Chapter 7

Recurrence Relations and Generating Functions

Many combinatorial counting problems depend on an integer parameter n. This parameter n often denotes the size of some underlying set or multiset in the problem, the size of subsets, the number of positions in permutations, and so on. Thus, a counting problem is often not one individual problem but a sequence of individual problems. For example, let h_n denote the number of permutations of $\{1, 2, \ldots, n\}$. We know that $h_n = n!$, and hence we obtain a sequence of numbers

$$h_0, h_1, h_2, \ldots, h_n, \ldots$$

for which the general term h_n equals $n!$. An instance of this problem is obtained by choosing n to be a specific integer. If we take $n = 5$, then we obtain $h_5 = 5!$ as the answer to the problem of determining the number of permutations of $\{1, 2, 3, 4, 5\}$.

As another example, let g_n denote the number of nonnegative integral solutions of the equation

$$x_1 + x_2 + x_3 + x_4 = n.$$

From Chapter 3, we know that the general term of the sequence

$$g_0, g_1, g_2, \ldots, g_n, \ldots$$

satisfies

$$g_n = \binom{n+3}{3}.$$

In this chapter, we develop algebraic methods for solving some counting problems involving an integer parameter n. Our methods lead either to an explicit formula or to a function, a *generating function*, the coefficients of whose power series give the answers to the counting problem.

7.1 Some Number Sequences

Let

$$h_0, h_1, h_2, \ldots, h_n, \ldots \tag{7.1}$$

denote a sequence of numbers. We call h_n the *general term* or *generic term* of the sequence. Two familiar types of sequences are

> *arithmetic sequences*, in which each term is a constant q more than the previous term,
>
> *geometric sequences*, in which each term is a constant multiple q of the previous term.

In both instances, the sequence is uniquely determined once the initial term h_0 and the constant q are specified:

(arithmetic sequence)

$$h_0, h_0 + q, h_0 + 2q, \ldots, h_0 + nq, \ldots$$

(geometric sequence)

$$h_0, qh_0, q^2h_0, \ldots, q^n h_0, \ldots.$$

In the case of an arithmetic sequence, we have the rule

$$h_n = h_{n-1} + q, \qquad (n \geq 1) \tag{7.2}$$

and the general term is

$$h_n = h_0 + nq, \qquad (n \geq 0).$$

In the case of a geometric sequence, we have the rule

$$h_n = qh_{n-1}, \qquad (n \geq 1) \tag{7.3}$$

and the general term is

$$h_n = h_0 q^n, \qquad (n \geq 0).$$

Example. *Arithmetic sequences*

(a) $h_0 = 1$, $q = 2: \; 1, 3, 5, \ldots, 1 + 2n, \ldots$

This is the sequence of odd positive integers: $h_n = 1 + 2n$ $(n \geq 0)$.

(b) $h_0 = 4$, $q = 0: \; 4, 4, 4, \ldots, 4, \ldots$

This is the constant sequence with each term equal to 4: $h_n = 4$ $(n \geq 0)$.

(c) $h_0 = 0$, $q = 1: \; 0, 1, 2, \ldots, n, \ldots$

This is the sequence of nonnegative integers (the counting numbers): $h_n = n$ $(n \geq 0)$. □

Example. *Geometric sequences*

(a) $h_0 = 1,\ q = 2:\ 1, 2, 2^2, \ldots, 2^n, \ldots$

$h_n = 2^n\ (n \geq 0)$

This is the sequence of nonnegative integral powers of 2. Its combinatorial significance is that it is the sequence for the counting problem that asks for the number of subsets of an n-element set. It is also the sequence used in determining the base 2 representation of a number.

(b) $h_0 = 5,\ q = 3:\ 5, 3 \times 5, 3^2 \times 5, \ldots, 3^n \times 5, \ldots$

$h_n = 3^n \times 5\ (n \geq 0)$

This is the sequence for the counting problem that asks for the number of combinations of the multiset consisting of $n+1$ different objects whose repetition numbers are given by $4, 2, 2, \ldots, 2$ (n 2s), respectively. □

The *partial sums* for a sequence (7.1) are the sums

$$\begin{array}{rcl} s_0 & = & h_0 \\ s_1 & = & h_0 + h_1 \\ s_2 & = & h_0 + h_1 + h_2 \\ & \vdots & \\ s_n & = & h_0 + h_1 + h_2 + \cdots + h_n = \sum_{k=0}^{n} h_k \\ & \vdots & . \end{array}$$

The partial sums form a new sequence $s_0, s_1, s_2, \ldots, s_n, \ldots$ with general term s_n.

The partial sums for an arithmetic sequence are

$$s_n = \sum_{k=0}^{n} (h_0 + kq) = (n+1)h_0 + \frac{qn(n+1)}{2}.$$

The partial sums for a geometric sequence are

$$s_n = \sum_{k=0}^{n} q^k h_0 = \begin{cases} \frac{q^{n+1}-1}{q-1} h_0 & (q \neq 1) \\ (n+1)h_0 & (q = 1). \end{cases}$$

The rules (7.2) and (7.3) for obtaining the next term in either an arithmetic sequence or geometric sequence are simple instances of linear recurrence relations. In our study of the derangement numbers in Chapter 6, we obtained two recurrence relations for D_n, namely

$$D_n = (n-1)(D_{n-2} + D_{n-1}), (n \geq 3) \text{ and } D_n = nD_{n-1} + (-1)^n, (n \geq 2).$$

In (7.2) and (7.3), the nth term h_n of the sequence is obtained from the $(n-1)$th term h_{n-1} and a constant q.

We defer the general definition of a linear recurrence relation until Section 7.4.

The remainder of this section concerns a counting sequence called the *Fibonacci sequence.* In his book *Liber Abaci,*[1] published in 1202, Leonardo of Pisa[2] posed a problem of determining how many pairs of rabbits are born from one pair in a year.

The problem posed by Leonardo [Fibonacci] is the following:

A newly born pair of rabbits of opposite sexes is placed in an enclosure at the beginning of a year. Beginning with the second month, the female gives birth to a pair of rabbits of opposite sexes each month. Each new pair also gives birth to a pair of rabbits each month starting with their second month.[3] *Determine the number of pairs of rabbits in the enclosure after one year.*

In the beginning, there is one pair of rabbits who mature during the first month, so that at the beginning of the second month there is also only one pair of rabbits in the enclosure. During the second month the original pair gives birth to a pair of rabbits, so that there will be two pairs of rabbits at the beginning of the third month. During the third month the newborn pair of rabbits is maturing and only the original pair gives birth. Therefore, at the beginning of the fourth month there will be a $2+1=3$ pairs of rabbits in the enclosure. In general, let f_n denote the number of pairs of rabbits in the enclosure at the beginning of month n (equivalently, at the end of month $n-1$). We have calculated that $f_1 = 1, f_2 = 1, f_3 = 2$, and $f_4 = 3$, and we are asked to determine f_{13}.

We derive a recurrence relation for f_n from which we can then easily calculate f_{13}. At the beginning of month n the pairs of rabbits in the enclosure can be partitioned into two parts: those present at the beginning of month $n-1$ and those born during month $n-1$. The number of pairs born during month $n-1$ is, because of the one-month maturation process, the number of pairs that there were at the beginning of month $n-2$. Thus, at the beginning of month n, there are $f_{n-1}+f_{n-2}$ pairs of rabbits, giving us the recurrence relation

$$f_n = f_{n-1} + f_{n-2}, \qquad (n \geq 3).$$

[1] Literally, a book about the abacus.

[2] Leonardo, better known by the name Fibonacci (meaning "son of Bonacci"), was largely responsible for the introduction of our present system of numeration in Western Europe.

[3] Admittedly, this doesn't sound very realistic, but it's just a mathematical puzzle to challenge one's mind.

Using this relation and the values for f_1, f_2, f_3, and f_4 computed, we now see that

$$\begin{array}{lllllll} f_5 & = & f_4+f_3 & = & 3+2 & = & 5 \\ f_6 & = & f_5+f_4 & = & 5+3 & = & 8 \\ f_7 & = & f_6+f_5 & = & 8+5 & = & 13 \\ f_8 & = & f_7+f_6 & = & 13+8 & = & 21 \\ f_9 & = & f_8+f_7 & = & 21+13 & = & 34 \\ f_{10} & = & f_9+f_8 & = & 34+21 & = & 55 \\ f_{11} & = & f_{10}+f_9 & = & 55+34 & = & 89 \\ f_{12} & = & f_{11}+f_{10} & = & 89+55 & = & 144 \\ f_{13} & = & f_{12}+f_{11} & = & 144+89 & = & 233. \end{array}$$

Consequently, after one year there are 233 pairs of rabbits in the enclosure. We define $f_0 = 0$ so that $f_2 = 1 = 1 + 0 = f_1 + f_0$. The sequence of numbers $f_0, f_1, f_2, f_3, \ldots$ satisfying the recurrence relation and initial conditions

$$\begin{aligned} f_n = f_{n-1} + f_{n-2} \qquad & (n \geq 2) \\ f_0 = 0, \quad f_1 = 1 \qquad & \end{aligned} \tag{7.4}$$

is called the *Fibonacci sequence*, and the terms of the sequence are called *Fibonacci numbers.* The recurrence relation in (7.4) is also called the *Fibonacci recurrence.* From our calculations, the first few terms of the Fibonacci sequence are

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, \ldots.$$

The Fibonacci sequence has many remarkable properties. We give two in the next two examples.

Example. The partial sums of the terms of the Fibonacci sequence are

$$s_n = f_0 + f_1 + f_2 + \cdots + f_n = f_{n+2} - 1. \tag{7.5}$$

In particular, the partial sums are one less than a Fibonacci number.

We prove (7.5) by induction on n. For $n = 0$, (7.5) reduces to $f_0 = f_2 - 1$, which is certainly valid since $0 = 1 - 1$. Now, let $n \geq 1$. We assume that (7.5) holds for n and then prove that it holds when n is replaced by $n + 1$:

$$\begin{aligned} f_0 + f_1 + f_2 + \cdots + f_{n+1} &= (f_0 + f_1 + f_2 \cdots + f_n) + f_{n+1} \\ &= (f_{n+2} - 1) + f_{n+1} \\ &\qquad \text{(by the induction assumption)} \\ &= f_{n+2} + f_{n+1} - 1 = f_{n+3} - 1 \\ &\qquad \text{(by the Fibonacci recurrence).} \end{aligned}$$

Thus, by induction, (7.5) holds for all $n \geq 0$. □

Example. The Fibonacci number f_n is even if and only if n is divisible by 3.

This certainly agrees with the values for the Fibonacci numbers f_0, f_1, f_2 (even, odd, odd). It follows in general because if we have

$$\text{even, odd, odd,}$$

then, applying the Fibonacci recurrence, we see that the next three numbers are also even, odd, odd:

$$\text{odd} + \text{odd} = \text{even},$$

$$\text{odd} + \text{even} = \text{odd},$$

and

$$\text{even} + \text{odd} = \text{odd}.$$

□

Several other properties of the Fibonacci numbers are given in the Exercises.

Our goal now is to obtain a formula for the Fibonacci numbers, and in doing so we illustrate a technique for solving recurrence relations that we develop in a later section.

Consider the Fibonacci recurrence relation in the form

$$f_n - f_{n-1} - f_{n-2} = 0, \qquad (n \geq 2), \tag{7.6}$$

and, for the moment, ignore any initial values for f_0 and f_1. One way to solve this recurrence relation is to look for a solution of the form

$$f_n = q^n,$$

where q is a nonzero number. Thus, we seek a solution among the familiar geometric sequences with first term equal to $q^0 = 1$. We observe that $f_n = q^n$ satisfies the Fibonacci recurrence relation if and only if

$$q^n - q^{n-1} - q^{n-2} = 0,$$

or, equivalently,

$$q^{n-2}(q^2 - q - 1) = 0, \qquad (n = 2, 3, 4, \ldots).$$

Since q is assumed to be different from zero, we conclude that $f_n = q^n$ is a solution of the Fibonacci recurrence relation if and only if $q^2 - q - 1 = 0$ or, equivalently, if and only if q is a root of the quadratic equation

$$x^2 - x - 1 = 0.$$

Using the quadratic formula, we find that the roots of this equation are

$$q_1 = \frac{1+\sqrt{5}}{2}, \qquad q_2 = \frac{1-\sqrt{5}}{2}.$$

Thus,

$$f_n = \left(\frac{1+\sqrt{5}}{2}\right)^n \quad \text{and} \quad f_n = \left(\frac{1-\sqrt{5}}{2}\right)^n$$

are both solutions of the Fibonacci recurrence relation. Since the Fibonacci recurrence relation is linear (there are no powers of f different from 1) and homogeneous (the right-hand side of (7.6) is 0), it follows by straightforward computation that

$$f_n = c_1\left(\frac{1+\sqrt{5}}{2}\right)^n + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n \tag{7.7}$$

is also a solution of the recurrence relation (7.6) for any choice of constants c_1 and c_2.

The Fibonacci sequence has the initial values $f_0 = 0$ and $f_1 = 1$. Can we choose c_1 and c_2 in (7.7) so that these initial values are attained? If so, then (7.7) will give a formula for the Fibonacci numbers. To satisfy these initial values, we must have

$$\begin{cases} (n=0) & c_1 + c_2 = 0, \\ (n=1) & c_1\left(\frac{1+\sqrt{5}}{2}\right) + c_2\left(\frac{1-\sqrt{5}}{2}\right) = 1. \end{cases}$$

This is a simultaneous system of two linear equations in the unknowns c_1 and c_2, whose unique solution is computed to be

$$c_1 = \frac{1}{\sqrt{5}}, \qquad c_2 = \frac{-1}{\sqrt{5}}.$$

Substituting into (7.7), we obtain the next formula.

Theorem 7.1.1 *The Fibonacci numbers satisfy the formula*

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n, \quad (n \geq 0). \tag{7.8}$$

$\square$

Even though the Fibonacci numbers are whole numbers, an explicit formula for them involves the irrational number $\sqrt{5}$. When the binomial theorem is used to expand the nth powers in (7.8), all of the $\sqrt{5}$'s miraculously cancel out.

The solution (7.7) is the general solution of the Fibonacci recurrence relation (7.6) in the sense that no matter what the initial values $f_0 = a$ and $f_1 = b$, constants c_1 and

c_2 can be determined so that the initial values hold. This is so because the matrix of coefficients of the linear system

$$\begin{cases} c_1 + c_2 & = a \\ c_1\left(\frac{1+\sqrt{5}}{2}\right) + c_2\left(\frac{1-\sqrt{5}}{2}\right) & = b \end{cases}$$

is invertible; its determinant,

$$\det \begin{bmatrix} 1 & 1 \\ \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \end{bmatrix} = -\sqrt{5},$$

is different from zero. Thus, no matter what the values of a and b, the linear system[4] can be solved uniquely for c_1 and c_2.

Example. Let $g_0, g_1, g_2, \ldots, g_n, \ldots$ be the sequence of numbers satisfying the Fibonacci recurrence relation and the initial conditions as follows:

$$\begin{aligned} &g_n = g_{n-1} + g_{n-2} \qquad (n \geq 2) \\ &g_0 = 2, \quad g_1 = -1. \end{aligned}$$

We would like to determine c_1 and c_2 that satisfy

$$\begin{cases} c_1 + c_2 & = 2, \\ c_1\left(\frac{1+\sqrt{5}}{2}\right) + c_2\left(\frac{1-\sqrt{5}}{2}\right) & = -1. \end{cases}$$

Solving this system, we obtain

$$c_1 = \frac{\sqrt{5}-2}{\sqrt{5}}, \qquad c_2 = \frac{\sqrt{5}+2}{\sqrt{5}}.$$

Thus, a formula for g_n is

$$g_n = \frac{\sqrt{5}-2}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{\sqrt{5}+2}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

□

The Fibonacci numbers also occur in other combinatorial problems.

Example. Determine the number h_n of ways to perfectly cover a 2-by-n board with dominoes. (See Chapter 1 for a definition of this.)

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[4]Here we use a little elementary linear algebra. By directly eliminating c_1 from the second equation, we can see that the system has one and only one solution for each choice of a and b.

We define $h_0 = 1$.[5] We also compute that $h_1 = 1$, $h_2 = 2$, and $h_3 = 3$. Let $n \geq 2$. We partition the perfect covers of a 2-by-n board into two parts A and B. In A we put those perfect covers in which there is a vertical domino covering the square in the upper-left-hand corner. In B we put the other perfect covers; that is, the perfect covers in which there is a horizontal domino covering the square in the upper-left-hand corner and thus another horizontal domino covering the square in the lower-left-hand corner. The perfect covers in A are equinumerous with the perfect covers of a 2-by-$(n-1)$ board. Thus, the number of perfect covers in A is

$$|A| = h_{n-1}.$$

The perfect covers in B are equinumerous with the perfect covers of a 2-by-$(n-2)$ board, and hence the number of perfect covers in B is

$$|B| = h_{n-2}.$$

We conclude that

$$h_n = |A| + |B| = h_{n-1} + h_{n-2}, \qquad (n \geq 2).$$

Since $h_0 = h_1 = 1$ (the values of the Fibonacci numbers f_1 and f_2) and $h_n = h_{n-1} + h_{n-2}$ $(n \geq 2)$ (the Fibonacci recurrence relation), we conclude that $h_0, h_1, h_2, \ldots, h_n, \ldots$ is the Fibonacci sequence $f_1, f_2, \ldots, f_n, \ldots$ with f_0 deleted. □

Example. Determine the number b_n of ways to perfectly cover a 1-by-n board with monominoes and dominoes.

If we take a perfect cover of a 2-by-n board with dominoes and look only at its first row, we see a perfect cover of a 1-by-n board with monominoes and dominoes. Conversely, each perfect cover of a 1-by-n board with monominoes and dominoes can be "extended" uniquely to a perfect cover of a 2-by-n board with dominoes. Thus, the number of perfect covers of a 1-by-n board with monominoes and dominoes equals the number of perfect covers of a 2-by-n board with dominoes. Therefore, $b_0, b_1, b_2, \ldots, b_n, \ldots$ is also the Fibonacci sequence with f_0 deleted. □

In the next theorem we show how the Fibonacci numbers occur as sums of binomial coefficients.

Theorem 7.1.2 *The sums of the binomial coefficients along the diagonals of Pascal's triangle running upward from the left are Fibonacci numbers. More precisely, the nth Fibonacci number f_n satisfies*

$$f_n = \binom{n-1}{0} + \binom{n-2}{1} + \binom{n-3}{2} + \cdots + \binom{n-t}{t-1},$$

where $t = \lfloor \frac{n+1}{2} \rfloor$ is the floor of $\frac{n+1}{2}$.

[5] A 2-by-0 board is empty and has exactly one perfect cover, namely the empty cover.

Proof. Define $g_0 = 0$ and

$$g_n = \binom{n-1}{0} + \binom{n-2}{1} + \cdots + \binom{n-t}{t-1}, \quad (n \geq 1).$$

Since $\binom{m}{p} = 0$ for each integer $p > m$, we also have

$$g_n = \binom{n-1}{0} + \binom{n-2}{1} + \binom{n-3}{2} + \cdots + \binom{0}{n-1}, \quad (n \geq 1),$$

or, using summation notation,

$$g_n = \sum_{p=0}^{n-1} \binom{n-1-p}{k}.$$

To prove the theorem, it will suffice to show that g_n satisfies the Fibonacci recurrence relation and has the same initial values as the Fibonacci sequence. We have

$$g_0 = \binom{0}{-1} = 0,$$

$$g_1 = \binom{0}{0} = 1,$$

$$g_2 = \binom{1}{0} + \binom{0}{1} = 1 + 0 = 1.$$

Using Pascal's formula, we see that, for each $n \geq 3$,

$$\begin{aligned}
g_{n-1} + g_{n-2} &= \sum_{k=0}^{n-2} \binom{n-2-k}{k} + \sum_{j=0}^{n-3} \binom{n-3-j}{j} \\
&= \binom{n-2}{0} + \sum_{k=1}^{n-2} \binom{n-2-k}{k} + \sum_{k=1}^{n-2} \binom{n-2-k}{k-1} \\
&= \binom{n-2}{0} + \sum_{k=1}^{n-2} \left(\binom{n-2-k}{k} + \binom{n-2-k}{k-1} \right) \\
&= \binom{n-2}{0} + \sum_{k=1}^{n-2} \binom{n-1-k}{k} \\
&= \binom{n-1}{0} + \sum_{k=1}^{n-2} \binom{n-1-k}{k} + \binom{0}{n-1} \\
&= \sum_{k=0}^{n-1} \binom{n-1-k}{k} = g_n.
\end{aligned}$$

Here, we have used the facts that

$$\binom{n-1}{0} = 1 = \binom{n-2}{0} \text{ and } \binom{0}{n-1} = 0, \quad (n \geq 2).$$

We conclude that $g_0, g_1, g_2, \ldots, g_n, \ldots$ is the Fibonacci sequence, and this proves the theorem. □

7.2 Generating Functions

In this section we discuss the method of generating functions as it pertains to solving counting problems. On one level, generating functions can be regarded as algebraic objects whose formal manipulation allows us to count the number of possibilities for a problem by means of algebra. On another level, generating functions are Taylor series (power series expansions) of infinitely differentiable functions. If we can find the function and its Taylor series, then the coefficients of the Taylor series give the solution to the problem. For the most part we keep questions of convergence in the background and manipulate power series on a formal basis.

Let

$$h_0, h_1, h_2, \ldots, h_n, \ldots \tag{7.9}$$

be an infinite sequence of numbers. Its *generating function* is defined to be the infinite series

$$g(x) = h_0 + h_1 x + h_2 x^2 + \cdots + h_n x^n + \cdots.$$

The coefficient of x^n in $g(x)$ is the nth term h_n of (7.9); thus, x^n acts as a placeholder for h_n. A finite sequence

$$h_0, h_1, h_2 \ldots, h_m$$

can be regarded as the infinite sequence

$$h_0, h_1, h_2, \ldots, h_m, 0, 0, \ldots$$

in which all but a finite number of terms equal 0. Hence, every finite sequence has a generating function

$$g(x) = h_0 + h_1 x + h_2 x^2 + \cdots + h_m x^m,$$

which is a polynomial.

Example. The generating function of the infinite sequence

$$1, 1, 1, \ldots, 1, \ldots,$$

each of whose terms equals 1, is

$$g(x) = 1 + x + x^2 + \cdots + x^n + \cdots.$$

This generating function $g(x)$ is the sum of a geometric series[6] with value

$$g(x) = \frac{1}{1-x}. \tag{7.10}$$

The formula (7.10) holds the information about the infinite sequence of all 1s in exceedingly compact form. □

Example. Let m be a positive integer. The generating function for the binomial coefficients

$$\binom{m}{0}, \binom{m}{1}, \binom{m}{2}, \ldots, \binom{m}{m}$$

is

$$g_m(x) = \binom{m}{0} + \binom{m}{1}x + \binom{m}{2}x^2 + \cdots + \binom{m}{m}x^m.$$

By the binomial theorem,

$$g_m(x) = (1+x)^m,$$

which also displays the information about the sequence of binomial coefficients in compact form. □

Example. Let α be a real number. By Newton's binomial theorem (see Section 5.6), the generating function for the infinite sequence of binomial coefficients

$$\binom{\alpha}{0}, \binom{\alpha}{1}, \binom{\alpha}{2}, \ldots, \binom{\alpha}{n}, \ldots$$

is

$$(1+x)^\alpha = \binom{\alpha}{0} + \binom{\alpha}{1}x + \binom{\alpha}{2}x^2 + \cdots + \binom{\alpha}{n}x^n + \cdots.$$

□

Example. Let k be an integer, and let the sequence

$$h_0, h_1, h_2, \ldots, h_n, \ldots$$

be defined by letting h_n equal the number of nonnegative integral solutions of

$$e_1 + e_2 + \cdots + e_k = n.$$

From Chapter 3, we know that

$$h_n = \binom{n+k-1}{k-1}, \qquad (n \geq 0).$$

[6]See Section 5.6.

The generating function (using summation notation now) is

$$g(x) = \sum_{n=0}^{\infty} \binom{n+k-1}{k-1} x^n.$$

From Chapter 5, we know that this generating function is

$$g(x) = \frac{1}{(1-x)^k}.$$

It is instructive to recall the derivation of this formula. We have

$$\begin{aligned} \frac{1}{(1-x)^k} &= \frac{1}{1-x} \times \frac{1}{1-x} \times \cdots \times \frac{1}{1-x} \qquad (k \text{ factors}) \\ &= (1+x+x^2+\cdots)(1+x+x^2+\cdots)\cdots(1+x+x^2+\cdots) \\ &= \left(\sum_{e_1=0}^{\infty} x^{e_1}\right)\left(\sum_{e_2=0}^{\infty} x^{e_2}\right)\cdots\left(\sum_{e_k=0}^{\infty} x^{e_k}\right). \end{aligned} \tag{7.11}$$

In the preceding notation, x^{e_1} is a typical term of the first factor, x^{e_2} is a typical term of the second factor, ... , x^{e_k} is a typical term of the kth factor. Multiplying these typical terms, we get

$$x^{e_1}x^{e_2}\cdots x^{e_k} = x^n, \text{ provided that}$$

$$e_1 + e_2 + \cdots + e_k = n. \tag{7.12}$$

Thus, the coefficient of x^n in (7.11) equals the number of nonnegative integral solutions of (7.12), and this number we know to be

$$\binom{n+k-1}{n}.$$

□

The ideas used in the previous example apply to more general circumstances.

Example. For what sequence is

$$(1+x+x^2+x^3+x^4+x^5)(1+x+x^2)(1+x+x^2+x^3+x^4)$$

the generating function?

Let x^{e_1}, $(0 \le e_1 \le 5)$, x^{e_2}, $(0 \le e_2 \le 2)$, and x^{e_3}, $(0 \le e_3 \le 4)$ denote typical terms in the first, second, and third factors, respectively. Multiplying, we obtain

$$x^{e_1}x^{e_2}x^{e_3} = x^n,$$

provided that

$$e_1 + e_2 + e_3 = n.$$

Thus, the coefficient of x^n in the product is the number h_n of integral solutions of $e_1 + e_2 + e_3 = n$ in which $0 \le e_1 \le 5$, $0 \le e_2 \le 2$, and $0 \le e_3 \le 4$. Note that $h_n = 0$ if $n > 5 + 2 + 4 = 11$. □

Example. Determine the generating function for the number of n-combinations of apples, bananas, oranges, and pears, where, in each n-combination, the number of apples is even, the number of bananas is odd, the number of oranges is between 0 and 4, and there is at least one pear.

First, we note that the problem is equivalent to finding the number h_n of nonnegative integral solutions of

$$e_1 + e_2 + e_3 + e_4 = n,$$

where e_1 is even (e_1 counts the number of apples), e_2 is odd (e_2 counts the number of bananas), $0 \le e_3 \le 4$ (e_3 counts the number of oranges), and $e_4 \ge 1$ (e_4 counts the number of pears). We create one factor for each type of fruit, where the exponents are the allowable numbers in the n-combinations for that type of fruit:

$$g(x) =$$

$$(1 + x^2 + x^4 + \cdots)(x + x^3 + x^5 + \cdots)(1 + x + x^2 + x^3 + x^4)(x + x^2 + x^3 + + \cdots).$$

The first factor is the "apple factor," the second is the "banana factor," and so on. We now notice that

$$\begin{aligned}
1 + x^2 + x^4 + \cdots &= 1 + x^2 + (x^2)^2 + \cdots &&= \frac{1}{1 - x^2} \\
x + x^3 + x^5 + \cdots &= x(1 + x^2 + x^4 + \cdots) &&= \frac{x}{1 - x^2} \\
&\quad 1 + x + x^2 + x^3 + x^4 &&= \frac{1 - x^5}{1 - x} \\
&\quad x + x^2 + x^3 + \cdots &&= x(1 + x + x^2 + \cdots) \\
& &&= \frac{x}{1 - x}.
\end{aligned}$$

Thus,

$$\begin{aligned}
g(x) &= \frac{1}{1 - x^2}\frac{x}{1 - x^2}\frac{1 - x^5}{1 - x}\frac{x}{1 - x} \\
&= \frac{x^2(1 - x^5)}{(1 - x^2)^2(1 - x)^2}.
\end{aligned}$$

Therefore, the coefficients in the Taylor series for this rational function count the number of combinations of the type considered. □

The next example shows how a counting problem can sometimes be explicitly solved by means of generating functions.

Example. Find the number h_n of bags of fruit that can be made out of apples, bananas, oranges, and pears, where, in each bag, the number of apples is even, the number of bananas is a multiple of 5, the number of oranges is at most 4, and the number of pears is 0 or 1.

We are asked to count certain n-combinations of apples, bananas, oranges, and pears. We determine the generating function $g(x)$ for the sequence $h_0, h_1, h_2, \ldots, h_n, \ldots$ We introduce a factor for each type of fruit, and we find that

$$\begin{aligned} g(x) &= (1+x^2+x^4+\cdots)(1+x^5+x^{10}+\cdots)\times \\ &\qquad (1+x+x^2+x^3+x^4)(1+x) \\ &= \frac{1}{1-x^2}\frac{1}{1-x^5}\frac{1-x^5}{1-x}(1+x) \\ &= \frac{1}{(1-x)^2} = \sum_{n=0}^{\infty}\binom{n+1}{n}x^n \\ &= \sum_{n=0}^{\infty}(n+1)x^n. \end{aligned}$$

Thus, we see that $h_n = n+1$. Notice how this formula for the counting number h_n was obtained merely by algebraic manipulation. □

Example. Determine the generating function for the number h_n of solutions of the equation

$$e_1 + e_2 + \cdots + e_k = n$$

in nonnegative *odd* integers $e_1, e_2, \ldots, e_k$.

We have

$$\begin{aligned} g(x) &= (x+x^3+x^5+\cdots)\cdots(x+x^3+x^5+\cdots) \quad (k \text{ factors}) \\ &= x(1+x^2+x^4+\cdots)\cdots x(1+x^2+x^4+\cdots) \\ &= \frac{x}{1-x^2}\cdots\frac{x}{1-x^2} \\ &= \frac{x^k}{(1-x^2)^k}. \end{aligned}$$

□

We know that the number h_n of nonnegative integral solutions of the equation

$$e_1 + e_2 + \cdots + e_k = n \tag{7.13}$$

is

$$h_n = \binom{n+k-1}{n},$$

and we have determined that

$$g(x) = \frac{1}{(1-x)^k}$$

is its generating function. It is much more difficult to determine an explicit formula for the number of nonnegative integral solutions of an equation obtained from (7.13) by putting arbitrary positive integral coefficients in front of the e_i. Nevertheless, the generating function for the number of solutions is readily obtained using the ideas we have already discussed. We illustrate with the next example.

Example. Let h_n denote the number of nonnegative integral solutions of the equation

$$3e_1 + 4e_2 + 2e_3 + 5e_4 = n.$$

Find the generating function $g(x)$ for $h_0, h_1, h_2, \ldots, h_n, \ldots$.

We introduce a change of variable by letting

$$f_1 = 3e_1, \ f_2 = 4e_2, \ f_3 = 2e_3, \text{ and } f_4 = 5e_4.$$

Then h_n also equals the number of nonnegative integral solutions of

$$f_1 + f_2 + f_3 + f_4 = n,$$

where f_1 is a multiple of 3, f_2 is a multiple of 4, f_3 is even, and f_4 is a multiple of 5. Equivalently, h_n is the number of n-combinations of apples, bananas, oranges, and pears in which the number of apples is a multiple of 3, the number of bananas is a multiple of 4, the number of oranges is even, and the number of pears is a multiple of 5. Hence,

$$\begin{aligned} g(x) &= (1+x^3+x^6+\cdots)(1+x^4+x^8+\cdots) \times \\ &\qquad\qquad (1+x^2+x^4+\cdots)(1+x^5+x^{10}+\cdots) \\ &= \frac{1}{1-x^3}\frac{1}{1-x^4}\frac{1}{1-x^2}\frac{1}{1-x^5}. \end{aligned}$$

□

We have the following example of a similar nature.

Example. There is available an unlimited number of pennies, nickels, dimes, quarters, and half-dollar pieces. Determine the generating function $g(x)$ for the number h_n of ways of making n cents with these pieces.

The number h_n equals the number of nonnegative integral solutions of the equation

$$e_1 + 5e_2 + 10e_3 + 25e_4 + 50e_5 = n.$$

The generating function is

$$g(x) = \frac{1}{1-x}\frac{1}{1-x^5}\frac{1}{1-x^{10}}\frac{1}{1-x^{25}}\frac{1}{1-x^{50}}.$$

□

We conclude this section with the following theorem concerning inversions in a permutation. Recall from Section 4.2 that an inversion in a permutation $\pi = i_1 i_2 \ldots i_n$ of $\{1, 2, \ldots, n\}$ is a pair (i_k, i_l) with $k < l$ and $i_k > i_l$. The total number of inversions in π is denoted by $\mathrm{inv}(\pi)$. As we know from Section 4.2, $0 \leq \mathrm{inv}(\pi) \leq n(n-1)/2$. For example, if $n = 6$ and $\pi = 315264$, then $\mathrm{inv}(\pi) = 5$. Let $h(n,t)$ denote the number of permutations of $\{1, 2, \ldots, n\}$ with t inversions. Then $h(n,t) \geq 1$ for $0 \leq t \leq n(n-1)/2$, and $h(n,t) = 0$ for $t > n(n-1)/2$. In the next theorem we identify the generating function

$$g_n(x) = h(n,0) + h(n,1)x + h(n,2)x^2 + \cdots + h(n, n(n-1)/2)x^{n(n-1)/2}$$

for the sequence

$$h(n,0), h(n,1), h(n,2), \ldots, h(n, n(n-1)/2).$$

Theorem 7.2.1 *Let n be a positive integer. Then*

$$\begin{aligned} g_n(x) &= 1(1+x)(1+x+x^2)(1+x+x^2+x^3)\cdots(1+x+x^2+\cdots+x^{n-1}) \\ &= \frac{\prod_{j=1}^{n}(1-x^j)}{(1-x)^n}. \end{aligned} \tag{7.14}$$

Proof. Denote the right side[7] of (7.14) by $q_n(x)$ so that we now have to prove that $q_n(x) = g_n(x)$. First notice that the degree of the polynomial $q_n(x)$ equals $1 + 2 + 3 + \cdots + (n-1) = n(n-1)/2$ as it should be if it is to equal $g_n(x)$. In multiplying out the formula for $q_n(x)$, we get exactly once each term of the form

$$x^{a_n} x^{a_{n-1}} x^{a_{n-2}} \cdots x^{a_1} = x^p,$$

[7] Of course, the initial factor of 1 on the right side of (7.14) can be omitted if $n \geq 2$, but if $n = 1$ it is the only factor.

where

$$p = a_n + a_{n-1} + a_{n-2} + \cdots + a_1 \tag{7.15}$$

and

$$0 \le a_n \le 0, 0 \le a_{n-1} \le 1, 0 \le a_{n-2} \le 2, \ldots, 0 \le a_1 \le n-1. \tag{7.16}$$

Thus the coefficient of x^p in $q_n(x)$ equals the number of solutions of the equation (7.15) satisfying (7.16). But we know from Section 4.2 that solutions of (7.16) are in one-to-one correspondence with the permutations of $\{1, 2, \ldots, n\}$, with the solutions of (7.16) satisfying (7.15) corresponding to the permutations with p inversions. Thus the coefficient of x^p in $q_n(x)$ equals $h(n,p)(x)$. Since this is true for all $p = 0, 1, 2, \ldots, n(n-1)/2$, $q_n(x) = g_n(x)$.

□

7.3 Exponential Generating Functions

In Section 7.2, we defined the generating function for a sequence $h_0, h_1, h_2, \ldots, h_n, \ldots$ by using the set of monomials

$$\{1, x, x^2, \ldots, x^k, \ldots\}.$$

This is particularly suited to some counting sequences, especially those involving binomial coefficients, because of the form of Newton's binomial theorem. However, for sequences whose terms count permutations, it is more useful to consider a generating function with respect to the monomials

$$\{1, x, \frac{x^2}{2!}, \ldots, \frac{x^n}{n!}, \ldots\}. \tag{7.17}$$

These monomials arise in the Taylor series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots.$$

Generating functions considered with respect to the monomials (7.17) are called *exponential generating functions.*[8] The *exponential generating function* for the sequence $h_0, h_1, h_2, \ldots, h_n, \ldots$ is defined to be

$$g^{(e)}(x) = \sum_{n=0}^{\infty} h_n \frac{x^n}{n!} = h_0 + h_1 x + h_2 \frac{x^2}{2!} + \cdots + h_n \frac{x^n}{n!} + \cdots.$$

[8]We reserve the phrase "generating function" or "ordinary generating function" for the case in which we use the monomials $\{1, x, x^2, \ldots, x^n, \ldots\}$.

Example. Let n be a positive integer. Determine the exponential generating function for the sequence of numbers

$$P(n,0), P(n,1), P(n,2), \ldots, P(n,n),$$

where $P(n,k)$ denotes the number of k-permutations of an n-element set, and thus has the value $n!/(n-k)!$ for $k = 0, 1, \ldots, n$. The exponential generating function is

$$\begin{aligned} g^{(e)}(x) &= P(n,0) + P(n,1)x + P(n,2)\frac{x^2}{2!} + \ldots + P(n,n)\frac{x^n}{n!} \\ &= 1 + nx + \frac{n!}{2!(n-2)!}x^2 + \cdots + \frac{n!}{n!0!}x^n \\ &= (1+x)^n. \end{aligned}$$

Thus, $(1+x)^n$ is the exponential generating function for the sequence of numbers $P(n,0), P(n,1), \ldots, P(n,n)$ and, as we saw in Section 7.5, the ordinary generating function for the sequence

$$\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}.$$

□

Example. The exponential generating function for the sequence

$$1, 1, 1, \ldots, 1, \ldots$$

is

$$g^{(e)}(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x.$$

More generally, if a is any real number, the exponential generating function for the sequence

$$a^0 = 1, a, a^2, \ldots, a^n, \ldots$$

is

$$g^{(e)}(x) = \sum_{n=0}^{\infty} a^n \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{(ax)^n}{n!} = e^{ax}.$$

We recall from Section 3.4 that, for a positive integer k, k^n represents the number of n-permutations of a multiset with objects of k different types, each with an infinite repetition number. Thus, the exponential generating function for this sequence of counting numbers is e^{kx}. □

For a multiset S with objects of k different types, each with a finite repetition number, the next theorem determines the exponential generating function for the number of n-permutations of S. This is the solution in the form of an exponential generating function that was promised at the end of Section 3.4. We define the number of 0-permutations of a multiset to be equal to 1.

Theorem 7.3.1 *Let S be the multiset $\{n_1 \cdot a_1, n_2 \cdot a_2, \ldots, n_k \cdot a_k\}$, where $n_1, n_2, \ldots, n_k$ are nonnegative integers. Let h_n be the number of n-permutations of S. Then the exponential generating function $g^{(e)}(x)$ for the sequence $h_0, h_1, h_2, \ldots, h_n, \ldots$ is given by*

$$g^{(e)}(x) = f_{n_1}(x) f_{n_2}(x) \cdots f_{n_k}(x), \tag{7.18}$$

where, for $i = 1, 2, \ldots, k$,

$$f_{n_i}(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{n_i}}{n_i!}. \tag{7.19}$$

Proof. Let

$$g^{(e)}(x) = h_0 + h_1 x + h_2 \frac{x^2}{2!} + \cdots + h_n \frac{x^n}{n!} + \cdots$$

be the exponential generating function for $h_0, h_1, h_2, \ldots, h_n, \ldots$. Note that $h_n = 0$ for $n > n_1 + n_2 + \cdots + n_k$, so that $g^{(e)}(x)$ is a finite sum. From (7.19), we see that, when (7.18) is multiplied out, we get terms of the form

$$\frac{x^{m_1}}{m_1!} \frac{x^{m_2}}{m_2!} \cdots \frac{x^{m_k}}{m_k!} = \frac{x^{m_1+m_2+\cdots+m_k}}{m_1! m_2! \cdots m_k!}, \tag{7.20}$$

where

$$0 \le m_1 \le n_1,\ 0 \le m_2 \le n_2, \ldots, 0 \le m_k \le n_k.$$

Let $n = m_1 + m_2 + \cdots + m_k$. Then the expression in (7.20) can be written as

$$\frac{x^n}{m_1! m_2! \cdots m_k!}, = \frac{n!}{m_1! m_2! \cdots m_k!} \frac{x^n}{n!}.$$

Thus, the coefficient of $x^n/n!$ in (7.18) is

$$\sum \frac{n!}{m_1! m_2! \cdots m_k!}, \tag{7.21}$$

where the summation extends over all integers $m_1, m_2, \ldots, m_k$, with

$$0 \le m_1 \le n_1, 0 \le m_2 \le n_2, \ldots, 0 \le m_k \le n_k,$$

$$m_1 + m_2 + \cdots + m_k = n.$$

But from Section 3.4 we know that the quantity

$$\frac{n!}{m_1! m_2! \cdots m_k!} \text{ with } n = m_1 + m_2 + \cdots + m_k$$

in the sum (7.21) equals the number of n-permutations (or, simply, permutations) of the combination $\{m_1 \cdot e_1, m_2 \cdot e_2, \ldots, m_k \cdot e_k\}$ of S. Since the number of n-permutations of S equals the number of permutations taken over all such combinations with $m_1 + m_2 + \cdots + m_k = n$, the number h_n equals the number in (7.21). Since this is also the coefficient of $x^n/n!$ in (7.18), we conclude that

$$g^{(e)}(x) = f_{n_1}(x) f_{n_2}(x) \cdots f_{n_k}(x). \qquad \square$$

Using the same type of reasoning as used in the proof of the preceding theorem, we can calculate the exponential generating function for sequences of numbers that count n-permutations of a multiset with additional restrictions. Let us first observe that if, in (7.19), we define

$$f_\infty(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^k}{k!} + \cdots = e^x,$$

then the theorem continues to hold if some of the repetition numbers $n_1, n_2, \ldots, n_k$ are equal to ∞.

Example. Let h_n denote the number of n-digit numbers with digits 1, 2, or 3, where the number of 1s is even, the number of 2s is at least three, and the number of 3s is at most four. Determine the exponential generating function $g^{(e)}(x)$ for the resulting sequence of numbers $h_0, h_1, h_2, \ldots, h_n, \ldots$.

The function $g^{(e)}(x)$ has a factor for each of the three digits 1, 2, and 3. The restrictions on the digits are reflected in the factors as follows: The factor of $g^{(e)}(x)$ corresponding to the digit 1 is

$$h_1(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots,$$

since the number of 1s is to be even. The factors of $g^{(e)}(x)$ corresponding to the digits 2 and 3 are, respectively,

$$h_2(x) = \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots,$$

and

$$h_3(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}.$$

The exponential generating function is the product of the preceding three factors:

$$g^{(e)}(x) = h_1(x) h_2(x) h_3(x).$$

$\square$

Exponential generating functions can sometimes be used to find explicit formulas for counting problems. We illustrate this with three examples.

Example. Determine the number of ways to color the squares of a 1-by-n chessboard, using the colors, red, white, and blue, if an even number of squares are to be colored red.

Let h_n denote the number of such colorings, where we define h_0 to be 1. Then h_n equals the number of n-permutations of a multiset of three colors (red, white, and blue), each with an infinite repetition number, in which red occurs an even number of times. Thus, the exponential generating function for $h_0, h_1, \ldots, h_n, \ldots$ is the product of red, white, and blue factors:

$$\begin{aligned}
g^{(e)} &= \left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots\right)\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots\right)\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots\right)\\
&= \frac{1}{2}(e^x+e^{-x})e^xe^x = \frac{1}{2}(e^{3x}+e^x)\\
&= \frac{1}{2}\left(\sum_{n=0}^{\infty}3^n\frac{x^n}{n!}+\sum_{n=0}^{\infty}\frac{x^n}{n!}\right)\\
&= \frac{1}{2}\sum_{n=0}^{\infty}(3^n+1)\frac{x^n}{n!}.
\end{aligned}$$

Hence, $h_n = (3^n+1)/2$.

The simple formula for h_n suggests there might be an alternative, more direct, way to solve this problem. First we note that $h_1 = 2$, since with only one square we can only color it white or blue. Let $n \geq 2$. If the first square is colored white or blue, there are h_{n-1} ways to complete the coloring. If the first square is colored red, then there must be an odd number of red squares among the remaining $n-1$ squares; hence we subtract the number h_{n-1} of ways to color with an even number of red squares from the total number 3^{n-1} ways to color in order to get the number $3^{n-1} - h_{n-1}$ ways to color with an odd number of red squares. Therefore, h_n satisfies the recurrence relation

$$h_n = 2h_{n-1} + (3^{n-1} - h_{n-1}) = h_{n-1} + 3^{n-1}, \quad (n \geq 2).$$

If we iterate the recurrence relation $h_n = h_{n-1} + 3^{n-1}$ and use $h_1 = 2$, we obtain

$$h_n = 1 + 3 + 3^2 + \cdots + 3^{n-1} = (3^n+1)/2.$$

□

Example. Determine the number h_n of n-digit numbers with each digit odd, where the digits 1 and 3 occur an even number of times.

Let $h_0 = 1$. The number h_n equals the number of n-permutations of the multiset $S = \{\infty \cdot 1, \infty \cdot 3, \infty \cdot 5, \infty \cdot 7, \infty \cdot 9\}$, in which 1 and 3 occur an even number of times. The exponential generating function for $h_0, h_1, h_2, \ldots, h_n, \ldots$ is a product of five factors, one for each of the allowable digits:

$$\begin{aligned}
g^{(e)}(x) &= \left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots\right)^2\left(1+x+\frac{x^2}{2!}+\cdots\right)^3 \\
&= \left(\frac{e^x+e^{-x}}{2}\right)^2 e^{3x} \\
&= \left(\frac{e^{2x}+1}{2}\right)^2 e^{x} \\
&= \frac{1}{4}(e^{4x}+2e^{2x}+1)e^x \\
&= \frac{1}{4}(e^{5x}+2e^{3x}+e^x) \\
&= \frac{1}{4}\left(\sum_{n=0}^{\infty}5^n\frac{x^n}{n!}+2\sum_{n=0}^{\infty}3^n\frac{x^n}{n!}+\sum_{n=0}^{\infty}\frac{x^n}{n!}\right) \\
&= \sum_{n=0}^{\infty}\left(\frac{5^n+2\times 3^n+1}{4}\right)\frac{x^n}{n!}.
\end{aligned}$$

Hence,

$$h_n = \frac{5^n+2\times 3^n+1}{4}, \qquad (n\ge 0).$$

□

Example. Determine the number h_n of ways to color the squares of a 1-by-n board with the colors red, white, and blue, where the number of red squares is even and there is at least one blue square.

The exponential generating function $g^{(e)}(x)$ is

$$\begin{aligned}
g^{(e)}(x) &= \left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots\right)\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots\right)\left(\frac{x}{1!}+\frac{x^2}{2!}+\cdots\right) \\
&= \frac{e^x+e^{-x}}{2}e^x(e^x-1) \\
&= \frac{e^{3x}-e^{2x}+e^x-1}{2} \\
&= -\frac{1}{2}+\sum_{n=0}^{\infty}\frac{3^n-2^n+1}{2}\frac{x^n}{n!}.
\end{aligned}$$

Thus,

$$h_0 = -\frac{1}{2}+\frac{3^0-2^0+1}{2} = -\frac{1}{2}+\frac{1}{2}=0$$

and

$$h_n = \frac{3^n-2^n+1}{2}, \qquad (n=1,2,\ldots).$$

Note that h_0 should be 0. A 1-by-0 board is empty, no squares get colored, and so we cannot satisfy the condition that the number of blue squares is at least 1. □

7.4 Solving Linear Homogeneous Recurrence Relations

In this section we give a formal definition of a certain class of recurrence relations for which there is a general method of solution. The application of the method is, however, limited by the fact that it requires us to find the roots of a polynomial equation whose degree may be large.

Let

$$h_0, h_1, h_2, \ldots, h_n, \ldots$$

be a sequence of numbers. This sequence is said to satisfy a *linear recurrence relation of order* k, provided that there exist quantities $a_1, a_2, \ldots, a_k$, with $a_k \neq 0$, and a quantity b_n (each of these quantities $a_1, a_2, \ldots, a_k, b_n$ may depend on n) such that

$$h_n = a_1 h_{n-1} + a_2 h_{n-2} + \cdots + a_k h_{n-k} + b_n, \quad (n \geq k). \tag{7.22}$$

Example. Our two recurrence relations for the sequence of derangement numbers $D_0, D_1, D_2, \ldots, D_n, \ldots$, namely,

$$D_n = (n-1)D_{n-1} + (n-1)D_{n-2} \ (n \geq 2) \quad \text{and}$$

$$D_n = nD_{n-1} + (-1)^n \ (n \geq 1),$$

are linear recurrence relations. The first has order 2, and we have $a_1 = n-1$, $a_2 = n-1$ and $b_n = 0$. The second has order 1, and we have $a_1 = n$ and $b_n = (-1)^n$. □

Example. The Fibonacci sequence $f_0, f_1, f_2, \ldots, f_n, \ldots$ satisfies the linear recurrence relation

$$f_n = f_{n-1} + f_{n-2} \qquad (n \geq 2)$$

of order 2 with $a_1 = 1, a_2 = 1$, and $b_n = 0$. □

Example. The factorial sequence $h_0, h_1, h_2, \ldots, h_n, \ldots$, where $h_n = n!$, satisfies the linear recurrence relation

$$h_n = nh_{n-1} \qquad (n \geq 1)$$

of order 1 with $a_1 = n$ and $b_n = 0$. □

Example. The geometric sequence $h_0, h_1, h_2, \ldots, h_n, \ldots$, where $h_n = q^n$, satisfies the linear recurrence relation

$$h_n = qh_{n-1} \qquad (n \geq 1)$$

of order 1 with $a_1 = q$ and $b_n = 0$. □

As these examples indicate, the quantities $a_1, a_2, \ldots, a_k$ in (7.22) may be constant or may depend on n. Similarly, the quantity b_n in (7.22) may be a constant (possibly zero) or also may depend on n.

The linear recurrence relation (7.22) is called *homogeneous* provided that b_n is the zero constant and is said to have *constant coefficients* provided that $a_1, a_2, \ldots, a_k$ are constants. In this section, we discuss a special method for solving linear homogeneous recurrence relations with constant coefficients—that is, recurrence relations of the form

$$h_n = a_1 h_{n-1} + a_2 h_{n-2} + \cdots + a_k h_{n-k}, \quad (n \geq k), \tag{7.23}$$

where $a_1, a_2, \ldots, a_k$ are constants and $a_k \neq 0$.[9] The success of the method to be described depends on being able to find the roots of a certain polynomial equation associated with (7.23).

The recurrence relation (7.23) can be rewritten in the form

$$h_n - a_1 h_{n-1} - a_2 h_{n-2} - \cdots - a_k h_{n-k} = 0, \quad (n \geq k). \tag{7.24}$$

A sequence of numbers $h_0, h_1, h_2, \ldots, h_n, \ldots$ satisfying the recurrence relation (7.24) (or, more generally, (7.22)) is uniquely determined once the values of $h_0, h_1, \ldots, h_{k-1}$, the so-called *initial values*, are prescribed. The recurrence relation (7.24) "kicks in" beginning with $n = k$. To begin with, we ignore the initial values and look for solutions of (7.24) without prescribed initial values. It turns out that we can find "enough" solutions by only considering solutions that form geometric sequences and suitably modifying such solutions.

Example.[10] In this example we recall a method for solving linear homogeneous differential equations with constant coefficients. Consider the differential equation

$$y'' - 5y' + 6y = 0. \tag{7.25}$$

Here y is a function of a real variable x. We seek solutions of this equation among the basic exponential functions $y = e^{qx}$. Let q be a constant. Since $y' = qe^{qx}$ and $y'' = q^2 e^{qx}$, it follows that $y = e^{qx}$ is a solution of (7.25) if and only if

$$q^2 e^{qx} - 5qe^{qx} + 6e^{qx} = 0.$$

Since the exponential function e^{qx} is never zero, it may be cancelled, and we obtain the following equation that does not depend on x:

$$q^2 - 5q + 6 = 0.$$

[9] If a_k were 0, we would delete the term $a_k h_{n-k}$ from (7.23) and obtain a lower order recurrence relation.

[10] For those who have not studied differential equations, this example can be safely ignored. It's only here to show the close similarity of the methods for recurrence relations (our interest) with those of differential equations that you may have studied.

This equation has two roots, namely, $q = 2$ and $q = 3$. Hence

$$y = e^{2x} \quad \text{and} \quad y = e^{3x}$$

are both solutions of (7.25). Since the differential equation is linear and homogeneous,

$$y = c_1e^{2x} + c_2e^{3x} \tag{7.26}$$

is also a solution of (7.25) for any choice of the constants c_1 and c_2.[11] Now we bring in initial conditions for (7.25). These are conditions that prescribe both the value of y and its first derivative when $x = 0$ that, with the differential equation (7.25), uniquely determine y. Suppose we prescribe the initial conditions

$$y(0) = a, \quad y'(0) = b, \tag{7.27}$$

where a and b are fixed but unspecified numbers. Then, in order that the solution (7.26) of the differential equation (7.25) satisfy these initial conditions, we must have

$$\begin{cases} y(0) = a: & c_1 + c_2 = a \\ y'(0) = b: & 2c_1 + 3c_2 = b. \end{cases}$$

This system of two equations has a unique solution for each choice of a and b, namely,

$$c_1 = 3a - b, \quad c_2 = b - 2a. \tag{7.28}$$

Thus, no matter what the initial conditions (7.27), we can choose c_1 and c_2 using (7.28) so that the function (7.26) is a solution of the differential equation (7.25). In this sense (7.26) is the *general solution* of the differential equation: Each solution of (7.25) with prescribed initial conditions can be written in the form (7.26) for suitable choice of the constants c_1 and c_2. □

The solution of linear homogeneous recurrence relations proceeds along similar lines with the role of the exponential function e^{qx} taken up by the discrete function q^n defined only for nonnegative integers n (the geometric sequences). We have already seen an example of this in our evaluation of the Fibonacci numbers in Section 7.1.

Theorem 7.4.1 *Let q be a nonzero number. Then $h_n = q^n$ is a solution of the linear homogeneous recurrence relation*

$$h_n - a_1h_{n-1} - a_2h_{n-2} - \cdots - a_kh_{n-k} = 0, \quad (a_k \neq 0, n \geq k) \tag{7.29}$$

with constant coefficients if and only if q is a root of the polynomial equation

$$x^k - a_1x^{k-1} - a_2x^{k-2} - \cdots - a_k = 0. \tag{7.30}$$

[11]This can be verified by computing y' and y'' and substituting into (7.25).

If the polynomial equation has k distinct roots $q_1, q_2, \ldots, q_k$, then

$$h_n = c_1 q_1^n + c_2 q_2^n + \cdots + c_k q_k^n \tag{7.31}$$

is the general solution of (7.29) *in the following sense: No matter what initial values for $h_0, h_1, \ldots, h_{k-1}$ are given, there are constants $c_1, c_2, \ldots, c_k$ so that* (7.31) *is the unique sequence which satisfies both the recurrence relation* (7.29) *and the initial values.*

Proof. We see that $h_n = q^n$ is a solution of (7.29) if and only if

$$q^n - a_1 q^{n-1} - a_2 q^{n-2} - \cdots - a_k q^{n-k} = 0$$

for all $n \geq k$. Since we assume $q \neq 0$, we may cancel q^{n-k}. Thus, these infinitely many equations (there is one for each $n \geq k$) reduce to only *one* equation:

$$q^k - a_1 q^{k-1} - a_2 q^{k-2} - \cdots - a_k = 0.$$

We conclude that $h_n = q^n$ is a solution of (7.29) if and only if q is a root of the polynomial equation (7.30).

Since a_k is assumed to be different from zero, 0 is not a root of (7.30). Hence, (7.30) has k roots, $q_1, q_2, \ldots, q_k$, all different from zero. These roots may be complex numbers. In general, $q_1, q_2, \ldots, q_k$ need not be distinct (the equation may have multiple roots), but we now assume that the roots $q_1, q_2, \ldots, q_k$ are distinct. Thus,

$$h_n = q_1^n, \quad h_n = q_2^n, \quad \ldots, \quad h_n = q_k^n$$

are k different solutions of (7.29). The linearity and the homogeneity of the recurrence relation (7.29) imply that, for any choice of constants $c_1, c_2, \ldots, c_k$,

$$h_n = c_1 q_1^n + c_2 q_2^n + \cdots + c_k q_k^n \tag{7.32}$$

is also a solution of (7.29).[12] We now show that (7.32) is the general solution of (7.29) in the sense given in the statement of the theorem.

Suppose we prescribe the initial values

$$h_0 = b_0, \quad h_1 = b_1, \quad \ldots, \quad \text{and } h_{k-1} = b_{k-1}.$$

Can we choose the constants $c_1, c_2, \ldots, c_k$ so that h_n as given in (7.32) satisfies these initial conditions? Equivalently, can we always solve the system of equations

$$\left\{\begin{array}{ll} (n=0) & c_1 + c_2 + \cdots + c_k = b_0 \\ (n=1) & c_1 q_1 + c_2 q_2 + \cdots + c_k q_k = b_1 \\ (n=2) & c_1 q_1^2 + c_2 q_2^2 + \cdots + c_k q_k^2 = b_2 \\ \vdots & \\ (n=k-1) & c_1 q_1^{k-1} + c_2 q_2^{k-1} + \cdots + c_k q_k^{k-1} = b_{k-1} \end{array}\right. \tag{7.33}$$

[12] This can be verified by direct substitution.

no matter what the choice of $b_0, b_1, \ldots, b_{k-1}$?

Now we need to rely on a basic fact from linear algebra. The coefficient matrix of this system of equations is

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ q_1 & q_2 & \cdots & q_k \\ q_1^2 & q_2^2 & \cdots & q_k^2 \\ \vdots & \vdots & \ddots & \vdots \\ q_1^{k-1} & q_2^{k-1} & \cdots & q_k^{k-1} \end{bmatrix}. \tag{7.34}$$

The matrix in (7.34) is an important matrix called the *Vandermonde matrix*. The Vandermonde matrix is an invertible matrix if and only if $q_1, q_2, \ldots, q_k$ are distinct. Indeed, its determinant equals

$$\prod_{1 \le i < j \le k} (q_j - q_i)$$

and hence is nonzero exactly when $q_1, q_2, \ldots, q_k$ are distinct.[13] Thus, our assumption of the distinctness of $q_1, q_2, \ldots, q_k$ implies that the system (7.33) has a unique solution for each choice of $b_0, b_1, \ldots, b_{k-1}$. Therefore, (7.32) is the general solution of (7.29), and the proof of the theorem is complete. □

The polynomial equation (7.30) is called the *characteristic equation* of the recurrence relation (7.29) and its k roots are the *characteristic roots*. By Theorem 7.4.1, if the characteristic roots are distinct, (7.31) is the general solution of (7.29).

Example. Solve the recurrence relation

$$h_n = 2h_{n-1} + h_{n-2} - 2h_{n-3}, \quad (n \ge 3),$$

subject to the initial values $h_0 = 1$, $h_1 = 2$, and $h_2 = 0$.

The characteristic equation of this recurrence relation is

$$x^3 - 2x^2 - x + 2 = 0,$$

and its three roots are $1, -1, 2$. By Theorem 7.4.1,

$$h_n = c_1 1^n + c_2(-1)^n + c_3 2^n = c_1 + c_2(-1)^n + c_3 2^n$$

is the general solution. We now want constants c_1, c_2, and c_3 so that

$$\begin{cases} (n=0) & c_1 + c_2 + c_3 = 1, \\ (n=1) & c_1 - c_2 + 2c_3 = 2, \\ (n=2) & c_1 + c_2 + 4c_3 = 0. \end{cases}$$

[13]The proof of this formula is elementary but nontrivial.

The unique solution of this system can be found by the usual elimination method to be $c_1 = 2, c_2 = -\frac{2}{3}, c_3 = -\frac{1}{3}$. Thus,

$$h_n = 2 - \frac{2}{3}(-1)^n - \frac{1}{3}2^n$$

is the solution of the given recurrence relation. □

Example. Words of length n, using only the three letters a, b, c, are to be transmitted over a communication channel subject to the condition that no word in which two a's appear consecutively is to be transmitted. Determine the number of words allowed by the communication channel.

Let h_n denote the number of allowed words of length n. We have $h_0 = 1$ (the empty word) and $h_1 = 3$. Let $n \geq 2$. If the first letter of the word is b or c, then the word can be completed in h_{n-1} ways. If the first letter of the word is a, then the second letter is b or c. If the second letter is b, the word can be completed in h_{n-2} ways. If the second letter is c, the word can also be completed in h_{n-2} ways. Hence, h_n satisfies the recurrence relation

$$h_n = 2h_{n-1} + 2h_{n-2}, \qquad (n \geq 2).$$

The characteristic equation is

$$x^2 - 2x - 2 = 0,$$

and the characteristic roots are

$$q_1 = 1 + \sqrt{3}, \qquad q_2 = 1 - \sqrt{3}.$$

Therefore, the general solution is

$$h_n = c_1(1 + \sqrt{3})^n + c_2(1 - \sqrt{3})^n, \qquad (n \geq 3).$$

To determine h_n, we find c_1 and c_2 such that the initial values $h_0 = 1$ and $h_1 = 3$ hold. This leads to the system of equations

$$\begin{cases} (n = 0) & c_1 + c_2 = 1 \\ (n = 1) & c_1(1 + \sqrt{3}) + c_2(1 - \sqrt{3}) = 3, \end{cases}$$

which has solution

$$c_1 = \frac{2 + \sqrt{3}}{2\sqrt{3}}, \quad c_2 = \frac{-2 + \sqrt{3}}{2\sqrt{3}}.$$

Therefore,

$$h_n = \frac{2 + \sqrt{3}}{2\sqrt{3}}(1 + \sqrt{3})^n + \frac{-2 + \sqrt{3}}{2\sqrt{3}}(1 - \sqrt{3})^n, \quad (n \geq 0)$$

is the number of words that can be transmitted over the communication channel with the restrictions as given. □

The method given for solving linear homogeneous recurrence relations with constant coefficients can be alternatively described in terms of generating functions. An important role is now played by Newton's binomial theorem. Specifically, the following case of Newton's binomial theorem will be used:

If n is a positive integer and r is a nonzero real number, then

$$(1-rx)^{-n} = \sum_{k=0}^{\infty} \binom{-n}{k} (-rx)^k,$$

or, equivalently,

$$\frac{1}{(1-rx)^n} = \sum_{k=0}^{\infty} (-1)^k \binom{-n}{k} r^k x^k, \quad \left(|x| < \frac{1}{|r|}\right).$$

We have seen in Section 5.6 that

$$\binom{-n}{k} = (-1)^k \binom{n+k-1}{k},$$

and hence we can write the formula for $1/(1-rx)^n$ as

$$\frac{1}{(1-rx)^n} = \sum_{k=0}^{\infty} \binom{n+k-1}{k} r^k x^k, \quad \left(|x| < \frac{1}{|r|}\right). \tag{7.35}$$

Example. Determine the generating function for the sequence of squares

$$0, 1, 4, \ldots, n^2, \ldots.$$

By (7.35), with $n = 2$ and $r = 1$,

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \cdots + nx^{n-1} + \cdots,$$

and hence

$$\frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + \cdots + nx^n + \cdots.$$

Differentiating and then multiplying by x, we get

$$\frac{1+x}{(1-x)^3} = 1 + 2^2x + 3^2x^2 + \cdots + n^2x^{n-1} + \cdots$$

and

$$\frac{x(1+x)}{(1-x)^3} = x + 2^2x^2 + 3^2x^3 + \cdots + n^2x^n + \cdots.$$

Therefore, $x(1+x)/(1-x)^3$ is the desired generating function. □

The next example illustrates how to use generating functions to solve linear homogeneous recurrence relations with constant coefficients.

Example. Solve the recurrence relation

$$h_n = 5h_{n-1} - 6h_{n-2} \quad (n \geq 2)$$

subject to the initial values $h_0 = 1$ and $h_1 = -2$.

We write the recurrence relation in the form

$$h_n - 5h_{n-1} + 6h_{n-2} \quad (n \geq 2).$$

Let $g(x) = h_0 + h_1x + h_2x^2 + \cdots + h_nx^n + \cdots$ be the generating function for the sequence $h_0, h_1, h_2, \ldots, h_n, \ldots$. We then have the following equations where the multipliers are chosen by looking at the recurrence relation:

$$\begin{array}{rcrcrcccrcc} g(x) & = & h_0 + h_1x & + & h_2x^2 & + & \cdots & + & h_nx^n & + & \cdots, \\ -5xg(x) & = & -5h_0x & - & 5h_1x^2 & - & \cdots & - & 5h_{n-1}x^n & + & \cdots, \\ 6x^2g(x) & = & & & 6h_0x^2 & + & \cdots & + & 6h_{n-2}x^n & + & \cdots. \end{array}$$

Adding these three equations, we obtain

$$\begin{aligned} (1 - 5x + 6x^2)g(x) &= h_0 + (h_1 - 5h_0)x + (h_2 - 5h_1 + 6h_0)x^2 + \cdots \\ &\quad + (h_n - 5h_{n-1} + 6h_{n-2})x^n + \cdots. \end{aligned}$$

Since $h_n - 5h_{n-1} + 6h_{n-2} = 0$ $(n \geq 2)$, and since $h_0 = 1$ and $h_1 = -2$, we have

$$(1 - 5x + 6x^2)g(x) = h_0 + (h_1 - 5h_0)x = 1 - 7x.$$

Thus,

$$g(x) = \frac{1 - 7x}{1 - 5x + 6x^2}.$$

From this closed formula for the generating function $g(x)$, we would like to be able to determine a formula for h_n. To obtain such a formula, we use the method of partial fractions along with (7.35). We observe that

$$1 - 5x + 6x^2 = (1 - 2x)(1 - 3x),$$

and thus it is possible to write

$$\frac{1 - 7x}{1 - 5x + 6x^2} = \frac{c_1}{1 - 2x} + \frac{c_2}{1 - 3x}$$

for some constants c_1 and c_2. We can determine c_1 and c_2 by multiplying both sides of this equation by $1-5x+6x^2$ to get

$$1-7x=(1-3x)c_1+(1-2x)c_2,$$

or

$$1-7x=(c_1+c_2)+(-3c_1-2c_2)x.$$

Hence,

$$\begin{cases} c_1+c_2 &= 1 \\ -3c_1-2c_2 &= -7. \end{cases}$$

Solving these equations simultaneously, we find that $c_1=5$ and $c_2=-4$. Thus,

$$g(x)=\frac{1-7x}{1-5x+6x^2}=\frac{5}{1-2x}-\frac{4}{1-3x}.$$

By (7.35),

$$\frac{1}{1-2x}=1+2x+2^2x^2+\cdots+2^nx^n+\cdots,$$

and

$$\frac{1}{1-3x}=1+3x+3^2x^2+\cdots+3^nx^n+\cdots.$$

Therefore,

$$\begin{aligned} g(x) &= 5(1+2x+2^2x^2+\cdots+2^nx^n+\cdots) \\ &\quad -4(1+3x+3^2x^2+\cdots+3^nx^n+\cdots) \\ &= 1+(-2)x+(-15)x^2+\cdots+(5\times 2^n-4\times 3^n)x^n+\cdots. \end{aligned}$$

Since this is the generating function for $h_0,h_1,h_2,\ldots,h_n,\ldots$, we obtain $h_n=5\times 2^n-4\times 3^n \quad (n=0,1,2,\ldots)$. □

If the roots $q_1,q_2,\ldots,q_k$ of the characteristic equation are not distinct, then

$$h_n=c_1q_1^n+c_2q_2^n+\cdots+c_kq_k^n \tag{7.36}$$

in Theorem 7.4.1 is not a general solution of the recurrence relation.

Example. The recurrence relation

$$h_n=4h_{n-1}-4h_{n-2} \qquad (n\geq 2)$$

has characteristic equation

$$x^2-4x+4=(x-2)^2=0.$$

Thus, 2 is a twofold characteristic root. In this case, (7.36) becomes

$$h_n=c_12^n+c_22^n=(c_1+c_2)2^n=c2^n,$$

where $c = c_1 + c_2$ is a new constant. Consequently, we have only a single constant to choose in order to satisfy two initial conditions, and it is not always possible to do so. For instance, suppose we prescribe the initial values $h_0 = 1$ and $h_1 = 3$. To satisfy these initial values, we must have

$$\begin{array}{ll} (n=0) & c = 1, \\ (n=1) & 2c = 3. \end{array}$$

But these equations are contradictory. Thus, $h_n = c2^n$ is not a general solution of the given recurrence relation. □

If, as in the preceding example, some characteristic root is repeated, we would like to find another solution associated with that root. The situation is similar to that which occurs in differential equations.

Example.[14] Solve

$$y'' - 4y' + 4y = 0.$$

We have that $y = e^{qx}$ is a solution if and only if

$$q^2 e^{qx} - 4qe^{qx} + 4e^{qx} = 0,$$

or, equivalently,

$$q^2 - 4q + 4 = 0.$$

The roots of this equation are $2, 2$ (2 is a double root) and lead directly to only one solution $y = e^{2x}$. But in this case, $y = xe^{2x}$ is also a solution:

$$y' = 2xe^{2x} + e^{2x}$$

$$y'' = 4xe^{2x} + 2e^{2x} + 2e^{2x} = 4xe^{2x} + 4e^{2x}$$

$$y'' - 4y' + 4y = (4xe^{2x} + 4e^{2x}) - 4(2xe^{2x} + e^{2x}) + 4xe^{2x} = 0.$$

Thus $y = e^{2x}$ and $y = xe^{2x}$ are both solutions of the differential equation, and hence so is

$$y = c_1 e^{2x} + c_2 xe^{2x}. \tag{7.37}$$

We now verify that (7.37) is the general solution. Suppose we prescribe the initial conditions $y(0) = a$ and $y'(0) = b$. In order for (7.37) to satisfy these initial conditions, we must have

$$y(0) = a: \quad c_1 = a$$

$$y'(0) = b: \quad 2c_1 + c_2 = b.$$

[14]Again, this example can be safely omitted by those who have not studied differential equations.

These equations have the unique solution $c_1 = a$ and $c_2 = b - 2a$. Hence, constants c_1 and c_2 can be uniquely chosen to satisfy any given initial conditions, and (7.37) is the general solution. □

Example. Find the general solution of the recurrence relation

$$h_n - 4h_{n-1} + 4h_{n-2} = 0, \qquad (n \geq 2).$$

The characteristic equation is

$$x^2 - 4x + 4 = (x-2)^2 = 0$$

and has roots $2, 2$. We know that $h_n = 2^n$ is a solution of the recurrence relation. We show that $h_n = n2^n$ is also a solution. We have

$$h_n = n2^n, \ h_{n-1} = (n-1)2^{n-1}, \ h_{n-2} = (n-2)2^{n-2};$$

hence,

$$\begin{aligned} h_n - 4h_{n-1} + 4h_{n-2} &= n2^n - 4(n-1)2^{n-1} + 4(n-2)2^{n-2} \\ &= 2^{n-2}(4n - 8(n-1) + 4(n-2)) \\ &= 2^{n-2}(0) = 0. \end{aligned}$$

We now conclude that

$$h_n = c_1 2^n + c_2 n 2^n \tag{7.38}$$

is a solution for each choice of constants c_1 and c_2. Now let us impose the initial conditions

$$h_0 = a \quad \text{and} \quad h_1 = b.$$

In order that these be satisfied, we must have

$$\begin{cases} (n=0) & c_1 = a \\ (n=1) & 2c_1 + 2c_2 = b. \end{cases}$$

These equations have the unique solution $c_1 = a$ and $c_2 = (b-2a)/2$. Hence, constants c_1 and c_2 can be uniquely chosen to satisfy the initial conditions, and we conclude that (7.38) is the general solution of the given recurrence relation. □

More generally, if a (possibly complex) number q is a root of multiplicity s of the characteristic equation of a linear homogeneous recurrence relation with constant coefficients, then it can be shown that each of

$$h_n = q^n, h_n = nq^n, h_n = n^2q^n, \ldots, h_n = n^{s-1}q^n$$

is a solution, and hence so is

$$h_n = c_1q^n + c_2nq^n + c_2n^2q^n + \cdots + c_sn^{s-1}q^n,$$

for each choice of constants $c_1, c_2, \ldots, c_s$.

The more general situation in which the characteristic equation has several roots of various multiplicities is treated in the next theorem, which we state without proof.

Theorem 7.4.2 *Let $q_1, q_2, \ldots, q_t$ be the distinct roots of the following characteristic equation of the linear homogeneous recurrence relation with constant coefficients:*

$$h_n = a_1 h_{n-1} + a_2 h_{n-2} + \cdots + a_k h_{n-k}, \quad a_k \neq 0, \quad (n \geq k). \tag{7.39}$$

If q_i is an s_i-fold root of the characteristic equation of (7.39)*, the part of the general solution of this recurrence relation corresponding to q_i is*

$$\begin{aligned} H_n^{(i)} &= c_1 q_i^n + c_2 n q_i^n + \cdots + c_{s_i} n^{s_i - 1} q_i^n \\ &= (c_1 + c_2 n + \cdots + c_{s_i} n^{s_i - 1}) q_i^n. \end{aligned}$$

The general solution of the recurrence relation is

$$h_n = H_n^{(1)} + H_n^{(2)} + \cdots + H_n^{(t)}.$$

Example. Solve the recurrence relation

$$h_n = -h_{n-1} + 3h_{n-2} + 5h_{n-3} + 2h_{n-4}, \qquad (n \geq 4)$$

subject to the initial values $h_0 = 1, h_1 = 0, h_2 = 1$, and $h_3 = 2$.

The characteristic equation of this recurrence relation is

$$x^4 + x^3 - 3x^2 - 5x - 2 = 0,$$

which has roots $-1, -1, -1, 2$. Thus, the part of the general solution corresponding to the root -1 is

$$H_n^{(1)} = c_1(-1)^n + c_2 n(-1)^n + c_3 n^2 (-1)^n,$$

while the part of a general solution corresponding to the root 2 is

$$H_n^{(2)} = c_4 2^n.$$

The general solution is

$$h_n = H_n^{(1)} + H_n^{(2)} = c_1(-1)^n + c_2 n(-1)^n + c_3 n^2(-1)^n + c_4 2^n.$$

We want to determine c_1, c_2, c_3, and c_4 so that the initial conditions hold. Thus the equations

$$\begin{cases} (n=0) & c_1 & & & & & + & c_4 & = 1 \\ (n=1) & -c_1 & - & c_2 & - & c_3 & + & 2c_4 & = 0 \\ (n=2) & c_1 & + & 2c_2 & + & 4c_3 & + & 4c_4 & = 1 \\ (n=3) & -c_1 & - & 3c_2 & - & 9c_3 & + & 8c_4 & = 2 \end{cases}$$

must hold. The unique solution of this system of equations is $c_1 = \frac{7}{9}$, $c_2 = -\frac{3}{9}$, $c_3 = 0$, $c_4 = \frac{2}{9}$. Thus, the solution is

$$h_n = \frac{7}{9}(-1)^n - \frac{3}{9}n(-1)^n + \frac{2}{9}2^n.$$

□

The practical application of the method discussed in this section is limited by the difficulty in finding all the roots of a polynomial equation.

We can also use generating functions to solve (at least theoretically) any linear homogeneous recurrence relation of order k with constant coefficients. The associated generating function will be of the form $p(x)/q(x)$, where $p(x)$ is a polynomial of degree less than k and where $q(x)$ is a polynomial of degree k having constant term equal to 1. To find a general formula for the terms of the sequence, we first use the method of partial fractions to express $p(x)/q(x)$ as a sum of algebraic fractions of the form

$$\frac{c}{(1-rx)^t},$$

where t is a positive integer, r is a real number, and c is a constant. We then use (7.35) to find a power series for $1/(1-rx)^t$. Combining like terms, we obtain a power series for the generating function, from which we can read off the terms of the sequence.

Example. Let $h_0, h_1, h_2, \ldots, h_n, \ldots$ be a sequence of numbers satisfying the recurrence relation

$$h_n + h_{n-1} - 16h_{n-2} + 20h_{n-3} = 0, \qquad (n \geq 3)$$

where $h_0 = 0$, $h_1 = 1$ and $h_2 = -1$. Find a general formula for h_n.

Let $g(x) = h_0 + h_1x + h_2x^2 + \cdots + h_nx^n + \cdots$ be the generating function for $h_0, h_1, h_2, \ldots, h_n, \ldots$. Adding the four equations,

$$\begin{aligned}
g(x) &= h_0 + h_1x + h_2x^2 + h_3x^3 + \cdots + h_nx^n + \cdots, \\
xg(x) &= h_0x + h_1x^2 + h_2x^3 + \cdots + h_{n-1}x^n + \cdots, \\
-16x^2g(x) &= -16h_0x^2 - 16h_1x^3 - \cdots - 16h_{n-2}x^n - \cdots, \\
20x^3g(x) &= 20h_0x^3 + \cdots + 20h_{n-3}x^n + \cdots,
\end{aligned}$$

we obtain

$$\begin{aligned}
(1 + x - 16x^2 + 20x^3)g(x) &= h_0 + (h_1 + h_0)x + (h_2 + h_1 - 16h_0)x^2 \\
&\quad +(h_3 + h_2 - 16h_1 + 20h_0)x^3 + \cdots \\
&\quad +(h_n + h_{n-1} - 16h_{n-2} + 20h_{n-3})x^n + \cdots.
\end{aligned}$$

Since $h_n + h_{n-1} - 16h_{n-2} + 20h_{n-3} = 0$, $(n \geq 3)$ and since $h_0 = 0$, $h_1 = 1$, and $h_2 = -1$, we get

$$(1 + x - 16x^2 + 20x^3)g(x) = x.$$

Hence,

$$g(x) = \frac{x}{1 + x - 16x^2 + 20x^3}.$$

We observe that $(1 + x - 16x^2 + 20x^3) = (1 - 2x)^2(1 + 5x)$. Thus, for some constants c_1, c_2, and c_3,

$$\frac{x}{1 + x - 16x^2 + 20x^3} = \frac{c_1}{1 - 2x} + \frac{c_2}{(1 - 2x)^2} + \frac{c_3}{1 + 5x}.$$

To determine the constants, we multiply both sides of this equation by $1 + x - 16x^2 + 20x^3$ to get

$$x = (1 - 2x)(1 + 5x)c_1 + (1 + 5x)c_2 + (1 - 2x)^2 c_3,$$

or, equivalently,

$$x = (c_1 + c_2 + c_3) + (3c_1 + 5c_2 - 4c_3)x + (-10c_1 + 4c_3)x^2.$$

Hence,

$$\begin{cases} c_1 + c_2 + c_3 = 0, \\ 3c_1 + 5c_2 - 4c_3 = 1, \\ -10c_1 + 4c_3 = 0. \end{cases}$$

Solving these equations simultaneously, we find that

$$c_1 = -\frac{2}{49}, \; c_2 = \frac{7}{49}, \text{ and } c_3 = -\frac{5}{49}.$$

Therefore,

$$g(x) = \frac{x}{1 + x - 16x^2 + 20x^3} = -\frac{2/49}{1 - 2x} + \frac{7/49}{(1 - 2x)^2} - \frac{5/49}{1 + 5x}.$$

By (7.35),

$$\begin{aligned} \frac{1}{1 - 2x} &= \sum_{k=0}^{\infty} 2^k x^k, \\ \frac{1}{(1 - 2x)^2} &= \sum_{k=0}^{\infty} \binom{k+1}{k} 2^k x^k = \sum_{k=0}^{\infty} (k+1) 2^k x^k, \\ \frac{1}{1 + 5x} &= \sum_{k=0}^{\infty} (-5)^k x^k. \end{aligned}$$

Consequently,

$$g(x) = -\frac{2}{49}\left(\sum_{k=0}^{\infty} 2^k x^k\right) + \frac{7}{49}\left(\sum_{k=0}^{\infty} (k+1) 2^k x^k\right) - \frac{5}{49}\left(\sum_{k=0}^{\infty} (-5)^k x^k\right)$$

$$= \sum_{k=0}^{\infty} \left[-\frac{2}{49}2^k + \frac{7}{49}(k+1)2^k - \frac{5}{49}(-5)^k \right] x^k.$$

Since $g(x)$ is the generating function for $h_0, h_1, h_2, \ldots, h_n, \ldots$, it follows that

$$h_n = -\frac{2}{49}2^n + \frac{7}{49}(n+1)2^n - \frac{5}{49}(-5)^n, \qquad (n = 0, 1, 2, \ldots).$$

□

The preceding formula for h_n should bring to mind the solution of recurrence relations using the roots of the characteristic equation. Indeed, the formula suggests that the roots of the characteristic equation for the given recurrence relation are 2, 2, and -5. The following discussion should clarify the relationship between the two methods.

In the foregoing example, we have expressed the generating function $g(x)$ in the form

$$g(x) = \frac{p(x)}{q(x)},$$

where

$$q(x) = 1 + x - 16x^2 + 20x^3.$$

Since the recurrence relation is

$$h_n + h_{n-1} - 16h_{n-2} + 20h_{n-3} = 0, \quad (n = 3, 4, 5, \ldots),$$

the associated characteristic equation is $r(x) = 0$, where

$$r(x) = x^3 + x^2 - 16x + 20.$$

If we replace x in $r(x)$ by $1/x$ (this amounts to the change in variable $y = 1/x$), we obtain

$$r(1/x) = \frac{1}{x^3} + \frac{1}{x^2} - 16\frac{1}{x} + 20,$$

or

$$x^3 r(1/x) = 1 + x - 16x^2 + 20x^3 = q(x).$$

The roots of the characteristic equation $r(x) = 0$ are $2, 2$, and -5. Since $r(x) = (x-2)^2(x+5)$, it follows that

$$q(x) = x^3 \left(\frac{1}{x} - 2\right)^2 \left(\frac{1}{x} + 5\right) = (1-2x)^2(1+5x),$$

which checks with our previous calculation.

The preceding relationships hold in general. Let $h_0, h_1, h_2, \ldots, h_n, \ldots$ be the sequence of numbers defined by the recurrence relation

$$h_n + a_1 h_{n-1} + \cdots + a_k h_{n-k} = 0, \ (n \geq k)$$

of order k and with initial values for $h_0, h_1, \ldots, h_{k-1}$. Recall that, since the recurrence relation has order k, a_k is assumed to be different from 0. Let $g(x)$ be the generating function for our sequence. Using the method given in the examples, we find that there are polynomials $p(x)$ and $q(x)$ such that

$$g(x) = \frac{p(x)}{q(x)},$$

where $q(x)$ has degree k and $p(x)$ has degree less than k. Indeed, we have

$$q(x) = 1 + a_1x + a_2x^2 + \cdots + a_kx^k$$

and

$$\begin{aligned} p(x) &= h_0 + (h_1 + a_1h_0)x + (h_2 + a_1h_1 + a_2h_0)x^2 \\ &\quad + \cdots + (h_{k-1} + a_1h_{k-2} + \cdots + a_{k-1}h_0)x^{k-1}. \end{aligned}$$

The characteristic equation for this recurrence relation is $r(x) = 0$, where

$$r(x) = x^k + a_1x^{k-1} + a_2x^{k-2} + \cdots + a_k.$$

Hence,

$$q(x) = x^k r(1/x).$$

Thus, if the roots of $r(x) = 0$ are $q_1, q_2, \ldots, q_k$, then

$$r(x) = (x - q_1)(x - q_2)\cdots(x - q_k) \quad \text{(with roots } q_1, q_2, \ldots, q_k)$$

and

$$q(x) = (1 - q_1x)(1 - q_2x)\cdots(1 - q_kx) \quad \text{(with roots } 1/q_1, 1/q_2, \ldots, 1/q_k).$$

Conversely, if we are given a polynomial

$$q(x) = b_0 + b_1x + \cdots + b_kx^k$$

of degree k with $b_0 \neq 0$ and a polynomial

$$p(x) = d_0 + d_1x + \cdots + d_{k-1}x^{k-1}$$

of degree less than k, then, using partial fractions and (7.35), we can find a power series[15] $h_0 + h_1x + \cdots + h_nx^n + \cdots$ such that

$$\frac{p(x)}{q(x)} = h_0 + h_1x + \cdots + h_nx^n + \cdots.$$

[15]This power series will converge to $p(x)/q(x)$ for all x with $|x| < t$, where t is the smallest absolute value of a root of $q(x) = 0$. Since we assume that $b_0 \neq 0$, 0 is not a root of $q(x) = 0$.

We can write the preceding equation in the form

$$\begin{aligned} d_0 + d_1 x + \cdots + d_{k-1}x^{k-1} &= (b_0 + b_1 x + \cdots + b_k x^k) \\ &\quad \times (h_0 + h_1 x + \cdots + h_n x^n + \cdots). \end{aligned}$$

Multiplying out the right side and comparing coefficients, we obtain

$$\begin{aligned} b_0 h_0 &= d_0, \\ b_0 h_1 + b_1 h_0 &= d_1, \\ &\vdots \\ b_0 h_{k-1} + b_1 h_{k-2} + \cdots + b_{k-1} h_0 &= d_{k-1}, \end{aligned} \tag{7.40}$$

and

$$b_0 h_n + b_1 h_{n-1} + \cdots + b_k h_{n-k} = 0, \qquad (n \geq k). \tag{7.41}$$

Since $b_0 \neq 0$, equation (7.41) can be written in the form

$$h_n + \frac{b_1}{b_0} h_{n-1} + \cdots + \frac{b_k}{b_0} h_{n-k} = 0, \qquad (n \geq k).$$

This is a linear homogeneous recurrence relation with constant coefficients that is satisfied by $h_0, h_1, h_2, \ldots, h_n, \ldots$. The initial values $h_0, h_1, \ldots, h_{k-1}$ can be determined by solving the triangular system of equations (7.40), using the fact that $b_0 \neq 0$. We summarize in the next theorem.

Theorem 7.4.3 *Let*

$$h_0, h_1, h_2, \ldots, h_n, \ldots$$

be a sequence of numbers that satisfies the linear homogeneous recurrence relation

$$h_n + c_1 h_{n-1} + \cdots + c_k h_{n-k} = 0, \quad c_k \neq 0, \quad (n \geq k) \tag{7.42}$$

of order k with constant coefficients. Then its generating function $g(x)$ is of the form

$$g(x) = \frac{p(x)}{q(x)}, \tag{7.43}$$

where $q(x)$ is a polynomial of degree k with a nonzero constant term and $p(x)$ is a polynomial of degree less than k. Conversely, given such polynomials $p(x)$ and $q(x)$, there is a sequence $h_0, h_1, h_2, \ldots, h_n, \ldots$ satisfying a linear homogeneous recurrence relation with constant coefficients of order k of the type (7.42) whose generating function is given by (7.43). □

7.5 Nonhomogeneous Recurrence Relations

Recurrence relations that are not homogeneous are, in general, more difficult to solve and can require special techniques depending on the nonhomogeneous part of the relation (the term b_n in (7.22)). In this section we consider several examples of linear nonhomogeneous recurrence relations with constant coefficients.

Our first example is a famous puzzle.

Example. *Towers of Hanoi puzzle.* There are three pegs and n circular disks of increasing size on one peg, with the largest disk on the bottom. These disks are to be transferred, one at a time, onto another of the pegs, with the provision that at no time is one allowed to place a larger disk on top of a smaller one. The problem is to determine the number of moves necessary for the transfer.

Let h_n be the number of moves required to transfer n disks. We verify that $h_0 = 0$, $h_1 = 1$ and $h_2 = 3$. Can we find a recurrence relation that is satisfied by h_n? To transfer n disks to another peg, we must first transfer the top $n-1$ disks to a peg, transfer the largest disk to the vacant peg, and then transfer the $n-1$ disks to the peg which now contains the largest disk. Thus, h_n satisfies

$$\begin{aligned} h_n &= 2h_{n-1}+1, \qquad (n \geq 1) \\ h_0 &= 0. \end{aligned} \tag{7.44}$$

This is a linear recurrence relation of order 1 with constant coefficients, but it is not homogeneous because of the presence of the quantity 1. To find h_n, we iterate (7.44):

$$\begin{aligned} h_n &= 2h_{n-1}+1 \\ &= 2(2h_{n-2}+1)+1 = 2^2h_{n-2}+2+1 \\ &= 2^2(2h_{n-3}+1)+2+1 = 2^3h_{n-3}+2^2+2+1 \\ &\vdots \\ &= 2^{n-1}(h_0+1)+2^{n-2}+\cdots+2^2+2+1 \\ &= 2^{n-1}+\cdots+2^2+2+1. \end{aligned}$$

Therefore, the numbers h_n are the partial sums of the geometric sequence

$$1, 2, 2^2, \ldots, 2^n, \ldots$$

and hence satisfy

$$h_n = \frac{2^n-1}{2-1} = 2^n - 1, \qquad (n \geq 0). \tag{7.45}$$

Now that we have a formula for h_n, it can easily be verified by mathematical induction making use of the recurrence relation (7.44). Here is how such a verification goes. Since

$h_0 = 0$, (7.45) holds for $n = 0$. Assume that (7.45) holds for n. We then show that it holds with n replaced by $n+1$; that is,

$$h_{n+1} = 2h_n + 1 = 2(2^n - 1) + 1 = 2^{n+1} - 1,$$

proving the formula (7.45).

With only two pegs and $n > 1$ disks, it is impossible to transfer the disks on one peg to the other, subject to the rule that a smaller disk is never below a larger disk. As we have just seen, with three pegs the minimum number of moves is $2^n - 1$. In the case of $k \geq 4$ pegs, it is an unsolved problem to determine the minimum number of moves needed to transfer n disks of different sizes on one peg onto a different peg, again subject to the rule that a smaller disk is never below a larger disk. The case $k = 4$ is sometimes called the *Brahma* or *Reve's puzzle*, and the puzzle is unsolved even in this case.[16] □

Our success in the solution of preceding example was made possible by the fact that, after we iterated the recurrence relation, we obtained a sum (in this case $2^{n-1} + \cdots + 2^2 + 2 + 1$) that we could evaluate. A similar situation occurred in Section 1.6 in our determination of the number of regions created by n mutually overlapping circles in general position. However, these are very special situations, and iteration of a recurrence relation does not usually lead to a simple formula.

The method of generating functions can also be used as a technique for solving nonhomogeneous recurrence relations.

Example. *Towers of Hanoi puzzle revisited.* Recall that h_n is the number of moves required to transfer n disks from one peg to a different peg and

$$h_n = 2h_{n-1} + 1, \ (n \geq 1), \quad h_0 = 0. \tag{7.46}$$

Let

$$g(x) = \sum_{n=0}^{\infty} h_n x^n$$

be the generating function of the sequence $h_0, h_1, \ldots, h_n, \ldots$. We then have

$$\begin{aligned} g(x) &= h_0 + h_1 x + h_2 x^2 + \cdots + h_n x^n + \cdots, \\ -2xg(x) &= \quad 2h_0 x + 2h_1 x^2 + \cdots + 2h_{n-1} x^n + \cdots. \end{aligned}$$

Subtracting these two equations and using (7.46), we see that

$$(1 - 2x)g(x) = x + x^2 + \ldots + x^n + \cdots = \frac{x}{1-x}.$$

[16] There is an algorithm—the Frame-Stewart algorithm—to transfer the n disks whose number of moves is conjectured to be minimal in this case. More information can be found in "Variations on the Four-Post Tower of Hanoi Puzzle" by P. K. Stockmeyer, *Congressus Numerantium*, 102 (1994), 3–12.

Hence
$$g(x) = \frac{x}{(1-x)(1-2x)}.$$
Using the method of partial fractions, we obtain
$$\begin{aligned} g(x) &= \frac{1}{1-2x} - \frac{1}{1-x} \\ &= \sum_{n=0}^{\infty}(2x)^n - \sum_{n=0}^{\infty} x^n \\ &= \sum_{n=0}^{\infty}(2^n-1)x^n. \end{aligned}$$
Hence we get $h_n = 2^n - 1$ as before. □

We now illustrate a technique for solving linear recurrence relations of order 1 with constant coefficients—that is, recurrence relations of the form
$$h_n = ah_{n-1} + b_n, \qquad (n \geq 1). \tag{7.47}$$

First we note that in the case $a = 1$, the recurrence relation (7.47) becomes
$$h_n = h_{n-1} + b_n, \qquad (n \geq 1), \tag{7.48}$$
and iteration yields
$$h_n = h_0 + (b_1 + b_2 + \cdots + b_n).$$
Thus, solving (7.48) is the same as summing the series
$$b_1 + b_2 + \cdots + b_n.$$

Thus we implicity assume that $a \neq 1$.

Example. Solve
$$\begin{aligned} &h_n = 3h_{n-1} - 4n, \qquad (n \geq 1) \\ &h_0 = 2. \end{aligned}$$

We first consider the corresponding homogeneous recurrence relation
$$h_n = 3h_{n-1}, \qquad (n \geq 1).$$
Its characteristic equation is
$$x - 3 = 0,$$
and hence it has one characteristic root $q = 3$, giving the general solution
$$h_n = c3^n, \qquad (n \geq 1). \tag{7.49}$$

We now seek a particular solution of the nonhomogeneous recurrence relation

$$h_n = 3h_{n-1} - 4n, \qquad (n \geq 1). \tag{7.50}$$

We try to find a solution of the form

$$h_n = rn + s \tag{7.51}$$

for appropriate numbers r and s. In order for (7.51) to satisfy (7.50), we must have

$$rn + s = 3(r(n-1) + s) - 4n$$

or, equivalently,

$$rn + s = (3r - 4)n + (-3r + 3s).$$

Equating the coefficients of n and the constant terms on both sides of this equation, we obtain

$$\begin{array}{lll} r = 3r - 4 & \text{or, equivalently,} & 2r = 4 \\ s = -3r + 3s & \text{or, equivalently,} & 2s = 3r. \end{array}$$

Hence, $r = 2$ and $s = 3$, and

$$h_n = 2n + 3 \tag{7.52}$$

satisfies (7.50). We now combine the general solution (7.49) of the homogeneous relation with the particular solution (7.52) of the nonhomogeneous relation to obtain

$$h_n = c3^n + 2n + 3. \tag{7.53}$$

In (7.53) we have, for each choice of the constant c, a solution of (7.50). Now we try to choose c so that the initial condition $h_0 = 2$ is satisfied:

$$(n = 0) \qquad 2 = c \times 3^0 + 2 \times 0 + 3.$$

This gives $c = -1$, and hence

$$h_n = -3^n + 2n + 3 \qquad (n \geq 0)$$

is the solution of the original problem. □

The preceding technique is the discrete analogue of a technique used to solve nonhomogeneous differential equations. It can be summarized as follows:

(1) Find the general solution of the homogeneous relation.

(2) Find a particular solution of the nonhomogeneous relation.

(3) Combine the general solution and the particular solution, and determine values of the constants arising in the general solution so that the combined solution satisfies the initial conditions.

The main difficulty (besides the difficulty in finding the roots of the characteristic equation) is finding a particular solution in step (2). For some nonhomogeneous parts b_n in (7.47), there are certain types of particular solutions to try.[17] We mention only two:

(a) If b_n is a polynomial of degree k in n, then look for a particular solution h_n that is also a polynomial of degree k in n. Thus, try

$$\begin{array}{lll} \text{(i)} & h_n = r \ \text{(a constant)} & \text{if } \ b_n = d \ \text{(a constant)}, \\ \text{(ii)} & h_n = rn + s & \text{if } \ b_n = dn + e, \\ \text{(iii)} & h_n = rn^2 + sn + t & \text{if } \ b_n = dn^2 + en + f. \end{array}$$

(b) If b_n is an exponential, then look for a particular solution that is also an exponential. Thus, try

$$h_n = pd^n \quad \text{if} \quad b_n = d^n.$$

The preceding example was of the type (a)(ii). By using generating functions, the problem of finding a particular solution can sometimes be avoided, as shown in the next example.

Example. Solve

$$\begin{aligned} h_n &= 2h_{n-1} + 3^n, \qquad (n \geq 1) \\ h_0 &= 2. \end{aligned}$$

First Solution: Since the homogeneous relation $h_n = 2h_{n-1} \ \ (n \geq 1)$ has only one characteristic root $q = 2$, its general solution is

$$h_n = c2^n, \qquad (n \geq 1).$$

For a particular solution of $h_n = 2h_{n-1} + 3^n \ \ (n \geq 1)$, we try

$$h_n = p3^n.$$

To be a solution, p must satisfy the equation

$$p3^n = 2p3^{n-1} + 3^n,$$

which, after cancellation, reduces to

$$3p = 2p + 3 \ \text{ or, equivalently, } \ p = 3.$$

Hence

$$h_n = c2^n + 3^{n+1}$$

[17] These are solutions to *try*. Whether or not they work depends on the characteristic polynomial.

is a solution for each choice of the constant c. We now want to determine c so that the initial condition $h_0 = 2$ is satisfied:

$$(n = 0) \qquad c2^0 + 3 = 2.$$

This gives $c = -1$, and the solution of the problem is

$$h_n = -2^n + 3^{n+1}, \qquad (n \geq 0).$$

Second Solution: Here we use generating functions. Let

$$g(x) = h_0 + h_1x + h_2x^2 + \cdots + h_nx^n + \cdots.$$

Using the recurrence and $h_0 = 2$, we see that

$$\begin{aligned} g(x) - 2xg(x) &= h_0 + (h_1 - 2h_0)x + (h_2 - 2h_1)x^2 + \cdots + (h_n - 2h_{n-1})x^n + \cdots \\ &= 2 + 3x + 3^2x^2 + \cdots + 3^nx^n + \cdots \\ &= 2 - 1 + (1 + 3x + 3^2x^2 + \cdots + 3^nx^n + \cdots) \\ &= 1 + \frac{1}{1-3x}. \end{aligned}$$

Hence

$$g(x) = \frac{1}{1-2x} + \frac{1}{(1-3x)(1-2x)}.$$

Using the method of partial fractions and the special case of (7.35) with $r = 3$ and $n = 1$, we get

$$\begin{aligned} g(x) &= \frac{1}{1-2x} + \frac{3}{1-3x} - \frac{2}{1-2x} \\ &= \sum_{n=0}^{\infty} 2^nx^n + \sum_{n=0}^{\infty} 3^{n+1}x^n - \sum_{n=0}^{\infty} 2^{n+1}x^n \\ &= \sum_{n=0}^{\infty} (2^n + 3^{n+1} - 2^{n+1})x^n \\ &= \sum_{n=0}^{\infty} (3^{n+1} - 2^n)x^n, \end{aligned}$$

and this agrees with our first solution. □

Example. Solve

$$\begin{aligned} &h_n = h_{n-1} + n^3, \qquad (n \geq 1) \\ &h_0 = 0. \end{aligned}$$

We have, after iteration,

$$h_n = 0^3 + 1^3 + 2^3 + \cdots + n^3,$$

the sum of the cubes of the first n positive integers.[18] We calculate that

$$\begin{array}{lllllllll}
h_0 & = & 0^3 & = & 0 & = & 0^2 & = & 0^2 \\
h_1 & = & 0+1^3 & = & 1 & = & 1^2 & = & (0+1)^2 \\
h_2 & = & 1+2^3 & = & 9 & = & 3^2 & = & (0+1+2)^2 \\
h_3 & = & 9+3^3 & = & 36 & = & 6^2 & = & (0+1+2+3)^2 \\
h_4 & = & 36+4^3 & = & 100 & = & 10^2 & = & (0+1+2+3+4)^2.
\end{array}$$

A reasonable conjecture is that

$$\begin{aligned}
h_n &= (0+1+2+3+\cdots+n)^2 = \left(\frac{n(n+1)}{2}\right)^2 \\
&= \frac{n^2(n+1)^2}{4}.
\end{aligned}$$

This formula can now be verified by induction on n as follows: Assuming that it holds for an integer n, we show that it also holds for $n+1$:

$$\begin{aligned}
h_{n+1} &= h_n + (n+1)^3 \\
&= \frac{n^2(n+1)^2}{4} + (n+1)^3 \\
&= \frac{(n+1)^2(n^2+4(n+1))}{4} \\
&= \frac{(n+1)^2(n+2)^2}{4}.
\end{aligned}$$

The latter is the formula with n replaced by $n+1$. Therefore, by mathematical induction,

$$h_n = \frac{n^2(n+1)^2}{4}, \qquad (n \geq 0). \qquad \square$$

Example. Solve

$$\begin{aligned}
&h_n = 3h_{n-1} + 3^n, \qquad (n \geq 1) \\
&h_0 = 2.
\end{aligned}$$

First Solution: The general solution of the corresponding homogeneous relation is

$$h_n = c3^n.$$

[18]In the next chapter we shall see how to sum the kth powers of the first n positive integers for each positive integer k.

We first try

$$h_n = p3^n$$

as a particular solution. Substituting, we get

$$p3^n = 3p3^{n-1} + 3^n,$$

which, after cancellation, gives

$$p = p + 1,$$

an impossibility. So instead we try, as a particular solution,

$$h_n = pn3^n.$$

Substituting, we now get

$$pn3^n = 3p(n-1)3^{n-1} + 3^n,$$

which, after cancellation, gives $p = 1$. Thus, $h_n = n3^n$ is a particular solution, and

$$h_n = c3^n + n3^n$$

is a solution for each choice of the constant c. To satisfy the initial condition $h_0 = 2$, we must choose c so that

$$(n = 0) \qquad c(3^0) + 0(3^0) = 2,$$

and this gives $c = 2$. Therefore,

$$h_n = 2 \times 3^n + n3^n = (2+n)3^n$$

is the solution.

Second Solution: Here we use generating functions. Let

$$g(x) = h_0 + h_1 x + h_2 x^2 + \cdots + h_n x^n + \cdots.$$

Using the given recurrence and $h_0 = 2$, we get that

$$\begin{aligned} g(x) - 3xg(x) &= h_0 + (h_1 - 3h_0)x + (h_2 - 3h_1)x^2 + \cdots + (h_n - 3h_{n-1})x^n + \cdots \\ &= 2 + 3x + 3^2x^2 + \cdots + 3^n x^n + \cdots \\ &= 2 - 1 + (1 + 3x + 3^2x^2 + \cdots + 3^n x^n + \cdots) \\ &= 1 + \frac{1}{1-3x}. \end{aligned}$$

Hence

$$g(x) = \frac{1}{1-3x} + \frac{1}{(1-3x)^2}.$$

Applying the special case of (7.35) with $r = 3$, and $n = 1$ and 2, we get

$$\begin{aligned} g(x) &= \sum_{n=0}^{\infty} 3^n x^n + \sum_{n=0}^{\infty} (n+1)3^n x^n \\ &= \sum_{n=0}^{\infty} (n+2)3^n, \end{aligned}$$

and this agrees with our first solution. □

7.6 A Geometry Example

A set K of points in the plane or in space is said to be *convex*, provided that for any two points p and q in K, all of the points on the line segment joining p and q are in K. Triangular regions, circular regions, and rectangular regions in the plane are all convex sets of points. On the other hand, the region on the left in Figure 7.1 is not convex since, for the two points p and q shown, the line segment joining p and q goes outside the region.

The regions in Figure 7.1 are examples of *polygonal regions*—that is, regions whose boundaries consist of a finite number of line segments, called their *sides*. Triangular regions and rectangular regions are polygonal, but circular regions are not. Any polygonal region must have at least three sides. The region on the right in Figure 7.1 is a convex polygonal region with six sides.

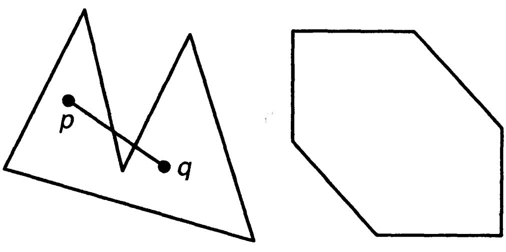

Figure 7.1

In a polygonal region, the points at which the sides meet are called *corners* (or *vertices*). A *diagonal* is a line segment joining two nonconsecutive corners.

Let K be a polygonal region with n sides. We can count the number of its diagonals as follows: Each corner is joined by a diagonal to $n - 3$ other corners. Thus, counting the number of diagonals at each corner and summing, we get $n(n - 3)$. Since each diagonal has two corners, each diagonal is counted twice in this sum. Hence, the number of diagonals is $n(n-3)/2$. We can arrive at this same number indirectly in the following way: There are

$$\binom{n}{2} = \frac{n(n-1)}{2}$$

line segments joining the n corners. Of these, n are sides of the polygonal region. The remaining ones are diagonals. Consequently, there are

$$\frac{n(n-1)}{2} - n = \frac{n(n-3)}{2}$$

diagonals.

Now assume that K is convex. Then each diagonal of K lies wholly within K. Thus, each diagonal of K divides K into one convex polygonal region with k sides and another with $n-k+2$ sides for some $k = 3, 4, \ldots, n-1$.

We can draw $n-3$ diagonals meeting a particular corner of K, and in doing so divide K into $n-2$ triangular regions. But, there are other ways of dividing the region into triangular regions by inserting $n-3$ diagonals no two of which intersect in the interior of K, as the example in Figure 7.2 shows for $n = 8$.

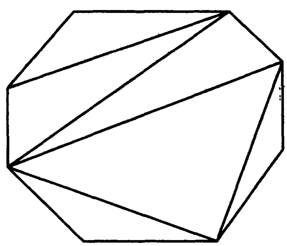

Figure 7.2

In the next theorem, we determine the number of different ways to divide a convex polygonal region into triangular regions by drawing diagonals that do not intersect in the interior. For notational convenience, we deal with a convex polygonal region of $n+1$ sides which is then divided into $n-1$ triangular regions by $n-2$ diagonals.

Theorem 7.6.1 *Let h_n denote the number of ways of dividing a convex polygonal region with $n+1$ sides into triangular regions by inserting diagonals that do not intersect in the interior. Define $h_1 = 1$. Then h_n satisfies the recurrence relation*

$$\begin{aligned} h_n &= h_1 h_{n-1} + h_2 h_{n-2} + \cdots + h_{n-1} h_1 \\ &= \sum_{k=1}^{n-1} h_k h_{n-k}, \quad (n \geq 2). \end{aligned} \tag{7.54}$$

The solution of this recurrence relation is

$$h_n = \frac{1}{n}\binom{2n-2}{n-1}, \qquad (n = 1, 2, 3, \ldots).$$

Proof. We have defined $h_1 = 1$, and we think of a line segment as a polygonal region with two sides and no interior. We have $h_2 = 1$, since a triangular region has no diagonals, and it cannot be further subdivided. The recurrence relation (7.54) holds for $n = 2$,[19] since

$$\sum_{k=1}^{2-1} h_k h_{2-k} = \sum_{k=1}^{1} h_k h_{2-k} = h_1 h_1 = 1.$$

Now let $n \geq 3$. Consider a convex polygonal region K with $n + 1 \geq 4$ sides. We distinguish one side of K and call it the *base*. In each division of K into triangular regions, the base is a side of one of the triangular regions T, and this triangular region divides the remainder of K into a polygonal region K_1 with $k+1$ sides and a polygonal region K_2 with $n - k + 1$ sides, for some $k = 1, 2, \ldots, n-1$. (See Figure 7.3.)

The further subdivision of K is accomplished by dividing K_1 and K_2 into triangular regions by inserting diagonals of K_1 and K_2, respectively, which do not intersect in the interior. Since K_1 has $k + 1$ sides, K_1 can be divided into triangular regions in h_k ways. Since K_2 has $n - k + 1$ sides, K_2 can be divided into triangular regions in h_{n-k} ways. Hence, for a particular choice of the triangular region T containing the base, there are $h_k h_{n-k}$ ways of dividing K into triangular regions by diagonals that do not intersect in the interior. Hence, there is a total of

$$h_n = \sum_{k=1}^{n-1} h_k h_{n-k}$$

ways to divide K into triangular regions in this way. This establishes the recurrence relation (7.54).

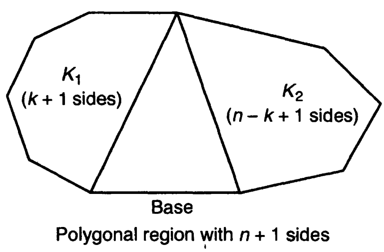

Figure 7.3

[19]This is why we defined $h_1 = 1$.

We now turn to the solution of (7.54) with the initial condition $h_1 = 1$. This recurrence relation is not linear. Moreover, h_n does not depend on a fixed number of values that come before it but on all the values $h_1, h_2, \ldots, h_{n-1}$ that come before it. Thus, none of our methods for solving recurrence relations apply. Let

$$g(x) = h_1x + h_2x^2 + \cdots + h_nx^n + \cdots$$

be the generating function for the sequence $h_1, h_2, h_3, \ldots, h_n, \ldots$. Multiplying $g(x)$ by itself, we find that

$$\begin{aligned}(g(x))^2 &= h_1^2x^2 + (h_1h_2 + h_2h_1)x^3 + (h_1h_3 + h_2h_2 + h_3h_1)x^4 \\ &\quad + \cdots + (h_1h_{n-1} + h_2h_{n-2} + \cdots + h_{n-1}h_1)x^n + \cdots.\end{aligned}$$

Using (7.54) and the fact that $h_1 = h_2 = 1$, we obtain

$$\begin{aligned}(g(x))^2 &= h_1^2x^2 + h_3x^3 + h_4x^4 + \cdots + h_nx^n + \cdots \\ &= h_2x^2 + h_3x^3 + h_4x^4 + \cdots + h_nx^n + \cdots \\ &= g(x) - h_1x = g(x) - x.\end{aligned}$$

Thus, $g(x)$ satisfies the equation

$$(g(x))^2 - g(x) + x = 0.$$

This is a quadratic equation for $g(x)$, so, by the quadratic formula,[20] $g(x) = g_1(x)$ or $g(x) = g_2(x)$, where

$$g_1(x) = \frac{1 + \sqrt{1-4x}}{2} \text{ and } g_2(x) = \frac{1 - \sqrt{1-4x}}{2}.$$

From the definition of $g(x)$, it follows that $g(0) = 0$. Since $g_1(0) = 1$ and $g_2(0) = 0$, we conclude that

$$g(x) = g_2(x) = \frac{1 - \sqrt{1-4x}}{2} = \frac{1}{2} - \frac{1}{2}(1-4x)^{1/2}.$$

By Newton's binomial theorem (see, in particular, the calculation done at the end of Section 5.6),

$$(1+z)^{1/2} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n \times 2^{2n-1}} \binom{2n-2}{n-1} z^n, \qquad (|z| < 1).$$

[20] We have omitted some subtleties.

If we replace z by $-4x$, we get

$$\begin{aligned}
(1-4x)^{1/2} &= 1+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n\times 2^{2n-1}}\binom{2n-2}{n-1}(-1)^n 4^n x^n \\
&= 1+\sum_{n=1}^{\infty}(-1)^{2n-1}\frac{2}{n}\binom{2n-2}{n-1}x^n \\
&= 1-2\sum_{n=1}^{\infty}\frac{1}{n}\binom{2n-2}{n-1}x^n, \quad (|x|<\frac{1}{4}).
\end{aligned}$$

Thus,

$$g(x)=\frac{1}{2}-\frac{1}{2}(1-4x)^{1/2}=\sum_{n=1}^{\infty}\frac{1}{n}\binom{2n-2}{n-1}x^n,$$

and hence,

$$h_n=\frac{1}{n}\binom{2n-2}{n-1}, \qquad (n\geq 1).$$

□

The numbers

$$\frac{1}{n}\binom{2n-2}{n-1}$$

in the previous theorem are the Catalan numbers, and these will be investigated more thoroughly in Chapter 8.

7.7 Exercises

1. Let $f_0, f_1, f_2, \ldots, f_n, \ldots$ denote the Fibonacci sequence. By evaluating each of the following expressions for small values of n, conjecture a general formula and then prove it, using mathematical induction and the Fibonacci recurrence:

   (a) $f_1+f_3+\cdots+f_{2n-1}$

   (b) $f_0+f_2+\cdots+f_{2n}$

   (c) $f_0-f_1+f_2-\cdots+(-1)^n f_n$

   (d) $f_0^2+f_1^2+\cdots+f_n^2$

2. Prove that the nth Fibonacci number f_n is the integer that is closest to the number

$$\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n.$$

3. Prove the following about the Fibonacci numbers:

(a) f_n is even if and only if n is divisible by 3.

(b) f_n is divisible by 3 if and only if n is divisible by 4.

(c) f_n is divisible by 4 if and only if n is divisible by 6.

4. Prove that the Fibonacci sequence is the solution of the recurrence relation

$$a_n = 5a_{n-4} + 3a_{n-5}, \quad (n \geq 5),$$

where $a_0 = 0, a_1 = 1, a_2 = 1, a_3 = 2$, and $a_4 = 3$. Then use this formula to show that the Fibonacci numbers satisfy the condition that f_n is divisible by 5 if and only if n is divisible by 5.

5. By examining the Fibonacci sequence, make a conjecture about when f_n is divisible by 7 and then prove your conjecture.

6. * Let m and n be positive integers. Prove that if m is divisible by n, then f_m is divisible by f_n.

7. * Let m and n be positive integers whose greatest common divisor is d. Prove that the greatest common divisor of the Fibonacci numbers f_m and f_n is the Fibonacci number f_d.

8. Consider a 1-by-n chessboard. Suppose we color each square of the chessboard with one of the two colors red and blue. Let h_n be the number of colorings in which no two squares that are colored red are adjacent. Find and verify a recurrence relation that h_n satisfies. Then derive a formula for h_n.

9. Let h_n equal the number of different ways in which the squares of a 1-by-n chessboard can be colored, using the colors red, white, and blue so that no two squares that are colored red are adjacent. Find and verify a recurrence relation that h_n satisfies. Then find a formula for h_n.

10. Suppose that, in his problem, Fibonacci had placed two pairs of rabbits in the enclosure at the beginning of a year. Find the number of pairs of rabbits in the enclosure after one year. More generally, find the number of pairs of rabbits in the enclosure after n months.

11. The *Lucas numbers* $l_0, l_1, l_2, \ldots, l_n \ldots$ are defined using the same recurrence relation defining the Fibonacci numbers, but with different initial conditions:

$$l_n = l_{n-1} + l_{n-2}, \ (n \geq 2), l_0 = 2, l_1 = 1.$$

Prove that

(a) $l_n = f_{n-1} + f_{n+1}$ for $n \geq 1$

(b) $l_0^2 + l_1^2 + \cdots + l_n^2 = l_n l_{n+1} + 2$ for $n \geq 0$

12. Let $h_0, h_1, h_2, \ldots, h_n, \ldots$ be the sequence defined by

$$h_n = n^3, \ (n \geq 0).$$

Show that $h_n = h_{n-1} + 3n^2 - 3n + 1$ is the recurrence relation for the sequence.

13. Determine the generating function for each of the following sequences:

(a) $c^0 = 1, c, c^2, \ldots, c^n, \ldots$

(b) $1, -1, 1, -1, \ldots, (-1)^n, \ldots$

(c) $\binom{\alpha}{0}, -\binom{\alpha}{1}, \binom{\alpha}{2}, \ldots, (-1)^n\binom{\alpha}{n}, \ldots,$ (α is a real number)

(d) $1, \frac{1}{1!}, \frac{1}{2!}, \ldots, \frac{1}{n!}, \ldots$

(e) $1, -\frac{1}{1!}, \frac{1}{2!}, \ldots, (-1)^n\frac{1}{n!}, \ldots$

14. Let S be the multiset $\{\infty \cdot e_1, \infty \cdot e_2, \infty \cdot e_3, \infty \cdot e_4\}$. Determine the generating function for the sequence $h_0, h_1, h_2, \ldots, h_n, \ldots$, where h_n is the number of n-combinations of S with the following added restrictions:

(a) Each e_i occurs an odd number of times.

(b) Each e_i occurs a multiple-of-3 number of times.

(c) The element e_1 does not occur, and e_2 occurs at most once.

(d) The element e_1 occurs $1, 3$, or 11 times, and the element e_2 occurs $2, 4$, or 5 times.

(e) Each e_i occurs at least 10 times.

15. Determine the generating function for the sequence of cubes

$$0, 1, 8, \ldots, n^3, \ldots.$$

16. Formulate a combinatorial problem for which the generating function is

$$(1 + x + x^2)(1 + x^2 + x^4 + x^6)(1 + x^2 + x^4 + \cdots)(x + x^2 + x^3 + \cdots).$$

17. Determine the generating function for the number h_n of bags of fruit of apples, oranges, bananas, and pears in which there are an even number of apples, at most two oranges, a multiple of three number of bananas, and at most one pear. Then find a formula for h_n from the generating function.

18. Determine the generating function for the number h_n of nonnegative integral solutions of

$$2e_1 + 5e_2 + e_3 + 7e_4 = n.$$

19. Let $h_0, h_1, h_2, \ldots, h_n, \ldots$ be the sequence defined by $h_n = \binom{n}{2}$, $(n \geq 0)$. Determine the generating function for the sequence.

20. Let $h_0, h_1, h_2, \ldots, h_n, \ldots$ be the sequence defined by $h_n = \binom{n}{3}$, $(n \geq 0)$. Determine the generating function for the sequence.

21. * Let h_n denote the number of regions into which a convex polygonal region with $n+2$ sides is divided by its diagonals, assuming no three diagonals have a common point. Define $h_0 = 0$. Show that

$$h_n = h_{n-1} + \binom{n+1}{3} + n, \quad (n \geq 1).$$

Then determine the generating function and obtain a formula for h_n.

22. Determine the exponential generating function for the sequence of factorials: $0!, 1!, 2!, 3!, \ldots, n!, \ldots$.

23. Let α be a real number. Let the sequence $h_0, h_1, h_2, \ldots, h_n, \ldots$ be defined by $h_0 = 1$, and $h_n = \alpha(\alpha-1)\cdots(\alpha-n+1)$, $(n \geq 1)$. Determine the exponential generating function for the sequence.

24. Let S denote the multiset $\{\infty \cdot e_1, \infty \cdot e_2, \ldots, \infty \cdot e_k\}$. Determine the exponential generating function for the sequence $h_0, h_1, h_2, \ldots, h_n, \ldots$, where $h_0 = 1$ and, for $n \geq 1$,

    (a) h_n equals the number of n-permutations of S in which each object occurs an odd number of times.

    (b) h_n equals the number of n-permutations of S in which each object occurs at least four times.

    (c) h_n equals the number of n-permutations of S in which e_1 occurs at least once, e_2 occurs at least twice, ... , e_k occurs at least k times.

    (d) h_n equals the number of n-permutations of S in which e_1 occurs at most once, e_2 occurs at most twice, ... , e_k occurs at most k times.

25. Let h_n denote the number of ways to color the squares of a 1-by-n board with the colors red, white, blue, and green in such a way that the number of squares colored red is even and the number of squares colored white is odd. Determine the exponential generating function for the sequence $h_0, h_1, \ldots, h_n, \ldots$, and then find a simple formula for h_n.

26. Determine the number of ways to color the squares of a 1-by-n chessboard, using the colors red, blue, green, and orange if an even number of squares is to be colored red and an even number is to be colored green.

27. Determine the number of n-digit numbers with all digits odd, such that 1 and 3 each occur a nonzero, even number of times.

28. Determine the number of n-digit numbers with all digits at least 4, such that 4 and 6 each occur an even number of times, and 5 and 7 each occur at least once, there being no restriction on the digits 8 and 9.

29. We have used exponential generating functions to show that the number h_n of n-digit numbers with each digit odd, where the digits 1 and 3 occur an even number of times, satisfies the formula

$$h_n = \frac{5^n + 2 \times 3^n + 1}{4}, \quad (n \geq 0).$$

Obtain an alternative derivation of this formula.

30. We have used exponential generating functions to show that the number h_n of ways to color the squares of a 1-by-n board with the colors red, white , and blue, where the number of red squares is even and there is at least one blue square, satisfies the formula

$$h_n = \frac{3^n - 2^n + 1}{2}, \quad (n \geq 1)$$

with $h_0 = 0$. Obtain an alternative derivation of this formula by finding a recurrence relation satisfied by h_n and then solving the recurrence relation.

31. Solve the recurrence relation $h_n = 4h_{n-2}$, $(n \geq 2)$ with initial values $h_0 = 0$ and $h_1 = 1$.

32. Solve the recurrence relation $h_n = (n+2)h_{n-1}$, $(n \geq 1)$ with initial value $h_0 = 2$.

33. Solve the recurrence relation $h_n = h_{n-1} + 9h_{n-2} - 9h_{n-3}$, $(n \geq 3)$ with initial values $h_0 = 0$, $h_1 = 1$, and $h_2 = 2$.

34. Solve the recurrence relation $h_n = 8h_{n-1} - 16h_{n-2}$, $(n \geq 2)$ with initial values $h_0 = -1$ and $h_1 = 0$.

35. Solve the recurrence relation $h_n = 3h_{n-2} - 2h_{n-3}$, $(n \geq 3)$ with initial values $h_0 = 1$, $h_1 = 0$, and $h_2 = 0$.

36. Solve the recurrence relation $h_n = 5h_{n-1} - 6h_{n-2} - 4h_{n-3} + 8h_{n-4}$, $(n \geq 4)$ with initial values $h_0 = 0$, $h_1 = 1$, $h_2 = 1$, and $h_3 = 2$.

37. Determine a recurrence relation for the number a_n of ternary strings (made up of 0s, 1s, and 2s) of length n that do not contain two consecutive 0's or two consecutive 1s. Then find a formula for a_n.

38. Solve the following recurrence relations by examining the first few values for a formula and then proving your conjectured formula by induction.

    (a) $h_n = 3h_{n-1}$, $(n \geq 1)$; $h_0 = 1$
    (b) $h_n = h_{n-1} - n + 3$, $(n \geq 1)$; $h_0 = 2$
    (c) $h_n = -h_{n-1} + 1$, $(n \geq 1)$; $h_0 = 0$
    (d) $h_n = -h_{n-1} + 2$, $(n \geq 1)$; $h_0 = 1$
    (e) $h_n = 2h_{n-1} + 1$, $(n \geq 1)$; $h_0 = 1$

39. Let h_n denote the number of ways to perfectly cover a 1-by-n board with monominoes and dominoes in such a way that no two dominoes are consecutive. Find, but do not solve, a recurrence relation and initial conditions satisfied by h_n.

40. Let a_n equal the number of ternary strings of length n made up of 0s, 1s, and 2s, such that the substrings 00, 01, 10, and 11 never occur. Prove that

$$a_n = a_{n-1} + 2a_{n-2}, \quad (n \geq 2),$$

with $a_0 = 1$ and a_1=3. Then find a formula for a_n.

41. * Let $2n$ equally spaced points be chosen on a circle. Let h_n denote the number of ways to join these points in pairs so that the resulting line segments do not intersect. Establish a recurrence relation for h_n.

42. Solve the nonhomogeneous recurrence relation

$$\begin{aligned} h_n &= 4h_{n-1} + 4^n, \quad (n \geq 1) \\ h_0 &= 3. \end{aligned}$$

43. Solve the nonhomogeneous recurrence relation

$$\begin{aligned} h_n &= 4h_{n-1} + 3 \times 2^n, \quad (n \geq 1) \\ h_0 &= 1. \end{aligned}$$

44. Solve the nonhomogeneous recurrence relation

$$\begin{aligned} h_n &= 3h_{n-1} - 2, \quad (n \geq 1) \\ h_0 &= 1. \end{aligned}$$

45. Solve the nonhomogeneous recurrence relation

$$\begin{aligned} h_n &= 2h_{n-1} + n, \quad (n \geq 1) \\ h_0 &= 1. \end{aligned}$$

46. Solve the nonhomogeneous recurrence relation

$$\begin{aligned} h_n &= 6h_{n-1} - 9h_{n-2} + 2n, \quad (n \geq 2) \\ h_0 &= 1 \\ h_1 &= 0. \end{aligned}$$

47. Solve the nonhomogeneous recurrence relation

$$\begin{aligned} h_n &= 4h_{n-1} - 4h_{n-2} + 3n + 1, \quad (n \geq 2) \\ h_0 &= 1 \\ h_1 &= 2. \end{aligned}$$

48. Solve the following recurrence relations by using the method of generating functions as described in Section 7.4:

    (a) $h_n = 4h_{n-2}$, $(n \geq 2)$; $h_0 = 0, h_1 = 1$
    (b) $h_n = h_{n-1} + h_{n-2}$, $(n \geq 2)$; $h_0 = 1, h_1 = 3$
    (c) $h_n = h_{n-1} + 9h_{n-2} - 9h_{n-3}$, $(n \geq 3)$; $h_0 = 0, h_1 = 1, h_2 = 2$
    (d) $h_n = 8h_{n-1} - 16h_{n-2}$, $(n \geq 2)$; $h_0 = -1, h_1 = 0$
    (e) $h_n = 3h_{n-2} - 2h_{n-3}$, $(n \geq 3)$; $h_0 = 1, h_1 = 0, h_2 = 0$
    (f) $h_n = 5h_{n-1} - 6h_{n-2} - 4h_{n-3} + 8h_{n-4}$, $(n \geq 4)$; $h_0 = 0, h_1 = 1, h_2 = 1, h_3 = 2$

49. (*q-binomial theorem*) Prove that

$$(x+y)(x+qy)(x+q^2y)\cdots(x+q^{n-1}y) = \sum_{k=0}^{n} \binom{n}{k}_q x^{n-k}y^k,$$

where

$$n!_q = \frac{\prod_{j=1}^{n}(1-q^j)}{(1-q)^n}$$

is the q-factorial (cf. Theorem 7.2.1 replacing q in (7.14) with x) and

$$\binom{n}{k}_q = \frac{n!_q}{k!_q(n-k)!_q}$$

is the *q-binomial coefficient.*

50. Call a subset S of the integers $\{1, 2, \ldots, n\}$ *extraordinary* provided its smallest integer equals its size:

$$\min\{x : x \in S\} = |S|.$$

For example, $S = \{3, 7, 8\}$ is extraordinary. Let g_n be the number of extraordinary subsets of $\{1, 2, \ldots, n\}$. Prove that

$$g_n = g_{n-1} + g_{n-2}, \quad (n \geq 3),$$

with $g_1 = 1$ and $g_2 = 1$.

51. Solve the recurrence relation

$$\begin{aligned} h_n &= 3h_{n-1} - 4n, \quad (n \geq 1) \\ h_0 &= 2 \end{aligned}$$

from Section 7.6 using generating functions.

52. Solve the following two recurrence relations:

   (a) $h_n = 2h_{n-1} + 5^n$, $(n \geq 1)$ with $h_0 = 3$

   (b) $h_n = 5h_{n-1} + 5^n$, $(n \geq 1)$ with $h_0 = 3$

53. Suppose you deposit \$500 in a bank account that pays 6% interest at the end of each year (compounded annually). Thereafter, at the beginning of each year you deposit \$100. Let h_n be the amount in your account after n years (so $h_0 = \$500$). Determine the generating function $g(x) = h_0 + h_1x + \cdots + h_nx^n + \cdots$ and then a formula for h_n.