10

Renewals

Summary. A renewal process is a recurrent-event process with independent identically distributed interevent times. The asymptotic behaviour of a renewal process is described by the renewal theorem and the elementary renewal theorem, and the key renewal theorem is often useful. The waiting-time paradox leads to a discussion of excess and current lifetimes, and their asymptotic distributions are found. Other renewal-type processes are studied, including alternating and delayed renewal processes, and the use of renewal is illustrated in applications to Markov chains and age-dependent branching processes. The asymptotic behaviour of renewal–reward processes is studied, and Little's formula is proved.

10.1 The renewal equation

We saw in Section 8.3 that renewal processes provide attractive models for many natural phenomena. Recall their definition.

(1) **Definition.** A **renewal process** $N = \{N(t) : t \geq 0\}$ is a process such that

$$N(t) = \max\{n : T_n \leq t\}$$

where $T_0 = 0$, $T_n = X_1 + X_2 + \cdots + X_n$ for $n \geq 1$, and $\{X_i\}$ is a sequence of independent identically distributed non-negative† random variables.

We commonly think of a renewal process $N(t)$ as representing the number of occurrences of some event in the time interval $[0, t]$; the event in question might be the arrival of a person or particle, or the failure of a light bulb. With this in mind, we shall speak of T_n as the 'time of the nth arrival' and X_n as the 'nth interarrival time'. We shall try to use the notation of (1) consistently throughout, denoting by X and T a typical interarrival time and a typical arrival time of the process N.

When is N an honest process, which is to say that $N(t) < \infty$ almost surely (see Definition (6.8.18))?

(2) **Theorem.** $\mathbb{P}(N(t) < \infty) = 1$ *for all t if and only if* $\mathbb{E}(X_1) > 0$.

†But soon we will impose the stronger condition that the X_i be *strictly* positive.

This amounts to saying that N is honest if and only if the interarrival times are not concentrated at zero. The proof is simple and relies upon the following important observation:

(3) $$N(t) \geq n \quad \text{if and only if} \quad T_n \leq t.$$

We shall make repeated use of (3). It provides a link between $N(t)$ and the sum T_n of independent variables; we know a lot about such sums already.

Proof of (2). Since the X_i are non-negative, if $\mathbb{E}(X_1) = 0$ then $\mathbb{P}(X_i = 0) = 1$ for all i. Therefore

$$\mathbb{P}\big(N(t) = \infty\big) = 1 \quad \text{for all } t > 0.$$

Conversely, suppose that $\mathbb{E}(X_1) > 0$. There exists $\epsilon > 0$ such that $\mathbb{P}(X_1 > \epsilon) = \delta > 0$. Let $A_i = \{X_i > \epsilon\}$, and let $A = \{X_i > \epsilon \text{ i.o.}\} = \limsup A_i$ be the event that infinitely many of the X_i exceed ϵ. We have that

$$\mathbb{P}(A^c) = \mathbb{P}\bigg(\bigcup_m \bigcap_{n>m} A_n^c\bigg) \leq \sum_m \lim_{n\to\infty} (1-\delta)^{n-m} = \sum_m 0 = 0.$$

Therefore, by (3),

$$\mathbb{P}\big(N(t) = \infty\big) = \mathbb{P}(T_n \leq t \text{ for all } n) \leq \mathbb{P}(A^c) = 0. \qquad \blacksquare$$

Thus N is honest if and only if X_1 is *not* concentrated at 0. Henceforth we shall assume not only that $\mathbb{P}(X_1 = 0) < 1$, but also impose the stronger condition that $\mathbb{P}(X_1 = 0) = 0$. That is, *we consider only the case when the X_i are strictly positive in that* $\mathbb{P}(X_1 > 0) = 1$.

It is easy in principle to find the distribution of $N(t)$ in terms of the distribution of a typical interarrival time. Let F be the distribution function of X_1, and let F_k be the distribution function of T_k.

(4) Lemma†. *We have that $F_1 = F$ and $F_{k+1}(x) = \int_0^x F_k(x-y)\,dF(y)$ for $k \geq 1$.*

Proof. Clearly $F_1 = F$. Also $T_{k+1} = T_k + X_{k+1}$, and Theorem (4.8.1) gives the result when suitably rewritten for independent variables of general type. ■

(5) Lemma. *We have that $\mathbb{P}(N(t) = k) = F_k(t) - F_{k+1}(t)$.*

Proof. $\{N(t) = k\} = \{N(t) \geq k\} \setminus \{N(t) \geq k+1\}$. Now use (3). ■

We shall be interested largely in the expected value of $N(t)$.

(6) Definition. The **renewal function** m is given by $m(t) = \mathbb{E}(N(t))$.

Again, it is easy to find m in terms of the F_k.

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†Readers of Section 5.6 may notice that the statement of this lemma violates our notation for the domain of an integral. We adopt the convention that expressions of the form $\int_a^b g(y)\,dF(y)$ denote integrals over the half-open interval $(a, b]$, with the left endpoint excluded.

(7) Lemma. *We have that* $m(t) = \sum_{k=1}^{\infty} F_k(t)$.

Proof. Define the indicator variables

$$I_k = \begin{cases} 1 & \text{if } T_k \le t, \\ 0 & \text{otherwise.} \end{cases}$$

Then $N(t) = \sum_{k=1}^{\infty} I_k$ and so

$$m(t) = \mathbb{E}\left(\sum_{k=1}^{\infty} I_k\right) = \sum_{k=1}^{\infty} \mathbb{E}(I_k) = \sum_{k=1}^{\infty} F_k(t). \qquad \blacksquare$$

An alternative approach to the renewal function is by way of conditional expectations and the 'renewal equation'. First note that m is the solution of a certain integral equation.

(8) Lemma. *The renewal function m satisfies the* renewal equation,

$$m(t) = F(t) + \int_0^t m(t-x)\,dF(x). \tag{9}$$

Proof. Use conditional expectation to obtain

$$m(t) = \mathbb{E}(N(t)) = \mathbb{E}\big(\mathbb{E}[N(t) \mid X_1]\big);$$

but, on the one hand,

$$\mathbb{E}\big(N(t) \bigm| X_1 = x\big) = 0 \quad \text{if} \quad t < x$$

since the first arrival occurs after time t. On the other hand,

$$\mathbb{E}\big(N(t) \bigm| X_1 = x\big) = 1 + \mathbb{E}\big(N(t-x)\big) \quad \text{if} \quad t \ge x$$

since the process of arrivals, starting from the epoch of the first arrival, is a copy of N itself. Thus

$$m(t) = \int_0^\infty \mathbb{E}\big(N(t) \bigm| X_1 = x\big)\,dF(x) = \int_0^t [1 + m(t-x)]\,dF(x). \qquad \blacksquare$$

We know from (7) that

$$m(t) = \sum_{k=1}^{\infty} F_k(t)$$

is a solution to the renewal equation (9). Actually, it is the unique solution to (9) which is bounded on finite intervals. This is a consequence of the next lemma. We shall encounter a more general form of (9) later, and it is appropriate to anticipate this now. The more general case involves solutions μ to the *renewal-type equation*

$$\mu(t) = H(t) + \int_0^t \mu(t-x)\,dF(x), \quad t \ge 0, \tag{10}$$

where H is a uniformly bounded function.

(11) Theorem. *The function μ, given by*

$$\mu(t) = H(t) + \int_0^t H(t-x)\,dm(x),$$

is a solution of the renewal-type equation (10). *If H is bounded on finite intervals then μ is bounded on finite intervals and is the unique solution of* (10) *with this property*†.

We shall make repeated use of this result, the proof of which is simple.

Proof. If $h : [0,\infty) \to \mathbb{R}$, define the functions $h * m$ and $h * F$ by

$$(h*m)(t) = \int_0^t h(t-x)\,dm(x), \quad (h*F)(t) = \int_0^t h(t-x)\,dF(x),$$

whenever these integrals exist. The operation $*$ is a type of convolution; do not confuse it with the related but different convolution operator of Sections 3.8 and 4.8. It can be shown that

$$(h*m)*F = h*(m*F),$$

and so we write $h*m*F$ for this double convolution. Note also that:

$$m = F + m*F \quad \text{by (9)}, \tag{12}$$

$$F_{k+1} = F_k * F = F * F_k \quad \text{by (4)}. \tag{13}$$

Using this notation, μ can be written as $\mu = H + H*m$. Convolve with F and use (12) to find that

$$\begin{aligned}\mu * F &= H*F + H*m*F = H*F + H*(m-F)\\ &= H*m = \mu - H,\end{aligned}$$

and so μ satisfies (10).

If H is bounded on finite intervals then

$$\begin{aligned}\sup_{0\le t\le T} |\mu(t)| &\le \sup_{0\le t\le T} |H(t)| + \sup_{0\le t\le T}\left|\int_0^t H(t-x)\,dm(x)\right|\\ &\le [1+m(T)] \sup_{0\le t\le T} |H(t)| < \infty,\end{aligned}$$

and so μ is indeed bounded on finite intervals; we have used the finiteness of m here (see Problem (10.6.1b)). To show that μ is the unique such solution of (10), suppose that μ_1 is another bounded solution and write $\delta(t) = \mu(t) - \mu_1(t)$; δ is a bounded function. Also $\delta = \delta * F$ by (10). Iterate this equation and use (13) to find that $\delta = \delta * F_k$ for all $k \ge 1$, which implies that

$$|\delta(t)| \le F_k(t) \sup_{0\le u\le t} |\delta(u)| \quad \text{for all } k \ge 1.$$

†Think of the integral in (11) as $\int H(t-x)m'(x)\,dx$ if you are unhappy about its present form.

Let $k \to \infty$ to find that $|\delta(t)| = 0$ for all t, since

$$F_k(t) = \mathbb{P}\big(N(t) \ge k\big) \to 0 \quad \text{as} \quad k \to \infty$$

by (2). The proof is complete. ■

The method of Laplace–Stieltjes transforms is often useful in renewal theory (see Definition (15) of Appendix I). For example, we can transform (10) to obtain the formula

$$\mu^*(\theta) = \frac{H^*(\theta)}{1 - F^*(\theta)} \quad \text{for} \quad \theta \neq 0,$$

an equation which links the Laplace–Stieltjes transforms of μ, H, and F. In particular, setting $H = F$, we find from (8) that

(14)
$$m^*(\theta) = \frac{F^*(\theta)}{1 - F^*(\theta)},$$

a formula which is directly derivable from (7) and (13). Hence there is a one–one correspondence between renewal functions m and distribution functions F of the interarrival times.

(15) Example. Poisson process. This is the only Markovian renewal process, and has exponentially distributed interarrival times with some parameter λ. The epoch T_k of the kth arrival is distributed as $\Gamma(\lambda, k)$; Lemma (7) gives that

$$m(t) = \sum_{k=1}^{\infty} \int_0^t \frac{\lambda(\lambda s)^{k-1} e^{-\lambda s}}{(k-1)!}\, ds = \int_0^t \lambda\, ds = \lambda t.$$

Alternatively, just remember that $N(t)$ has the Poisson distribution with parameter λt to obtain the same result. ●

Exercises for Section 10.1

1. Prove that $\mathbb{E}(e^{\theta N(t)}) < \infty$ for some strictly positive θ whenever $\mathbb{E}(X_1) > 0$. [Hint: Consider the renewal process with interarrival times $X_k' = \epsilon I_{\{X_k \ge \epsilon\}}$ for some suitable ϵ.]

2. Let N be a renewal process and let W be the waiting time until the length of some interarrival time has exceeded s. That is, $W = \inf\{t : C(t) > s\}$, where $C(t)$ is the time which has elapsed (at time t) since the last arrival. Show that

$$F_W(x) = \begin{cases} 0 & \text{if } x < s, \\ 1 - F(s) + \int_0^s F_W(x-u)\, dF(u) & \text{if } x \ge s, \end{cases}$$

where F is the distribution function of an interarrival time. If N is a Poisson process with intensity λ, show that

$$\mathbb{E}(e^{\theta W}) = \frac{\lambda - \theta}{\lambda - \theta e^{(\lambda-\theta)s}} \qquad \text{for } \theta < \lambda,$$

and $\mathbb{E}(W) = (e^{\lambda s} - 1)/\lambda$. You may find it useful to rewrite the above integral equation in the form of a renewal-type equation.

3. Find an expression for the mass function of $N(t)$ in a renewal process whose interarrival times are: (a) Poisson distributed with parameter λ, (b) gamma distributed, $\Gamma(\lambda, b)$.

4. Let the times between the events of a renewal process N be uniformly distributed on $(0, 1)$. Find the mean and variance of $N(t)$ for $0 \le t \le 1$.

10.2 Limit theorems

We study next the asymptotic behaviour of $N(t)$ and its renewal function $m(t)$ for large values of t. There are four main results here, two for each of N and m. For the renewal process N itself there is a law of large numbers and a central limit theorem; these rely upon the relation (10.1.3), which links N to the partial sums of independent variables. The two results for m deal also with first- and second-order properties. The first asserts that $m(t)$ is approximately linear in t; the second asserts that the gradient of m is asymptotically constant. The proofs are given later in the section.

How does $N(t)$ behave when t is large? Let $\mu = \mathbb{E}(X_1)$ be the mean of a typical interarrival time. Henceforth we assume that $\mu < \infty$.

(1) Theorem. $$\frac{1}{t}N(t) \xrightarrow{\text{a.s.}} \frac{1}{\mu} \quad \textit{as } t \to \infty.$$

(2) Theorem. *If* $\sigma^2 = \operatorname{var}(X_1)$ *satisfies* $0 < \sigma < \infty$, *then*

$$\frac{N(t) - t/\mu}{\sqrt{t\sigma^2/\mu^3}} \xrightarrow{\text{D}} N(0,1) \quad \textit{as} \quad t \to \infty.$$

It is not quite so easy to find the asymptotic behaviour of the renewal function.

(3) Elementary renewal theorem. $\frac{1}{t}m(t) \to \frac{1}{\mu}$ *as* $t \to \infty$.

The second-order properties of m are hard to find, and we require a preliminary definition.

(4) Definition. Call a random variable X and its distribution F_X **arithmetic with span** λ (> 0) if X takes values in the set $\{m\lambda : m = 0, \pm 1, \dots\}$ with probability 1, and λ is maximal with this property.

If the interarrival times of N are arithmetic, with span λ say, then so is T_k for each k. In this case $m(t)$ may be discontinuous at values of t which are multiples of λ, and this affects the second-order properties of m.

(5) Renewal theorem. *If* X_1 *is not arithmetic then*

$$m(t+h) - m(t) \to \frac{h}{\mu} \quad \textit{as} \quad t \to \infty \quad \textit{for all } h. \tag{6}$$

If X_1 *is arithmetic with span* λ, *then* (6) *holds whenever* h *is a multiple of* λ.

It is appropriate to make some remarks about these theorems before we set to their proofs. Theorems (1) and (2) are straightforward, and use the law of large numbers and the central limit theorem for partial sums of independent sequences. It is perhaps surprising that (3) is harder to demonstrate than (1) since it concerns only the mean value of $N(t)$; it has a suitably probabilistic proof which uses the method of truncation, a technique which proved useful in the proof of the strong law (7.5.1). On the other hand, the proof of (5) is difficult. The usual method of proof is largely an exercise in solving integral equations, and is not appropriate for inclusion here (see Feller 1971, p. 360). There is an alternative proof which is short, beautiful, and probabilistic, and uses 'coupling' arguments related to those in the proof of the ergodic theorem for discrete-time Markov chains. This method requires some results which appear later in this chapter, and so we defer a sketch of the argument until Example (10.4.21). In the case of arithmetic interarrival times, (5) is essentially the same as Theorem (5.2.24), a result about *integer-valued* random variables. There is an apparently more general form of (5) which is deducible from (5). It is called the 'key renewal theorem' because of its many applications.

In the rest of this chapter we shall commonly assume that the interarrival times are *not* arithmetic. Similar results often hold in the arithmetic case, but they are usually more complicated to state.

(7) Key renewal theorem. *If* $g : [0, \infty) \to [0, \infty)$ *is such that*:

(a) $g(t) \geq 0$ *for all* t,

(b) $\int_0^\infty g(t)\,dt < \infty$,

(c) g *is a non-increasing function*,

then

$$\int_0^t g(t-x)\,dm(x) \to \frac{1}{\mu}\int_0^\infty g(x)\,dx \quad \text{as} \quad t \to \infty$$

whenever X_1 *is not arithmetic.*

In order to deduce this theorem from the renewal theorem (5), first prove it for indicator functions of intervals, then for step functions, and finally for limits of increasing sequences of step functions. We omit the details.

Proof of (1). This is easy. Just note that

$$T_{N(t)} \leq t < T_{N(t)+1} \quad \text{for all } t. \tag{8}$$

Therefore, if $N(t) > 0$,

$$\frac{T_{N(t)}}{N(t)} \leq \frac{t}{N(t)} < \frac{T_{N(t)+1}}{N(t)+1}\left(1 + \frac{1}{N(t)}\right).$$

As $t \to \infty$, $N(t) \xrightarrow{\text{a.s.}} \infty$, and the strong law of large numbers gives

$$\mu \leq \lim_{t\to\infty}\left(\frac{t}{N(t)}\right) \leq \mu \quad \text{almost surely.}$$

■

Proof of (2). This is Problem (10.5.3). ■

In preparation for the proof of (3), we recall an important definition. Let M be a random variable taking values in the set $\{1, 2, \dots\}$. We call the random variable M a *stopping time* with respect to the sequence X_i of interarrival times if, for all $m \geq 1$, the event $\{M \leq m\}$ belongs to the σ-field of events generated by $X_1, X_2, \dots, X_m$. Note that $M = N(t) + 1$ is a stopping time for the X_i, since

$$\{M \leq m\} = \{N(t) \leq m - 1\} = \left\{\sum_{i=1}^m X_i > t\right\},$$

which is an event defined in terms of $X_1, X_2, \ldots, X_m$. The random variable $N(t)$ is *not* a stopping time.

(9) Lemma. Wald's equation. *Let $X_1, X_2, \ldots$ be independent identically distributed random variables with finite mean, and let M be a stopping time with respect to the X_i satisfying $\mathbb{E}(M) < \infty$. Then*

$$\mathbb{E}\left(\sum_{i=1}^{M} X_i\right) = \mathbb{E}(X_1)\mathbb{E}(M).$$

Applying Wald's equation to the sequence of interarrival times together with the stopping time $M = N(t) + 1$, we obtain

$$\mathbb{E}(T_{N(t)+1}) = \mu[m(t) + 1]. \tag{10}$$

Wald's equation may seem trite, but this is far from being the case. For example, it is not generally true that $\mathbb{E}(T_{N(t)}) = \mu m(t)$; the forthcoming Example (10.3.2) is an example of some of the dangers here.

Proof of Wald's equation (9). The basic calculation is elementary. Just note that

$$\sum_{i=1}^{M} X_i = \sum_{i=1}^{\infty} X_i I_{\{M \geq i\}},$$

so that (using dominated convergence or Exercise (5.6.2))

$$\mathbb{E}\left(\sum_{i=1}^{M} X_i\right) = \sum_{i=1}^{\infty} \mathbb{E}(X_i I_{\{M \geq i\}}) = \sum_{i=1}^{\infty} \mathbb{E}(X_i)\mathbb{P}(M \geq i) \qquad \text{by independence,}$$

since $\{M \geq i\} = \{M \leq i-1\}^{\mathrm{c}}$, an event definable in terms of $X_1, X_2, \ldots, X_{i-1}$ and therefore independent of X_i. The final sum equals

$$\mathbb{E}(X_1) \sum_{i=1}^{\infty} \mathbb{P}(M \geq i) = \mathbb{E}(X_1)\mathbb{E}(M). \qquad \blacksquare$$

Proof of (3). Half of this is easy. We have from (8) that $t < T_{N(t)+1}$; take expectations of this and use (10) to obtain

$$\frac{m(t)}{t} > \frac{1}{\mu} - \frac{1}{t}.$$

Letting $t \to \infty$, we obtain

$$\liminf_{t \to \infty} \frac{1}{t} m(t) \geq \frac{1}{\mu}. \tag{11}$$

We may be tempted to proceed as follows in order to bound $m(t)$ above. We have from (8) that $T_{N(t)} \leq t$, and so

$$t \geq \mathbb{E}(T_{N(t)}) = \mathbb{E}\big(T_{N(t)+1} - X_{N(t)+1}\big) = \mu[m(t) + 1] - \mathbb{E}(X_{N(t)+1}). \tag{12}$$

The problem is that $X_{N(t)+1}$ depends on $N(t)$, and so $\mathbb{E}(X_{N(t)+1}) \neq \mu$ in general. To cope with this, truncate the X_i at some $a > 0$ to obtain a new sequence

$$X_j^a = \begin{cases} X_j & \text{if } X_j < a, \\ a & \text{if } X_j \geq a. \end{cases}$$

Now consider the renewal process N^a with associated interarrival times $\{X_j^a\}$. Apply (12) to N^a, noting that $\mu^a = \mathbb{E}(X_j^a) \leq a$, to obtain

(13) $$t \geq \mu^a[\mathbb{E}(N^a(t)) + 1] - a.$$

However, $X_j^a \leq X_j$ for all j, and so $N^a(t) \geq N(t)$ for all t. Therefore

$$\mathbb{E}(N^a(t)) \geq \mathbb{E}(N(t)) = m(t)$$

and (13) becomes

$$\frac{m(t)}{t} \leq \frac{1}{\mu^a} + \frac{a - \mu^a}{\mu^a t}.$$

Let $t \to \infty$ to obtain

$$\limsup_{t\to\infty} \frac{1}{t} m(t) \leq \frac{1}{\mu^a};$$

now let $a \to \infty$ and use monotone convergence (5.6.12) to find that $\mu^a \to \mu$, and therefore

$$\limsup_{t\to\infty} \frac{1}{t} m(t) \leq \frac{1}{\mu}.$$

Combine this with (11) to obtain the result. ■

Exercises for Section 10.2

1. Planes land at Heathrow airport at the times of a renewal process with interarrival time distribution function F. Each plane contains a random number of people with a given common distribution and finite mean. Assuming as much independence as usual, find an expression for the rate of arrival of passengers over a long time period.

2. Let $Z_1, Z_2, \dots$ be independent identically distributed random variables with mean 0 and finite variance σ^2, and let $T_n = \sum_{i=1}^n Z_i$. Let M be a finite stopping time with respect to the Z_i such that $\mathbb{E}(M) < \infty$. Show that $\text{var}(T_M) = \mathbb{E}(M)\sigma^2$.

3. Show that $\mathbb{E}(T_{N(t)+k}) = \mu(m(t)+k)$ for all $k \geq 1$, but that it is not generally true that $\mathbb{E}(T_{N(t)}) = \mu m(t)$.

4. Show that, using the usual notation, the family $\{N(t)/t : 0 \leq t < \infty\}$ is uniformly integrable. How might one make use of this observation?

5. Consider a renewal process N having interarrival times with moment generating function M, and let T be a positive random variable which is independent of N. Find $\mathbb{E}(s^{N(T)})$ when:

(a) T is exponentially distributed with parameter ν,

(b) N is a Poisson process with intensity λ, in terms of the moment generating function of T. What is the distribution of $N(T)$ in this case, if T has the gamma distribution $\Gamma(\nu, b)$?

10.3 Excess life

Suppose that we begin to observe a renewal process N at some epoch t of time. A certain number $N(t)$ of arrivals have occurred by then, and the next arrival will be that numbered $N(t)+1$. That is to say, we have begun our observation at a point in the random interval $I_t = [T_{N(t)}, T_{N(t)+1})$, the endpoints of which are arrival times.

(1) Definition.

(a) The **excess lifetime** at t is $E(t) = T_{N(t)+1} - t$.

(b) The **current lifetime** (or **age**) at t is $C(t) = t - T_{N(t)}$.

(c) The **total lifetime** at t is $D(t) = E(t) + C(t) = X_{N(t)+1}$.

That is, $E(t)$ is the time which elapses before the next arrival, $C(t)$ is the elapsed time since the last arrival (with the convention that the zeroth arrival occurs at time 0), and $D(t)$ is the length of the interarrival time which contains t (see Figure 8.1 for a diagram of these random variables).

(2) Example. Waiting time paradox. Suppose that N is a Poisson process with parameter λ. How large is $\mathbb{E}(E(t))$? Consider the two following lines of reasoning.

(A) N is a Markov chain, and so the distribution of $E(t)$ does not depend on the arrivals prior to time t. Thus $E(t)$ has the same mean as $E(0) = X_1$, and so $\mathbb{E}(E(t)) = \lambda^{-1}$.

(B) If t is fairly large, then on average it lies near the midpoint of the interarrival interval I_t which contains it. That is

$$\mathbb{E}(E(t)) \simeq \frac{1}{2}\mathbb{E}\big(T_{N(t)+1} - T_{N(t)}\big) = \frac{1}{2}\mathbb{E}(X_{N(t)+1}) = \frac{1}{2\lambda}.$$

These arguments cannot both be correct. The reasoning of (B) is false, in that $X_{N(t)+1}$ does *not* have mean λ^{-1}; we have already observed this after (10.2.12). In fact, $X_{N(t)+1}$ is a very special interarrival time; longer intervals have a higher chance of catching t in their interiors than small intervals. In Problem (10.6.5) we shall see that $\mathbb{E}(X_{N(t)+1}) = (2 - e^{-\lambda t})/\lambda$. For this process, $E(t)$ and $C(t)$ are independent for any t; this property holds for no other renewal process with non-arithmetic interarrival times. ●

Now we find the distribution of the excess lifetime $E(t)$ for a general renewal process.

(3) Theorem. *The distribution function of the excess life $E(t)$ is given by*

$$\mathbb{P}\big(E(t) \le y\big) = F(t+y) - \int_0^t [1 - F(t+y-x)]\,dm(x).$$

Proof. Condition on X_1 in the usual way to obtain

$$\mathbb{P}\big(E(t) > y\big) = \mathbb{E}\big[\mathbb{P}\big(E(t) > y \,\big|\, X_1\big)\big].$$

However, you will see after a little thought that

$$\mathbb{P}\big(E(t) > y \,\big|\, X_1 = y\big) = \begin{cases} \mathbb{P}\big(E(t-x) > y\big) & \text{if } x \le t, \\ 0 & \text{if } t < x \le t+y, \\ 1 & \text{if } x > t+y, \end{cases}$$

since $E(t) > y$ if and only if no arrivals occur in $(t, t+y]$. Thus

$$\mathbb{P}\big(E(t) > y\big) = \int_0^\infty \mathbb{P}\big(E(t) > y \,\big|\, X_1 = x\big)\, dF(x)$$
$$= \int_0^t \mathbb{P}\big(E(t-x) > y\big)\, dF(x) + \int_{t+y}^\infty dF(x).$$

So $\mu(t) = \mathbb{P}(E(t) > y)$ satisfies (10.1.10) with $H(t) = 1 - F(t+y)$; use Theorem (10.1.11) to see that

$$\mu(t) = 1 - F(t+y) + \int_0^t [1 - F(t+y-x)]\, dm(x)$$

as required. ∎

(4) Corollary. *The distribution of the current life $C(t)$ is given by*

$$\mathbb{P}\big(C(t) \geq y\big) = \begin{cases} 0 & \textit{if } y > t, \\ 1 - F(t) + \displaystyle\int_0^{t-y} [1 - F(t-x)]\, dm(x) & \textit{if } y \leq t. \end{cases}$$

Proof. It is the case that $C(t) \geq y$ if and only if there are no arrivals in $(t-y, t]$. Thus

$$\mathbb{P}\big(C(t) \geq y\big) = \mathbb{P}\big(E(t-y) > y\big) \quad \text{if} \quad y \leq t$$

and the result follows from (3). ∎

Might the renewal process N have stationary increments, in the sense that the distribution of $N(t+s) - N(t)$ depends on s alone when $s \geq 0$? This is true for the Poisson process but fails in general. The reason is simple: generally speaking, the process of arrivals after time t depends on the age t of the process to date. When t is very large, however, it is plausible that the process may forget the date of its inception, thereby settling down into a stationary existence. Thus turns out to be the case. To show this asymptotic stationarity we need to demonstrate that the distribution of $N(t+s) - N(t)$ converges as $t \to \infty$. It is not difficult to see that this is equivalent to the assertion that the distribution of the excess life $E(t)$ settles down as $t \to \infty$, an easy consequence of the key renewal theorem (10.2.7) and Lemma (4.3.4).

(5) Theorem. *If X_1 is not arithmetic and $\mu = \mathbb{E}(X_1) < \infty$ then*

$$\mathbb{P}\big(E(t) \leq y\big) \to \frac{1}{\mu}\int_0^y [1 - F(x)]\, dx \quad \textit{as} \quad t \to \infty.$$

Some difficulties arise if X_1 is arithmetic. For example, if the X_j are concentrated at the value 1 then, as $n \to \infty$,

$$\mathbb{P}\big(E(n+c) \leq \tfrac{1}{2}\big) \to \begin{cases} 1 & \text{if } c = \tfrac{1}{2}, \\ 0 & \text{if } c = \tfrac{1}{4}. \end{cases}$$

Exercises for Section 10.3

1. Suppose that the distribution of the excess lifetime $E(t)$ does not depend on t. Show that the renewal process is a Poisson process.

2. Show that the current and excess lifetime processes, $C(t)$ and $E(t)$, are Markov processes.

3. Suppose that X_1 is non-arithmetic with finite mean μ.

(a) Show that $E(t)$ converges in distribution as $t \to \infty$, the limit distribution function being

$$H(x) = \int_0^x \frac{1}{\mu}[1 - F(y)]\,dy.$$

(b) Show that the rth moment of this limit distribution is given by

$$\int_0^\infty x^r\,dH(x) = \frac{\mathbb{E}(X_1^{r+1})}{\mu(r+1)},$$

assuming that this is finite.

(c) Show that

$$\mathbb{E}(E(t)^r) = \mathbb{E}\big(\{(X_1 - t)^+\}^r\big) + \int_0^t h(t-x)\,dm(x)$$

for some suitable function h to be found, and deduce by the key renewal theorem that $\mathbb{E}(E(t)^r) \to \mathbb{E}(X_1^{r+1})/\{\mu(r+1)\}$ as $t \to \infty$, assuming this limit is finite.

4. Find an expression for the mean value of the excess lifetime $E(t)$ conditional on the event that the current lifetime $C(t)$ equals x.

5. Let $M(t) = N(t) + 1$, and suppose that X_1 has finite non-zero variance σ^2.

(a) Show that $\text{var}(T_{M(t)} - \mu M(t)) = \sigma^2(m(t) + 1)$.

(b) In the non-arithmetic case, show that $\text{var}(M(t))/t \to \sigma^2/\mu^3$ as $t \to \infty$.

10.4 Applications

Here are some examples of the ways in which renewal theory can be applied.

(1) Example. Counters, and their dead periods. In Section 6.8 we used an idealized Geiger counter which was able to register radioactive particles, irrespective of the rate of their arrival. In practice, after the detection of a particle such counters require a certain interval of time in order to complete its registration. These intervals are called 'dead periods'; during its dead periods the counter is locked and fails to register arriving particles. There are two common types of counter.

Type 1. Each detected arrival locks the counter for a period of time, possibly of random length, during which it ignores all arrivals.

Type 2. Each arrival locks the counter for a period of time, possibly of random length, irrespective of whether the counter is already locked or not. The counter registers only those arrivals that occur whilst it is unlocked.

Genuine Geiger counters are of Type 1; this case might also be used to model the process in Example (8.3.1) describing the replacement of light bulbs in rented property when the landlord

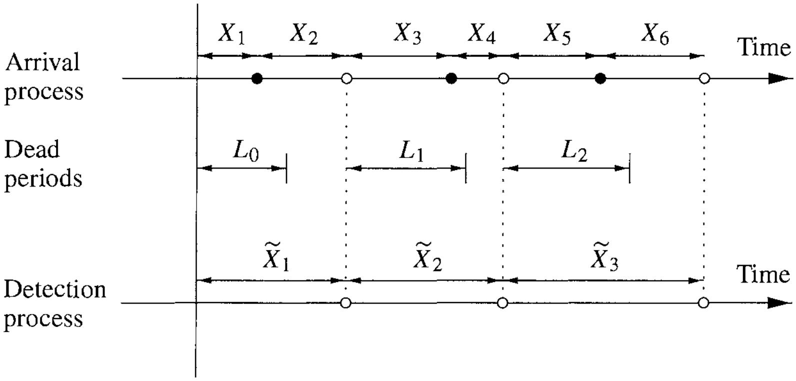

Figure 10.1. Arrivals and detections by a Type I counter; • indicates an undetected arrival, and ◦ indicates a detected arrival.

is either mean or lazy. We consider Type 1 counters briefly; Type 2 counters are harder to analyse, and so are left to the reader.

Suppose that arrivals occur as a renewal process N with renewal function m and interarrival times $X_1, X_2, \dots$ having distribution function F. Let L_n be the length of the dead period induced by the nth detected arrival. It is customary and convenient to suppose that an additional dead period, of length L_0, begins at time $t = 0$; the reason for this will soon be clear. We suppose that $\{L_n\}$ is a family of independent random variables with the common distribution function F_L, where $F_L(0) = 0$. Let $\widetilde{N}(t)$ be the number of arrivals detected by the Type 1 counter by time t. Then $\widetilde{N}$ is a stochastic process with interarrival times $\widetilde{X}_1, \widetilde{X}_2, \dots$ where $\widetilde{X}_{n+1} = L_n + E_n$ and E_n is the excess life of N at the end of the nth dead period (see Figure 10.1). The process $\widetilde{N}$ is *not* in general a renewal process, because the $\widetilde{X}_i$ need be neither independent nor identically distributed. In the very special case when N is a Poisson process, the E_n are independent exponential variables and $\widetilde{N}$ is a renewal process; it is easy to construct other examples for which this conclusion fails.

It is not difficult to find the elapsed time $\widetilde{X}_1$ until the first detection. Condition on L_0 to obtain

$$\mathbb{P}(\widetilde{X}_1 \le x) = \mathbb{E}\big(\mathbb{P}(\widetilde{X}_1 \le x \mid L_0)\big) = \int_0^x \mathbb{P}(L_0 + E_0 \le x \mid L_0 = l)\, dF_L(l).$$

However, $E_0 = E(L_0)$, the excess lifetime of N at L_0, and so

(2) $$\mathbb{P}(\widetilde{X}_1 \le x) = \int_0^x \mathbb{P}\big(E(l) \le x - l\big)\, dF_L(l).$$

Now use Theorem (10.3.3) and the integral representation

$$m(t) = F(t) + \int_0^t F(t-x)\, dm(x),$$

which follows from Theorem (10.1.11), to find that

(3) $$\begin{aligned}\mathbb{P}(\widetilde{X}_1 \le x) &= \int_0^x \left(\int_l^x [1 - F(x-y)]\, dm(y) \right) dF_L(l) \\ &= \int_0^x [1 - F(x-y)] F_L(y)\, dm(y).\end{aligned}$$

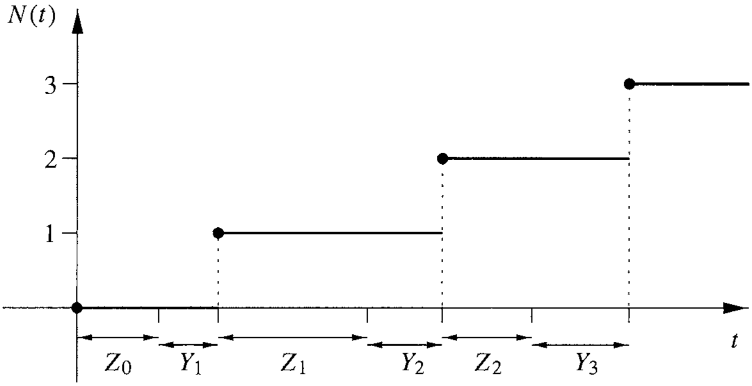

Figure 10.2. An alternating renewal process.

If N is a Poisson process with intensity λ, equation (2) becomes

$$\mathbb{P}(\widetilde{X}_1 \leq x) = \int_0^x (1 - e^{-\lambda(x-l)})\, dF_L(l).$$

$\widetilde{N}$ is now a renewal process, and this equation describes the common distribution of the interarrival times.

If the counter is registering the arrival of radioactive particles, then we may seek an estimate $\widehat{\lambda}$ of the unknown emission rate λ of the source based upon our knowledge of the mean length $\mathbb{E}(L)$ of a dead period and the counter reading $\widetilde{N}(t)$. Assume that the particles arrive in the manner of a Poisson process, and let $\gamma_t = \widetilde{N}(t)/t$ be the density of observed particles. Then

$$\gamma_t \simeq \frac{1}{\mathbb{E}(\widetilde{X}_1)} = \frac{1}{\mathbb{E}(L) + \lambda^{-1}} \quad \text{for large } t,$$

and so $\lambda \simeq \widehat{\lambda}$ where

$$\widehat{\lambda} = \frac{\gamma_t}{1 - \gamma_t \mathbb{E}(L)}.$$ ●

(4) Example. Alternating renewal process. A machine breaks down repeatedly. After the nth breakdown the repairman takes a period of time, length Y_n, to repair it; subsequently the machine runs for a period of length Z_n before it breaks down for the next time. We assume that the Y_m and the Z_n are independent of each other, the Y_m having common distribution function F_Y and the Z_n having common distribution function F_Z. Suppose that the machine was installed at time $t = 0$. Let $N(t)$ be the number of completed repairs by time t (see Figure 10.2). Then N is a renewal process with interarrival times $X_1, X_2, \dots$ given by $X_n = Z_{n-1} + Y_n$ and with distribution function

$$F(x) = \int_0^x F_Y(x - y)\, dF_Z(y).$$

Let $p(t)$ be the probability that the machine is working at time t.

(5) Lemma. *We have that*

$$p(t) = 1 - F_Z(t) + \int_0^t p(t-x)\,dF(x)$$

and hence

$$p(t) = 1 - F_Z(t) + \int_0^t [1 - F_Z(t-x)]\,dm(x)$$

where m is the renewal function of N.

Proof. The probability that the machine is on at time t satisfies

$$\begin{aligned}
p(t) &= \mathbb{P}(\text{on at } t) = \mathbb{P}(Z_0 > t) + \mathbb{P}(\text{on at } t,\ Z_0 \le t)\\
&= \mathbb{P}(Z_0 > t) + \mathbb{E}\big[\mathbb{P}(\text{on at } t,\ Z_0 \le t \mid X_1)\big]\\
&= \mathbb{P}(Z_0 > t) + \int_0^t \mathbb{P}(\text{on at } t \mid X_1 = x)\,dF(x)\\
&\qquad\qquad \text{since } \mathbb{P}(\text{on at } t,\ Z_0 \le t \mid X_1 > t) = 0\\
&= \mathbb{P}(Z_0 > t) + \int_0^t p(t-x)\,dF(x).
\end{aligned}$$

Now use Theorem (10.1.11). ■

(6) Corollary. *If X_1 is not arithmetic then $p(t) \to (1+\rho)^{-1}$ as $t \to \infty$, where $\rho = \mathbb{E}(Y)/\mathbb{E}(Z)$ is the ratio of the mean lengths of a typical repair period and a typical working period.*

Proof. Use the key renewal theorem (10.2.7). ■●

(7) Example. Superposition of renewal processes. Suppose that a room is illuminated by two lights, the bulbs of which fail independently of each other. On failure, they are replaced immediately. Let N_1 and N_2 be the renewal processes describing the occurrences of bulb failures in the first and second lights respectively, and suppose that these are independent processes with the same interarrival time distribution function F. Let $\widetilde{N}$ be the superposition of these two processes; that is, $\widetilde{N}(t) = N_1(t) + N_2(t)$ is the total number of failures by time t. In general $\widetilde{N}$ is not a renewal process. Let us assume for the sake of simplicity that the interarrival times of N_1 and N_2 are not arithmetic.

(8) Theorem. *$\widetilde{N}$ is a renewal process if and only if N_1 and N_2 are Poisson processes.*

Proof. It is easy to see that $\widetilde{N}$ is a Poisson process with intensity 2λ whenever N_1 and N_2 are Poisson processes with intensity λ. Conversely, suppose that $\widetilde{N}$ is a renewal process, and write $\{X_n(1)\}$, $\{X_n(2)\}$, and $\{\widetilde{X}_n\}$ for the interarrival times of N_1, N_2, and $\widetilde{N}$ respectively. Clearly $\widetilde{X}_1 = \min\{X_1(1), X_1(2)\}$, and so the distribution function $\widetilde{F}$ of $\widetilde{X}_1$ satisfies

$$1 - \widetilde{F}(y) = [1 - F(y)]^2. \tag{9}$$

Let $E_1(t)$, $E_2(t)$, and $\widetilde{E}(t)$ denote the excess lifetimes of N_1, N_2, and $\widetilde{N}$ respectively at time t. Clearly, $\widetilde{E}(t) = \min\{E_1(t), E_2(t)\}$, and so

$$\mathbb{P}\big(\widetilde{E}(t) > y\big) = \mathbb{P}\big(E_1(t) > y\big)^2.$$

Let $t \to \infty$ and use Theorem (10.3.5) to obtain

(10)
$$\frac{1}{\widetilde{\mu}} \int_y^\infty [1 - \widetilde{F}(x)]\, dx = \frac{1}{\mu^2} \left(\int_y^\infty [1 - F(x)]\, dx \right)^2$$

where $\widetilde{\mu} = \mathbb{E}(\widetilde{X}_1)$ and $\mu = \mathbb{E}(X_1(1))$. Differentiate (10) and use (9) to obtain

$$\begin{aligned}\frac{1}{\widetilde{\mu}}[1 - \widetilde{F}(y)] &= \frac{2}{\mu^2}[1 - F(y)] \int_y^\infty [1 - F(x)]\, dx \\ &= \frac{1}{\widetilde{\mu}}[1 - F(y)]^2\end{aligned}$$

(this step needs further justification if F is not continuous). Thus

$$1 - F(y) = \frac{2\widetilde{\mu}}{\mu^2} \int_y^\infty [1 - F(x)]\, dx$$

which is an integral equation with solution

$$F(y) = 1 - \exp\left(-\frac{2\widetilde{\mu}}{\mu^2} y \right). \qquad \blacksquare \bullet$$

(11) Example. Delayed renewal process. The Markov chain of Example (8.3.2) indicates that it is sometimes appropriate to allow the first interarrival time X_1 to have a distribution which differs from the shared distribution of $X_2, X_3, \dots$.

(12) Definition. Let $X_1, X_2, \dots$ be independent positive variables such that $X_2, X_3, \dots$ have the same distribution. Let

$$T_0 = 0, \quad T_n = \sum_1^n X_i, \quad N^{\mathrm{d}}(t) = \max\{n : T_n \le t\}.$$

Then N^{d} is called a **delayed** (or **modified**) **renewal process**.

Another example of a delayed renewal process is provided by a variation of the Type 1 counter of (1) with particles arriving in the manner of a Poisson process. It was convenient there to assume that the life of the counter began with a dead period in order that the process $\widetilde{N}$ of detections be a renewal process. In the absence of this assumption $\widetilde{N}$ is a delayed renewal process. The theory of delayed renewal processes is very similar to that of ordinary renewal processes and we do not explore it in detail. The renewal equation (10.1.9) becomes

$$m^{\mathrm{d}}(t) = F^{\mathrm{d}}(t) + \int_0^t m(t - x)\, dF^{\mathrm{d}}(x)$$

where F^{d} is the distribution function of X_1 and m is the renewal function of an ordinary renewal process N whose interarrival times are $X_2, X_3, \dots$. It is left to the reader to check that

(13)
$$m^{\mathrm{d}}(t) = F^{\mathrm{d}}(t) + \int_0^t m^{\mathrm{d}}(t - x)\, dF(x)$$

and

$$m^{\mathrm{d}}(t) = \sum_{k=1}^{\infty} F_k^{\mathrm{d}}(t) \tag{14}$$

where F_k^{d} is the distribution function of $T_k = X_1 + X_2 + \cdots + X_k$ and F is the shared distribution function of $X_2, X_3, \ldots$.

With our knowledge of the properties of m, it is not too hard to show that m^{d} satisfies the renewal theorems. Write μ for $\mathbb{E}(X_2)$.

(15) Theorem. *We have that:*
(a) $\dfrac{1}{t} m^{\mathrm{d}}(t) \to \dfrac{1}{\mu}$ *as* $t \to \infty$.
(b) *If* X_2 *is not arithmetic then*

$$m^{\mathrm{d}}(t+h) - m^{\mathrm{d}}(t) \to \frac{h}{\mu} \quad \text{as} \quad t \to \infty \quad \text{for any } h. \tag{16}$$

If X_2 *is arithmetic with span* λ *then* (16) *remains true whenever* h *is a multiple of* λ.

There is an important special case for the distribution function F^{d}.

(17) Theorem. *The process* N^{d} *has stationary increments if and only if*

$$F^{\mathrm{d}}(y) = \frac{1}{\mu} \int_0^y [1 - F(x)]\, dx. \tag{18}$$

If F^{d} is given by (18), then N^{d} is called a *stationary* (or *equilibrium*) *renewal process*. We should recognize (18) as the asymptotic distribution (10.3.5) of the excess lifetime of the ordinary renewal process N. So the result of (17) is no surprise since N^{d} starts off with this 'equilibrium' distribution. We shall see that in this case $m^{\mathrm{d}}(t) = t/\mu$ for all $t \geq 0$.

Proof of (17). Suppose that N^{d} has stationary increments. Then

$$\begin{aligned} m^{\mathrm{d}}(s+t) &= \mathbb{E}\big([N^{\mathrm{d}}(s+t) - N^{\mathrm{d}}(s)] + N^{\mathrm{d}}(s)\big) \\ &= \mathbb{E}(N^{\mathrm{d}}(t)) + \mathbb{E}(N^{\mathrm{d}}(s)) \\ &= m^{\mathrm{d}}(t) + m^{\mathrm{d}}(s). \end{aligned}$$

By monotonicity, $m^{\mathrm{d}}(t) = ct$ for some $c > 0$. Substitute into (13) to obtain

$$F^{\mathrm{d}}(t) = c \int_0^t [1 - F(x)]\, dx$$

and let $t \to \infty$ to obtain $c = 1/\mu$.

Conversely, suppose that F^{d} is given by (18). Substitute (18) into (13) and use the method of Laplace–Stieltjes transforms to deduce that

$$m^{\mathrm{d}}(t) = \frac{t}{\mu}. \tag{19}$$

Now, N^{d} has stationary increments if and only if the distribution of $E^{\mathrm{d}}(t)$, the excess lifetime of N^{d} at t, does not depend on t. But

$$\begin{aligned}
\mathbb{P}\big(E^{\mathrm{d}}(t) > y\big) &= \sum_{k=0}^{\infty} \mathbb{P}\big(E^{\mathrm{d}}(t) > y,\ N^{\mathrm{d}}(t) = k\big) \\
&= \mathbb{P}\big(E^{\mathrm{d}}(t) > y,\ N^{\mathrm{d}}(t) = 0\big) \\
&\quad + \sum_{k=1}^{\infty} \int_0^t \mathbb{P}\big(E^{\mathrm{d}}(t) > y,\ N^{\mathrm{d}}(t) = k \,\big|\, T_k = x\big)\, dF_k^{\mathrm{d}}(x) \\
&= 1 - F^{\mathrm{d}}(t+y) + \int_0^t [1 - F(t+y-x)]\, d\Big(\sum_{k=1}^{\infty} F_k^{\mathrm{d}}(x)\Big) \\
&= 1 - F^{\mathrm{d}}(t+y) + \int_0^t [1 - F(t+y-x)]\, dm^{\mathrm{d}}(x)
\end{aligned}$$

from (14). Now substitute (18) and (19) into this equation to obtain the result. ■●

(20) Example. Markov chains. Let $Y = \{Y_n : n \geq 0\}$ be a discrete-time Markov chain with countable state space S. At last we are able to prove the ergodic theorem (6.4.21) for Y, as a consequence of the renewal theorem (16). Suppose that $Y_0 = i$ and let j be an aperiodic state. We can suppose that j is persistent, since the result follows from Corollary (6.2.5) if j is transient. Observe the sequence of visits of Y to the state j. That is, let

$$T_0 = 0, \quad T_{n+1} = \min\{k > T_n : Y_k = j\} \quad \text{for} \quad n \geq 0.$$

T_1 may equal $+\infty$; actually $\mathbb{P}(T_1 < \infty) = f_{ij}$. Conditional on the event $\{T_1 < \infty\}$, the inter-visit times

$$X_n = T_n - T_{n-1} \quad \text{for} \quad n \geq 2$$

are independent and identically distributed; following Example (8.3.2), $N^{\mathrm{d}}(t) = \max\{n : T_n \leq t\}$ defines a delayed renewal process with a renewal function $m^{\mathrm{d}}(t) = \sum_{n=1}^{t} p_{ij}(n)$ for integral t. Now, adapt (16) to deal with the possibility that the first interarrival time $X_1 = T_1$ equals infinity, to obtain

$$p_{ij}(n) = m^{\mathrm{d}}(n) - m^{\mathrm{d}}(n-1) \to \frac{f_{ij}}{\mu_j} \quad \text{as} \quad n \to \infty$$

where μ_j is the mean recurrence time of j. ●

(21) Example. Sketch proof of the renewal theorem. There is an elegant proof of the renewal theorem (10.2.5) which proceeds by coupling the renewal process N to an independent delayed renewal process N^{d}; here is a sketch of the method. Let N be a renewal process with interarrival times $\{X_n\}$ and interarrival time distribution function F with mean μ. We suppose that F is non-arithmetic; the proof in the arithmetic case is easier. Let N^{d} be a stationary renewal process (see (17)) with interarrival times $\{Y_n\}$, where Y_1 has distribution function

$$F^{\mathrm{d}}(y) = \frac{1}{\mu} \int_0^y [1 - F(x)]\, dx$$

and $Y_2, Y_3, \dots$ have distribution function F; suppose further that the X_i are independent of the Y_i. The idea of the proof is as follows.

(a) For any $\epsilon > 0$, there must exist an arrival time $T_a = \sum_i^a X_i$ of N and an arrival time $T_b^{\mathrm{d}} = \sum_i^b Y_i$ of N^{d} such that $|T_a - T_b^{\mathrm{d}}| < \epsilon$.

(b) If we replace $X_{a+1}, X_{a+2}, \dots$ by $Y_{b+1}, Y_{b+2}, \dots$ in the construction of N, then the distributional properties of N are unchanged since all these variables are identically distributed.

(c) But the Y_i are the interarrival times of a stationary renewal process, for which (19) holds; this implies that $m^{\mathrm{d}}(t+h) - m^{\mathrm{d}}(t) = h/\mu$ for all t, h. However, $m(t)$ and $m^{\mathrm{d}}(t)$ are nearly the same for large t, by the previous remarks, and so $m(t+h) - m(t) \simeq h/\mu$ for large t.

The details of the proof are slightly too difficult for inclusion here (see Lindvall 1977). ●

(22) Example. Age-dependent branching process. Consider the branching process $Z(t)$ of Section 5.5 in which each individual lives for a random length of time before splitting into its offspring. We have seen that the expected number $m(t) = \mathbb{E}(Z(t))$ of individuals alive at time t satisfies the integral equation (5.5.4):

$$m(t) = \nu \int_0^t m(t-x)\, dF_T(x) + \int_t^\infty dF_T(x) \tag{23}$$

where F_T is the distribution function of a typical lifetime and ν is the mean number of offspring of an individual; we assume for simplicity that F_T is continuous. We have changed some of the notation of (5.5.4) for obvious reasons. Equation (23) reminds us of the renewal-type equation (10.1.10) but the factor ν must be assimilated before the solution can be found using the method of Theorem (10.1.11). This presents few difficulties in the supercritical case. If $\nu > 1$, there exists a unique $\beta > 0$ such that

$$F_T^*(\beta) = \int_0^\infty e^{-\beta x}\, dF_T(x) = \frac{1}{\nu};$$

this holds because the Laplace–Stieltjes transform $F_T^*(\theta)$ is a strictly decreasing continuous function of θ with

$$F_T^*(0) = 1, \quad F_T^*(\theta) \to 0 \quad \text{as} \quad \theta \to \infty.$$

Now, with this choice of β, define

$$\widetilde{F}(t) = \nu \int_0^t e^{-\beta x}\, dF_T(x), \quad g(t) = e^{-\beta t} m(t).$$

Multiply through (23) by $e^{-\beta t}$ to obtain

$$g(t) = h(t) + \int_0^t g(t-x)\, d\widetilde{F}(x) \tag{24}$$

where

$$h(t) = e^{-\beta t}[1 - F_T(t)];$$

(24) has the same general form as (10.1.10), since our choice for β ensures that $\widetilde{F}$ is the distribution function of a positive random variable. The behaviour of $g(t)$ for large t may be found by applying Theorem (10.1.11) and the key renewal theorem (10.2.7). ●

Exercise for Section 10.4

1. Find the distribution of the excess lifetime for a renewal process each of whose interarrival times is the sum of two independent exponentially distributed random variables having respective parameters λ and μ. Show that the excess lifetime has mean

$$\frac{1}{\mu} + \frac{\lambda e^{-(\lambda+\mu)t} + \mu}{\lambda(\lambda+\mu)}.$$

10.5 Renewal–reward processes

Renewal theory provides models for many situations in real life. In practice, there may be rewards and/or costs associated with such a process, and these may be introduced as follows.

Let $\{(X_i, R_i) : i \geq 1\}$ be independent and identically distributed pairs of random variables such that $X_i > 0$. For a typical pair (X, R), the quantity X is to be interpreted as an interarrival time of a renewal process, and the quantity R as a reward associated with that interarrival time; we do not assume that X and R are independent. Costs count as negative rewards. We now construct the renewal process N by $N(t) = \sup\{n : T_n \leq t\}$ where $T_n = X_1 + X_2 + \cdots + X_n$, and the 'cumulative reward process' C by

$$C(t) = \sum_{i=1}^{N(t)} R_i.$$

The *reward function* is $c(t) = \mathbb{E}C(t)$. The asymptotic properties of $C(t)$ and $c(t)$ are given by the following analogue of Theorems (10.2.1) and (10.2.3).

(1) Renewal–reward theorem. *Suppose that* $0 < \mathbb{E}X < \infty$ *and* $\mathbb{E}|R| < \infty$. *Then*:

(2) $$\frac{C(t)}{t} \xrightarrow{\text{a.s.}} \frac{\mathbb{E}R}{\mathbb{E}X} \quad \textit{as } t \to \infty,$$

(3) $$\frac{c(t)}{t} \to \frac{\mathbb{E}R}{\mathbb{E}X} \quad \textit{as } t \to \infty.$$

Proof. We have by the strong law of large numbers and Theorem (10.2.1) that

(4) $$\frac{C(t)}{t} = \frac{\sum_{i=1}^{N(t)} R_i}{N(t)} \cdot \frac{N(t)}{t} \xrightarrow{\text{a.s.}} \frac{\mathbb{E}R}{\mathbb{E}X}.$$

We saw prior to Lemma (10.2.9) that $N(t) + 1$ is a stopping time for the sequence $\{X_i : i \geq 1\}$, whence it is a stopping time for the sequence of pairs $\{(X_i, R_i) : i \geq 1\}$. By a straightforward generalization of Wald's equation (10.2.9),

$$c(t) = \mathbb{E}\left(\sum_{j=1}^{N(t)+1} R_j\right) - \mathbb{E}(R_{N(t)+1}) = \mathbb{E}\big(N(t)+1\big)\mathbb{E}(R) - \mathbb{E}(R_{N(t)+1}).$$

The result will follow once we have shown that $t^{-1}\mathbb{E}(R_{N(t)+1}) \to 0$ as $t \to \infty$.

By conditioning on X_1, as usual, we find that $r(t) = \mathbb{E}(R_{N(t)+1})$ satisfies the renewal equation

$$r(t) = H(t) + \int_0^t r(t-x)\,dF(x), \tag{5}$$

where F is the distribution function of X, $H(t) = \mathbb{E}(RI_{\{X>t\}})$, and (X, R) is a typical interarrival-time/reward pair. We note that

$$H(t) \to 0 \quad \text{as } t \to \infty, \qquad |H(t)| < \mathbb{E}|R| < \infty. \tag{6}$$

By Theorem (10.1.11), the renewal equation (5) has solution

$$r(t) = H(t) + \int_0^t H(t-x)\,dm(x)$$

where $m(t) = \mathbb{E}(N(t))$. By (6), for $\epsilon > 0$, there exists $M(\epsilon) < \infty$ such that $|H(t)| < \epsilon$ for $t \geq M(\epsilon)$. Therefore, when $t \geq M(\epsilon)$,

$$\begin{aligned}
\left|\frac{r(t)}{t}\right| &\leq \frac{1}{t}\left\{|H(t)| + \int_0^{t-M} |H(t-x)|\,dm(x) + \int_{t-M}^t |H(t-x)|\,dm(x)\right\} \\
&\leq \frac{1}{t}\left\{\epsilon + \epsilon m(t-M) + \mathbb{E}|R|\big(m(t) - m(t-M)\big)\right\} \\
&\to \frac{\epsilon}{\mathbb{E}X} \qquad \text{as } t \to \infty,
\end{aligned}$$

by the renewal theorems (10.2.3) and (10.2.5). The result follows on letting $\epsilon \downarrow 0$. ∎

The reward process C accumulates rewards at the rate of one reward per interarrival time. In practice, rewards may accumulate in a continuous manner, spread over the interval in question, in which case the accumulated reward $\widetilde{C}(t)$ at time t is obtained by adding to $C(t)$ that part of $R_{N(t)+1}$ arising from the already elapsed part of the interval in progress at time t. This makes no effective difference to the conclusion of the renewal–reward theorem so long as rewards accumulate in a monotone manner. Suppose then that the reward $\widetilde{C}(t)$ accumulated at time t necessarily lies between $C(t)$ and $C(t) + R_{N(t)+1}$. We have as in (4) that

$$\frac{1}{t}\big(C(t) + R_{N(t)+1}\big) = \frac{\sum_{i=1}^{N(t)+1} R_i}{N(t)+1} \cdot \frac{N(t)+1}{t} \xrightarrow{\text{a.s.}} \frac{\mathbb{E}R}{\mathbb{E}X}.$$

Taken with (4), this implies that

$$\frac{\widetilde{C}(t)}{t} \xrightarrow{\text{a.s.}} \frac{\mathbb{E}R}{\mathbb{E}X} \qquad \text{as } t \to \infty. \tag{7}$$

One has similarly that $\widetilde{c}(t) = \mathbb{E}(\widetilde{C}(t))$ satisfies $\widetilde{c}(t)/t \to \mathbb{E}R/\mathbb{E}X$ as $t \to \infty$.

(8) Example. A vital component of an aeroplane is replaced at a cost b whenever it reaches the given age A. If it fails earlier, the cost of replacement is a. The distribution function of the usual lifetime of a component of this type is F, which we assume to have density function f. At what rate does the cost of replacing the component accrue?

Let $X_1, X_2, \dots$ be the runtimes of the component and its replacements. The X_i may be assumed independent with common distribution function

$$H(x) = \begin{cases} F(x) & \text{if } x < A, \\ 1 & \text{if } x \ge A. \end{cases}$$

By Lemma (4.3.4), the mean of the X_i is

$$\mathbb{E}X = \int_0^A [1 - F(x)]\, dx.$$

The cost of replacing a component having runtime X is

$$S(X) = \begin{cases} a & \text{if } X < A, \\ b & \text{if } X \ge A, \end{cases}$$

whence $\mathbb{E}S = aF(A) + b[1 - F(A)]$.

By the renewal–reward theorem (1), the asymptotic cost per unit time is

$$\text{(9)} \qquad \frac{\mathbb{E}S}{\mathbb{E}X} = \frac{aF(A) + b[1 - F(A)]}{\int_0^A [1 - F(x)]\, dx}.$$

One may choose A to minimize this expression. ●

We give two major applications of the renewal–reward theorem, of which the first is to passage times of Markov chains. Let $X = \{X(t) : t \ge 0\}$ be an irreducible Markov chain in continuous time on the countable state space S, with transition semigroup $\{\mathbf{P}_t\}$ and generator $\mathbf{G} = (g_{ij})$. For simplicity we assume that X is the minimal process associated with its jump chain Z, as discussed prior to Theorem (6.9.24). Let $U = \inf\{t : X(t) \ne X(0)\}$ be the first 'holding time' of the chain, and define the 'first passage time' of the state i by $F_i = \inf\{t > U : X(t) = i\}$. We define the *mean recurrence time* of i by $\mu_i = \mathbb{E}(F_i \mid X(0) = i)$. In avoid to avoid a triviality, we assume $|S| \ge 2$, implying by the irreducibility of the chain that $g_i = -g_{ii} > 0$ for each i.

(10) Theorem. *Assume the above conditions, and let $X(0) = i$. If $\mu_i < \infty$, the proportion of time spent in state i, and the expectation of this amount, satisfy, as $t \to \infty$,*

$$\text{(11)} \qquad \frac{1}{t}\int_0^t I_{\{X(s)=i\}}\, ds \xrightarrow{\text{a.s.}} \frac{1}{\mu_i g_i},$$

$$\text{(12)} \qquad \frac{1}{t}\int_0^t p_{ii}(s)\, ds \to \frac{1}{\mu_i g_i}.$$

We note from Exercise (6.9.11b) that the limit in (11) and (12) is the stationary distribution of the chain.

Proof. We define the pairs (P_r, Q_r), $r \geq 0$, of times as follows. First, we let $P_0 = 0$ and $Q_0 = \inf\{t : X(t) \neq i\}$, and more generally

$$P_r = \inf\{t > P_{r-1} + Q_{r-1} : X(t) = i\},$$
$$Q_r = \inf\{s > 0 : X(P_r + s) \neq i\}.$$

That is, P_r is the time of the rth passage of X into the state i, and Q_r is the subsequent holding time in state i. The P_r may be viewed as the times of arrivals in a renewal process having interarrival times distributed as F_i conditional on $X(0) = i$, and we write $N(t) = \sup\{r : P_r \leq t\}$ for the associated renewal process. With the interarrival interval (P_r, P_{r+1}) we associate the reward Q_r.

We have that

$$\text{(13)} \qquad \frac{1}{t}\sum_{r=0}^{N(t)-1} Q_r \leq \frac{1}{t}\int_0^t I_{\{X(s)=i\}}\,ds \leq \frac{1}{t}\sum_{r=0}^{N(t)} Q_r.$$

Applying Theorem (1), we identify the limits in (11) and (12) as $\mathbb{E}(Q_0)/\mathbb{E}(P_1)$. Since Q_0 has the exponential distribution with parameter g_i, and P_1 is the first passage time of i, we see that $\mathbb{E}(Q_0)/\mathbb{E}(P_1) = (g_i\mu_i)^{-1}$ as required. ■

Another important and subtle application of the renewal–reward theorem is to queueing. A striking property of this application is its degree of generality, and only few assumptions are required of the queueing system. Specifically, we shall assume that:

(a) customers arrive one by one, and the nth customer spends a 'waiting time' V_n in the system before departing†;

(b) there exists almost surely a finite (random) time T (> 0) such that the process beginning at time T has the same distribution as the process beginning at time 0; the time T is called a 'regeneration point';

(c) the number $Q(t)$ of customers in the system at time t satisfies $Q(0) = Q(T) = 0$.

From (b) follows the almost-sure existence of an infinite sequence of times $0 = T_0 < T_1 < T_2 < \cdots$ each of which is a regeneration point of the process, and whose interarrival times $X_i = T_i - T_{i-1}$ are independent and identically distributed. That is, there exists a renewal process of regeneration points.

Examples of such systems are multifarious, and include various scenarios described in Chapter 11: a stable G/G/1 queue where the T_i are the times at which departing customers leave the queue empty, or alternatively the times at which an arriving customer finds no one waiting; a network of queues, with the regeneration points being stopping times at which the network is empty.

Let us assume that (a), (b), (c) hold. We call the time intervals $[T_{i-1}, T_i)$ *cycles* of the process, and note that the processes $P_i = \{Q(t) : T_{i-1} \leq t < T_i\}$, $i \geq 1$, are independent and identically distributed. We write N_i for the number of arriving customers during the cycle $[T_{i-1}, T_i)$, and $N = N_1$, $T = T_1$. In order to avoid a triviality, we assume also that the regeneration points are chosen in such a way that $N_i > 0$ for all i. We shall apply the renewal–reward theorem three times, and shall assume that

$$\text{(14)} \qquad \mathbb{E}T < \infty, \quad \mathbb{E}N < \infty, \quad \mathbb{E}(NT) < \infty.$$

†This waiting time includes any service time.

(A) Consider the renewal process with arrival times $T_0, T_1, T_2, \ldots$. The reward associated with the interarrival time $X_i = T_i - T_{i-1}$ is taken to be

$$R_i = \int_{T_{i-1}}^{T_i} Q(u)\,du.$$

The R_i have the same distribution as $R = R_1 = \int_0^T Q(u)\,du$; furthermore $Q(u) \le N$ when $0 \le u \le T$, whence $\mathbb{E}R \le \mathbb{E}(NT) < \infty$ by (14). By the renewal–reward theorem (1) and the discussion before (7),

(15) $$\frac{1}{t}\int_0^t Q(u)\,du \xrightarrow{\text{a.s.}} \frac{\mathbb{E}R}{\mathbb{E}T} \quad \text{as } t \to \infty.$$

The ratio $\mathbb{E}(R)/\mathbb{E}(T)$ is termed the 'long run average queue length' and is denoted by L.

(B) Consider now another renewal–reward process with arrival times $T_0, T_1, T_2, \ldots$. The reward associated with the interarrival time X_i is taken to be the number N_i of customers who arrive during the corresponding cycle. By the renewal–reward theorem and the discussion prior to (7), we have from hypothesis (14) that the number $N(t)$ of arrivals by time t satisfies

(16) $$\frac{N(t)}{t} \xrightarrow{\text{a.s.}} \frac{\mathbb{E}N}{\mathbb{E}T} \quad \text{as } t \to \infty.$$

The ratio $\mathbb{E}(N)/\mathbb{E}(T)$ is termed the 'long run rate of arrival' and is denoted by λ.

(C) Consider now the renewal–reward process with interarrival times $N_1, N_2, \ldots$, the reward S_i associated with the interarrival time N_i being the sum of the waiting times of customers arriving during the ith cycle of the queue. The mean reward is $\mathbb{E}S = \mathbb{E}\left(\sum_1^N V_i\right)$; this is no larger than $\mathbb{E}(NT)$ which by (14) is finite. Applying the renewal–reward theorem and the discussion prior to (7), we have that

(17) $$\frac{1}{n}\sum_{i=1}^{n} V_i \xrightarrow{\text{a.s.}} \frac{\mathbb{E}S}{\mathbb{E}N} \quad \text{as } n \to \infty.$$

The ratio $\mathbb{E}(S)/\mathbb{E}(N)$ is termed the 'long run average waiting time' and is denoted by W.

(18) Little's theorem. *Under the assumptions above, we have that $L = \lambda W$.*

Proof. We have that

$$\frac{L}{\lambda W} = \frac{\mathbb{E}R}{\mathbb{E}T} \cdot \frac{\mathbb{E}T}{\mathbb{E}N} \cdot \frac{\mathbb{E}N}{\mathbb{E}S} = \frac{\mathbb{E}\int_0^T Q(u)\,du}{\mathbb{E}\sum_1^N V_i}$$

so that the result will follow once we have shown that

(19) $$\mathbb{E}\left(\sum_1^N V_i\right) = \mathbb{E}\left(\int_0^T Q(u)\,du\right).$$

Each side of this equation is the mean amount of customer time spent during the first cycle of the system: the left side of (19) counts this by customer, and the right side counts it by unit of time. The required equality follows. ■

(20) Example. Cars arrive at a car wash in the manner of a Poisson process with rate ν. They wait in a line, while the car at the head of the line is washed by the unique car-wash machine. There is space in the line for exactly K cars, including any car currently being washed, and, when the line is full, arriving cars leave and never return. The wash times of cars are independent random variables with distribution function F and mean θ.

Let p_i denote the proportion of time that there are exactly i cars in the line, including any car being washed. Since the queue length is not a Markov chain (unless wash times are exponentially distributed), one should not talk of the system being 'in equilibrium'. Nevertheless, using renewal theory, one may see that there exists an asymptotic proportion p_i of time.

We shall apply Little's theorem to the smaller system comprising the location at the head of the line, that is, the car-wash machine itself. We take as regeneration points the times at which cars depart from the machine leaving the line empty.

The 'long run average queue length' is $L = 1 - p_0$, being the proportion of time that the machine is in use. The 'long run rate of arrival' λ is the rate at which cars enter this subsystem, and this equals the rate at which cars join the line. Since an arriving car joins the line with probability $1 - p_K$, and since cars arrive in the manner of a Poisson process with parameter ν, we deduce that $\lambda = \nu(1 - p_K)$. Finally, the 'long run average waiting time' W is the mean time taken by the machine to wash a car, so that $W = \theta$.

We have by Little's theorem (18) that $L = \lambda W$ which is to say that $1 - p_0 = \nu(1 - p_K)\theta$. This equation may be interpreted in terms of the cost of running the machine, which is proportional to $1 - p_0$, and the disappointment of customers who arrive to find the line full, which is proportional to νp_K. ●

Exercises for Section 10.5

1. If $X(t)$ is an irreducible persistent non-null Markov chain, and $u(\cdot)$ is a bounded function on the integers, show that

$$\frac{1}{t}\int_0^t u(X(s))\,ds \xrightarrow{\text{a.s.}} \sum_{i\in S} \pi_i u(i),$$

where $\boldsymbol{\pi}$ is the stationary distribution of $X(t)$.

2. Let $M(t)$ be an alternating renewal process, with interarrival pairs $\{X_r, Y_r : r \geq 1\}$. Show that

$$\frac{1}{t}\int_0^t I_{\{M(s)\text{ is even}\}}\,ds \xrightarrow{\text{a.s.}} \frac{\mathbb{E}X_1}{\mathbb{E}X_1 + \mathbb{E}Y_1} \quad \text{as } t \to \infty.$$

3. Let $C(s)$ be the current lifetime (or age) of a renewal process $N(t)$ with a typical interarrival time X. Show that

$$\frac{1}{t}\int_0^t C(s)\,ds \xrightarrow{\text{a.s.}} \frac{\mathbb{E}(X^2)}{2\mathbb{E}(X)} \quad \text{as } t \to \infty.$$

Find the corresponding limit for the excess lifetime.

4. Let j and k be distinct states of an irreducible discrete-time Markov chain X with stationary distribution $\boldsymbol{\pi}$. Show that

$$\mathbb{P}(T_j < T_k \mid X_0 = k) = \frac{1/\pi_k}{\mathbb{E}(T_j \mid X_0 = k) + \mathbb{E}(T_k \mid X_0 = j)}$$

where $T_i = \min\{n \geq 1 : X_n = i\}$ is the first passage time to the state i. [Hint: Consider the times of return to j having made an intermediate visit to k.]

10.6 Problems

In the absence of indications to the contrary, $\{X_n : n \geq 1\}$ denotes the sequence of interarrival times of either a renewal process N or a delayed renewal process N^{d}. In either case, F^{d} and F are the distribution functions of X_1 and X_2 respectively, though $F^{\mathrm{d}} \neq F$ only if the renewal process is delayed. We write $\mu = \mathbb{E}(X_2)$, and shall usually assume that $0 < \mu < \infty$. The functions m and m^{d} denote the renewal functions of N and N^{d}. We write $T_n = \sum_{i=1}^{n} X_i$, the time of the nth arrival.

1. (a) Show that $\mathbb{P}(N(t) \to \infty \text{ as } t \to \infty) = 1$.
(b) Show that $m(t) < \infty$ if $\mu \neq 0$.
(c) More generally show that, for all $k > 0$, $\mathbb{E}(N(t)^k) < \infty$ if $\mu \neq 0$.

2. Let $v(t) = \mathbb{E}(N(t)^2)$. Show that

$$v(t) = m(t) + 2\int_0^t m(t-s)\,dm(s).$$

Find $v(t)$ when N is a Poisson process.

3. Suppose that $\sigma^2 = \mathrm{var}(X_1) > 0$. Show that the renewal process N satisfies

$$\frac{N(t) - (t/\mu)}{\sqrt{t\sigma^2/\mu^3}} \xrightarrow{\mathrm{D}} N(0,1), \qquad \text{as } t \to \infty.$$

4. Find the asymptotic distribution of the current life $C(t)$ of N as $t \to \infty$ when X_1 is not arithmetic.

5. Let N be a Poisson process with intensity λ. Show that the total life $D(t)$ at time t has distribution function $\mathbb{P}(D(t) \leq x) = 1 - (1 + \lambda \min\{t, x\})e^{-\lambda x}$ for $x \geq 0$. Deduce that $\mathbb{E}(D(t)) = (2 - e^{-\lambda t})/\lambda$.

6. A Type 1 counter records the arrivals of radioactive particles. Suppose that the arrival process is Poisson with intensity λ, and that the counter is locked for a dead period of fixed length T after each detected arrival. Show that the detection process $\widetilde{N}$ is a renewal process with interarrival time distribution $\widetilde{F}(x) = 1 - e^{-\lambda(x-T)}$ if $x \geq T$. Find an expression for $\mathbb{P}(\widetilde{N}(t) \geq k)$.

7. Particles arrive at a Type 1 counter in the manner of a renewal process N; each detected arrival locks the counter for a dead period of random positive length. Show that

$$\mathbb{P}(\widetilde{X}_1 \leq x) = \int_0^x [1 - F(x-y)]F_L(y)\,dm(y)$$

where F_L is the distribution function of a typical dead period.

8. (a) Show that $m(t) = \frac{1}{2}\lambda t - \frac{1}{4}(1 - e^{-2\lambda t})$ if the interarrival times have the gamma distribution $\Gamma(\lambda, 2)$.
(b) Radioactive particles arrive like a Poisson process, intensity λ, at a counter. The counter fails to register the nth arrival whenever n is odd but suffers no dead periods. Find the renewal function $\widetilde{m}$ of the detection process $\widetilde{N}$.

9. Show that Poisson processes are the only renewal processes with non-arithmetic interarrival times having the property that the excess lifetime $E(t)$ and the current lifetime $C(t)$ are independent for each choice of t.

10. Let N_1 be a Poisson process, and let N_2 be a renewal process which is independent of N_1 with non-arithmetic interarrival times having finite mean. Show that $N(t) = N_1(t) + N_2(t)$ is a renewal process if and only if N_2 is a Poisson process.

11. Let N be a renewal process, and suppose that F is non-arithmetic and that $\sigma^2 = \mathrm{var}(X_1) < \infty$. Use the properties of the moment generating function $F^*(-\theta)$ of X_1 to deduce the formal expansion

$$m^*(\theta) = \frac{1}{\theta\mu} + \frac{\sigma^2 - \mu^2}{2\mu^2} + \mathrm{o}(1) \qquad \text{as } \theta \to 0.$$

Invert this Laplace–Stieltjes transform formally to obtain

$$m(t) = \frac{t}{\mu} + \frac{\sigma^2 - \mu^2}{2\mu^2} + o(1) \quad \text{as } t \to \infty.$$

Prove this rigorously by showing that

$$m(t) = \frac{t}{\mu} - F_E(t) + \int_0^t [1 - F_E(t-x)]\, dm(x),$$

where F_E is the asymptotic distribution function of the excess lifetime (see Exercise (10.3.3)), and applying the key renewal theorem. Compare the result with the renewal theorems.

12. Show that the renewal function m^{d} of a delayed renewal process satisfies

$$m^{\mathrm{d}}(t) = F^{\mathrm{d}}(t) + \int_0^t m^{\mathrm{d}}(t-x)\, dF(x).$$

Show that $v^{\mathrm{d}}(t) = \mathbb{E}(N^{\mathrm{d}}(t)^2)$ satisfies

$$v^{\mathrm{d}}(t) = m^{\mathrm{d}}(t) + 2\int_0^t m^{\mathrm{d}}(t-x)\, dm(x)$$

where m is the renewal function of the renewal process with interarrival times $X_2, X_3, \dots$.

13. Let $m(t)$ be the mean number of living individuals at time t in an age-dependent branching process with exponential lifetimes, parameter λ, and mean family size ν (> 1). Prove that $m(t) = Ie^{(\nu-1)\lambda t}$ where I is the number of initial members.

14. Alternating renewal process. The interarrival times of this process are $Z_0, Y_1, Z_1, Y_2, \dots$, where the Y_i and Z_j are independent with respective common moment generating functions M_Y and M_Z. Let $p(t)$ be the probability that the epoch t of time lies in an interval of type Z. Show that the Laplace–Stieltjes transform p^* of p satisfies

$$p^*(\theta) = \frac{1 - M_Z(-\theta)}{1 - M_Y(-\theta)M_Z(-\theta)}.$$

15. Type 2 counters. Particles are detected by a Type 2 counter of the following sort. The incoming particles constitute a Poisson process with intensity λ. The jth particle locks the counter for a length Y_j of time, and annuls any after-effect of its predecessors. Suppose that $Y_1, Y_2, \dots$ are independent of each other and of the Poisson process, each having distribution function G. The counter is unlocked at time 0.

Let L be the (maximal) length of the first interval of time during which the counter is locked. Show that $H(t) = \mathbb{P}(L > t)$ satisfies

$$H(t) = e^{-\lambda t}[1 - G(t)] + \int_0^t H(t-x)[1 - G(x)]\lambda e^{-\lambda x}\, dx.$$

Solve for H in terms of G, and evaluate the ensuing expression in the case $G(x) = 1 - e^{-\mu x}$ where $\mu > 0$.

16. Thinning. Consider a renewal process N, and suppose that each arrival is 'overlooked' with probability q, independently of all other arrivals. Let $M(t)$ be the number of arrivals which are detected up to time t/p where $p = 1 - q$.

(a) Show that M is a renewal process whose interarrival time distribution function F_p is given by $F_p(x) = \sum_{r=1}^{\infty} pq^{r-1} F_r(x/p)$, where F_n is the distribution function of the time of the nth arrival in the original process N.

(b) Find the characteristic function of F_p in terms of that of F, and use the continuity theorem to show that, as $p \downarrow 0$, $F_p(s) \to 1 - e^{-s/\mu}$ for $s > 0$, so long as the interarrival times in the original process have finite mean μ. Interpret!

(c) Suppose that $p < 1$, and M and N are processes with the same fdds. Show that N is a Poisson process.

17. (a) A PC keyboard has 100 different keys and a monkey is tapping them (uniformly) at random. Assuming no power failure, use the elementary renewal theorem to find the expected number of keys tapped until the first appearance of the sequence of fourteen characters 'W. Shakespeare'. Answer the same question for the sequence 'omo'.

(b) A coin comes up heads with probability p on each toss. Find the mean number of tosses until the first appearances of the sequences (i) HHH, and (ii) HTH.

18. Let N be a stationary renewal process. Let s be a fixed positive real number, and define $X(t) = N(s + t) - N(t)$ for $t \geq 0$. Show that X is a strongly stationary process.

19. Bears arrive in a village at the instants of a renewal process; they are captured and confined at a cost of \$$c$ per unit time per bear. When a given number B bears have been captured, an expedition (costing \$$d$) is organized to remove and release them a long way away. What is the long-run average cost of this policy?