Chapter 5

The Binomial Coefficients

The numbers $\binom{n}{k}$ count the number of k-subsets of a set of n elements. They have many fascinating properties and satisfy a number of interesting identities. Because of their appearance in the binomial theorem (see Section 5.2), they are called the *binomial coefficients*. In formulas arising in the analysis of algorithms in theoretical computer science, the binomial coefficients occur over and over again, so a facility for manipulating them is useful. In this chapter, we discuss some of their elementary properties and identities. We prove a useful theorem of Sperner and then continue our study of partially ordered sets and prove an important theorem of Dilworth.

5.1 Pascal's Triangle

The binomial coefficients $\binom{n}{k}$ have been defined in Section 2.3 for all nonnegative integers k and n. Recall that $\binom{n}{k} = 0$ if $k > n$ and that $\binom{n}{0} = 1$ for all n. If n is positive and $1 \leq k \leq n$, then

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\cdots(n-k+1)}{k(k-1)\cdots 1}. \tag{5.1}$$

In Section 2.3, we noted that

$$\binom{n}{k} = \binom{n}{n-k}.$$

This relation is valid for all integers k and n with $0 \leq k \leq n$. We also derived Pascal's formula, which asserts that

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}:$$

By using Pascal's formula and the initial information

$$\binom{n}{0} = 1 \text{ and } \binom{n}{n} = 1, \qquad (n \geq 0),$$

the binomial coefficients can be calculated without recourse to the formula (5.1). When the binomial coefficients are calculated in this way, the results are often displayed in an infinite array known as *Pascal's triangle.* This array, which appeared in Blaise Pascal's *Traité du triangle arithmétique* in 1653, is illustrated in Figure 5.1.

$n\backslash k$	0	1	2	3	4	5	6	7	8	$\cdots$
0	1									
1	1	1								
2	1	2	1							
3	1	3	3	1						
4	1	4	6	4	1					
5	1	5	10	10	5	1				
6	1	6	15	20	15	6	1			
7	1	7	21	35	35	21	7	1		
8	1	8	28	56	70	56	28	8	1	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Figure 5.1 Pascal's triangle

Each entry in the triangle, other than those equal to 1 occurring on the boundary of the triangle, is obtained by adding together two entries in the row above: the one directly above and the one immediately to the left. This is in accordance with Pascal's formula. For instance, in row $n = 8$, we have

$$\binom{8}{3} = 56 = 35 + 21 = \binom{7}{3} + \binom{7}{2}.$$

Many of the relations involving binomial coefficients can be discovered by careful examination of Pascal's triangle. The symmetry relation

$$\binom{n}{k} = \binom{n}{n-k}$$

is readily noticed in the triangle. The identity

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n$$

of Theorem 3.3.2 is discovered by adding the numbers in a row of Pascal's triangle. The numbers $\binom{n}{1} = n$ in column $k = 1$ are the counting numbers. The numbers $\binom{n}{2} = n(n-1)/2$ in column $k = 2$ are the so-called *triangular numbers*, which equal the number of dots in the triangular arrays of dots illustrated in Figure 5.2.

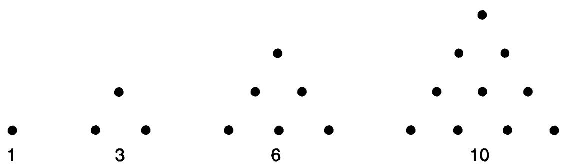

Figure 5.2

The numbers $\binom{n}{3} = n(n-1)(n-2)/3!$ in column $k = 3$ are the so-called *tetrahedral numbers*, and they equal the number of dots in tetrahedral arrays of dots (think of stacked cannon balls). Try now to examine Pascal's triangle for other relations involving binomial coefficients.

Another interpretation can be given to the entries of Pascal's triangle. Let n be a nonnegative integer and let k be an integer with $0 \le k \le n$. Define

$$p(n,k)$$

as the number of paths from the top left corner (the entry $\binom{0}{0} = 1$) to the entry $\binom{n}{k}$, where in each path we move from one entry to the entry in the next row immediately below it or immediately to its right. The two types of moves allowed in going from one entry to the next on the path are illustrated in Figure 5.3.

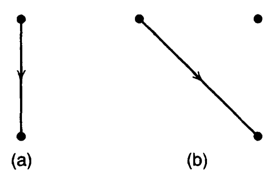

Figure 5.3

We define $p(0,0)$ to be 1, and, for each nonnegative integer n, we have

$$p(n,0) = 1 \qquad \text{(we must move straight down to reach } \binom{n}{0}\text{)}$$

and

$$p(n,n) = 1 \qquad \text{(we must move diagonally to reach } \binom{n}{n}\text{)}.$$

We note that each path from $\binom{0}{0}$ to $\binom{n}{k}$ is either

(1) a path from $\binom{0}{0}$ to $\binom{n-1}{k}$ followed by one vertical move of type (a) or

(2) a path from $\binom{0}{0}$ to $\binom{n-1}{k-1}$ followed by one diagonal move of type (b).

Thus, by the addition principle, we have

$$p(n,k) = p(n-1,k) + p(n-1,k-1),$$

a Pascal-type relation for the numbers $p(n,k)$. The numbers $p(n,k)$ are computed in exactly the same way as the binomial coefficients $\binom{n}{k}$, starting with the same initial values. Hence, for all integers n and k with $0 \le k \le n$,

$$p(n,k) = \binom{n}{k}.$$

Consequently, the value of an entry $\binom{n}{k}$ of Pascal's triangle represents the number of paths from the top left corner to that entry, using only moves of types (a) and (b). Therefore, we have another combinatorial interpretation of the numbers $\binom{n}{k}$.

5.2 The Binomial Theorem

The binomial coefficients receive their name from their appearance in the binomial theorem. The first few cases of this theorem should be familiar algebraic identities.

Theorem 5.2.1 *Let n be a positive integer. Then, for all x and y,*

$$(x+y)^n = x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-1}x^1y^{n-1} + y^n.$$

In summation notation,

$$(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k.$$

First proof. Write $(x+y)^n$ as the product

$$(x+y)(x+y)\cdots(x+y)$$

of n factors each equal to $x+y$. We completely expand this product, using the distributive law, and group like terms. Since, for each factor $(x+y)$, we can choose either x or y in multiplying out $(x+y)^n$, there are 2^n terms that result, and each can be arranged in the form $x^{n-k}y^k$ for some $k = 0, 1, \ldots, n$. We obtain the term $x^{n-k}y^k$ by choosing y in k of the n factors and x (by default) in the remaining $n-k$ factors.

Thus, the number of times the term $x^{n-k}y^k$ occurs in the expanded product equals the number $\binom{n}{k}$ of k-subsets of the set of n factors. Therefore,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k}y^k.$$

Second proof. The proof is by induction on n. It's more cumbersome and helps us appreciate the combinatorial viewpoint given in the first proof. If $n = 1$, the formula becomes

$$(x+y)^1 = \sum_{k=0}^{1} \binom{1}{k} x^{1-k}y^k = \binom{1}{0} x^1y^0 + \binom{1}{1} x^0y^1 = x+y,$$

and this is clearly true. We now assume that the formula is true for a positive integer n and prove that it is true when n is replaced by $n+1$. We write

$$(x+y)^{n+1} = (x+y)(x+y)^n,$$

which, by the induction assumption, becomes

$$\begin{aligned}
(x+y)^{n+1} &= (x+y)\left(\sum_{k=0}^{n} \binom{n}{k} x^{n-k}y^k\right) \\
&= x\left(\sum_{k=0}^{n} \binom{n}{k} x^{n-k}y^k\right) + y\left(\sum_{k=0}^{n} \binom{n}{k} x^{n-k}y^k\right) \\
&= \sum_{k=0}^{n} \binom{n}{k} x^{n+1-k}y^k + \sum_{k=0}^{n} \binom{n}{k} x^{n-k}y^{k+1} \\
&= \binom{n}{0} x^{n+1} + \sum_{k=1}^{n} \binom{n}{k} x^{n+1-k}y^k \\
&\quad + \sum_{k=0}^{n-1} \binom{n}{k} x^{n-k}y^{k+1} + \binom{n}{n} y^{n+1}.
\end{aligned}$$

Replacing k by $k-1$ in the last summation, we obtain

$$\sum_{k=0}^{n-1} \binom{n}{k} x^{n-k}y^{k+1} = \sum_{k=1}^{n} \binom{n}{k-1} x^{n+1-k}y^k.$$

Hence,

$$(x+y)^{n+1} = x^{n+1} + \sum_{k=1}^{n} \left[\binom{n}{k} + \binom{n}{k-1}\right] x^{n+1-k}y^k + y^{n+1},$$

which, using Pascal's formula, becomes

$$(x+y)^{n+1} = x^{n+1} + \sum_{k=1}^{n} \binom{n+1}{k} x^{n+1-k} y^k + y^{n+1}.$$

Since $\binom{n+1}{0} = \binom{n+1}{n+1} = 1$, we may rewrite this last equation and obtain

$$(x+y)^{n+1} = \sum_{k=0}^{n+1} \binom{n+1}{k} x^{n+1-k} y^k.$$

This is the binomial theorem with n replaced by $n+1$, and the theorem holds by induction. $\square$

The binomial theorem can be written in several other equivalent forms:

$$\begin{aligned}
(x+y)^n &= \sum_{k=0}^{n} \binom{n}{n-k} x^{n-k} y^k, \\
(x+y)^n &= \sum_{k=0}^{n} \binom{n}{n-k} x^k y^{n-k}, \\
(x+y)^n &= \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.
\end{aligned}$$

The first of these follows from Theorem 5.2.1 and the fact that

$$\binom{n}{k} = \binom{n}{n-k}, \quad (k = 0, 1, \ldots, n).$$

The other two follow by interchanging x with y.

The case $y = 1$ occurs sufficiently often to record it now as a special case.

Theorem 5.2.2 *Let n be a positive integer. Then, for all x,*

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k = \sum_{k=0}^{n} \binom{n}{n-k} x^k.$$

The special cases $n = 2, 3, 4$ of the binomial theorem are

$$\begin{aligned}
(x+y)^2 &= x^2 + 2xy + y^2, \\
(x+y)^3 &= x^3 + 3x^2y + 3xy^2 + y^3, \text{ and} \\
(x+y)^4 &= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4.
\end{aligned}$$

We note that the coefficients that occur in these expansions are the numbers in the row of Pascal's triangle. From Theorem 5.2.1 and the construction of Pascal's triangle, this is always the case.

We now consider some additional identities satisfied by the binomial coefficients. The identity

$$k\binom{n}{k} = n\binom{n-1}{k-1}, \qquad (n \text{ and } k \text{ positive integers}) \tag{5.2}$$

follows immediately from the fact that $\binom{n}{k} = 0$ if $k > n$ and

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k(k-1)\cdots 1} \qquad \text{for} \quad 1 \leq k \leq n.$$

The identity

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n, \quad (n \geq 0) \tag{5.3}$$

has already been proved as Theorem 3.3.2, but it also follows from the binomial theorem by setting $x = y = 1$. If we set $x = 1$, $y = -1$ in the binomial theorem, then we obtain the alternating sum

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n\binom{n}{n} = 0, \quad (n \geq 1). \tag{5.4}$$

Transposing the terms with a negative sign, we can also write this as

$$\binom{n}{0} + \binom{n}{2} + \cdots = \binom{n}{1} + \binom{n}{3} + \ldots, \quad (n \geq 1). \tag{5.5}$$

The identity (5.5) can be interpreted as follows: If S is a set of n elements, then the number of subsets of S with an even number of elements equals the number of subsets of S with an odd number of elements. Indeed, since the two sums are equal and, by (5.3), add up to 2^n, both have the value 2^{n-1}; that is,

$$\binom{n}{0} + \binom{n}{2} + \cdots = 2^{n-1}, \text{ and} \tag{5.6}$$

$$\binom{n}{1} + \binom{n}{3} + \cdots = 2^{n-1}. \tag{5.7}$$

We can verify these identities by combinatorial reasoning as follows: Let $S = \{x_1, x_2, \ldots, x_n\}$ be a set of n elements. We can think of subsets of S as resulting from the following decision process:

(1) we consider x_1 and decide either to put it in or leave it out (two choices);

(2) we consider x_2 and decide either to put it in or leave it out (two choices);

$\vdots$

(n) we consider x_n and decide either to put it in or leave it out (two choices).

We have n decisions to make each with two choices. Thus, there are 2^n subsets as we know by (5.3).

Now suppose we want to choose a subset with an even number of elements. Then as before we have two choices for each of $x_1, \ldots, x_{n-1}$. But when we get to x_n, we have only one choice. For if we have chosen an even number of the elements $x_1, x_2, \ldots, x_{n-1}$, we must leave x_n out; if we have chosen an odd number of the elements $x_1, x_2, \ldots, x_{n-1}$, we must put x_n in. Hence, the number of subsets of S with an even number of elements equals 2^{n-1}. Since the left side of (5.6) also counts the number of subsets of S with an even number of elements, (5.6) holds. In a similar way we verify (5.7). (However, now that we know that both (5.3) and (5.6) hold, so does (5.7).)

Using identities (5.2) and (5.3), we can derive the following identity:

$$1\binom{n}{1} + 2\binom{n}{2} + \cdots + n\binom{n}{n} = n2^{n-1}, (n \geq 1). \tag{5.8}$$

To see this, we first note that it follows from (5.2) that (5.8) is equivalent to

$$n\binom{n-1}{0} + n\binom{n-1}{1} + \cdots + n\binom{n-1}{n-1} = n2^{n-1}, (n \geq 1). \tag{5.9}$$

But now, by (5.3), with n replaced by $n-1$,

$$\begin{aligned} & n\binom{n-1}{0} + n\binom{n-1}{1} + \cdots + n\binom{n-1}{n-1} \\ = \;& n\left(\binom{n-1}{0} + \binom{n-1}{1} + \cdots + \binom{n-1}{n-1}\right) \\ = \;& n2^{n-1}. \end{aligned}$$

Thus, (5.9) and hence (5.8) hold. Another way to verify (5.8) is the following: By the binomial theorem,

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \cdots + \binom{n}{n}x^n.$$

If we differentiate both sides with respect to x, we obtain

$$n(1+x)^{n-1} = \binom{n}{1} + 2\binom{n}{2}x + 3\binom{n}{3}x^2 + \cdots + n\binom{n}{n}x^{n-1}.$$

Substituting $x = 1$, we get (5.8).

A number of interesting identities can be derived by successive differentiation and multiplication of the binomial expansion by x. For brevity we use the summation notation now. We begin with

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k. \tag{5.10}$$

Differentiating both sides of (5.10) with respect to x, we get

$$n(1+x)^{n-1} = \sum_{k=1}^{n} k\binom{n}{k} x^{k-1}. \tag{5.11}$$

Substituting $x = 1$ in (5.11), we get

$$n2^{n-1} = \sum_{k=1}^{n} k\binom{n}{k},$$

which is identity (5.8) again. Multiplying (5.11) by x, we get

$$nx(1+x)^{n-1} = \sum_{k=1}^{n} k\binom{n}{k} x^k. \tag{5.12}$$

Differentiating both sides of (5.12) with respect to x, we now get

$$n\left[(1+x)^{n-1} + (n-1)x(1+x)^{n-2}\right] = \sum_{k=1}^{n} k^2\binom{n}{k} x^{k-1}. \tag{5.13}$$

Substituting $x = 1$ in (5.13), we obtain

$$n\left[2^{n-1} + (n-1)2^{n-2}\right] = \sum_{k=1}^{n} k^2\binom{n}{k}; \tag{5.14}$$

hence,

$$n(n+1)2^{n-2} = \sum_{k=1}^{n} k^2\binom{n}{k}, \quad (n \geq 1). \tag{5.15}$$

By alternately differentiating with respect to x and multiplying by x, starting from (5.10), we can obtain an identity for

$$\sum_{k=1}^{n} k^p \binom{n}{k}$$

for any positive integer p, but this gets increasingly complicated as p gets large.

An identity for the sum of the squares of the numbers in the rows of Pascal's triangle is

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}, \quad (n \geq 0). \tag{5.16}$$

Identity (5.16) can be verified by combinatorial reasoning. Let S be a set with $2n$ elements. The right side of (5.16) counts the number of n-subsets of S. We partition S into two subsets, A and B, of n elements each. We use this partition of S to partition the n-subsets of S. Each n-subset of S contains a number k of elements of A, and the remaining $n-k$ elements come from B. Here, k may be any integer between 0 and n. We partition the n-subsets of S into $n+1$ parts,

$$C_0, C_1, C_2, \ldots, C_n,$$

where C_k consists of those n-subsets which contain k elements from A and $n-k$ elements from B. By the addition principle,

$$\binom{2n}{n} = |C_0| + |C_1| + |C_2| + \cdots + |C_n|. \tag{5.17}$$

An n-subset in C_k is obtained by choosing k elements from A (there are $\binom{n}{k}$ choices) and then $(n-k)$ elements from B (there are $\binom{n}{n-k}$ choices). Hence, by the multiplication principle,

$$|C_k| = \binom{n}{k}\binom{n}{n-k} = \binom{n}{k}^2, \quad (k = 0, 1, \ldots, n).$$

Substituting this into (5.17), we obtain

$$\binom{2n}{n} = \binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2,$$

and this proves (5.16). (A generalization of this identity, called the *Vandermonde convolution*, is given in Exercise 25.)

We now extend the domain of definition of the numbers $\binom{n}{k}$ to allow n to be any real number and k to be any integer (positive, negative, or zero).

Let r be a real number and let k be an integer. We then define the binomial coefficient $\binom{r}{k}$ by

$$\binom{r}{k} = \begin{cases} \dfrac{r(r-1)\cdots(r-k+1)}{k!} & \text{if } k \geq 1 \\ 1 & \text{if } k = 0 \\ 0 & \text{if } k \leq -1. \end{cases}$$

For instance,

$$\begin{aligned} \binom{5/2}{4} &= \frac{(5/2)(3/2)(1/2)(-1/2)}{4!} = \frac{-5}{128}, \\ \binom{-8}{2} &= \frac{(-8)(-9)}{2} = 36, \\ \binom{3.2}{0} &= 1, \text{ and} \\ \binom{3}{-2} &= 0. \end{aligned}$$

Pascal's formula and formula (5.2), namely,

$$\binom{r}{k} = \binom{r-1}{k} + \binom{r-1}{k-1} \quad \text{and} \quad k\binom{r}{k} = r\binom{r-1}{k-1},$$

are now valid for all r and k. Each of these formulas can be verified by direct substitution. By iteration of Pascal's formula, we can obtain two summation formulas for the binomial coefficients.

Consider Pascal's formula,

$$\binom{r}{k} = \binom{r-1}{k} + \binom{r-1}{k-1},$$

with k equal to a positive integer. We can apply Pascal's formula to either of the binomial coefficients on the right and obtain an expression for $\binom{r}{k}$ as a sum of three binomial coefficients. Suppose we repeatedly apply Pascal's formula to the last binomial coefficient that appears in it (the one with the smaller lower argument). We then obtain

$$\begin{aligned} \binom{r}{k} &= \binom{r-1}{k} + \binom{r-1}{k-1} \\ \binom{r}{k} &= \binom{r-1}{k} + \binom{r-2}{k-1} + \binom{r-2}{k-2} \\ \binom{r}{k} &= \binom{r-1}{k} + \binom{r-2}{k-1} + \binom{r-3}{k-2} + \binom{r-3}{k-3} \end{aligned}$$

$$\begin{aligned} &\vdots \\ \binom{r}{k} &= \binom{r-1}{k} + \binom{r-2}{k-1} + \binom{r-3}{k-2} + \cdots + \\ &\quad \binom{r-k}{1} + \binom{r-k-1}{0} + \binom{r-k-1}{-1}. \end{aligned}$$

The last term $\binom{r-k-1}{-1}$ has value 0 and can be deleted. If we replace r with $r+k+1$ in the summation above and transpose terms, we obtain

$$\binom{r}{0} + \binom{r+1}{1} + \cdots + \binom{r+k}{k} = \binom{r+k+1}{k}. \tag{5.18}$$

Identity (5.18) is valid for all real numbers r and all integers k. Notice that in (5.18) the upper argument starts with some number r, the lower argument starts with 0, and these arguments are successively increased by 1; the sum is then the binomial coefficient whose upper argument is 1 more than the last upper argument and whose lower argument is the last lower argument.

Now suppose we repeatedly apply Pascal's formula to the first binomial coefficient that appears in it. For simplicity, we now assume that r is a positive integer n, and we also assume that k is a positive integer.

$$\begin{aligned} \binom{n}{k} &= \binom{n-1}{k} + \binom{n-1}{k-1} \\ \binom{n}{k} &= \binom{n-2}{k} + \binom{n-2}{k-1} + \binom{n-1}{k-1} \\ \binom{n}{k} &= \binom{n-3}{k} + \binom{n-3}{k-1} + \binom{n-2}{k-1} + \binom{n-1}{k-1} \\ &\vdots \\ \binom{n}{k} &= \binom{0}{k} + \binom{0}{k-1} + \binom{1}{k-1} + \cdots + \binom{n-2}{k-1} + \binom{n-1}{k-1} \end{aligned}$$

Using the fact that $\binom{0}{k} = 0$ (and so we can drop this term), replacing n with $n+1$, and replacing k with $k+1$, we obtain

$$\binom{n+1}{k+1} = \binom{0}{k} + \binom{1}{k} + \cdots + \binom{n-1}{k} + \binom{n}{k}. \tag{5.19}$$

The identity (5.19) is valid for all positive integers k and n. It is important to understand that this identity is just an iterated form of Pascal's formula. Of course. the first nonzero term in (5.19) is $\binom{k}{k} = 1$.

If we take $k = 1$ in (5.19), we obtain

$$1 + 2 + \cdots + (n-1) + n = \frac{(n+1)n}{2},$$

the formula for the sum of the first n positive integers.

The identities (5.18) and (5.19) can be proved formally by mathematical induction and Pascal's formula. These are left as exercises. Some other identities for the binomial coefficients are given in the exercises.

5.3 Unimodality of Binomial Coefficients

If we examine the binomial coefficients in a row of Pascal's triangle, we notice that the numbers increase for a while and then decrease. A sequence of numbers with this property is called *unimodal.* Thus, the sequence $s_0, s_1, s_2, \ldots, s_n$ is unimodal, provided that there is an integer t with $0 \leq t \leq n$, such that

$$s_0 \leq s_1 \leq \cdots \leq s_t, \quad s_t \geq s_{t+1} \geq \cdots \geq s_n.$$

The number s_t is the largest number in the sequence. The integer t is not necessarily unique because the largest number may occur in the sequence more than once. For instance, if $s_0 = 1$, $s_1 = 3$, $s_2 = 3$, and $s_3 = 2$, then

$$s_0 \leq s_1 \leq s_2, \qquad s_2 \geq s_3, \qquad (t = 2)$$

but also

$$s_0 \leq s_1, \qquad s_1 \geq s_2 \geq s_3 \qquad (t = 1).$$

Theorem 5.3.1 *Let n be a positive integer. The sequence of binomial coefficients*

$$\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n}$$

is a unimodal sequence. More precisely, if n is even,

$$\binom{n}{0} < \binom{n}{1} < \cdots < \binom{n}{n/2},$$

$$\binom{n}{n/2} > \cdots > \binom{n}{n-1} > \binom{n}{n},$$

and if n is odd,

$$\binom{n}{0} < \binom{n}{1} < \cdots < \binom{n}{(n-1)/2} = \binom{n}{(n+1)/2},$$

$$\binom{n}{(n+1)/2} > \cdots > \binom{n}{n-1} > \binom{n}{n}.$$

Proof. We consider the quotient of successive binomial coefficients in the sequence. Let k be an integer with $1 \leq k \leq n$. Then

$$\frac{\binom{n}{k}}{\binom{n}{k-1}} = \frac{\frac{n!}{k!(n-k)!}}{\frac{n!}{(k-1)!(n-k+1)!}} = \frac{n-k+1}{k}.$$

Hence,

$$\binom{n}{k-1} < \binom{n}{k}, \quad \binom{n}{k-1} = \binom{n}{k} \text{ or } \binom{n}{k-1} > \binom{n}{k},$$

according to

$$k < n-k+1, \qquad k = n-k+1 \quad \text{or} \quad k > n-k+1.$$

Now, $k < n-k+1$ if and only if $k < (n+1)/2$. If n is even, then, since k is an integer, $k < (n+1)/2$ is equivalent to $k \leq n/2$. If n is odd, then $k < (n+1)/2$ is equivalent to $k \leq (n-1)/2$. Hence, the binomial coefficients increase as indicated in the statement of the theorem. We now observe that $k = n-k+1$ if and only if $2k = n+1$. If n is even, $2k \neq n+1$ for any k. If n is odd, then $2k = n+1$, for $k = (n+1)/2$. Thus, for n even, no two consecutive binomial coefficients in the sequence are equal. For n odd, the only two consecutive binomial coefficients of equal value are

$$\binom{n}{(n-1)/2} \quad \text{and} \quad \binom{n}{(n+1)/2}.$$

That the binomial coefficients decrease as indicated in the statement of the theorem follows in a similar way. □

For any real number x, let $\lfloor x \rfloor$ denote the greatest integer that is less than or equal to x. The integer $\lfloor x \rfloor$ is called the *floor* of x. Similarly, the *ceiling* of x is the smallest integer $\lceil x \rceil$ that is greater than or equal to x. For instance,

$$\lfloor 2.5 \rfloor = 2, \quad \lfloor 3 \rfloor = 3, \quad \lfloor -1.5 \rfloor = -2$$

and

$$\lceil 2.5 \rceil = 3, \quad \lceil 3 \rceil = 3, \quad \lceil -1.5 \rceil = -1.$$

We also have

$$\left\lfloor \frac{n}{2} \right\rfloor = \left\lceil \frac{n}{2} \right\rceil = \frac{n}{2}, \text{ if } n \text{ is even,}$$

and

$$\left\lfloor \frac{n}{2} \right\rfloor = \frac{n-1}{2} \text{ and } \left\lceil \frac{n}{2} \right\rceil = \frac{n+1}{2}, \text{ if } n \text{ is odd.}$$

Corollary 5.3.2 *For n a positive integer, the largest of the binomial coefficients*

$$\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n}$$

is

$$\binom{n}{\lfloor n/2 \rfloor} = \binom{n}{\lceil n/2 \rceil}.$$

Proof. The corollary follows from Theorem 5.3.1 and the preceding observations about the floor and ceiling functions. □

To conclude this section we discuss a generalization of Theorem 5.3.1 called Sperner' theorem.[1] Let S be a set of n elements. An *antichain*[2] of S is a collection $\mathcal{A}$ of subsets of S with the property that no subset in $\mathcal{A}$ is contained in another. For example, if $S = \{a, b, c, d\}$, then

$$\mathcal{A} = \{\{a, b\}, \{b, c, d\}, \{a, d\}, \{a, c\}\}$$

is an antichain. One way to obtain an antichain on a set S is to choose an integer $k \leq n$ and then take $\mathcal{A}_k$ to be the collection of all k-subsets of S. Since each subset in $\mathcal{A}_k$ has k elements, no subset in $\mathcal{A}_k$ can contain another; hence, $\mathcal{A}_k$ is an antichain. It follows from Corollary 5.3.2, that such an antichain contains at most

$$\binom{n}{\lfloor \frac{n}{2} \rfloor}$$

sets. For example, if $n = 4$ and $S = \{a, b, c, d\}$, the 2-subsets of S give the antichain

$$\mathcal{C}_2 = \{\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}\}$$

of size 6. Can we do better by choosing subsets of more than one size? The negative answer to this question is the conclusion of Sperner's theorem. Before stating that theorem, we introduce a new concept.

A collection C of subsets of S is a *chain* provided that for that each pair of subsets in $\mathcal{C}$, one is contained in the other:

$$A_1, A_2 \text{ in } \mathcal{C},\ A_1 \neq A_2 \text{ implies } A_1 \subset A_2 \text{ or } A_2 \subset A_1.$$

If $n = 5$ and $S = \{1, 2, 3, 4, 5\}$, examples of chains, written using the containment relation, are

$$\{2\} \subset \{2, 3, 5\} \subset \{1, 2, 3, 5\}$$

and

$$\emptyset \subset \{3\} \subset \{3, 4\} \subset \{1, 3, 4\} \subset \{1, 3, 4, 5\} \subset \{1, 2, 3, 4, 5\}.$$

[1] E. Sperner, Ein Satz über Untermengen einer endlichen Menger [A theorem about subsets of finite sets], *Math. Zeitschrift*, 27 (1928), 544–548.

[2] In anticipation of the concept of chain to be defined shortly.

The second example is an example of a maximal chain in that it contains one subset of S of each possible size; equivalently, it is not possible to squeeze more subsets into the chain. In general, if $S = \{1, 2, \ldots, n\}$, a *maximal chain* is a chain

$$A_0 = \emptyset \subset A_1 \subset A_2 \subset \cdots \subset A_n,$$

where $|A_i| = i$ for $i = 0, 1, 2 \ldots, n$. Each maximal chain of S is obtained as follows:

(0) Start with the empty set.

(1) Choose an element i_1 in S to form $A_1 = \{i_1\}$.

(2) Choose an element $i_2 \neq i_1$ to form $A_2 = \{i_1.i_2\}$.

(3) Choose an element $i_3 \neq i_1, i_2$ to form $A_3 = \{i_1, i_2, i_3\}$.

$\vdots$

(k) Choose an element $i_k \neq i_1, i_2, \ldots, i_{k-1}$ to form $A_k = \{i_1, i_2, \ldots, i_k\}$.

$\vdots$

(n) Choose an element $i_n \neq i_1, i_2, \ldots, i_{n-1}$ to form $A_n = \{i_1, i_2, \ldots, i_n\}$. Obviously, $A_n = \{1, 2, \ldots, n\}$.

Note that carrying out these steps is equivalent to choosing a permutation $i_1, i_2, \ldots, i_n$ of $\{1, 2, \ldots, n\}$, and there is a one-to-one correspondence between maximal chains of $S = \{1, 2, \ldots, n\}$ and permutations of $\{1, 2, \ldots, n\}$. In particular, the number of maximal chains equals $n!$. More generally, given any $A \subset S$ with $|S| = k$, the number of maximal chains containing A equals $k!(n-k)$ ($k!$ to get to A; $(n-k)!$ to get from A to $\{1, 2, \ldots, n\}$).

It is a consequence of the definitions of *chain* and *antichain* that a chain can contain at most one member of any antichain, that is, a chain and an antichain intersect in at most one member.

Theorem 5.3.3 *Let S be a set of n elements. Then an antichain on S contains at most $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ sets.*

Proof.[3] Let $\mathcal{A}$ be an antichain. We count in two different ways the number β of ordered pairs (A, C) such that A is in $\mathcal{A}$, and C is a maximal chain containing A. Focusing first on one maximal chain C, since each maximal chain contains at most one subset in the antichain $\mathcal{A}$, β is at most the number of maximal chains; that is, $\beta \leq n!$. Focusing now on one subset A in the antichain $\mathcal{A}$, we know that, if $|A| = k$,

[3] This elegant proof is due to D. Lubell, A Short Proof of Sperner's Theorem, *J. Combinatorial Theory*, 1 (1966), 299.

there are at most $k!(n-k)!$ maximal chains C containing A. Let α_k be the number of subsets in the antichain $\mathcal{A}$ of size k so that $|\mathcal{A}| = \sum_{k=0}^{n} \alpha_k$. Then

$$\beta = \sum_{k=0}^{n} \alpha_k k!(n-k)!,$$

and, since $\beta \leq n!$, we calculate that

$$\begin{aligned}
\sum_{k=0}^{n} \alpha_k k!(n-k)! &\leq n! \\
\sum_{k=0}^{n} \alpha_k \frac{k!(n-k)!}{n!} &\leq 1 \\
\sum_{k=0}^{n} \frac{\alpha_k}{\binom{n}{k}} &\leq 1.
\end{aligned}$$

By Corollary 5.3.2, $\binom{n}{k}$ is maximum when $k = \lfloor n/2 \rfloor$, and we get that

$$|\mathcal{A}| \leq \sum_{k=0}^{n} \alpha_k \leq \binom{n}{\lfloor \frac{n}{2} \rfloor},$$

as was to be proved. □

If n is even, it can be shown that the only antichain of size $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ is the antichain of all $\frac{n}{2}$-subsets of S. If n is odd, the only antichains of this size are the antichain of all $\frac{n-1}{2}$-subsets of S and the antichain of all $\frac{n+1}{2}$-subsets of S. See Exercises 30–32.

A stronger conclusion than that given in Theorem 5.3.3 can be obtained with a little more work. This is discussed in Section 5.6.

5.4 The Multinomial Theorem

The binomial theorem gives a formula for $(x+y)^n$ for each positive integer n. It can be generalized to give a formula for $(x+y+z)^n$ or, more generally, for the nth power of the sum of t real numbers: $(x_1 + x_2 + \cdots + x_t)^n$. In the general formula, the role of the binomial coefficients is taken over by numbers called the *multinomial coefficients*, which are defined by

$$\binom{n}{n_1\ n_2\ \cdots\ n_t} = \frac{n!}{n_1!n_2!\cdots n_t!}. \tag{5.20}$$

Here, $n_1, n_2, \ldots, n_t$ are nonnegative integers with

$$n_1 + n_2 + \cdots + n_t = n.$$

Recall from Section 3.4 that (5.20) represents the number of permutations of a multiset of objects of t different types with repetition numbers $n_1, n_2, \ldots, n_t$, respectively. The binomial coefficient $\binom{n}{k}$, for nonnegative n and k and having the value

$$\frac{n!}{k!(n-k)!}, \quad (k = 0, 1, \ldots, n)$$

in this notation becomes

$$\binom{n}{k \quad n-k}$$

and represents the number of permutations of a multiset of objects of two types with repetition numbers k and $n - k$, respectively.

In the same notation, Pascal's formula for the binomial coefficients with n and k positive is

$$\binom{n}{k \quad n-k} = \binom{n-1}{k \quad n-k-1} + \binom{n-1}{k-1 \quad n-k}.$$

Pascal's formula for the multinomial coefficients is

$$\binom{n}{n_1\, n_2 \,\cdots\, n_t} = \binom{n-1}{n_1 - 1\, n_2 \,\cdots\, n_t}$$

$$+\binom{n-1}{n_1\, n_2 - 1 \,\cdots\, n_t} + \cdots + \binom{n-1}{n_1\, n_2 \,\cdots\, n_t - 1}. \tag{5.21}$$

Formula (5.21) can be verified by direct substitution, using the value of the multinomial coefficients in (5.20). For instance, let $t = 3$ and let n_1, n_2, and n_3 be positive integers with $n_1 + n_2 + n_3 = n$. Then

$$\binom{n-1}{n_1 - 1\, n_2\, n_3} + \binom{n-1}{n_1\, n_2 - 1\, n_3} + \binom{n-1}{n_1\, n_2\, n_3 - 1}$$

$$= \frac{(n-1)!}{(n_1-1)!n_2!n_3!} + \frac{(n-1)!}{n_1!(n_2-1)!n_3!} + \frac{(n-1)!}{n_1!n_2!(n_3-1)!}$$

$$= \frac{n_1 \times (n-1)!}{n_1!n_2!n_3!} + \frac{n_2 \times (n-1)!}{n_1!n_2!n_3!} + \frac{n_3 \times (n-1)!}{n_1!n_2!n_3!}$$

$$= (n_1 + n_2 + n_3) \times \frac{(n-1)!}{n_1!n_2!n_3!} = n \times \frac{(n-1)!}{n_1!n_2!n_3!}$$

$$= \frac{n!}{n_1!n_2!n_3!} = \binom{n}{n_1\ n_2\ n_3}.$$

In the Exercises, a hint is given for a combinatorial verification of (5.21).

Before stating the general theorem, we first consider a special case. Let x_1, x_2, x_3 be real numbers. If we completely multiply out

$$(x_1 + x_2 + x_3)^3$$

and collect like terms (you are urged to do so), we obtain the sum

$$x_1^3 + x_2^3 + x_3^3 + 3x_1^2x_2 + 3x_1x_2^2 + 3x_1^2x_3 + 3x_1x_3^2 + 3x_2^2x_3 + 3x_2x_3^2 + 6x_1x_2x_3.$$

The terms that appear in the preceding sum are all the terms of the form $x_1^{n_1}x_2^{n_2}x_3^{n_3}$, where n_1, n_2, n_3 are nonnegative integers with $n_1 + n_2 + n_3 = 3$. The coefficient of $x_1^{n_1}x_2^{n_2}x_3^{n_3}$ in this expression is readily checked to be equal to

$$\binom{3}{n_1\ n_2\ n_3} = \frac{3!}{n_1!n_2!n_3!}.$$

More generally, we have the following *multinomial theorem*:

Theorem 5.4.1 *Let n be a positive integer. For all $x_1, x_2, \ldots, x_t$,*

$$(x_1 + x_2 + \cdots + x_t)^n = \sum \binom{n}{n_1\ n_2 \cdots n_t} x_1^{n_1}x_2^{n_2}\cdots x_t^{n_t},$$

where the summation extends over all nonnegative integral solutions $n_1, n_2, \ldots, n_t$ of $n_1 + n_2 + \cdots + n_t = n$.

Proof. We generalize the first proof of the binomial theorem. We write $(x_1 + x_2 + \cdots + x_t)^n$ as a product of n factors, each equal to $(x_1 + x_2 + \cdots + x_t)$. We completely expand this product, using the distributive law, and collect like terms. For each of the n factors, we choose one of the t numbers $x_1, x_2, \ldots, x_t$ and form their product. There are t^n terms that result in this way, and each can be arranged in the form $x_1^{n_1}x_2^{n_2}\cdots x_t^{n_t}$, where $n_1, n_2, \ldots, n_t$ are nonnegative integers summing to n. We obtain the term $x_1^{n_1}x_2^{n_2}\cdots x_t^{n_t}$ by choosing x_1 in n_1 of the n factors, x_2 in n_2 of the remaining $n - n_1$ factors, ..., x_t in n_t of the remaining $n - n_1 - \cdots - n_{t-1}$ factors. By the multiplication principle, the number of times the term $x_1^{n_1}x_2^{n_2}\cdots x_t^{n_t}$ occurs is given by

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\cdots\binom{n-n_1-\cdots-n_{t-1}}{n_t}.$$

We have already seen in Section 3.4 that this number equals the multinomial coefficient

$$\frac{n!}{n_1!n_2!\cdots n_t!},$$

and this proves the theorem. $\square$

Example. When $(x_1 + x_2 + x_3 + x_4 + x_5)^7$ is expanded, the coefficient of $x_1^2x_3x_4^3x_5$ equals

$$\binom{7}{2\ 0\ 1\ 3\ 1} = \frac{7!}{2!0!1!3!1!} = 420.$$

$\square$

Example. When $(2x_1 - 3x_2 + 5x_3)^6$ is expanded, the coefficient of $x_1^3x_2x_3^2$ equals

$$\binom{6}{3\ 1\ 2} 2^3(-3)(5)^2 = -36,000.$$

$\square$

The number of different terms that occur in the multinomial expansion of $(x_1 + x_2 + \cdots + x_t)^n$ equals the number of nonnegative integral solutions of

$$n_1 + n_2 + \cdots + n_t = n.$$

It follows from Section 3.5 that the number of these solutions equals

$$\binom{n+t-1}{n}.$$

For instance, $(x_1 + x_2 + x_3 + x_4)^6$ contains

$$\binom{6+4-1}{6} = \binom{9}{6} = 84$$

different terms if multiplied out completely. The total number of terms equals 4^6.

5.5 Newton's Binomial Theorem

In 1676, Isaac Newton generalized the binomial theorem given in Section 5.2 to obtain an expansion for $(x+y)^\alpha$, where α is any real number. For general exponents, however, the expansion becomes an infinite series, and questions of convergence need to be considered. We shall be satisfied with stating the theorem and considering some special cases. A proof of the theorem can be found in most advanced calculus texts.

Theorem 5.5.1 *Let α be a real number. Then, for all x and y with $0 \le |x| < |y|$,*

$$(x+y)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k y^{\alpha-k},$$

where

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}.$$

If α is a positive integer n, then for $k > n$, $\binom{n}{k} = 0$, and the preceding expansion becomes

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

This agrees with the binomial theorem of Section 5.2.

If we set $z = x/y$, then $(x+y)^\alpha = y^\alpha(z+1)^\alpha$. Thus, Theorem 5.5.1 can be stated in the equivalent form: For any z with $|z| < 1$,

$$(1+z)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} z^k.$$

Suppose that n is a positive integer and we choose α to be the negative integer $-n$. Then

$$\begin{aligned} \binom{\alpha}{k} = \binom{-n}{k} &= \frac{-n(-n-1)\cdots(-n-k+1)}{k!} \\ &= (-1)^k \frac{n(n+1)\cdots(n+k-1)}{k!} \\ &= (-1)^k \binom{n+k-1}{k}. \end{aligned}$$

Thus, for $|z| < 1$,

$$(1+z)^{-n} = \frac{1}{(1+z)^n} = \sum_{k=0}^{\infty} (-1)^k \binom{n+k-1}{k} z^k.$$

Replacing z by $-z$, we obtain

$$(1-z)^{-n} = \frac{1}{(1-z)^n} = \sum_{k=0}^{\infty} \binom{n+k-1}{k} z^k. \tag{5.22}$$

If $n = 1$, then $\binom{n+k-1}{k} = \binom{k}{k} = 1$, and we obtain

$$\frac{1}{1+z} = \sum_{k=0}^{\infty}(-1)^k z^k \qquad (|z| < 1)$$

and

$$\frac{1}{1-z} = \sum_{k=0}^{\infty} z^k \qquad (|z| < 1). \tag{5.23}$$

The binomial coefficient $\binom{n+k-1}{k}$ that occurs in the expansion (5.22) is of a type that has occurred before in counting problems, and this suggests a possible combinatorial derivation of (5.22). We start with the infinite geometric series (5.23). Then

$$\frac{1}{(1-z)^n} = (1+z+z^2+\cdots)\cdots(1+z+z^2+\cdots) \quad (n \text{ factors}). \tag{5.24}$$

We obtain a term z^k in this product by choosing z^{k_1} from the first factor, z^{k_2} from the second factor, ... , z^{k_n} from the nth factor, where $k_1, k_2, \ldots, k_n$ are nonnegative integers summing to k:

$$z^{k_1} z^{k_2} \ldots z^{k_n} = z^{k_1+k_2+\cdots+k_n} = z^k.$$

Thus, the number of different ways to get z^k, that is, the coefficient of z^k in (5.24), equals the number of nonnegative integral solutions of

$$k_1 + k_2 + \cdots + k_n = k,$$

and we know this to be

$$\binom{n+k-1}{k}.$$

The binomial theorem can be used to obtain square roots to any desired accuracy. If we take $\alpha = \frac{1}{2}$, then

$$\binom{\alpha}{0} = 1,$$

while, for $k > 0$,

$$\begin{aligned}
\binom{\alpha}{k} = \binom{1/2}{k} &= \frac{\frac{1}{2}(\frac{1}{2}-1)\cdots(\frac{1}{2}-k+1)}{k!} \\
&= \frac{(-1)^{k-1}}{2^k}\frac{1\times 2\times 3\times 4\times\cdots\times(2k-3)\times(2k-2)}{2\times 4\times\cdots\times(2k-2)\times(k!)} \\
&= \frac{(-1)^{k-1}}{k\times 2^{2k-1}}\frac{(2k-2)!}{(k-1)!^2} \\
&= \frac{(-1)^{k-1}}{k\times 2^{2k-1}}\binom{2k-2}{k-1}.
\end{aligned}$$

Thus, for $|z| < 1$,

$$\begin{aligned}\sqrt{1+z} &= (1+z)^{1/2} = 1 + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k \times 2^{2k-1}} \binom{2k-2}{k-1} z^k \\ &= 1 + \frac{1}{2}z - \frac{1}{2 \times 2^3}\binom{2}{1}z^2 + \frac{1}{3 \times 2^5}\binom{4}{2}z^3 - \cdots.\end{aligned}$$

For example,

$$\begin{aligned}\sqrt{20} &= \sqrt{16+4} = 4\sqrt{1+0.25} \\ &= 4\left(1 + \frac{1}{2}(0.25) - \frac{1}{8}(0.25)^2 + \frac{1}{16}(0.25)^3 - \ldots\right) \\ &= 4.472\ldots.\end{aligned}$$

In Chapter 7 we shall apply the general binomial theorem in the solution of certain recurrence relations by generating functions.

5.6 More on Partially Ordered Sets

In Section 5.4, we discussed the notions of antichain and chain in the special partially ordered set $\mathcal{P}(X)$ of all subsets of a set X. In the current section, we extend these notions to partially ordered sets in general, and prove some basic theorems.

Let $(X, \leq)$ be a finite partially ordered set. An *antichain* is a subset A of X no pair of whose elements is comparable. In contrast, a *chain* is a subset C of X each pair of whose elements is comparable. Thus, a chain C is a totally ordered subset of X, and hence, by Theorem 4.5.2, the elements of a chain can be linearly ordered: $x_1 < x_2 < \cdots < x_t$. We usually present a chain by writing it in a linear order in this way. It follows immediately from definitions that a subset of a chain is also a chain and that a subset of an antichain is also an antichain. The important relationship between antichains and chains, following from their definitions, is that

$$|A \cap C| \leq 1 \text{ if } A \text{ is an antichain and } C \text{ is a chain.}$$

Example. Let $X = \{1, 2, \ldots, 10\}$, and consider the partially ordered set $(X, |)$ whose partial order $|$ is "is divisible by." Then $\{4, 6, 7, 9, 10\}$ is an antichain of size 5 since no integer in this set is divisible by any other, while $1 \mid 2 \mid 4 \mid 8$ is a chain of size 4. There are no antichains of size 6 and no chains of size 5. □

Let $(X, \leq)$ be a finite partially ordered set. We now consider partitions of X into chains and also into antichains. Surely, if there is a chain C of size r, then, since no two elements of C can belong to the same antichain, X cannot be partitioned into fewer than r antichains. Similarly, if there is an antichain A of size s, then, since no two elements of A can belong to the same chain, X cannot be partitioned into fewer that s chains. Our primary goal in this section is to prove two theorems that makes more precise this connection between antichains and chains. In spite of the "duality" between chains and antichains,[4] the proof of one of these theorems is quite short and simple while that of the other is less so.

Recall that a *minimal element* of a partially ordered set is an element a such that no element x satisfies $x < a$. A *maximal element* is an element b such that no element y satisfies $b < y$. *The set of all minimal elements of a partially ordered set forms an antichain, as does the set of all maximal elements.*

Theorem 5.6.1 *Let $(X, \leq)$ be a finite partially ordered set, and let r be the largest size of a chain. Then X can be partitioned into r but no fewer antichains.*

Proof. As already noted, X cannot be partitioned into fewer than r antichains. Thus, *it suffices to show that X can be partitioned into r antichains.* Let $X_1 = X$ and let A_1 be the set of minimal elements of X. Delete the elements of A_1 from X_1 to get X_2. For each element of X_2, there is an element of A_1 that is below it in the partial order. Let A_2 be the set of minimal elements of X_2. Delete the elements of A_2 from X_2 to get X_3. For each element of X_3, there is an element of A_2 that is below it in the partial order. Let A_3 be the set of minimal elements of X_3. We continue like this until we get to the first integer p such that $X_p \neq \emptyset$ but $X_{p+1} = \emptyset$. Then $A_1, A_2, \ldots, A_p$ is a partition of X into antichains. Diagrammatically, we have

$$\begin{array}{c} A_p \\ — \\ A_{p-1} \\ — \\ \vdots \\ — \\ A_2 \\ — \\ A_1, \end{array}$$

where for each element of A_j there is an element of A_{j-1} below it in the partial order $(2 \leq j \leq p)$. Starting with an element a_p of A_p, we can obtain a chain

$$a_1 < a_2 < \cdots < a_p,$$

where a_1 is in A_1, a_2 is in A_2, ... , a_p is in A_p. Since r is the largest size of a chain, $r \geq p$. Since X is partitioned into p antichains, $r \leq p$. Hence $r = p$ and the theorem is proved. □

[4]In a chain every pair of elements is comparable; in an antichain every pair of elements is incomparable.

To illustrate Theorem 5.6.1, let $X = \{1, 2, \ldots, n\}$ and consider the partially ordered set of all subsets of X partially ordered by inclusion. Then the largest size of a chain is $n + 1$; in fact,

$$\emptyset \subset \{1\} \subset \{1, 2\} \subset \{1, 2, 3\} \subset \cdots \subset \{1, 2, \ldots, n\}$$

is such a chain. The collection $\mathcal{P}(X)$ of all subsets of X can be partitioned into $n + 1$ antichains, namely, the antichains consisting of all subsets of X of size k for $k = 0, 1, 2, \ldots, n$.

The "dual" theorem is generally known as *Dilworth's theorem.*

Theorem 5.6.2 *Let $(X, \leq)$ be a finite partially ordered set, and let m be the largest size of an antichain. Then X can be partitioned into m but no fewer chains.*

Proof.[5] As already noted, X cannot be partitioned into fewer than m chains. Thus *it suffices to show that X can be partitioned into m chains.* We prove this by induction on the number n of elements in X. If $n = 1$, then the conclusion holds trivially. Assume that $n > 1$.

We consider two cases:

Case 1. There is an antichain A of size m that is neither the set of all maximal elements nor the set of all minimal elements of X.

In this case, let

$$A^+ = \{x : x \text{ in } X \text{ with } a \leq x \text{ for some } a \text{ in } A\},$$

the set of elements of X at or above some element of A, and let

$$A^- = \{x : x \text{ in } X \text{ with } x \leq a \text{ for some } a \text{ in } A\},$$

the set of elements of X at or below some element of A. Thus, A^+ consists of all elements "above" A, and A^- consists of all elements "below" A. The following properties hold:

1. $A^+ \neq X$ (and thus $|A^+| < |X|$), since there is a minimal element not in A;

2. $A^- \neq X$ (and thus $|A^-| < |X|$), since there is a maximal element not in A;

3. $A^+ \cap A^- = A$, since, if there were an element x in $A^+ \cap A^-$ not in A, then we would have $a_1 < x < a_2$ for some elements a_1 and a_2 in A, contradicting the assumption that A is an antichain;

4. $A^+ \cup A^- = X$, since, if there were an element x not in $A^+ \cup A^-$, $A \cup \{x\}$ would be an antichain of larger size than A.

[5] This particularly simple proof is taken from M. A. Perles, A Proof of Dilworth's Decomposition Theorem for Partially Ordered Sets, *Israel J. Math.*, 1 (1963), 105–107.

We apply the induction assumption to the smaller partially ordered sets A^+ and A^- and conclude that A^+ can be partitioned into m chains $E_1, E_2, \ldots, E_m$, and A^- can be partitioned into m chains $F_1, F_2, \ldots, F_m$. The elements of A are the maximal elements of A^- and so the last elements on the chains $F_1, F_2, \ldots, F_m$; the elements of A are also the minimal elements of A^+ and so the first elements on the chains $E_1, E_2, \ldots, E_m$. We "glue" the chains together in pairs to form m chains that partition X.

Case 2. There are at most two antichains of size m, and these are one or both of the set of all maximal elements and the set of all minimal elements. Let x be a minimal element and y a maximal element with $x \le y$ (x may equal y). Then the largest size of an antichain of $X - \{x, y\}$ is $m - 1$. By the induction hypothesis, $X - \{x, y\}$ can be partitioned into $m - 1$ chains. These chains, together with the chain $x \le y$, give a partition of X into m chains. □

Now consider the partially ordered set $\mathcal{P}(X)$ of all subsets of a set $X = \{1, 2, \ldots, n\}$ of n elements. By Theorem 5.3.3. the largest size of an antichain of $\mathcal{P}(X)$ is the largest binomial coefficient $\binom{n}{\lfloor \frac{n}{2} \rfloor}$. Hence by Theorem 5.6.2, the collection of all subsets of X can be partitioned into $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ chains. Each chain will have to contain exactly one subset of X of size $\binom{n}{\lfloor \frac{n}{2} \rfloor}$. We now show how to construct such a partition into chains. Once we have done this, we will have another proof of Sperner's theorem.

Here are partitions into chains for $n = 1, 2, 3$:

$n = 1$:

$$\emptyset \subset \{1\};$$

$n = 2$:

$$\emptyset \subset \{1\} \subset \{1, 2\},$$
$$\{2\};$$

$n = 3$:

$$\emptyset \subset \{1\} \subset \{1, 2\} \subset \{1, 2, 3\},$$
$$\{2\} \subset \{2, 3\},$$
$$\{3\} \subset \{1, 3\}.$$

We can obtain a chain partition for the subsets of $\{1, 2, 3, 4\}$ from that shown for $\{1, 2, 3\}$ as follows: We take each chain with more than one subset in it (for $n = 3$ all chains shown have this property) and make two chains for $n = 4$:

(1) The first obtained by attaching at the end, the subset obtained by appending 4 to the last subset of the chain,

(2) The second obtained by appending 4 to all but the last subset of the chain (and deleting that last subset).

Thus, the chain

$$\emptyset \subset \{1\} \subset \{1,2\} \subset \{1,2,3\}$$

becomes

$$\emptyset \subset \{1\} \subset \{1,2\} \subset \{1,2,3\} \subset \{1,2,3,4\} \text{ and}$$

$$\{4\} \subset \{1,4\} \subset \{1,2,4\};$$

the chain

$$\{2\} \subset \{2,3\}$$

becomes

$$\{2\} \subset \{2,3\} \subset \{2,3,4\} \text{ and}$$

$$\{2,4\};$$

and the chain

$$\{3\} \subset \{1,3\}$$

becomes

$$\{3\} \subset \{1,3\} \subset \{1,3,4\} \text{ and}$$

$$\{3,4\}.$$

Consequently, we have a chain partition of $6 = \binom{4}{2}$ chains of the subsets of $\{1,2,3,4\}$. The chains in this partition for $n = 4$ have two properties: Each subset in a chain has one more element than the subset that precedes it (when there is a preceding subset). The size of the first subset in a chain plus the size of the last subset in the chain is $n = 4$. Similar properties hold for the chain partitions given for $n = 1, 2$ and, 3. A chain partition of the subsets of $\{1, 2, \ldots, n\}$ is a *symmetric chain partition*, provided that

(1) Each subset in a chain has one more element than the subset that precedes it in the chain; and

(2) The size of the first subset in a chain plus the size of the last subset in the chain equals n. (If the chain contains only one subset, then it is both first and last, so twice its size is n; that is, its size is $n/2$ and n is even.)

Each chain in a symmetric chain partition must contain exactly one $\lfloor n/2 \rfloor$-subset (and exactly one $\lceil n/2 \rceil$-subset); hence, the number of chains in a symmetric chain partition equals

$$\binom{n}{\lfloor \frac{n}{2} \rfloor} = \binom{n}{\lceil \frac{n}{2} \rceil}.$$

A symmetric chain decomposition for $\{1,2,\ldots,n\}$ can be obtained inductively from a symmetric chain decomposition of $\{1,2,\ldots,n-1\}$, as previously illustrated for $n=3$. We take each chain

$$A_1 \subset A_2 \subset \cdots \subset A_k, \text{ where } |A_1| + |A_k| = n-1$$

in a symmetric chain partition for $\{1,2,\ldots,n-1\}$ and, depending on whether $k = 1$ or > 1, obtain one or two chains for $\{1,2,\ldots,n\}$:

$$A_1 \subset A_2 \subset \cdots \subset A_k \subset A_k \cup \{n\}, \text{ where } |A_1| + |A_k \cup \{n\}| = n$$

and

$$A_1 \cup \{n\} \subset \cdots \subset A_{k-1} \cup \{n\} \text{ where } |A_1 \cup \{n\}| + |A_{k-1} \cup \{n\}| = n.$$

(If $k = 1$, the second chain does not occur.) Every subset of $\{1,2,\ldots,n\}$ occurs in exactly one of the chains constructed in this way; hence, the resulting collection of chains forms a symmetric chain partition for $\{1,2,\ldots,n\}$.

The number of chains in a symmetric chain partition of $\{1,2,\ldots,n\}$ is

$$\binom{n}{\lfloor n/2 \rfloor}.$$

Thus, the number of subsets in an antichain of $\{1,2,\ldots,n\}$ is at most equal to

$$\binom{n}{\lfloor n/2 \rfloor}.$$

Thus we have a more "constructive" proof of Sperner's theorem.

5.7 Exercises

1. Prove Pascal's formula by substituting the values of the binomial coefficients as given in equation (5.1).

2. Fill in the rows of Pascal's triangle corresponding to $n = 9$ and 10.

3. Consider the sum of the binomial coefficients along the diagonals of Pascal's triangle running upward from the left. The first few are $1, 1, 1+1 = 2, 1+2 = 3, 1+3+1 = 5, 1+4+3 = 8$. Compute several more of these diagonal sums, and determine how these sums are related. (Compare them with the values of the counting function f in Exercise 4 of Chapter 1.)

4. Expand $(x+y)^5$ and $(x+y)^6$ using the binomial theorem.

5. Expand $(2x-y)^7$ using the binomial theorem.

6. What is the coefficient of x^5y^{13} in the expansion of $(3x-2y)^{18}$? What is the coefficient of x^8y^9? (There is not a misprint in this last question!)

7. Use the binomial theorem to prove that

$$3^n = \sum_{k=0}^{n} \binom{n}{k} 2^k.$$

Generalize to find the sum

$$\sum_{k=0}^{n} \binom{n}{k} r^k$$

for any real number r.

8. Use the binomial theorem to prove that

$$2^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} 3^{n-k}.$$

9. Evaluate the sum

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} 10^k.$$

10. Use *combinatorial* reasoning to prove the identity (5.2).

11. Use *combinatorial* reasoning to prove the identity (in the form given)

$$\binom{n}{k} - \binom{n-3}{k} = \binom{n-1}{k-1} + \binom{n-2}{k-1} + \binom{n-3}{k-1}.$$

(*Hint*: Let S be a set with three distinguished elements a, b, and c and count certain k-subsets of S.)

12. Let n be a positive integer. Prove that

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k}^2 = \begin{cases} 0 & \text{if } n \text{ is odd} \\ (-1)^m \binom{2m}{m} & \text{if } n = 2m. \end{cases}$$

(*Hint*: For $n = 2m$, consider the coefficient of x^n in $(1-x^2)^n = (1+x)^n(1-x)^n$.)

13. Find one binomial coefficient equal to the following expression:

$$\binom{n}{k} + 3\binom{n}{k-1} + 3\binom{n}{k-2} + \binom{n}{k-3}.$$

14. Prove that

$$\binom{r}{k} = \frac{r}{r-k}\binom{r-1}{k}$$

for r a real number and k an integer with $r \neq k$.

15. Prove, that for every integer $n > 1$,

$$\binom{n}{1} - 2\binom{n}{2} + 3\binom{n}{3} + \cdots + (-1)^{n-1}n\binom{n}{n} = 0.$$

16. By integrating the binomial expansion, prove that, for a positive integer n,

$$1 + \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} + \cdots + \frac{1}{n+1}\binom{n}{n} = \frac{2^{n+1}-1}{n+1}.$$

17. Prove the identity in the previous exercise by using (5.2) and (5.3).

18. Evaluate the sum

$$1 - \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} - \frac{1}{4}\binom{n}{3} + \cdots + (-1)^n \frac{1}{n+1}\binom{n}{n}.$$

19. Sum the series $1^2 + 2^2 + 3^2 + \cdots + n^2$ by observing that

$$m^2 = 2\binom{m}{2} + \binom{m}{1}$$

and using the identity (5.19).

20. Find integers a, b, and c such that

$$m^3 = a\binom{m}{3} + b\binom{m}{2} + c\binom{m}{1}$$

for all m. Then sum the series $1^3 + 2^3 + 3^3 + \cdots + n^3$.

21. Prove that, for all real numbers r and all integers k,

$$\binom{-r}{k} = (-1)^k\binom{r+k-1}{k}.$$

22. Prove that, for all real numbers r and all integers k and m,

$$\binom{r}{m}\binom{m}{k} = \binom{r}{k}\binom{r-k}{m-k}.$$

23. Every day a student walks from her home to school, which is located 10 blocks east and 14 blocks north from home. She always takes a shortest walk of 24 blocks.

    (a) How many different walks are possible?

    (b) Suppose that four blocks east and five blocks north of her home lives her best friend, whom she meets each day on her way to school. Now how many different walks are possible?

    (c) Suppose, in addition, that three blocks east and six blocks north of her friend's house there is a park where the two girls stop each day to rest and play. Now how many different walks are there?

    (d) Stopping at a park to rest and play, the two students often get to school late. To avoid the temptation of the park, our two students decide never to pass the intersection where the park is. Now how many different walks are there?

24. Consider a three-dimensional grid whose dimensions are 10 by 15 by 20. You are at the front lower left corner of the grid and wish to get to the back upper right corner 45 "blocks" away. How many different routes are there in which you walk exactly 45 blocks?

25. Use a combinatorial argument to prove the *Vandermonde convolution* for the binomial coefficients: For all positive integers m_1, m_2, and n,

$$\sum_{k=0}^{n}\binom{m_1}{k}\binom{m_2}{n-k} = \binom{m_1+m_2}{n}.$$

    Deduce the identity (5.16) as a special case.

26. Let n and k be integers with $1 \leq k \leq n$. Prove that

$$\sum_{k=1}^{n}\binom{n}{k}\binom{n}{k-1} = \frac{1}{2}\binom{2n+1}{n+1} - \binom{2n}{n}.$$

27. Let n and k be positive integers. Give a combinatorial proof of the identity (5.15):

$$n(n+1)2^{n-2} = \sum_{k=1}^{n} k^2\binom{n}{k}.$$

28. Let n and k be positive integers. Give a combinatorial proof that

$$\sum_{k=1}^{n} k\binom{n}{k}^2 = n\binom{2n-1}{n-1}.$$

29. Find and prove a formula for

$$\sum_{\substack{r,s,t\geq 0\\ r+s+t=n}} \binom{m_1}{r}\binom{m_2}{s}\binom{m_3}{t},$$

where the summation extends over all nonnegative integers r, s and t with sum $r+s+t=n$.

30. Prove that the only antichain of $S = \{1,2,3,4\}$ of size 6 is the antichain of all 2-subsets of S.

31. Prove that there are only two antichains of $S = \{1,2,3,4,5\}$ of size 10 (10 is maximum by Sperner's theorem), namely, the antichain of all 2-subsets of S and the antichain of all 3-subsets.

32. * Let S be a set of n elements. Prove that, if n is even, the only antichain of size $\binom{n}{\lfloor \frac{n}{2}\rfloor}$ is the antichain of all $\frac{n}{2}$-subsets; if n is odd, prove that the only antichains of this size are the antichain of all $\frac{n-1}{2}$-subsets and the antichain of all $\frac{n+1}{2}$-subsets.

33. Construct a partition of the subsets of $\{1,2,3,4,5\}$ into symmetric chains.

34. In a partition of the subsets of $\{1,2,\ldots,n\}$ into symmetric chains, how many chains have only one subset in them? two subsets? k subsets?

35. A talk show host has just bought 10 new jokes. Each night he tells some of the jokes. What is the largest number of nights on which you can tune in so that you never hear on one night at least all the jokes you heard on *one* of the other nights? (Thus, for instance, it is acceptable that you hear jokes 1, 2, and 3 on one night, jokes 3 and 4 on another, and jokes 1, 2, and 4 on a third. It is not acceptable that you hear jokes 1 and 2 on one night and joke 2 on another night.)

36. Prove the identity of Exercise 25 using the binomial theorem and the relation $(1+x)^{m_1}(1+x)^{m_2} = (1+x)^{m_1+m_2}$.

37. Use the multinomial theorem to show that, for positive integers n and t,

$$t^n = \sum \binom{n}{n_1\, n_2 \cdots n_t},$$

where the summation extends over all nonnegative integral solutions $n_1, n_2, \ldots, n_t$ of $n_1 + n_2 + \cdots + n_t = n$.

38. Use the multinomial theorem to expand $(x_1 + x_2 + x_3)^4$.

39. Determine the coefficient of $x_1^3 x_2 x_3^4 x_5^2$ in the expansion of

$$(x_1 + x_2 + x_3 + x_4 + x_5)^{10}.$$

40. What is the coefficient of $x_1^3 x_2^3 x_3 x_4^2$ in the expansion of

$$(x_1 - x_2 + 2x_3 - 2x_4)^9?$$

41. Expand $(x_1 + x_2 + x_3)^n$ by observing that

$$(x_1 + x_2 + x_3)^n = ((x_1 + x_2) + x_3)^n$$

and then using the binomial theorem.

42. Prove the identity (5.21) by a combinatorial argument. (*Hint*: Consider the permutations of a multiset of objects of t different types with repetition numbers $n_1, n_2, \ldots, n_t$, respectively. Partition these permutations according to what type of object is in the first position.)

43. Prove by induction on n that, for n a positive integer,

$$\frac{1}{(1-z)^n} = \sum_{k=0}^{\infty} \binom{n+k-1}{k} z^k, \qquad |z| < 1.$$

Assume the validity of

$$\frac{1}{1-z} = \sum_{k=0}^{\infty} z^k, \qquad |z| < 1.$$

44. Prove that

$$\sum_{n_1+n_2+n_3=n} \binom{n}{n_1\ n_2\ n_3} (-1)^{n_1-n_2+n_3} = 1,$$

where the summation extends over all nonnegative integral solutions of $n_1 + n_2 + n_3 = n$.

45. Prove that

$$\sum_{n_1+n_2+n_3+n_4=n} \binom{n}{n_1\ n_2\ n_3\ n_4} (-1)^{n_1-n_2+n_3-n_4} = 0,$$

where the summation extends over all nonnegative integral solutions of $n_1 + n_2 + n_3 + n_4 = n$.

46. Use Newton's binomial theorem to approximate $\sqrt{30}$.

47. Use Newton's binomial theorem to approximate $10^{1/3}$.

48. Use Theorem 5.6.1 to show that, if m and n are positive integers, then a partially ordered set of $mn+1$ elements has a chain of size $m+1$ or an antichain of size $n+1$.

49. Use the result of the previous exercise to show that a sequence of $mn+1$ real numbers either contains an increasing subsequence of $m+1$ numbers or a decreasing subsequence of $n+1$ numbers (see Application 9 of Section 2.2).

50. Consider the partially ordered set $(X, |)$ on the set $X = \{1, 2, \ldots, 12\}$ of the first 12 positive integers, partially ordered by "is divisible by."

    (a) Determine a chain of largest size and a partition of X into the smallest number of antichains.

    (b) Determine an antichain of largest size and a partition of X into the smallest number of chains.

51. Let R and S be two partial orders on the same set X. Considering R and S as subsets of $X \times X$, we assume that $R \subseteq S$ but $R \neq S$. Show that there exists an ordered pair (p, q), where $(p, q) \in S$ and $(p, q) \notin R$ such that $R' = R \cup \{(p, q)\}$ is also a partial order on X. Show by example that not every such (p, q) has the property that R' is a partial order on X.