11

Abaci and Antisymmetric Polynomials

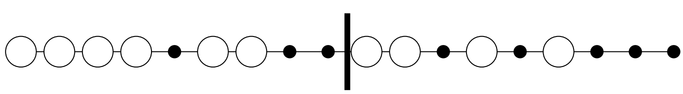
In the last chapter, we used combinatorial operations on tableaux to establish algebraic properties of Schur polynomials and symmetric polynomials. This chapter investigates the interplay between the combinatorics of abaci and the algebraic properties of antisymmetric polynomials. These concepts will be used to establish additional facts about integer partitions and symmetric polynomials. In particular, we will derive some formulas for expanding skew Schur polynomials in terms of various bases.

11.1 Abaci and Integer Partitions

An *abacus* is an instrument used in ancient times for performing arithmetical calculations. The abacus consists of one or more runners that contain sliding beads. The following combinatorial object gives a mathematical model of an abacus.

11.1. Definition: One-Runner Abacus. An *abacus with one runner* is a function $w : \mathbb{Z} \to \{0,1\}$ such that for some m, n, $w_i = 1$ for all $i \leq m$ and $w_i = 0$ for all $i \geq n$. We think of w as an infinite word $\cdots w_{-2}w_{-1}\underline{w_0}w_1w_2w_3\cdots$ that begins with an infinite string of 1's and ends with an infinite string of 0's. Each 1 is called a *bead*, and each 0 is called a *gap*. Let Abc denote the set of all one-runner abaci. An abacus w is called *justified at position* m iff $w_i = 1$ for all $i \leq m$ and $w_i = 0$ for all $i > m$. Intuitively, an abacus is justified iff all the beads have been pushed to the left as far as they will go. The *weight* of an abacus w, denoted $\mathrm{wt}(w)$, is the number of pairs $i < j$ with $w_i < w_j$ (or equivalently, $w_i = 0$ and $w_j = 1$).

11.2. Example. Here is a picture of a one-runner abacus:



This picture corresponds to the mathematical abacus

$$w = \cdots 111101100\underline{1}10101000\cdots,$$

where the underlined 1 is w_0. All positions to the left of the displayed region contain beads, and all positions to the right contain gaps.

Consider the actions required to transform w into a justified abacus. We begin with the bead following the leftmost gap, which slides one position to the left, producing

$$w' = \cdots 111110100\underline{1}10101000\cdots.$$

The next bead now slides into the position vacated by the previous bead, producing

$$w'' = \cdots 111111000\underline{1}10101000\cdots.$$

The next bead moves 3 positions to the left to give the abacus

$$w^{(3)} = \cdots 111111100\underline{0}10101000 \cdots .$$

In the next three steps, the remaining beads move left by 3, 4, and 5 positions, respectively, leading to the abacus

$$w^* = \cdots 111111111\underline{1}00000000 \cdots ,$$

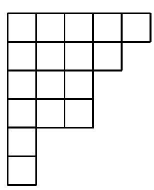
which is justified at position 0. If we list the number of positions that each bead moved, we obtain a weakly increasing sequence: $1 \leq 1 \leq 3 \leq 3 \leq 4 \leq 5$. This sequence can be identified with the integer partition $\lambda = (5, 4, 3, 3, 1, 1)$. Observe that $\mathrm{wt}(w) = 17 = |\lambda|$. This example generalizes as follows.

11.3. Theorem: Partitions vs. Abaci. Justification of abaci defines a bijection $J : \mathrm{Abc} \to \mathbb{Z} \times \mathrm{Par}$ with inverse $U : \mathbb{Z} \times \mathrm{Par} \to \mathrm{Abc}$. If $J(w) = (m, \lambda)$, then $\mathrm{wt}(w) = |\lambda|$.

Proof. Given an abacus w, let n be the least integer with $w_n = 0$ (the position of the leftmost gap), which exists since w begins with an infinite string of 1's. Since w ends with an infinite string of 0's, there are only finitely many $j > n$ with $w_j = 1$; let these indices be $j_1 < j_2 < \cdots < j_t$, where $n < j_1$. We justify the abacus by moving the bead at position j_1 left $\lambda_t = j_1 - n$ places. Then we move the bead at position j_2 left $\lambda_{t-1} = j_2 - (n+1)$ places. (We subtract $n+1$ since the leftmost gap is now at position $n+1$.) In general, at stage k we move the bead at position j_k left $\lambda_{t+1-k} = j_k - (n+k-1)$ places. After moving all t beads, we will have a justified abacus with the leftmost gap located at position $n+t$. Since $n < j_1 < j_2 < \cdots < j_t$, it follows that $0 < \lambda_t \leq \lambda_{t-1} \leq \cdots \leq \lambda_1$. We define $J(w) = (n+t-1, \lambda)$ where $\lambda = (\lambda_1, \ldots, \lambda_t)$. For all k, moving the bead at position j_k left λ_{t+1-k} places decreases the weight of the abacus by λ_{t+1-k}. Since a justified abacus has weight zero, it follows that the weight of the original abacus is precisely $\lambda_t + \cdots + \lambda_1 = |\lambda|$.

J is a bijection because "unjustification" is a two-sided inverse for J. More precisely, given $(m, \mu) \in \mathbb{Z} \times \mathrm{Par}$, we create an abacus $U(m, \mu)$ as follows. Start with an abacus justified at position m. Move the rightmost bead to the right μ_1 places, then move the next bead to the right μ_2 places, and so on. This process reverses the action of J. □

11.4. Remark: Computing U. The unjustification map U can also be computed using partition diagrams. We can reconstruct the bead-gap sequence in the abacus $U(m, \mu)$ by traversing the *frontier* of the diagram of μ (traveling northeast) and recording a gap (0) for each horizontal step and a bead (1) for each vertical step. For example, if $\mu = (5, 4, 3, 3, 1, 1)$, the diagram of μ is



and the bead-gap sequence is 01100110101. To obtain the abacus w, we prepend an infinite string of 1's, append an infinite string of zeroes, and finally use m to determine which symbol in the resulting string is considered to be w_0. One readily checks that this procedure produces the same abacus as the map U in the previous proof. We can also confirm that the map U is weight-preserving via the following bijection between the cells of the diagram of μ and the pairs $i < j$ with $w_i = 0$ and $w_j = 1$. Starting at a cell c, travel south to reach a horizontal edge on the frontier (encoded by some $w_i = 0$). Travel east from c to reach a vertical edge on the frontier (encoded by some $w_j = 1$ with $j > i$). For example, the cell in the second row and third column of the diagram above corresponds to the marked gap-bead pair in the associated abacus:

$$\cdots 0110\hat{0}110\hat{1}01 \cdots .$$

11.2 Jacobi Triple Product Identity

The *Jacobi triple product identity* is a partition identity that has several applications in combinatorics and number theory. We can give a bijective proof of this identity by using cleverly chosen weights on abaci.

11.5. Theorem: Jacobi Triple Product Identity. The following equation holds in the ring $\mathbb{Q}(u)[[q]]$:

$$\sum_{m\in\mathbb{Z}} q^{m(m+1)/2}u^m = \prod_{n\geq 1}(1+uq^n)\prod_{n\geq 0}(1+u^{-1}q^n)\prod_{n\geq 1}(1-q^n).$$

Proof. Since the formal power series $\prod_{n\geq 1}(1-q^n)$ is invertible, it suffices to prove the equivalent identity

$$\sum_{m\in\mathbb{Z}} q^{m(m+1)/2}u^m \prod_{n\geq 1}\frac{1}{1-q^n} = \prod_{n\geq 1}(1+uq^n)\prod_{n\geq 0}(1+u^{-1}q^n). \tag{11.1}$$

Let the weight of an integer m be $\mathrm{wt}(m) = q^{m(m+1)/2}u^m$, and let the weight of a partition μ be $q^{|\mu|}$. Since $\prod_{n\geq 1} 1/(1-q^n) = \sum_{\mu\in\mathrm{Par}} q^{|\mu|}$ by 8.17, the left side of (11.1) is

$$\sum_{(m,\mu)\in\mathbb{Z}\times\mathrm{Par}} \mathrm{wt}(m)\,\mathrm{wt}(\mu),$$

which is the generating function for the weighted set $\mathbb{Z}\times\mathrm{Par}$.

On the other hand, let us define new weights on the set Abc as follows. Given an abacus w, let $N(w) = \{i\leq 0 : w_i = 0\}$ be the set of nonpositive positions in w not containing a bead, and let $P(w) = \{i > 0 : w_i = 1\}$ be the set of positive positions in w containing a bead. Both $N(w)$ and $P(w)$ are finite sets. Define

$$\mathrm{wt}(w) = \prod_{i\in N(w)}(u^{-1}q^{|i|})\prod_{i\in P(w)}(u^1q^i).$$

We can build an abacus by choosing a bead or a gap in each nonpositive position (choosing "bead" all but finitely many times), and then choosing a bead or a gap in each positive position (choosing "gap" all but finitely many times). The generating function for the choice at position $i\leq 0$ is $1+u^{-1}q^{|i|}$, while the generating function for the choice at position $i>0$ is $1+u^1q^i$. By the product rule for weighted sets (see 8.9), the right side of (11.1) is $\sum_{w\in\mathrm{Abc}}\mathrm{wt}(w)$.

To complete the proof, it suffices to argue that the justification bijection $J : \mathrm{Abc}\to\mathbb{Z}\times\mathrm{Par}$ is weight-preserving. Suppose $J(w) = (m,\mu)$ for some abacus w. The map J converts w to an abacus w^*, justified at position m, by $|\mu|$ steps in which some bead moves one position to the left. *Claim 1:* The weight of the justified abacus w^* is $\mathrm{wt}(m) = u^mq^{m(m+1)/2}$. We prove this by considering three cases. When $m=0$, $N(w^*)=\emptyset=P(w^*)$, so $\mathrm{wt}(w^*)=1=\mathrm{wt}(0)$. When $m>0$, $N(w^*)=\emptyset$ and $P(w^*)=\{1,2,\ldots,m\}$, so

$$\mathrm{wt}(w^*) = u^mq^{1+2+\cdots+m} = u^mq^{m(m+1)/2} = \mathrm{wt}(m).$$

When $m<0$, $N(w^*) = \{0,-1,-2,\ldots,-(|m|-1)\}$ and $P(w^*)=\emptyset$, so

$$\mathrm{wt}(w^*) = u^{-|m|}q^{0+1+2+\cdots+(|m|-1)} = u^mq^{|m|(|m|-1)/2} = u^mq^{m(m+1)/2} = \mathrm{wt}(m).$$

Claim 2: If we move one bead one step left in a given abacus y, the u-weight stays the same and the q-weight drops by 1. Let i be the initial position of the moved bead, and let y' be the abacus obtained by moving the bead to position $i-1$. If $i > 1$, then $N(y') = N(y)$ and $P(y') = (P(y) \sim \{i\}) \cup \{i-1\}$, so $\mathrm{wt}(y') = \mathrm{wt}(y)/q$ as desired. If $i \leq 0$, then $P(y') = P(y)$ and $N(y') = (N(y) \sim \{i-1\}) \cup \{i\}$ (since the N-set records positions of gaps), and so $\mathrm{wt}(y') = \mathrm{wt}(y)q^{|i|}/q^{|i-1|} = \mathrm{wt}(y)/q$. If $i = 1$, then $P(y') = P(y) \sim \{1\}$ and $N(y') = N(y) \sim \{0\}$, so the total u-weight is preserved and the q-weight still drops by 1. Finally, combining the two claims gives

$$\mathrm{wt}(w) = \mathrm{wt}(w^*)q^{|\mu|} = \mathrm{wt}(m)\,\mathrm{wt}(\mu) = \mathrm{wt}(J(w)). \qquad \square$$

Variations of the preceding proof can be used to establish other partition identities. As an example, we now sketch a bijective proof of Euler's pentagonal number theorem. Unlike our earlier proof in §8.7, the current proof does not use an involution to cancel oppositely signed objects. We remark that Euler's identity also follows by suitably specializing the Jacobi triple product identity.

11.6. Euler's Pentagonal Number Theorem. In $\mathbb{Q}[[q]]$, we have

$$\prod_{n=1}^{\infty}(1-q^n) = \sum_{k\in\mathbb{Z}}(-1)^k q^{\frac{3}{2}k^2-\frac{1}{2}k}.$$

Proof. Note first that

$$\prod_{n=1}^{\infty}(1-q^n) = \prod_{i\geq 1}(1-q^{3i})\prod_{i\geq 1}(1-q^{3i-1})\prod_{i\geq 1}(1-q^{3i-2}).$$

It therefore suffices to prove the identity

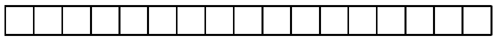
$$\prod_{i\geq 1}(1-q^{3i-1})\prod_{i\geq 1}(1-q^{3i-2}) = \sum_{k\in\mathbb{Z}}(-1)^k q^{(3k^2-k)/2}\prod_{i\geq 1}\frac{1}{1-q^{3i}} = \sum_{(k,\mu)\in\mathbb{Z}\times\mathrm{Par}}(-1)^k q^{3|\mu|+(3k^2-k)/2}. \tag{11.2}$$

Consider abaci $w = \{w_{3k+1} : k \in \mathbb{Z}\}$ whose positions are indexed by integers congruent to 1 mod 3. Define $N(w) = \{i \leq 0 : i \equiv 1 \pmod 3, w_i = 0\}$ and $P(w) = \{i > 0 : i \equiv 1 \pmod 3, w_i = 1\}$. Let $\mathrm{sgn}(w) = (-1)^{|N(w)|+|P(w)|}$ and $\mathrm{wt}(w) = \sum_{i\in N(w)\cup P(w)}|i|$. We can compute the generating function $\sum_w \mathrm{sgn}(w)q^{\mathrm{wt}(w)}$ in two ways. On one hand, placing a bead or a gap in each negative position and each positive position leads to the double product on the left side of (11.2). On the other hand, justifying the abacus transforms w into a pair $(3k-2, \mu)$ for some $k \in \mathbb{Z}$. As in the proof of the Jacobi triple product identity, one checks that the justified abacus associated to a given integer k has signed weight $(-1)^k q^{(3k^2-k)/2}$, while each of the $|\mu|$ bead moves in the justification process reduces the q-weight by 3 and preserves the sign. So the right side of (11.2) is also the generating function for these abaci, completing the proof. $\square$

11.3 Ribbons and k-Cores

Recall the following fact about division of integers: given integers $a \geq 0$ and $k > 0$, there exist a unique quotient q and remainder r satisfying $a = kq + r$ and $0 \leq r < k$. Our next goal is to develop an analogous operation for dividing an integer partition μ by a positive integer k. The result of this operation will consist of k "quotient partitions" together with a "remainder partition" with special properties. We begin by describing the calculation of the remainder, which is called a k-core. Abaci will then be used to establish the uniqueness of the remainder, and this will lead us to the definition of the k quotient partitions.

To motivate our construction, consider the following pictorial method for performing integer division. Suppose we wish to divide $a = 17$ by $k = 5$, obtaining quotient $q = 3$ and remainder $r = 2$. To find these answers geometrically, first draw a row of 17 boxes:



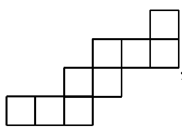
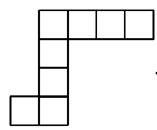
Now, starting at the right end, repeatedly remove strings of five consecutive cells until this is no longer possible. We depict this process by placing an i in every cell removed at stage i, and writing a star in any leftover cells:

$*$	$*$	3	3	3	3	3	2	2	2	2	2	1	1	1	1	1

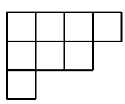
The quotient q is the number of 5-cell blocks we removed (here 3), and the remainder r is the number of leftover cells (here 2). This geometric procedure corresponds to the algebraic process of subtracting k from a repeatedly until a remainder less than k is reached. For the purposes of partition division, we now introduce a two-dimensional version of this strip-removal process.

11.7. Definition: Ribbons. A *ribbon* is a skew shape that can be formed by starting at a given square, repeatedly moving left or down one step at a time, and including all squares visited in this way. A ribbon consisting of k cells is called a *k-ribbon.* A *border ribbon* of a partition μ is a ribbon R contained in $\mathrm{dg}(\mu)$ such that $\mathrm{dg}(\mu) \sim R$ is also a partition diagram.

11.8. Example. Here are two examples of ribbons:

$(6,6,4,3)/(5,3,2) =$ , $(7,4,4,4)/(3,3,3,2) =$ .

The first ribbon is a 9-ribbon and a border ribbon of $(6,6,4,3)$. The partition $(4,3,1)$ with diagram



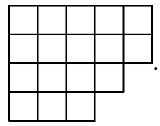
has exactly eight border ribbons, four of which begin at the cell $(1,4)$.

11.9. Definition: k-cores. Let k be a positive integer. An integer partition ν is called a *k-core* iff no border ribbon of ν is a k-ribbon.

For example, $(4,3,1)$ is a 5-core, but not a k-core for any $k < 5$.

Suppose μ is any partition and k is a positive integer. If μ has no border ribbons of size k, then μ is a k-core. Otherwise, we can pick one such ribbon and remove it from the diagram of μ to obtain a smaller partition diagram. We can iterate this process, repeatedly removing a border k-ribbon from the current partition diagram until this is no longer possible. Since the number of cells decreases at each step, the process will eventually terminate. The final partition ν (which may be empty) must be a k-core. This partition is the "remainder" when μ is divided by k.

11.10. Example. Consider the partition $\mu = (5,5,4,3)$ with diagram



Let us divide μ by $k = 4$. We record the removal of border 4-ribbons by entering an i in each square that is removed at stage i. Any leftover squares at the end are marked by a star. One possible removal sequence is the following:

*	*	*	*	1
*	3	3	1	1
3	3	2	1	
2	2	2		

Another possible sequence is:

*	*	*	*	2
*	3	2	2	2
3	3	1	1	
3	1	1		

Notice that the three 4-ribbons removed were different, but the final 4-core was the same, namely $\nu = (4,1)$.

We want to show that the k-core obtained when dividing μ by k depends only on μ and k, not on the choice of which border k-ribbon is removed at each stage. We now use abaci to prove this result.

11.11. Definition: Abacus with k Runners. A *k-runner abacus* is an ordered k-tuple of abaci. The set of all such objects is denoted Abc^k.

11.12. Theorem: Decimation of Abaci. For each $k \geq 1$, there are mutually inverse bijections $D_k : \mathrm{Abc} \to \mathrm{Abc}^k$ (decimation) and $I_k : \mathrm{Abc}^k \to \mathrm{Abc}$ (interleaving).

Proof. Given $w = (w_i : i \in \mathbb{Z}) \in \mathrm{Abc}$, set $D_k(w) = (w^0, w^1, \ldots, w^{k-1})$, where

$$w^r = (w_{qk+r} : q \in \mathbb{Z}) \qquad (0 \leq r < k).$$

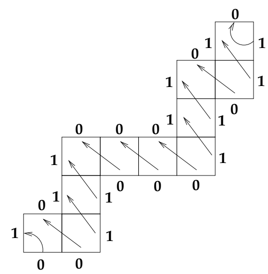
Thus, the abacus w^r is obtained by reading every kth symbol in the original abacus (in both directions), starting at position r. It is routine to check that each w^r is an abacus. The inverse map interleaves these abaci to reconstruct the original one-runner abacus. More precisely, given $v = (v^0, v^1, \ldots, v^{k-1}) \in \mathrm{Abc}^k$, let $I_k(v) = z$ where $z_{qk+r} = v^r_q$ for all $q \in \mathbb{Z}$ and $0 \leq r < k$. One readily checks that $I_k(v)$ is an abacus and that D_k and I_k are two-sided inverses. $\square$

By computing $D_k(U(-1,\mu))$, we can convert any partition into a k-runner abacus. We now show that moving one bead left one step on a k-runner abacus corresponds to removing a border k-ribbon from the associated partition diagram.

11.13. Theorem: Bead Motion vs. Ribbon Removal. Suppose a partition μ is encoded by a k-runner abacus $w = (w^0, w^1, \ldots, w^{k-1})$. Suppose that v is a k-runner abacus obtained from w by changing one substring $\ldots 01 \ldots$ to $\ldots 10 \ldots$ in some w^i. Then the partition ν associated to v can be obtained by removing one border k-ribbon from μ. Moreover, there is a bijection between the set of removable border k-ribbons in μ and the set of occurrences of the substring 01 in the components of w.

Proof. Recall from 11.4 that we can encode the frontier of a partition μ by writing a 0 (gap) for each horizontal step and writing a 1 (bead) for each vertical step. The word so obtained (when preceded by 1's and followed by 0's) is the 1-runner abacus associated to this partition, and w is the k-decimation of this abacus.

Let R be a border k-ribbon of μ. The southeast border of R, which is part of the frontier of μ, gets encoded as a string of $k+1$ symbols $r_0, r_1, \ldots, r_k$, where $r_0 = 0$ and $r_k = 1$. For instance, the first ribbon in 11.8 has southeast border 0001010011. Note that the *northwest* border of this ribbon is encoded by 1001010010, which is the string obtained by interchanging the initial 0 and the terminal 1 in the original string. The following picture suggests why this property holds for general k-ribbons.



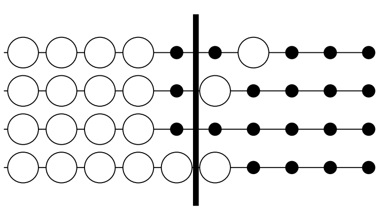
Since $r_0 = 0$ and $r_k = 1$ are separated by k positions in the 1-runner abacus, these two symbols map to two *consecutive* symbols 01 on one of the runners in the k-runner abacus for μ. Changing these symbols to 10 will interchange r_0 and r_k in the original word. Hence, the portion of the frontier of μ consisting of the southeast border of R gets replaced by the northwest border of R. So, this bead motion transforms μ into the partition ν obtained by removing the ribbon R.

Conversely, each substring 01 in the k-runner abacus for μ corresponds to a unique pair of symbols $0 \cdots 1$ in the 1-runner abacus that are k positions apart. This pair corresponds to a unique pair of steps H...V on the frontier that are k steps apart. Finally, this pair of steps corresponds to a unique removable border k-ribbon of μ. So, the map from these ribbons to occurrences of 01 on the runners of w is a bijection. $\square$

11.14. Example. Let us convert the partition $\mu = (5,5,4,3)$ from 11.10 to a 4-runner abacus. First, the 1-runner abacus $U(-1, \mu)$ is

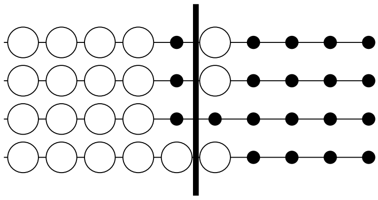
$$\cdots 111000 1\underline{0}1011000 \cdots .$$

Decimating by 4 produces the following 4-runner abacus:

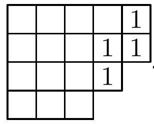


Note that the bead-gap pattern in this abacus can be read directly from the frontier of μ by filling in the runners one column at a time, working from left to right. For the purposes of ribbon removal, one may decide arbitrarily where to place the gap corresponding to the first step of the frontier; this decision determines the integer m in the expression $U(m, \mu)$.

Now let us start removing ribbons. Suppose we push the rightmost bead on the top runner left one position, producing the following abacus:



Reading down columns to recover the frontier of the new partition, we obtain the partition $\nu = (4, 3, 3, 3)$. We get ν from μ by removing one border 4-ribbon, as shown.



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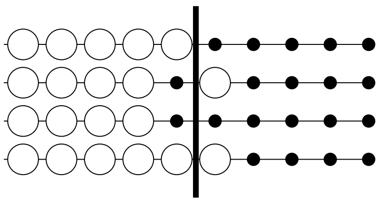
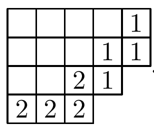
Pushing the same bead one more step on its runner produces the following abacus:



The new partition is $(4, 3, 2)$, which arises by removing one border 4-ribbon from ν:



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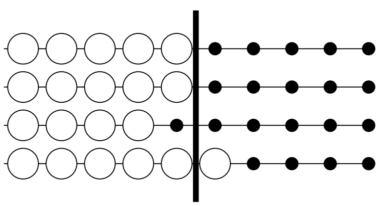
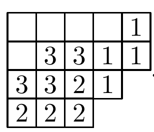
Finally, we push the rightmost bead on the second runner left one position to get the following abacus:



The associated partition is $(4, 1)$, as shown here:



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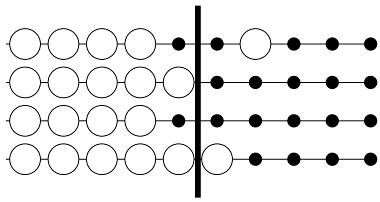
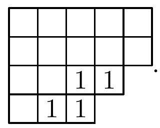
At this point, all runners on the abacus are justified, so no further bead motion is possible. This reflects the fact that we can remove no further border 4-ribbons from the 4-core $(4, 1)$.

Now return to the original partition μ and the associated 4-runner abacus. Suppose we start by moving the bead on the second runner left one position, producing the following abacus:

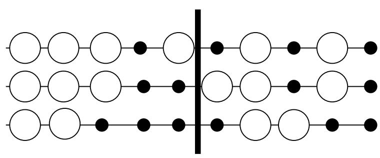


This corresponds to removing a different border 4-ribbon from μ:

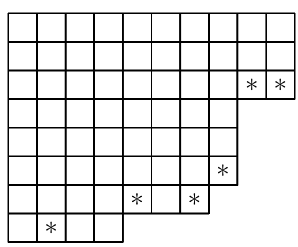


Observe that μ has exactly two removable border 4-ribbons, whereas the 4-runner abacus for μ has exactly two movable beads, in accordance with the last assertion of 11.13.

11.15. Example. Consider the following 3-runner abacus:



We count six beads on this abacus that can be moved one position left without bumping into another bead. Accordingly, we expect the associated partition to have exactly six removable border 3-ribbons. This is indeed the case, as shown below (we have marked the southwestmost cell of each removable ribbon with an asterisk):

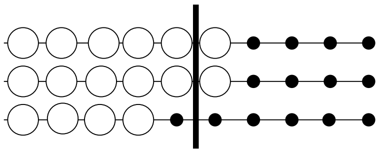


Now we can prove that the k-core obtained from a partition μ by repeated removal of border ribbons is uniquely determined by μ and k.

11.16. Theorem: Uniqueness of k-cores. Suppose μ is an integer partition and $k \geq 1$ is an integer. There is exactly one k-core ρ obtainable from μ by repeatedly removing border k-ribbons. We call ρ *the k-core of μ*.

Proof. Let w be a fixed k-runner abacus associated to μ (say $w = D_k(U(-1, \mu))$ for definiteness). As we have seen, a particular sequence of ribbon-removal operations on μ corresponds to a particular sequence of bead motions on w. The operations on μ terminate when we reach a k-core, whereas the corresponding operations on w terminate when the beads on all runners of w have been justified. Now ρ is uniquely determined by the justified k-runner abacus by applying I_k and then J. The key observation is that the justified abacus obtained from w does not depend on the order in which individual bead moves were made. Thus, the k-core ρ does not depend on the order in which border ribbons are removed from μ. □

11.17. Example. The theorem shows that we can calculate the k-core of μ by justifying any k-runner abacus associated to μ. For example, consider the partition $\mu = (10, 10, 10, 8, 8, 8, 7, 4)$ from 11.15. Justifying the 3-runner abacus in that example produces the following abacus:



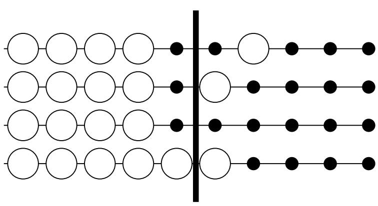
We find that the 3-core of μ is $(1, 1)$.

11.4 k-Quotients and Hooks

Each runner of a k-runner abacus can be regarded as a one-runner abacus, which corresponds (under the justification bijection J) to an element of $\mathbb{Z} \times \mathrm{Par}$. This observation leads to the definition of the k-quotients of a partition.

11.18. Definition: k-quotients of a partition. Let μ be a partition and $k \geq 1$ an integer. Consider the k-runner abacus $(w^0, w^1, \ldots, w^{k-1}) = D_k(U(-1, \mu))$. Write $J(w^i) = (m_i, \nu^i)$ for $0 \leq i < k$. The partitions appearing in the k-tuple $(\nu^0, \nu^1, \ldots, \nu^{k-1})$ are called the *k-quotients* of μ.

11.19. Example. Let $\mu = (5, 5, 4, 3)$. In 11.14, we computed the 4-runner abacus $D_k(U(-1, \mu))$:



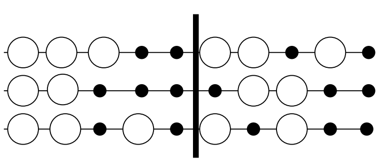
Justifying each runner and converting the resulting 4-runner abacus back to a partition produces the 4-core of μ, namely $(4, 1)$. On the other hand, converting each runner to a separate partition produces the 4-tuple of 4-quotients of μ, namely:

$$((2), (1), (0), (0)).$$

11.20. Example. Consider the partition $\mu = (10, 10, 10, 8, 8, 8, 7, 4)$ from 11.15. We compute

$$U(-1, \mu) = \cdots 111100001000\underline{1}01110011100 \cdots.$$

Decimation by 3 produces the 3-runner abacus shown here:



Justifying each runner shows that the 3-core of μ is $\rho = (1,1)$. On the other hand, by regarding each runner separately as a partition, we obtain the 3-tuple of 3-quotients of μ:

$$(\nu^0, \nu^1, \nu^2) = ((3,2,2), (4,4), (3,2,1)).$$

Observe that $|\mu| = 65 = 2 + 3 \cdot (7 + 8 + 6) = |\rho| + 3|\nu^0| + 3|\nu^1| + 3|\nu^2|$.

Now consider what would have happened if we had performed similar computations on the 3-runner abacus for μ displayed in 11.15, which is $D_3(U(0,\mu))$. The 3-core coming from this abacus is still $(1,1)$, but converting each runner to a partition produces the following 3-tuple:

$$((3,2,1), (3,2,2), (4,4)).$$

This 3-tuple arises by cyclically shifting the previous 3-tuple one step to the right. One can check that this holds in general: if the k-quotients for μ are $(\nu^0, \ldots, \nu^{k-1})$, then the k-quotients computed using $D_k(U(m,\mu))$ will be $(\nu^{k-m'}, \ldots, \nu^{k-1}, \nu^0, \nu^1, \ldots)$, where m' is the integer remainder when $m+1$ is divided by k.

11.21. Remark. Here is a way to compute $(w^0, w^1, \ldots, w^{k-1}) = D_k(U(-1,\mu))$ from the frontier of μ without writing down the intermediate abacus $U(-1,\mu)$. Draw a line of slope -1 starting at the northwest corner of the diagram of μ. The first step on the frontier of μ lying northeast of this line corresponds to position 0 of the zeroth runner w^0. The next step is position 0 on w^1, and so on. The step just southwest of the diagonal line is position -1 on w^{k-1}, the previous step is position -1 on w^{k-2}, and so on. To see that this works, one must check that the first step northeast of the diagonal line gets mapped to position 0 on the one-runner abacus $U(-1,\mu)$; we leave this as an exercise.

11.22. Theorem: Partition Division. Let Core(k) be the set of all k-cores. There is a bijection

$$\Delta_k : \mathrm{Par} \to \mathrm{Core}(k) \times \mathrm{Par}^k$$

such that $\Delta_k(\mu) = (\rho, \nu^0, \ldots, \nu^{k-1})$, where ρ is the k-core of μ and the ν^i are the k-quotients of μ. We have

$$|\mu| = |\rho| + k \sum_{i=0}^{k-1} |\nu^i|.$$

Proof. The function Δ_k is well-defined and maps into the stated codomain. To see that this function is a bijection, we describe its inverse. Given $(\rho, \nu^0, \ldots, \nu^{k-1}) \in \mathrm{Core}(k) \times \mathrm{Par}^k$, first compute the k-runner abacus $(w^0, \ldots, w^{k-1}) = D_k(U(-1,\rho))$. Each w^i is itself a justified one-runner abacus because ρ is a k-core; say w^i is justified at position m_i. Now replace each w^i by $v^i = U(m_i, \nu^i)$. Finally, let μ be the unique partition satisfying $J(I_k(v^0, \ldots, v^{k-1})) = (-1, \mu)$. This construction reverses the one used to produce k-cores and k-quotients, so μ is the unique partition mapped to $(\rho, \nu^0, \ldots, \nu^{k-1})$ by Δ_k.

To prove the formula for $|\mu|$, consider the bead movements used to justify the runners of the k-runner abacus $D_k(U(-1,\mu))$. On one hand, every time we move a bead one step left on this abacus, the area of μ drops by k since the bead motion removes one border k-ribbon. When we finish moving all the beads, we are left with the k-core ρ. It follows that $|\mu| = |\rho| + km$ where m is the total number of bead motions on all k runners. On the other hand, for $0 \le i < k$, let m_i be the number of times we move a bead one step left on runner i. Then $m = m_0 + m_1 + \cdots + m_{k-1}$, whereas $m_i = |\rho^i|$ by 11.3. Substituting these expressions into $|\mu| = |\rho| + km$ gives the desired formula. $\square$

We close our discussion of partition division by describing a way to compute the k-quotients of μ directly from the diagram of μ, without recourse to abaci. We will need the following device for labeling cells of $\mathrm{dg}(\mu)$ and steps on the frontier of μ by integers in $\{0, 1, \ldots, k-1\}$.

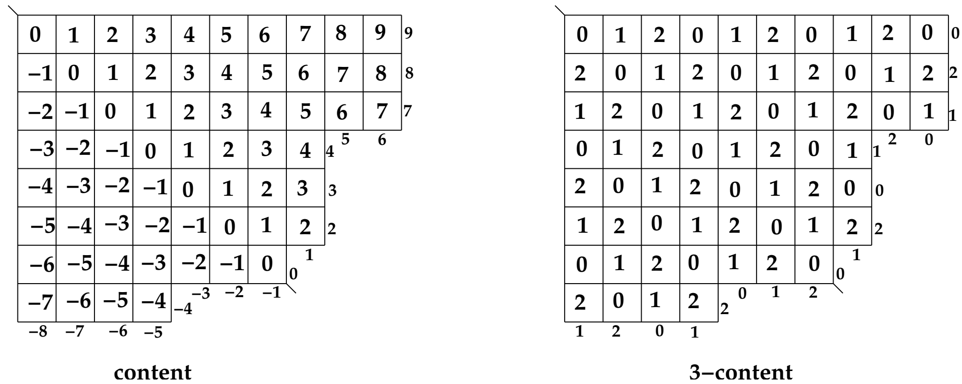
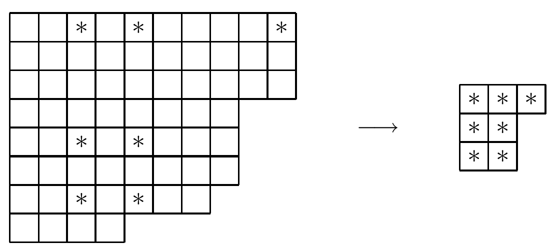


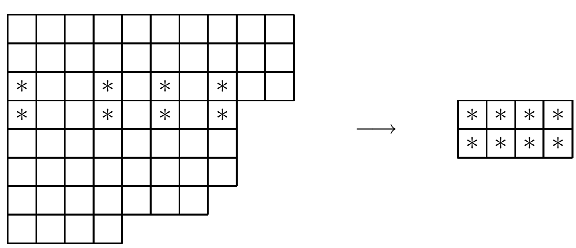
FIGURE 11.1
Content and 3-content of cells and steps.

11.23. Definition: Content and k-Content. Consider a partition diagram for μ, drawn with the longest row on top. Introduce a coordinate system so that the northwest corner of the diagram is $(0,0)$ and (i,j) is located i steps south and j steps east of the origin. The *content* of the point (i,j) is $c(i,j) = j - i$. The content of a cell in the diagram of μ is the content of its southeast corner. The content of a frontier step from (i,j) to $(i,j+1)$ is $j-i$. The content of a frontier step from (i,j) to $(i-1,j)$ is $j-i$. If z is a point, cell, or step in the diagram, then the k-content $c_k(z)$ is the unique value $r \in \{0,1,\ldots,k-1\}$ such that $c(z) \equiv r \pmod{k}$.

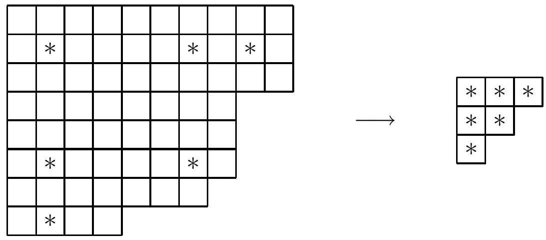
11.24. Example. The left side of Figure 11.1 shows the diagram of the partition $\mu = (10,10,10,8,8,8,7,4)$ with each cell and frontier step labeled by its content. On the right side of the figure, each cell and step is labeled by its 3-content. Given a cell in the diagram of μ, we obtain an associated pair of steps on the frontier of μ by traveling south (resp. east) from the cell in question. Suppose we mark all cells whose associated steps both have content zero. Then erase all other cells and shift the marked cells up and left as far as possible. The following diagram results:



This partition $(3,2,2)$ is precisely the zeroth 3-quotient of μ. Similarly, marking the cells whose associated steps both have 3-content equal to 1 produces the next 3-quotient of μ:



Finally, marking the cells whose associated steps both have 3-content equal to 2 produces the last 3-quotient of μ:



In general, to obtain the ith k-quotient ν^i of μ from the diagram of μ, label each row (resp. column) of the diagram with the k-content of the frontier step located in that row (resp. column). Erase all rows and columns not labeled i. The number of cells remaining in the jth unerased row is the jth part of ν^i. To see why this works, recall that the cells of ν^i correspond bijectively to the pairs of symbols $0 \cdots 1$ on the ith runner of the k-runner abacus for μ. In turn, these pairs correspond to pairs of symbols $w_s = 0$, $w_t = 1$ on the one-runner abacus for μ where $s < t$ and $s \equiv i \equiv t \pmod k$. The symbols in positions congruent to i mod k come from the steps on the frontier of μ whose k-content is i. Finally, the relevant pairs of steps on the frontier correspond to the unerased cells in the construction described above. Composing all these bijections, we see that the cells of ν^i are in one-to-one correspondence with the unerased cells of the construction. Furthermore, cells in row j of ν^i are mapped onto the unerased cells in the jth unerased row of μ. It follows that the construction at the beginning of this paragraph does indeed produce the k-quotient ν^i.

11.5 Antisymmetric Polynomials

We now define antisymmetric polynomials, which form a vector space analogous to the space of symmetric polynomials studied in the last chapter.

11.25. Definition: Antisymmetric Polynomials. Let K be a field containing $\mathbb{Q}$. A polynomial $f \in K[x_1, \dots, x_N]$ is called *antisymmetric* iff for all $w \in S_N$,

$$f(x_{w(1)}, x_{w(2)}, \dots, x_{w(N)}) = \operatorname{sgn}(w) f(x_1, x_2, \dots, x_N).$$

11.26. Remark. The group S_N acts on the set $\{x_1, \dots, x_N\}$ via $w \bullet x_i = x_{w(i)}$ for $w \in S_N$ and $1 \leq i \leq N$. This action extends (by the universal mapping property of polynomial rings) to an action of S_N on $K[x_1, \dots, x_N]$ such that $w \bullet f = f(x_{w(1)}, \dots, x_{w(N)})$ for $w \in S_N$ and $f \in K[x_1, \dots, x_N]$. The polynomial f is antisymmetric iff $w \bullet f = \operatorname{sgn}(w) f$ for all $w \in S_N$. It suffices to check this condition when w is a basic transposition $(i, i+1)$. For, any $w \in S_N$ can be written as a product of basic transpositions $w = t_1 t_2 \cdots t_k$. By hypothesis, $t_i \bullet (\pm f) = \operatorname{sgn}(t_i)(\pm f) = \mp f$ for all i, so

$$w \bullet f = t_1 \bullet \cdots \bullet (t_k \bullet f) = (-1)^k f = \operatorname{sgn}(w) f.$$

So $f \in K[x_1, \dots, x_N]$ is antisymmetric iff

$$f(x_1, \dots, x_{i+1}, x_i, \dots, x_N) = -f(x_1, \dots, x_i, x_{i+1}, \dots, x_N) \qquad \text{for all } i < N.$$

11.27. Example. The polynomial $f(x_1,\ldots,x_N)=\prod_{1\leq j<k\leq N}(x_j-x_k)$ is antisymmetric. To check this, consider what happens to the factors in the product when we interchange x_i and x_{i+1}. Factors not involving x_i or x_{i+1} are unchanged; factors of the form (x_i-x_k) with $k>i+1$ get interchanged with factors of the form $(x_{i+1}-x_k)$; and factors of the form (x_j-x_i) with $j<i$ get interchanged with factors of the form (x_j-x_{i+1}). Finally, the factor (x_i-x_{i+1}) becomes $(x_{i+1}-x_i)=-(x_i-x_{i+1})$. Thus, $(i,i+1)\bullet f=-f$ for all $i<N$, proving antisymmetry of f.

We remark that the polynomial $f=\prod_{j<k}(x_j-x_k)$ in the previous example is the *Vandermonde determinant*

$$\det||x_j^{N-i}||_{1\leq i,j\leq N}=\sum_{w\in S_N}\operatorname{sgn}(w)\prod_{i=1}^{N}x_{w(i)}^{N-i}.$$

(see §12.9 for a combinatorial proof of this assertion). We can use analogous determinants to manufacture additional examples of antisymmetric polynomials.

11.28. Definition: Monomial Antisymmetric Polynomials. Let $\mu=(\mu_1>\mu_2>\cdots>\mu_N)$ be a strictly decreasing sequence of N nonnegative integers. Define a polynomial $a_\mu(x_1,\ldots,x_N)$ by the formula

$$a_\mu(x_1,\ldots,x_N)=\det||x_j^{\mu_i}||_{1\leq i,j\leq N}=\sum_{w\in S_N}\operatorname{sgn}(w)\prod_{i=1}^{N}x_{w(i)}^{\mu_i}.$$

We call a_μ a *monomial antisymmetric polynomial indexed by* μ.

To see that a_μ really is antisymmetric, note that interchanging x_k and x_{k+1} has the effect of interchanging columns k and $k+1$ in the determinant defining a_μ. By 9.47, this column switch will change the sign of a_μ, as required.

11.29. Example. Let $N=3$ and $\mu=(5,4,2)$. Then

$$a_\mu(x_1,x_2,x_3)=+x_1^5x_2^4x_3^2+x_1^4x_2^2x_3^5+x_1^2x_2^5x_3^4-x_1^4x_2^5x_3^2-x_1^5x_2^2x_3^4-x_1^2x_2^4x_3^5.$$

As the previous example shows, $a_\mu(x_1,\ldots,x_N)$ is a sum of $N!$ distinct monomials obtained by rearranging the subscripts (or equivalently, the exponents) in the monomial $x_1^{\mu_1}x_2^{\mu_2}\cdots x_N^{\mu_N}$. Each monomial appears in the sum with sign $+1$ or -1, where the sign of $x_1^{e_1}\cdots x_N^{e_N}$ depends on the parity of the number of basic transpositions needed to transform the sequence $(e_1,\ldots,e_N)$ to the sorted sequence $(\mu_1,\ldots,\mu_N)$. It follows from these remarks that a_μ is a nonzero homogeneous polynomial of degree $|\mu|=\mu_1+\cdots+\mu_N$.

11.30. Definition: $\delta(N)$. For each $N\geq 1$, let $\delta(N)=(N-1,N-2,\ldots,2,1,0)$.

The *strictly* decreasing sequences $\mu=(\mu_1>\mu_2>\cdots>\mu_N)$ correspond bijectively to the *weakly* decreasing sequences $\lambda=(\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_N)$ via the maps $\mu\mapsto\mu-\delta(N)$ and $\lambda\mapsto\lambda+\delta(N)$. It follows that each polynomial a_μ can be written $a_{\lambda+\delta(N)}$ for a unique partition $\lambda\in\operatorname{Par}_N$. This indexing scheme will be used frequently below. Note that when $\lambda=(0,\ldots,0)$, we have $\mu=\delta(N)$ and $a_{\delta(N)}=\prod_{1\leq j<k\leq N}(x_j-x_k)$ (see 11.27). Observe that $a_{\delta(N)}$ is a homogeneous polynomial of degree $N(N-1)/2=\binom{N}{2}$.

11.31. Definition: Spaces of Antisymmetric Polynomials. For a given field K, let A_N be the set of all antisymmetric polynomials in $K[x_1,\ldots,x_N]$. Let A_N^n consist of those polynomials in A_N that are homogeneous of degree n, together with the zero polynomial.

One readily verifies that A_N is a vector subspace of $K[x_1,\ldots,x_N]$, and each A_N^n is a subspace of A_N. We now exhibit bases for these vector spaces involving monomial antisymmetric polynomials. We use the notation $\mathrm{Par}_N^d(n)$ to denote the set of all partitions of n into N distinct nonnegative parts, and $\mathrm{Par}_N^d = \bigcup_{n\geq\binom{N}{2}} \mathrm{Par}_N^d(n)$.

11.32. Theorem: Monomial Basis for A_N^n. Assume K is a field containing $\mathbb{Q}$. If $n<\binom{N}{2}$, then $A_N^n=\{0\}$. If $n\geq\binom{N}{2}$, then

$$\{a_\mu : \mu\in\mathrm{Par}_N^d(n)\} = \{a_{\lambda+\delta(N)} : \lambda\in\mathrm{Par}_N(n-N(N-1)/2)\}$$

is a basis of the K-vector space A_N^n. Hence, the collection

$$\{a_\mu : \mu\in\mathrm{Par}_N^d\} = \{a_{\lambda+\delta(N)} : \lambda\in\mathrm{Par}_N\}$$

is a basis of A_N.

Proof. Suppose $e=(e_1,\ldots,e_N)$ is any exponent sequence, $f\in A_N$ is an arbitrary antisymmetric polynomial, and $w\in S_N$. Let c be the coefficient of $x^e = x_1^{e_1}\cdots x_N^{e_N}$ in f, so

$$f = cx_1^{e_1}\cdots x_N^{e_N} + \text{other terms}.$$

Acting by w, we see that

$$\begin{aligned}\mathrm{sgn}(w)f = w\bullet f &= cx_{w(1)}^{e_1}\cdots x_{w(N)}^{e_N} + \text{other terms}\\ &= cx_1^{e_{w^{-1}(1)}}\cdots x_N^{e_{w^{-1}(N)}} + \text{other terms}\\ &= cx^{w*e} + \text{other terms},\end{aligned}$$

where $w*e = (e_{w^{-1}(1)},\ldots,e_{w^{-1}(N)})$. In other words, writing $f|_{x^\alpha}$ for the coefficient of x^α in f, we have $f|_{x^{w*e}} = \mathrm{sgn}(w)(f|_{x^e})$.

Let us apply this fact to an exponent sequence e such that $e_i=e_j$ for some $i\neq j$. Let $w=(i,j)$, so that $w*e=e$ and $\mathrm{sgn}(w)=-1$. It follows that $c=-c$, so $2c=0$. Because K contains $\mathbb{Q}$, we deduce $c=0$ in K. This means that no antisymmetric polynomial contains any monomial with a repeated value in its exponent vector. In particular, the smallest possible degree of a monomial that can appear with nonzero coefficient in any antisymmetric polynomial is $0+1+2+\cdots+(N-1)=\binom{N}{2}$. This proves the first assertion of the theorem.

For the second assertion, recall that $\lambda\mapsto\lambda+\delta(N)$ is a bijection from $\mathrm{Par}_N(n-\binom{N}{2})$ to $\mathrm{Par}_N^d(n)$. So we need only show that $\{a_\mu : \mu\in\mathrm{Par}_N^d(n)\}$ is a basis for A_N^n. To show that this set spans A_N^n, fix $f\in A_N^n$. By the previous paragraph, we can write $f=\sum_\alpha c_\alpha x^\alpha$ where we sum over all sequences $(\alpha_1,\ldots,\alpha_N)\in\mathbb{N}^N$ with *distinct* entries summing to n, and each c_α lies in K. We claim

$$f = \sum_{\nu\in\mathrm{Par}_N^d(n)} c_\nu a_\nu.$$

To prove this, we check the coefficient of x^α on each side. Choose $\mu\in\mathrm{Par}_N^d(n)$ and $w\in S_N$ such that $w*\mu=\alpha$ (μ consists of the entries of α sorted into decreasing order). By the first paragraph of the proof,

$$f|_{x^\alpha} = f|_{x^{w*\mu}} = \mathrm{sgn}(w)(f|_{x^\mu}) = \mathrm{sgn}(w)c_\mu.$$

On the other side, $a_\nu|_{x^\alpha}=0$ for all $\nu\neq\mu$ (since no rearrangement of ν equals α). For $\nu=\mu$, antisymmetry gives $a_\mu|_{x^\alpha} = \mathrm{sgn}(w)(a_\mu|_{x^\mu}) = \mathrm{sgn}(w)$. Multiplying by c_ν and summing over all ν, the coefficient of x^α in $\sum_\nu c_\nu a_\nu$ is $\mathrm{sgn}(w)c_\mu$, as desired.

To prove linear independence, suppose

$$0 = \sum_{\nu \in \mathrm{Par}_N^d(n)} d_\nu a_\nu \qquad (d_\nu \in K).$$

For a fixed $\mu \in \mathrm{Par}_N^d(n)$, a_μ is the only polynomial among the a_ν's that involves the monomial x^μ. Extracting this coefficient on both sides of the given equation, we find that $0 = d_\mu \cdot 1 = d_\mu$. Since μ was arbitrary, all d_μ's are zero. □

The next result explains the relationship between the various vector spaces Λ_N^k and A_N^n.

11.33. Theorem: Symmetric vs. Antisymmetric Polynomials. For each $k \geq 0$, the vector spaces Λ_N^k and $A_N^{k+\binom{N}{2}}$ are isomorphic, as are the vector spaces Λ_N and A_N. In each of these cases, an isomorphism is given by the formula $M(f) = f \cdot a_{\delta(N)}$ for $f \in \Lambda_N$, and the inverse isomorphism sends $g \in A_N$ to $g/a_{\delta(N)}$. In particular, every antisymmetric polynomial in N variables is divisible by the polynomial $a_{\delta(N)}$.

Proof. Fix $k \geq 0$, and consider the map $M = M_k : \Lambda_N^k \to K[x_1, \ldots, x_N]$ defined by $M(f) = f \cdot a_{\delta(N)}$ for $f \in \Lambda_N^k$. Note, first, that f is homogeneous of degree k and $a_{\delta(N)}$ is homogeneous of degree $\binom{N}{2}$, so $M(f)$ is homogeneous of degree $k + \binom{N}{2}$. Second, $M(f)$ is antisymmetric, since for any $w \in S_N$,

$$w \bullet (f a_{\delta(N)}) = (w \bullet f) \cdot (w \bullet a_{\delta(N)}) = f \cdot (\mathrm{sgn}(w) a_{\delta(N)}) = \mathrm{sgn}(w)(f a_{\delta(N)}).$$

So the map M takes values in the space $A_N^{k+\binom{N}{2}}$. Third, one immediately verifies that M is a K-linear map. Fourth, the kernel of this linear map is zero: $M(f) = 0$ implies $f \cdot a_{\delta(N)} = 0$, which implies $f = 0$ since $a_{\delta(N)}$ is a nonzero element of the integral domain $K[x_1, \ldots, x_N]$. So M is injective. Fifth, M must also be surjective since its domain and codomain are vector spaces having the same finite dimension $|\mathrm{Par}_N(k)|$. So each M_k is an isomorphism. Since Λ_N (resp. A_N) is the direct sum of subspaces Λ_N^k (resp. $A_N^{k+\binom{N}{2}}$), it follows that Λ_N and A_N are isomorphic as well. Finally, surjectivity of the map $f \mapsto f a_{\delta(N)}$ means that every antisymmetric polynomial g has the form $f a_{\delta(N)}$ for some symmetric polynomial f. So g is divisible by $a_{\delta(N)}$ in $K[x_1, \ldots, x_N]$. □

11.34. Remark. Suppose we apply the inverse of the isomorphism M_k to the basis $\{a_{\lambda+\delta(N)} : \lambda \in \mathrm{Par}_N(k)\}$ of $A_N^{k+\binom{N}{2}}$. We will obtain a basis $\{a_{\lambda+\delta(N)}/a_{\delta(N)} : \lambda \in \mathrm{Par}_N(k)\}$ of Λ_N^k. It turns out that $a_{\lambda+\delta(N)}/a_{\delta(N)}$ is none other than the Schur polynomial $s_\lambda(x_1, \ldots, x_N)$! To prove this fact and other properties of antisymmetric polynomials, we will use the *labeled abaci* introduced below.

11.6 Labeled Abaci

Given $\mu = (\mu_1 > \mu_2 > \cdots > \mu_N)$, recall that the monomial antisymmetric polynomial indexed by μ is defined by

$$a_\mu(x_1, \ldots, x_N) = \sum_{w \in S_N} \mathrm{sgn}(w) \prod_{i=1}^{N} x_{w(i)}^{\mu_i}.$$

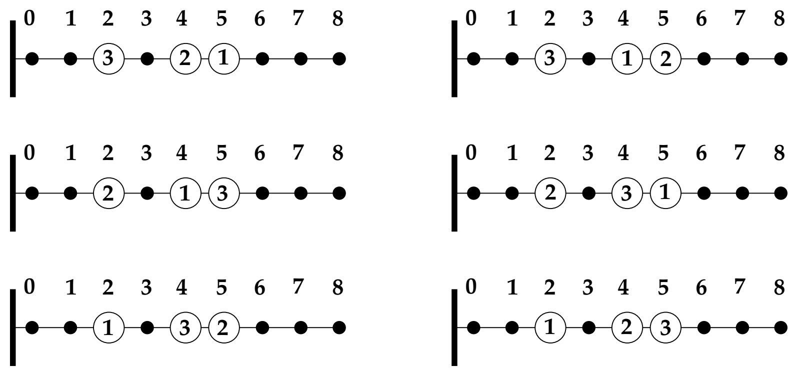


FIGURE 11.2
Labeled abaci.

The next definition introduces a set of signed, weighted combinatorial objects to model this formula.

11.35. Definition: Labeled Abaci. A *labeled abacus with N beads* is a word $v = (v_i : i \geq 0)$ such that each letter $1, \ldots, N$ appears exactly once in v, and all other letters of v are zero. We think of the indices i as positions on an abacus containing one runner that extends to infinity in the positive direction. When $v_i = 0$, there is a gap at position i on the abacus; when $v_i = j > 0$, there is a bead labeled j at position i. The *weight* of the abacus v is

$$\mathrm{wt}(v) = \prod_{i:\ v_i > 0} x_{v_i}^i.$$

So if bead j is located at position i, this bead contributes a factor of x_j^i to the weight.

We can encode a labeled abacus by specifying the positions occupied by the beads and the ordering of the bead labels. Formally, define $\mathrm{pos}(v) = (\mu_1 > \mu_2 > \cdots > \mu_N)$ to be the indices i such that $v_i > 0$. Then define $w(v) = (v_{\mu_1}, \ldots, v_{\mu_N}) \in S_N$. We define the *sign* of v to be the sign of the permutation $w(v)$, which is $(-1)^{\mathrm{inv}(w(v))}$. Let LAbc be the set of all labeled abaci, and for each $\mu \in \mathrm{Par}_N^d$, let

$$\mathrm{LAbc}(\mu) = \{v \in \mathrm{LAbc} : \mathrm{pos}(v) = \mu\}.$$

For each $\mu \in \mathrm{Par}_N^d$, there is a bijection between $\mathrm{LAbc}(\mu)$ and S_N given by $v \mapsto w(v)$. Furthermore, an abacus $v \in \mathrm{LAbc}(\mu)$ has sign $\mathrm{sgn}(w(v))$ and weight $\prod_{i=1}^N x_{w(v)_i}^{\mu_i}$. So

$$\sum_{v \in \mathrm{LAbc}(\mu)} \mathrm{sgn}(v)\,\mathrm{wt}(v) = \sum_{w \in S_N} \mathrm{sgn}(w) \prod_{i=1}^N x_{w(i)}^{\mu_i} = a_\mu(x_1, \ldots, x_N).$$

11.36. Example. Let $N = 3$ and $\nu = (5, 4, 2)$. Earlier, we computed

$$a_\nu(x_1, x_2, x_3) = +x_1^5 x_2^4 x_3^2 + x_1^4 x_2^2 x_3^5 + x_1^2 x_2^5 x_3^4 - x_1^4 x_2^5 x_3^2 - x_1^5 x_2^2 x_3^4 - x_1^2 x_2^4 x_3^5.$$

The six terms in this polynomial come from the six labeled abaci in $\mathrm{LAbc}(\nu)$ shown in Figure 11.2. Observe that we read labels from right to left in v to obtain the permutation $w(v)$. This is necessary so that the "leading term" $x_1^{\nu_1} \cdots x_N^{\nu_N}$ will correspond to the identity permutation and have a positive sign.

Informally, we *justify* a labeled abacus $v \in \mathrm{LAbc}(\mu)$ by moving all beads to the left as far as they will go. This produces a justified labeled abacus $J(v) = (w_N, \ldots, w_2, w_1, 0, 0, \ldots) \in \mathrm{LAbc}(\delta(N))$, where $(w_N, \ldots, w_1) = w(v)$. To recover v from $J(v)$, first write $\mu = \lambda + \delta(N)$ for some $\lambda \in \mathrm{Par}_N$. Move the rightmost bead (labeled w_1) to the right λ_1 positions from position $N-1$ to position $N-1+\lambda_1 = \mu_1$. Then move the next bead (labeled w_2) to the right λ_2 positions from position $N-2$ to position $N-2+\lambda_2 = \mu_2$, and so on.

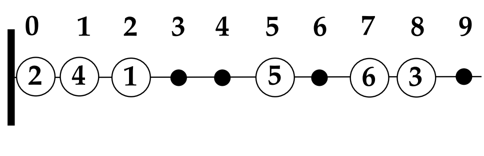
11.7 Pieri Rule for p_k

The product of an antisymmetric polynomial and a symmetric polynomial is an antisymmetric polynomial (see 11.96), which can be written as a linear combination of the monomial antisymmetric polynomials. In the next few sections, we will derive several "Pieri rules" for expressing a product $a_{\lambda+\delta(N)}g$ (where g is symmetric) in terms of the a_μ's. We begin by considering the case where $g = p_k(x_1, \ldots, x_N) = \sum_{i=1}^{N} x_i^k$ is a power-sum symmetric polynomial.

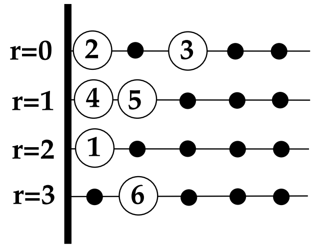
We know $a_{\lambda+\delta(N)}(x_1, \ldots, x_N)$ is a sum of signed terms, each of which represents a labeled abacus with beads in positions given by $\mu = \lambda + \delta(N)$. If we multiply some term in this sum by x_i^k, what happens to the associated abacus? Recalling that the power of x_i tells us where bead i is located, we see that this multiplication should move bead i to the right k positions. This bead motion occurs all at once, not one step at a time, so bead i is allowed to "jump over" any beads between its original position and its destination. However, there is a problem if the new position for bead i already contains a bead. In the proofs below, we will see that two objects of opposite sign cancel whenever a *bead collision* like this occurs. If there is no collision, the motion of bead i will produce a new labeled abacus whose x_i-weight has increased by k. However, the sign of the new abacus (compared to the original) depends on the parity of the number of beads that bead i "jumps over" when it moves to its new position.

To visualize these ideas more conveniently, we *decimate* our labeled abacus to obtain a labeled abacus with k runners. Formally, the k-decimation of the labeled abacus $v = (v_j : j \geq 0) \in \mathrm{LAbc}(\lambda+\delta(N))$ is the k-tuple $(v^0, v^1, \ldots, v^{k-1})$, where $v_q^r = v_{qk+r}$. Moving a bead from position j to position $j+k$ on the original abacus corresponds to moving a bead one position along its runner on the k-runner abacus. If there is already a bead in position $j+k$, we say that this bead move causes a *bead collision*. Otherwise, the bead motion produces a new labeled abacus in $\mathrm{LAbc}(\nu + \delta(N))$, for some $\nu \in \mathrm{Par}_N$. By ignoring the labels in the decimated abacus, we see that ν arises from λ by adding one k-ribbon at the border. The shape of this ribbon determines the sign change caused by the bead move, as illustrated in the following example.

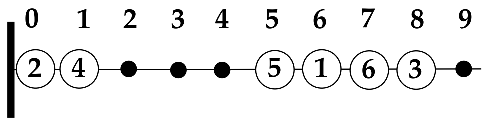
11.37. Example. Take $N = 6$, $k = 4$, $\lambda = (3,3,2,0,0,0)$, and $\mu = \lambda + \delta(6) = (8,7,5,2,1,0)$. Consider the following labeled abacus v in $\mathrm{LAbc}(\mu)$:



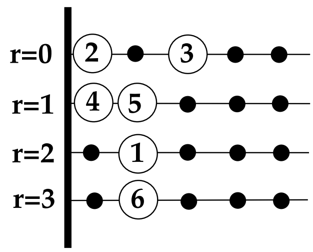
This abacus has weight $x_1^2 x_2^0 x_3^8 x_4^1 x_5^5 x_6^7$ and sign $\mathrm{sgn}(3,6,5,1,4,2) = (-1)^{10} = +1$. Decimation by 4 produces the following 4-runner abacus:



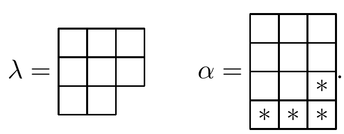
Suppose we move bead 1 four positions to the right in the original abacus, from position 2 to position 6:



The new abacus has weight $x_1^6 x_2^0 x_3^8 x_4^1 x_5^5 x_6^7 = \mathrm{wt}(v) x_1^4$ and sign $\mathrm{sgn}(3,6,1,5,4,2) = (-1)^9 = -1$. The change in weight arose since bead 1 moved 4 positions to the right. The change in sign arose since bead 1 passed one other bead (bead 5) to reach its new position, and one basic transposition is needed to transform the permutation $3,6,5,1,4,2$ into $3,6,1,5,4,2$. The decimation of the new abacus looks like:

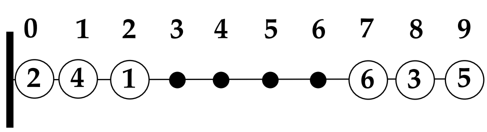


This abacus is in $\mathrm{LAbc}(\nu) = \mathrm{LAbc}(\alpha + \delta(6))$, where $\nu = (8,7,6,5,1,0)$ and $\alpha = (3,3,3,3,0,0)$. Compare the diagrams of the partitions λ and α:

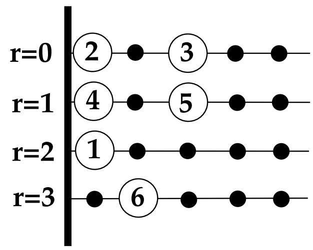


We obtain α from λ by adding a new border 4-ribbon. To go from λ to α, we change part of the frontier of λ from NEENE (where the first N step corresponds to bead 1) to EEENN (where the last N step corresponds to bead 1). There is one other N in this string, corresponding to the one bead (labeled 5) that bead 1 passes when it moves to position 6. Thus the number of passed beads (which is 1, here) is one less than the number of rows occupied by the new border ribbon (which is 2, here).

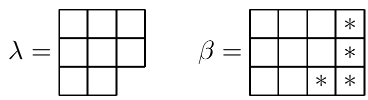
Let us return to the original abacus v and move bead 5 four positions, from position 5 to position 9:



This abacus has weight $x_1^2 x_2^0 x_3^8 x_4^1 x_5^9 x_6^7 = \mathrm{wt}(v)x_5^4$ and sign $\mathrm{sgn}(5,3,6,1,4,2) = (-1)^{10} = +1$. Note that the sign is unchanged since two basic transpositions are required to change the permutation $3,6,5,1,4,2$ into $5,3,6,1,4,2$. The decimation of the new abacus is:

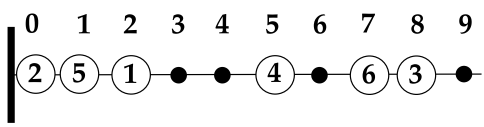


This abacus lies in $\mathrm{LAbc}(\beta + \delta(6))$ where $\beta = (4,4,4,0,0,0)$. The diagram of β arises by adding a border 4-ribbon to the diagram of λ:



This time the frontier changed from ...NENNE... (where the first N is bead 5) to ...EENNN... (where the last N is bead 5). The moved bead passed two other beads (beads 3 and 6), which is one less than the number of rows in the new ribbon (three). In general, the number of passed beads is one less than the number of N's in the frontier substring associated to the added ribbon, which is one less than the number of rows in the added ribbon.

Finally, consider what would happen if we tried to move bead 4 (in the original abacus) four positions to the right. A bead collision occurs with bead 5, so this move is impossible. Now consider the labeled abacus v' obtained by interchanging the labels 4 and 5 in v:



Moving bead 5 four positions to the right in v' causes a bead collision with bead 4. Notice that $\mathrm{sgn}(v') = -\mathrm{sgn}(v)$ since $[3,6,4,1,5,2] = (4,5)\circ[3,6,5,1,4,2]$. Also note that $\mathrm{wt}(v)x_4^4 = \mathrm{wt}(v')x_5^4$; this equality is valid precisely because of the bead collisions. The abaci v and v' are examples of a matched pair of oppositely signed objects that will cancel in the proof of the Pieri rule, given below.

The observations in the last example motivate the following definition.

11.38. Definition: Spin and Sign of Ribbons. The *spin* of a ribbon R, denoted $\mathrm{spin}(R)$, is one less than the number of rows occupied by the ribbon. The *sign* of R is $\mathrm{sgn}(R) = (-1)^{\mathrm{spin}(R)}$.

We now have all the combinatorial ingredients needed to prove the Pieri rule for multiplication by a power-sum polynomial.

11.39. Theorem: Antisymmetric Pieri Rule for p_k. For all $\lambda \in \mathrm{Par}_N$ and all $k \geq 1$, the following identity holds in $K[x_1,\ldots,x_N]$:

$$a_{\lambda+\delta(N)}(x_1,\ldots,x_N)p_k(x_1,\ldots,x_N) = \sum_{\substack{\beta\in\mathrm{Par}_N:\\ \beta/\lambda \text{ is a } k\text{-ribbon } R}} \mathrm{sgn}(R)a_{\beta+\delta(N)}(x_1,\ldots,x_N).$$

Proof. Let X be the set of pairs (v,i), where $v \in \mathrm{LAbc}(\lambda+\delta(N))$ and $1 \leq i \leq N$. For $(v,i) \in X$, set $\mathrm{sgn}(v,i) = \mathrm{sgn}(v)$ and $\mathrm{wt}(v,i) = \mathrm{wt}(v)x_i^k$. Then $a_{\lambda+\delta(N)}p_k = \sum_{z \in X} \mathrm{sgn}(z)\,\mathrm{wt}(z)$. We introduce a weight-preserving, sign-reversing involution I on X. Given (v,i) in X, try to move bead i to the right k positions in v. If this move causes a bead collision with bead j, let v' be v with beads i and j switched, and set $I(v,i) = (v',j)$. Otherwise, set $I(v,i) = (v,i)$. One verifies that I is an involution.

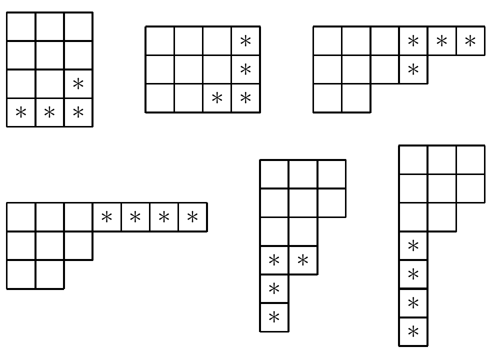
Consider the case where $I(v,i) = (v',j) \neq (v,i)$. Since the label permutation $w(v')$ is obtained from $w(v)$ by multiplying by the basic transposition (i,j), $\mathrm{sgn}(v',j) = \mathrm{sgn}(v') = -\mathrm{sgn}(v) = -\mathrm{sgn}(v,i)$. The weight of v must have the form $x_i^a x_j^{a+k}\cdots$ because of the bead collision, so $\mathrm{wt}(v') = x_j^a x_i^{a+k}\cdots$. It follows that $\mathrm{wt}(v,i) = \mathrm{wt}(v)x_i^k = \mathrm{wt}(v')x_j^k = \mathrm{wt}(v',j)$. Thus, I is a weight-preserving, sign-reversing map.

Now consider a fixed point (v,i) of I. Let v^* be the abacus obtained from v by moving bead i to the right k positions, so $\mathrm{wt}(v^*) = \mathrm{wt}(v)x_i^k = \mathrm{wt}(v,i)$. Since the unlabeled k-runner abacus for v^* arises from the unlabeled k-runner abacus for v by moving one bead one step along its runner, it follows that $v^* \in \mathrm{LAbc}(\beta+\delta(N))$ for a unique $\beta \in \mathrm{Par}_N$ such that $R = \beta/\lambda$ is a k-ribbon. As argued earlier, $\mathrm{sgn}(v^*)$ differs from $\mathrm{sgn}(v)$ by $\mathrm{sgn}(R) = (-1)^{\mathrm{spin}(R)}$, which is the number of beads that bead i passes over when it moves. Conversely, any abacus y counted by $a_{\beta+\delta(N)}$ (for some shape β as above) arises from a unique fixed point $(v,i) \in X$, since the moved bead i is uniquely determined by the shapes λ and β, and v is determined from y by moving the bead i back k positions. These remarks show that the sum appearing on the right side of the theorem is the generating function for the fixed point set of I, which completes the proof. $\square$

11.40. Example. When $N = 6$, we calculate

$$a_{(3,3,2)+\delta(6)}p_4 = -a_{(3,3,3,3)+\delta(6)} + a_{(4,4,4)+\delta(6)} - a_{(6,4,2)+\delta(6)} + a_{(7,3,2)+\delta(6)} + a_{(3,3,2,2,1,1)+\delta(6)}$$

by adding border 4-ribbons to the shape $(3,3,2)$, as shown here:



Observe that the last shape pictured does *not* contribute to the sum because it has more than N parts. An antisymmetric polynomial indexed by this shape would appear for $N \geq 7$.

11.8 Pieri Rule for e_k

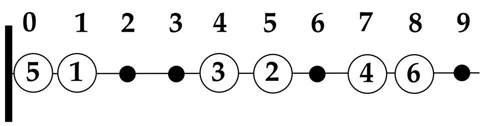
Next we derive Pieri rules for calculating $a_{\lambda+\delta(N)}e_k$ and $a_{\lambda+\delta(N)}h_k$. Our starting point is the following expression for the elementary symmetric polynomial e_k:

$$e_k(x_1,\ldots,x_N) = \sum_{\substack{S \subseteq \{1,2,\ldots,N\} \\ |S|=k}} \prod_{j \in S} x_j.$$

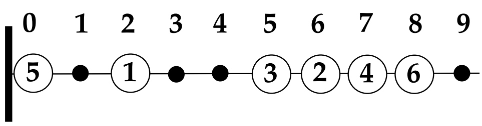
Let $S = \{j_1, \ldots, j_k\}$ be a fixed k-element subset of $\{1, 2, \ldots, N\}$. Then $\prod_{j \in S} x_j = x_{j_1} x_{j_2} \cdots x_{j_k}$ is a typical term in the polynomial e_k. On the other hand, a typical term in $a_{\lambda+\delta(N)}$ corresponds to a signed, weighted abacus v. Let us investigate what happens to the abacus when we multiply such a term by $x_{j_1} \cdots x_{j_k}$.

Since the power of x_j indicates which position bead j occupies, multiplication by $x_{j_1} \cdots x_{j_k}$ should cause each of the beads labeled $j_1, \ldots, j_k$ to move one position to the right. We execute this action by scanning the positions of v *from right to left.* Whenever we see a bead labeled j for some $j \in S$, we move this bead one step to the right, thus multiplying the weight by x_j. Bead collisions may occur, which will lead to object cancellations in the proof below. In the case where no bead collisions happen, we obtain a new abacus $v^* \in a_{\nu+\delta(N)}$. The beads on this abacus occur in the same order as on v, so $w(v^*) = w(v)$ and $\operatorname{sgn}(v^*) = \operatorname{sgn}(v)$. Recalling that the parts of λ (resp. ν) count the number of bead moves needed to justify the beads in v (resp. v^*), it follows that $\nu \in \mathrm{Par}_N$ is a partition obtained from $\lambda \in \mathrm{Par}_N$ by *adding 1 to k distinct parts of* λ. This means that the skew shape ν/λ is a vertical strip of size k.

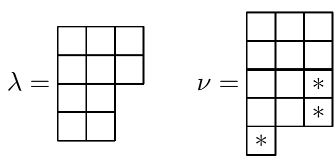
11.41. Example. Let $N = 6$ and $\lambda = (3, 3, 2, 2)$. Let v be the following abacus in $\mathrm{LAbc}(\lambda + \delta(6))$:



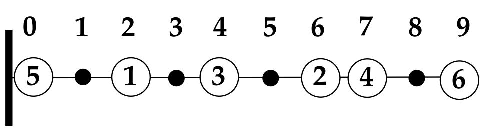
Suppose $k = 3$ and $S = \{1, 2, 3\}$. We move bead 2, then bead 3, then bead 1 one step right on the abacus. No bead collision occurs, and we get the following abacus:



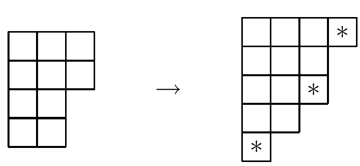
This abacus lies in $\mathrm{LAbc}(\nu + \delta(6))$, where $\nu = (3, 3, 3, 3, 1)$. Drawing the diagrams, we see that ν arises from λ by adding a vertical 3-strip:



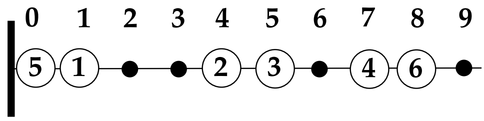
Suppose instead that $S = \{1, 2, 6\}$. This time we obtain the abacus



which is in $\mathrm{LAbc}((4, 3, 3, 2, 1) + \delta(6))$. Now the partition diagrams look like this:



However, suppose we start with the subset $S = \{3, 5, 6\}$. When we move bead 6, then bead 3, then bead 5 on the abacus v, bead 3 collides with bead 2. We can match the pair (v, S) to (w, T), where $T = \{2, 5, 6\}$ and w is this abacus:



Observe that $\operatorname{sgn}(w) = -\operatorname{sgn}(v)$ and $\operatorname{wt}(v)x_3x_5x_6 = \operatorname{wt}(w)x_2x_5x_6$. This example illustrates the cancellation idea used in the proof below.

11.42. Theorem: Antisymmetric Pieri Rule for e_k. For all $\lambda \in \operatorname{Par}_N$ and all $k \geq 1$, the following identity holds in $K[x_1, \ldots, x_N]$:

$$a_{\lambda+\delta(N)}(x_1, \ldots, x_N)e_k(x_1, \ldots, x_N) = \sum_{\substack{\beta \in \operatorname{Par}_N: \\ \beta/\lambda \text{ is a vertical } k\text{-strip}}} a_{\beta+\delta(N)}(x_1, \ldots, x_N).$$

Proof. Let X be the set of pairs (v, S) where $v \in \operatorname{LAbc}(\lambda + \delta(N))$ and S is a k-element subset of $\{1, 2, \ldots, N\}$. Letting $\operatorname{sgn}(v, S) = \operatorname{sgn}(v)$ and $\operatorname{wt}(v, S) = \operatorname{wt}(v) \prod_{j \in S} x_j$, we have

$$a_{\lambda+\delta(N)}e_k = \sum_{z \in X} \operatorname{sgn}(z) \operatorname{wt}(z).$$

Define an involution $I : X \to X$ as follows. Given $(v, S) \in X$, scan the abacus v from right to left and move each bead in S one step to the right. If this can be done with no bead collisions, we obtain an abacus v^* counted by the sum on the right side of the theorem, such that $\operatorname{sgn}(v) = \operatorname{sgn}(v^*)$ and $\operatorname{wt}(v, S) = \operatorname{wt}(v^*)$. In this case, (v, S) is a fixed point of I, and the bead motion rule defines a sign-preserving, weight-preserving bijection between these fixed points and the abaci counted by the right side of the theorem.

Now suppose a bead collision does occur. Then for some $j \in S$ and some $k \notin S$, bead k lies one step to the right of bead j in v. Take j to be the rightmost bead in v for which this is true. Let $I(v, S) = (v', S')$ where v' is v with beads j and k interchanged, and $S' = (S \sim \{j\}) \cup \{k\}$. It is immediately verified that $\operatorname{sgn}(v', S') = -\operatorname{sgn}(v, S)$, $\operatorname{wt}(v, S) = \operatorname{wt}(v', S')$, and $I(v', S') = (v, S)$. So I cancels all objects in which a bead collision occurs. $\square$

11.9 Pieri Rule for h_k

In the last section, we computed $a_{\lambda+\delta(N)}e_k$ by using a k-element *subset* of $\{1, 2, \ldots, N\}$ to move beads on a labeled abacus. Now we will compute $a_{\lambda+\delta(N)}h_k$ by moving beads based on a k-element *multiset*. This approach is motivated by the formula

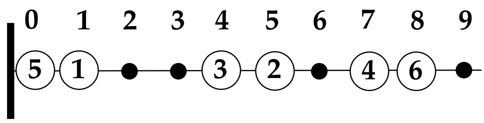
$$h_k(x_1, \ldots, x_N) = \sum_{\substack{k\text{-element multisets} \\ M \text{ of } \{1, \ldots, N\}}} \prod_{j \in M} x_j,$$

where the factor x_j is repeated as many times as j appears in M.

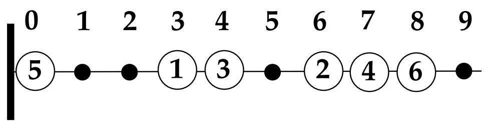
Suppose v is an abacus counted by $a_{\lambda+\delta(N)}$, and $x_1^{m_1} \cdots x_N^{m_N}$ is a typical term in h_k (so each $m_j \geq 0$ and $m_1 + \cdots + m_N = k$). Scan the beads in v *from left to right*. Whenever we encounter a bead labeled j, we move it right, one step at a time, for a total of m_j positions.

Bead collisions may occur and will lead to object cancellations later. If no collision occurs, we will have a new abacus $v^* \in \mathrm{LAbc}(\nu + \delta(N))$ with the same sign as v and weight $\mathrm{wt}(v^*) = \mathrm{wt}(v)x_1^{m_1} \cdots x_N^{m_N}$. It follows from the bead motion rule that the shape ν arises from λ by adding a *horizontal k-strip* to λ. Conversely, any abacus indexed by such a shape can be constructed from an abacus indexed by λ by a suitable choice of the bead multiset. These ideas are illustrated in the following example, which should be compared to the example in the preceding section.

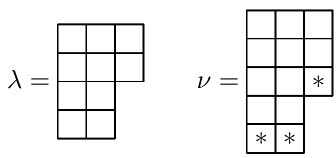
11.43. Example. Let $N = 6$ and $\lambda = (3, 3, 2, 2)$. Let v be the following abacus in $\mathrm{LAbc}(\lambda + \delta(6))$:



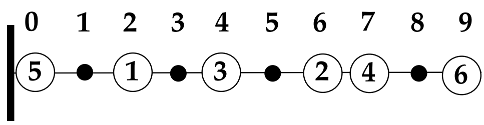
Let M be the multiset $[1, 1, 2]$. It is possible to move bead 1 to the right twice in a row, and then move bead 2 once, without causing any collisions. This produces the following abacus:



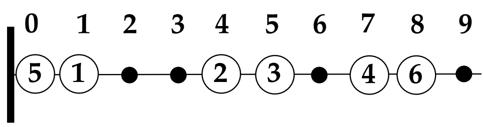
This abacus lies in $\mathrm{LAbc}(\nu + \delta(6))$, where $\nu = (3, 3, 3, 2, 2)$ arises from λ by adding a horizontal 3-strip:



If instead we take $M = [1, 2, 6]$, we move bead 1, then bead 2, then bead 6, leading to this abacus in $\mathrm{LAbc}((4, 3, 3, 2, 1) + \delta(6))$:



On the other hand, suppose we try to modify v using the multiset $M = [1, 2, 3]$. When scanning v from left to right, bead 3 moves before bead 2 and collides with bead 2. We match the pair (v, M) with the pair (w, N), where $N = [1, 2, 2]$ and w is the following abacus:



Observe that $\mathrm{sgn}(w) = -\mathrm{sgn}(v)$ and $\mathrm{wt}(v)x_1x_2x_3 = \mathrm{wt}(w)x_1x_2^2$. This example illustrates the cancellation idea used in the proof below.

11.44. Theorem: Antisymmetric Pieri Rule for h_k. For all $\lambda \in \mathrm{Par}_N$ and all $k \geq 1$, the following identity holds in $K[x_1, \ldots, x_N]$:

$$a_{\lambda+\delta(N)}(x_1, \ldots, x_N)h_k(x_1, \ldots, x_N) = \sum_{\substack{\beta \in \mathrm{Par}_N: \\ \beta/\lambda \text{ is a horizontal } k\text{-strip}}} a_{\beta+\delta(N)}(x_1, \ldots, x_N).$$

Proof. Let X be the set of pairs (v, M) where $v \in \mathrm{LAbc}(\lambda + \delta(N))$ and $M = [1^{m_1} 2^{m_2} \cdots N^{m_N}]$ is a k-element multiset. Putting $\mathrm{sgn}(v, M) = \mathrm{sgn}(v)$ and $\mathrm{wt}(v, M) = \mathrm{wt}(v) \prod_{j=1}^{N} x_j^{m_j}$, we have

$$a_{\lambda+\delta(N)} h_k = \sum_{z \in X} \mathrm{sgn}(z)\, \mathrm{wt}(z).$$

Define an involution $I : X \to X$ as follows. Given $(v, M) \in X$, scan the abacus v from left to right. When bead j is encountered in the scan, move it m_j steps right, one step at a time. If all bead motions are completed with no bead collisions, we obtain an abacus v^* counted by the sum on the right side of the theorem, such that $\mathrm{sgn}(v) = \mathrm{sgn}(v^*)$ and $\mathrm{wt}(v, M) = \mathrm{wt}(v^*)$. In this case, (v, M) is a fixed point of I, and the bead motion rule defines a sign-preserving, weight-preserving bijection between these fixed points and the abaci counted by the right side of the theorem.

Now consider the case where a bead collision does occur. Suppose the first collision occurs when bead j hits a bead k that is located $p \le m_j$ positions to the right of bead j's initial position. Define $I(v, M) = (v', M')$, where v' is v with beads j and k interchanged, and M' is obtained from M by letting j occur $m_j - p \ge 0$ times in M', letting k occur $m_k + p$ times in M', and leaving all other multiplicities the same. One may check that $\mathrm{sgn}(v', M') = -\,\mathrm{sgn}(v, M)$, $\mathrm{wt}(v, M) = \mathrm{wt}(v', M')$, and $I(v', M') = (v, M)$. So I cancels all objects in which a bead collision occurs. □

11.10 Antisymmetric Polynomials and Schur Polynomials

The Pieri rule for computing $a_{\lambda+\delta(N)} h_k$ closely resembles the rule for computing $s_\lambda h_k$ from §10.12. This resemblance leads to an algebraic proof of a formula expressing Schur polynomials as quotients of antisymmetric polynomials.

11.45. Theorem: Schur Polynomials and Antisymmetric Polynomials. For all $\lambda \in \mathrm{Par}_N$,

$$s_\lambda(x_1, \ldots, x_N) = \frac{a_{\lambda+\delta(N)}(x_1, \ldots, x_N)}{a_{\delta(N)}(x_1, \ldots, x_N)} = \frac{\det ||x_j^{\lambda_i + N - i}||_{1 \le i,j \le N}}{\det ||x_j^{N-i}||_{1 \le i,j \le N}}.$$

Proof. In 10.69, we iterated the Pieri rule

$$s_\nu(x_1, \ldots, x_N) h_k(x_1, \ldots, x_N) = \sum_{\substack{\beta \in \mathrm{Par}_N: \\ \beta/\nu \text{ is a horizontal } k\text{-strip}}} s_\beta(x_1, \ldots, x_N)$$

to deduce the formula

$$h_\mu(x_1, \ldots, x_N) = \sum_{\lambda \in \mathrm{Par}_N} K_{\lambda,\mu} s_\lambda(x_1, \ldots, x_N) \qquad (\mu \in \mathrm{Par}). \tag{11.3}$$

Recall that this derivation used semistandard tableaux to encode the sequence of horizontal strips that were added to go from the empty shape to the shape λ. Now, precisely the same idea can be applied to iterate the antisymmetric Pieri rule

$$a_{\nu+\delta(N)}(x_1, \ldots, x_N) h_k(x_1, \ldots, x_N) = \sum_{\substack{\beta \in \mathrm{Par}_N: \\ \beta/\nu \text{ is a horizontal } k\text{-strip}}} a_{\beta+\delta(N)}(x_1, \ldots, x_N).$$

If we start with $\nu = (0)$ and multiply successively by $h_{\mu_1}, h_{\mu_2}, \ldots$, we obtain the formula

$$a_{0+\delta(N)}(x_1,\ldots,x_N)h_\mu(x_1,\ldots,x_N) = \sum_{\lambda\in\mathrm{Par}_N} K_{\lambda,\mu}a_{\lambda+\delta(N)}(x_1,\ldots,x_N) \qquad (\mu \in \mathrm{Par}). \tag{11.4}$$

Now restrict attention to partitions $\lambda, \mu \in \mathrm{Par}_N(m)$. As in 10.72, we can write equations (11.3) in the form $\mathbf{H} = \mathbf{K}^t\mathbf{S}$, where $\mathbf{H} = (h_\mu : \mu \in \mathrm{Par}_N(m))$ and $\mathbf{S} = (s_\lambda : \lambda \in \mathrm{Par}_N(m))$ are column vectors, and $\mathbf{K}^t$ is the transpose of the Kostka matrix. Letting $\mathbf{A} = (a_{\lambda+\delta(N)}/a_{\delta(N)} : \lambda \in \mathrm{Par}_N(m))$, we can similarly write equations (11.4) in the form $\mathbf{H} = \mathbf{K}^t\mathbf{A}$. Finally, since the Kostka matrix is invertible (being unitriangular), we can conclude that

$$\mathbf{A} = (\mathbf{K}^t)^{-1}\mathbf{H} = \mathbf{S}.$$

Equating entries of these vectors gives the desired result. □

A purely combinatorial proof of the identity $a_{\lambda+\delta(N)} = s_\lambda a_{\delta(N)}$ will be given in §11.12.

11.11 Rim-Hook Tableaux

The connection between Schur polynomials and antisymmetric polynomials lets us deduce the following Pieri rule for calculating the product $s_\lambda p_k$.

11.46. Theorem: Symmetric Pieri Rule for p_k. For all $\lambda \in \mathrm{Par}_N$ and all $k \geq 1$, the following identity holds in $K[x_1,\ldots,x_N]$:

$$s_\lambda(x_1,\ldots,x_N)p_k(x_1,\ldots,x_N) = \sum_{\substack{\beta\in\mathrm{Par}_N:\\ \beta/\lambda \text{ is a } k\text{-ribbon } R}} \mathrm{sgn}(R)s_\beta(x_1,\ldots,x_N).$$

Proof. Start with the identity

$$a_{\lambda+\delta(N)}(x_1,\ldots,x_N)p_k(x_1,\ldots,x_N) = \sum_{\substack{\beta\in\mathrm{Par}_N:\\ \beta/\lambda \text{ is a } k\text{-ribbon } R}} \mathrm{sgn}(R)a_{\beta+\delta(N)}(x_1,\ldots,x_N)$$

(proved in 11.39), divide both sides by $a_{\delta(N)}$, and use 11.45. □

11.47. Example. Suppose we multiply $s_{(0)} = 1$ by p_4 using the Pieri rule. The result is a signed sum of Schur polynomials indexed by 4-ribbons:

$$p_4 = s_{(0)}p_4 = s_{(4)} - s_{(3,1)} + s_{(2,1,1)} - s_{(1,1,1,1)}.$$

To expand $p_{(4,3)}$ into Schur polynomials, first multiply both sides of the previous equation by p_3:

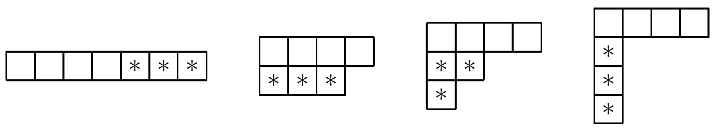
$$p_{(4,3)} = p_4p_3 = s_{(4)}p_3 - s_{(3,1)}p_3 + s_{(2,1,1)}p_3 - s_{(1,1,1,1)}p_3.$$

Now use the Pieri rule on each term on the right side. This leads to the diagrams shown in Figure 11.3. Taking signs into account, this leads to the formula

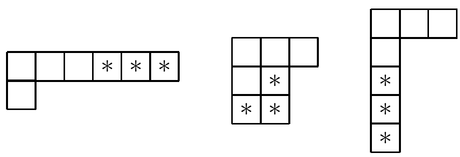
$$\begin{aligned}p_{(4,3)} \quad = \quad & s_{(7)} + s_{(4,3)} - s_{(4,2,1)} + s_{(4,1,1,1)}\\ & -s_{(6,1)} + s_{(3,2,2)} - s_{(3,1,1,1,1)}\\ & +s_{(5,1,1)} - s_{(3,3,1)} + s_{(2,1,1,1,1,1)}\\ & -s_{(4,1,1,1)} + s_{(3,2,1,1)} - s_{(2,2,2,1)} - s_{(1,1,1,1,1,1,1)}.\end{aligned}$$

Here we are assuming N (the number of variables) is at least 7.

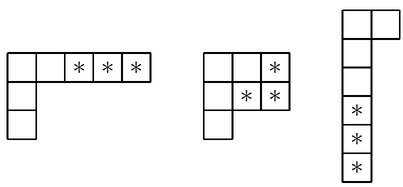
Shapes for $s_{(4)}p_3$:



Shapes for $-s_{(3,1)}p_3$:



Shapes for $s_{(2,1,1)}p_3$:



Shapes for $-s_{(1,1,1,1)}p_3$:

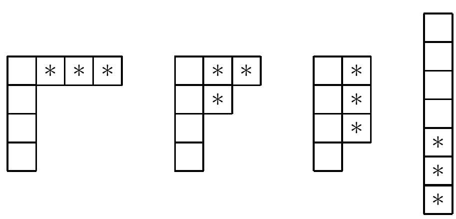


FIGURE 11.3
Adding k-ribbons to compute $s_\lambda p_k$.

Just as we used semistandard tableaux to encode successive additions of horizontal strips, we can use the following notion of a *rim-hook tableau* to encode successive additions of signed ribbons.

11.48. Definition: Rim-Hook Tableaux. Given a partition λ and a sequence $\alpha \in \mathbb{N}^s$, a *rim-hook tableau of shape λ and content α* is a sequence T of partitions

$$(0) = \nu^0 \subseteq \nu^1 \subseteq \nu^2 \subseteq \cdots \subseteq \nu^s = \lambda$$

such that ν^i/ν^{i-1} is an α_i-ribbon for $1 \leq i \leq s$. We represent this tableau pictorially by drawing the diagram of λ and entering the number i in each cell of the ribbon ν^i/ν^{i-1}. The *sign* of the rim-hook tableau T is the product of the signs of the ribbons ν^i/ν^{i-1}. (Recall that the sign of a ribbon occupying r rows is $(-1)^{r-1}$.) Let $\mathrm{RHT}(\lambda, \alpha)$ be the set of all rim-hook tableaux of shape λ and content α. Finally, define the integer

$$\chi_\alpha^\lambda = \sum_{T \in \mathrm{RHT}(\lambda,\alpha)} \mathrm{sgn}(T).$$

Rim-hook tableaux of skew shape λ/μ are defined analogously; now we require that $\nu^0 = \mu$, so that the cells of μ do not get filled with ribbons. The set $\mathrm{RHT}(\lambda/\mu, \alpha)$ and the integer $\chi_\alpha^{\lambda/\mu}$ are defined as above.

11.49. Example. Suppose we expand the product $p_4p_2p_3p_1$ into a sum of Schur polynomials. We can do this by applying the Pieri rule four times, starting with the empty shape. Each application of the Pieri rule will add a new border ribbon to the shape. The lengths of the ribbons are given by the content vector $\alpha = (4, 2, 3, 1)$. Here is one possible sequence of ribbon additions:

$$0 \to \begin{array}{cc} * & * \\ * & \\ * & \end{array} \to \begin{array}{cc} \square & \square \\ \square & * \\ \square & * \end{array} \to \begin{array}{cccc} \square & \square & * & * \\ \square & \square & * & \\ \square & \square & & \end{array} \to \begin{array}{cccc} \square & \square & \square & \square \\ \square & \square & \square & \\ \square & \square & * & \end{array}$$

This sequence of shapes defines a rim-hook tableau

$$T = ((0), (2, 1, 1), (2, 2, 2), (4, 3, 2), (4, 3, 3))$$

which can be visualized using the following diagram:

$$T = \begin{array}{cccc} 1 & 1 & 3 & 3 \\ 1 & 2 & 3 & \\ 1 & 2 & 4 & \end{array}.$$

Note that the ribbons we added have signs $+1$, -1, -1, and $+1$, so $\mathrm{sgn}(T) = +1$. This particular choice of ribbon additions will therefore produce a term $+s_{(4,3,3)}$ in the Schur expansion of $p_{(4,2,3,1)}$.

Now suppose we want to know the coefficient of $s_{(4,3,3)}$ in the Schur expansion of $p_4p_2p_3p_1$. The preceding discussion shows that we will obtain a term $\pm s_{(4,3,3)}$ for every rim-hook tableau of shape $(4, 3, 3)$ and content $(4, 2, 3, 1)$, where the sign of the term is the sign of the tableau. To find the desired coefficient, we must enumerate all the objects in $\mathrm{RHT}((4, 3, 3), (4, 2, 3, 1))$. In addition to the tableau T displayed above, we find the following tableaux:

$$\begin{array}{cccc} 1 & 1 & 3 & 4 \\ 1 & 2 & 3 & \\ 1 & 2 & 3 & \end{array} \qquad \begin{array}{cccc} 1 & 1 & 2 & 2 \\ 1 & 3 & 3 & \\ 1 & 3 & 4 & \end{array} \qquad \begin{array}{cccc} 1 & 1 & 1 & 4 \\ 1 & 2 & 2 & \\ 3 & 3 & 3 & \end{array} \qquad \begin{array}{cccc} 1 & 1 & 1 & 1 \\ 2 & 3 & 3 & \\ 2 & 3 & 4 & \end{array}$$

The signs of the new tableaux are -1, -1, -1, and $+1$, so the coefficient is $+1-1-1-1+1 = -1$.

The calculations in the preceding example generalize to give the following rule for expanding power-sum polynomials into sums of Schur polynomials.

11.50. Theorem: Schur Expansion of Power-Sum Polynomials. For all vectors $\alpha \in \mathbb{N}^t$ and all $N \geq 1$,

$$p_\alpha(x_1, \ldots, x_N) = \sum_{\lambda \in \mathrm{Par}_N} \chi_\alpha^\lambda s_\lambda(x_1, \ldots, x_N).$$

Proof. By iteration of the Pieri rule, the coefficient of s_λ in $p_\alpha = s_{(0)} p_{\alpha_1} \cdots p_{\alpha_t}$ is the signed sum of all sequences of partitions

$$0 = \nu^0 \subseteq \nu^1 \subseteq \nu^2 \subseteq \cdots \subseteq \nu^t = \lambda$$

such that the skew shape ν^i/ν^{i-1} is an α_i-ribbon for all i. By the very definition of rim-hook tableaux, this sum is precisely χ_α^λ. □

11.51. Theorem: Symmetry of χ_α^λ. If α and β are compositions with $\mathrm{sort}(\alpha) = \mathrm{sort}(\beta)$, then $\chi_\alpha^\lambda = \chi_\beta^\lambda$ for all partitions λ.

Proof. The hypothesis implies that the sequence $\alpha = (\alpha_1, \alpha_2, \ldots)$ can be rearranged to the sequence $\beta = (\beta_1, \beta_2, \ldots)$. It follows from this that $p_\alpha = \prod p_{\alpha_i} = \prod p_{\beta_i} = p_\beta$, since multiplication of polynomials is commutative. Let $k = \sum_i \alpha_i$ and take $N \geq k$. Two applications of the previous theorem give

$$\sum_{\lambda \in \mathrm{Par}(k)} \chi_\alpha^\lambda s_\lambda = p_\alpha = p_\beta = \sum_{\lambda \in \mathrm{Par}(k)} \chi_\beta^\lambda s_\lambda.$$

By linear independence of the Schur polynomials $\{s_\lambda(x_1, \ldots, x_N) : \lambda \in \mathrm{Par}(k)\}$, we conclude that $\chi_\alpha^\lambda = \chi_\beta^\lambda$ for all λ. □

11.52. Remark. The last theorem and corollary extend to skew shapes as follows. If μ is a partition, then

$$s_\mu p_\alpha = \sum_{\substack{\lambda \in \mathrm{Par}_N: \\ \mu \subseteq \lambda}} \chi_\alpha^{\lambda/\mu} s_\lambda.$$

Furthermore, if $\mathrm{sort}(\alpha) = \mathrm{sort}(\beta)$ then $\chi_\alpha^{\lambda/\mu} = \chi_\beta^{\lambda/\mu}$. The proof is the same as before, replacing (0) by μ throughout.

We have just seen how to expand power-sum symmetric polynomials into sums of Schur polynomials. Conversely, it is possible to express Schur polynomials in terms of the p_μ's. We can use the Hall scalar product from §10.26 to derive this expansion from the previous one.

11.53. Theorem: Power-Sum Expansion of Schur Polynomials. For $N \geq k$ and all $\lambda \in \mathrm{Par}(k)$,

$$s_\lambda(x_1, \ldots, x_N) = \sum_{\mu \in \mathrm{Par}(k)} \frac{\chi_\mu^\lambda}{z_\mu} p_\mu(x_1, \ldots, x_N).$$

Proof. For all $\mu \in \mathrm{Par}(k)$, we know that $p_\mu = \sum_{\nu \in \mathrm{Par}(k)} \chi_\mu^\nu s_\nu$. Therefore, for a given partition $\lambda \in \mathrm{Par}(k)$,

$$\langle p_\mu, s_\lambda \rangle = \sum_{\nu \in \mathrm{Par}(k)} \chi_\mu^\nu \langle s_\nu, s_\lambda \rangle = \chi_\mu^\lambda$$

since the Schur polynomials are orthonormal relative to the Hall scalar product. Now, since the p_μ's form a basis of Λ_N^k, we know there exist scalars $c_\nu \in \mathbb{Q}$ with $s_\lambda = \sum_\nu c_\nu p_\nu$. To find a given coefficient c_μ, we compute

$$\chi_\mu^\lambda = \langle p_\mu, s_\lambda \rangle = \sum_\nu c_\nu \langle p_\mu, p_\nu \rangle = c_\mu z_\mu,$$

where the last equality follows by definition of the Hall scalar product. We see that $c_\mu = \chi_\mu^\lambda / z_\mu$, as desired. □

11.12 Abaci and Tableaux

This section contains a combinatorial proof of the identity

$$a_{\delta(N)}(x_1, \ldots, x_N) s_\lambda(x_1, \ldots, x_N) = a_{\lambda+\delta(N)}(x_1, \ldots, x_N),$$

which we proved algebraically in §11.10.

Let X be the set of pairs (v, T), where v is a justified labeled abacus with N beads and T is a semistandard tableau using letters in $\{1, 2, \ldots, N\}$. It will be convenient to use the following non-standard total ordering on this alphabet that depends on v: $i <_v j$ iff bead i is to the right of bead j on the abacus v. Equivalently, we can describe the total order by writing

$$v_{N-1} <_v v_{N-2} <_v \cdots <_v v_1 <_v v_0.$$

Here are two examples of objects in X when $N = 7$ and $\lambda = (7, 7, 5, 3, 2)$:

$$(v^1, T^1) = \left(7654321000\cdots, \begin{array}{|c|c|c|c|c|c|c|} \hline 1&1&1&1&1&1&1 \\ \hline 2&2&2&2&2&2&2 \\ \hline 3&3&5&5&5 \\ \cline{1-5} 6&6&6 \\ \cline{1-3} 7&7 \\ \cline{1-2} \end{array} \right);$$

$$(v^2, T^2) = \left(2451763000\cdots, \begin{array}{|c|c|c|c|c|c|c|} \hline 3&3&3&3&3&3&3 \\ \hline 6&6&6&6&6&6&6 \\ \hline 7&7&5&5&5 \\ \cline{1-5} 4&4&4 \\ \cline{1-3} 2&2 \\ \cline{1-2} \end{array} \right).$$

Note that we can pass from the first tableau (which is semistandard under the usual ordering) to the second tableau (which is semistandard relative to one of the non-standard orderings) by applying the permutation $7 \mapsto 2, 6 \mapsto 4, \ldots$ to each entry in the first tableau. It follows that the generating function for the set $\mathrm{SSYT}_N(\lambda)$ relative to one of the orderings $<_v$ can be obtained from the usual generating function for semistandard tableaux (namely $s_\lambda(x_1, \ldots, x_N)$) by applying the permutation $x_7 \mapsto x_2, x_6 \mapsto x_4, \ldots$. Since Schur polynomials are symmetric, the answer is still $s_\lambda(x_1, \ldots, x_N)$. By the product rule for weighted sets, we conclude that

$$\sum_{(v,T) \in X} \mathrm{sgn}(v)\,\mathrm{wt}(v)\,\mathrm{wt}(T) = a_{\delta(N)}(x_1, \ldots, x_N) s_\lambda(x_1, \ldots, x_N).$$

On the other hand, the generating function for the set Y of N-bead labeled abaci with beads in positions $\lambda+\delta(N)$ is $a_{\lambda+\delta(N)}(x_1,\ldots,x_N)$. So it suffices to define a sign-reversing, weight-preserving involution $I : X \to X$ where the fixed point set of I corresponds bijectively to Y.

The main idea is that the tableau T encodes a sequence of bead motions on the abacus v. If performing these movements causes a bead collision, then (v,T) will cancel with some other object in X. Otherwise, the abacus obtained from v by the bead motions will be one of the objects in Y.

A tableau T specifies bead motions as follows. Define the *word of* T to be the word $w(T) = w_1w_2\cdots w_n$ (where $n = |\lambda|$) obtained by concatenating the rows of T from bottom to top. For example, the object (v^2, T^2) shown above has

$$w(T^2) = 22444775556666663333333.$$

Now, given $(v,T) \in X$, scan the symbols in $w(T)$ from right to left. When a symbol j is encountered, move the bead labeled j in v one step to the right.

Let us first determine which objects (v,T) have no bead collisions. Suppose $v = v_0\ldots v_{N-1}00\ldots$. Let i be the last entry in the top row of T, which is the rightmost letter in $w(T)$. We must first move bead i one step to the right. This move will already cause a collision (since v is justified) unless $i = v_{N-1}$. Since v_{N-1} is the smallest letter relative to $<_v$ and T is semistandard, $i = v_{N-1}$ iff all entries in the top row of T are equal to v_{N-1}. In this situation, we will move the rightmost bead v_{N-1} to the right λ_1 positions with no collisions.

Now we repeat the argument on the second row of T. The rightmost entry j in this row cannot be v_{N-1} (otherwise we would not have a strict increase in every column). The only way to avoid an immediate bead collision is when $j = v_{N-2}$, in which case all entries in the second row must equal v_{N-2}. In this situation, bead v_{N-2} will move to the right λ_2 positions with no collisions.

Continuing similarly, we see that (v,T) will have no collisions iff for all k, the kth row of T consists of λ_k copies of the kth smallest letter v_{N-k}. Moving the beads on v according to T has the effect of unjustifying v to an abacus $v^* \in Y = \mathrm{LAbc}(\lambda + \delta(N))$. Defining $I(v,T) = (v,T)$ in this case, we therefore have specified a bijection between the fixed points of I and Y. For example,

$$(v,T) = \left(2451763000\cdots, \begin{array}{|c|c|c|c|c|c|c|}\hline 3&3&3&3&3&3&3\\\hline 6&6&6&6&6&6&6\\\hline 7&7&7&7&7\\\cline{1-5} 1&1&1\\\cline{1-3} 5&5\\\cline{1-2}\end{array}\right) \mapsto v^* = 24005010070063000\cdots.$$

The map $(v,T) \mapsto v^*$ preserves signs and weights.

To complete the proof, we describe a cancellation mechanism to pair off objects (v,T) in which bead collisions do occur. Suppose the first bead collision for (v,T) occurs when some bead i moves to the right one step and bumps into bead j. Note that $i >_v j$, and i,j must be two adjacent letters in the total ordering $>_v$. Define $(v',T') = I(v,T)$ as follows. We obtain v' from v by interchanging the adjacent beads i and j, so that $\mathrm{sgn}(v') = -\mathrm{sgn}(v)$, $\mathrm{wt}(v')x_j = \mathrm{wt}(v)x_i$, and $<_{v'}$ agrees with $<_v$ except that now $i <_{v'} j$.

We obtain T' from T by modifying the occurrences of i and j in $w(T)$ by a procedure similar to the one used in §10.6. By the argument used to determine the fixed points of I, we know that the occurrence of i in T that caused the bead collision is the rightmost entry in some row of T, say the kth row; furthermore, for $1 \le l < k$, row l consists of λ_l copies of v_{N-l}. Now $i >_v v_{N-k}$ (or this entry of T would not cause a collision), and so $j \ge_v v_{N-k}$. This means that no entry in the first $k-1$ rows of T equals i or j, so these rows can be ignored in the following discussion.

We now describe how to change T into T'. Whenever j occurs directly above i in T (call these occurrences *matched pairs*), interchange these two symbols. Some rows of T will contain unmatched i's and j's, in which $a \geq 0$ copies of j are followed by $b \geq 0$ copies of i. In particular, row k will have $a \geq 0$ and $b > 0$, since the i at the end of the row cannot be matched with a j above it. In row k, replace the unmatched symbols $j^a i^b$ by $j^{a+1}i^{b-1}$. Then, in all rows containing unmatched i's and j's (including the new row k), replace the unmatched symbols $j^a i^b$ by $i^b j^a$. The following assertions can now be checked: T' is a semistandard tableau relative to $<_{v'}$; T' has one fewer i and one more j than T does; $\mathrm{wt}(T')x_i = \mathrm{wt}(T)x_j$; $\mathrm{wt}(v', T') = \mathrm{wt}(v, T)$; $\mathrm{sgn}(v', T') = -\mathrm{sgn}(v)$; the last symbol in row k of T' is an unmatched j; this unmatched j will cause the first bead collision when T' is used to move the beads on v'; and $I(v', T') = (v, T)$.

11.54. Example. Consider the object

$$(v,T) = \left(2451763000\cdots, \begin{array}{|c|c|c|c|c|c|c|}\hline 3&3&3&3&3&3&3\\\hline 6&6&6&6&6&6&6\\\hline 7&7&5&5&5\\\cline{1-5} 4&4&4\\\cline{1-3} 2&2\\\cline{1-2}\end{array}\right).$$

Processing the first two rows of T, we move bead 3 right seven positions, then move bead 6 right 7 positions with no collisions. But in row 3, the rightmost symbol $i = 5$ causes a collision with bead $j = 1$. There are no matched pairs of 5's and 1's in this tableau, so we first change the 555 in row 3 to 155, and then change this string to 551 to preserve semistandardness under the new ordering. We have

$$I(v,T) = \left(2415763000\cdots, \begin{array}{|c|c|c|c|c|c|c|}\hline 3&3&3&3&3&3&3\\\hline 6&6&6&6&6&6&6\\\hline 7&7&5&5&1\\\cline{1-5} 4&4&4\\\cline{1-3} 2&2\\\cline{1-2}\end{array}\right).$$

If we apply I to this object, bead 1 bumps into bead 5, and we find that $I(I(v,T)) = (v,T)$.

11.55. Example. Consider the object

$$(v,T) = \left(2451763000\cdots, \begin{array}{|c|c|c|c|c|c|c|}\hline 3&3&3&3&3&3&3\\\hline 6&6&6&7&7&7&7\\\hline 7&7&5&5&5\\\cline{1-5} 4&4&4\\\cline{1-3} 2&2\\\cline{1-2}\end{array}\right).$$

Now the first collision occurs when bead $i = 7$ bumps into bead $j = 6$ because of the 7 at the end of the second row of T. The first two 6's in that row are matched with 7's below, so the unmatched i's and j's in row 2 are 67777. We replace this string first by 66777, and then by 77766. Interchanging the matched 6's and 7's leads to

$$I(v,T) = \left(2451673000\cdots, \begin{array}{|c|c|c|c|c|c|c|}\hline 3&3&3&3&3&3&3\\\hline 7&7&7&7&7&6&6\\\hline 6&6&5&5&5\\\cline{1-5} 4&4&4\\\cline{1-3} 2&2\\\cline{1-2}\end{array}\right).$$

11.56. Example. The reader may check that

$$I\left(76543210\cdots, \begin{array}{|c|c|c|c|c|c|c|}\hline 1&1&1&2&2&3&4\\\hline 2&3&3&3&4&4&7\\\hline 3&4&5&5&5\\\cline{1-5}\end{array}\right) = \left(76534210\cdots, \begin{array}{|c|c|c|c|c|c|c|}\hline 1&1&1&2&2&4&3\\\hline 2&4&4&3&3&3&7\\\hline 3&3&5&5&5\\\cline{1-5}\end{array}\right).$$

11.13 Skew Schur Polynomials

In the remainder of this chapter, we will develop further combinatorial properties of skew Schur polynomials. Recall the definition from 10.14: for every skew shape λ/μ,

$$s_{\lambda/\mu}(x_1,\ldots,x_N) = \sum_{T\in\mathrm{SSYT}_N(\lambda/\mu)} x^{c(T)}.$$

In 10.35, we proved that skew Schur polynomials are symmetric. More precisely, we have the expansion in the monomial basis:

$$s_{\lambda/\mu}(x_1,\ldots,x_N) = \sum_{\nu\in\mathrm{Par}_N} K_{\lambda/\mu,\nu} m_\nu(x_1,\ldots,x_N),$$

where $K_{\lambda/\mu,\nu}$ is the number of semistandard tableaux of shape λ/μ and content ν. Our current goal is to find combinatorial formulas for the expansion of skew Schur polynomials relative to some other bases for Λ. We begin by proving an algebraic fact involving the Hall scalar product.

11.57. Theorem: Skew Schur Polynomials and the Hall Scalar Product. Suppose $\lambda,\mu\in\mathrm{Par}$, $k=|\lambda|-|\mu|$, $N\geq|\lambda|$, and $f\in\Lambda_N^k$. Then $\langle s_{\lambda/\mu},f\rangle = \langle s_\lambda, s_\mu f\rangle$.

Proof. We first prove the result for $f=h_\nu$, where $\nu\in\mathrm{Par}(k)$. On one hand, we have the expansion

$$s_{\lambda/\mu} = \sum_{\rho\in\mathrm{Par}(k)} K_{\lambda/\mu,\rho} m_\rho.$$

Taking the scalar product of both sides with h_ν gives $\langle s_{\lambda/\mu},h_\nu\rangle = K_{\lambda/\mu,\nu}$ (see 10.132).

On the other hand, the Pieri rule shows that

$$s_\mu h_\nu = \sum_\rho K_{\rho/\mu,\nu} s_\rho$$

(see 10.71). Taking the scalar product with s_λ gives $\langle s_\lambda, s_\mu h_\nu\rangle = K_{\lambda/\mu,\nu}$. Thus the result holds for every f in the complete homogeneous basis.

The general case now follows by linearity: given any $f\in\Lambda_N^k$, write $f=\sum_\nu c_\nu h_\nu$ for certain scalars $c_\nu\in K$. Then compute

$$\begin{aligned}\langle s_{\lambda/\mu},f\rangle &= \left\langle s_{\lambda/\mu},\sum_\nu c_\nu h_\nu\right\rangle = \sum_\nu c_\nu\langle s_{\lambda/\mu},h_\nu\rangle\\ &= \sum_\nu c_\nu\langle s_\lambda,s_\mu h_\nu\rangle = \left\langle s_\lambda,\sum_\nu c_\nu s_\mu h_\nu\right\rangle = \langle s_\lambda,s_\mu f\rangle. \quad\square\end{aligned}$$

We can use 11.57 to expand skew Schur polynomials in terms of power-sum symmetric polynomials.

11.58. Theorem: Power-Sum Expansion of Skew Schur Polynomials. Suppose $\mu\subseteq\lambda$ are partitions with $k=|\lambda|-|\mu|$. For all $N\geq|\lambda|$,

$$s_{\lambda/\mu}(x_1,\ldots,x_N) = \sum_{\nu\in\mathrm{Par}(k)} \frac{\chi_\nu^{\lambda/\mu}}{z_\nu} p_\nu(x_1,\ldots,x_N).$$

Proof. We imitate the proof of 11.53. Start with the expansion

$$s_\mu p_\nu = \sum_\lambda \chi_\nu^{\lambda/\mu} s_\lambda.$$

Now take the scalar product of both sides with a given partition λ:

$$\langle s_\lambda, s_\mu p_\nu \rangle = \chi_\nu^{\lambda/\mu}.$$

We know the symmetric polynomial $s_{\lambda/\mu}$ has some expansion in the power-sum basis, say $s_{\lambda/\mu} = \sum_\nu a_\nu p_\nu$ for some $a_\nu \in K$. To find a particular a_ν, take the scalar product with p_ν/z_ν to get

$$a_\nu = \left\langle s_{\lambda/\mu}, p_\nu/z_\nu \right\rangle = \langle s_\lambda, s_\mu p_\nu/z_\nu \rangle = \langle s_\lambda, s_\mu p_\nu \rangle / z_\nu = \chi_\nu^{\lambda/\mu}/z_\nu. \qquad \square$$

We also deduce the effect of the involution ω on skew Schur polynomials.

11.59. Theorem: Action of ω on Skew Schur Polynomials. For all partitions $\mu \subseteq \lambda$ and all $N \geq |\lambda|$,

$$\omega(s_{\lambda/\mu}(x_1, \ldots, x_N)) = s_{\lambda'/\mu'}(x_1, \ldots, x_N).$$

Proof. We already know that the involution ω is a ring homomorphism and isometry sending every s_α to $s_{\alpha'}$. For each partition ν of size $|\lambda| - |\mu|$, we can therefore write:

$$\begin{aligned} \left\langle \omega(s_{\lambda/\mu}), s_\nu \right\rangle &= \left\langle \omega^2(s_{\lambda/\mu}), \omega(s_\nu) \right\rangle = \left\langle s_{\lambda/\mu}, s_{\nu'} \right\rangle = \langle s_\lambda, s_\mu s_{\nu'} \rangle \\ &= \langle \omega(s_\lambda), \omega(s_\mu s_{\nu'}) \rangle = \langle s_{\lambda'}, s_{\mu'} s_\nu) \rangle = \left\langle s_{\lambda'/\mu'}, s_\nu \right\rangle. \end{aligned}$$

Thus $\omega(s_{\lambda/\mu})$ and $s_{\lambda'/\mu'}$ have the same expansion in the Schur basis and are therefore equal. $\square$

11.14 Jacobi-Trudi Formulas

Our next goal is to obtain formulas expressing skew Schur polynomials as determinants involving the complete symmetric polynomials h_k or the elementary symmetric polynomials e_k. To derive these results, we need a new combinatorial construction relating tableaux to collections of non-intersecting lattice paths.

We begin by interpreting $h_k(x_1, \ldots, x_N)$ in terms of lattice paths. Fix an integer a and consider the set S of lattice paths from $(a, 1)$ to $(a + k, N)$ that take unit steps up (u) and east (e). We can encode a path p in this set by listing the y-coordinates of the successive east steps of p. For example, the path eeuueuee corresponds to the sequence $1, 1, 3, 4, 4$. This gives a bijection from S to the set of weakly increasing sequences $1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq N$. Let us weight the path corresponding to this sequence by $x_{i_1} x_{i_2} \cdots x_{i_k}$. Comparing to the definition of h_k, we see that

$$h_k(x_1, \ldots, x_N) = \sum_{p \in S} \mathrm{wt}(p).$$

This formula holds for all integers k (possibly negative), provided we take $h_0 = 1$ and $h_k = 0$ for negative k.

Now let λ be a partition with $n \leq N$ parts, and let $\mu \subseteq \lambda$. Let X be the set of fillings of the skew shape λ/μ using letters in $\{1, 2, \ldots, N\}$ such that each row weakly increases. Let

Y be the set of sequences $P = (p_1, \ldots, p_n)$ where p_i is a lattice path from $(n - i + \mu_i, 1)$ to $(n - i + \lambda_i, N)$. Let $\mathrm{wt}(P) = \mathrm{wt}(p_1) \cdots \mathrm{wt}(p_n)$ record the y-coordinates of all the east steps of the paths in P. As explained above, we can encode each row i of a filling $U \in X$ as a lattice path p_i from $(a, 1)$ to $(a + \lambda_i - \mu_i, N)$, where $a = n - i + \mu_i$. The association $U \mapsto (p_1, \ldots, p_n)$ defines a weight-preserving bijection $f : X \to Y$. Some examples are shown in Figure 11.4.

We say that two lattice paths *intersect* iff they share a common edge or vertex. Let Y' be the set of $P \in Y$ such that no two paths in P intersect. Inspection of Figure 11.4 suggests that f restricts to a weight-preserving bijection from $\mathrm{SSYT}_N(\lambda/\mu)$ to Y'. To see why this holds, consider consecutive entries $U(i, j) = a$ and $U(i + 1, j) = b$ in column j of a filling $U \in X$. In $f(U)$, path p_i has an east step from $(n-i+\mu_i+(j-\mu_i)-1, a) = (n+j-i-1, a)$ to $(n + j - i, a)$, whereas p_{i+1} has an east step from $(n + j - i - 2, b)$ to $(n + j - i - 1, b)$. Suppose $a \geq b$. Since the beginning of p_i goes from $(n - i + \mu_i, 1)$ to $(n + j - i - 1, a)$, there is no way for p_{i+1} (which starts to the left of p_i) to reach the point $(n+j-i-1, b)$ without intersecting p_i. Conversely, suppose two paths intersect. Then there must exist i such that p_i and p_{i+1} intersect. The earliest intersection of these paths must occur when p_{i+1} "bumps into" p_i by taking an east step ending at some point $(n + j - i - 1, b)$. One may now check that there must exist an east step in p_i starting at $(n + j - i - 1, a)$ for some $a \geq b$, which shows that $U(i, j) \geq U(i + 1, j)$ in the filling U.

Now we are ready to prove the Jacobi-Trudi formulas. The idea is to introduce a large collection of signed, weighted sequences of paths that model the terms of a determinant. Cancellations will remove all sequences of intersecting paths, leaving only the objects in Y', which correspond to semistandard skew tableaux.

11.60. Theorem: Jacobi-Trudi Formula. Suppose λ is a partition with $n \leq N$ parts, and $\mu \subseteq \lambda$. Then

$$s_{\lambda/\mu}(x_1, \ldots, x_N) = \det ||h_{\lambda_i - \mu_j + j - i}(x_1, \ldots, x_N)||_{1 \leq i,j \leq n}.$$

Proof. By the definition of a determinant (see 9.37), the right side of the desired formula can be written

$$\sum_{w \in S_n} \mathrm{sgn}(w) \prod_{i=1}^{n} h_{\lambda_i - \mu_{w(i)} + w(i) - i}(x_1, \ldots, x_N).$$

This is the generating function for the following signed, weighted set. Let Z be the set of sequences $(w, p_1, \ldots, p_n)$ such that $w \in S_n$ and p_i is a path from $(n - w(i) + \mu_{w(i)}, 1)$ to $(\lambda_i + n - i, N)$. The weight of such a sequence is $\prod_{i=1}^{n} \mathrm{wt}(p_i)$, and the sign is $\mathrm{sgn}(w)$.

The following involution will cancel all objects $(w, p_1, \ldots, p_n)$ in which two or more paths intersect. Among all lattice points (u, v) where two paths intersect, choose the one for which u is minimized; if there are ties, choose the point that minimizes v. Let $i < j$ be the two least indices such that p_i and p_j pass through (u, v). Write $p_i = qr$ where q (resp. r) is the part of p_i before (resp. after) the point (u, v). Similarly write $p_j = st$. Now, pair the given object with the object $(w', p'_1, \ldots, p'_n)$ where $w' = w \circ (i, j)$, $p'_i = sr$, $p'_j = qt$, and $p'_k = p_k$ for all $k \neq i, j$. (Thus we have switched the initial segments of the two intersecting paths.) One may check that the new object lies in Z and has the same weight and opposite sign as the original object. One should also check that applying the map a second time will restore the original object, so we have an involution. Some examples are shown in Figure 11.5. (Note that path p_i goes from the $w(i)$th point from the right on the line $y = 1$ to the ith point from the right on the line $y = N$.) Let us consider an object $(w, p_1, \ldots, p_n)$ in Z that is not canceled by the involution. No two paths in this object can intersect. We claim that this forces $w = \mathrm{id}$. For otherwise, there would exist $i < j$ with $w(i) > w(j)$. But then p_i would start to the left of p_j on the line $y = 1$ and end to the right of p_j on the line $y = N$,

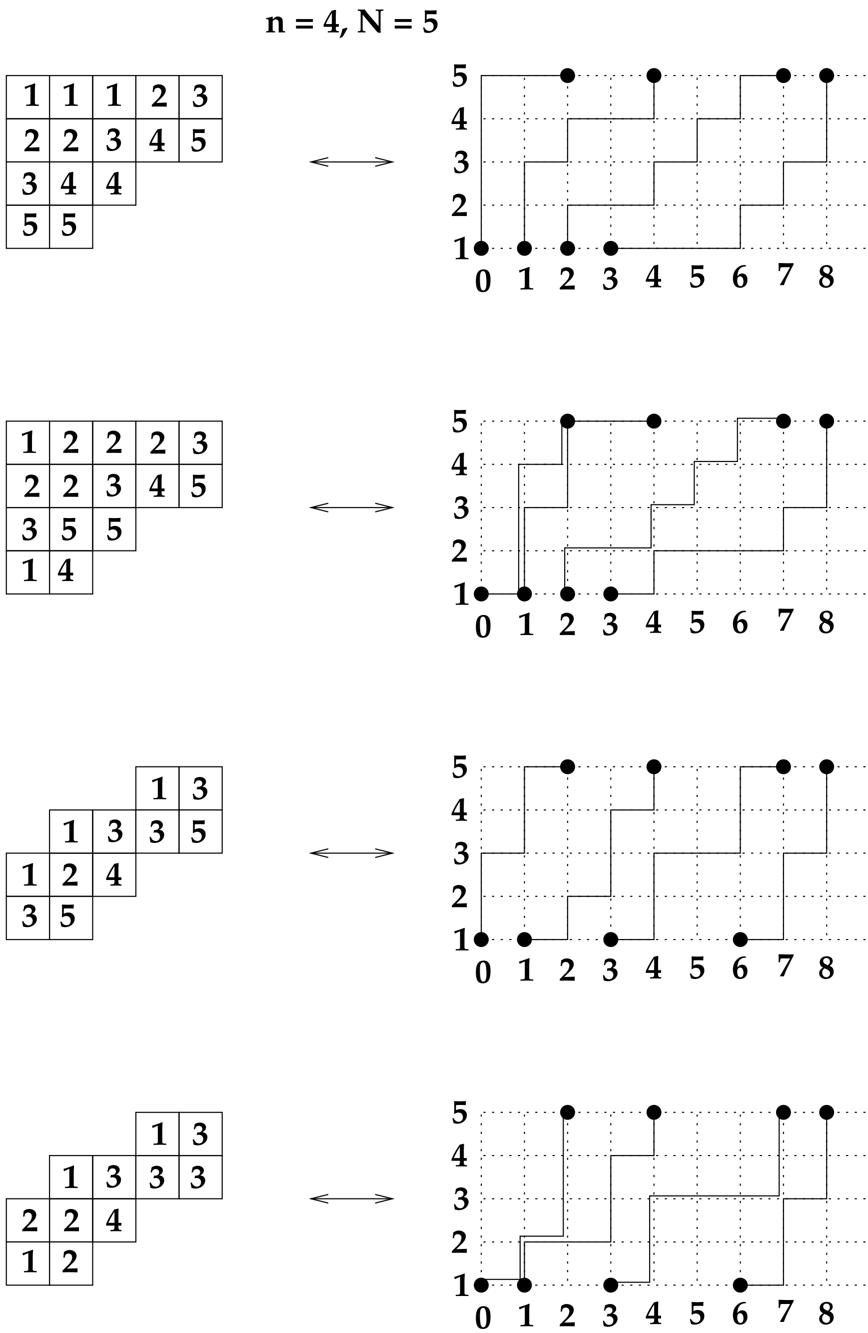


FIGURE 11.4
Encoding fillings of a skew shape by sequences of lattice paths.

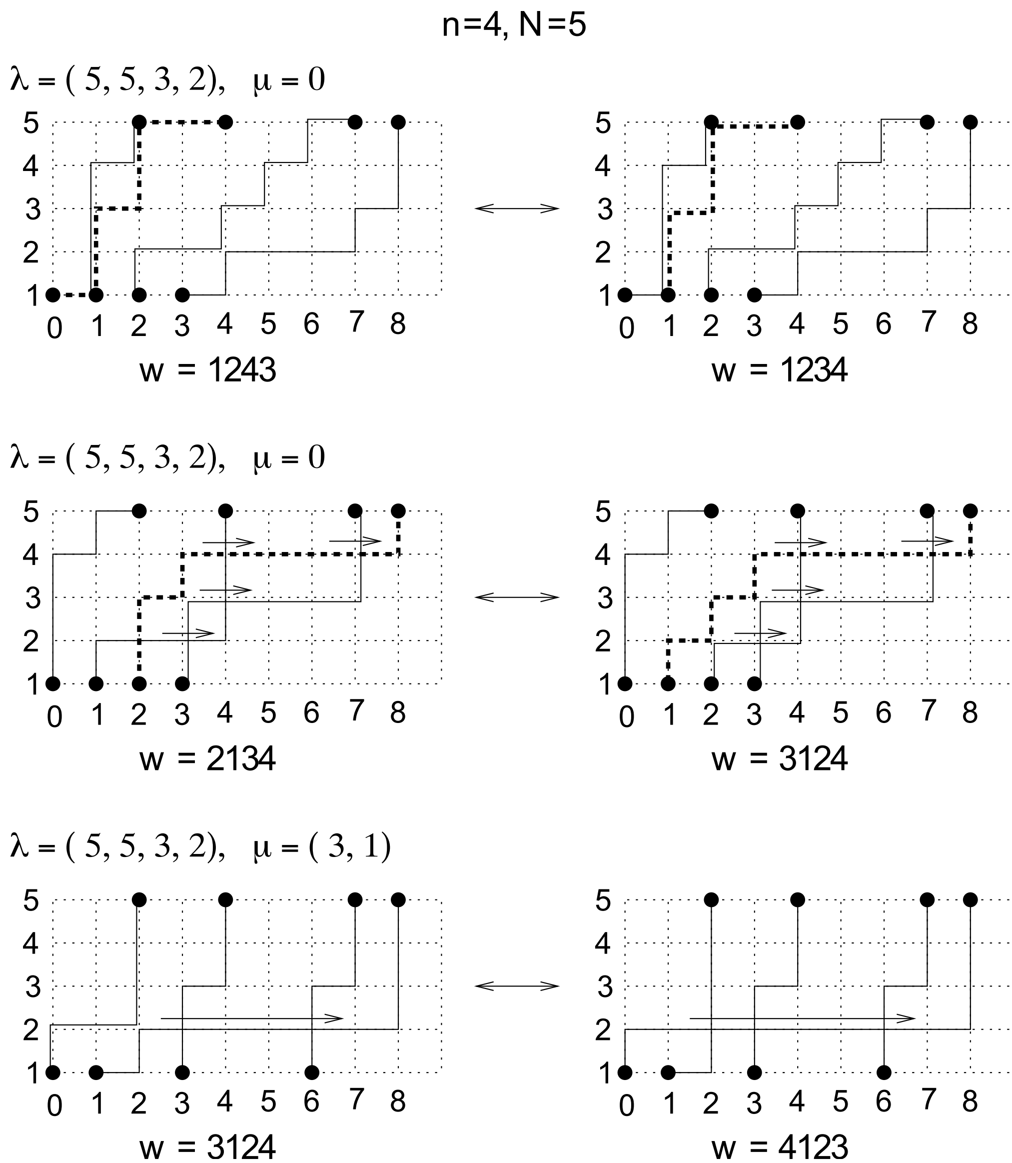


FIGURE 11.5
Cancellation mechanism for intersecting paths.

which would force p_i and p_j to intersect. So $w = \text{id}$. Erasing w maps the fixed points in Z bijectively to the set Y', which in turn maps bijectively to $\text{SSYT}_N(\lambda/\mu)$, as shown in the discussion preceding the theorem. $\square$

11.61. Theorem: Second Jacobi-Trudi Formula. Suppose λ is a partition with $\lambda_1 = n \leq N$, and $\mu \subseteq \lambda$. Then

$$s_{\lambda/\mu}(x_1, \ldots, x_N) = \det ||e_{\lambda'_i - \mu'_j + j - i}(x_1, \ldots, x_N)||_{1 \leq i,j \leq n}.$$

Proof. For all $f_{ij} \in \Lambda_N$, we have $\omega(\det ||f_{ij}||) = \det ||\omega(f_{ij})||$. This follows from the defining formula for determinants and the fact that ω is a ring homomorphism. Now 11.61 follows by applying ω to both sides of the first Jacobi-Trudi formula

$$s_{\lambda'/\mu'} = \det ||h_{\lambda'_i - \mu'_j + j - i}||.$$ $\square$

11.62. Example. According to the first Jacobi-Trudi formula,

$$s_{(3,3,1)} = \det \begin{bmatrix} h_3 & h_4 & h_5 \\ h_2 & h_3 & h_4 \\ 0 & 1 & h_1 \end{bmatrix} = h_{(3,3,1)} + h_{(5,2)} - h_{(4,3)} - h_{(4,2,1)}.$$

Note that the main diagonal entries in the formula for s_λ are $h_{\lambda_1}, h_{\lambda_2}, \ldots, h_{\lambda_n}$, and the subscripts increase by 1 (resp. decrease by 1) as we read to the right (resp. left) along each row. Similarly,

$$s_{(3,3,1)} = \det \begin{bmatrix} e_3 & e_4 & e_5 \\ e_1 & e_2 & e_3 \\ 1 & e_1 & e_2 \end{bmatrix} = e_{(3,2,2)} + e_{(4,3)} + e_{(5,1,1)} - e_{(5,2)} - e_{(3,3,1)} - e_{(4,2,1)}.$$

Here is a typical expansion of a skew Schur polynomial:

$$s_{(5,5,3)/(3,2,0)} = \det \begin{bmatrix} h_2 & h_4 & h_7 \\ h_1 & h_3 & h_6 \\ 0 & 1 & h_3 \end{bmatrix} = h_{(3,3,2)} + h_{(7,1)} - h_{(4,3,1)} - h_{(6,2)}.$$

11.15 Inverse Kostka Matrix

In Chapter 10, the Kostka matrix played a prominent role in relating the Schur basis of Λ_N to several other bases. More specifically, we proved the formulas

$$s_\lambda = \sum_\mu K_{\lambda,\mu} m_\mu, \qquad h_\mu = \sum_\lambda K_{\lambda,\mu} s_\lambda, \qquad e_\mu = \sum_\lambda K_{\lambda,\mu} s_{\lambda'},$$

where all symmetric polynomials have N variables and all summations extend over Par_N. Letting $\mathbf{K} = \mathbf{K}_N$ be the matrix of Kostka numbers with rows and columns indexed by elements of Par_N, these relations can also be written

$$\mathbf{s} = \mathbf{K}\mathbf{m}, \qquad \mathbf{h} = \mathbf{K}^t\mathbf{s}, \qquad \mathbf{e} = \mathbf{K}^t\omega(\mathbf{s}).$$

We know that the Kostka matrix is invertible (being unitriangular). Let $K'_{\lambda,\mu}$ be the entry in row λ and column μ of the inverse of the Kostka matrix. Inverting the relations above, we see that

$$m_\lambda = \sum_\mu K'_{\lambda,\mu} s_\mu, \qquad s_\mu = \sum_\lambda K'_{\lambda,\mu} h_\lambda, \qquad s_{\mu'} = \sum_\lambda K'_{\lambda,\mu} e_\lambda.$$

Observe that the determinant formulas in the previous section, which express Schur polynomials in terms of complete homogeneous symmetric polynomials, give algebraic interpretations for the coefficients $K'_{\lambda,\mu}$. Here we wish to derive combinatorial interpretations for these coefficients. To do this, we need the concept of a special rim-hook tableau.

11.63. Definition: Special Rim-hook Tableaux. For $\lambda, \mu \in \mathrm{Par}_N$, a *special rim-hook tableau of shape μ and type λ* is a rim-hook tableau S of shape μ and content α such that $\mathrm{sort}(\alpha) = \lambda$ and every nonzero rim-hook in S contains a cell in the leftmost column of the diagram of μ. The *sign* of such a tableau is defined as in 11.48. Let $\mathrm{SRHT}(\mu, \lambda)$ be the set of special rim-hook tableaux of shape μ and type λ.

11.64. Theorem: Combinatorial Interpretation of Inverse Kostka Matrix. For all $\lambda, \mu \in \mathrm{Par}_N$,

$$K'_{\lambda,\mu} = \sum_{S \in \mathrm{SRHT}(\mu,\lambda)} \mathrm{sgn}(S).$$

Proof. We intend to give a combinatorial proof of the identity

$$a_{\delta(N)}(x_1, \ldots, x_N) m_\lambda(x_1, \ldots, x_N) = \sum_{\mu \in \mathrm{Par}_N} \sum_{S \in \mathrm{SRHT}(\mu,\lambda)} \mathrm{sgn}(S) a_{\mu+\delta(N)}(x_1, \ldots, x_N).$$

Once this is done, the theorem will follow by dividing both sides by $a_{\delta(N)}$ and comparing the resulting identity to the known expansion $m_\lambda = \sum_\mu K'_{\lambda,\mu} s_\mu$.

To prove the identity, we study a combinatorial interpretation of the product $a_{\delta(N)} m_\lambda$ involving abaci. The polynomial $a_{\delta(N)}$ represents a justified abacus containing N beads labeled $w(N), \ldots, w(1)$ in positions $0, \ldots, N-1$ (respectively). Given such an abacus, we can view $m_\lambda(x_1, \ldots, x_N)$ as the sum of all distinct monomials $\prod_{i=1}^N x_{w(i)}^{e(i)}$ such that the exponent sequence $(e(1), \ldots, e(N))$ is a rearrangement of $(\lambda_1, \ldots, \lambda_N)$. (Here and below, we view elements of Par_N as partitions with *exactly* N parts, some of which may be zero.) The multiplication of $a_{\delta(N)}$ by one of these monomials can be implemented on the abacus as follows. Imagine moving the N justified beads from their current runner to a new, initially empty runner, by moving each bead $w(i)$ from position $N-i$ on the old runner to position $N-i+e(i)$ on the new runner. Call such a transformation of the justified abacus a *λ-move.* A given λ-move either causes a bead collision on the new runner, or else produces a new abacus, which is enumerated by a monomial in $a_{\mu+\delta(N)}(x_1, \ldots, x_N)$ for some $\mu \in \mathrm{Par}_N$.

Consider the situation where a bead collision occurs. Choose i minimal such that bead $w(i)$ collides with some other bead on the new runner, and then choose j minimal such that bead $w(i)$ collides with $w(j)$. Create a new object counted by $a_{\delta(N)} m_\lambda$ by switching beads $w(i)$ and $w(j)$ on the old abacus, and switching $e(i)$ and $e(j)$ in the exponent vector. This defines a sign-reversing, weight-preserving involution that cancels all objects in which bead collisions occur.

To complete the proof, we must find a sign-preserving, weight-preserving bijection ϕ from the set X of uncanceled objects counted by $a_{\delta(N)} m_\lambda$ to the signed weighted set

$$\bigcup_{\mu \in \mathrm{Par}_N} \mathrm{SRHT}(\mu, \lambda) \times \mathrm{LAbc}(\mu + \delta(N)).$$

$$S = \begin{array}{|c|c|c|c|c|c|c|}\hline 1&2&2&2&2&2&2\\ \hline 2&2&4&4&4 \\ \cline{1-5} 4&4&4&5 \\ \cline{1-4} 4&5&5&5 \\ \cline{1-4} 5&5 \\ \cline{1-2}\end{array}$$

FIGURE 11.6
A special rim-hook tableau.

For this purpose, let us fix $\mu \in \mathrm{Par}_N$ and consider the ways in which a justified abacus with N beads can be transformed into an abacus in $\mathrm{LAbc}(\mu + \delta(N))$ by means of a λ-move. Let us temporarily ignore bead labels and signs, concentrating at first only on the positions of the N beads. The positions of the N beads on the old runner are the entries in the sequence $\delta(N) = (N-1, N-2, \ldots, 2, 1, 0)$. A λ-move adds some rearrangement of the sequence $\lambda = (\lambda_1, \ldots, \lambda_N)$ to the sequence $\delta(N)$. We will obtain an abacus in $\mathrm{LAbc}(\mu + \delta(N))$ iff the sum of these sequences is some rearrangement of the sequence $\mu + \delta(N) = (\mu_1 + N - 1, \ldots, \mu_N + N - N)$.

We now show that the rearrangements of λ that produce abaci in $\mathrm{LAbc}(\mu+\delta(N))$ can be encoded by special rim-hook tableaux of shape μ and type λ. The proof will use induction on N. Let us first illustrate the idea of the proof by considering an example. Take $N = 5$, $\mu = (7, 5, 4, 4, 2)$, and $\lambda = (8, 7, 6, 1, 0)$. We seek rearrangements of the vector $(8, 7, 6, 1, 0)$ which, when added to the vector $(4, 3, 2, 1, 0)$, produce a rearrangement of $\mu+\delta(N) = (11, 8, 6, 5, 2)$. In this example, the only solution turns out to be $(1, 8, 0, 7, 6)+(4, 3, 2, 1, 0) = (5, 11, 2, 8, 6)$. We can visualize this solution using the special rim-hook tableau in Figure 11.6, in which the rim-hooks (from top to bottom) have lengths $(1, 8, 0, 7, 6)$. If we start with a labeled justified abacus $54321000\cdots$ and perform a λ-move using the rearrangement $(1, 8, 0, 7, 6)$, we obtain the abacus $003001504002000\cdots \in \mathrm{LAbc}(11, 8, 6, 5, 2)$. The sign of this abacus, namely $\mathrm{sgn}(24513) = -1$, differs from the sign of the original abacus, namely $\mathrm{sgn}(12345) = +1$, by a factor of $(-1)^5 = \mathrm{sgn}(S)$. A similar remark holds if the original abacus had involved some other permutation of the five labels.

With this example in mind, we return to the general proof. We are seeking permutations $j_1 \cdots j_N$ and $k_1 \cdots k_N$ satisfying the system of equations

$$\begin{array}{rcl} 0 + \lambda_{j_N} &=& \mu_{k_N} + N - k_N \\ 1 + \lambda_{j_{N-1}} &=& \mu_{k_{N-1}} + N - k_{N-1} \\ \cdots && \cdots \\ N - 1 + \lambda_{j_1} &=& \mu_{k_1} + N - k_1 \end{array} \tag{11.5}$$

In particular, to satisfy the first equation, we need an index $j = j_N$ and an index $k = k_N$ such that $\lambda_j = \mu_k + N - k$. If such an index exists, we encode it by drawing the unique border ribbon of length λ_j starting in the leftmost cell of row N of μ. By choice of j and k, this border ribbon must end in the rightmost cell of row k of μ. In terms of the abaci, the λ-move encoded by $j_1 \cdots j_N$ moves the bead in position 0 on the old runner (the Nth bead from the right) to position $\mu_k + N - k$ on the new runner (which will become the kth bead from the right). Thus this bead "moves past" $N - k$ other beads during the λ-move, which causes a sign change of $(-1)^{N-k}$ for any choice of labels. But $N - k$ is precisely the spin of the border ribbon we just drew.

To finish solving system (11.5), let λ^* be the partition obtained by dropping one part λ_j from λ, and let μ^* be the partition in Par_{N-1} obtained by erasing the cells of μ occupied by the ribbon that starts in row N. Suppose we ignore the first equation in the system (11.5) and subtract 1 from both sides of the remaining $N-1$ equations. One may check that the resulting system of $N-1$ equations is precisely the system we must solve to change a justified abacus to an abacus in $\mathrm{LAbc}(\mu^*+\delta(N-1))$ by means of a λ^*-move. (For instance, in the example considered earlier, after we move a bead from position 0 to position 6 [accounting for the lowest rim-hook in the displayed tableau], we have $\lambda^*=(8,7,1,0)$ and $\mu^*=(7,5,3,1)$. Having moved one bead, we are left with the task of moving beads from positions $(4,3,2,1)=(1,1,1,1)+\delta(4)$ to positions $(11,8,5,2)=(1,1,1,1)+\mu^*+\delta(4)$ using the moves in $\lambda^*=(8,7,1,0)$.) By induction on N, the solutions of the reduced system are encoded by special rim-hook tableaux S^* of shape μ^* and type λ^*; and furthermore, the net sign change going from the old abacus to the new abacus (disregarding the bead originally in position 0) is $\mathrm{sgn}(S^*)$. It follows that all solutions of the original system are encoded by special rim-hook tableaux S of shape μ and type λ; and furthermore, the net sign change going from the old abacus to the new abacus (taking all beads into account) is $\mathrm{sgn}(S)$.

The preceding discussion contains an implicit recursive definition of the desired bijection ϕ. More explicitly, suppose $z=(w(N)\cdots w(1)000\cdots, e(N)\cdots e(1))\in X$ is an uncanceled object counted by $a_{\delta(N)}m_\lambda$. Then $\phi(z)=(S,v)$ where $v\in\mathrm{LAbc}(\mu+\delta(N))$ is obtained from the first component of z by moving bead $w(i)$ right $e(i)$ positions for all i, and S is the unique special rim-hook tableau (of shape μ determined by v) that has a rim-hook of length $e(i)$ starting in the leftmost cell of row i of the diagram. The preceding arguments show that ϕ preserves signs and weights. To compute $\phi^{-1}(S,v)$, it suffices to note that the sequence $(e(1),\ldots,e(N))$ is the content of the rim-hook tableau S. Knowledge of this sequence allows us to reverse the λ-move and recover $w(N)\cdots w(1)$. Thus, ϕ is a bijection. □

11.65. Remark. An alternate approach to the theorem is to *define*

$$K'_{\lambda,\mu}=\sum_{S\in\mathrm{SRHT}(\mu,\lambda)}\mathrm{sgn}(S)$$

and then give a combinatorial proof of the matrix identity $\mathbf{KK'}=\mathbf{I}$ (see 11.127). Since $\mathbf{K}$ is known to be invertible, it follows that $\mathbf{K'}$ must be the (two-sided) matrix inverse of $\mathbf{K}$.

11.16 Schur Expansion of Skew Schur Polynomials

We now consider the expansion of skew Schur polynomials as linear combinations of ordinary Schur polynomials. Since the ordinary Schur polynomials are a basis of Λ_N and the skew Schur polynomials are in this vector space, we know there exist unique scalars $c^{\lambda}_{\nu,\mu}\in\mathbb{Q}$ such that

$$s_{\lambda/\nu}(x_1,\ldots,x_N)=\sum_{\mu}c^{\lambda}_{\nu,\mu}s_\mu(x_1,\ldots,x_N), \tag{11.6}$$

where it suffices to sum over partitions μ of size $|\lambda/\nu|$. The scalars $c^{\lambda}_{\nu,\mu}$ are called *Littlewood-Richardson coefficients.* The following result shows that these coefficients are all nonnegative integers. Recall that, for a semistandard tableau T of any shape, the *word of* T is obtained by concatenating the rows of T from bottom to top.

11.66. Theorem: Littlewood-Richardson Rule for Skew Schur Polynomials. For all partitions λ,μ,ν, $c^{\lambda}_{\nu,\mu}$ is the number of semistandard tableaux T of shape λ/ν and content μ such that every suffix of the word of T has partition content. In other words, writing $w(T) = w_1w_2\cdots w_n$, we require that for all $k \le n$ and all $i \ge 1$, the number of i's in the suffix $w_kw_{k+1}\cdots w_n$ equals or exceeds the number of $i+1$'s in this suffix.

Proof. Multiplying both sides of (11.6) by $a_{\delta(N)}$, it suffices to prove the identity

$$a_{\delta(N)}(x_1,\ldots,x_N)s_{\lambda/\nu}(x_1,\ldots,x_N) = \sum_{\mu} c^{\lambda}_{\nu,\mu} a_{\mu+\delta(N)}(x_1,\ldots,x_N).$$

The idea is to generalize the proof of the special case $\nu = (0)$ which we gave in §11.12. Model the left side of the desired identity by the set X of pairs (v,T), where v is a justified N-bead labeled abacus and T is a semistandard tableau of shape λ/ν over the alphabet $\{1,2,\ldots,N\}$ ordered by $<_v$. Since skew Schur polynomials are symmetric, the generating function for the signed, weighted set X is $a_{\delta(N)}s_{\lambda/\nu}$.

We now define an involution $I : X \to X$. Given $(v,T) \in X$, T determines a sequence of bead motions on v by reading $w(T)$ from right to left and moving bead k one step to the right each time the symbol k is seen. If these bead motions cause a collision, define $I(v,T) = (v',T')$ by the following steps. Suppose the first collision occurs when bead i bumps into bead j (where $i >_v j$ are adjacent beads in v). Let v' be v with beads i and j switched, so $\operatorname{sgn}(v') = -\operatorname{sgn}(v)$ and $\operatorname{wt}(v')x_j = \operatorname{wt}(v)x_i$.

Next, we calculate T' from T as follows. Starting with the word of T, replace each i by a left parenthesis, each j by a right parenthesis, and ignore all other symbols. Match left and right parentheses in the resulting string of parentheses, and ignore these matched pairs of parentheses hereafter. The remaining unmatched parentheses must consist of a string of $a \ge 0$ right parentheses followed by a string of $b \ge 0$ left parentheses, since if a left parenthesis appeared somewhere to the left of a right parenthesis we could find another matched pair of parentheses.

Note that $b > 0$, since otherwise bead i would never bump into bead j. Indeed, the first bead collision occurs when we reach the rightmost unmatched left parenthesis (occurrence of i) in the word of T. Now, change the subword of unmatched parentheses from "$)^a(^b$" to "$)^{b-1}(^{a+1}$", and then convert all left parentheses to j's and all right parentheses to i's. One may verify that the new word is the word of a tableau $T' \in \mathrm{SSYT}_N(\lambda/\nu)$, relative to the ordering $<_{v'}$, using the facts that i and j are adjacent relative to the orderings $<_v$ and $<_{v'}$, and the status of a given parenthesis symbol in T' (matched or unmatched) is the same as its status in T. See the example following the proof for more discussion of this point.

Because T' has one less i than T and T' has one more j than T, we have $\operatorname{wt}(T')x_i = \operatorname{wt}(T)x_j$. Since we also had $\operatorname{wt}(v')x_j = \operatorname{wt}(v)x_i$, we see that $\operatorname{wt}(v',T') = \operatorname{wt}(v,T)$. Thus I is sign-reversing and weight-preserving. Finally, to check that I is an involution, consider what happens when we use T' to move the beads on v'. Bead j on v' moves the same way as bead i did on v (and vice versa) until we reach the rightmost unmatched parenthesis (relative to i and j) in $w(T')$. When this symbol is reached, bead j bumps into bead i on v', just as bead i bumped into bead j on v. To compute $I(v',T')$, we will therefore apply the parenthesis modification rule to the i's and j's appearing in $w(T')$. This rule will change the unmatched parentheses from "$)^{b-1}(^{a+1}$" back to "$)^a(^b$", which shows that $I(v',T') = (v,T)$. So I is an involution.

All that remains is to analyze the fixed points of I, which are (by definition) the pairs (v,T) for which no bead collision occurs. Recall that we are starting with a justified abacus v, scanning the symbols in $w(T) = w_1\cdots w_n$ from right to left, and moving the corresponding beads on v. Suppose all suffixes of T have partition content relative to the ordering $<_v$ (which means the rightmost bead label occurs at least as often in each suffix as the next bead label, and so on). We see from the description of the bead motion that no collision will occur. Conversely, if the condition is first violated by some suffix $w_kw_{k+1}\cdots w_n$, then a collision will occur at this point in the scan. Thus the fixed points of I are the pairs (v,T)

such that each suffix of T has partition content relative to $<_v$. We map each such fixed point to the abacus v^* obtained from v by performing the bead motions specified by T. The abacus v^* lies in the set $\mathrm{LAbc}(\mu+\delta(N))$, where μ is the content of T (calculating content relative to $<_v$, so μ_1 is the number of times the rightmost bead moves, etc.).

We can obtain all the fixed points of I from fixed points of the form (v^0, T), where $v^0 = (N, N-1, \ldots, 1, 0, 0, \ldots)$, $<_{v^0}$ is the usual ordering on integers, and T is a semistandard tableau satisfying the conditions in the theorem statement. We need only permute the bead labels in v^0 by any $w \in S_N$, and permute the entries of T in the same way. The object (v^0, T) thereby generates $N!$ fixed points which together contribute one copy of $a_{\mu+\delta(N)}(x_1, \ldots, x_N)$ to the generating function for the fixed points of I. The total number of times this term appears in the generating function is the total number of semistandard tableaux T of content μ satisfying the conditions in the theorem. Since the generating function for X must equal the generating function for the fixed point set of I, the proof is complete. $\square$

11.67. Example. To illustrate the parenthesis construction, we compute $I(5432100\cdots, T)$, where

$$T = \begin{array}{cccccccccccccccc} &&&&&&&&&&&&&1&1&1\\ &&&&&&&&&&1&2&2&2&2&\\ &&&&&&1&1&1&1&2&&&&&\\ &1&1&1&2&2&2&2&&&&&&&&\\ 1&2&2&&&&&&&&&&&&& \end{array}.$$

The word of T is 12211122221111212222111. The suffix 2222111 of $w(T)$ does not have partition content, so this object will cancel with some object $(5431200\cdots, T')$. To find T', first convert 1's to right parentheses and 2's to left parentheses in $w(T)$:

```
12211122221111212222111
)(()))(((())))()((((()))
```

Now we balance parentheses and mark the remaining unmatched symbols:

```
)(()))(((())))()((((()))
*     *              *
```

The substring of unmatched parentheses is "))(." Observe that the rightmost symbol in this substring is a left parenthesis corresponding to the first 2 in the offending suffix 2222111, and this 2 is the symbol in $w(T)$ causing the first bead collision. As directed by the proof, we convert the unmatched parenthesis string to "(((" and then replace left parentheses by 1's and right parentheses by 2's:

```
*     *              *
((())((((())))()((((()))
11122111112222121111222
```

This new word is $w(T')$, so finally

$$T' = \begin{array}{cccccccccccccccc} &&&&&&&&&&&&&2&2&2\\ &&&&&&&&&&2&1&1&1&1&\\ &&&&&&2&2&2&2&1&&&&&\\ &2&2&1&1&1&1&1&&&&&&&&\\ 1&1&1&&&&&&&&&&&&& \end{array}.$$

Observe that T' is a semistandard tableau relative to the ordering $5 > 4 > 3 > 1 > 2$. In particular, columns of T' strictly increase because whenever 1 appears above 2 in T, these occurrences of 1 and 2 become matched parentheses. Rearranging the unmatched parentheses does not affect these symbols, so in the end we will get a 2 above a 1 in T'. Also, rows of T' weakly increase since a strict decrease in some row would be encoded as a matched parenthesis pair in $w(T')$, which would have also been matched in $w(T)$, implying that T had a strict decrease in some row. But T is a semistandard tableau so this cannot happen. Finally, note that the shortest suffix of T' that does not have partition content (relative to the new ordering) is 1111222, where the leftmost 1 corresponds to the rightmost unbalanced parenthesis in $w(T')$. Consequently, $I(5431200\cdots, T') = (5432100\cdots, T)$. Observe that these two objects have opposite sign, but both have weight $x_1^{16}x_2^{14}x_3^2x_4^1x_5^0$.

11.68. Example. Let us compute $I(v^0, T)$, where

$$v_0 = 5432100\cdots, \qquad T = \begin{array}{ccccccc} &&&&1&1&1\\ &&&2&3&3&\\ 1&1&2&4&4&5&\\ 2&2&3&5&&&\\ 3&5&&&&& \end{array}$$

Moving beads on v^0 according to the word $w(T) = 352235112445233111$, bead 3 bumps into bead 2 when we have scanned the suffix 3111 (which is the shortest suffix without partition content). We therefore modify the 2's and 3's in the word as follows:

```
352235112445233111
3 223    2    233
( ))(    )    )((
   *          ***
( ))(    )    (((
2 332    3    222
253325113445222111
```

Therefore $I(v^0, T) = (v', T')$, where

$$v' = 54231, \qquad T' = \begin{array}{ccccccc} &&&&1&1&1\\ &&&2&2&2&\\ 1&1&3&4&4&5&\\ 3&3&2&5&&&\\ 2&5&&&&& \end{array}$$

Observe that $\mathrm{wt}(v^0, T) = \mathrm{wt}(v', T') = x_1^9x_2^7x_3^6x_4^3x_5^3$, $\mathrm{sgn}(v', T') = -\mathrm{sgn}(v^0, T)$, and $I(v', T') = (v^0, T)$.

11.69. Example. Let us compute $c^{\lambda}_{\nu,\mu}$ when $\lambda = (5, 4, 4, 1)$, $\nu = (3, 1)$, and $\mu = (4, 4, 2)$. We draw the semistandard tableaux of shape λ/ν whose words have the required suffix property. The following two tableaux are the only ones, so $c^{\lambda}_{\nu,\mu} = 2$:

$$T_1 = \begin{array}{ccccc} &&&1&1\\ &1&1&2&\\ 2&2&2&3&\\ 3&&&& \end{array}, \qquad T_2 = \begin{array}{ccccc} &&&1&1\\ &1&2&2&\\ 1&2&3&3&\\ 2&&&& \end{array}$$

Let us see how these tableaux correspond to fixed points of I when $N = 5$. The first tableau changes the standard abacus $v^0 = (5432100\cdots)$ to the abacus $(54003002100\cdots)$ counted by $\mathrm{LAbc}(\mu + \delta(5))$ by moving bead 1 twice, then bead 2 once, then bead 1 twice, and so on. Permuting the labels gives the other 119 signed objects that make up one copy of $a_{\mu+\delta(5)}(x_1, \ldots, x_5)$; for instance,

$$\left(3425100\cdots, \begin{array}{ccccc} &&&1&1\\ &1&1&5&\\ 5&5&5&2&\\ 2&&&& \end{array}\right) \quad \mapsto \quad (34002005100\cdots).$$

On the other hand, the second tableau changes the standard abacus $(5432100\cdots)$ to the abacus $(54003002100\cdots)$ via a different sequence of collision-free bead moves: move bead 1 twice, then bead 2 twice, then bead 1 once, and so on. This pair and its permutations produce another copy of the generating function $a_{\mu+\delta(5)}(x_1,\ldots,x_5)$. Dividing by $a_{\delta(5)}$, we conclude that

$$s_{\lambda/\nu} = 2s_\mu + \cdots.$$

Now let us compute $c^\lambda_{\mu,\nu}$. The required skew tableaux, which have shape $(5,4,4,1)/(4,4,2)$ and content $(3,1)$, are:

$$\begin{array}{ccccc} & & & & \boxed{1} \\ & & & & \\ & & \boxed{1} & \boxed{1} & \\ \boxed{2} & & & & \end{array}, \qquad \begin{array}{ccccc} & & & & \boxed{1} \\ & & & & \\ & & \boxed{1} & \boxed{2} & \\ \boxed{1} & & & & \end{array}.$$

So $c^\lambda_{\mu,\nu} = 2$. This illustrates the general symmetry property $c^\lambda_{\nu,\mu} = c^\lambda_{\mu,\nu}$, which is true but not immediately evident from our combinatorial description of these coefficients. We will prove this property later when we discuss products of Schur polynomials.

11.70. Example. For $N \geq 7$, let us find the Schur expansion of the skew Schur polynomial $s_{(3,3,2,2)/(2,1)}$ in N variables. This expansion is found by enumerating all semistandard skew tableaux of shape $(3,3,2,2)/(2,1)$ satisfying the required suffix property. Each such tableau of content μ contributes one term s_μ to the expansion. The relevant tableaux are shown here:

$$\begin{array}{ccc} & & \boxed{1} \\ & \boxed{1} & \boxed{2} \\ \boxed{1} & \boxed{2} & \\ \boxed{2} & \boxed{3} & \end{array} \qquad \begin{array}{ccc} & & \boxed{1} \\ & \boxed{1} & \boxed{2} \\ \boxed{1} & \boxed{2} & \\ \boxed{3} & \boxed{3} & \end{array} \qquad \begin{array}{ccc} & & \boxed{1} \\ & \boxed{1} & \boxed{2} \\ \boxed{1} & \boxed{3} & \\ \boxed{2} & \boxed{4} & \end{array} \qquad \begin{array}{ccc} & & \boxed{1} \\ & \boxed{1} & \boxed{2} \\ \boxed{2} & \boxed{3} & \\ \boxed{3} & \boxed{4} & \end{array}$$

We conclude that

$$s_{(3,3,2,2)/(2,1)} = 1s_{(3,3,1)} + 1s_{(3,2,2)} + 1s_{(3,2,1,1)} + 1s_{(2,2,2,1)}.$$

11.17 Products of Schur Polynomials

Given partitions $\mu \in \mathrm{Par}_N(m)$ and $\nu \in \mathrm{Par}_N(n)$, the product $s_\mu(x_1,\ldots,x_N)s_\nu(x_1,\ldots,x_N)$ is a symmetric polynomial, so it can be expressed uniquely in terms of Schur polynomials $s_\lambda(x_1,\ldots,x_N)$ indexed by $\lambda \in \mathrm{Par}_N(m+n)$:

$$s_\mu s_\nu = \sum_\lambda a(\lambda,\mu,\nu)s_\lambda \qquad (a(\lambda,\mu,\nu) \in \mathbb{Q}). \tag{11.7}$$

Now, 11.57 shows that the coefficients here are precisely the Littlewood-Richardson numbers:

$$a(\lambda,\mu,\nu) = \langle s_\mu s_\nu, s_\lambda\rangle = \left\langle s_\mu, s_{\lambda/\nu}\right\rangle = c^\lambda_{\nu,\mu}.$$

Since $s_\mu s_\nu = s_\nu s_\mu$ (because multiplication of polynomials is commutative), we deduce the symmetry

$$c^\lambda_{\nu,\mu} = c^\lambda_{\mu,\nu}.$$

We now derive another combinatorial expression for these integers by viewing the product $s_\mu s_\nu$ as a skew Schur polynomial. We claim that $s_\mu s_\nu = s_{\alpha/\beta}$, where

$$\alpha = (\mu_1+\nu_1, \mu_1+\nu_2, \ldots, \mu_1+\nu_N, \mu_1, \ldots, \mu_N), \qquad \beta = (\mu_1^N).$$

This follows since the skew shape α/β consists of two disconnected pieces, one of shape ν and one of shape μ. A semistandard skew tableau of this shape can be formed by choosing a semistandard tableau of shape ν and independently choosing a semistandard tableau of shape μ; thus the result follows from the product rule for weighted sets. We conclude that

$$c^{\lambda}_{\mu,\nu} = c^{\alpha}_{\beta,\lambda} \qquad \text{(with } \alpha,\beta \text{ as above).}$$

This formula is illustrated in the next example.

11.71. Example. Let us compute the Schur expansion of $s_{(2,1)}s_{(2,1)}$ using the observation $s_{(2,1)}s_{(2,1)} = s_{(4,3,2,1)/(2,2)}$. The following skew tableaux have words such that all suffixes have partition content:

$$\begin{array}{cccc} & & 1 & 1 \\ & & 2 & \\ 1 & 1 & & \\ 2 & & & \end{array} \qquad \begin{array}{cccc} & & 1 & 1 \\ & & 2 & \\ 1 & 1 & & \\ 3 & & & \end{array} \qquad \begin{array}{cccc} & & 1 & 1 \\ & & 2 & \\ 1 & 2 & & \\ 3 & & & \end{array} \qquad \begin{array}{cccc} & & 1 & 1 \\ & & 2 & \\ 1 & 3 & & \\ 2 & & & \end{array}$$

$$\begin{array}{cccc} & & 1 & 1 \\ & & 2 & \\ 1 & 2 & & \\ 2 & & & \end{array} \qquad \begin{array}{cccc} & & 1 & 1 \\ & & 2 & \\ 2 & 3 & & \\ 3 & & & \end{array} \qquad \begin{array}{cccc} & & 1 & 1 \\ & & 2 & \\ 2 & 3 & & \\ 4 & & & \end{array} \qquad \begin{array}{cccc} & & 1 & 1 \\ & & 2 & \\ 1 & 3 & & \\ 4 & & & \end{array}$$

Looking at contents, we conclude that

$$s_{(2,1)}s_{(2,1)} = s_{(4,2)} + s_{(4,1,1)} + 2s_{(3,2,1)} + s_{(3,3)} + s_{(2,2,2)} + s_{(2,2,1,1)} + s_{(3,1,1,1)}.$$

Observe that the upper-right portion of the skew tableau could only be filled in one way. So we could ignore this part of the tableau and just consider suitable fillings of the lower shape. Generalizing this remark leads to the following prescription for the Littlewood-Richardson coefficients.

11.72. Theorem: Alternate Formula for Littlewood-Richardson Coefficients. For all partitions $\lambda,\mu,\nu \in \mathrm{Par}_N$, the coefficent $c^{\lambda}_{\mu,\nu} = c^{\lambda}_{\nu,\mu}$ is the number of semistandard tableaux T of shape μ and content $\lambda-\nu = (\lambda_i - \nu_i : 1 \leq i \leq N)$ such that $w(T) = w_1 \cdots w_n$ satisfies the following condition: for all $k \leq n$, the exponent vector of the monomial $\prod_{j=1}^{N} x_j^{\nu_j} \prod_{i=k}^{n} x_{w_i}$ is a partition. (This condition means that for all $k \leq n$ and $j < N$, if there are a copies of j and b copies of $j+1$ in the suffix $w_k w_{k+1} \cdots w_n$, then $\nu_j + a \geq \nu_{j+1} + b$.)

Proof. We already know that $c^{\lambda}_{\mu,\nu} = c^{\alpha}_{\beta,\lambda}$ where the skew shape α/β consists of an upper part of shape ν and a lower part of shape μ. We also know that $c^{\alpha}_{\beta,\lambda}$ is the number of skew tableaux U of shape α/β and content λ such that every suffix of $w(U)$ has partition content. Consider the last $|\nu|$ symbols in $w(U)$. The last letter is the label in the rightmost cell of the first row of the skew shape α/β. The partition content condition forces this letter to be 1, and then all letters in the first row of the skew tableau U must be 1. The letter at the end of the second row must be strictly greater than 1, so it is 2 (by the partition content condition), and then every entry in the second row must be 2. Proceeding in this way, we see that for $k \leq N$, every entry in row k of U must equal k. Equivalently, the last $|\nu|$ symbols of $w(U)$ must be $N^{\nu_N} \cdots 2^{\nu_2} 1^{\nu_1}$. Call this suffix z.

Now we must fill the lower part of the shape α/β by choosing a semistandard tableau T of shape μ. Because the upper part of U has content ν, the content of the entire skew tableau U will be λ iff the content of the lower part T is $\lambda-\nu$. The other condition imposed on T is that, for every suffix y of $w(T)$, the suffix yz of $w(U)$ has partition content. Given the formula for z above, this condition is equivalent to the condition on $w(T)$ in the theorem statement. $\square$

TABLE 11.1
Formulas for manipulating antisymmetric and symmetric polynomials.

Pieri rules:	$a_{\lambda+\delta(N)}p_k = \sum_{\beta:\ \beta/\lambda \text{ is a } k\text{-ribbon } R} \operatorname{sgn}(R)a_{\beta+\delta(N)};$
	$a_{\lambda+\delta(N)}e_k = \sum_{\beta:\ \beta/\lambda \text{ is a vertical } k\text{-strip}} a_{\beta+\delta(N)};$
	$a_{\lambda+\delta(N)}h_k = \sum_{\beta:\ \beta/\lambda \text{ is a horizontal } k\text{-strip}} a_{\beta+\delta(N)};$
	$s_\lambda p_k = \sum_{\beta:\ \beta/\lambda \text{ is a } k\text{-ribbon } R} \operatorname{sgn}(R)s_\beta;$
	$s_\mu p_\alpha = \sum_\lambda \chi_\alpha^{\lambda/\mu} s_\lambda.$
Determinant formula for s_λ:	$s_\lambda = \frac{a_{\lambda+\delta(N)}}{a_{\delta(N)}} = \frac{\det \lvert\lvert x_j^{\lambda_i+N-i} \rvert\rvert_{1\le i,j\le N}}{\det \lvert\lvert x_j^{N-i} \rvert\rvert_{1\le i,j\le N}}.$
Schur expansion of power-sums:	$p_\alpha = \sum_\lambda \chi_\alpha^\lambda s_\lambda.$
Power-sum expansion of Schur polys.:	$s_\lambda = \sum_\mu \frac{\chi_\mu^\lambda}{z_\mu} p_\mu.$
Formulas for skew Schur polynomials:	$s_{\lambda/\mu} = \sum_\nu \frac{\chi_\nu^{\lambda/\mu}}{z_\nu} p_\nu;$
	$\left\langle s_{\lambda/\mu}, f\right\rangle = \langle s_\lambda, s_\mu f\rangle$ for $f \in \Lambda_N;$
	$\omega(s_{\lambda/\mu}) = s_{\lambda'/\mu'};$
	$s_{\lambda/\mu} = \det \lvert\lvert h_{\lambda_i-\mu_j+j-i} \rvert\rvert_{1\le i,j\le \ell(\lambda)};$
	$s_{\lambda/\mu} = \det \lvert\lvert e_{\lambda'_i-\mu'_j+j-i} \rvert\rvert_{1\le i,j\le \lambda_1};$
	$s_{\lambda/\mu} = \sum_\nu c_{\mu,\nu}^\lambda s_\nu;$
	$s_\mu s_\nu = \sum_\lambda c_{\mu,\nu}^\lambda s_\lambda.$
Inverse Kostka formulas:	$m_\lambda = \sum_\mu K'_{\lambda,\mu} s_\mu;$
$(K'_{\lambda,\mu} = \sum_{S\in \mathrm{SRHT}(\mu,\lambda)} \operatorname{sgn}(S))$	$s_\mu = \sum_\lambda K'_{\lambda,\mu} h_\lambda;$
	$s_{\mu'} = \sum_\lambda K'_{\lambda,\mu} e_\lambda.$

Summary

Table 11.1 contains formulas derived in this chapter for computing with antisymmetric and symmetric polynomials.

- *Unlabeled Abaci.* An abacus is a function $w : \mathbb{Z} \to \{0, 1\}$ with $w(i) = 1$ for all small enough i and $w(j) = 0$ for all large enough j. Justification of abaci gives a bijection to pairs $(m, \lambda) \in \mathbb{Z} \times \mathrm{Par}$. The inverse bijection can be computed by traversing the frontier of $\mathrm{dg}(\lambda)$, converting north steps to beads (1's) and east steps to gaps (0's), and using m to decide which step on the frontier corresponds to position 0 of w.

- *Jacobi Triple Product Identity.* Abaci can be used to prove

$$\sum_{m\in\mathbb{Z}} q^{m(m+1)/2}u^m = \prod_{n\ge1}(1+uq^n)\prod_{n\ge0}(1+u^{-1}q^n)\prod_{n\ge1}(1-q^n).$$

One consequence is the formula $\prod_{n\geq 1}(1-q^n)=\sum_{k\in\mathbb{Z}}(-1)^k q^{(3k^2-k)/2}$.

- *k-cores and k-quotients.* Given a partition μ, repeated removal of border ribbons of size k (in any order) will lead to a unique partition from which no further ribbons of this kind can be removed. This partition is called the k-core of μ. We can also find the k-core by converting μ to an abacus, decimating the abacus to give a k-runner abacus, justifying all runners, and converting back to a partition. Each ribbon removal corresponds to moving one bead one step to the left on the k-runner abacus. Justifying each separate runner on the k-runner abacus for μ produces the k-quotients $(\nu^0,\ldots,\nu^{k-1})$ of μ. Alternatively, $\mathrm{dg}(\nu^i)$ can be found by taking the cells of $\mathrm{dg}(\mu)$ lying due north and due west of steps of k-content i on the frontier of μ. We get a bijection $\Delta_k:\mathrm{Par}\to\mathrm{Core}(k)\times\mathrm{Par}^k$ by mapping μ to its k-core and k-quotients.

- *Labeled Abaci and Antisymmetric Polynomials.* A polynomial f in N variables is antisymmetric iff interchanging any two adjacent variables changes the sign of f. For each $\mu=(\mu_1>\mu_2>\cdots>\mu_N\geq 0)$, the polynomial $a_\mu(x_1,\ldots,x_N)=\det||x_j^{\mu_i}||_{1\leq i,j\leq N}$ is antisymmetric. Writing $\delta(N)=(N-1,N-2,\ldots,2,1,0)$, the set $\{a_{\lambda+\delta(N)}:\lambda\in\mathrm{Par}_N\}$ is a basis for the space A_N of antisymmetric polynomials. Division by $a_{\delta(N)}=\prod_{1\leq i<j\leq N}(x_i-x_j)$ gives a vector space isomorphism from A_N to Λ_N sending $a_{\lambda+\delta(N)}$ to the Schur polynomial $s_\lambda=a_{\lambda+\delta(N)}/a_{\delta(N)}$. To model the terms in $a_{\lambda+\delta(N)}$, we use the $N!$ labeled abaci consisting of beads $1,2,\ldots,N$ (in any order) at positions given by $\lambda+\delta(N)$.

- *Rim-Hook Tableaux.* A rim-hook tableau of shape λ/μ and content α is obtained by enlarging the diagram of μ using border ribbons of lengths $\alpha_1,\alpha_2,\ldots$ (in this order) until the diagram of λ is obtained. The set of such tableaux is denoted $\mathrm{RHT}(\lambda/\mu,\alpha)$. A ribbon occupying r rows has sign $(-1)^{r-1}$, and the sign of a rim-hook tableau is the product of the signs of its ribbons. We write $\chi_\alpha^{\lambda/\mu}=\sum_{T\in\mathrm{RHT}(\lambda/\mu,\alpha)}\mathrm{sgn}(T)$. We have $\chi_\alpha^{\lambda/\mu}=\chi_\beta^{\lambda/\mu}$ whenever $\mathrm{sort}(\alpha)=\mathrm{sort}(\beta)$.

- *Interactions between Abaci and Tableaux.* One can give combinatorial proofs of several identities in Table 11.1 by using the word of a tableau to encode bead motions on abaci. When these motions lead to bead collisions, one obtains two objects of opposite sign and equal weight that cancel terms on one side of the formula to be proved. Objects with no collisions are fixed points that can be reorganized to give the other side of the formula.

- *Inverse Kostka Matrix.* A rim-hook tableau is called special iff each ribbon in the tableau begins in the leftmost column; $\mathrm{SRHT}(\mu,\lambda)$ is the set of such tableaux of shape μ and content α with $\mathrm{sort}(\alpha)=\lambda$. Letting $K'_{\lambda,\mu}=\sum_{S\in\mathrm{SRHT}(\mu,\lambda)}\mathrm{sgn}(S)$, we have $\mathbf{KK'}=\mathbf{I}$.

- *Littlewood-Richardson Coefficients.* The scalars $c_{\nu,\mu}^\lambda=c_{\mu,\nu}^\lambda$ appearing in the Schur expansions of $s_{\lambda/\nu}$ and $s_\mu s_\nu$ count semistandard tableaux T of shape λ/ν and content μ such that every suffix of the word of T has partition content. The scalars $c_{\nu,\mu}^\lambda$ also count semistandard tableaux T of shape μ and content $\lambda-\nu$ such that $w(T)=w_1\cdots w_n$ satisfies the following condition: for all $k\leq n$, the exponent vector of $\prod_j x_j^{\nu_j}\prod_{i=k}^n x_{w_i}$ is a partition.

Exercises

11.73. Let $w = \cdots 110110111\underline{1}0101001100 \cdots$. Compute $\mathrm{wt}(w)$ and $J(w)$.

11.74. Compute $U(-1, \mu)$ for each $\mu \in \mathrm{Par}(5)$.

11.75. In the computation of $U(m, \mu)$ in 11.4, describe in detail how to use m to decide which symbol in the bead-gap sequence is w_0.

11.76. Given $\mu \in \mathrm{Par}$, what is the relationship between the abaci $U(-1, \mu)$ and $U(-1, \mu')$?

11.77. Show that the abacus w in 11.2 and its justification $J(w)$ have the same weight, if we use the weights defined in the proof of 11.5.

11.78. Show how to deduce Euler's pentagonal number theorem as an algebraic consequence of the Jacobi triple product identity.

11.79. Fill in all the details in the proof of 11.6.

11.80. Use 11.5 to simplify the product $\prod_{n \geq 0}(1 - x^{5n+1})^{-1} \prod_{n \geq 0}(1 - x^{5n+4})^{-1}$ appearing in one of the Rogers-Ramanujan identities. Can you give a direct proof of the resulting identity using abaci?

11.81. Use 11.4 to find a bijective proof of 11.5 that makes no reference to abaci, instead using combinatorial operations on partition diagrams and their frontiers.

11.82. Complete the proof of 11.12 by verifying that $D_k(w) \in \mathrm{Abc}^k$, $I_k(v) \in \mathrm{Abc}$, and D_k and I_k are two-sided inverses.

11.83. (a) Verify that the 3-core of $\mu = (10, 10, 10, 8, 8, 8, 7, 4)$ is $(1, 1)$ by removing border 3-ribbons from μ in several different orders. (b) Use the 3-runner abacus encoding μ to determine exactly how many ways there are to change μ into $(1, 1)$ by removing an ordered sequence of border 3-ribbons.

11.84. Let $\mu = (8, 7, 6, 4, 4, 4, 3, 1, 1, 1)$. Use abaci to compute the k-core and k-quotients of μ for $1 \leq k \leq 6$.

11.85. Find all integer partitions that are 2-cores, and draw some of their diagrams.

11.86. Find all 3-cores with at most 8 cells.

11.87. Verify the assertion in the last sentence of 11.20.

11.88. Let $\mu = (8, 8, 8, 8, 8, 8, 8, 8, 8)$. (a) Use abaci to compute the k-core and k-quotients of μ for $3 \leq k \leq 8$. (b) Use the construction at the end of §11.4 to compute the k-quotients of μ (for $3 \leq k \leq 8$) directly from the diagram of μ.

11.89. Compute the k-quotients of $\mu = (6, 6, 6, 3, 3, 2, 2, 2, 1, 1)$ without using abaci, for $k = 3, 4, 5$.

11.90. Consider the construction at the end of §11.4 for computing k-quotients of μ. Show that the hook-length of each unerased cell is divisible by k, and these are the only cells in the diagram of μ whose hook-lengths are divisible by k.

11.91. For each $k \geq 1$, find a formula for the generating function $\sum_{\mu \in \mathrm{Core}(k)} q^{|\mu|}$.

11.92. Given that μ has k-core ρ and k-quotients $\nu^0, \ldots, \nu^{k-1}$, find a formula for the number of ways we can go from μ to ρ by removing an ordered sequence of border k-ribbons.

11.93. Compute $a_\mu(x_1, x_2, x_3)$ and $a_{\lambda+\delta(3)}(x_1, x_2, x_3)$ for $\mu = (6, 3, 1)$ and $\lambda = (2, 2, 1)$.

11.94. Verify by direct calculation that, for $N = 3$ and $\lambda = (2, 1, 0)$, $a_{\lambda+\delta(N)}$ is divisible by $a_{\delta(N)}$ and $a_{\lambda+\delta(N)}/a_{\delta(N)} = s_\lambda(x_1, \ldots, x_N)$.

11.95. Verify that A_N and A_N^n are subspaces of $K[x_1, \ldots, x_N]$, and that the map $f \mapsto fa_{\delta(N)}$ (for $f \in \Lambda_N$) is K-linear.

11.96. (a) Show that the product of two antisymmetric polynomials is symmetric. (b) Show that the product of a symmetric polynomial and an antisymmetric polynomial is antisymmetric.

11.97. Define a map $T : K[x_1, \ldots, x_N] \to K[x_1, \ldots, x_N]$ by setting

$$T(f) = \frac{1}{N!} \sum_{w \in S_N} \operatorname{sgn}(w) f(x_{w(1)}, \ldots, x_{w(N)}).$$

Show that T is a K-linear map with image A_N whose restriction to A_N is the identity map. Can you describe the kernel of T?

11.98. Let v be the labeled abacus $v = 0041000300502600\cdots$. Compute $\operatorname{wt}(v)$, $w(v)$, $\operatorname{pos}(v)$, and $\operatorname{sgn}(v)$. For which λ is v in $\operatorname{LAbc}(\lambda + \delta(6))$?

11.99. Draw all the labeled abaci in $\operatorname{LAbc}(6, 5, 1)$, and compute the sign of each abacus.

11.100. Using $N = 6$ variables, compute:
(a) $a_{(4,2,1)+\delta(6)}p_4$; (b) $a_{(3,3,3)+\delta(6)}p_3$; (c) $a_{(1,1,1,1,1)+\delta(6)}p_2$.
How would the answers change if we changed N?

11.101. Let $v = 0310040206500\cdots \in \operatorname{LAbc}(\lambda + \delta(6))$ and $k = 4$. For $1 \le i \le 6$, compute $I(v, i)$ where I is the involution in the proof of 11.39. For any fixed points that arise, compute v^* and indicate which border 4-ribbon is added to $\operatorname{dg}(\lambda)$ in the passage from v to v^*.

11.102. Using $N = 6$ variables, compute:
(a) $a_{(4,2,1)+\delta(6)}e_3$; (b) $a_{(3,3,3)+\delta(6)}e_2$; (c) $a_{(5,4,3,1,1)+\delta(6)}e_4$.
How would the answers change if we changed N?

11.103. Let $v = 0310040206500\cdots \in \operatorname{LAbc}(\lambda + \delta(6))$. Compute $I(v, S)$ for $S = \{2, 5, 6\}$, $S = \{1, 4, 5\}$, $S = \{1, 3, 4\}$, and $S = \{3, 4, 6\}$, where I is the involution in the proof of 11.42. For any fixed points that arise, compute v^* and indicate which vertical strip is added to $\operatorname{dg}(\lambda)$ in the passage from v to v^*.

11.104. Using $N = 5$ variables, compute:
(a) $a_{(4,2,1)+\delta(6)}h_3$; (b) $a_{(3,3,3)+\delta(6)}h_3$; (c) $a_{(5,4,3,1,1)+\delta(6)}h_4$.
How would the answers change if we changed N?

11.105. Let $v = 0310040206500\cdots \in \operatorname{LAbc}(\lambda+\delta(6))$. Compute $I(v, M)$ for $M = [1, 1, 4, 5]$, $M = [2, 2, 5, 6]$, $M = [2, 4, 5, 5]$, and $M = [1, 2, 3, 4]$, where I is the involution in the proof of 11.44. For any fixed points that arise, compute v^* and indicate which horizontal strip is added to $\operatorname{dg}(\lambda)$ in the passage from v to v^*.

11.106. Explain in detail why the bead motion rule in §11.9 leads to the addition of a horizontal k-strip to the shape λ, assuming no bead collision occurs.

11.107. In the proof of 11.44, check in detail that I reverses signs, preserves weights, and is an involution.

11.108. Reprove 11.45 by comparing the symmetric and antisymmetric Pieri rules for multiplication by e_k.

11.109. Expand the following symmetric polynomials into linear combinations of Schur polynomials: (a) $s_{(3,3,2)}p_3$; (b) $p_{(3,1,3)}$; (c) $s_{(2,2)}p_{(2,1)}$.

11.110. Compute the coefficients of the following Schur polynomials in the Schur expansion of $p_{(3,3,2,1)}$: (a) $s_{(9)}$; (b) $s_{(3,3,3)}$; (c) $s_{(4,4,1)}$; (d) $s_{(1^9)}$.

11.111. Show that, for $\lambda \in \mathrm{Par}(n)$, $\chi^{\lambda}_{(1^n)} = |\,\mathrm{SYT}(\lambda)|$.

11.112. Write $s_{(3,2,1)}$ as a linear combination of power-sum polynomials.

11.113. For each $\mu \in \mathrm{Par}(4)$, write p_μ in terms of Schur polynomials, and write s_μ in terms of power-sum polynomials.

11.114. Let I be the involution in §11.12. For each $(v, T) \in X$ given below, compute $I(v, T)$. If (v, T) is a fixed point, compute $v^* \in Y$.

$$\text{(a) } v = 5432100\cdots,\ T = \begin{array}{|c|c|c|c|c|}\hline 1&1&1&2&2\\ \hline 2&3&4 \\ \cline{1-3} 3&5 \\ \cline{1-2}\end{array}\ .$$

$$\text{(b) } v = 2431500\cdots,\ T = \begin{array}{|c|c|c|c|c|}\hline 5&5&5&5&5\\ \hline 1&1&1 \\ \cline{1-3} 3&3 \\ \cline{1-2}\end{array}\ .$$

$$\text{(c) } v = 3452100\cdots,\ T = \begin{array}{|c|c|c|c|c|}\hline 1&1&1&1&1\\ \hline 2&2&4 \\ \cline{1-3} 5&3 \\ \cline{1-2}\end{array}\ .$$

11.115. Let I, X, and Y be defined as in §11.12. Take $N = 3$ and $\lambda = (2, 1, 0)$. List all the elements of X and Y, compute the action of I on X, and show how the fixed points of I map bijectively to Y.

11.116. Verify all the assertions stated just before 11.54.

11.117. Express $s_{(4,3,1)/(2,1)}$ as a linear combination of power-sums.

11.118. Explain why the formulas $\omega(h_\mu) = e_\mu$ and $\omega(e_\mu) = h_\mu$ are special cases of 11.59.

11.119. For $N \geq k$, two linear operators S and T on Λ^k_N are called *adjoint* iff $\langle S(f), g\rangle = \langle f, T(g)\rangle$ for all $f, g \in \Lambda^k_N$. Prove that this condition holds for all such f, g iff it holds for all f in some basis of Λ^k_N and all g in some (possibly different) basis of Λ^k_N.

11.120. Write the following Schur polynomials in terms of the complete symmetric polynomials h_μ: (a) $s_{(5,3)}$; (b) $s_{(4,1,1)}$; (c) $s_{(5,5,2,2)}$.

11.121. Write the following Schur polynomials in terms of the elementary symmetric polynomials e_μ: (a) $s_{(2,2,2,2)}$; (b) $s_{(3,2,1)}$; (c) $s_{(4,2)}$.

11.122. Write the skew Schur polynomial $s_{(4,4,3)/(2,1,1)}$ in terms of: (a) the h_μ's; (b) the e_μ's; (c) the p_μ's; (d) the m_μ's.

11.123. Modify the definition of the involution used in the proof of 11.60 as follows. If two or more paths in $(w, p_1, \ldots, p_n)$ intersect, choose i minimal and then j minimal such that p_i and p_j intersect. Let (u, v) be the earliest vertex on p_i that is also a vertex of p_j, and switch the initial segments of these two paths as in the original proof. Show that the map just defined is *not* always an involution.

11.124. Can you find a way to rephrase the combinatorial proof of 11.60 in terms of abaci?

11.125. Enumerate special rim-hook tableaux to compute $K'_{\mu,\lambda}$ for all partitions λ, μ with at most 4 cells. Use this to confirm by direct calculation that $\mathbf{KK}' = \mathbf{I}$.

11.126. Find and prove a Pieri-type rule giving the Schur expansion of a product $s_\nu m_\lambda$.

11.127. Let $\mathbf{K}'$ be the matrix defined combinatorially by $K'_{\lambda,\mu} = \sum_{S \in \mathrm{SRHT}(\mu,\lambda)} \mathrm{sgn}(S)$. Find involutions that prove $\mathbf{KK}' = \mathbf{I}$.

11.128. Let $\mathbf{K}'$ be the inverse Kostka matrix, defined using special rim-hook tableaux. Can you prove the identity $\mathbf{K}'\mathbf{K} = \mathbf{I}$ combinatorially?

11.129. Let I be the involution in the proof of 11.66. (a) Compute $I(v^0, T)$, where

$$v_0 = 5432100\cdots, \qquad T = \begin{array}{ccccc} & & 1 & 1 & 1 \\ & & 2 & 2 & 3 \\ 1 & 2 & 3 & 4 & 4 \\ 2 & 3 & 5 & & \end{array}.$$

(b) Answer (a) if the last 1 in the top row of T is changed to a 2. (c) Answer (a) if the last 3 in row 2 of T is changed to a 2.

11.130. Compute $c^\lambda_{\nu,\mu}$ and $c^\lambda_{\mu,\nu}$ using 11.66, where: (a) $\lambda = (5,3,1,1)$, $\mu = (3,1)$, $\nu = (3,2,1)$; (b) $\lambda = (5,4,4,3,1)$, $\mu = (4,3,3,1)$, $\nu = (3,1,1,1)$.

11.131. Repeat the previous exercise, but use 11.72 to compute the Littlewood-Richardson coefficients.

11.132. Continuing 11.69, find the expansion of $s_{(5,4,4,1)/(3,1)}$ into a sum of Schur polynomials.

11.133. Expand the following skew Schur polynomials into sums of Schur polynomials: (a) $s_{(3,3,3)/(2,1)}$; (b) $s_{(5,4)/(2)}$; (c) $s_{(4,3,2,1)/(1,1,1)}$.

11.134. Expand $s_{(3,2)}s_{(2,2)}$ into a sum of Schur polynomials.

11.135. In the Schur expansion of $s^2_{(3,2,1,1)}$, find the coefficients of: (a) $s_{(5,4,2,2,1)}$; (b) $s_{(5,3,3,1,1,1)}$; (c) $s_{(4,3,3,2,1,1)}$.

11.136. Give a combinatorial proof of 11.72 based on abaci.

Notes

The proof of the Jacobi triple product identity in §11.2 is adapted from a lecture of Richard Borcherds. One source for material on unlabeled abaci, k-cores, and k-quotients is the book by James and Kerber [72]; for labeled abaci, see Loehr [84]. Gessel and Viennot [53] have used intersecting lattice path models to prove many enumeration results. The combinatorial interpretation of the inverse Kostka matrix is due to Eğecioğlu and Remmel [33]. The proof of the Littlewood-Richardson rule given in §11.16 may be viewed as a combinatorialization of the algebraic proof in Remmel and Shimozono [111]. Many other proofs of this rule may be found in the literature; see, e.g., the bibliographic notes in Fulton [46] and Stanley [127, Ch. 7].