Chapter 3

The Pigeonhole Principle and Ramsey Numbers

3.1. Introduction

You may have come across the following statements which are of the same mathematical nature:

"Among any group of two or more people, there are two who have the same number of friends in the group."

"Among any 5 points in an equilateral triangle of unit length, there are two whose distance is at most half a unit apart."

"Given any set A of 5 numbers, there are 3 distinct elements in A whose sum is divisible by 3."

"Given a sequence of 10 distinct numbers, there exists either a decreasing subsequence of 4 terms or an increasing subsequence of 4 terms."

This type of problems concerns with the existence of a certain kind of quantity, pattern or arrangement. In this chapter, we shall introduce a fundamental principle in combinatorics, known as the Pigeonhole Principle, which deals with a class of problems of this type. We shall also see how the principle can be applied to study some problems that give rise to a class of numbers, called Ramsey numbers.

3.2. The Pigeonhole Principle

If three pigeons are to be put into two compartments, then you will certainly agree that one of the compartments will accommodate at least two pigeons. A much more general statement of this simple observation, known as the *Pigeonhole Principle*, is given below.

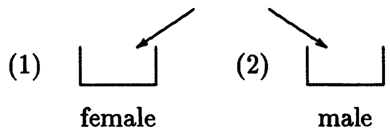
The Pigeonhole Principle (PP). Let k and n be any two positive integers. If at least $kn+1$ objects are distributed among n boxes, then one of the boxes must contain at least $k+1$ objects. In particular, if at least $n+1$ objects are to be put into n boxes, then one of the boxes must contain at least two objects.

The proof of (PP) is easy. If no boxes contain $k+1$ or more objects, then every box contains at most k objects. This implies that the total number of objects put into the n boxes is at most kn, a contradiction. ∎

(PP) is also known as the *Dirichlet Drawer Principle*, after the German mathematician Peter G.L. Dirichlet (1805-1859) who had used it to prove some results in number theory. (PP) looks almost trivial; however, as we shall witness in this and the following two sections, it is a surprisingly useful and powerful device that proves many "existence" statements in mathematics.

Example 3.2.1. Among any group of 7 people, there must be at least 4 of the same sex.

Let us see how (PP) can be applied to this example. We treat the 7 people as 7 objects, and create two "boxes": box (1) for "female" and box (2) for "male".



If a person of the group is a lady (resp., gentleman), then she (resp., he) is put into box (1) (resp., box (2)). Thus the $7(=3\cdot 2+1=kn+1)$ "objects" are put into $2(=n)$ "boxes" and so by (PP), there is a box which contains at least $4(=3+1=k+1)$ objects; i.e., there are at least 4 people among the group who are of the same sex. ∎

By applying (PP) in a similar way, you should be able to prove the following.

Example 3.2.2. Among any group of 13 people, there must be at least 2 whose birthdays are in the same month.

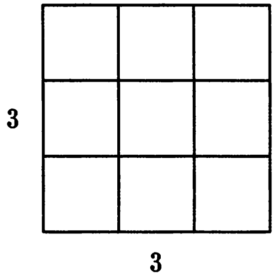
Example 3.2.3. Among any group of 3000 people, there are at least 9 who have the same birthday.

As we have just seen, in applying (PP), we have to identify what the "objects" and what the "boxes" are. Moreover, we must know the values of k and n (number of boxes) involved in (PP), and to make sure that the number of objects is at least $kn + 1$.

Example 3.2.4. Show that for any set of 10 points chosen within a square whose sides are of length 3 units, there are two points in the set whose distance apart is at most $\sqrt{2}$.

What are the objects? What are the boxes? These are the two questions we have to ask beforehand. It is fairly clear that we should treat the 10 given points in the set as our "objects". The conclusion we wish to arrive at is the existence of "2 points" from the set which are "close" to each other (i.e. their distance apart is at most $\sqrt{2}$ units). This indicates that "$k + 1 = 2$" (i.e., $k = 1$), and suggests also that we should partition the 3×3 square into n smaller regions, $n < 10$, so that the distance between any 2 points in a region is at most $\sqrt{2}$.

Solution. Divide the 3×3 square into 9 unit squares as shown below:

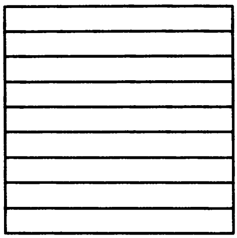


Let A be any set of 10 points (our objects) chosen from the 3×3 square. Since each point in A is contained in (at least) one of the 9 unit squares (our boxes), and since $10 > 9$, by (PP), there is a unit square (box) which contains at least 2 points (objects) of A. Let these 2 points be u and v. It is easy to verify that the distance between u and v does not exceed the length of a diagonal of the unit square, which is $\sqrt{1^2 + 1^2} = \sqrt{2}$. ∎

Remarks. **(1)** If the hypothesis of (PP) is satisfied, then the conclusion of (PP) guarantees that there is a (certain) box which contains at least $k+1$ objects. It should be noted, however, that (PP) does not tell us "which box it is" and "which $k+1$ objects it contains".

(2) In Example 3.2.4, one might divide the 3×3 square into the 9 rectangles as shown below, and apply (PP) to reach the conclusion that there are 2 points in A contained in one of the 9 rectangles.

3



3

In this case, however, one would not be able to draw the conclusion that the distance between these 2 points is at most $\sqrt{2}$. So, does it mean that (PP) is invalid? Certainly not! It simply reveals the fact that the "boxes" we create here are not appropriate!

3.3. More Examples

In the preceding section, we gave some simple examples where (PP) can be applied. They are "simple" since the identification of the "objects" and "boxes" in these problems is rather straightforward. This is not always the case in general. In this section, we shall present more difficult and sophisticated problems from different areas where (PP) can be applied (but in a nontrivial way). Through the discussion of these problems, it is hoped that readers will appreciate, as a mathematical tool in problem solving, how powerful and useful (PP) is.

Example 3.3.1. Let $A = \{a_1, a_2, ..., a_5\}$ be a set of 5 positive integers. Show that for any permutation $a_{i_1}a_{i_2}a_{i_3}a_{i_4}a_{i_5}$ of A, the product

$$(a_{i_1} - a_1)(a_{i_2} - a_2)\cdots(a_{i_5} - a_5)$$

is always even.

For instance, if

$$\begin{array}{lllll} a_1 = 2, & a_2 = 5, & a_3 = 7, & a_4 = 3, & a_5 = 8, \\ a_{i_1} = a_4, & a_{i_2} = a_3, & a_{i_3} = a_5, & a_{i_4} = a_1, & a_{i_5} = a_2, \end{array}$$

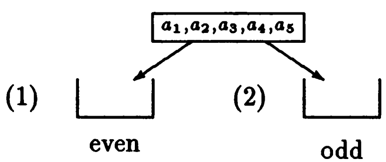
then the product

$$\begin{aligned} &(a_4 - a_1)(a_3 - a_2)(a_5 - a_3)(a_1 - a_4)(a_2 - a_5) \\ = &(3-2)(7-5)(8-7)(2-3)(5-8) = 6, \end{aligned}$$

which is even.

Unlike those examples given in Section 3.2, it is not apparent how (PP) can be applied here. Let us first analyze the problem. To show that the product is even, it suffices to show the "existence" of an even factor, say $(a_{i_k} - a_k)$. We note that the number $a_{i_k} - a_k$ is even if and only if a_{i_k} and a_k are both even or both odd (in this case, we say that a_{i_k} and a_k have the same parity). Thus we see that it may have something to do with the parity of the 5 numbers in A. In view of this, we create 2 "boxes", one for "even numbers" and one for "odd numbers".

Solution. We have $|A| = 5$. By (PP), there exist at least 3 elements of A (say, a_1, a_2, a_3) which are of the same parity (in this case $k = n = 2$).



Observe that $\{a_{i_1}, a_{i_2}, a_{i_3}\} \cap \{a_1, a_2, a_3\} \neq \emptyset$ (otherwise, $|A|$ will be at least $6 = |\{a_{i_1}, a_{i_2}, a_{i_3}, a_1, a_2, a_3\}|$). Thus we may assume, say $a_1 = a_{i_3}$. This implies that $a_{i_3} - a_3 = a_1 - a_3$, the latter being even as a_1 and a_3 are of the same parity. Thus the factor $(a_{i_3} - a_3)$ is even, which completes the proof. ∎

Remarks. **(1)** The above proof can be extended in a natural way to prove the same result for any odd number of positive integers $a_1, a_2, \ldots, a_{2p+1}$ (see Problem 3.6).

(2) The conclusion of Example 3.3.1 is no longer true if $|A|$ is even.

Example 3.3.2. Ten players took part in a round robin chess tournament (i.e., each player must play exactly one game against every other player). According to the rules, a player scores 1 point if he wins a game; -1 point if he loses; and 0 point if the game ends in a draw. When the tournament was over, it was found that more than 70% of the games ended in a draw. Show that there were two players who had the same total score.

Judging from the last statement of the problem, it seems that we should treat the 10 players as our "10 objects", and create "boxes" for "total scores". However, what are the possible total scores? How many different total scores are there? (Don't forget, we have to ensure that the number of objects is bigger than the number of boxes.) These questions seem not easy to answer. Let us try an "indirect" way.

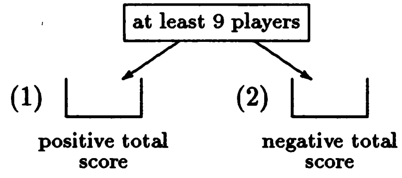
Solution. Since every 2 players played one and only one game against each other, there were $\binom{10}{2} = 45$ games held during the tournament. Among these games, there were at least

$$\lceil 45 \times 70\% \rceil = \lceil 31.5 \rceil = 32$$

games that ended in a draw, where $\lceil x \rceil$ denotes the least integer not less than the real x. Hence

(*) there were at most $45 - 32 = 13$ games which did not end in a draw.

Now, suppose to the contrary that the 10 players had 10 different total scores. This implies particularly that at most one of the players had total score "0". Thus at least 9 players had either "positive" or "negative" total scores. Treat these players as our "objects", and create 2 "boxes", one for "positive" and one for "negative". By (PP), at least 5 players had positive total scores or at least 5 players had negative total scores.



By symmetry, we may assume the former. Since the total scores are assumed to be different, the sum of these positive total scores must be at least

$$1 + 2 + 3 + 4 + 5 = 15.$$

But this implies that there must be at least 15 games which did not end in a draw, contradicting the statement $(*)$. Hence there were 2 players who had the same total score. ∎

Example 3.3.3. Let A be a set of m positive integers where $m \geq 1$. Show that there exists a nonempty subset B of A such that the sum $\sum(x \mid x \in B)$ is divisible by m.

For example, if $A = \{3, 9, 14, 18, 23\}$, then $m = |A| = 5$, and we take $B = \{3, 14, 18\}$ (there are other choices for B). Observe that

$$\sum(x \mid x \in B) = 3 + 14 + 18 = 35,$$

which is divisible by 5.

Since the conclusion of the problem involves the divisibility of m, one possible way to tackle the problem is to make use of the congruence relation modulo m. (Recall that $a \equiv b \pmod{m}$ iff $m|(a-b)$.) A basic property of "$\equiv$" says that if $a \equiv r \pmod{m}$ and $b \equiv r \pmod{m}$, then $a \equiv b \pmod{m}$, i.e., $m|(a-b)$. This observation suggests the following solution.

Solution. Let $A = \{a_1, a_2, ..., a_m\}$. Consider the following m subsets of A and their respective sums:

$$\begin{array}{cccc} A_1 = \{a_1\}, & A_2 = \{a_1, a_2\}, & \cdots, & A_m = \{a_1, a_2, ..., a_m\}; \\ a_1, & a_1 + a_2, & \cdots, & a_1 + a_2 + \cdots + a_m. \end{array}$$

If one of the sums (say $a_1 + a_2 + a_3$) is divisible by m, then we take B to be the respective set (in this case, $B = A_3$), and we are through. Thus we may assume that no sums above are divisible by m, and we have

$$\begin{aligned} a_1 &\equiv r_1 \pmod{m}, \\ a_1 + a_2 &\equiv r_2 \pmod{m}, \\ &\vdots \\ a_1 + a_2 + \cdots + a_m &\equiv r_m \pmod{m}, \end{aligned}$$

where $r_i \in \{1, 2, ..., m-1\}$ for each $i = 1, 2, ..., m$.

Now, treat the m sums as our m "objects" and create $m-1$ "boxes" for the $m-1$ residue classes modulo m:

(1) $\sqcup$	(2) $\sqcup$	$\cdots$	$(m-1)$ $\sqcup$
$x \equiv 1 \pmod{m}$	$x \equiv 2 \pmod{m}$		$x \equiv m-1 \pmod{m}$

By (PP), there are 2 sums, say

$$a_1 + a_2 + \cdots + a_i \quad \text{and} \quad a_1 + a_2 + \cdots + a_i + \cdots + a_j,$$

where $i < j$, that are in the same residue class modulo m; i.e.,

$$\begin{aligned} a_1 + a_2 + \cdots + a_i &\equiv r \pmod{m} \\ \text{and} \quad a_1 + a_2 + \cdots + a_i + \cdots + a_j &\equiv r \pmod{m} \end{aligned}$$

for some $r \in \{1, 2, ..., m-1\}$. This implies that

$$\begin{aligned} &a_1 + a_2 + \cdots + a_i + \cdots + a_j \equiv a_1 + a_2 + \cdots + a_i \pmod{m}, \\ \text{i.e. } \; &m \mid ((a_1 + \cdots + a_i + \cdots + a_j) - (a_1 + a_2 + \cdots + a_i)) \\ \text{or } \; &m \mid (a_{i+1} + a_{i+2} + \cdots + a_j). \end{aligned}$$

Thus, $B = \{a_{i+1}, a_{i+2}, ..., a_j\}$ $(= A_j \backslash A_i)$ is a required subset of A. ∎

The following example is an IMO problem (IMO, 1972/1) with the original statement rephrased.

Example 3.3.4. Let $X \subseteq \{1, 2, ..., 99\}$ and $|X| = 10$. Show that it is possible to select two disjoint nonempty proper subsets Y, Z of X such that $\sum(y \mid y \in Y) = \sum(z \mid z \in Z)$.

For instance, if $X = \{2, 7, 15, 19, 23, 50, 56, 60, 66, 99\}$, take $Y = \{19, 50\}$ and $Z = \{2, 7, 60\}$, and check that

$$\sum(y \mid y \in Y) = 19 + 50 = 2 + 7 + 60 = \sum(z \mid z \in Z).$$

The required conclusion suggests that we may treat the nonempty proper subsets of X as our "objects", and create the "boxes" for their possible sums. If the number of "objects" is larger than the number of "boxes", then there are two nonempty proper subsets of X which have the same sum. This conclusion is very close to what we want. Now, how are we going to estimate the number of "objects" and the number of "boxes"? These are two crucial questions that we have to answer.

Solution. Since $|X| = 10$, by Example 1.5.2, the number of nonempty proper subsets of X (excluding $\emptyset$, X) is

$$2^{10} - 2 = 1022.$$

On the other hand, for each nonempty proper subset A of X,

$$1 \leq \sum(a \mid a \in A) \leq 91 + 92 + \cdots + 99 = 885;$$

that is, the sum of numbers in each A lies inclusively between 1 and 855.

Now, treat the 1022 nonempty proper subsets of X as our "1022 objects", and create "855 boxes" for the sums "$1, 2, ..., 855$". Since $1022 > 855$, by (PP), there are two distinct nonempty proper subsets B, C of X which have the same sum; i.e.,

$$\sum(b \mid b \in B) = \sum(c \mid c \in C).$$

(Note that B and C need not be disjoint and thus they may not be the desired subsets of X.) Clearly, $B \not\subseteq C$ and $C \not\subseteq B$. Let $Y = B \backslash (B \cap C)$ and $Z = C \backslash (B \cap C)$. Then we have $\sum(y \mid y \in Y) = \sum(z \mid z \in Z)$ (why?) and so Y and Z are two desired subsets of X. ∎

Example 3.3.5. **(IMO, 1983/4)** Let ABC be an equilateral triangle and $\mathcal{E}$ the set of all points contained in the 3 segments AB, BC, CA (including A, B and C). Show that, for every partition of $\mathcal{E}$ into 2 disjoint subsets, at least one of the 2 subsets contains the vertices of a right-angled triangle.

For instance, if the set $\mathcal{E}$ is partitioned into 2 subsets $\mathcal{E}_1$ and $\mathcal{E}_2$ as shown in Figure 3.3.1(a), then it is not difficult to find 3 points all in $\mathcal{E}_1$ or $\mathcal{E}_2$ (in this case both) which form a right-angled triangle (see Figure 3.3.1(b)).

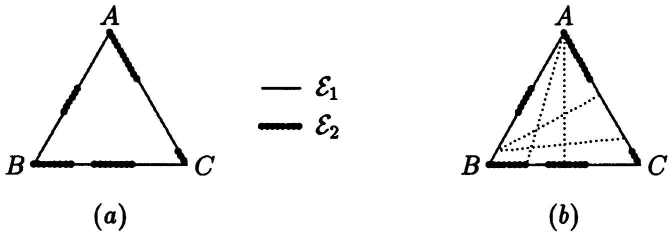


Figure 3.3.1.

Solution. We prove it by contradiction. Suppose to the contrary that there is a partition of $\mathcal{E}$ into 2 disjoint subsets $\mathcal{E}_1$ and $\mathcal{E}_2$ such that

(*) no three points in $\mathcal{E}_1$ (resp., $\mathcal{E}_2$) form a right-angled triangle.

Let X, Y and Z be the points on AB, BC and CA respectively such that

$$\frac{AX}{XB} = \frac{BY}{YC} = \frac{CZ}{ZA} = 2.$$

Consider ΔAZX. We claim that $\angle AZX = 90°$ (see Figure 3.3.2.).

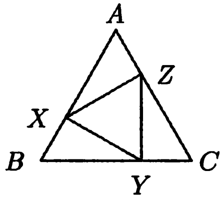


Figure 3.3.2

By cosine law,

$$\begin{aligned}(XZ)^2 &= (AZ)^2 + (AX)^2 - 2(AX)(AZ)\cos\angle XAZ \\ &= (\frac{1}{3}AC)^2 + (\frac{2}{3}AB)^2 - 2(\frac{1}{3}AC)(\frac{2}{3}AB)\cos 60° \\ &= \frac{1}{3}(AB)^2.\end{aligned}$$

Thus $(XZ)^2 + (AZ)^2 = \frac{1}{3}(AB)^2 + \frac{1}{9}(AC)^2 = (\frac{2}{3}AB)^2 = (AX)^2$, and by Pythagolas's theorem, $\angle AZX = 90°$, as claimed.

Similarly, we have $\angle BXY = \angle CYZ = 90°$.

Treat the points X, Y and Z as "3 objects", and create "2 boxes": one for $\mathcal{E}_1$ and one for $\mathcal{E}_2$. By (PP), at least 2 of the points are all in $\mathcal{E}_1$ or all in $\mathcal{E}_2$. By symmetry, say $X, Y \in \mathcal{E}_1$.

Since $YX \perp AB$ and since we assume the condition (*), no points in $AB\backslash\{X\}$ can be in $\mathcal{E}_1$; i.e., all points in $AB\backslash\{X\}$ must be in $\mathcal{E}_2$. The latter, in turn, implies that $C \notin \mathcal{E}_2$ and $Z \notin \mathcal{E}_2$ (why?); i.e., $C, Z \in \mathcal{E}_1$. But then we have $\{C, Z, Y\} \subseteq \mathcal{E}_1$, and they form a right-angled triangle. This, however, contradicts (*). The proof is thus complete. ∎

For more examples and a special application of (PP) to map-colouring problems, the reader may like to read the excellent expository article [Re] by Rebman.

3.4. Ramsey Type Problems and Ramsey Numbers

In this section, we shall continue to apply (PP) to solve a class of problems that are of a different flavour from the previous ones. The study of these problems leads to the introduction of a famous class of numbers, called Ramsey numbers.

To begin with, we first state an IMO problem, that was proposed by Hungarian representatives in the 6th IMO, held in Russia in 1964.

Example 3.4.1 (IMO, 1964/4) Seventeen people correspond by mail with one another – each one with all the rest. In their letters only three different topics are discussed. Each pair of correspondents deals with only one of these topics. Prove that there are at least three people who write to one another about the same topic.

We shall solve this problem later on. To make it easier to follow the solution, let us first mention two simpler but related problems.

Example 3.4.2. Prove that at a gathering of any six people. Some three of them are either mutual acquaintances or complete strangers to one another.

The above problem was proposed by Bostwick of Maryland, USA, in 1958 as Problem E 1321 in the American Mathematical Monthly, **65** (1958), 446. As shown in the American Mathematical Monthly, **66** (1959), 141-142, Example 3.4.2 received much attention, and various solutions were provided. It was further pointed out by someone there that Example 3.4.2 could indeed be reduced to the following problem.

Example 3.4.3. Six points are in general position in space (no three in a line, no four in a plane). The fifteen line segments joining them in pairs are drawn, and then painted with some segments red and the rest blue. Prove that some triangle has all its sides the same colour.

Example 3.4.3 first appeared as a competition problem in Hungary in 1947, and was also a problem in the William Lowell Putnam Mathematical Competition (for undergraduates in North America) that was held in 1953.

Let us first prove Example 3.4.3 by apply (PP).

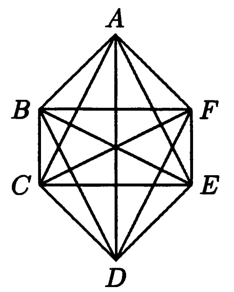


Figure 3.4.1.

Proof of Example 3.4.3. Figure 3.4.1 shows a configuration with 6 vertices (or points) in which every two are joined by an edge (or a line segment). Each of these $\binom{6}{2} = 15$ edges is coloured by one of the two colours: blue and red.

Consider the vertex A. The 5 edges incident with A (namely, AB, AC, AD, AE and AF) are each coloured by either blue or red. By (PP), one of the colours, say blue, is used to colour at least 3 of these 5 edges. By symmetry, assume that AB, AC and AD are coloured blue. Consider now the 3 edges BC, BD and CD. If any one of them (say BC) is coloured blue, then we have a "blue triangle" (namely, ABC). If none of them is coloured blue, then the 3 edges must be coloured red, and in this case, we have a "red triangle" BCD. ∎

We now proceed to prove Example 3.4.2.

Proof of Example 3.4.2. Represent the 6 people by 6 vertices $A, B, ..., F$ as shown in the configuration of Figure 3.4.1. Given any 2 people X and Y, the edge joining X and Y is coloured blue (resp., red) if X and Y are acquaintances (resp., strangers). By Example 3.4.3, there exists in the resulting configuration either a "blue triangle" or a "red triangle". In other words, there are 3 mutual acquaintances or 3 mutual strangers. ∎

We are now in a position to solve the IMO problem mentioned earlier.

Proof of Example 3.4.1. We represent the 17 people by 17 vertices $A, B, C, \ldots$, in which every two are joined by an edge. An edge joining two vertices X and Y is coloured blue (resp., red and yellow) if X and Y discuss topic I (resp., II and II). Consider the vertex A. By assumption, the 16 edges incident with A are each coloured by exactly one of the 3 colours: blue, red and yellow. Since $16 = 5 \cdot 3 + 1$, by (PP), one of the colours, say blue, is used to colour at least $5 + 1 (= 6)$ edges. By symmetry, assume that AB, AC, AD, AE, AF and AG are coloured blue. Consider now the configuration consisting of 6 vertices $B, C, \ldots, G$, together with the 15 edges joining all pairs of the 6 vertices. If any one these edges, say BC, is coloured blue, then we have a "blue triangle", namely ABC. If none of them is coloured blue, then the 15 edges must be coloured red or yellow. But then by Example 3.4.3, there is in this configuration a "red triangle" or a "yellow triangle". In any event, there is a triangle having all its edges the same colour. This means that there are at least 3 people who discuss the same topic with one another. ∎

Though the IMO problem has been solved, we cannot help but carry on the story.

Let us revisit Example 3.4.3. Suppose now we have only 5 vertices (instead of 6) in the problem. Is the conclusion still valid? To see this, consider the configuration of Figure 3.4.2 which consists of 5 vertices and 10 edges. If the edges AB, BC, CD, DE and EA are coloured blue while the rest red as shown, then the resulting configuration contains neither blue triangles nor red triangles. This shows that 6 is the minimum number of vertices that are needed in a configuration in order to ensure the existence of a triangle coloured by the same colour if each edge of the configuration is coloured by one of the two given colours.

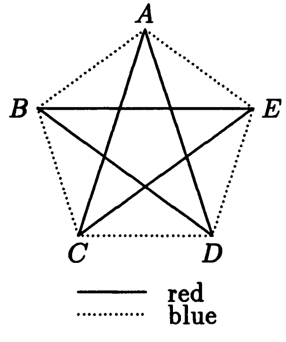


Figure 3.4.2.

The above discussion serves as a motivation for us to introduce a class of numbers, known as Ramsey numbers.

A *clique* is a configuration consisting of a finite set of vertices together with edges joining all pairs of vertices. A *k-clique* is a clique which has exactly k vertices. Thus a 1-clique is a vertex, a 2-clique is an edge joining 2 vertices and a 3-clique looks like a triangle. The configuration of Figure 3.4.2 is a 5-clique and that of Figure 3.4.1 is a 6-clique. Given $p, q \in \mathbf{N}$, let $R(p,q)$ denote the smallest natural number "n" such that for any colouring of the edges of an n-clique by 2 colours: blue or red (one colour for each edge), there exists either a "blue p-clique" or a "red q-clique".

Thus, as shown above, we have $R(3,3) = 6$. The following equalities follow directly from the definition:

$$\begin{cases} R(p,q) = R(q,p) \\ R(1,q) = 1 \\ R(2,q) = q. \end{cases} \tag{3.4.1}$$

The numbers $R(p,q)$ are called *Ramsey numbers*, in honour of the English philosopher Frank P. Ramsey (1903-1930), who proved around 1928 a remarkable existence theorem [Ra] that includes the following as a special case.

Theorem 3.4.1. (Ramsey's Theorem) *For all integers $p, q \geq 2$, the number $R(p,q)$ always exists.*

Ramsey died of complications following an abdominal operation in 1930 before his 27th birthday. In 1983, a special issue of the Journal of Graph Theory (Vol.7 No.1) was dedicated to the memory of Ramsey on the occasion of the 80th anniversary of his birth.

3.5. Bounds for Ramsey Numbers

The determination of the exact values of $R(p,q)$, where p and q are large, is far beyond our research. In this section, we present some bounds for $R(p,q)$ that may be useful in estimating these numbers. The following recursive upper bound for $R(p,q)$ was obtained by two Hungarian mathematicians Erdös and Szekeres [ES] (see also Greenwood and Gleason [GG]).

Theorem 3.5.1. *For all integers $p, q \geq 2$,*

$$R(p,q) \leq R(p-1,q) + R(p,q-1).$$

Before proving this theorem, we state the following *Generalized Pigeonhole Principle*:

The Generalized Pigeonhole Principle (GPP).

Let $n, k_1, k_2, \ldots, k_n \in \mathbf{N}$. If $k_1 + k_2 + \cdots k_n - (n-1)$ or more objects are put into n boxes, then either the first box contains at least k_1 objects, or the second box contains at least k_2 objects, ..., or the nth box contains at least k_n objects.

Proof of Theorem 3.5.1. Let $n = R(p-1,q) + R(p,q-1)$. Since $R(p,q)$ always exists by Theorem 3.4.1, to show that $R(p,q) \leq n$, we need only to prove that for any colouring of the edges of an n-clique K_n by 2 colours: blue and red, there exists either a "blue p-clique" or a "red q-clique".

Fix a vertex, say v in K_n. Then v is incident with $n-1 = R(p-1,q) + R(p,q-1) - 1$ edges in K_n. By (GPP), either $R(p-1,q)$ of the edges are coloured blue or $R(p,q-1)$ of the edges are coloured red, say the former.

Let X be the set of vertices of K_n, other than v, which are incident with these $R(p-1,q)$ blue edges. Since $|X| = R(p-1,q)$, by definition, the clique induced by X contains either a blue $(p-1)$-clique or a red q-clique. If it contains a red q-clique, then we are through. If it contains a blue $(p-1)$-clique, then the clique induced by $X \cup \{v\}$ contains a blue p-clique. ∎

The inequality in Theorem 3.5.1 can be slightly improved under an additional condition.

Theorem 3.5.2. [GG] *For all integers $p, q \geq 2$, if $R(p-1,q)$ and $R(p,q-1)$ are even, then*

$$R(p,q) \leq R(p-1,q) + R(p,q-1) - 1.$$

Proof. Let $m = R(p-1,q)+R(p,q-1)-1$ and let K_m be an m-clique in which the $\binom{m}{2}$ edges are coloured blue or red. Fix an arbitrary vertex w in K_m. Then w is incident with $m-1$ edges. If $R(p-1,q)$ or more of the edges are blue, then as shown in the above proof, there is either a blue p-clique or a red q-clique in K_m. Likewise, if $R(p,q-1)$ or more of the edges are red, then we are again through. It remains to consider the following case: there are exactly $R(p-1,q)-1$ blue edges and $R(p,q-1)-1$ red edges incident with *each* vertex v in K_m. We claim that this is impossible. Indeed, if this were the case, then the number of blue edges in K_m would be

$$\frac{m}{2} \cdot \{R(p-1,q)-1\}.$$

Since $R(p-1,q)$ and $R(p,q-1)$ are even, both m and $R(p-1,q)-1$ are odd, and so the above number is not an integer, a contradiction. The proof is thus complete. ■

By applying Theorem 3.5.1, it is not hard to prove, by induction on $p+q$, the following result:

$$\text{If } p,q \geq 2, \text{ then } R(p,q) \leq \binom{p+q-2}{p-1}. \tag{3.5.1}$$

When $p=3$, inequality (3.5.1) becomes

$$R(3,q) \leq \frac{1}{2}(q^2+q).$$

However, by applying Theorems 3.5.1 and 3.5.2, the following sharper bound can be derived (see Problem 3.33):

$$\text{For } q \in \mathbf{N},\ R(3,q) \leq \tfrac{1}{2}(q^2+3). \tag{3.5.2}$$

When $p = q \geq 3$, we also have:

$$\frac{p \cdot 2^{\frac{p}{2}}}{\sqrt{2}e} < R(p,p) \leq 4R(p-2,p)+2. \tag{3.5.3}$$

The above upper bound for $R(p,p)$ was given by Walker [W] while the lower bound was proved by Erdös [E] using a probabilistic method.

As an example, let us show how Theorem 3.5.2 can be applied to obtain the exact value of $R(3,4)$.

We know that $R(2,4) = 4$ by (3.4.1) and $R(3,3) = 6$. Since both numbers are even, we have by Theorem 3.5.2

$$R(3,4) \leq R(2,4) + R(3,3) - 1 = 9.$$

We claim that $R(3,4) \geq 9$. To see this, consider the 8-clique of Figure 3.5.1. By colouring the edges $AB, BC, CD, DE, EF, FG, GH, HA, AE, BF, CG$ and DH blue and the rest red, it can be checked that the resulting configuration contains neither a "blue 3-clique" nor a "red 4-clique". This shows that $R(3,4) \geq 9$. Hence we have $R(3,4) = 9$.

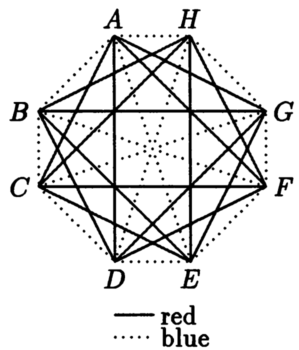


Figure 3.5.1.

Note. The following points should be borne in mind. To show that $R(p,q) \leq n$, we may apply known inequalities or show by definition that every n-clique in which the edges are coloured blue or red contains either a blue p-clique or a red q-clique. On the other hand, to show that $R(p,q) > n$, we may construct an n-clique K_n and colour the $\binom{n}{2}$ edges blue or red in a specific way so that K_n contains neither blue p-clique nor red q-clique.

Some known exact values or bounds for $R(p,q)$ when p and q are small are contained in Table 3.5.1 (see [CG], [GRS] and [MZ]). Grinstead and Roberts [GR] showed that $28 \leq R(3,8) \leq 29$. In the recent article [MZ], McKay and Zhang proved that $R(3,8) = 28$. It was also reported in [CL] that $R(4,8) \geq 52$.

p \ q	3	4	5	6	7	8	9	10
3	6	9	14	18	23	28	36	40-43
4		18	25-28	34-44				
5			42-55	57-94				
6				102-169				
7					126-586			

Table 3.5.1

A Generalization. The definition of the Ramsey number $R(p,q)$ with 2 parameters can be generalized in a natural way to the Ramsey number $R(p_1,p_2,...,p_k)$ with k parameters as follows. Let $k,p_1,p_2,...,p_k \in \mathrm{N}$ with $k \geq 3$. The *Ramsey number* $R(p_1,p_2,...,p_k)$ is the smallest natural number n such that for any colouring of the edges of an n-clique by k colours: colour 1, colour 2, ..., colour k, there exist a colour i ($i = 1,2,...,k$) and a p_i-clique in the resulting configuration such that all edges in the p_i-clique are coloured by colour i.

The result of Example 3.4.1 shows that $R(3,3,3) \leq 17$. In 1955, Greenwood and Gleason [GG] proved by construction that $R(3,3,3) \geq 17$. Thus $R(3,3,3) = 17$. Surprisingly enough, this is the only exact value known up to date for $R(p_1,p_2,...,p_k)$ when $k \geq 3$ and $p_i \geq 3$ for each $i = 1,2,...,k$.

Final remarks. We have by now introduced some very basic knowledge of Ramsey numbers and shown how they are linked to (PP) and (GPP). The theory of Ramsey numbers forms in fact a tiny part of the more profound and more general Ramsey theory of structures. One may obtain a rough scope of this general theory from the book [GRS] by Graham, Rothschild and Spencer. Just like other theories in combinatorics, the theory of Ramsey numbers can also find applications in other areas. An introduction of applications of Ramsey numbers to areas such as number theory, geometry, computer science, communication, decision making, etc. can be found in Chapter 8 of Roberts' book [12]. Readers are also encouraged to read the two expository articles [G] and [GS] on Ramsey Theory.

As a result of numerous contributions from Ramsey theoreticians, Ramsey theory has now been recognized as a cohesive, established and growing branch of combinatorics and graph theory, that in no way could have been anticipated by Frank Ramsey when he read to the London Mathematical Society his celebrated article [Ra] in 1928 at the age of 24, two years before he left the world.

Exercise 3

1. Show that among any 5 points in an equilateral triangle of unit side length, there are 2 whose distance is at most $\frac{1}{2}$ units apart.

2. Given any set C of $n+1$ distinct points ($n \in \mathbf{N}$) on the circumference of a unit circle, show that there exist $a, b \in C, a \neq b$, such that the distance between them does not exceed $2\sin\frac{\pi}{n}$.

3. Given any set S of 9 points within a unit square, show that there always exist 3 distinct points in S such that the area of the triangle formed by these 3 points is less than or equal to $\frac{1}{8}$. (Beijing Math. Competition, 1963)

4. Show that given any set of 5 numbers, there are 3 numbers in the set whose sum is divisible by 3.

5. Let A be a set of $n+1$ elements, where $n \in \mathbf{N}$. Show that there exist $a, b \in A$ with $a \neq b$ such that $n|(a-b)$.

6. Let $A = \{a_1, a_2, \cdots, a_{2k+1}\}$, where $k \geq 1$, be a set of $2k+1$ positive integers. Show that for any permutation $a_{i_1}a_{i_2}\ldots a_{i_{2k+1}}$ of A, the product
$$\prod_{j=1}^{2k+1}(a_{i_j} - a_j)$$
is always even.

7. Let $A \subseteq \{1, 2, \ldots, 2n\}$ such that $|A| = n+1$, where $n \in \mathbf{N}$. Show that there exist $a, b \in A$, with $a \neq b$ such that $a|b$.

8. Let A be a subset of $\{1, 2, \ldots, 2n\}$ such that $|A| = n+1$. Show that there exist $a, b \in A$ such that a and b are coprime.

9. Show that among any group of n people, where $n \geq 2$, there are at least two people who know exactly the same number of people in the group (assuming that "knowing" is a symmetry relation).

10. Let $C = \{r_1, r_2, \ldots, r_{n+1}\}$ be a set of $n+1$ real numbers, where $0 \leq r_i < 1$ for each $i = 1, 2, \ldots, n+1$. Show that there exist r_p, r_q in C, where $p \neq q$, such that $|r_p - r_q| < \frac{1}{n}$.

11. Show that given any set A of 13 distinct real numbers, there exist $x, y \in A$ such that
$$0 < \frac{x-y}{1+xy} \leq 2 - \sqrt{3}.$$

12. Consider a set of $2n$ points in space, $n > 1$. Suppose they are joined by at least n^2+1 segments. Show that at least one triangle is formed. Show that for each n it is possible to have $2n$ points joined by n^2 segments without any triangles being formed. (Putnam, 1956)

13. Let there be given nine lattice points (points with integral coordinates) in the three dimensional Euclidean space. Show that there is a lattice point on the interior of one of the line segments joining two of these points. (Putnam, 1971)

14. (i) A point (a_1, a_2) in the $x-y$ plane is called a *lattice point* if both a_1 and a_2 are integers. Given any set L_2 of 5 lattice points in the $x-y$ plane, show that there exist 2 distinct members in L_2 whose midpoint is also a lattice point (not necessarily in L_2).

    More generally, we have:

    (ii) A point $(a_1, a_2, \ldots, a_n)$ in the space $\mathbf{R}^n$, where $n \geq 2$ is an integer, is called a *lattice point* if all the a_i's are integers. Show that given any set L_n of $2^n + 1$ lattice points in $\mathbf{R}^n$, there exist 2 distinct members in L_n whose midpoint is also a lattice point (but not necessarily in L_n).

15. Let A be any set of 20 distinct integers chosen from the arithmetic progression $1, 4, 7, \ldots, 100$. Prove that there must be two distinct integers in A whose sum is 104. (Putnam, 1978)

16. Let A be a set of 6 points in a plane such that no 3 are collinear. Show that there exist 3 points in A which form a triangle having an interior angle not exceeding $30°$. (26th Moscow MO)

17. Let $n \geq 3$ be an odd number. Show that there is a number in the set
$$\{2^1 - 1, 2^2 - 1, \ldots, 2^{n-1} - 1\}$$
which is divisible by n. (USSR MO, 1980)

18. There are n people at a party. Prove that there are two people such that, of the remaining $n-2$ people, there are at least $\lfloor n/2 \rfloor - 1$ of them, each of whom either knows both or else knows neither of the two. Assume that "knowing" is a symmetric relation, and that $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x. (USA MO, 1985/4)

19. For a finite set A of integers, denote by $s(A)$ the sum of numbers in A. Let S be a subset of $\{1, 2, 3, \ldots, 14, 15\}$ such that $s(B) \neq s(C)$ for any 2 disjoint subsets B, C of S. Show that $|S| \leq 5$. (USA MO, 1986)

20. In the rectangular array

$$\begin{array}{cccc} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{array}$$

of $m \times n$ real numbers, the difference between the maximum and the minimum element in each row is at most d, where $d > 0$. Each column is then rearranged in decreasing order so that the maximum element of the column occurs in the first row, and the minimum element occurs in the last row. Show that in the rearranged array the difference between the maximum and the minimum elements in each row is still at most d. (Swedish Math. Competition, 1986)

21. We are given a regular decagon with all diagonals drawn. The number "+1" is attached to each vertex and to each point where diagonals intersect (we consider only internal points of intersection). We can decide at any time to simultaneously change the sign of all such numbers along a given side or a given diagonal. Is it possible after a certain number of such operations to have changed all the signs to negative? (International Mathematics Tournament of the Towns, Senior, 1984)

22. In a football tournament of one round (each team plays each other once, 2 points for win, 1 point for draw and 0 points for loss), 28 teams compete. During the tournament more than 75% of the matches finished in a draw. Prove that there were two teams who finished with the same number of points. (International Mathematics Tournament of the Towns, Junior, 1986)

23. Fifteen problems, numbered 1 through 15, are posed on a certain examination. No student answers two consecutive problems correctly. If 1600 candidates sit the test, must at least two of them have the identical answer patterns? (Assume each question has only 2 possible answers, right or wrong, and assume that no student leaves any question unanswered.) (24th Spanish MO, 1989)

24. Suppose that $a_1 \leq a_2 \leq \cdots \leq a_n$ are natural numbers such that $a_1 + \cdots + a_n = 2n$ and such that $a_n \neq n+1$. Show that if n is even, then for some subset K of $\{1, 2, \ldots, n\}$ it is true that $\sum_{i \in K} a_i = n$. Show that this is true also if n is odd when we make the additional assumption that $a_n \neq 2$. (Proposed by J. Q. Longyear, see *Amer. Math. Monthly*, **80** (1973), 946-947.)

25. Let X be a nonempty set having n elements and C be a colour set with $p \geq 1$ elements. Find the greatest number p satisfying the following property: If we colour in an arbitrary way each subset of X with colours from C such that each subset receives only one colour, then there exist two distinct subsets A, B of X such that the sets $A, B, A \cup B, A \cap B$ have the same colour. (Proposed by I. Tomescu, see *Amer. Math. Monthly*, **95** (1988), 876-877.)

26. Consider the system of p equations in $q = 2p$ unknowns $x_1, x_2, \ldots, x_q$:

$$\begin{array}{ll} a_{11}x_1 + a_{12}x_2 + \cdots & +a_{1q}x_q = 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots & +a_{2q}x_q = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots \\ a_{p1}x_1 + a_{p2}x_2 + \cdots & +a_{pq}x_q = 0 \end{array}$$

with every coefficient a_{ij} a member of the set $\{-1, 0, 1\}$. Prove that the system has a solution $(x_1, x_2, \ldots, x_q)$ such that

(a) all x_j $(j = 1, 2, \ldots, q)$ are integers,

(b) there is at least one value of j for which $x_j \neq 0$,

(c) $|x_j| \leq q \quad (j = 1, 2, \ldots, q)$.

(IMO, 1976/5)

27. An international society has its members from six different countries. The list of members contains 1978 names, numbered $1, 2, \ldots, 1978$. Prove that there is at least one member whose number is the sum of the numbers of two members from his own country, or twice as large as the number of one member from his own country. (IMO, 1978/6)

28. Let $p, q \in \mathbf{N}$. Show that in any given sequence of $R(p,q)$ distinct integers, there is either an increasing subsequence of p terms or a decreasing subsequence of q terms.

29. Show that given any sequence of $pq+1$ distinct real numbers, where p and q are nonnegative integers, there is either an increasing subsequence of $p+1$ terms or a decreasing subsequence of $q+1$ terms. (P. Erdös and G. Szekeres (1935))

30. Show that

    (a) $R(p,q) = R(q,p)$, for all $p, q \in \mathbf{N}$;

    (b) $R(2,q) = q$, for all $q \in \mathbf{N}$.

31. Let $p, p', q, q' \in \mathbf{N}$ with $p' \le p$ and $q' \le q$. Show that

    (i) $R(p', q') \le R(p,q)$;

    (ii) $R(p-1, q) \le R(p,q) - 1$ for $p \ge 2$;

    (iii) $R(p', q') = R(p,q)$ iff $p' = p$ and $q' = q$.

32. For $p, q \in \mathbf{N}$, show that

$$R(p,q) \le \binom{p+q-2}{p-1}.$$

33. Show that

$$R(3,q) \le \frac{1}{2}(q^2+3)$$

    for $q \ge 1$.

34. Show that $R(3,5) = 14$.

35. Show that

    (a) $R(4,4) \le 18$,

    (b) $R(3,6) \le 19$.

36. Show that

    (a) $R(p_1, p_2, \ldots, p_k) = 1$ if $p_i = 1$ for some $i \in \{1, 2, \ldots, k\}$;

    (b) $R(p, 2, 2, \ldots, 2) = p$ for $p \ge 2$.

37. Let $k, p_1, p_2, \ldots, p_k \in \mathbf{N}$ with $k \ge 2$. Show that

$$R(p_1, p_2, \ldots, p_k) = R(p_1, p_2, \ldots, p_k, 2).$$

38. Given any k integers $p_i \geq 2$, $i = 1, 2, \ldots, k$, where $k \geq 2$, show that

$$R(p_1, p_2, \ldots, p_k) \leq \sum_{i=1}^{k} R(p_1, \ldots, p_{i-1}, p_i - 1, p_{i+1}, \ldots, p_k) - (k-2).$$

39. Let $k \in \mathbf{N}, p_1, p_2, \ldots, p_k \in \mathbf{N}^*$ and $p = \sum_{i=1}^{k} p_i$. Show by induction on p that

$$R(p_1 + 1, p_2 + 1, \ldots, p_k + 1) \leq \frac{p!}{p_1! p_2! \ldots p_k!}.$$

40. For $k \in \mathbf{N}$ with $k \geq 2$, let R_k denote $R(\underbrace{3, 3, \ldots, 3}_{k})$. Show that

(a) (i) $R_k \leq k(R_{k-1} - 1) + 2$ for $k \geq 3$;

(ii) $R_k \leq \lfloor k!e \rfloor + 1$;

(iii) $R_4 \leq 66$.

(R. E. Greenwood and A. M. Gleason, *Canad. J. Math.*, 7 (1955), 1-7.)

(b) $R_k \geq 2^k + 1$.

41. Let $k \in \mathbf{N}$ and let $\{S_1, S_2, \ldots, S_k\}$ be any partition of the set $\mathbf{N}_n = \{1, 2, \ldots, n\}$, where $n = R(\underbrace{3, 3, \ldots, 3}_{k})$. Show that there exist $i \in \{1, 2, \ldots, k\}$, and some integers a, b, c (not necessarily distinct) in S_i such that $a + b = c$.

42. Show that

(i) $R(3, 3, 2) = 6$,

(ii) $R(3, 3, 3) \leq 17$. (See also Example 3.4.1.)

43. A p-clique is *monochromatic* if all its edges are coloured by the same colour.

(a) Show that for any colouring of the edges of the 6-clique K_6 by 2 colours: blue or red, there are at least two monochromatic 3-cliques (not necessarily disjoint).

(b) Give a colouring of the edges of K_6 by 2 colours such that there are no three monochromatic 3-cliques.

44. The edges of the 7-clique K_7 are coloured by 2 colours: blue or red. Show that there are at least four monochromatic 3-cliques in the resulting configuration.

45. Given any colouring of the edges of an n-clique $K_n (n \in \mathrm{N}, n \geq 3)$ by 2 colours, let $T(n)$ denote the number of monochromatic 3-cliques in the resulting configuration. Show that

$$T(n) \geq \begin{cases} \frac{1}{3}k(k-1)(k-2) & \text{if } n = 2k, \\ \frac{2}{3}k(k-1)(4k+1) & \text{if } n = 4k+1, \\ \frac{2}{3}k(k+1)(4k-1) & \text{if } n = 4k+3. \end{cases}$$

46. Each of the 36 line segments joining 9 distinct points on a circle is coloured either red or blue. Suppose that each triangle determined by 3 of the 9 points contains at least one red side. Prove that there are four points such that the 6 segments connecting them are red. (Canadian MO, 1976)

References.

[CG] F.R.K. Chung and C.M. Grinstead, A survey of bounds for classical Ramsey numbers, *J. Graph Theory,* **7** (1983), 25-37.

[CL] V. Chokbowornpaishand and V. Longani, Lower bounds for some Ramsey numbers, *Southeast Asian Bulletin of Maths.,* **15** (1991), 105-107.

[E] P. Erdös, Some remarks on the theory of graphs, *Bull. Amer. Math. Soc.* **53** (1947), 292-294.

[ES] P. Erdös and G. Szekeres, A combinatorial problem in geometry, *Coposito Math..* **2** (1935), 464-470.

[G] M. Gardner, Mathematical Games, *Scientific American,* **237(5)** (November, 1977), 18-28.

[GR] C.M. Grinstead and S.M. Roberts, On the Ramsey numbers $R(3,8)$ and $R(3,9)$, *J. Combinatorial Theory Ser. B,* **33** (1982), 27-51.

[GRS] R.L. Graham, B.L. Rothschild and J.H. Spencer, *Ramsey theory,* Second Edition, John Wiley & Sons, Inc., 1990.

[GS] R.L. Graham and J.H. Spencer, Ramsey Theory, *Scientific American,* July (1990), 112 -117.

[GG] R.E. Greenwood and A.M. Gleason, Combinatorial relations and chromatic graphs, *Canad. J. Math.* **7** (1955), 1-7.

[MZ] B.D. McKay and K.M. Zhang, The value of the Ramsey number $R(3,8)$, *J. Graph Theory,* **16** (1992), 99-105.

[Ra] F.P. Ramsey, On a problem of formal logic, *Proc. London Math. Soc.* **30** (1930), 264-286.

[Re] K.R. Rebman, The Pigeonhole Principle, Mathematical Gems, *Two-Year College Mathematics Journal,* **10** (1979), 3-12.

[W] K. Walker, Dichromatic graphs and ramsey numbers, *J. Combinatorial Theory,* **5** (1968), 238-243.