Chapter 6

The Inclusion–Exclusion Principle and Applications

In this chapter we derive an important counting formula called the inclusion–exclusion principle. Recall that the addition principle gives a formula for counting the number of objects in a union of sets, *provided that the sets do not overlap* (i.e., provided that the sets determine a partition). The inclusion–exclusion principle gives a formula for the most general of circumstances in which the sets are free to overlap without restriction. The formula is necessarily more complicated but, as a result, it is more widely applicable. We give several applications, in particular, to counting permutations with forbidden positions. We also derive a generalization of the inclusion–exclusion principle for general partially ordered sets, called Möbius inversion.

6.1 The Inclusion–Exclusion Principle

In Chapter 3 we saw several examples in which it is easier to make an indirect count of the number of objects in a set rather than to count the objects directly; that is, to use the subtraction principle. We now give two more examples.

Example. Count the permutations $i_1 i_2 \ldots i_n$ of $\{1, 2, \ldots, n\}$ in which 1 is not in the first position (that is, $i_1 \neq 1$).

We could make a direct count by observing that the permutations with 1 not in the first position can be divided into $n-1$ parts according to which of the $n-1$ integers k from $\{2, 3, \ldots, n\}$ is in the first position. A permutation with k in the first position consists of k followed by a permutation of the $(n-1)$-element set $\{1, \ldots, k-1, k+1, \ldots, n\}$. Hence, there are $(n-1)!$ permutations of $\{1, 2, \ldots, n\}$ with k in the first position. By the addition principle, there are $(n-1) \cdot (n-1)!$ permutations of $\{1, 2, \ldots, n\}$ with 1 not in the first position.

Alternatively, we could use the subtraction principle by observing that the number of permutations of $\{1, 2, \ldots, n\}$ with 1 in the first position is the same as the number $(n-1)!$ of permutations of $\{2, 3, \ldots, n\}$. Since the total number of permutations of $\{1, 2, \ldots, n\}$ is $n!$, the number of permutations of $\{1, 2, \ldots, n\}$ in which 1 is not in the first position is $n! - (n-1)! = (n-1) \cdot (n-1)!$. □

Example. Count the number of integers between 1 and 600, inclusive, which are not divisible by 6.

We can do this by the subtraction principle as follows. Since every sixth integer is divisible by 6, the number of integers between 1 and 600 which are divisible by 6 is $600/6 = 100$. Hence $600 - 100 = 500$ of the integers between 1 and 600 are not divisible by 6. □

The subtraction principle is the simplest instance of the inclusion–exclusion principle. We shall formulate the inclusion–exclusion principle in a manner in which it is convenient to apply.

As a first generalization of the subtraction principle, let S be a finite set of objects, and let P_1 and P_2 be two "properties" that each object in S may or may not possess. We wish to count the number of objects in S that have *neither* of the properties P_1 and P_2. Extending the reasoning behind the subtraction principle, we can do this by first including all objects of S in our count, then excluding all objects that have property P_1 and excluding all objects that have property P_2, and then, noting that we have excluded objects having both properties P_1 and P_2 twice, readmitting all such objects once. We can write this symbolically as follows: Let A_1 be the subset of objects of S that have property P_1, and let A_2 be the subset of objects of S that have property P_2. Then $\overline{A}_1$ consists of those objects of S not having property P_1, and $\overline{A}_2$ consists of those objects of S not having property P_2. The objects of the set $\overline{A}_1 \cap \overline{A}_2$ are those having neither property P_1 nor property P_2. We then have

$$|\overline{A}_1 \cap \overline{A}_2| = |S| - |A_1| - |A_2| + |A_1 \cap A_2|. \tag{6.1}$$

To verify (6.1) formally, we argue as follows. Since the left side of (6.1) counts the number of objects of S that have neither of the properties P_1 and P_2, we can establish its validity by showing that an object with neither of the two properties P_1 and P_2 makes a net contribution of 1 to the right side, and every other object makes a net contribution of 0. If x is an object with neither of the properties P_1 and P_2, it is counted among the objects of S, not counted among the objects of A_1 or of A_2, and not counted among the objects of $A_1 \cap A_2$. Hence, its net contribution to the right side of equation (6.1) is

$$1 - 0 - 0 + 0 = 1.$$

If x has only the property P_1, it contributes

$$1 - 1 - 0 + 0 = 0$$

to the right side, while if it has only the property P_2, it contributes

$$1 - 0 - 1 + 0 = 0$$

to the right side. Finally, if x has both properties P_1 and P_2, it contributes

$$1 - 1 - 1 + 1 = 0$$

to the right side of (6.1). Thus, the right side of equation (6.1) also counts the number of objects of S with neither property P_1 nor property P_2.

This inclusion–exclusion principle for two properties extends to any number of properties. Let $P_1, P_2, \ldots, P_m$ be m properties referring to the objects in S, and let

$$A_i = \{x : x \text{ in } S \text{ and } x \text{ has property } P_i\}, \quad (i = 1, 2, \ldots, m)$$

be the subset of objects of S that have property P_i (and possibly other properties). Then $A_i \cap A_j$ is the subset of objects that have both properties P_i and P_j (and possibly others), $A_i \cap A_j \cap A_k$ is the subset of objects that have properties P_i, P_j, and P_k, and so on. The subset of objects having none of the properties is $\overline{A}_1 \cap \overline{A}_2 \cap \cdots \cap \overline{A}_m$. The inclusion–exclusion principle shows how to count the number of objects in this set by counting objects according to the properties they *do* have. Thus, in this sense, the inclusion–exclusion principle "inverts" the counting process.

Theorem 6.1.1 *The number of objects of the set S that have none of the properties $P_1, P_2, \ldots, P_m$ is given by the alternating expression*

$$\begin{aligned} |\overline{A}_1 \cap \overline{A}_2 \cap \cdots \cap \overline{A}_m| &= |S| - \Sigma|A_i| + \Sigma|A_i \cap A_j| - \Sigma|A_i \cap A_j \cap A_k| \\ &\quad + \cdots + (-1)^m |A_1 \cap A_2 \cap \cdots \cap A_m|, \end{aligned} \tag{6.2}$$

where the first sum is over all 1-subsets $\{i\}$ of $\{1, 2, \ldots, m\}$, the second sum is over all 2-subsets $\{i, j\}$ of $\{1, 2, \ldots, m\}$, the third sum is over all 3-subsets $\{i, j, k\}$ of $\{1, 2, \ldots, m\}$, and so on until the mth sum over all m-subsets of $\{1, 2, \ldots, m\}$ of which the only one is itself.

If $m = 3$, (6.2) becomes

$$\begin{aligned} |\overline{A}_1 \cap \overline{A}_2 \cap \overline{A}_3| &= |S| - (|A_1| + |A_2| + |A_3|) + \\ &\quad (|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|) \\ &\quad - |A_1 \cap A_2 \cap A_3|. \end{aligned}$$

Note that there are $1 + 3 + 3 + 1 = 8$ terms on the right side. If $m = 4$, then equation (6.2) becomes

$$|\overline{A}_1 \cap \overline{A}_2 \cap \overline{A}_3 \cap \overline{A}_4| = |S| - (|A_1| + |A_2| + |A_3| + |A_4|)$$

$$\begin{aligned}
&+(|A_1 \cap A_2| + |A_1 \cap A_3| + |A_1 \cap A_4| \\
&+|A_2 \cap A_3| + |A_2 \cap A_4| + |A_3 \cap A_4|) \\
&-(|A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| \\
&+|A_1 \cap A_3 \cap A_4| + |A_2 \cap A_3 \cap A_4|) \\
&+|A_1 \cap A_2 \cap A_3 \cap A_4|.
\end{aligned}$$

In this case there are $1+4+6+4+1 = 16$ terms on the right side. In the general case, the number of terms on the right side of (6.2) is by Theorem 2.3.4,

$$\binom{m}{0} + \binom{m}{1} + \binom{m}{2} + \binom{m}{3} + \cdots + \binom{m}{m} = 2^m.$$

Proof of Theorem 6.1.1. The left side of equation (6.2) counts the number of objects of S with none of the properties. As in the special case $m = 2$ already treated, we can establish the validity of the equation by showing that an object with none of the properties $P_1, P_2, \ldots, P_m$ makes a net contribution of 1 to the right side, and an object with at least one of the properties makes a net contribution of 0. First, consider an object x with none of the properties. Its contribution to the right side of (6.2) is

$$1 - 0 + 0 - 0 + \cdots + (-1)^m 0 = 1,$$

since it is in S but in none of the other sets. Now consider an object y with exactly $n \geq 1$ of the properties. The contribution of y to $|S|$ is $1 = \binom{n}{0}$. Its contribution to $\Sigma|A_i|$ is $n = \binom{n}{1}$ since it has exactly n of the properties and so is a member of exactly n of the sets $A_1, A_2, \ldots, A_m$. The contribution of y to $\Sigma|A_i \cap A_j|$ is $\binom{n}{2}$ since we may select a pair of the properties y has in $\binom{n}{2}$ ways, and so y is a member of exactly $\binom{n}{2}$ of the sets $A_i \cap A_j$. The contribution of y to $\Sigma|A_i \cap A_j \cap A_k|$ is $\binom{n}{3}$, and so on. Thus, the net contribution of y to the right side of (6.2) is

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots + (-1)^m \binom{n}{m},$$

which equals

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots + (-1)^n \binom{n}{n},$$

because $n \leq m$ and $\binom{n}{k} = 0$ if $k > n$. Since this last expression equals 0 according to the identity (5.4), the net contribution of y to the right side of (6.2) is 0 if y has at least one of the properties. $\square$

Theorem 6.1.1 implies a formula for the number of objects in the union of sets that are free to overlap.

Corollary 6.1.2 *The number of objects of S which have at least one of the properties $P_1, P_2, \ldots, P_m$ is given by*

$$\begin{aligned} |A_1 \cup A_2 \cup \cdots \cup A_m| &= \Sigma|A_i| - \Sigma|A_i \cap A_j| + \Sigma|A_i \cap A_j \cap A_k| - \cdots \\ &\quad +(-1)^{m+1}|A_1 \cap A_2 \cap \cdots \cap A_m|, \end{aligned} \tag{6.3}$$

where the summations are as specified in Theorem 6.1.1.

Proof. The set $A_1 \cup A_2 \cup \cdots \cup A_m$ consists of all those objects in S which possess at least one of the properties. Also,

$$|A_1 \cup A_2 \cup \cdots \cup A_m| = |S| - |\overline{A_1 \cup A_2 \cup \cdots \cup A_m}|.$$

Since, as is readily verified,[1]

$$\overline{A_1 \cup A_2 \cup \cdots \cup A_m} = \overline{A}_1 \cap \overline{A}_2 \cap \cdots \cap \overline{A}_m,$$

we have

$$|A_1 \cup A_2 \cup \cdots \cup A_m| = |S| - |\overline{A}_1 \cap \overline{A}_2 \cap \cdots \cap \overline{A}_m|.$$

Combining this equation with equation (6.2), we get equation (6.3). □

Example. Find the number of integers between 1 and 1000, inclusive, that are not divisible by 5, 6, and 8.

To solve this problem, we introduce some notation. For a real number r, recall that $\lfloor r \rfloor$ stands for the largest integer that does not exceed r. Also, we shall abbreviate the least common multiple of two integers, a, b, or three integers, a, b, c, by $\operatorname{lcm}\{a, b\}$ and $\operatorname{lcm}\{a, b, c\}$, respectively. Let P_1 be the property that an integer is divisible by 5, P_2 the property that an integer is divisible by 6, and P_3 the property that an integer is divisible by 8. Let S be the set consisting of the first thousand positive integers. For $i = 1, 2, 3$, let A_i be the set consisting of those integers in S with property P_i. We wish to find the number of integers in $\overline{A}_1 \cap \overline{A}_2 \cap \overline{A}_3$.

We first see that

$$|A_1| = \lfloor \tfrac{1000}{5} \rfloor = 200,$$

$$|A_2| = \lfloor \tfrac{1000}{6} \rfloor = 166,$$

$$|A_3| = \lfloor \tfrac{1000}{8} \rfloor = 125.$$

Integers in the set $A_1 \cap A_2$ are divisible by both 5 and 6. But an integer is divisible by both 5 and 6 if and only if it is divisible by $\operatorname{lcm}\{5, 6\}$. Since $\operatorname{lcm}\{5, 6\} = 30$,

[1] This is one of DeMorgan's rules.

$\text{lcm}\{5, 8\} = 40$, and $\text{lcm}\{6, 8\} = 24$, we see that

$$|A_1 \cap A_2| = \lfloor \tfrac{1000}{30} \rfloor = 33,$$

$$|A_1 \cap A_3| = \lfloor \tfrac{1000}{40} \rfloor = 25,$$

$$|A_2 \cap A_3| = \lfloor \tfrac{1000}{24} \rfloor = 41.$$

Because $\text{lcm}\{5, 6, 8\} = 120$, we conclude that

$$|A_1 \cap A_2 \cap A_3| = \left\lfloor \frac{1000}{120} \right\rfloor = 8.$$

Thus, by the inclusion–exclusion principle, the number of integers between 1 and 1000 that are not divisible by 5, 6, and 8 equals

$$\begin{aligned} |\overline{A}_1 \cap \overline{A}_2 \cap \overline{A}_3| &= 1000 - (200 + 166 + 125) + (33 + 25 + 41) - 8 \\ &= 600. \end{aligned}$$

□

Example. How many permutations of the letters

$$M, A, T, H, I, S, F, U, N$$

are there such that none of the words MATH, IS, and FUN occur as consecutive letters? (Thus, for instance, the permutation MATHISFUN is not allowed, nor are the permutations INUMATHSF and ISMATHFUN.)

We apply the inclusion–exclusion principle (6.2). First, we identify the set S as the set of all permutations of the 9 letters given. We then let P_1 be the property that a permutation in S contains the word MATH as consecutive letters, let P_2 be the property that a permutation contains the word IS as consecutive letters, and let P_3 be the property that a permutation contains the word FUN as consecutive letters. For $i = 1, 2, 3$, let A_i be the set of those permutations in S satisfying property P_i. We wish to find the number of permutations in $\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}$.

We have $|S| = 9! = 362,880$. The permutations in A_1 can be thought of as permutations of the six symbols

$$MATH, I, S, F, U, N$$

treating $MATH$ as one symbol. Hence,

$$|A_1| = 6! = 720.$$

Similarly, the permutations in A_2 are permutations of the eight symbols

$$M, A, T, H, IS, F, U, N,$$

so

$$|A_2| = 8! = 40,320,$$

and the permutations in A_3 are permutations of the seven symbols

$$M, A, T, H, I, S, FUN,$$

so

$$|A_3| = 7! = 5040.$$

The permutations in $A_1 \cap A_2$ are permutations of the five symbols

$$MATH, IS, F, U, N;$$

the permutations in $A_1 \cap A_3$ are permutations of the four symbols

$$MATH, I, S, FUN;$$

and the permutations in $A_2 \cap A_3$ are permutations of the six symbols

$$M, A, T, H, IS, FUN.$$

Hence, we have

$$|A_1 \cap A_2| = 5! = 120, |A_1 \cap A_3| = 4! = 24, \text{ and } |A_2 \cap A_3| = 6! = 720.$$

Finally, $A_1 \cap A_2 \cap A_3$ consists of the permutations of the three symbols $MATH, IS, FUN$ therefore,

$$|A_1 \cap A_2 \cap A_3| = 3! = 6.$$

Substituting into (6.2), we obtain

$$\begin{aligned}|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}| &= 362,880 - 720 - 40,320 - 5040 \\ &\quad +120 + 24 + 720 - 6 = 317,658.\end{aligned}$$

□

In later sections we consider applications of the inclusion–exclusion principle to some general problems. The following special case of the inclusion–exclusion principle will be useful:

Assume that the size of the set $A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}$ that occurs in the inclusion–exclusion principle depends only on k and not on which k sets are used in the intersection. Thus, there are constants $\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_n$ such that

$$\begin{aligned}
\alpha_0 &= |S| \\
\alpha_1 &= |A_1| = |A_2| = \cdots = |A_m| \\
\alpha_2 &= |A_1 \cap A_2| = \cdots = |A_{m-1} \cap A_m| \\
\alpha_3 &= |A_1 \cap A_2 \cap A_3| = \cdots = |A_{m-2} \cap A_{m-1} \cap A_m| \\
&\vdots \\
\alpha_m &= |A_1 \cap A_2 \cap \cdots \cap A_m|.
\end{aligned}$$

In this case, the inclusion–exclusion principle simplifies to

$$\begin{aligned}|\overline{A}_1 \cap \overline{A}_2 \cap \cdots \cap \overline{A}_m| &= \alpha_0 - \binom{m}{1}\alpha_1 + \binom{m}{2}\alpha_2 - \binom{m}{3}\alpha_3 + \cdots + \\ &\quad (-1)^k\binom{m}{k}\alpha_k + \cdots + (-1)^m\alpha_m. \qquad (6.4)\end{aligned}$$

This is because the kth summation that occurs in the inclusion–exclusion principle contains $\binom{m}{k}$ summands, each equal to α_k.

Example. How many integers between 0 and 99,999 (inclusive) have among their digits each of 2, 5, and 8?

Let S be the set of integers between 0 and 99,999. Each integer in S has 5 digits including possible leading 0s. (Thus we think of the integers in S as the 5-permutations of the multiset in which each digit $0, 1, 2, \ldots, 9$ has repetition number 5 or greater.) Let P_1 be the property that an integer does not contain the digit 2, let P_2 be the property that an integer does not contain the digit 5, and let P_3 be the property that an integer does not contain the digit 8. For $i = 1, 2, 3$, let A_i be the set consisting of those integers in S with property P_i. We wish to count the number of integers in $\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}$.

Using the notation in the preceding example, we have

$$\begin{aligned}\alpha_0 &= 10^5\\ \alpha_1 &= 9^5\\ \alpha_2 &= 8^5\\ \alpha_3 &= 7^5.\end{aligned}$$

For instance, the number of integers between 0 and 99,999 that do not contain the digit 2 and that do not contain the digit 5, the size of $|A_1 \cap A_2|$, equals the number of 5-permutations of the multiset

$$\{5 \cdot 0, 5 \cdot 1, 5 \cdot 3, 5 \cdot 4, 5 \cdot 6, 5 \cdot 7, 5 \cdot 8, 5 \cdot 9\},$$

containing 8 different symbols each with repetition number equal to 5, and this equals 8^5. By (6.3), we obtain the answer

$$10^5 - 3 \times 9^5 + 3 \times 8^5 - 7^5.$$

□

6.2 Combinations with Repetition

In Sections 2.3 and 2.5, showed that the number of r-subsets of a set of n distinct elements is

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

and that the number of r-combinations of a multiset with k distinct objects, each with an infinite repetition number, equals

$$\binom{r+k-1}{r}.$$

In this section we show how the latter formula, in conjunction with the inclusion–exclusion principle, gives a method for finding the number of r-combinations of a multiset without any restrictions on its repetition numbers.

Suppose T is a multiset and an object x of T of a certain type has a repetition number that is greater than r. The number of r-combinations of T equals the number of r-combinations of the multiset obtained from T by replacing the repetition number of x by r. This is so because the number of times x can be used in an r-combination of T cannot exceed r. Therefore, any repetition number that is greater than r can be replaced by r. For example, the number of 8-combinations of the multiset $\{3\cdot a, \infty\cdot b, 6\cdot c, 10\cdot d, \infty\cdot e\}$ equals the number of 8-combinations of the multiset $\{3\cdot a, 8\cdot b, 6\cdot c, 8\cdot d, 8\cdot e\}$. We can summarize by saying that we have determined the number of r-combinations of a multiset $T = \{n_1 \cdot a_1, n_2 \cdot a_2, \ldots, n_k \cdot a_k\}$ in the two "extreme" cases:

(1) $n_1 = n_2 = \cdots = n_k = 1$; (i.e., T is a set) and

(2) $n_1 = n_2 = \cdots = n_k = r$.

We shall illustrate how the inclusion–exclusion principle can be applied to obtain solutions for the remaining cases. Although we shall take a specific example, it should be clear that the method works in general.

Example. Determine the number of 10-combinations of the multiset $T = \{3 \cdot a, 4 \cdot b, 5 \cdot c\}$.

We shall apply the inclusion–exclusion principle to the set S of all 10-combinations of the multiset $T^* = \{\infty \cdot a, \infty \cdot b, \infty \cdot c\}$ (or $\{10 \cdot a, 10 \cdot b, 10 \cdot c\}$. Let P_1 be the property that a 10-combination of T^* has more than three a's. Let P_2 be the property that a 10-combination of T^* has more than four b's. Finally, let P_3 be the property that a 10-combination of T^* has more than five c's. The number of 10-combinations of T is then the number of 10-combinations of T^* that have none of the properties P_1, P_2, and P_3. As usual, let A_i consist of those 10-combinations of T^* which have property P_i, $(i = 1, 2, 3)$. We wish to determine the size of the set $\overline{A}_1 \cap \overline{A}_2 \cap \overline{A}_3$. By the inclusion–exclusion principle,

$$\begin{aligned} |\overline{A}_1 \cap \overline{A}_2 \cap \overline{A}_3| &= |S| - (|A_1| + |A_2| + |A_3|) \\ &\quad + (|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|) \\ &\quad - |A_1 \cap A_2 \cap A_3|. \end{aligned}$$

By Theorem 3.5.1,

$$|S| = \binom{10+3-1}{10} = \binom{12}{10} = 66.$$

The set A_1 consists of all 10-combinations of T^* in which a occurs at least four times. If we take any one of these 10-combinations in A_1 and remove four a's, we are left with a 6-combination of T^*. Conversely, if we take a 6-combination of T^* and add four a's to it, we get a 10-combination of T^* in which a occurs at least 4 times. Thus, the number of 10-combinations in A_1 equals the number of 6-combinations of T^*. Hence,

$$|A_1| = \binom{6+3-1}{6} = \binom{8}{6} = 28.$$

In a similar way we see that the number of 10-combinations in A_2 equals the number of 5-combinations of T^*, and the number of 10-combinations in A_3 equals the number of 4-combinations of T^*. Consequently,

$$|A_2| = \binom{5+3-1}{5} = \binom{7}{5} = 21 \text{ and } |A_3| = \binom{4+3-1}{4} = \binom{6}{4} = 15.$$

The set $A_1 \cap A_2$ consists of all 10-combinations of T^* in which a occurs at least four times and b occurs at least five times. If, from any of these 10-combinations, we remove four a's and five b's, we are left with a 1-combination of T^*. Conversely, if to a 1-combination of T^* we add four a's and five b's we obtain a 10-combination in which a occurs at least four times and b occurs at least five times. Thus, the number of 10-combinations in $A_1 \cap A_2$ equals the number of 1-combinations of T^*, so that

$$|A_1 \cap A_2| = \binom{1+3-1}{1} = \binom{3}{1} = 3.$$

We can deduce in a similar way that the number of 10-combinations in $A_1 \cap A_3$ equals the number of 0-combinations in T^* and that there are no 10-combinations in $A_2 \cap A_3$. Therefore,

$$|A_1 \cap A_3| = \binom{0+3-1}{0} = \binom{2}{0} = 1$$

and

$$|A_2 \cap A_3| = 0.$$

Also,

$$|A_1 \cap A_2 \cap A_3| = 0.$$

Putting all these counts into the inclusion–exclusion principle, we obtain

$$\begin{aligned} |\overline{A}_1 \cap \overline{A}_2 \cap \overline{A}_3| &= 66 - (28+21+15) + (3+1+0) - 0 \\ &= 6. \end{aligned}$$

(We should say "all that work for just six combinations" rather than "all those combinations." Can you now list the six 10-combinations?) □

In the proof of Theorem 2.5.1, we pointed out the connection between r-combination and solutions of equations in integers. The number of r-combinations of the multiset $\{n_1 \cdot a_1, n_2 \cdot a_2, \cdots, n_k \cdot a_k\}$ equals the number of integral solutions of the equation

$$x_1 + x_2 + \cdots + x_k = r$$

that satisfy

$$0 \leq x_i \leq n_i \qquad (i = 1, 2, \ldots, k).$$

Thus, the number of these solutions can be calculated by the method just illustrated.

Example. What is the number of integral solutions of the equation

$$x_1 + x_2 + x_3 + x_4 = 18$$

that satisfy

$$1 \leq x_1 \leq 5, \quad -2 \leq x_2 \leq 4, \quad 0 \leq x_3 \leq 5, \quad 3 \leq x_4 \leq 9?$$

We introduce new variables

$$y_1 = x_1 - 1,\ y_2 = x_2 + 2,\ y_3 = x_3, \ \text{ and } y_4 = x_4 - 3,$$

and our equation becomes

$$y_1 + y_2 + y_3 + y_4 = 16. \tag{6.5}$$

The inequalities on the x_i's are satisfied if and only if

$$0 \leq y_1 \leq 4, \quad 0 \leq y_2 \leq 6, \quad 0 \leq y_3 \leq 5, \quad 0 \leq y_4 \leq 6.$$

Let S be the set of all nonnegative integral solutions of equation (6.5). The size of S is

$$|S| = \binom{16+4-1}{16} = \binom{19}{16} = 969.$$

Let P_1 be the property that $y_1 \geq 5$, P_2 the property that $y_2 \geq 7$, P_3 the property that $y_3 \geq 6$, and P_4 the property that $y_4 \geq 7$. Let A_i denote the subset of S consisting of the solutions satisfying property P_i, $(i = 1, 2, 3, 4)$. We wish to evaluate the size of the set $\overline{A}_1 \cap \overline{A}_2 \cap \overline{A}_3 \cap \overline{A}_4$, and we do so by applying the inclusion–exclusion principle. The set A_1 consists of all those solutions in S for which $y_1 \geq 5$. Performing a change in variable $(z_1 = y_1 - 5, z_2 = y_2, z_3 = y_3, z_4 = y_4)$, we see that the number of solutions in A_1 is the same as the number of nonnegative integral solutions of

$$z_1 + z_2 + z_3 + z_4 = 11.$$

Hence,

$$|A_1| = \binom{14}{11} = 364.$$

In a similar way, we obtain

$$|A_2| = \binom{12}{9} = 220,\ |A_3| = \binom{13}{10} = 286,\ |A_4| = \binom{12}{9} = 220.$$

The set $A_1 \cap A_2$ consists of all those solutions in S for which $y_1 \geq 5$ and $y_2 \geq 7$. Performing a change in variable ($u_1 = y_1 - 5, u_2 = y_2 - 7, u_3 = y_3, u_4 = y_4$), we see that the number of solutions in $A_1 \cap A_2$ is the same as the number of nonnegative integral solutions of

$$u_1 + u_2 + u_3 + u_4 = 4.$$

Hence,

$$|A_1 \cap A_2| = \binom{7}{4} = 35.$$

Similarly, we get

$$|A_1 \cap A_3| = \binom{8}{5} = 56,\ |A_1 \cap A_4| = \binom{7}{4} = 35,$$

$$|A_2 \cap A_3| = \binom{6}{3} = 20,\ |A_2 \cap A_4| = \binom{5}{2} = 10,$$

$$\text{and } |A_3 \cap A_4| = \binom{6}{3} = 20.$$

The intersection of any three of the sets A_1, A_2, A_3, A_4 is empty. We now apply the inclusion–exclusion principle to obtain

$$\begin{aligned} |\overline{A}_1 \cap \overline{A}_2 \cap \overline{A}_3 \cap \overline{A}_4| &= 969 - (364 + 220 + 286 + 220) \\ &\quad +(35 + 56 + 35 + 20 + 10 + 20) \\ &= 55. \end{aligned}$$

□

6.3 Derangements

At a party, 10 gentlemen check their hats. In how many ways can their hats be returned so that no gentleman gets the hat with which he arrived? The eight spark plugs of a V-8 engine are removed from their cylinders for cleaning. In how many ways can they be returned to the cylinders so that no spark plug goes into the cylinder whence it came? In how many ways can the letters M,A,D,I,S,O,N be written down so that the "word" spelled disagrees completely with the spelling of the word MADISON in the sense that no letter occupies the same position as it does in the word MADISON? Each of these questions is an instance of the following general problem.

We are given an n-element set X in which each element has a specified location, and we are asked to find the number of permutations of the set X in which no element is in its specified location. In the first question, the set X is the set of 10 hats, and the specified location of a hat is (the head of) the gentleman to which it belongs. In the second question, X is the set of spark plugs, and the location of a spark plug is the cylinder which contained it. In the third question, $X = \{\text{M,A,D,I,S,O,N}\}$, and the location of a letter is that specified by the word MADISON.

Since the actual nature of the objects is irrelevant, we may take X to be the set $\{1, 2, \ldots, n\}$ in which the location of each of the integers is that specified by its position in the sequence $1, 2, \ldots, n$. A *derangement* of $\{1, 2, \ldots, n\}$ is a permutation $i_1 i_2 \ldots i_n$ of $\{1, 2, \ldots, n\}$ such that $i_1 \neq 1$, $i_2 \neq 2, \ldots, i_n \neq n$. Thus, a derangement of $\{1, 2, \ldots, n\}$ is a permutation $i_1 i_2 \cdots i_n$ of $\{1, 2, \ldots, n\}$ in which no integer is in its natural position:

$$\underline{i_1 \neq 1} \quad \underline{i_2 \neq 2} \quad \cdots \quad \underline{i_n \neq n}.$$

We denote by D_n the number of derangements of $\{1, 2, \ldots, n\}$. The preceding questions ask us to evaluate, respectively, D_{10}, D_8, and D_7. For $n = 1$, there are no derangements. The only derangement for $n = 2$ is 2 1. For $n = 3$, there are two derangements, namely, 2 3 1 and 3 1 2. The derangements for $n = 4$ are as follows:

2 1 4 3	3 1 4 2	4 1 2 3
2 3 4 1	3 4 1 2	4 3 1 2
2 4 1 3	3 4 2 1	4 3 2 1.

Thus, we have $D_1 = 0$, $D_2 = 1$, $D_3 = 2$, and $D_4 = 9$.

The inclusion–exclusion principle enables us to get a formula for the derangement numbers D_n.

Theorem 6.3.1 *For* $n \geq 1$,

$$D_n = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right).$$

Proof. Let S be the set of all $n!$ permutations of $\{1, 2, \ldots, n\}$. For $j = 1, 2, \ldots, n$, let P_j be the property that, in a permutation, j is in its natural position. Thus, the permutation $i_1 i_2 \cdots i_n$ of $\{1, 2, \ldots, n\}$ has property P_j provided $i_j = j$. A permutation of $\{1, 2, \ldots, n\}$ is a derangement if and only if it has none of the properties $P_1, P_2, \ldots, P_n$. Let A_j denote the set of permutations of $\{1, 2, \ldots, n\}$ with property P_j, $(j = 1, 2, \ldots, n)$. The derangements of $\{1, 2, \ldots, n\}$ are precisely those permutations in $\overline{A}_1 \cap \overline{A}_2 \cap \cdots \cap \overline{A}_n$. Hence,

$$D_n = |\overline{A}_1 \cap \overline{A}_2 \cap \cdots \cap \overline{A}_n|,$$

and we use the inclusion–exclusion principle to evaluate D_n. The permutations in A_1 are of the form $1i_2\cdots i_n$, where $i_2\cdots i_n$ is a permutation of $\{2,\ldots,n\}$. Thus, $|A_1| = (n-1)!$, and, more generally we have $|A_j| = (n-1)!$ for $j = 1,2,\ldots,n$. The permutations in $A_1 \cap A_2$ are of the form $1\ 2\ i_3\cdots i_n$, where $i_3\cdots i_n$ is a permutation of $\{3,\ldots,n\}$. Therefore, $|A_1 \cap A_2| = (n-2)!$, and more generally we have $|A_i \cap A_j| = (n-2)!$ for any 2-subset $\{i,j\}$ of $\{1,2,\ldots,n\}$. For any integer k with $1 \le k \le n$, the permutations in $A_1 \cap A_2 \cap \cdots \cap A_k$ are of the form $1\ 2\cdots k i_{k+1}\cdots i_n$, where $i_{k+1}\cdots i_n$ is a permutation of $\{k+1,\ldots,n\}$. Consequently, $|A_1 \cap A_2 \cap \cdots \cap A_k| = (n-k)!$, and more generally,

$$|A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}| = (n-k)!$$

for any k-subset $\{i_1,i_2,\ldots,i_k\}$ of $\{1,2,\ldots,n\}$. Since there are $\binom{n}{k}$ k-subsets of $\{1,2,\ldots,n\}$, applying the inclusion–exclusion principle (see (6.4) at the end of Section 6.1), we obtain

$$\begin{aligned} D_n &= n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - \binom{n}{3}(n-3)! \\ &\quad + \cdots + (-1)^n\binom{n}{n}0! \\ &= n! - \frac{n!}{1!} + \frac{n!}{2!} - \frac{n!}{3!} + \cdots + (-1)^n\frac{n!}{n!} \\ &= n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n\frac{1}{n!}\right). \end{aligned}$$

Thus, the theorem is proved. □

We can use the formula obtained to calculate that

$$D_5 = 5!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}\right) = 44.$$

In a similar way, we can calculate that

$$D_6 = 265,\ D_7 = 1854,\ \text{ and } D_8 = 14{,}833.$$

Recalling the infinite series expansion

$$e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots,$$

we may write

$$e^{-1} = \frac{D_n}{n!} + (-1)^{n+1}\frac{1}{(n+1)!} + (-1)^{n+2}\frac{1}{(n+2)!} + \cdots.$$

From elementary facts about alternating infinite series, we conclude that e^{-1} and $D_n/n!$ differ by less than $1/(n+1)!$; in fact, D_n is the integer closest to $n!/e$. A calculation shows that, for $n \geq 7$, e^{-1} and $D_n/n!$ agree to at least three decimal places. Thus, from a practical point of view, e^{-1} and $D_n/n!$ are the same for $n \geq 7$. The number $D_n/n!$ is the ratio of the number of derangements of $\{1, 2, \ldots, n\}$ to the total number of permutations of $\{1, 2, \ldots, n\}$. Consider the experiment of selecting a permutation of $\{1, 2, \ldots, n\}$ at random, and the event E that no integer in the permutation is in its natural position; that is, that the permutation selected is a derangement. Thus $|E| = D_n$, and the probability of E is

$$\text{Prob}(E) = \frac{D_n}{n!}.$$

Returning to the hat question posed at the beginning of this section, if the hats are returned to the gentlemen at random, the probability that no gentleman receives his own hat is $D_{10}/10!$, and this is effectively e^{-1}. From the preceding remarks, it is apparent (and perhaps quite surprising) that the probability that no gentleman receives his own hat would be essentially the same if the number of gentlemen were 1,000,000.

The derangement numbers D_n satisfy other relations that facilitate their evaluation. The first of these that we discuss is

$$D_n = (n-1)(D_{n-2} + D_{n-1}), \quad (n = 3, 4, 5, \ldots). \tag{6.6}$$

This formula is an example of a linear recurrence relation.[2] Starting with the initial information $D_1 = 0$, $D_2 = 1$, we can use (6.6) to calculate D_n for any positive integer n. For instance,

$$\begin{aligned} D_3 &= 2(D_1 + D_2) = 2(0+1) = 2, \\ D_4 &= 3(D_2 + D_3) = 3(1+2) = 9, \\ D_5 &= 4(D_3 + D_4) = 4(2+9) = 44, \text{ and} \\ D_6 &= 5(D_4 + D_5) = 5(9+44) = 265. \end{aligned}$$

In the next chapter we show how to solve linear recurrence relations with constant coefficients. The techniques introduced there will not apply here, however, since the formula (6.6) has a variable coefficient $n-1$.

We can verify the formula (6.6) combinatorially as follows: Let $n \geq 3$, and consider the D_n derangements of $\{1, 2, \ldots, n\}$. These derangements can be partitioned into $n-1$ parts according to which of the integers $2, 3, \ldots, n$ is in the first position of the permutation. It should be clear that each part contains the same number of derangements. Thus, D_n equals $(n-1)d_n$, where d_n is the number of derangements in which 2 is in the first position. Such derangements are of the form

$$2i_2i_3 \cdots i_n, \qquad i_2 \neq 2, i_3 \neq 3, \ldots, i_n \neq n.$$

[2]Recurrence relations are discussed in Chapter 7.

These d_n derangements can be partitioned further into two subparts according to whether $i_2 = 1$ or $i_2 \neq 1$. Let d'_n be the number of derangements of the form

$$21i_3i_4\cdots i_n, \qquad i_3 \neq 3, \ldots, i_n \neq n.$$

Let d''_n be the number of derangements of the form

$$2i_2i_3\cdots i_n, \qquad i_2 \neq 1, i_3 \neq 3, \ldots, i_n \neq n.$$

Then $d_n = d'_n + d''_n$, and it follows that

$$D_n = (n-1)d_n = (n-1)(d'_n + d''_n).$$

We first observe that d'_n is the same as the number of permutations $i_3i_4\cdots i_n$ of $\{3, 4, \ldots, n\}$ in which $i_3 \neq 3, i_4 \neq 4, \ldots, i_n \neq n$. In other words, d'_n is the number of permutations of $\{3, 4, \ldots, n\}$ in which 3 is not in the first position, 4 is not in the second position, and so on. Thus, $d'_n = D_{n-2}$. We next observe that d''_n equals the number of permutations $i_2i_3\cdots i_n$ of $\{1, 3, \ldots, n\}$ in which 1 is not in the first position, 3 is not in the second position, $\ldots$, n is not in the $(n-1)$th position. Hence, $d''_n = D_{n-1}$, and we conclude that

$$D_n = (n-1)(d'_n + d''_n) = (n-1)(D_{n-2} + D_{n-1}),$$

giving us equation (6.6).

Formula (6.6) can be rewritten as

$$D_n - nD_{n-1} = -[D_{n-1} - (n-1)D_{n-2}], \qquad (n \geq 3). \tag{6.7}$$

The expression in the brackets on the right side is the same as the expression on the left side with n replaced by $n-1$. Thus we can apply (6.7) recursively[3] to get

$$\begin{aligned} D_n - nD_{n-1} &= -[D_{n-1} - (n-1)D_{n-2}] \\ &= (-1)^2[D_{n-2} - (n-2)D_{n-3}] \\ &= (-1)^3[D_{n-3} - (n-3)D_{n-4}] \\ &= \cdots \\ &= (-1)^{n-2}(D_2 - 2D_1). \end{aligned}$$

Since $D_2 = 1$ and $D_1 = 0$, we obtain the simpler recurrence relation:

$$D_n = nD_{n-1} + (-1)^{n-2}$$

for the derangement numbers, or, equivalently,

$$D_n = nD_{n-1} + (-1)^n \qquad \text{for} \quad n = 2, 3, 4, \ldots. \tag{6.8}$$

[3]That is, over and over again, with smaller and smaller values of n.

(Strictly speaking, our verification applies only for $n = 3, 4, \ldots$, but it is simple to check that (6.8) holds also when $n = 2$.) Using (6.8) and the value $D_6 = 265$ previously computed, we see that

$$D_7 = 7D_6 + (-1)^7 = 7 \times 265 - 1 = 1854.$$

By repeated application of the formula (6.8), or using it and mathematical induction, we can obtain a different proof of Theorem 6.3.1. (See Exercise 20.) Since (6.8) follows from (6.6), which was given an independent combinatorial proof, this will provide a proof of Theorem 6.3.1 without using the inclusion–exclusion principle.

The formulas (6.6) and (6.8) are similar to formulas that hold for factorials:

$$\begin{aligned} n! &= (n-1)\left((n-2)! + (n-1)!\right), \qquad (n = 3, 4, 5, \ldots) \\ n! &= n(n-1)!, \qquad (n = 2, 3, 4, \ldots). \end{aligned}$$

Example. At a party there are n men and n women. In how many ways can the n women choose male partners for the first dance? How many ways are there for the second dance if everyone has to change partners?

For the first dance there are $n!$ possibilities. For the second dance, each woman has to choose as a partner a man other than the one with whom she first danced. The number of possibilities is the nth derangement number D_n. □

Example. Suppose the n men and the n women at the party check their hats before the dance. At the end of the party their hats are returned randomly. In how many ways can they be returned if each man gets a male hat and each woman gets a female hat, but no one gets the hat he or she checked?

With no restrictions, the hats can be returned in $(2n)!$ ways. With the restriction that each man gets a male hat and each women gets a female hat, there are $n! \times n!$ ways. With the additional restriction that no one gets the correct hat, there are $D_n \times D_n$ ways. □

6.4 Permutations with Forbidden Positions

In this section we consider the general problem of counting permutations of $\{1, 2, \ldots, n\}$ with restrictions on which integers can occupy each place of the permutation.

Let

$$X_1, X_2, \ldots, X_n$$

be (possibly empty) subsets of $\{1, 2, \ldots, n\}$. We denote by

$$P(X_1, X_2, \ldots, X_n)$$

the set of all permutations $i_1 i_2 \cdots i_n$ of $\{1, 2, \ldots, n\}$ such that

$$\begin{array}{c} i_1 \text{ is not in } X_1, \\ i_2 \text{ is not in } X_2, \\ \vdots \\ i_n \text{ is not in } X_n. \end{array}$$

Thus, for each $j = 1, 2, \ldots, n$, only the integers in $\overline{X_j}$ can occupy the jth position in the permutations being considered. A permutation of $\{1, 2, \ldots, n\}$ belongs to the set $P(X_1, X_2, \ldots, X_n)$ provided that an element of X_1 does not occupy the first place (thus, the only elements that can be in the first place are those in the complement $\overline{X_1}$ of X_1), an element of X_2 does not occupy the second place, ... , and an element of X_n does not occupy the nth place. The number of permutations in $P(X_1, X_2, \ldots, X_n)$ is denoted by

$$p(X_1, X_2, \ldots, X_n) = |P(X_1, X_2, \ldots, X_n)|.$$

Example. Let $n = 4$ and let $X_1 = \{1, 2\}$, $X_2 = \{2, 3\}$, $X_3 = \{3, 4\}$, $X_4 = \{1, 4\}$. Then $P(X_1, X_2, X_3, X_4)$ consists of all permutations $i_1 i_2 i_3 i_4$ of $\{1, 2, 3, 4\}$ such that

$$i_1 \neq 1, 2; \; i_2 \neq 2, 3; \; i_3 \neq 3, 4; \;\; \text{and } i_4 \neq 1, 4.$$

Equivalently, $i_1 = 3$ or 4, $i_2 = 1$ or 4, $i_3 = 1$ or 2, and $i_4 = 2$ or 3. The set $P(X_1, X_2, X_3, X_4)$ contains only the two permutations 3 4 1 2 and 4 1 2 3. Thus we have $p(X_1, X_2, X_3, X_4) = 2$. □

Example. Let $X_1 = \{1\}, X_2 = \{2\}, \ldots, X_n = \{n\}$. Then the set $P(X_1, X_2, \ldots, X_n)$ equals the set of all permutations $i_1 i_2 \cdots i_n$ of $\{1, 2, \ldots, n\}$ for which $i_1 \neq 1, i_2 \neq 2, \ldots, i_n \neq n$. We conclude that $P(X_1, X_2, \ldots, X_n)$ is the set of derangements of $\{1, 2, \ldots, n\}$, and we have $p(X_1, X_2, \ldots, X_n) = D_n$. □

As seen in Section 3.4 there is a one-to-one correspondence between permutations of $\{1, 2, \ldots, n\}$ and placements of n nonattacking, indistinguishable rooks on an n-by-n board. The permutation $i_1 i_2 \cdots i_n$ of $\{1, 2, \ldots, n\}$ corresponds to the placement of n rooks on the board in the squares with coordinates $(1, i_1), (2, i_2), \ldots, (n, i_n)$. (Recall that the square with coordinates (k, l) is the square occupying the kth row and the lth column of the board.) The permutations in $P(X_1, X_2, \ldots, X_n)$ correspond to placements of n nonattacking rooks on an n-by-n board in which there are certain squares in which it is forbidden to put a rook.

Example. Let $n = 5$ and let $X_1 = \{1, 4\}, X_2 = \{3\}, X_3 = \emptyset, X_4 = \{1, 5\}, X_5 = \{2, 5\}$. Then the permutations in $P(X_1, X_2, X_3, X_4, X_5)$ are in one-to-one correspondence with the placements of five nonattacking rooks on the board with forbidden positions as shown.

	1	2	3	4	5
1	×			×	
2			×		
3					
4	×				×
5		×			×

□

Generalizing the derivation of the formula for the number D_n of derangements of $\{1, 2, \ldots, n\}$, we apply the inclusion–exclusion principle to obtain a formula for $p(X_1, X_2, \ldots, X_n)$. However, as we will point out later, this formula is not always of computational value. For convenience, our argument will be couched in the language of nonattacking rooks on an n-by-n board.

Let S be the set of all $n!$ placements of n nonattacking rooks on an n-by-n board. We say that such a placement of n nonattacking rooks satisfies property P_j provided that the rook in the jth row is in a column belonging to X_j, $(j = 1, 2, \ldots, n)$. As usual, A_j denotes the set of rook placements satisfying property P_j, $(j = 1, 2, \ldots, n)$. The set $P(X_1, X_2, \ldots, X_n)$ consists of all the placements of n nonattacking rooks that satisfy none of the properties $P_1, P_2, \ldots, P_n$. Hence,

$$\begin{aligned} p(X_1, X_2, \ldots, X_n) &= |\overline{A}_1 \cap \overline{A}_2 \cap \cdots \cap \overline{A}_n| \\ &= n! - \Sigma|A_i| + \Sigma|A_i \cap A_j| \\ &\quad - \cdots + (-1)^k \Sigma|A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}| \\ &\quad + \cdots + (-1)^n |A_1 \cap A_2 \cap \cdots \cap A_n|, \end{aligned} \tag{6.9}$$

where the kth summation is over all k-subsets of $\{1, 2, \ldots, n\}$. We now evaluate the n sums in the preceding formula.

What does, for instance, $|A_1|$ count? It counts the number of ways to place n nonattacking rooks on the board where the rook in row 1 is in one of the columns in X_1. We can choose the column of that rook in $|X_1|$ ways and then place the remaining $n - 1$ nonattacking rooks in $(n-1)!$ ways. Thus, $|A_1| = |X_1|(n-1)!$ and, more generally,

$$|A_i| = |X_i|(n-1)!, \qquad (i = 1, 2, \ldots, n).$$

Hence,

$$\Sigma|A_i| = (|X_1| + |X_2| + \cdots + |X_n|)(n-1)!.$$

We let $r_1 = |X_1| + |X_2| + \cdots + |X_n|$ and obtain

$$\Sigma|A_i| = r_1(n-1)!.$$

The number r_1 equals the number of forbidden squares of the board. Equivalently, r_1 equals the number of ways to place one rook on the board *in* a forbidden square.

Now consider $|A_1 \cap A_2|$. This number counts the number of ways to place n nonattacking rooks on the board where the rooks in row 1 and row 2 are both in forbidden positions (in X_1 and X_2, respectively). Each placement of two nonattacking rooks in rows 1 and 2 in forbidden positions can be completed to n nonattacking rooks in $(n-2)!$ ways. Similar considerations hold for any $|A_i \cap A_j|$, and we obtain the following: Let r_2 equal the number of ways to place two nonattacking rooks on the board *in* the forbidden positions. Then

$$\Sigma|A_i \cap A_j| = r_2(n-2)!.$$

We may directly generalize the preceding argument and evaluate the kth sum in (6.9). We define r_k as follows:

> r_k is the number of ways to place k nonattacking rooks on the n-by-n board where each of the k rooks is in a forbidden position, $(k = 1, 2 \ldots, n)$.

Then

$$\Sigma|A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}| = r_k(n-k)!, \quad (k = 1, 2, \ldots, n).$$

Substituting this formula into (6.9), we obtain the next theorem.

Theorem 6.4.1 *The number of ways to place n nonattacking, indistinguishable rooks on an n-by-n board with forbidden positions equals*

$$n! - r_1(n-1)! + r_2(n-2)! - \cdots + (-1)^k r_k(n-k)! + \cdots + (-1)^n r_n.$$

□

Example. Determine the number of ways to place six nonattacking rooks on the following 6-by-6 board, with forbidden positions as shown.

×					
×	×				
		×	×		
		×	×		

Since r_1 equals the number of forbidden positions, we have $r_1 = 7$. Before evaluating $r_2, r_3, \ldots, r_6$, we note that the set of forbidden positions can be partitioned into two "independent" parts, one part F_1 containing the three positions closest to the upper left corner, and the other part F_2 containing the other four positions in a

2-by-2 square. Here by "independent" we mean that squares in different parts do not belong to a common row or column, and hence a rook in F_1 cannot attack a rook in F_2. We now evaluate r_2, the number of ways to place 2 nonattacking rooks in forbidden positions. The rooks may be both in F_1, both in F_2, or one in F_1 and one in F_2. In the last case they are automatically nonattacking because F_1 and F_2 are independent. Counting in this way, we obtain

$$r_2 = 1 + 2 + 3 \times 4 = 15.$$

For r_3 we need two nonattacking rooks in F_1 and one rook in F_2, or one rook in F_1 and two nonattacking rooks in F_2. Thus,

$$r_3 = 1 \times 4 + 3 \times 2 = 10.$$

For r_4 we need two nonattacking rooks in F_1 and two nonattacking rooks in F_2; hence,

$$r_4 = 1 \times 2 = 2.$$

Clearly, $r_5 = r_6 = 0$, and, by Theorem 6.4.1, the number of ways to place six nonattacking rooks on the board so that no rook occupies a forbidden position equals

$$6! - 7 \times 5! + 15 \times 4! - 10 \times 3! + 2 \times 2! = 184.$$

□

In conclusion, we note that the formula in Theorem 6.4.1 is of computational value only if it is easier to evaluate the numbers $r_1, r_2, \ldots, r_n$ than to evaluate directly the number of ways to place n nonattacking rooks on an n-by-n board with forbidden positions. Note that the number r_n equals the number of ways to place n nonattacking rooks on the n-by-n "complementary" board, obtained by interchanging the forbidden and nonforbidden positions. If there are a lot of forbidden squares, then it may be more difficult to evaluate r_n than it is to count directly the number of ways to place n nonattacking rooks on the board.

6.5 Another Forbidden Position Problem

In Sections 6.3 and 6.4 we counted permutations of $\{1, 2, \ldots, n\}$ in which there are certain absolute forbidden positions. In this section we consider a problem of counting permutations in which there are certain *relative* forbidden positions and show how the inclusion–exclusion principle can be used to count the number of these permutations.

We introduce the problem as follows: Suppose a class of eight boys takes a walk every day. The students walk in a line of eight so that every boy except the first is preceded by another. In order that a child not see the same person in front of him, on the second day the students decide to switch positions so that no boy is preceded by the same boy who preceded him on the first day. In how many ways can they switch positions?

One possibility is to reverse the order of the boys so that the first boy is now last, and so on, but there are other possibilities. If we assign to the boys the numbers $1, 2, \ldots, 8$, with the last boy in the column of the first day receiving the number 1, the next to last boy receiving the number 2, ... , and the first boy receiving the number 8, as in

$$1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8,$$

then we are asked to determine the number of permutations of the set $\{1, 2, \ldots, 8\}$ in which the patterns 12, 23, ... , 78 do not occur. Thus, 31542876 is an allowable permutation, but 84312657 is not. For each positive integer n, we let Q_n denote the number of permutations of $\{1, 2, \ldots, n\}$ in which none of the patterns 12, 23, ... , $(n-1)n$ occurs. We use the inclusion–exclusion principle to evaluate Q_n. If $n = 1$, 1 is an allowable permutation. If $n = 2$, 21 is an allowable permutation. If $n = 3$, the allowable permutations are 213, 321, and 132, while if $n = 4$, they are as follows:

4 1 3 2	4 3 2 1	4 2 1 3
3 2 1 4	3 2 4 1	2 1 4 3
2 4 3 1	2 4 1 3	3 1 4 2.
1 3 2 4	1 4 3 2	

Hence, $Q_1 = 1$, $Q_2 = 1$, $Q_3 = 3$, and $Q_4 = 11$.

Theorem 6.5.1 *For $n \geq 1$,*

$$\begin{aligned} Q_n &= n! - \binom{n-1}{1}(n-1)! + \binom{n-1}{2}(n-2)! \\ &\quad - \binom{n-1}{3}(n-3)! + \cdots + (-1)^{n-1}\binom{n-1}{n-1}1!. \end{aligned}$$

Proof. Let S be the set of all $n!$ permutations of $\{1, 2, \ldots, n\}$. Let P_j be the property that, in a permutation, the pattern $j(j+1)$ does occur, $(j = 1, 2, \ldots, n-1)$. Thus, a permutation of $\{1, 2, \ldots, n\}$ is counted in the number Q_n if and only if it has none of the properties $P_1, P_2, \ldots, P_{n-1}$. As usual, let A_j denote the set of permutations of $\{1, 2, \ldots, n\}$ that satisfy property P_j, $(j = 1, 2, \ldots, n-1)$. Then

$$Q_n = |\overline{A}_1 \cap \overline{A}_2 \cap \cdots \cap \overline{A}_{n-1}|,$$

and we apply the inclusion–exclusion principle to evaluate Q_n. We first calculate the number of permutations in A_1. A permutation is in A_1 if and only if the pattern 12 occurs in it. Thus, a permutation in A_1 may be regarded as a permutation of the $n-1$ symbols $\{12, 3, 4, \ldots, n\}$. We conclude that $|A_1| = (n-1)!$, and in general we see that

$$|A_j| = (n-1)! \qquad (j = 1, 2, \ldots, n-1).$$

Permutations that are in two of the sets $A_1, A_2, \ldots, A_{n-1}$ contain two patterns. These patterns either share an element, such as the patterns 12 and 23, or have no element in common, such as the patterns 12 and 34 . A permutation which contains the two patterns 12 and 34 can be regarded as a permutation of the $n-2$ symbols $\{12, 34, 5, \ldots, n\}$. Thus, $|A_1 \cap A_3| = (n-2)!$. A permutation that contains the two patterns 12 and 23 contains the pattern 123 and thus can be regarded as a permutation of the $n-2$ symbols $\{123, 4, \ldots, n\}$. Hence, $|A_1 \cap A_2| = (n-2)!$. In general, we see that

$$|A_i \cap A_j| = (n-2)!$$

for each 2-subset $\{i, j\}$ of $\{1, 2, \ldots, n-1\}$. More generally, we see that a permutation which contains k specified patterns from the list $12, 23, \ldots, (n-1)n$ can be regarded as a permutation of $n-k$ symbols, and thus that

$$|A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}| = (n-k)!$$

for each k-subset $\{i_1, i_2, \ldots, i_k\}$ of $\{1, 2, \ldots, n-1\}$. Since, for each $k = 1, 2, \ldots, n-1$, there are $\binom{n-1}{k}$ k-subsetss of $\{1, 2, \ldots, n-1\}$, applying the inclusion–exclusion principle we obtain the formula in the theorem. $\square$

Using the formula of Theorem 6.5.1, we calculate that

$$Q_5 = 5! - \binom{4}{1}4! + \binom{4}{2}3! - \binom{4}{3}2! + \binom{4}{4}1! = 53.$$

The numbers $Q_1, Q_2, Q_3, \ldots$ are closely related to the derangement numbers. Indeed, we have $Q_n = D_n + D_{n-1}$, $(n \geq 2)$. (See Exercise 23.) Thus, knowing the derangement numbers, we can calculate all the numbers Q_n, $(n \geq 2)$. Since we have already seen in the preceding section that $D_5 = 44$, $D_6 = 265$, we conclude that $Q_6 = D_6 + D_5 = 265 + 44 = 309$.

6.6 Möbius Inversion

This section includes more sophisticated mathematics than the other sections in this chapter.

The inclusion–exclusion principle is an instance of Möbius inversion on a finite[4] partially ordered set. In order to set the stage for the generality of Möbius inversion, we first discuss a somewhat more general version of the inclusion–exclusion principle.

Let n be a positive integer and consider the set $X_n = \{1, 2, \ldots, n\}$ of n elements, and the partially ordered set $(\mathcal{P}(X_n), \subseteq)$ of all subsets of X_n partially ordered by containment. Let

$$F : \mathcal{P}(X_n) \to \Re$$

[4]One can replace the property of being finite by a weaker property called locally finite, which asserts that, for all a and b with $a \leq b$, the *interval* $\{x : a \leq x \leq b\}$ is a finite set.

be a real-valued function defined on $\mathcal{P}(X_n)$. We use F to define a new function

$$G : \mathcal{P}(X_n) \to \Re$$

by

$$G(K) = \sum_{L \subseteq K} F(L), \quad (K \subseteq X_n), \tag{6.10}$$

where, as indicated, K is a subset of X_n and the summation extends over all subsets L of K. Möbius inversion allows one to *invert* equation (6.10) and to recover F from G; specifically, we have

$$F(K) = \sum_{L \subseteq K} (-1)^{|K|-|L|} G(L), \quad (K \subseteq X_n). \tag{6.11}$$

Notice that F is obtained from G in (6.11) in a way similar to that in which G is obtained from F in (6.10); the only difference is that in (6.11) we insert in front of each term of the summation either a 1 or -1 depending on whether $|K| - |L|$ is even or odd.

Let $A_1, A_2, \ldots, A_n$ be subsets of a finite set S, and for a set $K \subseteq \{1, 2, \ldots, n\}$, define $F(K)$ to be the number of elements of S that belong to *exactly* those sets A_i with $i \notin K$. Thus, for $s \in S$, s is counted by $F(K)$ if and only if

$$\begin{array}{ll} s \notin A_i, & \text{for each } i \in K, \text{ and} \\ s \in A_j, & \text{for each } j \notin K. \end{array}$$

Then

$$G(K) = \sum_{L \subseteq K} F(L)$$

counts the number of elements of S that belong to all of the sets A_j with j not in K and possibly other sets as well. Thus,

$$G(K) = |\cap_{i \notin K} A_i|.$$

By (6.11),

$$F(K) = \sum_{L \subseteq K} (-1)^{|K|-|L|} G(L). \tag{6.12}$$

Taking $K = \{1, 2, \ldots, n\}$ in (6.12), we get

$$F(X_n) = \sum_{L \subseteq X_n} (-1)^{n-|L|} G(L). \tag{6.13}$$

Now, $F(X_n)$ counts the number of elements of S that belong only to those sets A_i with $i \notin X_n$; that is, $F(X_n)$ is the number of elements of S that belong to none of the sets

$A_1, A_2, \ldots, A_n$ and thus equals the number of elements contained in $\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}$. Substituting into (6.13), we obtain

$$|\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}| = \sum_{L \subseteq X_n} (-1)^{n-|L|} |\cap_{i \notin L} A_i|,$$

or, equivalently, by replacing L with its complement in X_n and calling it J,

$$|\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}| = \sum_{J \subseteq X_n} (-1)^{|J|} |\cap_{i \in J} A_i|. \tag{6.14}$$

Equation (6.14) is equivalent to the formula for the inclusion–exclusion principle as given in Theorem 6.1.1.

We now replace $(\mathcal{P}(X_n), \subseteq)$ with an arbitrary finite partially ordered set $(X, \leq)$. To derive the formula for Möbius inversion, we first consider functions of two variables.

Let $\mathcal{F}(X)$ be the collection of all real-valued functions

$$f : X \times X \to \Re,$$

with the property that $f(x, y) = 0$ whenever $x \not\leq y$. Thus, $f(x, y)$ can be different from 0 only when $x \leq y$. We define the *convolution product* $h = f * g$ of two functions f and g in $\mathcal{F}(X)$ by

$$h(x,y) = \begin{cases} \sum_{\{z : x \leq z \leq y\}} f(x, z) g(z, y), & \text{if } x \leq y, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, in the convolution product, to compute $h(x, y)$ when $x \leq y$, we add up all products $f(x, z)g(z, y)$ as z varies over all elements z between x and y in the given partial order. We leave it as an exercise to verify that the convolution product satisfies the associative law:

$$f * (g * h) = (f * g) * h, \quad (f, g, h \text{ in } \mathcal{F}(X)).$$

There are three special functions in $\mathcal{F}(X)$ of interest to us. The first is the *Kronecker delta function* δ, given by

$$\delta(x,y) = \begin{cases} 1, & \text{if } x = y \\ 0, & \text{otherwise.} \end{cases}$$

Note that $\delta * f = f * \delta = f$ for all functions $f \in \mathcal{F}(X)$, and thus δ acts as an identity function with respect to convolution product. The second is the *zeta function* ζ defined by

$$\zeta(x,y) = \begin{cases} 1, & \text{if } x \leq y \\ 0, & \text{otherwise.} \end{cases}$$

The zeta function is a representation of the poset $(X, \leq)$ in that it contains all the information about which pairs x, y of elements satisfy $x \leq y$.

Let f be a function in $\mathcal{F}(X)$ such that $f(y, y) \neq 0$ for all y in X. We can inductively define a function g in $\mathcal{F}(X)$ by first letting

$$g(y, y) = \frac{1}{f(y, y)}, \quad (y \in X), \tag{6.15}$$

and then letting

$$g(x, y) = -\frac{1}{f(y, y)} \sum_{\{z : x \leq z < y\}} g(x, z) f(z, y), \quad (x < y). \tag{6.16}$$

From (6.16), we get

$$\sum_{\{z : x \leq z \leq y\}} g(x, z) f(z, y) = \delta(x, y), \quad (x \leq y). \tag{6.17}$$

Equation (6.17) tells us that

$$g * f = \delta,$$

and therefore g is a *left-inverse function* of f with respect to the convolution product. In a similar way, we can show that f has a *right-inverse function* h satisfying

$$f * h = \delta.$$

Using the associative law for convolution product, we get

$$g = g * \delta = g * (f * h) = (g * f) * h = \delta * h = h.$$

Thus, $g = h$ and g is an *inverse function* of f. In sum, every function $f \in \mathcal{F}(X)$ with $f(y, y) \neq 0$ for all y in X has an inverse function g, inductively defined by (6.15) and (6.16), satisfying

$$g * f = f * g = \delta.$$

The third special function we define is the *Möbius function* μ. Since $\zeta(y, y) = 1$ for all $y \in X$, ζ has an inverse, and we define μ to be its inverse. Therefore,

$$\mu * \zeta = \delta,$$

and so, applying (6.17) with $f = \zeta$ and $g = \mu$, we get

$$\sum_{\{z : x \leq z \leq y\}} \mu(x, z) \zeta(z, y) = \delta(x, y), \quad (x \leq y),$$

or, equivalently,

$$\sum_{\{z : x \leq z \leq y\}} \mu(x, z) = \delta(x, y), \quad (x \leq y). \tag{6.18}$$

Equation (6.18) implies that

$$\mu(x, x) = 1 \text{ for all } x \tag{6.19}$$

and

$$\mu(x, y) = -\sum_{\{z: x \leq z < y\}} \mu(x, z), \quad (x < y). \tag{6.20}$$

Example. In this example, we compute the Möbius function of the partially ordered set $(\mathcal{P}(X_n), \subseteq)$, where $X_n = \{1, 2, \ldots, n\}$. Let A and B be subsets of X_n with $A \subseteq B$. We prove by induction on $|B| - |A|$ that

$$\mu(A, B) = (-1)^{|B|-|A|}. \tag{6.21}$$

We have from (6.19) that $\mu(A, A) = 1$ and hence (6.21) holds if $B = A$. Suppose that $B \neq A$, and let $p = |B \setminus A| = |B| - |A|$. Then, from (6.20) and the induction hypothesis, we get

$$\begin{aligned} \mu(A, B) &= -\sum_{\{C: A \subseteq C \subset B\}} \mu(A, C) \\ &= -\sum_{\{C: A \subseteq C \subset B\}} (-1)^{|C|-|A|} \\ &= -\sum_{k=0}^{p-1} (-1)^k \binom{p}{k}. \end{aligned} \tag{6.22}$$

The last equality is a consequence of the fact that, for each integer k with $0 \leq k \leq p-1$, there are as many sets C satisfying $A \subseteq C \subset B$ and $|C| - |A| = k$ as there are subsets of cardinality k contained in the set $B \setminus A$ of cardinality p. By the binomial theorem, we have

$$0 = (1-1)^p = \sum_{k=0}^{p} (-1)^k \binom{p}{k},$$

and so

$$\sum_{k=0}^{p-1} (-1)^k \binom{p}{k} = -(-1)^p \binom{p}{p}.$$

Substituting in equation (6.22), we obtain

$$\mu(A, B) = (-1)^p \binom{p}{p} = (-1)^p = (-1)^{|B|-|A|}, \tag{6.23}$$

a formula for the Möbius function of $(\mathcal{P}(X_n), \subseteq)$, □

Example. In this example we compute the Möbius function of a linearly ordered set. Let $X_n = \{1, 2, \ldots, n\}$ and consider the linearly ordered set $(X_n, \leq)$, where $1 < 2 < \cdots < n$. We have $\mu(k, k) = 1$ for $k = 1, 2, \ldots, n$, and $\mu(k, l) = 0$ for $1 \leq l < k \leq n$. Suppose that $l = k + 1$, where $1 \leq k \leq n - 1$. Then

$$\sum_{\{j:k\leq j\leq k+1\}} \mu(k, j) = 0;$$

hence,

$$\mu(k, k) + \mu(k, k + 1) = 0,$$

and this implies that $\mu(k, k+1) = -\mu(k, k) = -1$. We now assume that $1 \leq k \leq n-2$. Then

$$\mu(k, k) + \mu(k, k + 1) + \mu(k, k + 2) = 0;$$

therefore,

$$\mu(k, k + 2) = -(\mu(k, k) + \mu(k, k + 1)) = -(1 + (-1)) = 0.$$

Continuing like this, or using induction, we see that the Möbius function of a linearly ordered set $1 < 2 < \cdots < n$ satisfies

$$\mu(k, l) = \begin{cases} 1, & \text{if } l = k, \\ -1, & \text{if } l = k + 1, \\ 0, & \text{otherwise.} \end{cases}$$

□

We now state and prove the general *Möbius inversion formula* for functions defined on a finite partially ordered set. In this theorem, we assume that $(X, \leq)$ has a *smallest element*—that is, an element 0 such that $0 \leq x$ for all $x \in X$. This holds, for instance, for the partially ordered set $(\mathcal{P}(X_n), \subseteq)$, where the smallest element is the empty set.

Theorem 6.6.1 *Let $(X, \leq)$ be a partially ordered set with a smallest element 0. Let μ be its Möbius function, and let $F : X \to \Re$ be a real-valued function defined on X. Let the function $G : X \to \Re$ be defined by*

$$G(x) = \sum_{\{z:z\leq x\}} F(z), \quad (x \in X).$$

Then

$$F(x) = \sum_{\{y:y\leq x\}} G(y)\mu(y, x), \quad (x \in X).$$

Proof. Let ζ be the zeta function of $(X, \leq)$. Using the properties of ζ and μ previously discussed, we calculate as follows for x an arbitrary element in X:

$$\begin{aligned}
\sum_{\{y:y\leq x\}} G(y)\mu(y,x) &= \sum_{\{y:y\leq x\}} \sum_{\{z:z\leq y\}} F(z)\mu(y,x) \\
&= \sum_{\{y:y\leq x\}} \mu(y,x) \sum_{\{z:z\in X\}} \zeta(z,y)F(z) \\
&= \sum_{\{z:z\in X\}} \sum_{\{y:y\leq x\}} \zeta(z,y)\mu(y,x)F(z) \\
&= \sum_{\{z:z\in X\}} \left(\sum_{\{y:z\leq y\leq x\}} \zeta(z,y)\mu(y,x) \right) F(z) \\
&= \sum_{\{z:z\in X\}} \delta(z,x)F(z) \\
&= F(x).
\end{aligned}$$

□

As a corollary, we get the general inclusion–exclusion principle as formulated in equations (6.10) and (6.11).

Corollary 6.6.2 *Let $X_n = \{1, 2, \ldots, n\}$ and let $F : \mathcal{P}(X_n) \to \Re$ be a function defined on the subsets of X_n. Let $G : \mathcal{P}(X_n) \to \Re$ be the function defined by*

$$G(K) = \sum_{L\subseteq K} F(L), \quad (K \subseteq X_n).$$

Then

$$F(K) = \sum_{L\subseteq K} (-1)^{|K|-|L|} G(L), \quad (K \subseteq X_n).$$

Proof. The corollary follows from Theorem 6.6.1 and the evaluation of the Möbius function of $(\mathcal{P}(X_n), \subseteq)$ as given in (6.23). □

Example. We use Möbius inversion to obtain a formula for the number of ways to place n nonattacking rooks on an n-by-n board with forbidden positions, which is different from that given in Theorem 6.4.1. To facilitate our discussion, we now model an n-by-n board as an n-by-n matrix

$$A = [a_{ij} : 1 \leq i, j \leq n]$$

of 0s and 1s. We put a 0 in each position that is forbidden and a 1 in each position that is not. For example, the board

×		×	
			×
	×		
		×	

(6.24)

corresponds to the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}. \tag{6.25}$$

A collection of four nonattacking rooks on the board corresponds to a collection of four 1s in A with the property that each row and column contains exactly one of these 1s (equivalently, no repeated 1 in a row or in a column). For example, the four 1s

$$a_{14} = 1, a_{23} = 1, a_{31} = 1, \text{ and } a_{42} = 1$$

correspond to four nonattacking rooks in positions

$$(1,4), (2,3), (3,1), (4,2).$$

These four 1s correspond to the permutation 4, 3, 1, 2 of $\{1,2,3,4\}$, or, equivalently, to the *bijection*[5]

$$f : \{1,2,3,4\} \to \{1,2,3,4\},$$

with

$$f(1) = 4, f(2) = 3, f(3) = 1, \text{ and } f(4) = 2.$$

Returning to the general case, we let $X_n = \{1, 2, \ldots, n\}$ and let $\mathcal{P}_n$ denote the set of all $n!$ bijections $f : X_n \to X_n$. In general, n nonattacking rooks on an n-by-n board correspond to n 1s in the matrix with exactly one 1 in each row and in each column. This, in turn, corresponds to a bijection

$$f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$$

in $\mathcal{P}_n$ with $a_{if(i)} = 1$ for $i = 1, 2, \ldots, n$, or, equivalently, with

$$\prod_{i=1}^{n} a_{if(i)} = a_{1f(1)} a_{2f(2)} \cdots a_{nf(n)} = 1.$$

[5]In this section, we use *bijection* (or bijective function) to mean a function that is both one-to-one and onto. An *injection* (or injective function) means a one-to-one function. A *surjection* (or surjective function) means an onto function. So a bijection is a function that is both a surjection and an injection.

If f is a bijection for which $a_{if(i)} = 0$ for some i, then

$$\prod_{i=1}^{n} a_{if(i)} = a_{1f(1)}a_{2f(2)} \cdots a_{nf(n)} = 0.$$

Therefore, we conclude that the number of ways to place n nonattacking rooks on an n-by-n board with the associated n-by-n matrix $A = [a_{ij}]$ of 0s and 1s equals

$$\sum_{f \in \mathcal{P}_n} \prod_{i=1}^{n} a_{if(i)}. \tag{6.26}$$

(The expression in (6.26) is an important combinatorial function of a matrix A; it's called the *permanent* of A.) .

Consider the partially ordered set $(\mathcal{P}(X_n), \subseteq)$. Each subset S of cardinality k of X_n picks out a set of k columns of A, and we denote the n-by-k submatrix formed by these columns by $A[S]$. Let $\mathcal{F}_n(S)$ denote the set of all functions $f : \{1, 2, \ldots, n\} \to S$, and let $\mathcal{G}_n(S)$ denote the subset of *surjective* functions. We then have

$$\mathcal{F}_n(S) = \cup_{T \subseteq S} \mathcal{G}_n(T).$$

Define the function $F : \mathcal{P}(X_n) \to \Re$ by

$$F(S) = \sum_{f \in \mathcal{G}_n(S)} \prod_{i=1} a_{if(i)}, \quad (S \subseteq X_n).$$

(Here, if $S = \emptyset$, then $F(S) = 0$.) Notice that $F(X_n)$ is equal to (6.26), since a surjective function $f : X_n \to X_n$ is a bijection. Thus, our goal is to calculate $F(X_n)$.

Let

$$G(S) = \sum_{T \subseteq S} F(T), \quad (S \subseteq X_n).$$

Then

$$G(S) = \sum_{g \in \mathcal{F}_n(S)} \prod_{i=1} a_{ig(i)}, \quad (S \subseteq X_n).$$

From Corollary 6.6.2, we get

$$F(X_n) = \sum_{S \subseteq X_n} (-1)^{n-|S|} G(S). \tag{6.27}$$

$G(S)$, being the summation of $a_{1g(1)}a_{2g(2)} \cdots a_{ng(n)}$ over *all* functions $g : X_n \to S$, is just the product

$$\prod_{i=1}^{n} \left(\sum_{j \in S} a_{ij} \right);$$

that is, $G(S)$ is the product of the sums of the elements in each row of $A[S]$. Thus, (6.27) becomes

$$F(X_n) = \sum_{S \subseteq X_n} (-1)^{n-|S|} \prod_{i=1}^{n} \left(\sum_{j \in S} a_{ij} \right), \tag{6.28}$$

and this gives a way to calculate the number of ways to place n nonattacking rooks on an n-by-n board: We pick a subset of columns, evaluate the sum of the elements of each row in those columns, multiply these sums together, affix the appropriate sign, and add the results over all choices of subsets of columns. The number of summands equals the number of subsets of a set of size n and hence equals 2^n.

Applying formula (6.27) to the board in (6.24) with associated 4-by-4 matrix (6.25), we get by a tedious calculation that the number of ways to place four nonattacking rooks on the board (6.24) equals 6. In this case, with a small $n = 4$, it would be easier to arrive at this number 6 directly, but that's not the point. The point is that we have a way to count that depends only on simple, arithmetical calculations, even though there may be exponentially many of them.

□

The next example makes use of the direct product construction for partially ordered sets (see Exercise 38 of Chapter 4), which we review here. Let $(X, \leq_1)$ and $(Y, \leq_2)$ be partially ordered sets. Define the relation $\leq$ on the set

$$X \times Y = \{(x, y) : x \text{ in } X, y \text{ in } Y\}$$

by

$$(x, y) \leq (x', y') \text{ if and only if } x \leq_1 x' \text{ and } y \leq_2 y'.$$

It is straightforward to check that $(X \times Y, \leq)$ is a partially ordered set, called the *direct product* of $(X, \leq_1)$ with $(Y, \leq_2)$. We may generalize this direct product construction to any number of partially ordered sets.

The next theorem shows how the Möbius function of a direct product is determined from the Möbius functions of its component partially ordered sets.

Theorem 6.6.3 *Let $(X, \leq_1)$ and $(Y, \leq_2)$ be two finite partially ordered sets with Möbius functions μ_1 and μ_2, respectively. Let μ be the Möbius function of the direct product of $(X, \leq_1)$ and $(Y, \leq_2)$. Then*

$$\mu((x, y), (x', y')) = \mu(x, x')\mu(y, y'), \quad ((x, y), (x', y') \text{ in } X \times Y). \tag{6.29}$$

Proof. If $(x, y) \not\leq (x', y')$, then $\mu((x, y), (x', y')) = 0$, and either $x \not\leq_1 y$ or $x' \not\leq y'$, implying that either $\mu_1(x, x') = 0$ or $\mu_2(y, y') = 0$. Hence, (6.29) holds in this case.

Now suppose that $(x, y) \leq (x', y')$. We prove that (6.29) holds by induction on the number of pairs (u, v) that lie between (x, y) and (x', y') in the partial order. We have

$x \leq_1 x'$ and $y \leq_2 y'$. If $(x,y) = (x',y')$, then $x = x'$ and $y = y'$ and both sides of (6.29) have value equal to 1. We assume that $(x,y) \neq (x',y')$ and proceed by induction:

$$\begin{aligned}
\mu((x,y),(x',y')) &= -\sum_{\{(u,v):(x,y)\leq(u,v)<(x',y')\}} \mu((u,v),(x',y')) \\
&= -\sum_{\{(u,v):(x,y)\leq(u,v)<(x',y')\}} \mu_1(u,x')\mu_2(v,y') \\
&\qquad \text{(by the inductive assumption)} \\
&= -\left(\sum_{\{u:x\leq_1 u\leq_1 x'\}} \mu_1(u,x')\right)\left(\sum_{\{v:y\leq_2 v\leq_2 v'\}} \mu_2(v,y')\right) \\
&\qquad +\mu_1(x,x')\mu_2(y,y') \\
&= (0)(0) + \mu_1(x,x')\mu_2(y,y').
\end{aligned}$$

Thus, the theorem holds by induction. □

We can express Theorem 6.6.3 by saying the Möbius function of the direct product of two partially ordered sets is the product of their Möbius functions. More generally, the Möbius function of the direct product of a finite number of finite partially ordered sets is the product of their Möbius functions.

Example. Let n be a positive integer and again let $X_n = \{1,2,\ldots,n\}$. We now consider the partially ordered set $D_n = (X_n, \mid\,)$, where the partial order is that given by divisibility: $a \mid b$ if and only if a is a factor of b. For clarity, we use the divisibility symbol "|" rather than the general symbol "$\leq$" for a partial order. Our goal is to compute $\mu(1,n)$ for this partially ordered set. From this, we can then compute $\mu(a,b)$ for any integers a and b in X_n by $\mu(a,b) = \mu(1,\frac{b}{a})$ if $a \mid b$. (See the Exercises.)

The integer n has a unique factorization into primes, and thus

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k},$$

where $p_1,p_2,\ldots,p_k$ are distinct primes and $\alpha_1,\alpha_2,\ldots,\alpha_k$ are positive integers.[6] Since $\mu(1,n)$ is given inductively by

$$\mu(1,n) = -\sum_{\{m\geq 1:m|n,m\neq n\}} \mu(1,m),$$

we need consider only $(X_n^*, \mid\,)$, where X_n^* is the subset of X_n consisting of all positive integers k such that $k \mid n$. Let r and s be integers in X_n^*. We have

$$r = p_1^{\beta_1} p_2^{\beta_2} \cdots p_k^{\beta_k} \text{ and } s = p_1^{\gamma_1} p_2^{\gamma_2} \cdots p_k^{\gamma_k},$$

[6]The factorization is unique apart from the order in which the primes are written down.

where $0 \leq \beta_i, \gamma_i \leq \alpha_i, (i = 1, 2, \ldots, k)$.[7] Then $r|s$ if and only if $\beta_i \leq \gamma_i, (i = 1, 2, \ldots, k)$. Thus, the partially ordered set $(X_n^*, \mid)$ is just the direct product of k linear orders of sizes $\alpha_1 + 1, \alpha_2 + 1, \ldots, \alpha_k + 1$, respectively. From Theorem 6.6.3, we get

$$\mu(1, n) = \prod_{i=1}^{k} \mu(1, p_i^{\alpha_i}).$$

From our evaluation of the Möbius function of a linear order, we see that

$$\mu(1, p_i^{\alpha_i}) = \begin{cases} 1, & \text{if } \alpha_i = 0, \\ -1, & \text{if } \alpha_i = 1, \\ 0, & \text{if } \alpha_i \geq 2. \end{cases}$$

Hence,

$$\mu(1, n) = \begin{cases} 1, & \text{if } n = 1, \\ (-1)^k, & \text{if } n \text{ is a product of distinct primes,} \\ 0, & \text{otherwise.} \end{cases}$$

□

We now obtain the classical Möbius inversion formula.

Theorem 6.6.4 *Let F be a real-valued function defined on the set of positive integers. Define a real-valued function G on the positive integers by*

$$G(n) = \sum_{k:k|n} F(k).$$

Then, for each positive integer n, we have

$$F(n) = \sum_{k:k|n} \mu(n/k) G(k),$$

where we write $\mu(n/k)$ for $\mu(1, n/k)$.

Proof. Since, for any fixed n, the definition of $G(n)$ depends only on the values of F on the set $X_n = \{1, 2, \ldots, n\}$, we may confine our attention to the partially ordered set $(X_n, \mid)$. By Theorem 6.6.1, we have

$$F(n) = \sum_{\{k:k|n\}} \mu(k, n) G(k) = \sum_{\{k:k|n\}} \mu(1, n/k) G(k).$$

□

In the next two examples we apply Theorem 6.6.4 to solve two counting problems.

[7] In order to have the same primes in these factorizations of r and s, we allow some of the exponents to be 0.

Example. In this example, we compute the value of the *Euler ϕ function* defined for a positive integer n by $\phi(n) = |S_n|$, where

$$S_n = \{k : 1 \leq k \leq n, \mathrm{GCD}(k, n) = 1\}.$$

Thus, $\phi(n)$ equals the number of positive integers not exceeding n that are relatively prime to n. For example, $\phi(1) = 1$,

$$\phi(9) = |\{1, 2, 4, 5, 7, 8\}| = 6,$$

and $\phi(13) = 12$ (the value of ϕ at a prime number p is always $p - 1$). Let

$$S_n^d = \{k : 1 \leq k \leq n, \mathrm{GCD}(k, n) = d\}, \quad (d \text{ a positive divisor of } n).$$

Then $S_n = S_n^1$. We also have $|S_n^d| = \phi(n/d)$, since any integer k with $\mathrm{GCD}(k, n) = 1$ is of the form $k = dk'$, where $1 \leq k' \leq n/d$ and $\mathrm{GCD}(k', n/d) = 1$. We take the function F in Möbius inversion to be the Euler ϕ function, and we define

$$G(n) = \sum_{\{d:d|n\}} \phi(d).$$

Since $\phi(d)$ equals the number of integers k between 1 and n such that $\mathrm{GCD}(k, n) = d$, and since, for each such integer k, $\mathrm{GCD}(k, n) = d$ for some integer d with $d \mid n$, we conclude that $G(n) = n$. Thus, we have

$$n = \sum_{\{d:d|n\}} \phi(d),$$

and, inverting this equation, we get

$$\phi(n) = \sum_{\{d:d|n\}} \mu(n/d)d = \sum_{\{d:d|n\}} \mu(d)\, n/d. \tag{6.30}$$

Now, $\mu(d)$ is nonzero if and only if $d = 1$ or d is a product of distinct primes; in the latter case, $\mu(d) = (-1)^r$, where r is the number of distinct primes in d. Let the distinct primes dividing n be $p_1, p_2, \ldots, p_r$. Then (6.30) implies that $\phi(n)$ equals

$$n - \left(\frac{n}{p_1} + \frac{n}{p_2} + \cdots\right) + \left(\frac{n}{p_1p_2} + \frac{n}{p_1p_3} + \cdots\right) + \cdots +$$
$$(-1)^r \frac{n}{p_1p_2\cdots p_r},$$

and this is just the product expansion

$$n\prod_{i=1}^{r}\left(1 - \frac{1}{p_i}\right).$$

Thus,

$$\phi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right),$$

where the product is over all distinct primes p dividing n. □

We conclude this section with an application of classical Möbius inversion.

Example. We count the number of circular n-permutations of k different symbols $a_1, a_2, \ldots, a_k$, where each symbol can be used any number of times; equivalently, we count the number of circular n-permutations of the multiset $\{n{\cdot}a_1, n{\cdot}a_2, \ldots, n{\cdot}a_k\}$. We define the *period* of such a circular permutation to be the smallest positive number d of clockwise, circular shifts by one position required to leave the circular word unchanged. For example,

$$\begin{array}{ccc} & a_1 & \\ a_2 & & a_2 \\ & a_1 & \end{array}$$

has period 2, since

$$\begin{array}{ccccccccccc} & a_1 & & & & a_2 & & & & a_1 & \\ a_2 & & a_2 & \to & a_1 & & a_1 & \to & a_2 & & a_2. \\ & a_1 & & & & a_2 & & & & a_1 & \end{array}$$

The circular permutation

$$\begin{array}{ccc} & a_1 & \\ a_2 & & a_1 \\ & a_3 & \end{array}$$

has period 4, since we don't return to it until we have made a complete revolution (four position shifts). The period d of a circular n-permutation satisfies $1 \leq d \leq n$ and $d \mid n$, since period d implies that a particular pattern is repeated n/d times. We can consider a circular permutation as a linear string of symbols in which the first symbol is regarded as following the last symbol. Thus, a_1, a_2, a_1, a_2 corresponds to the first circular permutation just considered. Shifting, we get the string a_2, a_1, a_2, a_1; one more shift gets us back to a_1, a_2, a_1, a_2. The string

$$a_1, a_2, a_3, a_1, a_2, a_3$$

corresponds to a circular 6-permutation of period 3. Shifting three times we get

$$a_1, a_2, a_3, a_1, a_2, a_3 \rightarrow a_3, a_1, a_2, a_3, a_1, a_2 \rightarrow a_2, a_3, a_1, a_2, a_3, a_1 \rightarrow$$
$$a_1, a_2, a_3, a_1, a_2, a_3,$$

and we are back to the original string for the first time. In general, a circular n-permutation of period d corresponds in this way to exactly d different linear strings, each of period d.

Let $h(n)$ be the number of circular n-words possible using the symbols $a_1, a_2, \ldots, a_k$.[8] For m a positive integer, let $f(m)$ equal the number of strings of length m possible using the symbols $a_1, a_2, \ldots, a_k$. Since each string has a period d, where $d \mid n$, it follows that

$$h(n) = \sum_{\{d:d|n\}} \frac{f(d)}{d}. \tag{6.31}$$

Therefore, if we can calculate the number of strings of length n of each possible period d, then we can calculate $h(n)$. Let

$$g(m) = \sum_{\{e:e|m\}} f(e).$$

Then $g(m)$ is the total number of strings of length m, and so $g(m) = k^m$. By classical Möbius inversion (i.e. Theorem 6.6.4) we get

$$f(m) = \sum_{\{e:e|m\}} \mu(m/e)g(e) = \sum_{\{e:e|m\}} \mu(m/e)k^e. \tag{6.32}$$

Using (6.32) in (6.31), we obtain

$$\begin{aligned} h(n) &= \sum_{\{d:d|n\}} \frac{f(d)}{d} \\ &= \sum_{\{d:d|n\}} \frac{1}{d} \sum_{\{e:e|d\}} \mu(d/e)k^e \\ &= \sum_{\{e:e|n\}} \left(\sum_{\{m:m|n/e\}} \frac{1}{me}\mu(m) \right) k^e \\ &\quad \text{(since } e \mid d \text{ and } d \mid n \text{, we have } d = me, \\ &\quad \text{where } me \mid n \text{ and so } m \mid n/e) \end{aligned}$$

[8] $h(n)$ depends on k, but this is not reflected in our notation.

$$\begin{aligned}
&= \sum_{\{e:e|n\}} \left(\sum_{\{r:r|n/e\}} \frac{r}{n}\mu((n/e)/r) \right) k^e \\
&= \sum_{\{e:e|n\}} \frac{\phi(n/e)}{n} k^e \\
&= \frac{1}{n} \sum_{\{e:e|n\}} \phi(n/e) k^e.
\end{aligned}$$

Therefore, the number of circular n-words that can be made from an alphabet of size k equals

$$\frac{1}{n} \sum_{\{e:e|n\}} \phi(n/e) k^e.$$

□

6.7 Exercises

1. Find the number of integers between 1 and 10,000 inclusive that are not divisible by 4, 5, or 6.

2. Find the number of integers between 1 and 10,000 inclusive that are not divisible by 4, 6, 7, or 10.

3. Find the number of integers between 1 and 10,000 that are neither perfect squares nor perfect cubes.

4. Determine the number of 12-combinations of the multiset

$$S = \{4 \cdot a, 3 \cdot b, 4 \cdot c, 5 \cdot d\}.$$

5. Determine the number of 10-combinations of the multiset

$$S = \{\infty \cdot a, 4 \cdot b, 5 \cdot c, 7 \cdot d\}.$$

6. A bakery sells chocolate, cinnamon, and plain doughnuts and at a particular time has 6 chocolate, 6 cinnamon, and 3 plain. If a box contains 12 doughnuts, how many different options are there for a box of doughnuts?

7. Determine the number of solutions of the equation $x_1 + x_2 + x_3 + x_4 = 14$ in nonnegative integers x_1, x_2, x_3, and x_4 not exceeding 8.

8. Determine the number of solutions of the equation $x_1 + x_2 + x_3 + x_4 + x_5 = 14$ in positive integers x_1, x_2, x_3, x_4 and x_5 not exceeding 5.

9. Determine the number of integral solutions of the equation

$$x_1 + x_2 + x_3 + x_4 = 20$$

that satisfy

$$1 \leq x_1 \leq 6,\ 0 \leq x_2 \leq 7,\ 4 \leq x_3 \leq 8,\ 2 \leq x_4 \leq 6.$$

10. Let S be a multiset with k distinct objects with given repetition numbers $n_1, n_2, \ldots, n_k$, respectively. Let r be a positive integer such that there is at least one r-combination of S. Show that, in applying the inclusion–exclusion principle to determine the number of r-combinations of S, one has $A_1 \cap A_2 \cap \cdots \cap A_k = \emptyset$.

11. Determine the number of permutations of $\{1, 2, \ldots, 8\}$ in which no even integer is in its natural position.

12. Determine the number of permutations of $\{1, 2, \ldots, 8\}$ in which exactly four integers are in their natural positions.

13. Determine the number of permutations of $\{1, 2, \ldots, 9\}$ in which at least one odd integer is in its natural position.

14. Determine a general formula for the number of permutations of the set $\{1, 2, \ldots, n\}$ in which exactly k integers are in their natural positions.

15. At a party, seven gentlemen check their hats. In how many ways can their hats be returned so that

    (a) no gentleman receives his own hat?
    (b) at least one of the gentlemen receives his own hat?
    (c) at least two of the gentlemen receive their own hats?

16. Use combinatorial reasoning to derive the identity

$$\begin{aligned} n! &= \binom{n}{0}D_n + \binom{n}{1}D_{n-1} + \binom{n}{2}D_{n-2} \\ &\quad + \cdots + \binom{n}{n-1}D_1 + \binom{n}{n}D_0. \end{aligned}$$

(Here, D_0 is defined to be 1.)

17. Determine the number of permutations of the multiset

$$S = \{3 \cdot a, 4 \cdot b, 2 \cdot c\},$$

where, for each type of letter, the letters of the same type do not appear consecutively. (Thus *abbbbcaca* is not allowed, but *abbbacacb* is.)

18. Verify the factorial formula

$$n! = (n-1)((n-2)! + (n-1)!), \qquad (n = 2, 3, 4, \ldots).$$

19. Using the evaluation of the derangement numbers as given in Theorem 6.3.1, provide a proof of the relation

$$D_n = (n-1)(D_{n-2} + D_{n-1}), \qquad (n = 3, 4, 5, \ldots).$$

20. Starting from the formula $D_n = nD_{n-1} + (-1)^n$, $(n = 2, 3, 4, \ldots)$, give a proof of Theorem 6.3.1.

21. Prove that D_n is an even number if and only if n is an odd number.

22. Show that the numbers Q_n of Section 6.5 can be rewritten in the form

$$Q_n = (n-1)!\left(n - \frac{n-1}{1!} + \frac{n-2}{2!} - \frac{n-3}{3!} + \cdots + \frac{(-1)^{n-1}}{(n-1)!}\right).$$

23. (Continuation of Exercise 22.) Use the identity

$$(-1)^k \frac{n-k}{k!} = (-1)^k \frac{n}{k!} + (-1)^{k-1} \frac{1}{(k-1)!}$$

to prove that $Q_n = D_n + D_{n-1}$, $(n = 2, 3, \ldots)$.

24. What is the number of ways to place six nonattacking rooks on the 6-by-6 boards with forbidden positions as shown?

(a)

×	×				
		×	×		
				×	×

(b)

×	×				
×	×				
		×	×		
		×	×		
				×	×
				×	×

(c)

×	×				
	×	×			
		×			
				×	×
					×

25. Count the permutations $i_1i_2i_3i_4i_5i_6$ of $\{1, 2, 3, 4, 5, 6\}$, where $i_1 \neq 1, 5$; $i_3 \neq 2, 3, 5$; $i_4 \neq 4$; and $i_6 \neq 5, 6$.

26. Count the permutations $i_1i_2i_3i_4i_5i_6$ of $\{1, 2, 3, 4, 5, 6\}$, where $i_1 \neq 1, 2, 3$; $i_2 \neq 1$; $i_3 \neq 1$; $i_5 \neq 5, 6$; and $i_6 \neq 5, 6$.

27. A carousel has eight seats, each representing a different animal. Eight girls are seated on the carousel facing forward (each girl looks at another girl's back). In how many ways can the girls change seats so that each has a different girl in front of her? How does the problem change if all the seats are identical?

28. A carousel has eight seats, each representing a different animal. Eight boys are seated on the carousel but facing inward, so that each boy faces another (each boy looks at another boy's front). In how many ways can the boys change seats so that each faces a different boy? How does the problem change if all the seats are identical?

29. A subway has six stops on its route from its base location. There are 10 people on the subway as it departs its base location. Each person exits the subway at one of its six stops, and at each stop at least one person exits. In how many ways can this happen?

30. How many circular permutations are there of the multiset

$$\{3 \cdot a, 4 \cdot b, 2 \cdot c, 1 \cdot d\},$$

where, for each type of letter, all letters of that type do not appear consecutively?

31. How many circular permutations are there of the multiset

$$\{2 \cdot a, 3 \cdot b, 4 \cdot c, 5 \cdot d\},$$

where, for each type of letter, all letters of that type do not appear consecutively?

32. Let n be a positive integer and let $p_1, p_2, \ldots, p_k$ be all the different prime numbers that divide n. Consider the Euler function ϕ defined by

$$\phi(n) = |\{k : 1 \leq k \leq n, \mathrm{GCD}\{k, n\} = 1\}|.$$

Use the inclusion–exclusion principle to show that

$$\phi(n) = n\prod_{i=1}^{k}(1 - \frac{1}{p_i}).$$

33. * Let n and k be positive integers with $k \leq n$. Let $a(n,k)$ be the number of ways to place k nonattacking rooks on an n-by-n board in which the positions $(1,1),(2,2),\ldots,(n,n)$ and $(1,2),(2,3),\ldots,(n-1,n),(n,1)$ are forbidden. For example, if $n = 6$ the board is

×	×				
	×	×			
		×	×		
			×	×	
				×	×
×					×

.

prove that

$$a(n,k) = \frac{2n}{2n-k}\binom{2n-k}{k}.$$

Note that $a(n,k)$ is the number of ways to choose k children from a group of $2n$ children arranged in a circle so that no two consecutive children are chosen.

34. Prove that the convolution product satisfies the associative law: $f*(g*h) = (f*g)*h$.

35. Consider the linearly ordered set $1 < 2 < \cdots < n$, and let $F : \{1,2,\ldots,n\} \to \Re$ be a function. Let the function $G : \{1,2,\ldots,n\} \to \Re$ be defined by

$$G(m) = \sum_{k=1}^{m} F(k), \quad (1 \leq k \leq n).$$

Apply Möbius inversion to get F in terms of G.

36. Consider the board with forbidden positions as shown:

	×	×	
×			
			×
	×		

Use formula (6.28) to compute the number of ways to place four nonattacking rooks on this board.

37. Consider the partially ordered set $(\mathcal{P}(X_3), \subseteq)$ of subsets of $\{1,2,3\}$ partially ordered by containment. Let a function f in $\mathcal{F}(\mathcal{P}(X))$ be defined by

$$f(A,B) = \begin{cases} 1, & \text{if } A = B, \\ 2, & \text{if } A \subset B \text{ and } |B| - |A| = 1, \\ 1, & \text{if } A \subset B \text{ and } |B| - |A| = 2, \\ -1, & \text{if } A \subset B \text{ and } |B| - |A| = 3. \end{cases}$$

Find the inverse of f with respect to the convolution product.

38. Recall the partially ordered set Π_n of all partitions of $\{1, 2, \ldots n\}$, where the partial order is that of refinement (see Exercise 47 of Chapter 4). Determine the Möbius functions of Π_3 and Π_4.

39. Let n be a positive integer and consider the partially ordered set $(X_n, \mid)$, where $X_n = \{1, 2, \ldots, n\}$ and the partial order is that of divisibility. Let a and b be positive integers in X_n, where $a|b$. Prove that $\mu(a,b) = \mu(1, b/a)$.

40. Consider the multiset $X = \{n_1 \cdot a_1, n_2 \cdot a_2, \ldots, n_k \cdot a_k\}$ of k distinct elements with positive repetition numbers $n_1, n_2, \ldots, n_k$. We introduce a partial order on the combinations of X by stating the following relationship: If $A = \{p_1 \cdot a_1, p_2 \cdot a_2, \ldots, p_k \cdot a_k\}$ and $B = \{q_1 \cdot a_1, q_2 \cdot a_2, \ldots, q_k \cdot a_k\}$ are combinations of X, then $A \leq B$ provided that $p_i \leq q_i$ for $i = 1, 2, \ldots, k$. Prove that this statement defines a partial order on X and then compute its Möbius function.