# Chapter 14

# Pólya Counting

Suppose you wish to color the four corners of a regular tetrahedron, and you have just two colors, red and blue. How many different colorings are there? One answer to this question is $2^4 = 16$, since a tetrahedron has four corners, and each corner can be colored with either of the two colors. But should we regard all of the 16 colorings to be different? If the tetrahedron is fixed in space, then each corner is distinguished from the others by its position, and it matters which color each corner gets. Thus, in this case, all 16 colorings are different. Now suppose that we are allowed to "move the tetrahedron around." Then, because it is so symmetrical, it matters not which corners are colored red and which are colored blue. The only way two colorings can be distinguished from one another is by the number of corners of each color. Hence, there is one coloring with all red corners, one with three red corners, one with two red corners, one with one red corner, and one with no red corners, giving a total of five different colorings.

Now suppose we color the four corners of a square with the colors red and blue. Again, we have 16 different colorings, provided the square is regarded as fixed in position. How many different colorings are there if we allow the square to move around? The square is also a highly symmetrical figure, although it does not possess the "complete symmetry" of the tetrahedron. As shown in Figure 14.1, there is one coloring with all red corners, one with three red corners, two with two red corners (the red corners can either be consecutive or separated by a blue corner), one with one red corner, and one with no red corners, giving a total of six different colorings.

For both the tetrahedron and square, if allowed to move around freely, the $2^4 = 16$ ways to color its corners are partitioned into parts in such a way that two colorings in the same part are regarded as the same (the colorings are *equivalent*), and two colorings in different parts are regarded as different (the colorings are *nonequivalent*). The number of nonequivalent colorings is thus the number of different parts. The purpose of this chapter is to develop and illustrate a technique for counting nonequivalent colorings in the presence of symmetries.

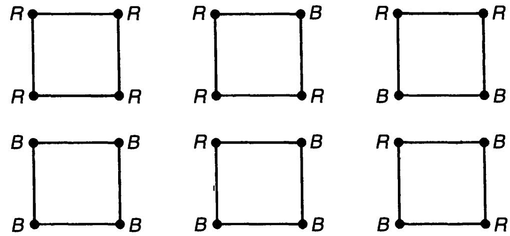


**Figure 14.1**

## 14.1 Permutation and Symmetry Groups

Let $X$ be a finite set. Without loss of generality, we take $X$ to be the set $\{1, 2, \ldots, n\}$, consisting of the first $n$ positive integers. Each permutation $i_1, i_2, \ldots, i_n$ of $X$ can be viewed as a one-to-one function from $X$ to itself defined by

$$f : X \to X$$

where

$$f(1) = i_1, f(2) = i_2, \ldots, f(n) = i_n.$$

By the pigeonhole principle, each one-to-one function $f : X \to X$ is onto.[1] To emphasize the view that a permutation can also be viewed as a function, we also denote this permutation by the 2-by-$n$ array

$$\begin{pmatrix} 1 & 2 & \cdots & n \\ i_1 & i_2 & \cdots & i_n \end{pmatrix}. \tag{14.1}$$

In (14.1), the value $i_k$ of the function at the integer $k$ is written below $k$.

**Example.** The $3! = 6$ permutations of $\{1, 2, 3\}$, regarded as functions, are

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}.$$

□

[1]Thus, one-to-one functions from $X$ to $X$ are one-to-one correspondences.

We denote the set of all $n!$ permutations of $\{1, 2, \ldots, n\}$ by $S_n$. Thus, $S_3$ consists of the six permutations listed in the previous example. Since permutations are now functions, they can be combined, using composition; that is, following one by another. If

$$f = \begin{pmatrix} 1 & 2 & \cdots & n \\ i_1 & i_2 & \cdots & i_n \end{pmatrix}$$

and

$$g = \begin{pmatrix} 1 & 2 & \cdots & n \\ j_1 & j_2 & \cdots & j_n \end{pmatrix}$$

are two permutations of $\{1, 2, \ldots, n\}$, then their *composition*, in the order $f$ followed by $g$, is the permutation

$$g \circ f = \begin{pmatrix} 1 & 2 & \cdots & n \\ j_1 & j_2 & \cdots & j_n \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & \cdots & n \\ i_1 & i_2 & \cdots & i_n, \end{pmatrix},$$

where

$$(g \circ f)(k) = g(f(k)) = j_{i_k}.$$

Composition of functions defines a *binary operation* on $S_n$: *If $f$ and $g$ are in $S_n$, then $g \circ f$ is also in $S_n$.*

**Example.** Let $f$ and $g$ be the permutations in $S_4$ defined by

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix} \qquad g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix}.$$

Then

$$(g \circ f)(1) = 3, \quad (g \circ f)(2) = 4, \quad (g \circ f)(3) = 1, \quad (g \circ f)(4) = 2.$$

Thus,

$$g \circ f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}.$$

We also have

$$f \circ g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}.$$

□

The binary operation $\circ$ of composition of permutations in $S_n$ satisfies the *associative law*[2]

$$(f \circ g) \circ h = f \circ (g \circ h),$$

[2]Composition of functions is always associative.

but as the previous example shows, it does not satisfy the commutative law. In general,

$$f \circ g \neq g \circ f,$$

although equality may hold in some instances. We use the usual power notation to denote compositions of a permutation with itself:

$$f^1 = f,\ f^2 = f \circ f,\ f^3 = f \circ f \circ f, \ldots, f^k = f \circ f \circ \cdots \circ f \ (k\ f\text{'s}).$$

The *identity permutation* is the permutation $\iota$ of $\{1, 2, \ldots, n\}$ that takes each integer to itself:

$$\iota(k) = k \text{ for all } k = 1, 2, \ldots, n;$$

equivalently,

$$\iota = \begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & 2 & \cdots & n \end{pmatrix}.$$

Obviously,

$$\iota \circ f = f \circ \iota = f$$

for all permutations $f$ in $S_n$. Each permutation in $S_n$, since it is a one-to-one function, has an inverse $f^{-1}$ that is also a permutation in $S_n$:

$$f^{-1}(k) = s, \text{ provided that } f(s) = k.$$

The 2-by-$n$ array for $f^{-1}$ can be gotten from the 2-by-$n$ array for $f$ by interchanging rows 1 and 2 and then rearranging columns so that the integers $1, 2, \ldots, n$ occur in the natural order in the first row. For each permutation $f$ we define $f^0 = \iota$. The inverse of the identity permutation is itself: $\iota^{-1} = \iota$.

**Example.** Consider the permutation in $S_6$ given by

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 6 & 3 & 1 & 2 & 4 \end{pmatrix}.$$

Then, interchanging rows 1 and 2, we get

$$\begin{pmatrix} 5 & 6 & 3 & 1 & 2 & 4 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}.$$

Rearranging columns, we get

$$f^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 5 & 3 & 6 & 1 & 2 \end{pmatrix}.$$

$\square$

The definition of inverse implies that, for all $f$ in $S_n$, we have

$$f \circ f^{-1} = f^{-1} \circ f = \iota.$$

A *group of permutations of* $X$, (in abbreviated form, a *permutation group*), is defined to be a nonempty subset $G$ of permutations in $S_n$ satisfying the following three properties:

(1) *closure under composition*: For all permutations $f$ and $g$ in $G$, $f \circ g$ is also in $G$.

(2) *identity*: The identity permutation $\iota$ of $S_n$ belongs to $G$.

(3) *closure under inverses*: For each permutation $f$ in $G$ the inverse $f^{-1}$ is also in $G$.

The set $S_n$ of all permutations of $X = \{1, 2, \ldots, n\}$ is a permutation group, called the *symmetric group of order n*. At the other extreme, the set $G = \{\iota\}$ consisting only of the identity permutation is a permutation group.

Every permutation group satisfies the *cancellation law*

$$f \circ g = f \circ h \text{ implies that } g = h.$$

This is because we may apply $f^{-1}$ to both sides of this equation and, using the associative law, obtain

$$\begin{aligned} f^{-1} \circ (f \circ g) &= f^{-1} \circ (f \circ h) \\ (f^{-1} \circ f) \circ g &= (f^{-1} \circ f) \circ h \\ \iota \circ g &= \iota \circ h \\ g &= h. \end{aligned}$$

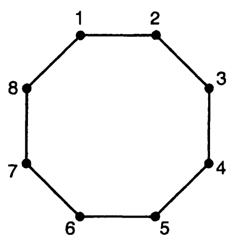


**Figure 14.2**

**Example.** Let $n$ be a positive integer, and let $\rho_n$ denote the permutation of $\{1, 2, \ldots, n$ defined by

$$\rho_n = \begin{pmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ 2 & 3 & 4 & \cdots & n & 1 \end{pmatrix}.$$

Thus, $\rho_n(i) = i+1$ for $i = 1, 2, \ldots, n-1$, and $\rho_n(n) = 1$. Think of the integers from 1 to $n$ as evenly spaced around a circle or on the corners of a regular $n$-gon, as shown, for $n = 8$, in Figure 14.2. Then $\rho_n$ sends each integer to the integer that follows it in the clockwise direction. Indeed, we may consider $\rho_n$ as the rotation of the circle by an angle of $360/n$ degrees. The permutation $\rho_n^2$ is then the rotation by $2 \times (360/n)$ degrees, and, more generally, for each nonnegative integer $k$, $\rho_n^k$ is the rotation by $k \times (360/n)$ degrees. This implies that

$$\rho_n^k = \begin{pmatrix} 1 & 2 & \cdots & n-k & n-k+1 & \cdots & n \\ k+1 & k+2 & \cdots & n & 1 & \cdots & k \end{pmatrix}.$$

In particular, if $r$ equals $k$ mod $n$, then $\rho_n^r = \rho_n^k$. Thus, there are only $n$ distinct powers of $\rho_n$, namely,

$$\rho_n^0 = \iota,\ \rho_n,\ \rho_n^2,\ \ldots, \rho_n^{n-1}.$$

Also,

$$\rho_n^{-1} = \rho_n^{n-1},$$

and, more generally,

$$(\rho_n^k)^{-1} = \rho_n^{n-k} \text{ for } k = 0, 1, \ldots, n-1.$$

We thus conclude that

$$C_n = \{\rho_n^0 = \iota, \rho_n, \rho_n^2, \ldots, \rho_n^{n-1}\}$$

is a permutation group.[3] It is an example of a *cyclic group* of order $n$. As you may realize, this is the group that was implicitly used for calculating the number of ways to arrange $n$ distinct objects in a circle. More about this later. □

Let $\Omega$ be a geometrical figure. A *symmetry* of $\Omega$ is a (geometric) motion or congruence that brings the figure $\Omega$ onto itself. The geometric figures that we consider, like a square, a tetrahedron, and a cube, are composed of corners (or vertices) and edges, and in the case of three-dimensional figures, of faces (or sides). As a result, each symmetry acts as a permutation on the corners, on the edges, and, in the case of three dimensional figures, on the faces. A symmetry of $\Omega$ followed by another (that is, the composition of two symmetries) is again a symmetry. Similarly, the inverse of a symmetry is also a symmetry. Finally, the motion that leaves everything fixed[4] is a symmetry, the identity symmetry. Hence, we conclude that the symmetries of $\Omega$ act as a permutation group $G_C$ on its corners, a permutation group $G_E$ on its edges, and,

[3] In more formal language, the permutation group $C_n$ is isomorphic to the additive group of the integers mod $n$ as discussed in Section 10.1.

[4] So nothing actually moves in this motion!

in the case where $\Omega$ is three-dimensional, a permutation group $G_F$ on its faces.[5] As a result, a set of permutations that results by considering all the symmetries of a figure is automatically a permutation group. Thus, we have a *corner-symmetry group*, an *edge-symmetry group*, a *face-symmetry group*, and so on.

**Example.** Consider a square $\Omega$ with its corners labeled 1, 2, 3, and 4 and its edges labeled $a$, $b$, $c$, and $d$, as in Figure 14.3. There are eight symmetries of $\Omega$ and they are of two types. There are the four rotations about the center of the square through the angles of 0, 90, 180, and 270 degrees. These four symmetries constitute the *planar symmetries* of $\Omega$, the symmetries where the motion takes place in the plane containing $\Omega$. The planar symmetries by themselves form a group. The other symmetries are the four reflections about the lines joining opposite corners and the lines joining the midpoints of opposite sides. For these symmetries the motion takes place in space since to "flip" the square we need to go outside the plane containing it.

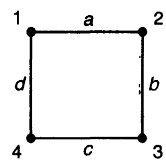


**Figure 14.3**

The rotations acting on the corners give the four permutations

$$\rho_4^0 = \iota = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} \quad \rho_4 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$$

$$\rho_4^2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} \quad \rho_4^3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}.$$

[5]There is an abstract concept of a group, which is defined to be a nonempty set with a binary operation, which satisfies the associative law and also (1) closure under composition, (2) identity, and (3) closure under inverses. Permutation groups are groups since the associative law is automatic for composition of functions. The symmetries of a figure $\Omega$ form a group under this definition, but, as indicated, these symmetries can act as a permutation group of its corners, a permutation group of its edges, and so on.

The reflections acting on the corners give the four permutations[6]

$$\tau_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix} \quad \tau_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix}$$

$$\tau_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \quad \tau_4 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}.$$

Thus, the corner-symmetry group of a square is

$$G_C = \{\rho_4^0 = \iota, \rho_4, \rho_4^2, \rho_4^3, \tau_1, \tau_2, \tau_3, \tau_4\}.$$

We check that

$$\tau_3 = \rho_4 \circ \tau_1, \ \tau_2 = \rho_4^2 \circ \tau_1, \ \text{and } \tau_4 = \rho_4^3 \circ \tau_1.$$

Hence, we can also write

$$G_C = \{\rho_4^0 = \iota, \rho_4, \rho_4^2, \rho_4^3, \tau_1, \rho_4 \circ \tau_1, \rho_4^2 \circ \tau_1, \rho_4^3 \circ \tau_1\}.$$

Consider the edges of $\Omega$ to be labeled $a$, $b$, $c$, and $d$, as in Figure 14.3. The edge-symmetry group $G_E$ is obtained by letting the symmetries of $\Omega$ act on the edges. For example, the reflection about the line joining the corners 2 and 4 gives the following permutation of the edges:

$$\begin{pmatrix} a & b & c & d \\ b & a & d & c \end{pmatrix}.$$

The other permutation of the edges in $G_C$ can be obtained in a similar way. □

In a similar way we can obtain the symmetry group of a regular $n$-gon for any $n \geq 3$. Besides the $n$ rotations $\rho_n^0 = \iota, \rho, \ldots, \rho_n^{n-1}$, we have $n$ reflections $\tau_1, \tau_2, \ldots, \tau_n$. If $n$ is even, then there are $n/2$ reflections about opposite corners and $n/2$ reflections about the lines joining the midpoints of opposite sides. If $n$ is odd, then the reflections are the $n$ reflections about the lines joining a corner to the side opposite it. The resulting group

$$D_n = \{\rho_n^0 = \iota, \rho, \ldots, \rho_n^{n-1}, \tau_1, \tau_2, \ldots, \tau_n\}$$

of $2n$ permutations of $\{1, 2, \ldots, n\}$ is an instance of a *dihedral group* of order $2n$. In the next example we compute $D_5$.

**Example.** *The dihedral group of order* 10. Consider the regular pentagon with its vertices labeled $1, 2, 3, 4,$ and 5, as in Figure 14.4. Its (corner) symmetry group $D_5$ contains five rotations and five reflections. The five rotations are

$$\rho_5^0 = \iota = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix} \quad \rho_5^1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix}$$

[6] $\tau_1$ comes from the reflection about the line joining vertices 1 and 3, $\tau_2$ comes from the reflection about the line joining vertices 2 and 4, $\tau_3$ comes from the reflection about the line joining the midpoints of the lines $a$ and $c$, and $\tau_4$ comes from the reflection about the line joining the midpoints of the lines $b$ and $d$.

$$\rho_5^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 1 & 2 \end{pmatrix} \quad \rho_5^3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 1 & 2 & 3 \end{pmatrix}$$

$$\rho_5^4 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 3 & 4 \end{pmatrix}.$$

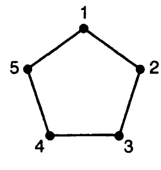


**Figure 14.4**

Let $\tau_i$ denote the reflection about the line joining corner $i$ to the side opposite it ($i = 1, 2, 3, 4, 5$). Then we have

$$\tau_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 5 & 4 & 3 & 2 \end{pmatrix} \quad \tau_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 1 & 5 & 4 \end{pmatrix}$$

$$\tau_3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 3 & 2 & 1 \end{pmatrix} \quad \tau_4 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 5 & 4 & 3 \end{pmatrix}$$

$$\tau_5 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 2 & 1 & 5 \end{pmatrix}.$$

□

Suppose we have a group $G$ of permutations of a set $X$, where $X$ is again taken to be the set $\{1, 2, \ldots, n\}$ of the first $n$ positive integers. A *coloring* of $X$ is an assignment of a color to each element of $X$. Let $\mathcal{C}$ be a collection of colorings of $X$. Usually we have a number of colors, say red and blue, and $\mathcal{C}$ consists of all colorings of $X$ with these colors. But this need not be the case. The set $\mathcal{C}$ can be *any* collection of colorings of $X$ as long as $G$ takes a coloring in $\mathcal{C}$ to another coloring in $\mathcal{C}$ in the manner to be described now.

Let $\mathbf{c}$ be a coloring of $X$ and let the colors of $1, 2, \ldots, n$ be $c(1), c(2), \ldots, c(n)$, respectively. Let

$$f = \begin{pmatrix} 1 & 2 & \cdots & n \\ i_1 & i_2 & \cdots & i_n \end{pmatrix}$$

be a permutation in $G$. Then $f * \mathbf{c}$ is defined to be the coloring in which $i_k$ has the color $c(k)$, that is,

$$(f * \mathbf{c})(i_k) = c(k), \quad (k = 1, 2, \ldots, n). \tag{14.2}$$

In words, since $f$ moves $k$ to $i_k$, the color of $k$, namely $\mathbf{c}(k)$, moves to $f(k) = i_k$ and becomes the color of $i_k$. Using the inverse of $f$, we can write (14.2) as

$$(f * \mathbf{c})(l) = c(f^{-1}(l)), \quad (l = 1, 2, \ldots, n).$$

The set $\mathcal{C}$ of colorings is required to have the property:

*For all $f$ in $G$ and all $\mathbf{c}$ in $\mathcal{C}$, $f * \mathbf{c}$ is also in $\mathcal{C}$.*

This implies that $f$ moves each coloring in $\mathcal{C}$ to another (possibly the same) coloring in $\mathcal{C}$; $f * \mathbf{c}$ denotes the coloring in $\mathcal{C}$ into which $\mathbf{c}$ is sent by $f$. Note that, if $\mathcal{C}$ is the set of *all* colorings of $X$ for a given set of colors or if $\mathcal{C}$ is the set of all colorings of $X$ with a specified number of elements of $X$ of each color, then $\mathcal{C}$ automatically has the required property.

The basic relationship that holds between the two operations $\circ$ (composition of permutations in $G$) and $*$ (action of permutations in $G$ on colorings in $\mathcal{C}$) is

$$(g \circ f) * \mathbf{c} = g * (f * \mathbf{c}). \tag{14.3}$$

The left side of equation (14.3) is the coloring in which the color of $k$ moves to $(g \circ f)(k)$. The right side is the coloring in which the color of $k$ moves to $f(k)$ and then moves to $g(f(k))$. Since $(g \circ f)(k) = g(f(k))$ by the definition of composition, we have verified (14.3).

**Example.** We continue with the earlier example in which $\Omega$ is the square in Figure 14.3 and $G_C$ is the corner-symmetry group of $\Omega$. Let $\mathcal{C}$ be the set of all colorings of the corners $1, 2, 3, 4$ of $\Omega$ in which the colors are either red or blue. The permutation group $G_C$ contains eight permutations, and there are 16 colorings in $\mathcal{C}$. Let us denote a coloring by writing the colors of the corners in the order $1, 2, 3, 4$, using $R$ to denote red and $B$ to denote blue. For instance,

$$(R, B, B, R) \tag{14.4}$$

is the coloring in which corner 1 is red, corner 2 is blue, corner 3 is blue, and corner 4 is red. The permutation $\rho_4$ sends this coloring into the coloring

$$(R, R, B, B),$$

in which corners 1 and 2 are red and corners 3 and 4 are blue. In the following table, we list the effect of each permutation in $G_C$ on the coloring (14.4).

Notice that the permutation $\tau_4$ doesn't change the coloring (14.4); that is, $\tau_4$ fixes the coloring (14.4). Of course, the identity $\iota$ also doesn't change it. In fact, each coloring on the list appears exactly twice. Let us say that two colorings are equivalent, provided that there is a permutation in $G_C$ which sends one to the other. Thus, the coloring $(R, B, B, R)$ is equivalent to each of

$$(R, B, B, R), \ (R, R, B, B), \ (B, R, R, B), \text{ and } (B, B, R, R).$$

| Permutation in $G_C$ | Effect on the Coloring $(R, B, B, R)$ |
|---|---|
| $\rho_4^0 = \iota$ | $(R, B, B, R)$ |
| $\rho_4$ | $(R, R, B, B)$ |
| $\rho_4^2$ | $(B, R, R, B)$ |
| $\rho_4^3$ | $(B, B, R, R)$ |
| $\tau_1$ | $(R, R, B, B)$ |
| $\tau_2$ | $(B, B, R, R)$ |
| $\tau_3$ | $(B, R, R, B)$ |
| $\tau_4$ | $(R, B, B, R)$ |

Since a permutation cannot change the number of corners of each of the colors, a necessary—but not, in general sufficient—condition for two colorings to be equivalent is that they contain the same number of $R$'s and the same number of $B$'s.[7] The coloring $(R, B, R, B)$ also has two $R$'s and two $B$'s but is not equivalent to $(R, B, B, R)$. Indeed, as can now be checked, $(R, B, R, B)$ is equivalent only to $(R, B, R, B)$ and $(B, R, B, R)$, and each of these colorings arises four times as we examine the effect of all the permutations in $G_C$ on it. In particular, we can now conclude that there are two nonequivalent colorings among all the colorings with two red and two blue corners. The coloring $(R, R, R, R)$ is clearly equivalent only to itself, as is the coloring $(B, B, B, B)$. Consider the coloring $(R, B, B, B)$ with one red and three blue corners. This coloring is equivalent, by a rotation, to each of the colorings $(R, B, B, B), (B, R, B, B), (B, B, R, B)$, and $(B, B, B, R)$, and hence all colorings with one red are equivalent. Similarly, all colorings with three red (and therefore one blue) are equivalent by a rotation. *Consequently, there are* $2 + 1 + 1 + 1 + 1 = 6$ *nonequivalent ways to color the corners of a square with two colors, under the action of the corner-symmetry group* $G_C$ *of the square.* If we don't allow the full symmetry group of the square, but only the group of symmetries consisting of the four rotations $\rho_0 = \iota$, $\rho_4$, $\rho_4^2$, and $\rho_4^3$, then the number of nonequivalent colorings is still 6. This is because if two colorings are equivalent by a symmetry of the square, then they are equivalent by a rotation. □

We now give the general definition of equivalent colorings. Let $G$ be a group of permutations acting on a set $X$, as usual taken to be the set $\{1, 2, \ldots, n\}$ of the first $n$ positive integers. Let $\mathcal{C}$ be a collection of colorings of $X$, such that for all $f$ in $G$ and

[7] Of course, if two colorings have the same number of $R$'s, they must have the same number of $B$'s.

all $\mathbf{c}$ in $\mathcal{C}$, the coloring $f * \mathbf{c}$ of $X$ is also in $\mathcal{C}$. Thus $G$ acts on $\mathcal{C}$ in the sense that it takes colorings in $\mathcal{C}$ to colorings in $\mathcal{C}$. Let $\mathbf{c}_1$ and $\mathbf{c}_2$ be two colorings in $\mathcal{C}$. We define a relation called *equivalence*, denoted by $\overset{G}{\sim}$ (or, more briefly, by $\sim$) on $\mathcal{C}$ as follows: $\mathbf{c}_1$ is *equivalent* (*under the action of* $G$) to $\mathbf{c}_2$, provided that there is a permutation $f$ in $G$ such that

$$f * \mathbf{c}_1 = \mathbf{c}_2.$$

Two colorings are *nonequivalent*, provided that they are not equivalent. We have the following:

(1) *reflexive property*: $\mathbf{c} \sim \mathbf{c}$ for each coloring $\mathbf{c}$ (because $\iota * \mathbf{c} = \mathbf{c}$).

(2) *symmetry property*: If $\mathbf{c}_1 \sim \mathbf{c}_2$, then $\mathbf{c}_2 \sim \mathbf{c}_1$
(if $f * \mathbf{c}_1 = \mathbf{c}_2$ for some $f$ in $G$, then $f^{-1} * \mathbf{c}_2 = \mathbf{c}_1$).

(3) *transitive property*: If $\mathbf{c}_1 \sim \mathbf{c}_2$ and $\mathbf{c}_2 \sim \mathbf{c}_3$, then $\mathbf{c}_1 \sim \mathbf{c}_3$)
(if $f * \mathbf{c}_1 = \mathbf{c}_2$ and $g * \mathbf{c}_2 = \mathbf{c}_3$, then $(g \circ f) * \mathbf{c}_1 = \mathbf{c}_3$).

It thus follows that $\sim$ is an equivalence relation on $\mathcal{C}$ in the sense defined in Section 4.5, which justifies our use of the term *equivalence*.

Notice how the three basic properties of a permutation group—namely, identity, closure under inverses, and closure under composition—are used in the verification of (1)-(3). By Theorem 4.5.3 of Chapter 4, equivalence partitions the colorings of $\mathcal{C}$ into parts, with two colorings being in the same part if and only if they are equivalent colorings. In the next section we derive a general formula for the number of parts—that is, for the number of nonequivalent colorings—of $\mathcal{C}$ under the action of the permutation group $G$.

## 14.2 Burnside's Theorem

In this section we derive and apply a formula of Burnside[8] for counting the number of nonequivalent colorings of a set $X$ under the action of a group of permutations of $X$.

Let $G$ be a group of permutations of $X$ and let $\mathcal{C}$ be a set of colorings of $X$ such that $G$ acts on $\mathcal{C}$. Recall that this means that

$$f * \mathbf{c}$$

[8]That's what it is commonly called because of its appearance in the book by W. Burnside, *Theory of Groups of Finite Order*, 2nd edition, Cambridge University Press, London, 1911 (reprinted by Dover, New York, 1955), p. 191. As discovered in the paper by P. M. Neumann, A Lemma That Is Not Burnside's, *Math. Sci.*, 4 (1979), 133–141, it appeared earlier in works of Cauchy (1845) and Frobenius (1887).

is in $\mathcal{C}$ for all $f$ in $G$ and all $\mathbf{c}$ in $\mathcal{C}$, and each $f$ in $G$ permutes the colorings in $\mathcal{C}$. It is possible, that for an appropriate choice of $f$ and of $\mathbf{c}$, we have

$$f * \mathbf{c} = \mathbf{c}. \tag{14.5}$$

For example, in Figure 14.3, if we color corners 1 and 3 of the square red and the corners 2 and 4 blue, then reflecting about the line through 1 and 3 or the line through 2 and 4, or rotating by 180 degrees, does not alter the coloring; each of these motions fixes the color of each corner and hence fixes the coloring. If, in (14.5), we allow either $f$ to vary over all permutations in $G$ or $\mathbf{c}$ to vary over all colorings in $\mathcal{C}$, then we get

$$G(\mathbf{c}) = \{f : f \text{ in } G, f * \mathbf{c} = \mathbf{c}\},$$

the set of all permutations in $G$ that fix the coloring $\mathbf{c}$, and

$$\mathcal{C}(f) = \{\mathbf{c} : \mathbf{c} \text{ in } \mathcal{C}, f * \mathbf{c} = \mathbf{c}\},$$

the set of all colorings in $\mathcal{C}$ that are fixed by $f$. The set $G(\mathbf{c})$ of all permutations that fix the coloring $\mathbf{c}$ is called the *stabilizer*[9] of $\mathbf{c}$. The stabilizer of any coloring also forms a group of permutations.

**Theorem 14.2.1** *For each coloring* $\mathbf{c}$, *the stabilizer* $G(\mathbf{c})$ *of* $\mathbf{c}$ *is a permutation group. Moreover, for any permutations* $f$ *and* $g$ *in* $G$, $g * \mathbf{c} = f * \mathbf{c}$ *if and only if* $f^{-1} \circ g$ *is in* $G(\mathbf{c})$.

**Proof.** If $f$ and $g$ both fix $\mathbf{c}$, then $f$ followed by $g$ fixes $\mathbf{c}$; that is, $(g \circ f)(\mathbf{c}) = \mathbf{c}$. Thus, $G(\mathbf{c})$ is closed under composition. Clearly, the identity $\iota$ fixes $\mathbf{c}$ since it fixes every coloring. Also, if $f$ fixes $\mathbf{c}$, then so does $f^{-1}$, and hence $G(\mathbf{c})$ is closed under inverses. All of the defining properties of a permutation group are satisfied; therefore, $G(\mathbf{c})$ is a permutation group.

Suppose that $f * \mathbf{c} = g * \mathbf{c}$. By the basic relationship (14.3), we get

$$(f^{-1} \circ g) * \mathbf{c} = f^{-1} * (g * \mathbf{c}) = f^{-1} * (f * \mathbf{c}) = (f^{-1} \circ f) * \mathbf{c} = \iota * \mathbf{c} = \mathbf{c}.$$

It follows that $f^{-1} \circ g$ fixes $\mathbf{c}$, and hence $f^{-1} \circ g$ is in $G(\mathbf{c})$. Conversely, suppose that $f^{-1} \circ g$ is in $G(\mathbf{c})$. Then a similar calculation shows that $f * \mathbf{c} = g * \mathbf{c}$. □

As a corollary of Theorem 14.2.1, starting from a given coloring $\mathbf{c}$, we can determine the number of different colorings we can get under the action of $G$.

**Corollary 14.2.2** *Let* $\mathbf{c}$ *be a coloring in* $\mathcal{C}$. *The number*

$$|\{f * \mathbf{c} : f \text{ in } G\}|$$

[9] A synonym for *fixed* is *stable*.

*of different colorings that are equivalent to* $\mathbf{c}$ *equals the number*

$$\frac{|G|}{|G(\mathbf{c})|}$$

*obtained by dividing the number of permutations in* $G$ *by the number of permutations in the stabilizer of* $\mathbf{c}$.

**Proof.** Let $f$ be a permutation in $G$. By Theorem 14.2.1, the permutations $g$ that satisfy

$$g * \mathbf{c} = f * \mathbf{c}$$

are precisely the permutations in

$$\{f \circ h : h \text{ in } G(\mathbf{c})\}. \tag{14.6}$$

By the cancellation law, $f \circ h = f \circ h'$ implies $h = h'$. Hence, the number of permutations in the set (14.6) equals the number $|G(\mathbf{c})|$ of permutations $h$ in $G(\mathbf{c})$. Thus, for each permutation $f$, there are exactly $|G(\mathbf{c})|$ permutations that have the same effect on $\mathbf{c}$ as $f$. Since there are $|G|$ permutations overall, the number

$$|\{f * \mathbf{c} : f \text{ in } G\}|$$

of colorings equivalent to $\mathbf{c}$ equals, by the division principle,

$$\frac{|G|}{|G(\mathbf{c})|},$$

proving the corollary. □

The next theorem of Burnside gives a formula for counting the number of nonequivalent colorings.

**Theorem 14.2.3** *Let* $G$ *be a group of permutations of* $X$ *and let* $C$ *be a set of colorings of* $X$ *such that* $f * \mathbf{c}$ *is in* $C$ *for all* $f$ *in* $G$ *and all* $\mathbf{c}$ *in* $C$. *Then the number* $N(G, C)$ *of nonequivalent colorings in* $C$ *is given by*

$$N(G, C) = \frac{1}{|G|} \sum_{f \in G} |C(f)|. \tag{14.7}$$

*In words, the number of nonequivalent colorings in* $C$ *equals the average of the number of colorings fixed by the permutations in* $G$.

**Proof.** With the information we now have, the proof is a simple application of a technique we have experienced many times, namely, counting in two different ways and then equating counts. What do we count? We count the number of pairs $(f, \mathbf{c})$

such that $f$ fixes $\mathbf{c}$; that is, such that $f * \mathbf{c} = \mathbf{c}$. One way to count is to consider each $f$ in $G$ and compute the number of colorings that $f$ fixes, and then add up all quantities. Counting in this way, we get

$$\sum_{f \in G} |\mathcal{C}(f)|,$$

since $\mathcal{C}(f)$ is the set of colorings that are fixed by $f$.

Another way to count is to consider each $\mathbf{c}$ in $\mathcal{C}$ and compute the number of permutations $f$ such that $f * \mathbf{c} = \mathbf{c}$, and then add up all the quantities. For each coloring $\mathbf{c}$, the set of all $f$ such that $f * \mathbf{c} = \mathbf{c}$ is what we have called the stabilizer $G(\mathbf{c})$ of $\mathbf{c}$. Thus, each $\mathbf{c}$ contributes

$$|G(\mathbf{c})|$$

to the sum. Counting in this way, we get

$$\sum_{c \in \mathcal{C}} |G(\mathbf{c})|.$$

Putting these two counts together, we get

$$\sum_{f \in G} |\mathcal{C}(f)| = \sum_{c \in \mathcal{C}} |G(\mathbf{c})|. \tag{14.8}$$

Now, by Corollary 14.2.2,

$$|G(\mathbf{c})| = \frac{|G|}{(\text{the number of colorings equivalent to } \mathbf{c})}. \tag{14.9}$$

Hence we get

$$\sum_{c \in \mathcal{C}} |G(\mathbf{c})| = |G| \sum_{c \in \mathcal{C}} \frac{1}{(\text{the number of colorings equivalent to } \mathbf{c})}. \tag{14.10}$$

The second summation in (14.10) can be simplified if we group the colorings by equivalence class. Two colorings in the equivalence class of $\mathbf{c}$ contribute the same amount

$$\frac{1}{(\text{the number of colorings equivalent to } \mathbf{c})}$$

to this sum. Thus the total contribution of every equivalence class is 1. Consequently, (14.10) equals

$$N(G, \mathcal{C}) \times |G|, \tag{14.11}$$

since the number of equivalence classes is the number $N(G, \mathcal{C})$ of nonequivalent colorings. Substituting into equation (14.8), we get

$$\sum_{f \in G} |\mathcal{C}(f)| = N(G, \mathcal{C}) \times |G|;$$

solving for $N(G, \mathcal{C})$, we obtain (14.7). □

In the remainder of this section we illustrate Burnside's theorem with several examples.

**Example.** *Counting circular permutations.* How many ways are there to arrange $n$ distinct objects in a circle?

As already hinted at in Section 14.1, the answer is the number of ways to color the corners of a regular $n$-gon $\Omega$ with $n$ different colors that are nonequivalent with respect to the group of rotations of $\Omega$. Let $\mathcal{C}$ consist of all $n!$ ways to color the $n$ corners of $\Omega$ in which each of the $n$ colors occurs once. Then the cyclic group

$$C_n = \{\rho_n^0 = \iota, \rho_n, \ldots, \rho_n^{n-1}\}$$

acts[10] on $\mathcal{C}$, and the number of circular permutations equals the number of nonequivalent colorings in $\mathcal{C}$. The identity permutation $\iota$ in $C_n$ fixes all $n!$ of the colorings in $\mathcal{C}$. Every other permutation in $\mathcal{C}$ does not fix any coloring in $\mathcal{C}$, since, in the colorings of $\mathcal{C}$, every corner has a different color.[11] Hence, using (14.7) of Theorem 14.2.3, we see that the number of nonequivalent colorings is

$$N(C_n, \mathcal{C}) = \frac{1}{n}(n! + 0 + \cdots + 0) = (n-1)!.$$

□

**Example.** *Counting necklaces.* How many ways are there to arrange $n \geq 3$ differently colored beads in a necklace?

We have almost the same situation as described in the previous example, except since necklaces can be flipped over, the group $G$ of permutations now has to be taken to be the entire vertex-symmetry group of a regular $n$-gon. Thus, in this case, $G$ is the dihedral group $D_n$ of order $2n$. The only permutation that can fix a coloring is the identity and it fixes all $n!$ colorings. Hence, the number of nonequivalent colorings—that is, the number of different necklaces—is, by (14.7),

$$N(D_n, \mathcal{C}) = \frac{1}{2n}(n! + 0 + \cdots + 0) = \frac{(n-1)!}{2}.$$ □

[10]Recall that $\rho_n$ is the rotation by $360/n$ degrees.

[11]In fact, no permutation different from the identity can fix any coloring if all colors are different. This is because, for a permutation different from the identity, at least one color has to move, and hence the coloring is changed.

**Example.** How many nonequivalent ways are there to color the corners of a regular 5-gon with the colors red and blue?

The group of symmetries of a regular 5-gon is the dihedral group

$$D_5 = \{\rho_5^0 = \iota, \rho_5, \rho_5^2, \rho_5^3, \rho_5^4, \tau_1, \tau_2, \tau_3, \tau_4, \tau_5\},$$

where, as in Section 14.1, $\tau_j$ is the reflection about the line joining corner $j$ with the midpoint of the opposite side ($j = 1, 2, 3, 4, 5$). Let $\mathcal{C}$ be the set of all $2^5 = 32$ colorings of the corners of a regular 5-gon. We compute the number of colorings left fixed by each permutation in $D_5$ and then apply Theorem 14.2.3. The identity $\iota$ fixes all colorings. Each of the other four rotations fixes only two colorings, namely, the colorings in which all corners are red and all corners are blue. Thus,

$$|\mathcal{C}(\rho_5^i)| = \begin{cases} 32 & \text{if } i = 0, \\ 2 & \text{if } i = 1, 2, 3, 4. \end{cases}$$

Now consider any of the reflections $\tau_j$, say, $\tau_1$. For a coloring to be fixed by $\tau_1$, corners 2 and 5 must have the same color and corners 3 and 4 must have the same color. Hence, the colorings fixed by $\tau_1$ are obtained by picking a color for corner 1 (two choices), picking a color for corners 2 and 5 (two choices), and picking a color for corners 3 and 4 (again two choices). Therefore, the number of colorings fixed by $\tau_1$ equals $2 \times 2 \times 2 = 8$. A similar calculation holds for each reflection, and we have

$$|\mathcal{C}(\tau_j)| = 8 \quad \text{for each } j = 1, 2, 3, 4, 5.$$

Therefore, by (14.7), the number of nonequivalent colorings is

$$N(D_5, \mathcal{C}) = \frac{1}{10}(32 + 2 + 2 + 2 + 2 + 8 + 8 + 8 + 8 + 8) = 8.$$

□

**Example.** How many nonequivalent ways are there to color the corners of a regular 5-gon now with the three colors red, blue, and green?

The set $\mathcal{C}$ of colorings of the corners of a regular 5-gon numbers $3^5 = 243$. The identity fixes all 243 colorings. The other rotations fix three colorings. The reflections fix $3 \times 3 \times 3 = 27$ colorings. Thus, the number of nonequivalent colorings is

$$N(D_5, \mathcal{C}) = \frac{1}{10}(243 + 3 + 3 + 3 + 3 + 27 + 27 + 27 + 27 + 27) = 39.$$

Generalizing the preceding calculations using $p$ colors, we get

$$N(D_5, \mathcal{C}) = \frac{1}{10}(p^5 + 4 \times p + 5 \times p^3) = \frac{p(p^2 + 4)(p^2 + 1)}{10}.$$

□

**Example.** Let $S = \{\infty \cdot r, \infty \cdot b, \infty \cdot g, \infty \cdot y\}$ be a multiset of four distinct objects $r, b, g, y$, each with an infinite repetition number. How many $n$-permutations of $S$ are there if we do not distinguish between a permutation read from left to right and a permutation read from right to left? Thus, for instance, $r, g, g, g, b, y, y$ is regarded as equivalent to $y, y, b, g, g, g, r$.

The answer is the number of nonequivalent ways to color the integers from 1 to $n$ with the four colors red, blue, green, and yellow under the action of the group of permutations

$$G = \{\iota, \tau\},$$

where

$$\iota = \begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & 2 & \cdots & n \end{pmatrix} \text{ and } \tau = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ n & n-1 & \cdots & 2 & 1 \end{pmatrix}.$$

Here, $\iota$ is, as usual, the identity permutation. The permutation $\tau$ is obtained by listing the integers from 1 to $n$ in reverse order. Note that $G$ does form a group, since $\tau \circ \tau = \iota$ and hence $\tau^{-1} = \tau$.[12] Let $\mathcal{C}$ be the set of all $4^n$ ways to color the integers from 1 to $n$ with the given four colors. Then $\iota$ fixes all colorings in $\mathcal{C}$. The number of colorings fixed by $\tau$ depends on whether $n$ is even or odd. First, suppose that $n$ is even. Then a coloring is fixed by $\tau$ if and only if 1 and $n$ have the same color, 2 and $n-1$ have the same color, ... , and $n/2$ and $(n/2)+1$ have the same color. Hence, $\tau$ fixes $4^{n/2}$ colorings in $\mathcal{C}$. Now suppose that $n$ is odd. Then a coloring is fixed by $\tau$ if and only if 1 and $n$ have the same color, 2 and $n-1$ have the same color, ... , and $(n-1)/2$ and $(n+3)/2$ have the same color, there being no restriction on the color of $(n+1)/2$. Thus, the number of colorings fixed by $\tau$ is $4^{(n-1)/2} \times 4 = 4^{(n+1)/2}$. Using the floor function, we can combine both cases and obtain

$$|\mathcal{C}(\tau)| = 4^{\lfloor \frac{n+1}{2} \rfloor}.$$

Applying Burnside's formula (14.7), we find that the number of nonequivalent colorings is

$$N(G, \mathcal{C}) = \frac{4^n + 4^{\lfloor \frac{(n+1)}{2} \rfloor}}{2}.$$

If instead of four colors, we have $p$ colors, the number of nonequivalent colorings is

$$N(G, \mathcal{C}) = \frac{p^n + p^{\lfloor \frac{(n+1)}{2} \rfloor}}{2}.$$

□

In the next section, we develop a little more theory that will enable us to solve more easily more difficult counting problems using Theorem 14.2.3.

[12]Think of a line segment consisting of $n$ equally spaced points that are labeled $1, 2, \ldots, n$. Then $\tau$ is a rotation of this line segment by 180 degrees. Equivalently, $\tau$ is a reflection of this line segment about its perpendicular bisector.

## 14.3 Pólya's Counting Formula

The counting formula discussed in this section was developed (and extensively applied) by Pólya in an important, long, and very influential paper.[13] Around 1960 it was recognized that 10 years before Pólya's famous paper was published, Redfield published a paper[14] in which he anticipated the basic technique of Pólya.

As we have seen in the previous section, success in using Burnside's theorem for counting the number of nonequivalent colorings in the presence of a permutation group $G$ acting on a set $\mathcal{C}$ of colorings is dependent on being able to compute the number $|\mathcal{C}(f)|$ of colorings in $\mathcal{C}$ fixed by a permutation $f$ in $G$. This computation can be facilitated by consideration of the cyclic structure of a permutation.

Let $f$ be a permutation of $X = \{1, 2, \ldots, n\}$. Let $D_f = (X, A_f)$ be the digraph whose set of vertices is $X$ and whose set of arcs is

$$A_f = \{(i, f(i)) : i \text{ in } X\}.$$

The digraph has $n$ vertices and $n$ arcs. Moreover, the indegree and outdegree of each vertex equal 1. As shown in Corollary 11.8.8, the set $A_f$ of arcs can be partitioned into directed cycles, with each vertex belonging to exactly one directed cycle. The reason is simply that, starting at any vertex $j$, we proceed along the unique arc leaving $j$ and arrive at another vertex $k$; we now repeat with $k$ and continue until we arrive back at vertex $i$, thereby creating a directed cycle. We must eventually arrive at our starting vertex $i$ since each vertex has indegree and outdegree equal to 1. We remove the vertices and arcs of the directed cycle so obtained and continue until we exhaust all the vertices and arcs of $D_f$, thereby partitioning both the vertices and arcs of $D_f$ into directed cycles.

**Example.** Let

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 8 & 5 & 4 & 1 & 3 & 2 & 7 \end{pmatrix}$$

be a permutation of $\{1, 2, \ldots, 8\}$. Then, applying the foregoing procedure, we obtain the following partition of $D_f$ into directed cycles:

$$1 \to 6 \to 3 \to 5 \to 1, \quad 2 \to 8 \to 7 \to 2, \quad 4 \to 4.$$

Let us write

$$[1\ 6\ 3\ 5]$$

[13] G. Pólya, Kombinatorische Anzahlbestimmungen für Gruppen, Graphen und chemische Verbindungen, *Acta Mathematica*, 68 (1937), 145–254.

[14] J. H. Redfield, The Theory of Group-Reduced Distributions, *American Journal of Mathematics*, 49 (1927), 433–455.

for the permutation of $\{1,2,3,4,5,6,7,8\}$ that sends 1 to 6, 6 to 3, 3 to 5, and 5 to 1, and that fixes the remaining integers.[15] Thus,

$$[1\ 6\ 3\ 5] = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 2 & 5 & 4 & 1 & 3 & 7 & 8 \end{pmatrix}.$$

The digraph corresponding to the permutation $[1\ 6\ 3\ 5]$ is the digraph consisting of the directed cycles

$$1 \to 6 \to 3 \to 5 \to 1, \quad 2 \to 2, \quad 4 \to 4, \quad 7 \to 7, \quad 8 \to 8.$$

We call such a permutation, in which certain of the elements are permuted in a cycle and the remaining elements, if any, are fixed, a *cycle permutation* or, more briefly, a *cycle*. If the number of elements in the cycle is $k$, then we call it a $k$-*cycle*. Thus, $[1\ 6\ 3\ 5]$ is a 4-cycle. The other directed cycles in the partition of $D_f$ give the following cycles:

$$[2\ 8\ 7] \text{ and } [4].$$

We now observe that the partition of $D_f$ into directed cycles corresponds to a factorization (with respect to the composition $\circ$) of $f$ into permutation cycles:

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 8 & 5 & 4 & 1 & 3 & 2 & 7 \end{pmatrix} = [1\ 6\ 3\ 5] \circ [2\ 8\ 7] \circ [4]. \tag{14.12}$$

The reason is that each integer in the permutation $f$ moves in, at most, one of the cycles in the factorization.

We make two observations about this factorization. The first is that it doesn't matter in which order we write the cycles.[16] This is because each element occurs in exactly one cycle. The second is that the 1-cycle $[4]$ is just the identity permutation[17] and thus could be omitted in (14.12) without affecting its validity. But we choose to leave it there since, for our counting problems, it is useful to include all 1-cycles. □

Let $f$ be any permutation of the set $X$. Then, generalizing from the previous example, we see that, with respect to the operation of composition, $f$ has a factorization

$$f = [i_1\ i_2\ \cdots\ i_p] \circ [j_1\ j_2\ \cdots\ j_q] \circ \cdots \circ [l_1\ l_2\ \cdots\ l_r] \tag{14.13}$$

into cycles, where each integer in $X$ occurs in exactly one of the cycles. We call (14.13) the *cycle factorization* of $f$. The cycle factorization of $f$ is unique, apart from the order

[15]The notation is a little ambiguous because we cannot determine from it the set of elements being permuted. All we can conclude is that the set at least contains $1, 3, 5$, and 6. But there should be no confusion, since the the set will be implicit in the particular problem treated.

[16]That is, "disjoint cycles" satisfy the commutative law.

[17]Recall what $[4]$ means here: 4 goes to 4, and every other integer is fixed. This means that every integer, including 4, is fixed, and hence we have the identity permutation. If the permutation $f$ in this example were the identity permutation, then we would write $f = [1] \circ [2] \circ \cdots \circ [8]$.

in which the cycles appear, and this order is arbitrary. In the cycle factorization of a permutation of $X$, every element of $X$ occurs exactly once.

**Example.** Determine the cycle factorization of each permutation in the dihedral group $D_4$ of order 8 (the corner-symmetry group of a square).

The permutations in $D_4$ were computed in Section 13.1. The cycle factorization of each is given in the next table:

| $D_4$ | Cycle Factorization |
|---|---|
| $\rho_4^0 = \iota$ | $[1] \circ [2] \circ [3] \circ [4]$ |
| $\rho_4$ | $[1\ 2\ 3\ 4]$ |
| $\rho_4^2$ | $[1\ 3] \circ [2\ 4]$ |
| $\rho_4^3$ | $[1\ 4\ 3\ 2]$ |
| $\tau_1$ | $[1] \circ [2\ 4] \circ [3]$ |
| $\tau_2$ | $[1\ 3] \circ [2] \circ [4]$ |
| $\tau_3$ | $[1\ 2] \circ [3\ 4]$ |
| $\tau_4$ | $[1\ 4] \circ [2\ 3]$ |

.

Notice that, in the cycle factorization of the identity permutation $\iota$, all cycles are 1-cycles. This is in agreement with the fact that the identity permutation fixes all elements. In the cycle factorizations of the reflections $\tau_1$ and $\tau_2$, two 1-cycles occur, since each of these reflections is about a line joining two opposite corners of the square, and these corners are thus fixed. For $\tau_3$ and $\tau_4$ we get two 2-cycles, since these are reflections about the line joining the midpoints of opposite sides. The reflections in the corner-symmetry group of a regular $n$-gon with $n$ even behave similarly. Half of them have two 1-cycles and $((n/2) - 1)$ 2-cycles, and half have $(n/2)$ 2-cycles. □

**Example.** Determine the cycle factorization of each permutation in the dihedral group $D_5$ of order 10 (the corner-symmetry group of a regular 5-gon).

The permutations in $D_5$ were computed in Section 13.1. The cycle factorization of each is given in the following table:

| $D_5$ | Cycle Factorization |
|---|---|
| $\rho_5^0 = \iota$ | $[1] \circ [2] \circ [3] \circ [4] \circ [5]$ |
| $\rho_5$ | $[1\ 2\ 3\ 4\ 5]$ |
| $\rho_5^2$ | $[1\ 3\ 5\ 2\ 4]$ |
| $\rho_5^3$ | $[1\ 4\ 2\ 5\ 3]$ |
| $\rho_5^4$ | $[1\ 5\ 4\ 3\ 2]$ |
| $\tau_1$ | $[1] \circ [2\ 5] \circ [3\ 4]$ |
| $\tau_2$ | $[1\ 3] \circ [2] \circ [4\ 5]$ |
| $\tau_3$ | $[1\ 5] \circ [3] \circ [2\ 4]$ |
| $\tau_4$ | $[1\ 2] \circ [3\ 5] \circ [4]$ |
| $\tau_5$ | $[1\ 4] \circ [2\ 3] \circ [5]$ |

Notice that, in the cycle factorizations of the reflections $\tau_i$, exactly one 1-cycle occurs since each such reflection is about a line joining a corner to the midpoint of the opposite side, and hence only the one corner is fixed. The reflections in the corner-symmetry group of a regular $n$-gon with $n$ odd behave similarly. Each has one 1-cycle and $(n-1)/2$ 2-cycles. □

The importance of the cycle decomposition in counting nonequivalent colorings is illustrated by the next example.

**Example.** Let $f$ be the permutation of $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ defined by

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 9 & 1 & 7 & 6 & 5 & 3 & 8 & 2 \end{pmatrix}.$$

The cycle factorization of $f$ is

$$f = [1\ 4\ 7\ 3] \circ [2\ 9] \circ [5\ 6] \circ [8].$$

Suppose that we color the elements of $X$ with the colors red, white, and blue, and let $\mathcal{C}$ be the set of all such colorings. What is the number $|\mathcal{C}(f)|$ of colorings in $\mathcal{C}$ that are left fixed by $f$?

Let $\mathbf{c}$ be a coloring such that $f * \mathbf{c} = \mathbf{c}$. First, consider the 4-cycle [1 4 7 3]. This 4-cycle moves the color of 1 to 4, the color of 4 to 7, the color of 7 to 3, and the color of 3 to 1. Since the coloring $\mathbf{c}$ is fixed by $f$, following through on this cycle, we see that

$$\text{color of } 1 = \text{ color of } 4 = \text{ color of } 7 =$$

$$\text{color of } 3 = \text{ color of } 1.$$

This means that 1, 4, 7, and 3 have the same color. In a similar way, we see that the elements 2 and 9 of the 2-cycle [2 9] have the same color, and the elements 5 and 6 of the 2-cycle [5 6] have the same color. There is no restriction placed on 8, since it belongs to a 1-cycle. So how many colorings $\mathbf{c}$ are there which are fixed by $f$—that is, which satisfy $f * \mathbf{c} = \mathbf{c}$? The answer is clear: We pick any one of the three colors red, white, and blue for $\{1,4,7,3\}$ (three choices), any of the three colors for $\{2,9\}$ (three choices), any of the three colors for $\{5,6\}$ (three choices), and any of the three colors for $\{8\}$ (three choices), for a total of

$$3^4 = 81$$

colorings. Note that the exponent 4 in the answer is the *number* of cycles of $f$ in its cycle factorization, and the answer is independent of the sizes of the cycles. □

The analysis in the preceding example is quite general. It can be used to find the number of colorings fixed by any permutation no matter what the number of colors available is. We record the result in the next theorem. We denote by

$$\#(f)$$

the *number of cycles in the cycle factorization* of a permutation $f$.

**Theorem 14.3.1** *Let $f$ be a permutation of a set $X$. Suppose we have $k$ colors available with which to color the elements of $X$. Let $\mathcal{C}$ be the set of all colorings of $X$. Then the number of colorings that are fixed by $f$ satisfies*

$$|\mathcal{C}(f)| = k^{\#(f)}.$$

□

**Example.** How many nonequivalent ways are there to color the corners of a square with the colors red, white, and blue?

Let $\mathcal{C}$ be the set of all $3^4 = 81$ colorings of the corners of a square with the colors red, white, and blue. The corner-symmetry group of a square is the dihedral group $D_4$, the cycle factorization of whose elements was already computed. We repeat the results in the following table, with additional columns indicating $\#(f)$ and the number $|\mathcal{C}(f)|$ of colorings left fixed by $f$ for each of the permutations $f$ in $D_4$.

| $f$ in $D_4$ | Cycle Factorization | $\#(f)$ | $\lvert\mathcal{C}(f)\rvert$ |
|---|---|---|---|
| $\rho_4^0 = \iota$ | $[1] \circ [2] \circ [3] \circ [4]$ | 4 | $3^4 = 81$ |
| $\rho_4$ | $[1\ 2\ 3\ 4]$ | 1 | $3^1 = 3$ |
| $\rho_4^2$ | $[1\ 3] \circ [2\ 4]$ | 2 | $3^2 = 9$ |
| $\rho_4^3$ | $[1\ 4\ 3\ 2]$ | 1 | $3^1 = 3$ . |
| $\tau_1$ | $[1] \circ [2\ 4] \circ [3]$ | 3 | $3^3 = 27$ |
| $\tau_2$ | $[1\ 3] \circ [2] \circ [4]$ | 3 | $3^3 = 27$ |
| $\tau_3$ | $[1\ 2] \circ [3\ 4]$ | 2 | $3^2 = 9$ |
| $\tau_4$ | $[1\ 4] \circ [2\ 3]$ | 2 | $3^2 = 9$ |

Hence, by Theorem 14.2.3, the number of nonequivalent colorings is

$$N(D_4, \mathcal{C}) = \frac{81 + 3 + 9 + 3 + 27 + 27 + 9 + 9}{8} = 21.$$

□

Theorems 14.2.3 and 14.3.1 give us a method to compute, in the presence of a group $G$ of permutations of a set $X$, the number of nonequivalent colorings in the set $\mathcal{C}$ of all colorings of $X$ with a given set of colors. This method requires that we be able to compute the cycle factorization (or at least the number of cycles in the cycle factorization) of each permutation in $G$. To compute the number of nonequivalent colorings for more general sets $\mathcal{C}$ of colorings, we introduce a generating function for the number of permutations in $G$ whose cycle factorizations have the same number of cycles of each size.

Let $f$ be a permutation of $X$ where $X$ has $n$ elements. Suppose that the cycle factorization of $f$ has $e_1$ 1-cycles, $e_2$ 2-cycles, ... , and $e_n$ $n$-cycles. Since each element of $X$ occurs in exactly one cycle in the cycle factorization of $f$, the numbers $e_1, e_2, \cdots, e_n$ are nonnegative integers satisfying

$$1e_1 + 2e_2 + \cdots + ne_n = n. \tag{14.14}$$

We call the $n$-tuple $(e_1, e_2, \ldots, e_n)$ the *type* of the permutation $f$ and write

$$\text{type}(f) = (e_1, e_2, \ldots, e_n).$$

Note that the number of cycles in the cycle factorization of $f$ is

$$\#(f) = e_1 + e_2 + \cdots + e_n.$$

Different permutations may have the same type, since the type of a permutation depends only on the size of the cycles in its cycle factorization and not on which elements are in which cycles. Since we now want to distinguish permutations only by type, we introduce $n$ indeterminates

$$z_1, z_2, \ldots, z_n,$$

where $z_k$ corresponds to a $k$-cycle ($k = 1, 2, \ldots, n$). To each permutation $f$ with $\text{type}(f) = (e_1, e_2, \ldots, e_n)$, we associate the *monomial of $f$*:

$$\text{mon}(f) = z_1^{e_1} z_2^{e_2} \cdots z_n^{e_n}.$$

Notice that the total degree of the monomial of $f$ is the number $\#(f)$ of cycles in the cycle factorization of $f$.

Let $G$ be a group of permutations of $X$. Summing these monomials for each $f$ in $G$, we get the generating function

$$\sum_{f \in G} \text{mon}(f) = \sum_{f \in G} z_1^{e_1} z_2^{e_2} \cdots z_n^{e_n} \tag{14.15}$$

for the permutations in $G$ according to type. If we combine like terms in (14.15), the coefficient of $z_1^{e_1} z_2^{e_2} \cdots z_n^{e_n}$ equals the number of permutations in $G$ of type $(e_1, e_2, \ldots, e_n)$. The *cycle index*

$$P_G(z_1, z_2, \ldots, z_n) = \frac{1}{|G|} \sum_{f \in G} z_1^{e_1} z_2^{e_2} \cdots z_n^{e_n}$$

of $G$ is this generating function divided by the number $|G|$ of permutations in $G$.

**Example.** Determine the cycle index of the dihedral group $D_4$.

In the example just after Theorem 14.3.1, we gave a table that included the cycle factorization of each permutation in $D_4$. Using those factorizations, we give the type of each permutation and its associated monomial in the following table:

| $D_4$ | Cycle Factorization | Type | Monomial |
|---|---|---|---|
| $\rho_4^0 = \iota$ | $[1] \circ [2] \circ [3] \circ [4]$ | $(4,0,0,0)$ | $z_1^4 z_2^0 z_3^0 z_4^0 = z_1^4$ |
| $\rho_4$ | $[1\ 2\ 3\ 4]$ | $(0,0,0,1)$ | $z_1^0 z_2^0 z_3^0 z_4^1 = z_4$ |
| $\rho_4^2$ | $[1\ 3] \circ [2\ 4]$ | $(0,2,0,0)$ | $z_1^0 z_2^2 z_3^0 z_4^0 = z_2^2$ |
| $\rho_4^3$ | $[1\ 4\ 3\ 2]$ | $(0,0,0,1)$ | $z_1^0 z_2^0 z_3^0 z_4^1 = z_4$ |
| $\tau_1$ | $[1] \circ [2\ 4] \circ [3]$ | $(2,1,0,0)$ | $z_1^2 z_2^1 z_3^0 z_4^0 = z_1^2 z_2$ |
| $\tau_2$ | $[1\ 3] \circ [2] \circ [4]$ | $(2,1,0,0)$ | $z_1^2 z_2^1 z_3^0 z_4^0 = z_1^2 z_2$ |
| $\tau_3$ | $[1\ 2] \circ [3\ 4]$ | $(0,2,0,0)$ | $z_1^0 z_2^2 z_3^0 z_4^0 = z_2^2$ |
| $\tau_4$ | $[1\ 4] \circ [2\ 3]$ | $(0,2,0,0)$ | $z_1^0 z_2^2 z_3^0 z_4^0 = z_2^2$ |

The cycle index of $D_4$ is

$$P_{D_4}(z_1, z_2, z_3, z_4) = \frac{1}{8}(z_1^4 + 2z_4 + 3z_2^2 + 2z_1^2 z_2).$$

□

We can now determine the number of nonequivalent colorings among all the colorings of a set $X$, using a specified set of colors, provided that we know the cycle index of the group $G$ of permutations of $X$.

**Theorem 14.3.2** *Let $X$ be a set of $n$ elements, and suppose we have a set of $k$ colors available with which to color the elements of $X$. Let $\mathcal{C}$ be the set of all $k^n$ colorings of $X$. Let $G$ be a group of permutations of $X$. Then the number of nonequivalent colorings is the number*

$$N(G, \mathcal{C}) = P_G(k, k, \ldots, k)$$

*obtained by substituting $z_i = k$, $(i = 1, 2, \ldots, n)$ into the cycle index of $G$.*

**Proof.** This theorem is a consequence of Theorems 14.2.3 and 14.3.1. The cycle index of $G$ is the average

$$P_G(z_1, z_2, \ldots, z_n) = \frac{1}{|G|} \sum_{f \in G} z_1^{e_1} z_2^{e_2} \cdots z_n^{e_n}$$

of the sum of the monomials associated with the permutations $f$ in $G$. By Theorem 14.3.1, the number of colorings in $\mathcal{C}$ that are fixed by $f$ equals

$$k^{\#(f)} = k^{e_1+e_2+\cdots+e_n} = k^{e_1}k^{e_2}\cdots k^{e_n},$$

where $(e_1, e_2, \ldots, e_n)$ is the type of $f$. By Theorem 14.2.3, the number of nonequivalent colorings is

$$N(G, \mathcal{C}) = \frac{1}{|G|}\sum_{f\in G} k^{e_1}k^{e_2}\cdots k^{e_n} = P_G(k, k, \ldots, k).$$

□

**Example.** We are given a set of $k$ colors. What is the number of nonequivalent ways to color the corners of a square?

The cycle index of the dihedral group $D_4$ has already been determined to be

$$P_{D_4}(z_1, z_2, z_3, z_4) = \frac{1}{8}(z_1^4 + 2z_4 + 3z_2^2 + 2z_1^2 z_2).$$

Hence, by Theorem 14.3.2, the number of nonequivalent colorings is

$$P_{D_4}(k, k, k, k) = \frac{k^4 + 2k + 3k^2 + 2k^2k}{8} = \frac{k^4 + 2k^3 + 3k^2 + 2k}{8}.$$

If the number of colors is $k = 6$, then the number of nonequivalent colorings is

$$P_{D_4}(6, 6, 6, 6) = \frac{6^4 + 26^3 + 36^2 + 2 \times 6}{8} = 231.$$

□

Theorem 14.3.2 gives a satisfactory way to count the number of nonequivalent colorings in $\mathcal{C}$, provided that $\mathcal{C}$ is the set of *all* colorings possible with $k$ given colors. The formula in the theorem requires that we know the number of permutations of each type in the group $G$ of permutations, and so can be difficult to apply. But it is as simple as we could expect, given that $G$ can be any permutation group on the set $X$ of objects being colored. Our final concern is with more general sets $\mathcal{C}$ of colorings. Recall that, in Theorem 14.2.3, the only restriction on $\mathcal{C}$ is that for every coloring $\mathbf{c}$ in $\mathcal{C}$ and every permutation $f$ in $G$, $f * \mathbf{c}$ is also in $\mathcal{C}$, that is, each permutation $f$ in $G$ takes a coloring $\mathbf{c}$ of $\mathcal{C}$ to another coloring $f * \mathbf{c}$ of $\mathcal{C}$. Under these more general circumstances, the most we might expect is to have some formal way to determine the nonequivalent colorings.

We now show how the cycle index of $G$ can be used to determine the number of nonequivalent colorings where the number of times each color is used is specified.

Let $\mathcal{C}$ be the set of all colorings of $X$ in which the number of elements in $X$ of each color has been specified. For each permutation $f$ of $X$ and each coloring $\mathbf{c}$ in $\mathcal{C}$, the number of times a particular color appears in $\mathbf{c}$ is the same as the number of times that color appears in $f * \mathbf{c}$. Put another way, permuting the objects in $X$ along with their colors does not change the number of colors of each kind. This means that any group $G$ of permutations of $X$ acts as a permutation group on such a set $\mathcal{C}$ of colorings.

**Example.** How many nonequivalent colorings are there of the corners of a regular 5-gon in which three corners are colored red and two are colored blue?

Let $\mathcal{C}$ be the set of all colorings of the corners of a 5-gon with three corners colored red and two colored blue. The number of colorings in $\mathcal{C}$ is 10, since we can select three corners to be colored red in 10 ways and then color the other two corners blue. The corner-symmetry group $D_5$ acts as a permutation group on $\mathcal{C}$. We have previously computed the cycle factorization of each permutation in $G$. In the following table, we again list those factorizations, along with the number of colorings in $\mathcal{C}$ fixed by the permutations in $D_5$.

| $D_5$ | Cycle Factorization | Number of Fixed Colorings |
|---|---|---|
| $\rho_5^0 = \iota$ | $[1] \circ [2] \circ [3] \circ [4] \circ [5]$ | 10 |
| $\rho_5$ | $[1\ 2\ 3\ 4\ 5]$ | 0 |
| $\rho_5^2$ | $[1\ 3\ 5\ 2\ 4]$ | 0 |
| $\rho_5^3$ | $[1\ 4\ 2\ 5\ 3]$ | 0 |
| $\rho_5^4$ | $[1\ 5\ 4\ 3\ 2]$ | 0 |
| $\tau_1$ | $[1] \circ [2\ 5] \circ [3\ 4]$ | 2 |
| $\tau_2$ | $[1\ 3] \circ [2] \circ [4\ 5]$ | 2 |
| $\tau_3$ | $[1\ 5] \circ [3] \circ [2\ 4]$ | 2 |
| $\tau_4$ | $[2] \circ [3\ 5] \circ [4]$ | 2 |
| $\tau_5$ | $[1\ 4] \circ [2\ 3] \circ [5]$ | 2 |

The reason that none of the rotations different from the identity fixes any coloring is that, for such a rotation to fix a coloring, all colors in the coloring must be the same (and so we do not have three red and two blue colors as specified). Each reflection fixes two colorings in $\mathcal{C}$. This is because, for the 5-gon, each of the reflections has type $(1, 2, 0, 0, 0)$. To have two blue corners in a fixed coloring, we must color blue the corners in one of the two 2-cycles in the factorization. Applying Theorem 14.2.3, we find that the number of nonequivalent colorings of the type being counted is

$$\frac{10+0+0+0+0+2+2+2+2+2}{10} = 2.$$

This answer can easily be arrived at directly, the two nonequivalent colorings are the one with two blue corners that are consecutive and the other with two blue corners that are not consecutive. □

To apply Burnside's theorem to determine the number of nonequivalent colorings when the number of occurrences of each color is specified, we must be able to determine the number of such colorings fixed by a permutation. Let $f$ be a permutation of the set $X$, and suppose that

$$\text{type}(f) = (e_1, e_2, \ldots, e_n)$$

and

$$\text{mon}(f) = z_1^{e_1} z_2^{e_2} \cdots z_n^{e_n}.$$

Thus, $f$ has $e_1$ 1-cycles, $e_2$ 2-cycles, ... , and $e_n$ $n$-cycles in its cycle factorization. To keep our discussion simple initially, suppose we have only two colors: red and blue. Let

$$\mathcal{C}_{p,q}$$

denote the set of all colorings of $X$ with $p$ elements colored red and $q = n - p$ elements colored blue. A coloring in $\mathcal{C}_{p,q}$ is fixed by $f$ if and only if, for each cycle in the cycle factorization of $f$, all of the elements have the same color. Thus, to determine the number of colorings in $\mathcal{C}_{p,q}$ fixed by $f$, we can think of assigning colors to cycles in such a way that the number of *elements* that get assigned the color red is $p$ (and hence the number assigned the color blue is $n - p = q$). Suppose that $t_1$ of the 1-cycles get assigned red, $t_2$ of the 2-cycles get red, ... , and $t_n$ of the $n$-cycles get red. For the number of elements assigned red to be $p$ we must have

$$p = t_1 1 + t_2 2 + \cdots + t_n n. \tag{14.16}$$

Hence, the number $|\mathcal{C}_{p,q}(f)|$ of colorings in $\mathcal{C}_{p,q}$ that are fixed by $f$ is obtained as follows: Choose a solution of (14.16) in integers $t_1, t_2, \ldots, t_n$ satisfying

$$0 \le t_1 \le e_1,\ 0 \le t_2 \le e_2,\ \cdots,0 \le t_n \le e_n \tag{14.17}$$

(to determine *how many* cycles of each length are assigned the color red), and then multiply such a solution by

$$\binom{e_1}{t_1}\binom{e_2}{t_2}\cdots\binom{e_n}{t_n}$$

(to determine *which* cycles of each of the lengths $1, 2, \ldots, n$ are assigned the color red). Now, consider the color red as a variable $r$ and the color blue as a variable $b$ that we can manipulate algebraically in the usual way. Then the number of solutions of (14.16) satisfying (14.17) is the coefficient of $r^p b^q$ in the expression

$$(r+b)^{e_1}(r^2+b^2)^{e_2}\cdots(r^n+b^n)^{e_n},$$

obtained by making the substitutions

$$z_1 = r+b,\ z_2 = r^2+b^2,\ \cdots, z_n = r^n+b^n \tag{14.18}$$

in the monomial of $f$. The cycle index of a permutation group $G$ is the average of the monomials of the permutations $f$ in $G$. Hence, by Theorem 14.2.3, the number of nonequivalent colorings in $\mathcal{C}(p,q)$ equals the coefficient of $r^p b^q$ in the expression

$$P_G(r+b, r^2+b^2, \cdots, r^n+b^n), \tag{14.19}$$

obtained by making the substitutions (14.18) in the cycle index of $G$. This means that (14.19) is a two-variable generating function for the number of nonequivalent colorings in $\mathcal{C}(p,q)$ with a specified number of elements colored with each color.[18]

The preceding discussion applies for any number of colors, and it enables us to give a generating function for the number of nonequivalent colorings in which the number of colors of each kind is specified. This provides us with the *final theorem* in this book.[19] This theorem is commonly called *Pólya's theorem*, and its motivation, derivation, and application have been the primary purpose of this chapter.

As with the case of two colors, we need to think of the colors as variables $u_1, u_2, \ldots, u$ to be manipulated algebraically. The only change in the preceding argument is the change from two colors to any number $k$ of colors.

**Theorem 14.3.3** *Let $X$ be a set of elements and let $G$ be a group of permutations of $X$. Let $\{u_1, u_2, \ldots, u_k\}$ be a set of $k$ colors, and let $\mathcal{C}$ be a set of all colorings of $X$. Then the generating function for the number of nonequivalent colorings of $\mathcal{C}$ according to the number of colors of each kind is the expression*

$$P_G(u_1+\cdots+u_k, u_1^2+\cdots+u_k^2, \ldots, u_1^n+\cdots+u_k^n), \tag{14.20}$$

[18]The two variables in the generating function are $r$ and $b$. We could get a one-variable generating function by setting $b = 1$. Nothing is lost by doing so, since as we have already remarked, once the number of reds is specified, the number of blues is whatever is left. However, since we are about to write down the generating function for any number of colors where we cannot reduce the generating function to one variable, it is better here to use two variables.

[19]If you started on page 1 and worked your way here doing many of the exercises, then *congratulations*! You know a lot about combinatorics and graph theory. But there is a lot more to know, and the amount of information increases every day. Research articles on the wide variety of topics within combinatorics and graph theory continue to be published in journals at a substantial rate. But that is not too surprising since, as I hope that you have discovered, the subject is exciting, fascinating, and indeed fun. In addition, its applicability in the biological and physical world is increasing. Following the exercises for this chapter, we include a list of books for further study.

*obtained from the cycle index* $P_G(z_1, z_2, \ldots, z_n)$ *by making the substitutions*

$$z_j = u_1^j + \cdots + u_k^j \quad (j = 1, 2, \ldots, n).$$

*In other words, the coefficient of*

$$u_1^{p_1} u_2^{p_2} \cdots u_k^{p_k}$$

*in* (14.20) *equals the number of nonequivalent colorings in* $C$ *with* $p_1$ *elements of* $X$ *colored* $u_1$, $p_2$ *elements colored* $u_2$, $\ldots$ , $p_k$ *elements colored* $u_k$.

Substituting $u_i = 1$ for $i = 1, 2, \ldots, k$ in (14.20), we get the sum of its coefficients and hence the total number of nonequivalent colorings of $X$ with $k$ available colors. Since this substitution yields

$$P_G(k, k, \ldots, k),$$

it follows that Theorem 14.3.3 is a refinement of Theorem 14.3.2. Theorem 14.3.3 contains more detailed information than Theorem 14.3.2, which is subsequently lost upon replacing each $u_i$ with 1.

**Example.** Determine the generating function for the number of nonequivalent colorings of the corners of a square with two colors and also those with three colors.

The cycle index of $D_4$, the corner-symmetry group of the square, has been previously computed to be

$$P_{D_4}(z_1, z_2, z_3, z_4) = \frac{1}{8}(z_1^4 + 2z_4 + 3z_2^2 + 2z_1^2 z_2).$$

Let the two colors be $r$ and $b$. Then the generating function is

$$P_{D_4}(r + b, r^2 + b^2, r^3 + b^3, r^4 + b^4) =$$

$$\frac{1}{8}((r + b)^4 + 2(r^4 + b^4) + 3(r^2 + b^2)^2 + 2(r + b)^2(r^2 + b^2)$$

$$= \frac{1}{8}(8r^4 + 8r^3 b + 16r^2 b^2 + 8rb^3 + 8b^4).$$

Hence, we have

$$P_{D_4}(r + b, r^2 + b^2, r^3 + b^3, r^4 + b^4) = r^4 + r^3 b + 2r^2 b^2 + rb^3 + b^4. \qquad (14.21)$$

Thus, there is one nonequivalent coloring with all corners red and one with all corners blue. There is also one with three corners red and one blue, and one with one corner red and three blue. Finally, there are two with two corners of each color. The total number of nonequivalent colorings, the sum of the coefficients in (14.21), is 6.

Now suppose that we have three colors $r$, $b$, and $g$. The generating function for the number of nonequivalent colorings is

$$P_{D_4}(r+b+g, r^2+b^2+g^2, r^3+b^3+g^3, r^4+b^4+g^4)$$

$$= \frac{1}{8}\left((r+b+g)^4 + 2(r^4+b^4+g^4) + 3(r^2+b^2+g^2)^2 + 2(r+b+g)^2(r^2+b^2+g^2)\right).$$

This expression can be calculated using the multinomial theorem in Chapter 5. For instance, the coefficient of $r^1b^2g^1$ equals

$$\frac{1}{8}(12+0+0+4) = 2.$$

Thus, there are 2 nonequivalent colorings that have one red, two blue, and one green corner(s). The total number of nonequivalent colorings equals

$$P_{D_4}(3,3,3) = 21.$$

□

**Example.** Determine the generating function for the number of nonequivalent colorings of the corners of a regular 5-gon with two colors and also those with three colors.

From our previous calculations, the cycle index of $D_5$ is

$$P_{D_5}(z_1, z_2, z_3, z_4, z_5) = \frac{1}{10}(z_1^5 + 4z_5 + 5z_1z_2^2).$$

Notice that neither $z_3$ nor $z_4$ appear in any nonzero term in the cycle index. This is because no permutation in $D_5$ has either a 3-cycle or 4-cycle in its cycle factorization. Suppose that we have two colors $r$ and $b$. Then the generating function for the number of nonequivalent colorings is

$$\begin{aligned} &P_{D_5}(r+b, r^2+b^2, r^3+b^3, r^4+b^4, r^5+b^5) \\ &= \frac{1}{10}((r+b)^5 + 4(r^5+b^5) + 5(r+b)(r^2+b^2)^2) \\ &= r^5 + r^4b + 2r^3b^2 + 2r^2b^3 + rb^4 + b^5. \end{aligned}$$

The total number of nonequivalent colorings equals

$$1+1+2+2+1+1 = 8.$$

The generating function for the number of nonequivalent colorings for three colors is

$$\frac{1}{10}((r+b+g)^5 + 4(r^5+b^5+g^5) + 5(r+b+g)(r^2+b^2+g^2)^2).$$

The total number of nonequivalent colorings equals

$$\frac{1}{10}(3^5 + 4(3) + 5(3)(3^2)) = 39.$$

□

**Example.** *Coloring the corners and faces of a cube.* Determine the symmetry group of a cube and the number of nonequivalent ways to color the corners and faces of a cube with a specified number of colors.

There are 24 symmetries of a cube, and they are rotations of four different kinds:

(1) The identity rotation $\iota$ (number is 1).

(2) The rotations about the centers of the three pairs of opposite faces by

(a) 90 degrees (number is 3).
(b) 180 degrees (number is 3).
(c) 270 degrees (number is 3).

(3) The rotations by 180 degrees about midpoints of opposite edges (number is 6).

(4) The rotations about opposite corners by

(a) 120 degrees (number is 4).
(b) 240 degrees (number is 4).

The total number of symmetries of a cube is 24.

In the next table, we give the type of each symmetry as both a permutation of its eight corners (as a member of the corner-symmetry group of the cube) and as a permutation of its six faces (as a member of the face-symmetry group of the cube). In this table, we refer to the classification of the symmetries previously given.

| Kind of Symmetry | Number of | Corner Type | Face Type |
|---|---|---|---|
| (i) | 1 | $(8,0,0,0,0,0,0,0)$ | $(6,0,0,0,0,0)$ |
| (ii)(a) | 3 | $(0,0,0,2,0,0,0,0)$ | $(2,0,0,1,0,0)$ |
| (ii)(b) | 3 | $(0,4,0,0,0,0,0,0)$ | $(2,2,0,0,0,0)$ |
| (ii)(c) | 3 | $(0,0,0,2,0,0,0,0)$ | $(2,0,0,1,0,0)$ |
| (iii) | 6 | $(0,4,0,0,0,0,0,0)$ | $(0,3,0,0,0,0)$ |
| (iv)(a) | 4 | $(2,0,2,0,0,0,0,0)$ | $(0,0,2,0,0,0)$ |
| (iv)(b) | 4 | $(2,0,2,0,0,0,0,0)$ | $(0,0,2,0,0,0)$ |

From the table, we see that the cycle index of the corner-symmetry group $G_C$ of the cube is

$$P_{G_C}(z_1, z_2, \ldots, z_8) = \frac{1}{24}(z_1^8 + 6z_4^2 + 9z_2^4 + 8z_1^2 z_3^2),$$

and that of the face-symmetry group $G_F$ is

$$P_{G_F}(z_1, z_2, \ldots, z_6) = \frac{1}{24}(z_1^6 + 6z_1^2 z_4 + 3z_1^2 z_2^2 + 6z_2^3 + 8z_3^2).$$

The generating function for the number of nonequivalent colorings of the corners of a cube with the colors red and blue is

$$P_{G_C}(r + b, r^2 + b^2, \ldots, r^8 + b^8)$$
$$= \frac{1}{24}((r + b)^8 + 6(r^4 + b^4)^2 + 9(r^2 + b^2)^4 + 8(r + b)^2(r^3 + b^3)^2).$$

For the faces of the cube, the generating function is

$$P_{G_F}(r + b, r^2 + b^2, \ldots, r^6 + b^6) =$$
$$\frac{1}{24}((r + b)^6 + 6(r + b)^2(r^4 + b^4) + 3(r + b)^2(r^2 + b^2)^2 + 6((r^2 + b^2)^3 + 8(r^3 + b^3)^2).$$

Some algebraic calculation now shows that the generating function for the number of nonequivalent colorings of the corners is

$$r^8 + r^7 b + 3r^6 b^2 + 3r^5 b^3 + 7r^4 b^4 + 3r^3 b^5 + 3r^2 b^6 + rb^7 + b^8$$

and, for the faces, is

$$r^6 + r^5 b + 2r^4 b^2 + 2r^3 b^3 + 2r^2 b^4 + rb^5 + b^6.$$

The total number of nonequivalent colorings for the corners is 23, and for the faces the total number is 10.

If we have $k$ colors, the number of nonequivalent corner colorings is

$$\frac{1}{24}(k^8 + 6k^2 + 9k^4 + 8k^2 k^2) = \frac{1}{24}(k^8 + 17k^4 + 6k^2),$$

and the number of nonequivalent face colorings is

$$\frac{1}{24}(k^6 + 6k^2 k + 3k^2 k^2 + 6k^3 + 8k^2) = \frac{1}{24}(k^6 + 3k^4 + 12k^3 + 8k^2).$$

□

In our final example we illustrate how Theorem 14.3.3 can be applied to determine the number of nonisomorphic graphs of order $n$ with a specified number of edges.

**Example.** Determine the number of nonisomorphic graphs of order 4 with each possible number of edges.

The number 4 is small enough for us to solve this problem without recourse to Theorem 14.3.3. But our purpose in this example is to illustrate how to apply Theorem 14.3.3 to count graphs.

Let $\mathcal{G}_4$ be the set of all graphs of order 4 with vertex set $V = \{1,2,3,4\}$. We seek the generating function for the number of nonisomorphic graphs in $\mathcal{G}_4$ with a specified number of edges. The set $E$ of edges of a graph $H_1 = (V, E_1)$ in $\mathcal{G}_4$ is a subset of the set

$$X = \{\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\}\}.$$

We can think of $H_1$ as a coloring of the edges in the set $X$, with two colors "yes" (or y) and "no" (or n), where the edges in $E_1$ get the color yes and the edges not in $E_1$ get the color no. Let $\mathcal{C}$ be the set of all colorings of $X$ with the two colors $y$ and $n$. Thus, the graphs in $\mathcal{G}_4$ are exactly the colorings in $\mathcal{C}$. This is the first important observation for obtaining our solution.

Let $H_2 = (V, E_2)$ be another graph in $\mathcal{G}_4$. Then $H_1$ and $H_2$ are isomorphic if and only if there is a permutation $f$ of $V = \{1,2,3,4\}$ (so, a permutation in $S_4$), such that $\{i,j\}$ is an edge in $E_1$ if and only if $\{f(i), f(j)\}$ is an edge in $E_2$. Each of the 24 permutations $f$ in $S_4$ also permutes the edges in $X$, using the rule

$$\{i,j\} \to \{f(i), f(j)\} \quad (\{i,j\} \text{ in } X).$$

For example, let

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix}.$$

Then $f$ permutes the edges as follows:

$$\begin{pmatrix} \{1,2\} & \{1,3\} & \{1,4\} & \{2,3\} & \{2,4\} & \{3,4\} \\ \{2,3\} & \{3,4\} & \{1,3\} & \{2,4\} & \{1,2\} & \{1,4\} \end{pmatrix}.$$

Let $S_4^{(2)}$ be the group of permutations of $X$ obtained in this way from $S_4$.[20] Our second important observation is that two graphs in $\mathcal{G}_4$ are isomorphic if and only if, as colorings of $X$, they are equivalent. This observation is an immediate consequence of the definitions of isomorphic graphs and equivalent colorings.

We have thus reduced our problem to counting the number of colorings in $\mathcal{C}$ that are nonequivalent with respect to the permutation group $S_4^{(2)}$, according to the number of y's and n's. This is exactly the setup of Theorem 14.3.3. It only remains to compute the cycle index of $S_4^{(2)}$. To do this we must compute the type of each of the 24 permutations in $S_4^{(2)}$. The results are summarized in the following table.

[20]Since $S_4$ is a group of permutations, it follows readily that $S_4^{(2)}$ is also a group of permutations. $S_4$ and $S_4^{(2)}$ are isomorphic as abstract groups but not as permutation groups.

| Type | Monomial | Number of Permutations in $S_4^{(2)}$ |
|---|---|---|
| $(6,0,0,0,0,0)$ | $z_1^6$ | 1 |
| $(2,2,0,0,0,0)$ | $z_1^2 z_2^2$ | 9 |
| $(0,0,2,0,0,0)$ | $z_3^2$ | 8 |
| $(0,1,0,1,0,0)$ | $z_2 z_4$ | 6 |

The cycle index of $S_4^{(2)}$ is

$$P_{S_4^{(2)}}(z_1, z_2, z_3, z_4, z_5, z_6) = \frac{1}{24}(z_1^6 + 9z_1^2 z_2^2 + 8z_3^2 + 6z_2 z_4). \tag{14.22}$$

By Theorem 14.3.3, the generating function for the number of nonequivalent colorings in $C$ is obtained by making the substitutions

$$z_j = y^j + n^j \quad (j = 1, 2, 3, 4, 5, 6)$$

in (14.22). A little calculation shows that the result is

$$y^6 + y^5 n + 2y^4 n^2 + 3y^3 n^3 + 2y^2 n^4 + yn^5 + n^6.$$

Remembering that the number of $y$'s equals the number of edges, we see that the number of nonisomorphic graphs of order 4, according to the number of edges, is given as follows:

| Number of Edges | Number of Nonisomorphic Graphs |
|---|---|
| 6 | 1 |
| 5 | 1 |
| 4 | 2 |
| 3 | 3 |
| 2 | 2 |
| 1 | 1 |
| 0 | 1 |

In particular, the total number of nonisomorphic graphs of order 4 equals 11.

## 14.4 Exercises

1. Let

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 4 & 2 & 1 & 5 & 3 \end{pmatrix} \text{ and } g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 6 & 2 & 4 & 1 \end{pmatrix}.$$

Determine

(a) $f \circ g$ and $g \circ f$

(b) $f^{-1}$ and $g^{-1}$

(c) $f^2$, $f^5$

(d) $f \circ g \circ f$

(e) $g^3$ and $f \circ g^3 \circ f^{-1}$

2. Prove that permutation composition is associative: $(f \circ g) \circ h = f \circ (g \circ h)$.

3. Determine the symmetry group and corner-symmetry group of an equilateral triangle.

4. Determine the symmetry group and corner-symmetry group of a triangle that is isoceles but not equilateral.

5. Determine the symmetry group and corner-symmetry group of a triangle that is neither equilateral nor isoceles.

6. Determine the symmetry group of a regular tetrahedron. (*Hint*: There are 12 symmetries.)

7. Determine the corner-symmetry group of a regular tetrahedron.

8. Determine the edge-symmetry group of a regular tetrahedron.

9. Determine the face-symmetry group of a regular tetrehedron.

10. Determine the symmetry group and the corner-symmetry group of a rectangle that is not a square.

11. Compute the corner-symmetry group of a regular hexagon (the dihedral group $D_6$ of order 12).

12. Determine all the permutations in the edge-symmetry group of a square.

13. Let $f$ and $g$ be the permutations in Exercise 1. Consider the coloring $\mathbf{c} = (R, B, B, R, R, R)$ of $1, 2, 3, 4, 5, 6$ with the colors $R$ and $B$. Determine the following actions on $\mathbf{c}$:

(a) $f * \mathbf{c}$

(b) $f^{-1} * \mathbf{c}$

(c) $g * \mathbf{c}$

(d) $(g \circ f) * \mathbf{c}$ and $(f \circ g) * \mathbf{c}$

(e) $(g^2 \circ f) * \mathbf{c}$

14. By examining all possibilities, determine the number of nonequivalent colorings of the corners of an equilateral triangle with the colors red and blue. (Then do so with the colors red, white, and blue.)

15. By examining all possibilities, determine the number of nonequivalent colorings of the corners of a regular tetrahedron with the colors red and blue. (Then do so with the colors red, white, and blue.)

16. Characterize the cycle factorizations of those permutations $f$ in $S_n$ for which $f^{-1} = f$, that is, for which $f^2 = \iota$.

17. In Section 14.2 it is established that there are eight nonequivalent colorings of the corners of a regular pentagon with the colors red and blue. Explicitly determine eight nonequivalent colorings.

18. Use Theorem 14.2.3 to determine the number of nonequivalent colorings of the corners of a square with $p$ colors.

19. Use Theorem 14.2.3 to determine the number of nonequivalent colorings of the corners of an equilateral triangle with the colors red and blue. Do the same with $p$ colors (cf. Exercise 3).

20. Use Theorem 14.2.3 to determine the number of nonequivalent colorings of the corners of a triangle that is isoceles, but not equilateral, with the colors red and blue. Do the same with $p$ colors (cf. Exercise 4).

21. Use Theorem 14.2.3 to determine the number of nonequivalent colorings of the corners of a triangle that is neither equilateral nor isoceles, with the colors red and blue. Do the same With $p$ colors (cf. Exercise 5).

22. Use Theorem 14.2.3 to determine the number of nonequivalent colorings of the corners of a rectangle that is not a square with the colors red and blue. Do the same with $p$ colors (cf. Exercise 10).

23. A (one-sided) *marked domino* is a piece consisting of two squares joined along an edge, where each square on one side of the piece is marked with $0, 1, 2, 3, 4, 5,$ or 6 dots. The two squares of a marked domino may receive the same number of dots.

    (a) Use Theorem 14.2.3 to determine the number of different marked dominoes.

    (b) How many different marked dominoes are there if we are allowed to mark the squares with $0, 1, \ldots, p-1,$ or $p$ dots?

24. A *two-sided marked domino* is a piece consisting of two squares joined along an edge, where each square on both sides of the piece is marked with $0, 1, 2, 3,$ $4, 5,$ or 6 dots.

(a) Use Theorem 14.2.3 to determine the number of different two sided marked-dominoes.

(b) How many different two-sided marked dominoes are there if we are allowed to mark the squares with $0, 1, \ldots, p-1$, or $p$ dots?

25. How many different necklaces are there that contain three red and two blue beads?

26. How many different necklaces are there that contain four red and three blue beads?

27. Determine the cycle factorization of the permutations $f$ and $g$ in Exercise 1.

28. Let $f$ be a permutation of a set $X$. Give a simple algorithm for finding the cycle factorization of $f^{-1}$ from the cycle factorization of $f$.

29. Determine the cycle factorization of each permutation in the dihedral group $D_6$ (cf. Exercise 11).

30. Determine permutations $f$ and $g$ of the same set $X$ such that $f$ and $g$ each have two cycles in their cycle factorizations but $f \circ g$ has only one.

31. Show that the number of nonequivalent colorings of the corners of a regular 5-gon with $p$ colors is
$$\frac{p(p^2+4)(p^2+1)}{10}.$$

32. Determine the number of nonequivalent colorings of the corners of a regular hexagon with the colors red, white and blue (cf. Exercise 29).

33. Prove that a permutation and its inverse have the same type (cf. Exercise 28).

34. Let $e_1, e_2, \ldots, e_n$ be nonnegative integers such that $1e_1 + 2e_2 + \cdots + ne_n = n$. Show how to construct a permutation $f$ of the set $\{1, 2, \ldots, n\}$ such that $\text{type}(f) = (e_1, e_2, \ldots, e_n)$.

35. Determine the number of nonequivalent colorings of the corners of a regular 6-gon with $k$ colors (cf. Exercise 29).

36. Determine the number of nonequivalent colorings of the corners of a regular 5-gon with the colors red, white, and blue in which two corners are colored red, two are colored white, and one is colored blue.

37. Determine the number of nonequivalent colorings of the corners of a regular 8-gon with colors red, white, and blue under the action of the corner symmetry group of the 8-gon.

38. A two-sided triomino is a 1 by 3 board of three squares with each square (six in in all because of the two sides) colored with one of the colors red, white, blue, green, and yellow (squares on opposite sides may be colored differently). How many nonequivalent two-sided triominoes are there?

39. A two-sided 4-omino is a 1-by-4 board of four squares with each square (eight in in all because of the two sides) colored with one of the colors red, white, blue, green, and yellow (squares on opposite sides may be colored differently). How many nonequivalent two-sided 4-ominoes are there?

40. A two-sided $n$-omino is a 1-by-$n$ board of $n$ squares with each square ($2n$ in in all because of the two sides) colored with one of $p$ given colors (squares on opposite sides may be colored differently). How many nonequivalent two-sided $n$-ominoes are there?

41. Determine the cycle index of the dihedral group $D_6$ (cf. Exercise 29).

42. Determine the generating function for nonequivalent colorings of the corners of a regular hexagon with two colors and also with three colors (cf. Exercise 41).

43. Determine the cycle index of the edge-symmetry group of a square.

44. Determine the generating function for nonequivalent colorings of the edges of a square with the colors red and blue. How many nonequivalent colorings are there with $k$ colors (cf. Exercise 43)?

45. Let $n$ be an odd prime number. Prove that each of the permutations, $\rho_n, \rho_n^2, \ldots, \rho_n^n$ of $\{1, 2, \ldots, n\}$ is an $n$-cycle. (Recall that $\rho_n$ is the permutation that sends 1 to 2, 2 to 3, $\ldots$, $n-1$ to $n$, and $n$ to 1.)

46. Let $n$ be a prime number. Determine the number of different necklaces that can be made from $n$ beads of $k$ different colors.

47. The nine squares of a 3-by-3 chessboard are to be colored red and blue. The chessboard is free to rotate but cannot be flipped over. Determine the generating function for the number of nonequivalent colorings and the total number of nonequivalent colorings.

48. A stained glass window in the form of a 3-by-3 chessboard has nine squares, each of which is colored red or blue (the colors are transparent and the window can be looked at from either side). Determine the generating function for the number of different stained glass windows and the total number of stained glass windows.

49. Repeat Exercise 48 for stained glass windows in the form of a 4-by-4 chessboard with 16 squares.

50. Find the generating function for the different necklaces that can be made with $p$ beads each of color red or blue if $p$ is a prime number (cf. Exercise 46).

51. Determine the cycle index of the dihedral group $D_{2p}$, where $p$ is a prime number.

52. Find the generating function for the different necklaces that can be made with $2p$ beads each of color red or blue if $p$ is a prime number.

53. Ten balls are stacked in a triangular array with 1 atop 2 atop 3 atop 4. (Think of billiards.) The triangular array is free to rotate. Find the generating function for the number of nonequivalent colorings with the colors red and blue. Find the generating function if we are also allowed to turn over the array.

54. Use Theorem 14.3.3 to determine the generating function for nonisomorphic graphs of order 5. (*Hint*: This exercise will require some work and is a fitting last exercise. We need to obtain the cycle index of the group $S_5^{(2)}$ of permutations of the set $X$ of 10 unordered pairs of distinct integers from $\{1, 2, 3, 4, 5\}$ (the possible edges of a graph of order 5). First, compute the number of permutations $f$ of $S_5$ of each type. Then use the fact that the type of $f$ as a permutation of $X$ depends only on the type of $f$ as a permutation of $\{1, 2, 3, 4, 5\}$.)