}(F)$. Then the determinant of A is nonzero if and only if A is invertible. Moreover, if A is invertible, then*

$$A^{-1} = \frac{1}{\det(A)}\begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix}.$$

Proof. If $\det(A) \neq 0$, then we can define a matrix

$$M = \frac{1}{\det(A)}\begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix}.$$

A straightforward calculation shows that $AM = MA = I$, and so A is invertible and $M = A^{-1}$.

Conversely, suppose that A is invertible. A remark on page 152 shows that the rank of

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

must be 2. Hence $A_{11} \neq 0$ or $A_{21} \neq 0$. If $A_{11} \neq 0$, add $-A_{21}/A_{11}$ times row 1 of A to row 2 to obtain the matrix

$$\begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} - \dfrac{A_{12}A_{21}}{A_{11}} \end{pmatrix}.$$

Because elementary row operations are rank-preserving by the corollary to Theorem 3.4 (p. 153), it follows that

$$A_{22} - \frac{A_{12}A_{21}}{A_{11}} \neq 0.$$

Therefore $\det(A) = A_{11}A_{22} - A_{12}A_{21} \neq 0$. On the other hand, if $A_{21} \neq 0$, we see that $\det(A) \neq 0$ by adding $-A_{11}/A_{21}$ times row 2 of A to row 1 and applying a similar argument. Thus, in either case, $\det(A) \neq 0$. ∎

In Sections 4.2 and 4.3, we extend the definition of the determinant to $n \times n$ matrices and show that Theorem 4.2 remains true in this more general context. In the remainder of this section, which can be omitted if desired, we explore the geometric significance of the determinant of a 2×2 matrix. In particular, we show the importance of the sign of the determinant in the study of orientation.

The Area of a Parallelogram

By the **angle** between two vectors in R^2, we mean the angle with measure θ ($0 \leq \theta < \pi$) that is formed by the vectors having the same magnitude and direction as the given vectors but emanating from the origin. (See Figure 4.1.)

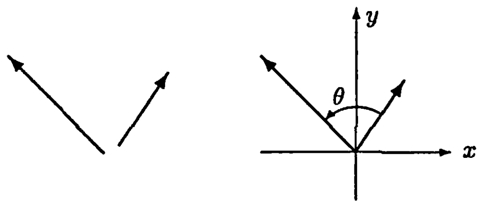

Figure 4.1: Angle between two vectors in R^2

If $\beta = \{u, v\}$ is an ordered basis for R^2, we define the **orientation** of β to be the real number

$$\mathsf{O}\begin{pmatrix} u \\ v \end{pmatrix} = \frac{\det\begin{pmatrix} u \\ v \end{pmatrix}}{\left|\det\begin{pmatrix} u \\ v \end{pmatrix}\right|}.$$

(The denominator of this fraction is nonzero by Theorem 4.2.) Clearly

$$\mathsf{O}\begin{pmatrix} u \\ v \end{pmatrix} = \pm 1.$$

Notice that

$$\mathsf{O}\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = 1 \quad \text{and} \quad \mathsf{O}\begin{pmatrix} e_1 \\ -e_2 \end{pmatrix} = -1.$$

Recall that a coordinate system $\{u, v\}$ is called **right-handed** if u can be rotated in a counterclockwise direction through an angle θ ($0 < \theta < \pi$)

to coincide with v. Otherwise $\{u, v\}$ is called a **left-handed** system. (See Figure 4.2.) In general (see Exercise 12),

A right-handed coordinate system A left-handed coordinate system

Figure 4.2

$$\mathsf{O}\begin{pmatrix} u \\ v \end{pmatrix} = 1$$

if and only if the ordered basis $\{u, v\}$ forms a right-handed coordinate system. For convenience, we also define

$$\mathsf{O}\begin{pmatrix} u \\ v \end{pmatrix} = 1$$

if $\{u, v\}$ is linearly dependent.

Any ordered set $\{u, v\}$ in R^2 determines a parallelogram in the following manner. Regarding u and v as arrows emanating from the origin of R^2, we call the parallelogram having u and v as adjacent sides the **parallelogram determined by** u **and** v. (See Figure 4.3.) Observe that if the set $\{u, v\}$

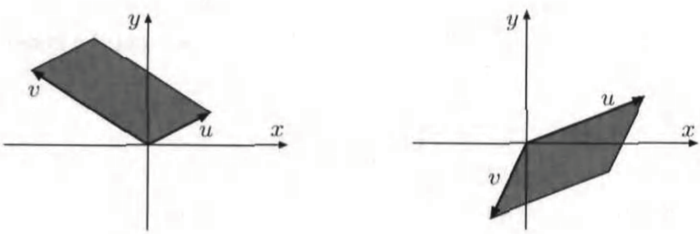

Figure 4.3: Parallelograms determined by u and v

is linearly dependent (i.e., if u and v are parallel), then the "parallelogram" determined by u and v is actually a line segment, which we consider to be a degenerate parallelogram having area zero.

There is an interesting relationship between

$$\mathrm{A}\begin{pmatrix} u \\ v \end{pmatrix},$$

the area of the parallelogram determined by u and v, and

$$\det\begin{pmatrix} u \\ v \end{pmatrix},$$

which we now investigate. Observe first, however, that since

$$\det\begin{pmatrix} u \\ v \end{pmatrix}$$

may be negative, we cannot expect that

$$\mathrm{A}\begin{pmatrix} u \\ v \end{pmatrix} = \det\begin{pmatrix} u \\ v \end{pmatrix}.$$

But we can prove that

$$\mathrm{A}\begin{pmatrix} u \\ v \end{pmatrix} = \mathrm{O}\begin{pmatrix} u \\ v \end{pmatrix} \cdot \det\begin{pmatrix} u \\ v \end{pmatrix},$$

from which it follows that

$$\mathrm{A}\begin{pmatrix} u \\ v \end{pmatrix} = \left|\det\begin{pmatrix} u \\ v \end{pmatrix}\right|.$$

Our argument that

$$\mathrm{A}\begin{pmatrix} u \\ v \end{pmatrix} = \mathrm{O}\begin{pmatrix} u \\ v \end{pmatrix} \cdot \det\begin{pmatrix} u \\ v \end{pmatrix}$$

employs a technique that, although somewhat indirect, can be generalized to R^n. First, since

$$\mathrm{O}\begin{pmatrix} u \\ v \end{pmatrix} = \pm 1,$$

we may multiply both sides of the desired equation by

$$\mathrm{O}\begin{pmatrix} u \\ v \end{pmatrix}$$

to obtain the equivalent form

$$\mathrm{O}\begin{pmatrix} u \\ v \end{pmatrix} \cdot \mathrm{A}\begin{pmatrix} u \\ v \end{pmatrix} = \det\begin{pmatrix} u \\ v \end{pmatrix}.$$

We establish this equation by verifying that the three conditions of Exercise 11 are satisfied by the function

$$\delta\begin{pmatrix} u \\ v \end{pmatrix} = \text{O}\begin{pmatrix} u \\ v \end{pmatrix} \cdot \text{A}\begin{pmatrix} u \\ v \end{pmatrix}.$$

(a) We begin by showing that for any real number c

$$\delta\begin{pmatrix} u \\ cv \end{pmatrix} = c \cdot \delta\begin{pmatrix} u \\ v \end{pmatrix}.$$

Observe that this equation is valid if $c = 0$ because

$$\delta\begin{pmatrix} u \\ cv \end{pmatrix} = \text{O}\begin{pmatrix} u \\ 0 \end{pmatrix} \cdot \text{A}\begin{pmatrix} u \\ 0 \end{pmatrix} = 1 \cdot 0 = 0.$$

So assume that $c \neq 0$. Regarding cv as the base of the parallelogram determined by u and cv, we see that

$$\text{A}\begin{pmatrix} u \\ cv \end{pmatrix} = \text{base} \times \text{altitude} = |c|(\text{length of } v)(\text{altitude}) = |c| \cdot \text{A}\begin{pmatrix} u \\ v \end{pmatrix},$$

since the altitude h of the parallelogram determined by u and cv is the same as that in the parallelogram determined by u and v. (See Figure 4.4.) Hence

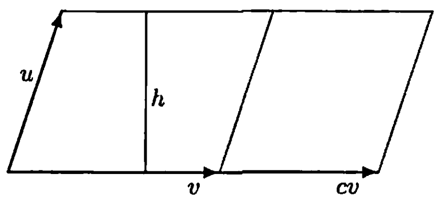

Figure 4.4

$$\delta\begin{pmatrix} u \\ cv \end{pmatrix} = \text{O}\begin{pmatrix} u \\ cv \end{pmatrix} \cdot \text{A}\begin{pmatrix} u \\ cv \end{pmatrix} = \left[\frac{c}{|c|} \cdot \text{O}\begin{pmatrix} u \\ v \end{pmatrix}\right]\left[|c| \cdot \text{A}\begin{pmatrix} u \\ v \end{pmatrix}\right]$$

$$= c \cdot \text{O}\begin{pmatrix} u \\ v \end{pmatrix} \cdot \text{A}\begin{pmatrix} u \\ v \end{pmatrix} = c \cdot \delta\begin{pmatrix} u \\ v \end{pmatrix}.$$

A similar argument shows that

$$\delta\begin{pmatrix} cu \\ v \end{pmatrix} = c \cdot \delta\begin{pmatrix} u \\ v \end{pmatrix}.$$

We next prove that

$$\delta\begin{pmatrix} u \\ au+bw \end{pmatrix} = b\cdot\delta\begin{pmatrix} u \\ w \end{pmatrix}$$

for any $u, w \in \mathsf{R}^2$ and any real numbers a and b. Because the parallelograms determined by u and w and by u and $u+w$ have a common base u and the same altitude (see Figure 4.5), it follows that

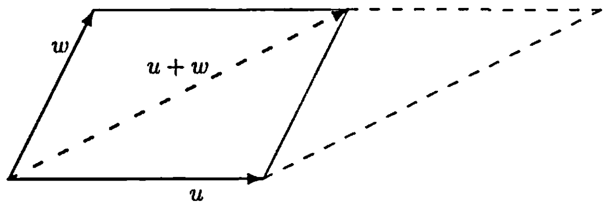

Figure 4.5

$$\mathrm{A}\begin{pmatrix} u \\ w \end{pmatrix} = \mathrm{A}\begin{pmatrix} u \\ u+w \end{pmatrix}.$$

If $a = 0$, then

$$\delta\begin{pmatrix} u \\ au+bw \end{pmatrix} = \delta\begin{pmatrix} u \\ bw \end{pmatrix} = b\cdot\delta\begin{pmatrix} u \\ w \end{pmatrix}$$

by the first paragraph of (a). Otherwise, if $a \neq 0$, then

$$\delta\begin{pmatrix} u \\ au+bw \end{pmatrix} = a\cdot\delta\begin{pmatrix} u \\ u+\dfrac{b}{a}w \end{pmatrix} = a\cdot\delta\begin{pmatrix} u \\ \dfrac{b}{a}w \end{pmatrix} = b\cdot\delta\begin{pmatrix} u \\ w \end{pmatrix}.$$

So the desired conclusion is obtained in either case.

We are now able to show that

$$\delta\begin{pmatrix} u \\ v_1+v_2 \end{pmatrix} = \delta\begin{pmatrix} u \\ v_1 \end{pmatrix} + \delta\begin{pmatrix} u \\ v_2 \end{pmatrix}$$

for all $u, v_1, v_2 \in \mathsf{R}^2$. Since the result is immediate if $u = 0$, we assume that $u \neq 0$. Choose any vector $w \in \mathsf{R}^2$ such that $\{u, w\}$ is linearly independent. Then for any vectors $v_1, v_2 \in \mathsf{R}^2$ there exist scalars a_i and b_i such that $v_i = a_i u + b_i w$ $(i = 1, 2)$. Thus

$$\delta\begin{pmatrix} u \\ v_1+v_2 \end{pmatrix} = \delta\begin{pmatrix} u \\ (a_1+a_2)u+(b_1+b_2)w \end{pmatrix} = (b_1+b_2)\delta\begin{pmatrix} u \\ w \end{pmatrix}$$

$$= \delta\begin{pmatrix} u \\ a_1u + b_1w \end{pmatrix} + \delta\begin{pmatrix} u \\ a_2u + b_2w \end{pmatrix} = \delta\begin{pmatrix} u \\ v_1 \end{pmatrix} + \delta\begin{pmatrix} u \\ v_2 \end{pmatrix}.$$

A similar argument shows that

$$\delta\begin{pmatrix} u_1 + u_2 \\ v \end{pmatrix} = \delta\begin{pmatrix} u_1 \\ v \end{pmatrix} + \delta\begin{pmatrix} u_2 \\ v \end{pmatrix}$$

for all $u_1, u_2, v \in \mathsf{R}^2$.

(b) Since

$$\mathrm{A}\begin{pmatrix} u \\ u \end{pmatrix} = 0, \quad \text{it follows that} \quad \delta\begin{pmatrix} u \\ u \end{pmatrix} = \mathrm{O}\begin{pmatrix} u \\ u \end{pmatrix} \cdot \mathrm{A}\begin{pmatrix} u \\ u \end{pmatrix} = 0$$

for any $u \in \mathsf{R}^2$.

(c) Because the parallelogram determined by e_1 and e_2 is the unit square,

$$\delta\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \mathrm{O}\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \cdot \mathrm{A}\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = 1 \cdot 1 = 1.$$

Therefore δ satisfies the three conditions of Exercise 11, and hence $\delta = \det$. So the area of the parallelogram determined by u and v equals

$$\mathrm{O}\begin{pmatrix} u \\ v \end{pmatrix} \cdot \det\begin{pmatrix} u \\ v \end{pmatrix}.$$

Thus we see, for example, that the area of the parallelogram determined by $u = (-1, 5)$ and $v = (4, -2)$ is

$$\left|\det\begin{pmatrix} u \\ v \end{pmatrix}\right| = \left|\det\begin{pmatrix} -1 & 5 \\ 4 & -2 \end{pmatrix}\right| = 18.$$

EXERCISES

1. Label the following statements as true or false.
   (a) The function $\det\colon \mathsf{M}_{2\times 2}(F) \to F$ is a linear transformation.
   (b) The determinant of a 2×2 matrix is a linear function of each row of the matrix when the other row is held fixed.
   (c) If $A \in \mathsf{M}_{2\times 2}(F)$ and $\det(A) = 0$, then A is invertible.
   (d) If u and v are vectors in R^2 emanating from the origin, then the area of the parallelogram having u and v as adjacent sides is

$$\det\begin{pmatrix} u \\ v \end{pmatrix}.$$

(e) A coordinate system is right-handed if and only if its orientation equals 1.

2. Compute the determinants of the following matrices in $\mathsf{M}_{2\times 2}(R)$.

(a) $\begin{pmatrix} 6 & -3 \\ 2 & 4 \end{pmatrix}$ (b) $\begin{pmatrix} -5 & 2 \\ 6 & 1 \end{pmatrix}$ (c) $\begin{pmatrix} 8 & 0 \\ 3 & -1 \end{pmatrix}$

3. Compute the determinants of the following matrices in $\mathsf{M}_{2\times 2}(C)$.

(a) $\begin{pmatrix} -1+i & 1-4i \\ 3+2i & 2-3i \end{pmatrix}$ (b) $\begin{pmatrix} 5-2i & 6+4i \\ -3+i & 7i \end{pmatrix}$ (c) $\begin{pmatrix} 2i & 3 \\ 4 & 6i \end{pmatrix}$

4. For each of the following pairs of vectors u and v in R^2, compute the area of the parallelogram determined by u and v.

(a) $u = (3, -2)$ and $v = (2, 5)$
(b) $u = (1, 3)$ and $v = (-3, 1)$
(c) $u = (4, -1)$ and $v = (-6, -2)$
(d) $u = (3, 4)$ and $v = (2, -6)$

5. Prove that if B is the matrix obtained by interchanging the rows of a 2×2 matrix A, then $\det(B) = -\det(A)$.

6. Prove that if the two columns of $A \in \mathsf{M}_{2\times 2}(F)$ are identical, then $\det(A) = 0$.

7. Prove that $\det(A^t) = \det(A)$ for any $A \in \mathsf{M}_{2\times 2}(F)$.

8. Prove that if $A \in \mathsf{M}_{2\times 2}(F)$ is upper triangular, then $\det(A)$ equals the product of the diagonal entries of A.

9. Prove that $\det(AB) = \det(A) \cdot \det(B)$ for any $A, B \in \mathsf{M}_{2\times 2}(F)$.

10. The **classical adjoint** of a 2×2 matrix $A \in \mathsf{M}_{2\times 2}(F)$ is the matrix

$$C = \begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix}.$$

Prove that

(a) $CA = AC = [\det(A)]I$.
(b) $\det(C) = \det(A)$.
(c) The classical adjoint of A^t is C^t.
(d) If A is invertible, then $A^{-1} = [\det(A)]^{-1}C$.

11. Let $\delta\colon \mathsf{M}_{2\times 2}(F) \to F$ be a function with the following three properties.

(i) δ is a linear function of each row of the matrix when the other row is held fixed.
(ii) If the two rows of $A \in \mathsf{M}_{2\times 2}(F)$ are identical, then $\delta(A) = 0$.

(iii) If I is the 2×2 identity matrix, then $\delta(I) = 1$.

Prove that $\delta(A) = \det(A)$ for all $A \in \mathsf{M}_{2\times 2}(F)$. (This result is generalized in Section 4.5.)

12. Let $\{u, v\}$ be an ordered basis for R^2. Prove that

$$\mathsf{O}\begin{pmatrix} u \\ v \end{pmatrix} = 1$$

if and only if $\{u, v\}$ forms a right-handed coordinate system. *Hint:* Recall the definition of a rotation given in Example 2 of Section 2.1.

4.2 DETERMINANTS OF ORDER n

In this section, we extend the definition of the determinant to $n \times n$ matrices for $n \geq 3$. For this definition, it is convenient to introduce the following notation: Given $A \in \mathsf{M}_{n\times n}(F)$, for $n \geq 2$, denote the $(n-1) \times (n-1)$ matrix obtained from A by deleting row i and column j by $\tilde{A}_{ij}$. Thus for

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \in \mathsf{M}_{3\times 3}(R),$$

we have

$$\tilde{A}_{11} = \begin{pmatrix} 5 & 6 \\ 8 & 9 \end{pmatrix}, \qquad \tilde{A}_{13} = \begin{pmatrix} 4 & 5 \\ 7 & 8 \end{pmatrix}, \quad \text{and} \quad \tilde{A}_{32} = \begin{pmatrix} 1 & 3 \\ 4 & 6 \end{pmatrix},$$

and for

$$B = \begin{pmatrix} 1 & -1 & 2 & -1 \\ -3 & 4 & 1 & -1 \\ 2 & -5 & -3 & 8 \\ -2 & 6 & -4 & 1 \end{pmatrix} \in \mathsf{M}_{4\times 4}(R),$$

we have

$$\tilde{B}_{23} = \begin{pmatrix} 1 & -1 & -1 \\ 2 & -5 & 8 \\ -2 & 6 & 1 \end{pmatrix} \quad \text{and} \quad \tilde{B}_{42} = \begin{pmatrix} 1 & 2 & -1 \\ -3 & 1 & -1 \\ 2 & -3 & 8 \end{pmatrix}.$$

Definitions. *Let $A \in \mathsf{M}_{n\times n}(F)$. If $n = 1$, so that $A = (A_{11})$, we define $\det(A) = A_{11}$. For $n \geq 2$, we define $\det(A)$ recursively as*

$$\det(A) = \sum_{j=1}^{n} (-1)^{1+j} A_{1j} \cdot \det(\tilde{A}_{1j}).$$

The scalar $\det(A)$ *is called the* **determinant** *of A and is also denoted by* $|A|$. *The scalar*

$$(-1)^{i+j}\det(\tilde{A}_{ij})$$

is called the **cofactor** *of the entry of A in row i, column j.*

Letting

$$c_{ij} = (-1)^{i+j}\det(\tilde{A}_{ij})$$

denote the cofactor of the row i, column j entry of A, we can express the formula for the determinant of A as

$$\det(A) = A_{11}c_{11} + A_{12}c_{12} + \cdots + A_{1n}c_{1n}.$$

Thus the determinant of A equals the sum of the products of each entry in row 1 of A multiplied by its cofactor. This formula is called **cofactor expansion along the first row** of A. Note that, for 2×2 matrices, this definition of the determinant of A agrees with the one given in Section 4.1 because

$$\det(A) = A_{11}(-1)^{1+1}\det(\tilde{A}_{11}) + A_{12}(-1)^{1+2}\det(\tilde{A}_{12}) = A_{11}A_{22} - A_{12}A_{21}.$$

Example 1

Let

$$A = \begin{pmatrix} 1 & 3 & -3 \\ -3 & -5 & 2 \\ -4 & 4 & -6 \end{pmatrix} \in \mathsf{M}_{3\times 3}(R).$$

Using cofactor expansion along the first row of A, we obtain

$$\begin{aligned}
\det(A) &= (-1)^{1+1}A_{11}\cdot\det(\tilde{A}_{11}) + (-1)^{1+2}A_{12}\cdot\det(\tilde{A}_{12}) \\
&\qquad + (-1)^{1+3}A_{13}\cdot\det(\tilde{A}_{13}) \\
&= (-1)^2(1)\cdot\det\begin{pmatrix} -5 & 2 \\ 4 & -6 \end{pmatrix} + (-1)^3(3)\cdot\det\begin{pmatrix} -3 & 2 \\ -4 & -6 \end{pmatrix} \\
&\qquad + (-1)^4(-3)\cdot\det\begin{pmatrix} -3 & -5 \\ -4 & 4 \end{pmatrix} \\
&= 1\,[-5(-6) - 2(4)] - 3\,[-3(-6) - 2(-4)] - 3\,[-3(4) - (-5)(-4)] \\
&= 1(22) - 3(26) - 3(-32) \\
&= 40. \quad \blacklozenge
\end{aligned}$$

Example 2

Let

$$B = \begin{pmatrix} 0 & 1 & 3 \\ -2 & -3 & -5 \\ 4 & -4 & 4 \end{pmatrix} \in \mathsf{M}_{3\times 3}(R).$$

Using cofactor expansion along the first row of B, we obtain

$$\begin{aligned}
\det(B) &= (-1)^{1+1}B_{11}\cdot\det(\tilde{B}_{11}) + (-1)^{1+2}B_{12}\cdot\det(\tilde{B}_{12}) \\
&\qquad + (-1)^{1+3}B_{13}\cdot\det(\tilde{B}_{13}) \\
&= (-1)^2(0)\cdot\det\begin{pmatrix} -3 & -5 \\ -4 & 4 \end{pmatrix} + (-1)^3(1)\cdot\det\begin{pmatrix} -2 & -5 \\ 4 & 4 \end{pmatrix} \\
&\qquad + (-1)^4(3)\cdot\det\begin{pmatrix} -2 & -3 \\ 4 & -4 \end{pmatrix} \\
&= 0 - 1[-2(4) - (-5)(4)] + 3[-2(-4) - (-3)(4)] \\
&= 0 - 1(12) + 3(20) \\
&= 48. \quad \blacklozenge
\end{aligned}$$

Example 3

Let

$$C = \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 3 & -3 \\ -2 & -3 & -5 & 2 \\ 4 & -4 & 4 & -6 \end{pmatrix} \in \mathsf{M}_{4\times 4}(R).$$

Using cofactor expansion along the first row of C and the results of Examples 1 and 2, we obtain

$$\begin{aligned}
\det(C) &= (-1)^2(2)\cdot\det(\tilde{C}_{11}) + (-1)^3(0)\cdot\det(\tilde{C}_{12}) \\
&\qquad + (-1)^4(0)\cdot\det(\tilde{C}_{13}) + (-1)^5(1)\cdot\det(\tilde{C}_{14}) \\
&= (-1)^2(2)\cdot\det\begin{pmatrix} 1 & 3 & -3 \\ -3 & -5 & 2 \\ -4 & 4 & -6 \end{pmatrix} + 0 + 0 \\
&\qquad + (-1)^5(1)\cdot\det\begin{pmatrix} 0 & 1 & 3 \\ -2 & -3 & -5 \\ 4 & -4 & 4 \end{pmatrix} \\
&= 2(40) + 0 + 0 - 1(48) \\
&= 32. \quad \blacklozenge
\end{aligned}$$

Example 4

The determinant of the $n \times n$ identity matrix is 1. We prove this assertion by mathematical induction on n. The result is clearly true for the 1×1 identity matrix. Assume that the determinant of the $(n-1) \times (n-1)$ identity matrix is 1 for some $n \geq 2$, and let I denote the $n \times n$ identity matrix. Using cofactor expansion along the first row of I, we obtain

$$\begin{aligned}\det(I) &= (-1)^2(1)\cdot\det(\tilde{I}_{11}) + (-1)^3(0)\cdot\det(\tilde{I}_{12}) + \cdots \\ &\qquad + (-1)^{1+n}(0)\cdot\det(\tilde{I}_{1n}) \\ &= 1(1) + 0 + \cdots + 0 \\ &= 1\end{aligned}$$

because $\tilde{I}_{11}$ is the $(n-1) \times (n-1)$ identity matrix. This shows that the determinant of the $n \times n$ identity matrix is 1, and so the determinant of any identity matrix is 1 by the principle of mathematical induction. ◆

As is illustrated in Example 3, the calculation of a determinant using the recursive definition is extremely tedious, even for matrices as small as 4×4. Later in this section, we present a more efficient method for evaluating determinants, but we must first learn more about them.

Recall from Theorem 4.1 (p. 200) that, although the determinant of a 2×2 matrix is *not* a linear transformation, it is a linear function of each row when the other row is held fixed. We now show that a similar property is true for determinants of any size.

Theorem 4.3. *The determinant of an $n \times n$ matrix is a linear function of each row when the remaining rows are held fixed. That is, for $1 \leq r \leq n$, we have*

$$\det\begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ u + kv \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix} = \det\begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ u \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix} + k\det\begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ v \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix}$$

whenever k is a scalar and u, v, and each a_i are row vectors in F^n.

Proof. The proof is by mathematical induction on n. The result is immediate if $n = 1$. Assume that for some integer $n \geq 2$ the determinant of any $(n-1) \times (n-1)$ matrix is a linear function of each row when the remaining rows are held fixed. Let A be an $n \times n$ matrix with rows $a_1, a_2, \ldots, a_n$, respectively, and suppose that for some r $(1 \le r \le n)$, we have $a_r = u + kv$ for some $u, v \in \mathsf{F}^n$ and some scalar k. Let $u = (b_1, b_2, \ldots, b_n)$ and $v = (c_1, c_2, \ldots, c_n)$, and let B and C be the matrices obtained from A by replacing row r of A by u and v, respectively. We must prove that $\det(A) = \det(B) + k\det(C)$. We leave the proof of this fact to the reader for the case $r = 1$. For $r > 1$ and $1 \le j \le n$, the rows of $\tilde{A}_{1j}$, $\tilde{B}_{1j}$, and $\tilde{C}_{1j}$ are the same except for row $r-1$. Moreover, row $r-1$ of $\tilde{A}_{1j}$ is

$$(b_1 + kc_1, \ldots, b_{j-1} + kc_{j-1}, b_{j+1} + kc_{j+1}, \ldots, b_n + kc_n),$$

which is the sum of row $r-1$ of $\tilde{B}_{1j}$ and k times row $r-1$ of $\tilde{C}_{1j}$. Since $\tilde{B}_{1j}$ and $\tilde{C}_{1j}$ are $(n-1) \times (n-1)$ matrices, we have

$$\det(\tilde{A}_{1j}) = \det(\tilde{B}_{1j}) + k\det(\tilde{C}_{1j})$$

by the induction hypothesis. Thus since $A_{1j} = B_{1j} = C_{1j}$, we have

$$\begin{aligned}
\det(A) &= \sum_{j=1}^{n} (-1)^{1+j} A_{1j} \cdot \det(\tilde{A}_{1j}) \\
&= \sum_{j=1}^{n} (-1)^{1+j} A_{1j} \cdot \left[\det(\tilde{B}_{1j}) + k\det(\tilde{C}_{1j})\right] \\
&= \sum_{j=1}^{n} (-1)^{1+j} A_{1j} \cdot \det(\tilde{B}_{1j}) + k\sum_{j=1}^{n} (-1)^{1+j} A_{1j} \cdot \det(\tilde{C}_{1j}) \\
&= \det(B) + k\det(C).
\end{aligned}$$

This shows that the theorem is true for $n \times n$ matrices, and so the theorem is true for all square matrices by mathematical induction. ∎

Corollary. *If $A \in \mathsf{M}_{n\times n}(F)$ has a row consisting entirely of zeros, then* $\det(A) = 0$.

Proof. See Exercise 24. ∎

The definition of a determinant requires that the determinant of a matrix be evaluated by cofactor expansion along the first row. Our next theorem shows that the determinant of a square matrix can be evaluated by cofactor expansion along *any* row. Its proof requires the following technical result.

Lemma. *Let $B \in \mathsf{M}_{n\times n}(F)$, where $n \ge 2$. If row i of B equals e_k for some k $(1 \le k \le n)$, then $\det(B) = (-1)^{i+k}\det(\tilde{B}_{ik})$.*

Proof. The proof is by mathematical induction on n. The lemma is easily proved for $n = 2$. Assume that for some integer $n \geq 3$, the lemma is true for $(n-1) \times (n-1)$ matrices, and let B be an $n \times n$ matrix in which row i of B equals e_k for some k $(1 \leq k \leq n)$. The result follows immediately from the definition of the determinant if $i = 1$. Suppose therefore that $1 < i \leq n$. For each $j \neq k$ $(1 \leq j \leq n)$, let C_{ij} denote the $(n-2) \times (n-2)$ matrix obtained from B by deleting rows 1 and i and columns j and k. For each j, row $i-1$ of $\tilde{B}_{1j}$ is the following vector in F^{n-1}:

$$\begin{cases} e_{k-1} & \text{if } j < k \\ \mathit{0} & \text{if } j = k \\ e_k & \text{if } j > k. \end{cases}$$

Hence by the induction hypothesis and the corollary to Theorem 4.3, we have

$$\det(\tilde{B}_{1j}) = \begin{cases} (-1)^{(i-1)+(k-1)} \det(C_{ij}) & \text{if } j < k \\ 0 & \text{if } j = k \\ (-1)^{(i-1)+k} \det(C_{ij}) & \text{if } j > k. \end{cases}$$

Therefore

$$\begin{aligned} \det(B) &= \sum_{j=1}^{n} (-1)^{1+j} B_{1j} \cdot \det(\tilde{B}_{1j}) \\ &= \sum_{j<k} (-1)^{1+j} B_{1j} \cdot \det(\tilde{B}_{1j}) + \sum_{j>k} (-1)^{1+j} B_{1j} \cdot \det(\tilde{B}_{1j}) \\ &= \sum_{j<k} (-1)^{1+j} B_{1j} \cdot \left[(-1)^{(i-1)+(k-1)} \det(C_{ij})\right] \\ &\qquad + \sum_{j>k} (-1)^{1+j} B_{1j} \cdot \left[(-1)^{(i-1)+k} \det(C_{ij})\right] \\ &= (-1)^{i+k} \left[\sum_{j<k} (-1)^{1+j} B_{1j} \cdot \det(C_{ij}) \right. \\ &\qquad \left. + \sum_{j>k} (-1)^{1+(j-1)} B_{1j} \cdot \det(C_{ij}) \right]. \end{aligned}$$

Because the expression inside the preceding bracket is the cofactor expansion of $\tilde{B}_{ik}$ along the first row, it follows that

$$\det(B) = (-1)^{i+k} \det(\tilde{B}_{ik}).$$

This shows that the lemma is true for $n \times n$ matrices, and so the lemma is true for all square matrices by mathematical induction. ∎

We are now able to prove that cofactor expansion along any row can be used to evaluate the determinant of a square matrix.

Theorem 4.4. *The determinant of a square matrix can be evaluated by cofactor expansion along any row. That is, if $A \in \mathsf{M}_{n\times n}(F)$, then for any integer i $(1 \le i \le n)$,*

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} A_{ij} \cdot \det(\tilde{A}_{ij}).$$

Proof. Cofactor expansion along the first row of A gives the determinant of A by definition. So the result is true if $i = 1$. Fix $i > 1$. Row i of A can be written as $\sum_{j=1}^{n} A_{ij} e_j$. For $1 \le j \le n$, let B_j denote the matrix obtained from A by replacing row i of A by e_j. Then by Theorem 4.3 and the lemma, we have

$$\det(A) = \sum_{j=1}^{n} A_{ij} \det(B_j) = \sum_{j=1}^{n} (-1)^{i+j} A_{ij} \cdot \det(\tilde{A}_{ij}). \qquad \blacksquare$$

Corollary. *If $A \in \mathsf{M}_{n\times n}(F)$ has two identical rows, then* $\det(A) = 0$.

Proof. The proof is by mathematical induction on n. We leave the proof of the result to the reader in the case that $n = 2$. Assume that for some integer $n \ge 3$, it is true for $(n-1) \times (n-1)$ matrices, and let rows r and s of $A \in \mathsf{M}_{n\times n}(F)$ be identical for $r \ne s$. Because $n \ge 3$, we can choose an integer i $(1 \le i \le n)$ other than r and s. Now

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} A_{ij} \cdot \det(\tilde{A}_{ij})$$

by Theorem 4.4. Since each $\tilde{A}_{ij}$ is an $(n-1) \times (n-1)$ matrix with two identical rows, the induction hypothesis implies that each $\det(\tilde{A}_{ij}) = 0$, and hence $\det(A) = 0$. This completes the proof for $n \times n$ matrices, and so the lemma is true for all square matrices by mathematical induction. $\blacksquare$

It is possible to evaluate determinants more efficiently by combining cofactor expansion with the use of elementary row operations. Before such a process can be developed, we need to learn what happens to the determinant of a matrix if we perform an elementary row operation on that matrix. Theorem 4.3 provides this information for elementary row operations of type 2 (those in which a row is multiplied by a nonzero scalar). Next we turn our attention to elementary row operations of type 1 (those in which two rows are interchanged).

Theorem 4.5. *If $A \in \mathsf{M}_{n\times n}(F)$ and B is a matrix obtained from A by interchanging any two rows of A, then $\det(B) = -\det(A)$.*

Proof. Let the rows of $A \in \mathsf{M}_{n\times n}(F)$ be $a_1, a_2, \ldots, a_n$, and let B be the matrix obtained from A by interchanging rows r and s, where $r < s$. Thus

$$A = \begin{pmatrix} a_1 \\ \vdots \\ a_r \\ \vdots \\ a_s \\ \vdots \\ a_n \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} a_1 \\ \vdots \\ a_s \\ \vdots \\ a_r \\ \vdots \\ a_n \end{pmatrix}.$$

Consider the matrix obtained from A by replacing rows r and s by $a_r + a_s$. By the corollary to Theorem 4.4 and Theorem 4.3, we have

$$0 = \det\begin{pmatrix} a_1 \\ \vdots \\ a_r + a_s \\ \vdots \\ a_r + a_s \\ \vdots \\ a_n \end{pmatrix} = \det\begin{pmatrix} a_1 \\ \vdots \\ a_r \\ \vdots \\ a_r + a_s \\ \vdots \\ a_n \end{pmatrix} + \det\begin{pmatrix} a_1 \\ \vdots \\ a_s \\ \vdots \\ a_r + a_s \\ \vdots \\ a_n \end{pmatrix}$$

$$= \det\begin{pmatrix} a_1 \\ \vdots \\ a_r \\ \vdots \\ a_r \\ \vdots \\ a_n \end{pmatrix} + \det\begin{pmatrix} a_1 \\ \vdots \\ a_r \\ \vdots \\ a_s \\ \vdots \\ a_n \end{pmatrix} + \det\begin{pmatrix} a_1 \\ \vdots \\ a_s \\ \vdots \\ a_r \\ \vdots \\ a_n \end{pmatrix} + \det\begin{pmatrix} a_1 \\ \vdots \\ a_s \\ \vdots \\ a_s \\ \vdots \\ a_n \end{pmatrix}$$

$$= 0 + \det(A) + \det(B) + 0.$$

Therefore $\det(B) = -\det(A)$. ∎

We now complete our investigation of how an elementary row operation affects the determinant of a matrix by showing that elementary row operations of type 3 do not change the determinant of a matrix.

Theorem 4.6. *Let $A \in \mathsf{M}_{n\times n}(F)$, and let B be a matrix obtained by adding a multiple of one row of A to another row of A. Then $\det(B) = \det(A)$.*

Proof. Suppose that B is the $n \times n$ matrix obtained from A by adding k times row r to row s, where $r \neq s$. Let the rows of A be $a_1, a_2, \ldots, a_n$, and the rows of B be $b_1, b_2, \ldots, b_n$. Then $b_i = a_i$ for $i \neq s$ and $b_s = a_s + ka_r$. Let C be the matrix obtained from A by replacing row s with a_r. Applying Theorem 4.3 to row s of B, we obtain

$$\det(B) = \det(A) + k\det(C) = \det(A)$$

because $\det(C) = 0$ by the corollary to Theorem 4.4. ∎

In Theorem 4.2 (p. 201), we proved that a 2×2 matrix is invertible if and only if its determinant is nonzero. As a consequence of Theorem 4.6, we can prove half of the promised generalization of this result in the following corollary. The converse is proved in the corollary to Theorem 4.7.

Corollary. *If $A \in \mathsf{M}_{n\times n}(F)$ has rank less than n, then* $\det(A) = 0$.

Proof. If the rank of A is less than n, then the rows $a_1, a_2, \ldots, a_n$ of A are linearly dependent. By Exercise 14 of Section 1.5, some row of A, say, row r, is a linear combination of the other rows. So there exist scalars c_i such that

$$a_r = c_1 a_1 + \cdots + c_{r-1} a_{r-1} + c_{r+1} a_{r+1} + \cdots + c_n a_n.$$

Let B be the matrix obtained from A by adding $-c_i$ times row i to row r for each $i \neq r$. Then row r of B consists entirely of zeros, and so $\det(B) = 0$. But by Theorem 4.6, $\det(B) = \det(A)$. Hence $\det(A) = 0$. ∎

The following rules summarize the effect of an elementary row operation on the determinant of a matrix $A \in \mathsf{M}_{n\times n}(F)$.

(a) If B is a matrix obtained by interchanging any two rows of A, then $\det(B) = -\det(A)$.
(b) If B is a matrix obtained by multiplying a row of A by a nonzero scalar k, then $\det(B) = k\det(A)$.
(c) If B is a matrix obtained by adding a multiple of one row of A to another row of A, then $\det(B) = \det(A)$.

These facts can be used to simplify the evaluation of a determinant. Consider, for instance, the matrix in Example 1:

$$A = \begin{pmatrix} 1 & 3 & -3 \\ -3 & -5 & 2 \\ -4 & 4 & -6 \end{pmatrix}.$$

Adding 3 times row 1 of A to row 2 and 4 times row 1 to row 3, we obtain

$$M = \begin{pmatrix} 1 & 3 & -3 \\ 0 & 4 & -7 \\ 0 & 16 & -18 \end{pmatrix}.$$

Since M was obtained by performing two type 3 elementary row operations on A, we have $\det(A) = \det(M)$. The cofactor expansion of M along the first row gives

$$\det(M) = (-1)^{1+1}(1)\cdot\det(\tilde{M}_{11}) + (-1)^{1+2}(3)\cdot\det(\tilde{M}_{12}) + (-1)^{1+3}(-3)\cdot\det(\tilde{M}_{13}).$$

Both $\tilde{M}_{12}$ and $\tilde{M}_{13}$ have a column consisting entirely of zeros, and so $\det(\tilde{M}_{12}) = \det(\tilde{M}_{13}) = 0$ by the corollary to Theorem 4.6. Hence

$$\begin{aligned}\det(M) &= (-1)^{1+1}(1)\cdot\det(\tilde{M}_{11})\\ &= (-1)^{1+1}(1)\cdot\det\begin{pmatrix}4 & -7\\ 16 & -18\end{pmatrix}\\ &= 1[4(-18) - (-7)(16)] = 40.\end{aligned}$$

Thus with the use of two elementary row operations of type 3, we have reduced the computation of $\det(A)$ to the evaluation of one determinant of a 2×2 matrix.

But we can do even better. If we add -4 times row 2 of M to row 3 (another elementary row operation of type 3), we obtain

$$P = \begin{pmatrix}1 & 3 & -3\\ 0 & 4 & -7\\ 0 & 0 & 10\end{pmatrix}.$$

Evaluating $\det(P)$ by cofactor expansion along the first row, we have

$$\begin{aligned}\det(P) &= (-1)^{1+1}(1)\cdot\det(\tilde{P}_{11})\\ &= (-1)^{1+1}(1)\cdot\det\begin{pmatrix}4 & -7\\ 0 & 10\end{pmatrix} = 1\cdot 4\cdot 10 = 40,\end{aligned}$$

as described earlier. Since $\det(A) = \det(M) = \det(P)$, it follows that $\det(A) = 40$.

The preceding calculation of $\det(P)$ illustrates an important general fact. *The determinant of an upper triangular matrix is the product of its diagonal entries.* (See Exercise 23.) By using elementary row operations of types 1 and 3 only, we can transform any square matrix into an upper triangular matrix, and so we can easily evaluate the determinant of any square matrix. The next two examples illustrate this technique.

Example 5

To evaluate the determinant of the matrix

$$B = \begin{pmatrix}0 & 1 & 3\\ -2 & -3 & -5\\ 4 & -4 & 4\end{pmatrix}$$

in Example 2, we must begin with a row interchange. Interchanging rows 1 and 2 of B produces

$$C = \begin{pmatrix} -2 & -3 & -5 \\ 0 & 1 & 3 \\ 4 & -4 & 4 \end{pmatrix}.$$

By means of a sequence of elementary row operations of type 3, we can transform C into an upper triangular matrix:

$$\begin{pmatrix} -2 & -3 & -5 \\ 0 & 1 & 3 \\ 4 & -4 & 4 \end{pmatrix} \longrightarrow \begin{pmatrix} -2 & -3 & -5 \\ 0 & 1 & 3 \\ 0 & -10 & -6 \end{pmatrix} \longrightarrow \begin{pmatrix} -2 & -3 & -5 \\ 0 & 1 & 3 \\ 0 & 0 & 24 \end{pmatrix}.$$

Thus $\det(C) = -2 \cdot 1 \cdot 24 = -48$. Since C was obtained from B by an interchange of rows, it follows that

$$\det(B) = -\det(C) = 48. \quad \blacklozenge$$

Example 6

The technique in Example 5 can be used to evaluate the determinant of the matrix

$$C = \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 3 & -3 \\ -2 & -3 & -5 & 2 \\ 4 & -4 & 4 & -6 \end{pmatrix}$$

in Example 3. This matrix can be transformed into an upper triangular matrix by means of the following sequence of elementary row operations of type 3:

$$\begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 3 & -3 \\ -2 & -3 & -5 & 2 \\ 4 & -4 & 4 & -6 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 3 & -3 \\ 0 & -3 & -5 & 3 \\ 0 & -4 & 4 & -8 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 3 & -3 \\ 0 & 0 & 4 & -6 \\ 0 & 0 & 16 & -20 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 3 & -3 \\ 0 & 0 & 4 & -6 \\ 0 & 0 & 0 & 4 \end{pmatrix}.$$

Thus $\det(C) = 2 \cdot 1 \cdot 4 \cdot 4 = 32$. $\blacklozenge$

Using elementary row operations to evaluate the determinant of a matrix, as illustrated in Example 6, is far more efficient than using cofactor expansion. Consider first the evaluation of a 2×2 matrix. Since

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc,$$

the evaluation of the determinant of a 2×2 matrix requires 2 multiplications (and 1 subtraction). For $n \geq 3$, evaluating the determinant of an $n \times n$ matrix by cofactor expansion along any row expresses the determinant as a sum of n products involving determinants of $(n-1) \times (n-1)$ matrices. Thus in all, the evaluation of the determinant of an $n \times n$ matrix by cofactor expansion along any row requires over $n!$ multiplications, whereas evaluating the determinant of an $n \times n$ matrix by elementary row operations as in Examples 5 and 6 can be shown to require only $(n^3 + 2n - 3)/3$ multiplications. To evaluate the determinant of a 20×20 matrix, which is not large by present standards, cofactor expansion along a row requires over $20! \approx 2.4 \times 10^{18}$ multiplications. Thus it would take a computer performing one billion multiplications per second over 77 years to evaluate the determinant of a 20×20 matrix by this method. By contrast, the method using elementary row operations requires only 2679 multiplications for this calculation and would take the same computer less than three-millionths of a second! It is easy to see why most computer programs for evaluating the determinant of an arbitrary matrix do not use cofactor expansion.

In this section, we have defined the determinant of a square matrix in terms of cofactor expansion along the first row. We then showed that the determinant of a square matrix can be evaluated using cofactor expansion along *any* row. In addition, we showed that the determinant possesses a number of special properties, including properties that enable us to calculate $\det(B)$ from $\det(A)$ whenever B is a matrix obtained from A by means of an elementary row operation. These properties enable us to evaluate determinants much more efficiently. In the next section, we continue this approach to discover additional properties of determinants.

EXERCISES

1. Label the following statements as true or false.
   - (a) The function $\det\colon \mathsf{M}_{n\times n}(F) \to F$ is a linear transformation.
   - (b) The determinant of a square matrix can be evaluated by cofactor expansion along any row.
   - (c) If two rows of a square matrix A are identical, then $\det(A) = 0$.
   - (d) If B is a matrix obtained from a square matrix A by interchanging any two rows, then $\det(B) = -\det(A)$.
   - (e) If B is a matrix obtained from a square matrix A by multiplying a row of A by a scalar, then $\det(B) = \det(A)$.
   - (f) If B is a matrix obtained from a square matrix A by adding k times row i to row j, then $\det(B) = k\det(A)$.
   - (g) If $A \in \mathsf{M}_{n\times n}(F)$ has rank n, then $\det(A) = 0$.
   - (h) The determinant of an upper triangular matrix equals the product of its diagonal entries.

2. Find the value of k that satisfies the following equation:

$$\det\begin{pmatrix} 3a_1 & 3a_2 & 3a_3 \\ 3b_1 & 3b_2 & 3b_3 \\ 3c_1 & 3c_2 & 3c_3 \end{pmatrix} = k\det\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}.$$

3. Find the value of k that satisfies the following equation:

$$\det\begin{pmatrix} 2a_1 & 2a_2 & 2a_3 \\ 3b_1+5c_1 & 3b_2+5c_2 & 3b_3+5c_3 \\ 7c_1 & 7c_2 & 7c_3 \end{pmatrix} = k\det\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}.$$

4. Find the value of k that satisfies the following equation:

$$\det\begin{pmatrix} b_1+c_1 & b_2+c_2 & b_3+c_3 \\ a_1+c_1 & a_2+c_2 & a_3+c_3 \\ a_1+b_1 & a_2+b_2 & a_3+b_3 \end{pmatrix} = k\det\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}.$$

In Exercises 5–12, evaluate the determinant of the given matrix by cofactor expansion along the indicated row.

5. $\begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ 2 & 3 & 0 \end{pmatrix}$
along the first row

6. $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 5 \\ -1 & 3 & 0 \end{pmatrix}$
along the first row

7. $\begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ 2 & 3 & 0 \end{pmatrix}$
along the second row

8. $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 5 \\ -1 & 3 & 0 \end{pmatrix}$
along the third row

9. $\begin{pmatrix} 0 & 1+i & 2 \\ -2i & 0 & 1-i \\ 3 & 4i & 0 \end{pmatrix}$
along the third row

10. $\begin{pmatrix} i & 2+i & 0 \\ -1 & 3 & 2i \\ 0 & -1 & 1-i \end{pmatrix}$
along the second row

11. $\begin{pmatrix} 0 & 2 & 1 & 3 \\ 1 & 0 & -2 & 2 \\ 3 & -1 & 0 & 1 \\ -1 & 1 & 2 & 0 \end{pmatrix}$
along the fourth row

12. $\begin{pmatrix} 1 & -1 & 2 & -1 \\ -3 & 4 & 1 & -1 \\ 2 & -5 & -3 & 8 \\ -2 & 6 & -4 & 1 \end{pmatrix}$
along the fourth row

In Exercises 13–22, evaluate the determinant of the given matrix by any legitimate method.

13. $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$

14. $\begin{pmatrix} 2 & 3 & 4 \\ 5 & 6 & 0 \\ 7 & 0 & 0 \end{pmatrix}$

15. $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$

16. $\begin{pmatrix} -1 & 3 & 2 \\ 4 & -8 & 1 \\ 2 & 2 & 5 \end{pmatrix}$

17. $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 2 & -5 \\ 6 & -4 & 3 \end{pmatrix}$

18. $\begin{pmatrix} 1 & -2 & 3 \\ -1 & 2 & -5 \\ 3 & -1 & 2 \end{pmatrix}$

19. $\begin{pmatrix} i & 2 & -1 \\ 3 & 1+i & 2 \\ -2i & 1 & 4-i \end{pmatrix}$

20. $\begin{pmatrix} -1 & 2+i & 3 \\ 1-i & i & 1 \\ 3i & 2 & -1+i \end{pmatrix}$

21. $\begin{pmatrix} 1 & 0 & -2 & 3 \\ -3 & 1 & 1 & 2 \\ 0 & 4 & -1 & 1 \\ 2 & 3 & 0 & 1 \end{pmatrix}$

22. $\begin{pmatrix} 1 & -2 & 3 & -12 \\ -5 & 12 & -14 & 19 \\ -9 & 22 & -20 & 31 \\ -4 & 9 & -14 & 15 \end{pmatrix}$

23. Prove that the determinant of an upper triangular matrix is the product of its diagonal entries.

24. Prove the corollary to Theorem 4.3.

25. Prove that $\det(kA) = k^n \det(A)$ for any $A \in \mathsf{M}_{n\times n}(F)$.

26. Let $A \in \mathsf{M}_{n\times n}(F)$. Under what conditions is $\det(-A) = \det(A)$?

27. Prove that if $A \in \mathsf{M}_{n\times n}(F)$ has two identical columns, then $\det(A) = 0$.

28. Compute $\det(E_i)$ if E_i is an elementary matrix of type i.

29.† Prove that if E is an elementary matrix, then $\det(E^t) = \det(E)$.

30. Let the rows of $A \in \mathsf{M}_{n\times n}(F)$ be $a_1, a_2, \ldots, a_n$, and let B be the matrix in which the rows are $a_n, a_{n-1}, \ldots, a_1$. Calculate $\det(B)$ in terms of $\det(A)$.

4.3 PROPERTIES OF DETERMINANTS

In Theorem 3.1, we saw that performing an elementary row operation on a matrix can be accomplished by multiplying the matrix by an elementary matrix. This result is very useful in studying the effects on the determinant of applying a sequence of elementary row operations. Because the determinant of the $n\times n$ identity matrix is 1 (see Example 4 in Section 4.2), we can interpret the statements on page 217 as the following facts about the determinants of elementary matrices.

(a) If E is an elementary matrix obtained by interchanging any two rows of I, then $\det(E) = -1$.
(b) If E is an elementary matrix obtained by multiplying some row of I by the nonzero scalar k, then $\det(E) = k$.
(c) If E is an elementary matrix obtained by adding a multiple of some row *of I to another row, then $\det(E) = 1$.*

We now apply these facts about determinants of elementary matrices to prove that the determinant is a *multiplicative* function.

Theorem 4.7. *For any $A, B \in \mathsf{M}_{n\times n}(F)$, $\det(AB) = \det(A)\cdot\det(B)$.*

Proof. We begin by establishing the result when A is an elementary matrix. If A is an elementary matrix obtained by interchanging two rows of I, then $\det(A) = -1$. But by Theorem 3.1 (p. 149), AB is a matrix obtained by interchanging two rows of B. Hence by Theorem 4.5 (p. 216), $\det(AB) = -\det(B) = \det(A)\cdot\det(B)$. Similar arguments establish the result when A is an elementary matrix of type 2 or type 3. (See Exercise 18.)

If A is an $n\times n$ matrix with rank less than n, then $\det(A) = 0$ by the corollary to Theorem 4.6 (p. 216). Since $\operatorname{rank}(AB) \le \operatorname{rank}(A) < n$ by Theorem 3.7 (p. 159), we have $\det(AB) = 0$. Thus $\det(AB) = \det(A)\cdot\det(B)$ in this case.

On the other hand, if A has rank n, then A is invertible and hence is the product of elementary matrices (Corollary 3 to Theorem 3.6 p. 159), say, $A = E_m\cdots E_2E_1$. The first paragraph of this proof shows that

$$\begin{aligned}\det(AB) &= \det(E_m\cdots E_2E_1B)\\ &= \det(E_m)\cdot\det(E_{m-1}\cdots E_2E_1B)\\ &\;\;\vdots\\ &= \det(E_m)\cdot\cdots\cdot\det(E_2)\cdot\det(E_1)\cdot\det(B)\\ &= \det(E_m\cdots E_2E_1)\cdot\det(B)\\ &= \det(A)\cdot\det(B).\end{aligned}$$

∎

Corollary. *A matrix $A \in \mathsf{M}_{n\times n}(F)$ is invertible if and only if $\det(A) \neq 0$. Furthermore, if A is invertible, then $\det(A^{-1}) = \dfrac{1}{\det(A)}$.*

Proof. If $A \in \mathsf{M}_{n\times n}(F)$ is not invertible, then the rank of A is less than n. So $\det(A) = 0$ by the corollary to Theorem 4.6 (p, 217). On the other hand, if $A \in M_{n\times n}(F)$ is invertible, then

$$\det(A)\cdot\det(A^{-1}) = \det(AA^{-1}) = \det(I) = 1$$

by Theorem 4.7. Hence $\det(A) \neq 0$ and $\det(A^{-1}) = \dfrac{1}{\det(A)}$. ∎

In our discussion of determinants until now, we have used only the rows of a matrix. For example, the recursive definition of a determinant involved cofactor expansion along a *row*, and the more efficient method developed in Section 4.2 used elementary *row* operations. Our next result shows that the determinants of A and A^t are always equal. Since the rows of A are the columns of A^t, this fact enables us to translate any statement about determinants that involves the rows of a matrix into a corresponding statement that involves its columns.

Theorem 4.8. *For any $A \in \mathsf{M}_{n\times n}(F)$, $\det(A^t) = \det(A)$.*

Proof. If A is not invertible, then $\operatorname{rank}(A) < n$. But $\operatorname{rank}(A^t) = \operatorname{rank}(A)$ by Corollary 2 to Theorem 3.6 (p. 158), and so A^t is not invertible. Thus $\det(A^t) = 0 = \det(A)$ in this case.

On the other hand, if A is invertible, then A is a product of elementary matrices, say $A = E_m \cdots E_2E_1$. Since $\det(E_i) = \det(E_i^t)$ for every i by Exercise 29 of Section 4.2, by Theorem 4.7 we have

$$\begin{aligned}\det(A^t) &= \det(E_1^tE_2^t\cdots E_m^t)\\ &= \det(E_1^t)\cdot\det(E_2^t)\cdot\cdots\cdot\det(E_m^t)\\ &= \det(E_1)\cdot\det(E_2)\cdot\cdots\cdot\det(E_m)\\ &= \det(E_m)\cdot\cdots\cdot\det(E_2)\cdot\det(E_1)\\ &= \det(E_m\cdots E_2E_1)\\ &= \det(A).\end{aligned}$$

Thus, in either case, $\det(A^t) = \det(A)$. ∎

Among the many consequences of Theorem 4.8 are that determinants can be evaluated by cofactor expansion along a column, and that elementary column operations can be used as well as elementary row operations in evaluating a determinant. (The effect on the determinant of performing an elementary column operation is the same as the effect of performing the corresponding elementary row operation.) We conclude our discussion of determinant properties with a well-known result that relates determinants to the solutions of certain types of systems of linear equations.

Theorem 4.9 (Cramer's Rule). *Let $Ax = b$ be the matrix form of a system of n linear equations in n unknowns, where $x = (x_1, x_2, \ldots, x_n)^t$. If $\det(A) \neq 0$, then this system has a unique solution, and for each k ($k = 1, 2, \ldots, n$),*

$$x_k = \frac{\det(M_k)}{\det(A)},$$

where M_k is the $n \times n$ matrix obtained from A by replacing column k of A by b.

Proof. If $\det(A) \neq 0$, then the system $Ax = b$ has a unique solution by the corollary to Theorem 4.7 and Theorem 3.10 (p. 174). For each integer k ($1 \leq k \leq n$), let a_k denote the kth column of A and X_k denote the matrix obtained from the $n \times n$ identity matrix by replacing column k by x. Then by Theorem 2.13 (p. 90), AX_k is the $n \times n$ matrix whose ith column is

$$Ae_i = a_i \text{ if } i \neq k \qquad \text{and} \qquad Ax = b \text{ if } i = k.$$

Thus $AX_k = M_k$. Evaluating X_k by cofactor expansion along row k produces

$$\det(X_k) = x_k \cdot \det(I_{n-1}) = x_k.$$

Hence by Theorem 4.7,

$$\det(M_k) = \det(AX_k) = \det(A) \cdot \det(X_k) = \det(A) \cdot x_k.$$

Therefore

$$x_k = [\det(A)]^{-1} \cdot \det(M_k). \qquad \blacksquare$$

Example 1

We illustrate Theorem 4.9 by using Cramer's rule to solve the following system of linear equations:

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 2 \\ x_1 + x_3 &= 3 \\ x_1 + x_2 - x_3 &= 1. \end{aligned}$$

The matrix form of this system of linear equations is $Ax = b$, where

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 1 & 1 & -1 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}.$$

Because $\det(A) = 6 \neq 0$, Cramer's rule applies. Using the notation of Theorem 4.9, we have

$$x_1 = \frac{\det(M_1)}{\det(A)} = \frac{\det\begin{pmatrix} 2 & 2 & 3 \\ 3 & 0 & 1 \\ 1 & 1 & -1 \end{pmatrix}}{\det(A)} = \frac{15}{6} = \frac{5}{2},$$

$$x_2 = \frac{\det(M_2)}{\det(A)} = \frac{\det\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 1 \\ 1 & 1 & -1 \end{pmatrix}}{\det(A)} = \frac{-6}{6} = -1,$$

and

$$x_3 = \frac{\det(M_3)}{\det(A)} = \frac{\det\begin{pmatrix} 1 & 2 & 2 \\ 1 & 0 & 3 \\ 1 & 1 & 1 \end{pmatrix}}{\det(A)} = \frac{3}{6} = \frac{1}{2}.$$

Thus the unique solution to the given system of linear equations is

$$(x_1, x_2, x_3) = \left(\frac{5}{2}, -1, \frac{1}{2}\right). \quad \blacklozenge$$

In applications involving systems of linear equations, we sometimes need to know that there is a solution in which the unknowns are integers. In this situation, Cramer's rule can be useful because it implies that a system of linear equations with integral coefficients has an integral solution if the determinant of its coefficient matrix is ± 1. On the other hand, Cramer's rule is not useful for computation because it requires evaluating $n+1$ determinants of $n \times n$ matrices to solve a system of n linear equations in n unknowns. The amount of computation to do this is far greater than that required to solve the system by the method of Gaussian elimination, which was discussed in Section 3.4. Thus Cramer's rule is primarily of theoretical and aesthetic interest, rather than of computational value.

As in Section 4.1, it is possible to interpret the determinant of a matrix $A \in \mathsf{M}_{n\times n}(R)$ geometrically. If the rows of A are $a_1, a_2, \ldots, a_n$, respectively, then $|\det(A)|$ is the **n-dimensional volume** (the generalization of area in R^2 and volume in R^3) of the parallelepiped having the vectors $a_1, a_2, \ldots, a_n$ as adjacent sides. (For a proof of a more generalized result, see Jerrold E. Marsden and Michael J. Hoffman, *Elementary Classical Analysis*, W.H. Freeman and Company, New York, 1993, p. 524.)

Example 2

The volume of the parallelepiped having the vectors $a_1 = (1, -2, 1)$, $a_2 = (1, 0, -1)$, and $a_3 = (1, 1, 1)$ as adjacent sides is

$$\left|\det\begin{pmatrix} 1 & -2 & 1 \\ 1 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix}\right| = 6.$$

Note that the object in question is a rectangular parallelepiped (see Figure 4.6) with sides of lengths $\sqrt{6}$, $\sqrt{2}$, and $\sqrt{3}$. Hence by the familiar formula for volume, its volume should be $\sqrt{6}\cdot\sqrt{2}\cdot\sqrt{3} = 6$, as the determinant calculation shows. $\blacklozenge$

In our earlier discussion of the geometric significance of the determinant formed from the vectors in an ordered basis for R^2, we also saw that this

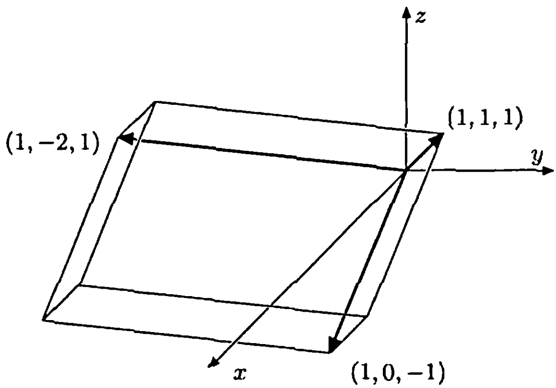

Figure 4.6: Parallelepiped determined by three vectors in R^3.

determinant is positive if and only if the basis induces a right-handed coordinate system. A similar statement is true in R^n. Specifically, if γ is any ordered basis for R^n and β is the standard ordered basis for R^n, then γ induces a *right-handed coordinate system* if and only if $\det(Q) > 0$, where Q is the change of coordinate matrix changing γ-coordinates into β-coordinates. Thus, for instance,

$$\gamma = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

induces a left-handed coordinate system in R^3 because

$$\det \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = -2 < 0,$$

whereas

$$\gamma' = \left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

induces a right-handed coordinate system in R^3 because

$$\det \begin{pmatrix} 1 & -2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 5 > 0.$$

More generally, if β and γ are two ordered bases for R^n, then the coordinate systems induced by β and γ have the same **orientation** (either both are right-handed or both are left-handed) if and only if $\det(Q) > 0$, where Q is *the change of coordinate matrix changing γ-coordinates into β-coordinates.*

EXERCISES

1. Label the following statements as true or false.

(a) If E is an elementary matrix, then $\det(E) = \pm 1$.
(b) For any $A, B \in \mathsf{M}_{n\times n}(F)$, $\det(AB) = \det(A)\cdot\det(B)$.
(c) A matrix $M \in \mathsf{M}_{n\times n}(F)$ is invertible if and only if $\det(M) = 0$.
(d) A matrix $M \in \mathsf{M}_{n\times n}(F)$ has rank n if and only if $\det(M) \neq 0$.
(e) For any $A \in \mathsf{M}_{n\times n}(F)$, $\det(A^t) = -\det(A)$.
(f) The determinant of a square matrix can be evaluated by cofactor expansion along any column.
(g) Every system of n linear equations in n unknowns can be solved by Cramer's rule.
(h) Let $Ax = b$ be the matrix form of a system of n linear equations in n unknowns, where $x = (x_1, x_2, \ldots, x_n)^t$. If $\det(A) \neq 0$ and if M_k is the $n \times n$ matrix obtained from A by replacing row k of A by b^t, then the unique solution of $Ax = b$ is

$$x_k = \frac{\det(M_k)}{\det(A)} \quad \text{for } k = 1, 2, \ldots, n.$$

In Exercises 2–7, use Cramer's rule to solve the given system of linear equations.

2. $$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned}$$
where $a_{11}a_{22} - a_{12}a_{21} \neq 0$

3. $$\begin{aligned} 2x_1 + x_2 - 3x_3 &= 5 \\ x_1 - 2x_2 + x_3 &= 10 \\ 3x_1 + 4x_2 - 2x_3 &= 0 \end{aligned}$$

4. $$\begin{aligned} 2x_1 + x_2 - 3x_3 &= 1 \\ x_1 - 2x_2 + x_3 &= 0 \\ 3x_1 + 4x_2 - 2x_3 &= -5 \end{aligned}$$

5. $$\begin{aligned} x_1 - x_2 + 4x_3 &= -4 \\ -8x_1 + 3x_2 + x_3 &= 8 \\ 2x_1 - x_2 + x_3 &= 0 \end{aligned}$$

6. $$\begin{aligned} x_1 - x_2 + 4x_3 &= -2 \\ -8x_1 + 3x_2 + x_3 &= 0 \\ 2x_1 - x_2 + x_3 &= 6 \end{aligned}$$

7. $$\begin{aligned} 3x_1 + x_2 + x_3 &= 4 \\ -2x_1 - x_2 &= 12 \\ x_1 + 2x_2 + x_3 &= -8 \end{aligned}$$

8. Use Theorem 4.8 to prove a result analogous to Theorem 4.3 (p. 212), but for columns.

9. Prove that an upper triangular $n \times n$ matrix is invertible if and only if all its diagonal entries are nonzero.

10. A matrix $M \in \mathsf{M}_{n\times n}(C)$ is called **nilpotent** if, for some positive integer k, $M^k = O$, where O is the $n \times n$ zero matrix. Prove that if M is nilpotent, then $\det(M) = 0$.

11. A matrix $M \in \mathsf{M}_{n\times n}(C)$ is called **skew-symmetric** if $M^t = -M$. Prove that if M is skew-symmetric and n is odd, then M is not invertible. What happens if n is even?

12. A matrix $Q \in \mathsf{M}_{n\times n}(R)$ is called **orthogonal** if $QQ^t = I$. Prove that if Q is orthogonal, then $\det(Q) = \pm 1$.

13. For $M \in \mathsf{M}_{n\times n}(C)$, let $\overline{M}$ be the matrix such that $(\overline{M})_{ij} = \overline{M_{ij}}$ for all i, j, where $\overline{M_{ij}}$ is the complex conjugate of M_{ij}.

(a) Prove that $\det(\overline{M}) = \overline{\det(M)}$.
(b) A matrix $Q \in \mathsf{M}_{n\times n}(C)$ is called **unitary** if $QQ^* = I$, where $Q^* = \overline{Q^t}$. Prove that if Q is a unitary matrix, then $|\det(Q)| = 1$.

14. Let $\beta = \{u_1, u_2, \ldots, u_n\}$ be a subset of F^n containing n distinct vectors, and let B be the matrix in $\mathsf{M}_{n\times n}(F)$ having u_j as column j. Prove that β is a basis for F^n if and only if $\det(B) \neq 0$.

15.† Prove that if $A, B \in \mathsf{M}_{n\times n}(F)$ are similar, then $\det(A) = \det(B)$.

16. Use determinants to prove that if $A, B \in \mathsf{M}_{n\times n}(F)$ are such that $AB = I$, then A is invertible (and hence $B = A^{-1}$).

17. Let $A, B \in \mathsf{M}_{n\times n}(F)$ be such that $AB = -BA$. Prove that if n is odd and F is not a field of characteristic two, then A or B is not invertible.

18. Complete the proof of Theorem 4.7 by showing that if A is an elementary matrix of type 2 or type 3, then $\det(AB) = \det(A) \cdot \det(B)$.

19. A matrix $A \in \mathsf{M}_{n\times n}(F)$ is called **lower triangular** if $A_{ij} = 0$ for $1 \leq i < j \leq n$. Suppose that A is a lower triangular matrix. Describe $\det(A)$ in terms of the entries of A.

20. Suppose that $M \in \mathsf{M}_{n\times n}(F)$ can be written in the form

$$M = \begin{pmatrix} A & B \\ O & I \end{pmatrix},$$

where A is a square matrix. Prove that $\det(M) = \det(A)$.

21.† Prove that if $M \in \mathsf{M}_{n\times n}(F)$ can be written in the form

$$M = \begin{pmatrix} A & B \\ O & C \end{pmatrix},$$

where A and C are square matrices, then $\det(M) = \det(A) \cdot \det(C)$.

22. Let $\mathsf{T}\colon \mathsf{P}_n(F) \to \mathsf{F}^{n+1}$ be the linear transformation defined in Exercise 22 of Section 2.4 by $\mathsf{T}(f) = (f(c_0), f(c_1), \ldots, f(c_n))$, where $c_0, c_1, \ldots, c_n$ are distinct scalars in an infinite field F. Let β be the standard ordered basis for $\mathsf{P}_n(F)$ and γ be the standard ordered basis for F^{n+1}.

(a) Show that $M = [\mathsf{T}]_\beta^\gamma$ has the form

$$\begin{pmatrix} 1 & c_0 & c_0^2 & \cdots & c_0^n \\ 1 & c_1 & c_1^2 & \cdots & c_1^n \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & c_n & c_n^2 & \cdots & c_n^n \end{pmatrix}.$$

A matrix with this form is called a **Vandermonde matrix**.

(b) Use Exercise 22 of Section 2.4 to prove that $\det(M) \neq 0$.

(c) Prove that

$$\det(M) = \prod_{0 \leq i < j \leq n} (c_j - c_i),$$

the product of all terms of the form $c_j - c_i$ for $0 \leq i < j \leq n$.

23. Let $A \in \mathsf{M}_{n\times n}(F)$ be nonzero. For any m $(1 \leq m \leq n)$, an $m \times m$ **submatrix** is obtained by deleting any $n - m$ rows and any $n - m$ columns of A.

(a) Let k $(1 \leq k \leq n)$ denote the largest integer such that some $k \times k$ submatrix has a nonzero determinant. Prove that $\operatorname{rank}(A) = k$.

(b) Conversely, suppose that $\operatorname{rank}(A) = k$. Prove that there exists a $k \times k$ submatrix with a nonzero determinant.

24. Let $A \in \mathsf{M}_{n\times n}(F)$ have the form

$$A = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & a_0 \\ -1 & 0 & 0 & \cdots & 0 & a_1 \\ 0 & -1 & 0 & \cdots & 0 & a_2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & a_{n-1} \end{pmatrix}.$$

Compute $\det(A + tI)$, where I is the $n \times n$ identity matrix.

25. Let c_{jk} denote the cofactor of the row j, column k entry of the matrix $A \in \mathsf{M}_{n\times n}(F)$.

(a) Prove that if B is the matrix obtained from A by replacing column k by e_j, then $\det(B) = c_{jk}$.

(b) Show that for $1 \le j \le n$, we have

$$A\begin{pmatrix} c_{j1} \\ c_{j2} \\ \vdots \\ c_{jn} \end{pmatrix} = \det(A)\cdot e_j.$$

Hint: Apply Cramer's rule to $Ax = e_j$.

(c) Deduce that if C is the $n \times n$ matrix such that $C_{ij} = c_{ji}$, then $AC = [\det(A)]I$.

(d) Show that if $\det(A) \neq 0$, then $A^{-1} = [\det(A)]^{-1}C$.

The following definition is used in Exercises 26–27.

Definition. *The* ***classical adjoint*** *of a square matrix A is the transpose of the matrix whose ij-entry is the ij-cofactor of A.*

26. Find the classical adjoint of each of the following matrices.

(a) $\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ (b) $\begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$

(c) $\begin{pmatrix} -4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix}$ (d) $\begin{pmatrix} 3 & 6 & 7 \\ 0 & 4 & 8 \\ 0 & 0 & 5 \end{pmatrix}$

(e) $\begin{pmatrix} 1-i & 0 & 0 \\ 4 & 3i & 0 \\ 2i & 1+4i & -1 \end{pmatrix}$ (f) $\begin{pmatrix} 7 & 1 & 4 \\ 6 & -3 & 0 \\ -3 & 5 & -2 \end{pmatrix}$

(g) $\begin{pmatrix} -1 & 2 & 5 \\ 8 & 0 & -3 \\ 4 & 6 & 1 \end{pmatrix}$ (h) $\begin{pmatrix} 3 & 2+i & 0 \\ -1+i & 0 & i \\ 0 & 1 & 3-2i \end{pmatrix}$

27. Let C be the classical adjoint of $A \in \mathsf{M}_{n\times n}(F)$. Prove the following statements.

(a) $\det(C) = [\det(A)]^{n-1}$.

(b) C^t is the classical adjoint of A^t.

(c) If A is an invertible upper triangular matrix, then C and A^{-1} are both upper triangular matrices.

28. Let $y_1, y_2, \ldots, y_n$ be linearly independent functions in C^∞. For each $y \in \mathsf{C}^\infty$, define $\mathsf{T}(y) \in \mathsf{C}^\infty$ by

$$[\mathsf{T}(y)](t) = \det\begin{pmatrix} y(t) & y_1(t) & y_2(t) & \cdots & y_n(t) \\ y'(t) & y_1'(t) & y_2'(t) & \cdots & y_n'(t) \\ \vdots & \vdots & \vdots & & \vdots \\ y^{(n)}(t) & y_1^{(n)}(t) & y_2^{(n)}(t) & \cdots & y_n^{(n)}(t) \end{pmatrix}.$$

The preceding determinant is called the **Wronskian** of $y, y_1, \ldots, y_n$.

(a) Prove that $\mathsf{T}\colon \mathsf{C}^\infty \to \mathsf{C}^\infty$ is a linear transformation.
(b) Prove that $\mathsf{N(T)}$ contains $\operatorname{span}(\{y_1, y_2, \ldots, y_n\})$.

4.4 SUMMARY—IMPORTANT FACTS ABOUT DETERMINANTS

In this section, we summarize the important properties of the determinant needed for the remainder of the text. The results contained in this section have been derived in Sections 4.2 and 4.3; consequently, the facts presented here are stated without proofs.

The **determinant** of an $n \times n$ matrix A having entries from a field F is a scalar in F, denoted by $\det(A)$ or $|A|$, and can be computed in the following manner:

1. If A is 1×1, then $\det(A) = A_{11}$, the single entry of A.

2. If A is 2×2, then $\det(A) = A_{11}A_{22} - A_{12}A_{21}$. For example,

$$\det\begin{pmatrix} -1 & 2 \\ 5 & 3 \end{pmatrix} = (-1)(3) - (2)(5) = -13.$$

3. If A is $n \times n$ for $n > 2$, then

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} A_{ij} \cdot \det(\tilde{A}_{ij})$$

(if the determinant is evaluated by the entries of row i of A) or

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} A_{ij} \cdot \det(\tilde{A}_{ij})$$

(if the determinant is evaluated by the entries of column j of A), where $\tilde{A}_{ij}$ is the $(n-1) \times (n-1)$ matrix obtained by deleting row i and column j from A.

In the formulas above, the scalar $(-1)^{i+j}\det(\tilde{A}_{ij})$ is called the **cofactor** of the row i column j entry of A. In this language, the determinant of A is evaluated as the sum of terms obtained by multiplying each entry of some row or column of A by the cofactor of that entry. Thus $\det(A)$ is expressed in terms of n determinants of $(n-1) \times (n-1)$ matrices. These determinants are then evaluated in terms of determinants of $(n-2) \times (n-2)$ matrices, and so forth, until 2×2 matrices are obtained. The determinants of the 2×2 matrices are then evaluated as in item 2.

Let us consider two examples of this technique in evaluating the determinant of the 4×4 matrix

$$A = \begin{pmatrix} 2 & 1 & 1 & 5 \\ 1 & 1 & -4 & -1 \\ 2 & 0 & -3 & 1 \\ 3 & 6 & 1 & 2 \end{pmatrix}.$$

To evaluate the determinant of A by expanding along the fourth row, we must know the cofactors of each entry of that row. The cofactor of $A_{41} = 3$ is $(-1)^{4+1}\det(B)$, where

$$B = \begin{pmatrix} 1 & 1 & 5 \\ 1 & -4 & -1 \\ 0 & -3 & 1 \end{pmatrix}.$$

Let us evaluate this determinant by expanding along the first column. We have

$$\begin{aligned} \det(B) &= (-1)^{1+1}(1)\det\begin{pmatrix} -4 & -1 \\ -3 & 1 \end{pmatrix} + (-1)^{2+1}(1)\det\begin{pmatrix} 1 & 5 \\ -3 & 1 \end{pmatrix} \\ &\qquad + (-1)^{3+1}(0)\det\begin{pmatrix} 1 & 5 \\ -4 & -1 \end{pmatrix} \\ &= 1(1)[(-4)(1) - (-1)(-3)] + (-1)(1)[(1)(1) - (5)(-3)] + 0 \\ &= -7 - 16 + 0 = -23. \end{aligned}$$

Thus the cofactor of A_{41} is $(-1)^5(-23) = 23$. Similarly, the cofactors of A_{42}, A_{43}, and A_{44} are 8, 11, and -13, respectively. We can now evaluate the determinant of A by multiplying each entry of the fourth row by its cofactor; this gives

$$\det(A) = 3(23) + 6(8) + 1(11) + 2(-13) = 102.$$

For the sake of comparison, let us also compute the determinant of A by expansion along the second column. The reader should verify that the cofactors of A_{12}, A_{22}, and A_{42} are -14, 40, and 8, respectively. Thus

$$\begin{aligned} \det(A) &= (-1)^{1+2}(1)\det\begin{pmatrix} 1 & -4 & -1 \\ 2 & -3 & 1 \\ 3 & 1 & 2 \end{pmatrix} + (-1)^{2+2}(1)\det\begin{pmatrix} 2 & 1 & 5 \\ 2 & -3 & 1 \\ 3 & 1 & 2 \end{pmatrix} \\ &\quad + (-1)^{3+2}(0)\det\begin{pmatrix} 2 & 1 & 5 \\ 1 & -4 & -1 \\ 3 & 1 & 2 \end{pmatrix} + (-1)^{4+2}(6)\det\begin{pmatrix} 2 & 1 & 5 \\ 1 & -4 & -1 \\ 2 & -3 & 1 \end{pmatrix} \\ &= 14 + 40 + 0 + 48 = 102. \end{aligned}$$

Of course, the fact that the value 102 is obtained again is no surprise since the value of the determinant of A is independent of the choice of row or column used in the expansion.

Observe that the computation of $\det(A)$ is easier when expanded along the second column than when expanded along the fourth row. The difference is the presence of a zero in the second column, which makes it unnecessary to evaluate one of the cofactors (the cofactor of A_{32}). For this reason, it is beneficial to evaluate the determinant of a matrix by expanding along a row or column of the matrix that contains the largest number of zero entries. In fact, it is often helpful to introduce zeros into the matrix by means of elementary row operations before computing the determinant. This technique utilizes the first three properties of the determinant.

Properties of the Determinant

1. If B is a matrix obtained by interchanging any two rows or interchanging any two columns of an $n \times n$ matrix A, then $\det(B) = -\det(A)$.
2. If B is a matrix obtained by multiplying each entry of some row or column of an $n \times n$ matrix A by a scalar k, then $\det(B) = k \cdot \det(A)$.
3. If B is a matrix obtained from an $n \times n$ matrix A by adding a multiple of row i to row j or a multiple of column i to column j for $i \neq j$, then $\det(B) = \det(A)$.

As an example of the use of these three properties in evaluating determinants, let us compute the determinant of the 4×4 matrix A considered previously. Our procedure is to introduce zeros into the second column of A by employing property 3, and then to expand along that column. (The elementary row operations used here consist of adding multiples of row 1 to rows 2 and 4.) This procedure yields

$$\det(A) = \det\begin{pmatrix} 2 & 1 & 1 & 5 \\ 1 & 1 & -4 & -1 \\ 2 & 0 & -3 & 1 \\ 3 & 6 & 1 & 2 \end{pmatrix} = \det\begin{pmatrix} 2 & 1 & 1 & 5 \\ -1 & 0 & -5 & -6 \\ 2 & 0 & -3 & 1 \\ -9 & 0 & -5 & -28 \end{pmatrix}$$

$$= 1(-1)^{1+2}\det\begin{pmatrix} -1 & -5 & -6 \\ 2 & -3 & 1 \\ -9 & -5 & -28 \end{pmatrix}.$$

The resulting determinant of a 3×3 matrix can be evaluated in the same manner: Use type 3 elementary row operations to introduce two zeros into the first column, and then expand along that column. This results in the value -102. Therefore

$$\det(A) = 1(-1)^{1+2}(-102) = 102.$$

The reader should compare this calculation of $\det(A)$ with the preceding ones to see how much less work is required when properties 1, 2, and 3 are employed.

In the chapters that follow, we often have to evaluate the determinant of matrices having special forms. The next two properties of the determinant are useful in this regard:

4. The determinant of an upper triangular matrix is the product of its diagonal entries. In particular, $\det(I) = 1$.
5. If two rows (or columns) of a matrix are identical, then the determinant of the matrix is zero.

As an illustration of property 4, notice that

$$\det\begin{pmatrix} -3 & 1 & 2 \\ 0 & 4 & 5 \\ 0 & 0 & -6 \end{pmatrix} = (-3)(4)(-6) = 72.$$

Property 4 provides an efficient method for evaluating the determinant of a matrix:

(a) Use Gaussian elimination and properties 1, 2, and 3 above to reduce the matrix to an upper triangular matrix.

(b) Compute the product of the diagonal entries.

For instance,

$$\det\begin{pmatrix} 1 & -1 & 2 & 1 \\ 2 & -1 & -1 & 4 \\ -4 & 5 & -10 & -6 \\ 3 & -2 & 10 & -1 \end{pmatrix} = \det\begin{pmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -5 & 2 \\ 0 & 1 & -2 & -2 \\ 0 & 1 & 4 & -4 \end{pmatrix}$$

$$= \det\begin{pmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -5 & 2 \\ 0 & 0 & 3 & -4 \\ 0 & 0 & 9 & -6 \end{pmatrix} = \det\begin{pmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -5 & 2 \\ 0 & 0 & 3 & -4 \\ 0 & 0 & 0 & 6 \end{pmatrix}$$

$$= 1 \cdot 1 \cdot 3 \cdot 6 = 18.$$

The next three properties of the determinant are used frequently in later chapters. Indeed, perhaps the most significant property of the determinant is that it provides a simple characterization of invertible matrices. (See property 7.)

6. For any $n \times n$ matrices A and B, $\det(AB) = \det(A) \cdot \det(B)$.

7. An $n \times n$ matrix A is invertible if and only if $\det(A) \neq 0$. Furthermore, if A is invertible, then $\det(A^{-1}) = \dfrac{1}{\det(A)}$.

8. For any $n \times n$ matrix A, the determinants of A and A^t are equal.

For example, property 7 guarantees that the matrix A on page 233 is invertible because $\det(A) = 102$.

The final property, stated as Exercise 15 of Section 4.3, is used in Chapter 5. It is a simple consequence of properties 6 and 7.

9. If A and B are similar matrices, then $\det(A) = \det(B)$.

EXERCISES

1. Label the following statements as true or false.

(a) The determinant of a square matrix may be computed by expanding the matrix along any row or column.
(b) In evaluating the determinant of a matrix, it is wise to expand along a row or column containing the largest number of zero entries.
(c) If two rows or columns of A are identical, then $\det(A) = 0$.
(d) If B is a matrix obtained by interchanging two rows or two columns of A, then $\det(B) = \det(A)$.
(e) If B is a matrix obtained by multiplying each entry of some row or column of A by a scalar, then $\det(B) = \det(A)$.
(f) If B is a matrix obtained from A by adding a multiple of some row to a different row, then $\det(B) = \det(A)$.
(g) The determinant of an upper triangular $n \times n$ matrix is the product of its diagonal entries.
(h) For every $A \in \mathsf{M}_{n\times n}(F)$, $\det(A^t) = -\det(A)$.
(i) If $A, B \in \mathsf{M}_{n\times n}(F)$, then $\det(AB) = \det(A) \cdot \det(B)$.
(j) If Q is an invertible matrix, then $\det(Q^{-1}) = [\det(Q)]^{-1}$.
(k) A matrix Q is invertible if and only if $\det(Q) \neq 0$.

2. Evaluate the determinant of the following 2×2 matrices.

(a) $\begin{pmatrix} 4 & -5 \\ 2 & 3 \end{pmatrix}$ **(b)** $\begin{pmatrix} -1 & 7 \\ 3 & 8 \end{pmatrix}$

(c) $\begin{pmatrix} 2+i & -1+3i \\ 1-2i & 3-i \end{pmatrix}$ **(d)** $\begin{pmatrix} 3 & 4i \\ -6i & 2i \end{pmatrix}$

3. Evaluate the determinant of the following matrices in the manner indicated.

(a) $\begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ 2 & 3 & 0 \end{pmatrix}$ along the first row

(b) $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 5 \\ -1 & 3 & 0 \end{pmatrix}$ along the first column

(c) $\begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ 2 & 3 & 0 \end{pmatrix}$ along the second column

(d) $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 5 \\ -1 & 3 & 0 \end{pmatrix}$ along the third row

(e) $\begin{pmatrix} 0 & 1+i & 2 \\ -2i & 0 & 1-i \\ 3 & 4i & 0 \end{pmatrix}$ along the third row

(f) $\begin{pmatrix} i & 2+i & 0 \\ -1 & 3 & 2i \\ 0 & -1 & 1-i \end{pmatrix}$ along the third column

(g) $\begin{pmatrix} 0 & 2 & 1 & 3 \\ 1 & 0 & -2 & 2 \\ 3 & -1 & 0 & 1 \\ -1 & 1 & 2 & 0 \end{pmatrix}$ along the fourth column

(h) $\begin{pmatrix} 1 & -1 & 2 & -1 \\ -3 & 4 & 1 & -1 \\ 2 & -5 & -3 & 8 \\ -2 & 6 & -4 & 1 \end{pmatrix}$ along the fourth row

4. Evaluate the determinant of the following matrices by any legitimate method.

(a) $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$

(b) $\begin{pmatrix} -1 & 3 & 2 \\ 4 & -8 & 1 \\ 2 & 2 & 5 \end{pmatrix}$

(c) $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 2 & -5 \\ 6 & -4 & 3 \end{pmatrix}$

(d) $\begin{pmatrix} 1 & -2 & 3 \\ -1 & 2 & -5 \\ 3 & -1 & 2 \end{pmatrix}$

(e) $\begin{pmatrix} i & 2 & -1 \\ 3 & 1+i & 2 \\ -2i & 1 & 4-i \end{pmatrix}$

(f) $\begin{pmatrix} -1 & 2+i & 3 \\ 1-i & i & 1 \\ 3i & 2 & -1+i \end{pmatrix}$

(g) $\begin{pmatrix} 1 & 0 & -2 & 3 \\ -3 & 1 & 1 & 2 \\ 0 & 4 & -1 & 1 \\ 2 & 3 & 0 & 1 \end{pmatrix}$

(h) $\begin{pmatrix} 1 & -2 & 3 & -12 \\ -5 & 12 & -14 & 19 \\ -9 & 22 & -20 & 31 \\ -4 & 9 & -14 & 15 \end{pmatrix}$

5. Suppose that $M \in \mathsf{M}_{n\times n}(F)$ can be written in the form

$$M = \begin{pmatrix} A & B \\ O & I \end{pmatrix},$$

where A is a square matrix. Prove that $\det(M) = \det(A)$.

6.† Prove that if $M \in \mathsf{M}_{n\times n}(F)$ can be written in the form

$$M = \begin{pmatrix} A & B \\ O & C \end{pmatrix},$$

where A and C are square matrices, then $\det(M) = \det(A)\cdot\det(C)$.

4.5* A CHARACTERIZATION OF THE DETERMINANT

In Sections 4.2 and 4.3, we showed that the determinant possesses a number of properties. In this section, we show that three of these properties completely characterize the determinant; that is, the only function $\delta\colon \mathsf{M}_{n\times n}(F) \to F$ having these three properties is the determinant. This characterization of the determinant is the one used in Section 4.1 to establish the relationship between $\det\begin{pmatrix} u \\ v \end{pmatrix}$ and the area of the parallelogram determined by u and v. The first of these properties that characterize the determinant is the one described in Theorem 4.3 (p. 212).

Definition. *A function $\delta\colon \mathsf{M}_{n\times n}(F) \to F$ is called an* ***n*-linear function** *if it is a linear function of each row of an $n\times n$ matrix when the remaining $n-1$ rows are held fixed, that is, δ is n-linear if, for every $r = 1, 2, \ldots, n$, we have*

$$\delta\begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ u+kv \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix} = \delta\begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ u \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix} + k\delta\begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ v \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix}$$

whenever k is a scalar and u, v, and each a_i are vectors in F^n.

Example 1

The function $\delta\colon \mathsf{M}_{n\times n}(F) \to F$ defined by $\delta(A) = 0$ for each $A \in \mathsf{M}_{n\times n}(F)$ is an n-linear function. ♦

Example 2

For $1 \le j \le n$, define $\delta_j\colon \mathsf{M}_{n\times n}(F) \to F$ by $\delta_j(A) = A_{1j}A_{2j}\cdots A_{nj}$ for each $A \in \mathsf{M}_{n\times n}(F)$; that is, $\delta_j(A)$ equals the product of the entries of column j of

A. Let $A \in \mathsf{M}_{n\times n}(F)$, $a_i = (A_{i1}, A_{i2}, \ldots, A_{in})$, and $v = (b_1, b_2, \ldots, b_n) \in \mathsf{F}^n$. Then each δ_j is an n-linear function because, for any scalar k, we have

$$\delta\begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ a_r + kv \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix} = A_{1j}\cdots A_{(r-1)j}(A_{rj} + kb_j)A_{(r+1)j}\cdots A_{nj}$$

$$\begin{aligned} &= A_{1j}\cdots A_{(r-1)j}A_{rj}A_{(r+1)j}\cdots A_{nj} \\ &\qquad + A_{1j}\cdots A_{(r-1)j}(kb_j)A_{(r+1)j}\cdots A_{nj} \\ &= A_{1j}\cdots A_{(r-1)j}A_{rj}A_{(r+1)j}\cdots A_{nj} \\ &\qquad + k(A_{1j}\cdots A_{(r-1)j}b_jA_{(r+1)j}\cdots A_{nj}) \\ &= \delta\begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ a_r \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix} + k\delta\begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ v \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix}. \quad \blacklozenge \end{aligned}$$

Example 3

The function $\delta\colon \mathsf{M}_{n\times n}(F) \to F$ defined for each $A \in \mathsf{M}_{n\times n}(F)$ by $\delta(A) = A_{11}A_{22}\cdots A_{nn}$ (i.e., $\delta(A)$ equals the product of the diagonal entries of A) is an n-linear function. ◆

Example 4

The function $\delta\colon \mathsf{M}_{n\times n}(R) \to R$ defined for each $A \in \mathsf{M}_{n\times n}(R)$ by $\delta(A) = \operatorname{tr}(A)$ is not an n-linear function for $n \geq 2$. For if I is the $n \times n$ identity matrix and A is the matrix obtained by multiplying the first row of I by 2, then $\delta(A) = n + 1 \neq 2n = 2\cdot\delta(I)$. ◆

Theorem 4.3 (p. 212) asserts that the determinant is an n-linear function. For our purposes this is the most important example of an n-linear function. Now we introduce the second of the properties used in the characterization of the determinant.

Definition. *An n-linear function $\delta\colon \mathsf{M}_{n\times n}(F) \to F$ is called* **alternating** *if, for each $A \in \mathsf{M}_{n\times n}(F)$, we have $\delta(A) = 0$ whenever two adjacent rows of A are identical.*

Theorem 4.10. *Let $\delta\colon \mathsf{M}_{n\times n}(F) \to F$ be an alternating n-linear function.*

(a) *If $A \in \mathsf{M}_{n\times n}(F)$ and B is a matrix obtained from A by interchanging any two rows of A, then $\delta(B) = -\delta(A)$.*

(b) *If $A \in \mathsf{M}_{n\times n}(F)$ has two identical rows, then $\delta(A) = 0$.*

Proof. (a) Let $A \in \mathsf{M}_{n\times n}(F)$, and let B be the matrix obtained from A by interchanging rows r and s, where $r < s$. We first establish the result in the case that $s = r + 1$. Because $\delta\colon \mathsf{M}_{n\times n}(F) \to F$ is an n-linear function that is alternating, we have

$$0 = \delta\begin{pmatrix} a_1 \\ \vdots \\ a_r + a_{r+1} \\ a_r + a_{r+1} \\ \vdots \\ a_n \end{pmatrix} = \delta\begin{pmatrix} a_1 \\ \vdots \\ a_r \\ a_r + a_{r+1} \\ \vdots \\ a_n \end{pmatrix} + \delta\begin{pmatrix} a_1 \\ \vdots \\ a_{r+1} \\ a_r + a_{r+1} \\ \vdots \\ a_n \end{pmatrix}$$

$$= \delta\begin{pmatrix} a_1 \\ \vdots \\ a_r \\ a_r \\ \vdots \\ a_n \end{pmatrix} + \delta\begin{pmatrix} a_1 \\ \vdots \\ a_r \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix} + \delta\begin{pmatrix} a_1 \\ \vdots \\ a_{r+1} \\ a_r \\ \vdots \\ a_n \end{pmatrix} + \delta\begin{pmatrix} a_1 \\ \vdots \\ a_{r+1} \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix}$$

$$= 0 + \delta(A) + \delta(B) + 0.$$

Thus $\delta(B) = -\delta(A)$.

Next suppose that $s > r + 1$, and let the rows of A be $a_1, a_2, \ldots, a_n$. Beginning with a_r and a_{r+1}, successively interchange a_r with the row that follows it until the rows are in the sequence

$$a_1, a_2, \ldots, a_{r-1}, a_{r+1}, \ldots, a_s, a_r, a_{s+1}, \ldots, a_n.$$

In all, $s-r$ interchanges of adjacent rows are needed to produce this sequence. Then successively interchange a_s with the row that precedes it until the rows are in the order

$$a_1, a_2, \ldots, a_{r-1}, a_s, a_{r+1}, \ldots, a_{s-1}, a_r, a_{s+1}, \ldots, a_n.$$

This process requires an additional $s - r - 1$ interchanges of adjacent rows and produces the matrix B. It follows from the preceding paragraph that

$$\delta(B) = (-1)^{(s-r)+(s-r-1)}\delta(A) = -\delta(A).$$

(b) Suppose that rows r and s of $A \in \mathsf{M}_{n\times n}(F)$ are identical, where $r < s$. If $s = r + 1$, then $\delta(A) = 0$ because δ is alternating and two adjacent rows of A are identical. If $s > r+1$, let B be the matrix obtained from A by interchanging rows $r+1$ and s. Then $\delta(B) = 0$ because two adjacent rows of B are identical. But $\delta(B) = -\delta(A)$ by (a). Hence $\delta(A) = 0$. ∎

Corollary 1. *Let $\delta\colon \mathsf{M}_{n\times n}(F) \to F$ be an alternating n-linear function. If B is a matrix obtained from $A \in \mathsf{M}_{n\times n}(F)$ by adding a multiple of some row of A to another row, then $\delta(B) = \delta(A)$.*

Proof. Let B be obtained from $A \in \mathsf{M}_{n\times n}(F)$ by adding k times row i of A to row j, where $j \neq i$, and let C be obtained from A by replacing row j of A by row i of A. Then the rows of A, B, and C are identical except for row j. Moreover, row j of B is the sum of row j of A and k times row j of C. Since δ is an n-linear function and C has two identical rows, it follows that

$$\delta(B) = \delta(A) + k\delta(C) = \delta(A) + k\cdot 0 = \delta(A).$$ ∎

The next result now follows as in the proof of the corollary to Theorem 4.6 (p. 216). (See Exercise 11.)

Corollary 2. *Let $\delta\colon \mathsf{M}_{n\times n}(F) \to F$ be an alternating n-linear function. If $M \in \mathsf{M}_{n\times n}(F)$ has rank less than n, then $\delta(M) = 0$.*

Proof. Exercise. ∎

Corollary 3. *Let $\delta\colon \mathsf{M}_{n\times n}(F) \to F$ be an alternating n-linear function, and let E_1, E_2, and E_3 in $\mathsf{M}_{n\times n}(F)$ be elementary matrices of types 1, 2, and 3, respectively. Suppose that E_2 is obtained by multiplying some row of I by the nonzero scalar k. Then $\delta(E_1) = -\delta(I)$, $\delta(E_2) = k\cdot\delta(I)$, and $\delta(E_3) = \delta(I)$.*

Proof. Exercise. ∎

We wish to show that under certain circumstances, the only alternating n-linear function $\delta\colon \mathsf{M}_{n\times n}(F) \to F$ is the determinant, that is, $\delta(A) = \det(A)$ for all $A \in \mathsf{M}_{n\times n}(F)$. In view of Corollary 3 to Theorem 4.10 and the facts on page 223 about the determinant of an elementary matrix, this can happen only if $\delta(I) = 1$. Hence the third condition that is used in the characterization of the determinant is that the determinant of the $n \times n$ identity matrix is 1. Before we can establish the desired characterization of the determinant, we must first show that an alternating n-linear function δ such that $\delta(I) = 1$ is a multiplicative function. The proof of this result is identical to the proof of Theorem 4.7 (p. 223), and so it is omitted. (See Exercise 12.)

Theorem 4.11. *Let $\delta\colon \mathsf{M}_{n\times n}(F) \to F$ be an alternating n-linear function such that $\delta(I) = 1$. For any $A, B \in \mathsf{M}_{n\times n}(F)$, we have $\delta(AB) = \delta(A)\cdot\delta(B)$.*

Proof. Exercise. ∎

Theorem 4.12. *If $\delta\colon \mathsf{M}_{n\times n}(F) \to F$ is an alternating n-linear function such that $\delta(I) = 1$, then $\delta(A) = \det(A)$ for every $A \in \mathsf{M}_{n\times n}(F)$.*

Proof. Let $\delta\colon \mathsf{M}_{n\times n}(F) \to F$ be an alternating n-linear function such that $\delta(I) = 1$, and let $A \in \mathsf{M}_{n\times n}(F)$. If A has rank less than n, then by Corollary 2 to Theorem 4.10, $\delta(A) = 0$. Since the corollary to Theorem 4.6 (p. 217) gives $\det(A) = 0$, we have $\delta(A) = \det(A)$ in this case. If, on the other hand, A has rank n, then A is invertible and hence is the product of elementary matrices (Corollary 3 to Theorem 3.6 p. 159), say $A = E_m \cdots E_2E_1$. Since $\delta(I) = 1$, it follows from Corollary 3 to Theorem 4.10 and the facts on page 223 that $\delta(E) = \det(E)$ for every elementary matrix E. Hence by Theorems 4.11 and 4.7 (p. 223), we have

$$\begin{aligned}\delta(A) &= \delta(E_m \cdots E_2E_1)\\ &= \delta(E_m)\cdot \cdots \cdot\delta(E_2)\cdot\delta(E_1)\\ &= \det(E_m)\cdot \cdots \cdot \det(E_2)\cdot \det(E_1)\\ &= \det(E_m \cdots E_2E_1)\\ &= \det(A).\end{aligned}$$ ∎

Theorem 4.12 provides the desired characterization of the determinant: It is the unique function $\delta\colon \mathsf{M}_{n\times n}(F) \to F$ that is n-linear, is alternating, and has the property that $\delta(I) = 1$.

EXERCISES

1. Label the following statements as true or false.
   (a) Any n-linear function $\delta\colon \mathsf{M}_{n\times n}(F) \to F$ is a linear transformation.
   (b) Any n-linear function $\delta\colon \mathsf{M}_{n\times n}(F) \to F$ is a linear function of each row of an $n \times n$ matrix when the other $n-1$ rows are held fixed.
   (c) If $\delta\colon \mathsf{M}_{n\times n}(F) \to F$ is an alternating n-linear function and the matrix $A \in \mathsf{M}_{n\times n}(F)$ has two identical rows, then $\delta(A) = 0$.
   (d) If $\delta\colon \mathsf{M}_{n\times n}(F) \to F$ is an alternating n-linear function and B is obtained from $A \in \mathsf{M}_{n\times n}(F)$ by interchanging two rows of A, then $\delta(B) = \delta(A)$.
   (e) There is a unique alternating n-linear function $\delta\colon \mathsf{M}_{n\times n}(F) \to F$.
   (f) The function $\delta\colon \mathsf{M}_{n\times n}(F) \to F$ defined by $\delta(A) = 0$ for every $A \in \mathsf{M}_{n\times n}(F)$ is an alternating n-linear function.
2. Determine all the 1-linear functions $\delta\colon \mathsf{M}_{1\times 1}(F) \to F$.

Determine which of the functions $\delta\colon \mathsf{M}_{3\times 3}(F) \to F$ in Exercises 3–10 are 3-linear functions. Justify each answer.

3. $\delta(A) = k$, where k is any nonzero scalar

4. $\delta(A) = A_{22}$

5. $\delta(A) = A_{11}A_{23}A_{32}$

6. $\delta(A) = A_{11} + A_{23} + A_{32}$

7. $\delta(A) = A_{11}A_{21}A_{32}$

8. $\delta(A) = A_{11}A_{31}A_{32}$

9. $\delta(A) = A_{11}^2A_{22}^2A_{33}^2$

10. $\delta(A) = A_{11}A_{22}A_{33} - A_{11}A_{21}A_{32}$

11. Prove Corollaries 2 and 3 of Theorem 4.10.

12. Prove Theorem 4.11.

13. Prove that det: $\mathsf{M}_{2\times 2}(F) \to F$ is a 2-linear function of the *columns* of a matrix.

14. Let $a, b, c, d \in F$. Prove that the function $\delta\colon \mathsf{M}_{2\times 2}(F) \to F$ defined by $\delta(A) = A_{11}A_{22}a + A_{11}A_{21}b + A_{12}A_{22}c + A_{12}A_{21}d$ is a 2-linear function.

15. Prove that $\delta\colon \mathsf{M}_{2\times 2}(F) \to F$ is a 2-linear function if and only if it has the form

$$\delta(A) = A_{11}A_{22}a + A_{11}A_{21}b + A_{12}A_{22}c + A_{12}A_{21}d$$

for some scalars $a, b, c, d \in F$.

16. Prove that if $\delta\colon \mathsf{M}_{n\times n}(F) \to F$ is an alternating n-linear function, then there exists a scalar k such that $\delta(A) = k\det(A)$ for all $A \in \mathsf{M}_{n\times n}(F)$.

17. Prove that a linear combination of two n-linear functions is an n-linear function, where the sum and scalar product of n-linear functions are as defined in Example 3 of Section 1.2 (p. 9).

18. Prove that the set of all n-linear functions over a field F is a vector space over F under the operations of function addition and scalar multiplication as defined in Example 3 of Section 1.2 (p. 9).

19. Let $\delta\colon \mathsf{M}_{n\times n}(F) \to F$ be an n-linear function and F a field that does not have characteristic two. Prove that if $\delta(B) = -\delta(A)$ whenever B is obtained from $A \in \mathsf{M}_{n\times n}(F)$ by interchanging any two rows of A, then $\delta(M) = 0$ whenever $M \in \mathsf{M}_{n\times n}(F)$ has two identical rows.

20. Give an example to show that the implication in Exercise 19 need not hold if F has characteristic two.

INDEX OF DEFINITIONS FOR CHAPTER 4