10

Tableaux and Symmetric Polynomials

In this chapter, we study combinatorial objects called *tableaux*. Informally, a tableau is a filling of the cells in the diagram of an integer partition with labels that may be subject to certain ordering conditions. We use tableaux to give a combinatorial definition of *Schur polynomials*, which are examples of *symmetric polynomials*. The theory of symmetric polynomials nicely demonstrates the interplay between combinatorics and algebra. We give a brief introduction to this vast subject in this chapter, stressing bijective proofs throughout.

10.1 Partition Diagrams and Skew Shapes

The reader may find it helpful at this point to review the basic definitions concerning integer partitions (see §2.8). Table 10.1 summarizes the notation used in this chapter to discuss integer partitions. In combinatorial arguments, we usually visualize the diagram $\mathrm{dg}(\mu)$ as a collection of unit boxes, where $(i,j) \in \mathrm{dg}(\mu)$ corresponds to the box in row i and column j. The conjugate partition μ' is the partition whose diagram is obtained from $\mathrm{dg}(\mu)$ by interchanging the roles of rows and columns.

Before defining tableaux, we need the notion of a *skew shape*.

10.1. Definition: Skew Shapes. Let μ and ν be two integer partitions such that $\mathrm{dg}(\nu) \subseteq \mathrm{dg}(\mu)$, or equivalently, $\nu_i \leq \mu_i$ for all $i \geq 1$. In this situation, we define the *skew shape*

$$\mu/\nu = \mathrm{dg}(\mu) \sim \mathrm{dg}(\nu) = \{(i,j) : 1 \leq i \leq \ell(\mu), \nu_i < j \leq \mu_i\}.$$

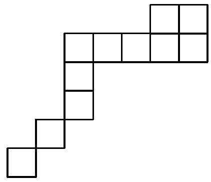
We can visualize μ/ν as the collection of unit squares obtained by starting with the diagram of μ and erasing the squares in the diagram of ν. If $\nu = 0 = (0,0,\ldots)$ is the zero

TABLE 10.1
Notation related to integer partitions.

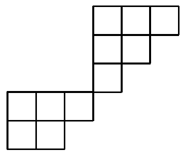
Notation	Definition
$\mathrm{Par}(k)$	set of integer partitions of k
$p(k)$	number of integer partitions of k
$\mathrm{Par}_N(k)$	set of integer partitions of k with at most N parts
$\mu \vdash k$	μ is an integer partition of k
μ_i	the ith largest part of the partition μ
$\ell(\mu)$	the number of nonzero parts of the partition μ
$\mathrm{dg}(\mu)$	the diagram of μ, i.e., $\{(i,j) \in \mathbb{N} \times \mathbb{N} : 1 \leq i \leq \ell(\mu), 1 \leq j \leq \mu_i\}$
μ'	conjugate partition to μ
$(1^{a_1}2^{a_2}\cdots k^{a_k}\cdots)$	the partition with a_k parts equal to k for $k \geq 1$

partition, then $\mu/0 = \text{dg}(\mu)$. A skew shape of the form $\mu/0$ is sometimes called a *straight shape*.

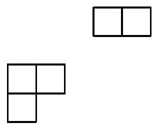
10.2. Example. If $\mu = (7,7,3,3,2,1)$ and $\nu = (5,2,2,2,1)$, then

$\mu/\nu =$ 

Similarly,

$(6,5,4,3,2)/(3,3,3) =$ 

Skew shapes need not be connected; for instance,

$(5,2,2,1)/(3,2) =$ 

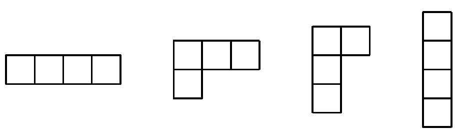
The skew shape μ/ν does not always determine μ and ν uniquely; for example,

$$(5,2,2,1)/(3,2) = (5,3,2,1)/(3,3).$$

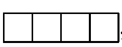
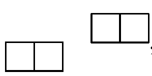
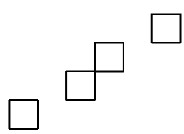
Some special skew shapes will arise frequently in the sequel.

10.3. Definition: Hooks and Strips. A *hook* is a skew shape of the form $(a, 1^{n-a})/0$ for some $a \leq n$. A *horizontal strip* is a skew shape that contains at most one cell in each column. A *vertical strip* is a skew shape that contains at most one cell in each row.

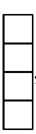
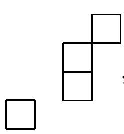
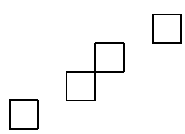
10.4. Example. The following picture displays the four hooks of size 4.



The following skew shapes are horizontal strips of size 4.

, $(5,2)/(3) =$ , $(7,5,4,2)/(6,4,3,1) =$ 

The following skew shapes are vertical strips of size 4.

, $(4,3,3,1)/(3,2,2) =$ , $(7,5,4,2)/(6,4,3,1) =$ 

10.2 Tableaux

Now we are ready to define tableaux.

10.5. Definition: Tableaux. Let μ/ν be a skew shape, and let X be a set. A *tableau of shape μ/ν with values in the alphabet* X is a function $T : \mu/\nu \to X$.

Informally, we obtain a tableau from the skew shape μ/ν by filling each box $c \in \mu/\nu$ with a letter $T(c) \in X$. We often take $\nu = 0$, in which case T is called a *tableau of shape μ*. Note that the plural form of "tableau" is "tableaux"; both words are pronounced *tab–loh*.

10.6. Example. The following picture displays a tableau T of shape $(5, 5, 2)$ with values in $\mathbb{N}$:

$$\begin{array}{|c|c|c|c|c|}\hline 4&3&3&7&2\\\hline 1&1&3&9&1\\\hline 5&6\\\cline{1-2}\end{array}$$

Formally, T is the function with domain $\mathrm{dg}((5, 5, 2))$ such that

$$T((1, 1)) = 4,\ T((1, 2)) = T((1, 3)) = 3,\ \ldots,\ T((3, 1)) = 5,\ T((3, 2)) = 6.$$

As another example, here is a tableau of shape $(2, 2, 2, 2)$ with values in $\{a, b, c, d\}$:

$$\begin{array}{|c|c|}\hline a&b\\\hline b&d\\\hline d&a\\\hline c&c\\\hline\end{array}$$

Here is a tableau of shape $(3, 3, 3)/(2, 1)$ with values in $\mathbb{Z}$:

$$\begin{array}{ccc}\cline{3-3} & & \multicolumn{1}{|c|}{-1}\\\cline{2-3} & \multicolumn{1}{|c|}{0} & \multicolumn{1}{c|}{7}\\\hline \multicolumn{1}{|c|}{4} & \multicolumn{1}{c|}{4} & \multicolumn{1}{c|}{-1}\\\hline\end{array}$$

In most discussions of tableaux, we take the alphabet to be either $\{1, 2, \ldots, N\}$ for some fixed N, or $\mathbb{N}^+ = \{1, 2, 3, \ldots\}$, or $\mathbb{Z}$.

10.7. Definition: Semistandard Tableaux and Standard Tableaux. Let T be a tableau of shape μ/ν taking values in an *ordered* set $(X, \leq)$. T is *semistandard* iff $T((i, j)) \leq T((i, j + 1))$ for all i, j such that (i, j) and $(i, j + 1)$ both belong to μ/ν; and $T((i, j)) < T((i + 1, j))$ for all i, j such that (i, j) and $(i + 1, j)$ both belong to μ/ν. A *standard* tableau is a bijection $T : \mu/\nu \to \{1, 2, \ldots, n\}$ that is also a semistandard tableau, where $n = |\mu/\nu|$.

Less formally, a tableau T is semistandard iff the entries in each row of T weakly increase from left to right, and the entries in each column of T strictly increase from top to bottom. A semistandard tableau is standard iff it contains each number from 1 to n exactly once. The alphabet X is usually a subset of $\mathbb{Z}$ with the usual ordering. Semistandard tableaux are sometimes called *Young tableaux* (in honor of Alfred Young, one of the pioneers in the subject) or *column-strict tableaux*.

10.8. Example. Consider the following three tableaux of shape $(3, 2, 2)$:

$$T_1 = \begin{array}{|c|c|c|}\hline 1&1&3\\\hline 3&4\\\cline{1-2} 5&5\\\cline{1-2}\end{array} \qquad T_2 = \begin{array}{|c|c|c|}\hline 1&2&6\\\hline 3&5\\\cline{1-2} 4&7\\\cline{1-2}\end{array} \qquad T_3 = \begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&2\\\cline{1-2} 4&5\\\cline{1-2}\end{array}\ .$$

T_1 is semistandard but not standard; T_2 is both standard and semistandard; T_3 is neither standard nor semistandard. T_3 is not semistandard because of the strict decrease $3 > 2$ in row 2, and also because of the weak increase $2 \leq 2$ in column 2.

10.9. Example. There are five standard tableaux of shape $(3, 2)$, namely:

$$S_1 = \begin{array}{ccc} 1&2&3\\ 4&5& \end{array} \qquad S_2 = \begin{array}{ccc} 1&2&4\\ 3&5& \end{array} \qquad S_3 = \begin{array}{ccc} 1&2&5\\ 3&4& \end{array} \qquad S_4 = \begin{array}{ccc} 1&3&4\\ 2&5& \end{array} \qquad S_5 = \begin{array}{ccc} 1&3&5\\ 2&4& \end{array}.$$

As mentioned in the introduction, there is an amazing formula for counting the number of standard tableaux of a given shape μ. This formula is proved in §12.10.

10.10. Example. Here is a semistandard tableau of shape $(6, 5, 5, 3)/(3, 2, 2)$:

$$S = \begin{array}{cccccc} &&&1&1&4\\ &&2&3&4&\\ &&4&4&5&\\ 3&7&7&&& \end{array}$$

It will be convenient to have notation for certain sets of tableaux.

10.11. Definition: $\mathrm{SSYT}_X(\mu/\nu)$ and $\mathrm{SYT}(\mu/\nu)$. For every skew shape μ/ν and every ordered alphabet X, let $\mathrm{SSYT}_X(\mu/\nu)$ be the set of all semistandard tableaux with shape μ/ν taking values in X. When $X = \{1, 2, \ldots, N\}$, we abbreviate the notation to $\mathrm{SSYT}_N(\mu/\nu)$. Let $\mathrm{SYT}(\mu/\nu)$ be the set of all standard tableaux of shape μ/ν.

If $\nu = 0$, then we omit it from the notation; for instance, 10.9 displays the five elements in the set $\mathrm{SYT}((3, 2))$. Observe that $\mathrm{SYT}(\mu/\nu)$ is a finite set of tableaux. On the other hand, $\mathrm{SSYT}_X(\mu/\nu)$ is finite iff X is finite.

10.3 Schur Polynomials

We now introduce a weight function on tableaux that keeps track of the number of times each label is used.

10.12. Definition: Content of a Tableau. Let T be a tableau of shape μ/ν with values in $\mathbb{N}^+$. The *content* of T is the infinite sequence $c(T) = (c_1, c_2, \ldots)$, where c_k is the number of times the label k appears in T. Formally, $c_k = |\{(i, j) \in \mathrm{dg}(\mu) : T((i, j)) = k\}|$. Every c_k is a nonnegative integer, and the sum of all c_k's is $|\mu/\nu|$. Given variables (indeterminates) $x_1, x_2, \ldots$, the *content monomial* of T is

$$x^{c(T)} = x_1^{c_1} x_2^{c_2} \cdots x_k^{c_k} \cdots = \prod_{u \in \mu/\nu} x_{T(u)}.$$

10.13. Example. Consider the tableaux from 10.8. The content of T_1 is $c(T_1) = (2, 0, 2, 1, 2, 0, 0, \ldots)$, and the content monomial of T_1 is $x^{c(T_1)} = x_1^2 x_3^2 x_4 x_5^2$. Similarly,

$$x^{c(T_2)} = x_1 x_2 x_3 x_4 x_5 x_6 x_7, \qquad x^{c(T_3)} = x_1 x_2^2 x_3 x_4 x_5^2.$$

All five standard tableaux in 10.9 have content monomial $x_1 x_2 x_3 x_4 x_5$. More generally, the content monomial of any $S \in \mathrm{SYT}(\mu/\nu)$ will be $\prod_{i=1}^{n} x_i$, where $n = |\mu/\nu|$. The tableau S shown in 10.10 has content $c(S) = (2, 1, 2, 4, 1, 0, 2, 0, 0, \ldots)$.

We can now define the Schur polynomials, which are essentially generating functions for semistandard tableaux weighted by content. Recall (§7.16) that $\mathbb{Q}[x_1, \ldots, x_N]$ is the ring of all formal polynomials in the variables $x_1, \ldots, x_N$ with rational coefficients.

10.14. Definition: Schur Polynomials and Skew Schur Polynomials. For each integer $N \geq 1$ and every integer partition μ, define the *Schur polynomial for μ in N variables* by setting

$$s_\mu(x_1, \ldots, x_N) = \sum_{T \in \mathrm{SSYT}_N(\mu)} x^{c(T)}.$$

More generally, for any skew shape μ/ν, define the *skew Schur polynomial for μ/ν* by setting

$$s_{\mu/\nu}(x_1, \ldots, x_N) = \sum_{T \in \mathrm{SSYT}_N(\mu/\nu)} x^{c(T)}.$$

10.15. Example. Let us compute the Schur polynomials $s_\mu(x_1, x_2, x_3)$ for all partitions of 3. First, when $\mu = (3)$, we have the following semistandard tableaux of shape (3) using the alphabet $\{1, 2, 3\}$:

$$\begin{array}{|c|c|c|}\hline 1&1&1\\\hline\end{array}\quad \begin{array}{|c|c|c|}\hline 1&1&2\\\hline\end{array}\quad \begin{array}{|c|c|c|}\hline 1&1&3\\\hline\end{array}\quad \begin{array}{|c|c|c|}\hline 1&2&2\\\hline\end{array}\quad \begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}$$

$$\begin{array}{|c|c|c|}\hline 1&3&3\\\hline\end{array}\quad \begin{array}{|c|c|c|}\hline 2&2&2\\\hline\end{array}\quad \begin{array}{|c|c|c|}\hline 2&2&3\\\hline\end{array}\quad \begin{array}{|c|c|c|}\hline 2&3&3\\\hline\end{array}\quad \begin{array}{|c|c|c|}\hline 3&3&3\\\hline\end{array}$$

It follows that

$$s_{(3)}(x_1, x_2, x_3) = x_1^3 + x_1^2x_2 + x_1^2x_3 + x_1x_2^2 + x_1x_2x_3 + x_1x_3^2 + x_2^3 + x_2^2x_3 + x_2x_3^2 + x_3^3.$$

Second, when $\mu = (2, 1)$, we obtain the following semistandard tableaux:

$$\begin{array}{|c|c|}\hline 1&1\\\hline \multicolumn{1}{|c|}{2}\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 1&1\\\hline \multicolumn{1}{|c|}{3}\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 2&2\\\hline \multicolumn{1}{|c|}{3}\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 1&2\\\hline \multicolumn{1}{|c|}{2}\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 1&2\\\hline \multicolumn{1}{|c|}{3}\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 1&3\\\hline \multicolumn{1}{|c|}{2}\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 1&3\\\hline \multicolumn{1}{|c|}{3}\\\cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 2&3\\\hline \multicolumn{1}{|c|}{3}\\\cline{1-1}\end{array}$$

So $s_{(2,1)}(x_1, x_2, x_3) = x_1^2x_2 + x_1^2x_3 + x_2^2x_3 + x_1x_2^2 + 2x_1x_2x_3 + x_1x_3^2 + x_2x_3^2$. Third, when $\mu = (1, 1, 1)$, we see that $s_{(1,1,1)}(x_1, x_2, x_3) = x_1x_2x_3$, since there is only one semistandard tableau in this case.

Now consider what happens when we change the number of variables. Suppose first that we use $N = 2$ instead of $N = 3$. This means that the allowed alphabet for the tableaux has changed to $\{1, 2\}$. Consulting the tableaux just computed, but disregarding those that use the letter 3, we conclude that

$$s_{(3)}(x_1, x_2) = x_1^3+x_1^2x_2+x_1x_2^2+x_2^3; \qquad s_{(2,1)}(x_1, x_2) = x_1^2x_2+x_1x_2^2; \qquad s_{(1,1,1)}(x_1, x_2) = 0.$$

In these examples, note that we can obtain the polynomial $s_\mu(x_1, x_2)$ from $s_\mu(x_1, x_2, x_3)$ by setting $x_3 = 0$. More generally, we claim that for any μ and any $N > M$, we can obtain $s_\mu(x_1, \ldots, x_M)$ from $s_\mu(x_1, \ldots, x_N)$ by setting the last $N - M$ variables equal to zero. To verify this, consider the defining formula

$$s_\mu(x_1, x_2, \ldots, x_N) = \sum_{T \in \mathrm{SSYT}_N(\mu)} x^{c(T)}.$$

Upon setting $x_{M+1} = \cdots = x_N = 0$ in this formula, the terms coming from tableaux T that use letters larger than M will become zero. We are left with the sum over $T \in \mathrm{SSYT}_M(\mu)$, which is precisely $s_\mu(x_1, x_2, \ldots, x_M)$.

Suppose instead that we increase the number of variables from $N = 3$ to $N = 5$. Here we must draw new tableaux to find the new Schur polynomial. For instance, the tableaux for $\mu = (1,1,1)$ are:

$$\ytableaushort{1,2,3} \quad \ytableaushort{1,2,4} \quad \ytableaushort{1,2,5} \quad \ytableaushort{1,3,4} \quad \ytableaushort{1,3,5} \quad \ytableaushort{1,4,5} \quad \ytableaushort{2,3,4} \quad \ytableaushort{2,3,5} \quad \ytableaushort{2,4,5} \quad \ytableaushort{3,4,5}$$

Accordingly,

$$s_{(1,1,1)}(x_1, x_2, x_3, x_4, x_5) = x_1x_2x_3 + x_1x_2x_4 + x_1x_2x_5 + \cdots + x_3x_4x_5.$$

10.16. Example. A semistandard tableau of shape (1^k) using the alphabet $X = \{1, 2, \ldots, N\}$ is essentially a strictly increasing sequence of k elements of X, which can be identified with a k-element subset of X. Combining this remark with the definition of Schur polynomials, we conclude that

$$s_{(1^k)}(x_1, \ldots, x_N) = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le N} x_{i_1} x_{i_2} \cdots x_{i_k}. \tag{10.1}$$

Similarly, a semistandard tableau of shape (k) is a weakly increasing sequence of k elements of X, which can be identified with a k-element multiset using letters in X. So

$$s_{(k)}(x_1, \ldots, x_N) = \sum_{1 \le i_1 \le i_2 \le \cdots \le i_k \le N} x_{i_1} x_{i_2} \cdots x_{i_k}. \tag{10.2}$$

10.17. Example. Given any integer $N \ge 4$, what is the coefficient of $x_1^2 x_2^2 x_3 x_4$ in the skew Schur polynomial $s_{(4,3)/(1)}(x_1, \ldots, x_N)$? The answer is the number of semistandard tableaux of shape $(4,3)/(1)$ using exactly two 1's, two 2's, one 3, and one 4. Equivalently, we seek the semistandard tableaux with content $(2,2,1,1)$. The required tableaux are shown here:

$$\ytableaushort{\none 112, 234} \quad \ytableaushort{\none 113, 224} \quad \ytableaushort{\none 114, 223} \quad \ytableaushort{\none 122, 134} \quad \ytableaushort{\none 123, 124} \quad \ytableaushort{\none 124, 123}.$$

So the desired coefficient is 6. Next, what is the coefficient of $x_1 x_2 x_3^2 x_4^2$? Now we must find the tableaux of content $(1,1,2,2)$, which are the following:

$$\ytableaushort{\none 123, 344} \quad \ytableaushort{\none 124, 334} \quad \ytableaushort{\none 133, 244} \quad \ytableaushort{\none 134, 234} \quad \ytableaushort{\none 233, 144} \quad \ytableaushort{\none 234, 134}.$$

Again there are six tableaux, so the coefficient of $x_1 x_2 x_3^2 x_4^2$ is 6. Finally, what is the coefficient of $x_1^2 x_2 x_3 x_4^2$? Drawing the tableaux of content $(2,1,1,2)$ produces the following list:

$$\ytableaushort{\none 112, 344} \quad \ytableaushort{\none 113, 244} \quad \ytableaushort{\none 114, 234} \quad \ytableaushort{\none 124, 134} \quad \ytableaushort{\none 134, 124} \quad \ytableaushort{\none 123, 144}.$$

The coefficient is 6 again! One may check that for any rearrangement of the vector $(2,1,1,2,0,0,\ldots)$, the number of semistandard tableaux of shape $(4,3)/(1)$ having this content is always 6. This is not a coincidence; it is a consequence of the fact that Schur polynomials are *symmetric*, which we will prove shortly (§10.6).

10.18. Remark. We have presented a combinatorial definition of Schur polynomials using semistandard tableaux. One can also define Schur polynomials algebraically as a quotient of two determinants; see 11.45. Alternatively, one can define Schur polynomials using determinants involving the elementary or homogeneous symmetric polynomials to be defined below; see 11.60 and 11.61. Many properties of Schur polynomials can be established either combinatorially or algebraically. In this text, we prefer to give the combinatorial proofs.

10.4 Symmetric Polynomials

The examples of Schur polynomials computed in the last section were all *symmetric*; in other words, permuting the subscripts of the x-variables in any fashion did not change the answer. This section begins our examination of the general theory of symmetric polynomials. Throughout the discussion, we assume that K is a field containing $\mathbb{Q}$ (for instance, $\mathbb{Q}$ or $\mathbb{R}$ or $\mathbb{C}$), and N is a fixed positive integer. We will be working in the polynomial ring $R = K[x_1, \ldots, x_N]$ consisting of all polynomials in N variables with coefficients in K (see §7.16).

One property of polynomial rings such as R is that we can substitute arbitrary ring elements for each of the variables x_i. A formal statement of this "universal mapping property" of R was given in 7.102. Suppose that $\sigma \in S_N$ is a given permutation of the subscripts of the x-variables. According to 7.102, there is a unique ring homomorphism $E : R \to R$ such that $E(c) = c$ for all $c \in K$ and $E(x_i) = x_{\sigma(i)}$ for all i. For any polynomial $f \in R$, we often denote $E(f)$ by $f(x_{\sigma(1)}, \ldots, x_{\sigma(N)})$. Note, in particular, that $f(x_1, \ldots, x_n) = f$. Informally, we compute $E(f)$ by starting with a symbolic expression for f as a sum of products of x_i's, and then replacing each symbol x_i by $x_{\sigma(i)}$. With this notation in hand, we can now give the formal definition of symmetric polynomials.

10.19. Definition: Symmetric Polynomials. A polynomial $f \in K[x_1, \ldots, x_N]$ is *symmetric* iff

$$f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(N)}) = f(x_1, \ldots, x_N) \qquad \text{for all } \sigma \in S_N.$$

In other words, any permutation of the variables x_i leaves f unchanged. Since any permutation can be achieved by a finite sequence of basic transpositions (see 9.29), f is symmetric iff for every $i < N$, interchanging x_i and x_{i+1} in f leaves f unchanged.

We now introduce special names for some commonly used symmetric polynomials.

10.20. Definition: Power Sums. For every $k \geq 1$, the polynomial

$$p_k(x_1, x_2, \ldots, x_N) = x_1^k + x_2^k + \cdots + x_N^k$$

is evidently symmetric. This polynomial is called the kth *power-sum* in N variables.

For example, $p_3(x_1, x_2, x_3, x_4, x_5) = x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3$.

10.21. Definition: Elementary Symmetric Polynomials. For fixed k with $1 \leq k \leq N$, define the polynomial

$$e_k(x_1, x_2, \ldots, x_N) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq N} x_{i_1} x_{i_2} \cdots x_{i_k}.$$

The polynomial e_k is called an *elementary symmetric polynomial* in N variables. One may check that e_k is indeed symmetric. We also set $e_0(x_1, \ldots, x_N) = 1$ and $e_k(x_1, \ldots, x_N) = 0$ for all $k > N$.

For example, $e_2(x_1, x_2, x_3, x_4) = x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4$. By formula (10.1), we see that $e_k(x_1, \ldots, x_N) = s_{(1^k)}(x_1, \ldots, x_N)$, so that elementary symmetric polynomials are special cases of Schur polynomials.

10.22. Definition: Complete Symmetric Polynomials. For fixed $k \geq 1$, define the polynomial

$$h_k(x_1, x_2, \ldots, x_N) = \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq N} x_{i_1} x_{i_2} \cdots x_{i_k}.$$

One may verify that h_k really is symmetric. We also set $h_0(x_1,\ldots,x_N)=1$. The polynomials h_k are called *complete homogeneous symmetric polynomials* in N variables.

We call h_k "complete" because it is the sum of *all* monomials of degree k in the given variables. For example, $h_2(x_1,x_2,x_3)=x_1^2+x_2^2+x_3^2+x_1x_2+x_1x_3+x_2x_3$. By formula (10.2), we see that $h_k(x_1,\ldots,x_N)=s_{(k)}(x_1,\ldots,x_N)$, so that complete symmetric polynomials are also special cases of Schur polynomials.

The polynomial $q(x_1,\ldots,x_N)=\sum_{i\neq j}x_i^2x_j^3$ is readily seen to be symmetric. This example can be generalized as follows.

10.23. Definition: Monomial Symmetric Polynomials. Let μ be an integer partition with at most N nonzero parts. Write $\mu=(\mu_1\geq\mu_2\geq\cdots\geq\mu_N)\in\mathbb{N}^N$ by adding zero parts if necessary. For any $\alpha\in\mathbb{N}^N$, write $x^\alpha=x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_N^{\alpha_N}$. Let $\mathrm{sort}(\alpha)\in\mathbb{N}^N$ be the unique partition obtained by sorting the entries of α into weakly decreasing order. Next, let $M(\mu)=\{\alpha\in\mathbb{N}^N:\mathrm{sort}(\alpha)=\mu\}$. Finally, define the *monomial symmetric polynomial indexed by* μ to be

$$m_\mu(x_1,\ldots,x_N)=\sum_{\alpha\in M(\mu)}x^\alpha.$$

Informally, $m_\mu(x_1,\ldots,x_N)$ is the sum of all distinct monomials $x_1^{\alpha_1}\cdots x_N^{\alpha_N}$ whose exponent vector can be rearranged to give μ. In this notation, the polynomial q above is $m_{(3,2)}(x_1,\ldots,x_N)$. Some of our previous examples are instances of monomial symmetric polynomials. Namely, we have $p_k(x_1,\ldots,x_N)=m_{(k)}(x_1,\ldots,x_N)$ and $e_k(x_1,\ldots,x_N)=m_{(1^k)}(x_1,\ldots,x_N)$.

Let us check that m_μ really is symmetric. Given $\sigma\in S_N$, we have

$$m_\mu(x_{\sigma(1)},\ldots,x_{\sigma(N)})=\sum_{\alpha\in M(\mu)}x_{\sigma(1)}^{\alpha_1}\cdots x_{\sigma(N)}^{\alpha_N}=\sum_{\alpha\in M(\mu)}\prod_{i=1}^N x_{\sigma(i)}^{\alpha_i}=\sum_{\alpha\in M(\mu)}\prod_{j=1}^N x_j^{\alpha_{\sigma^{-1}(j)}}.$$

The last step follows by setting $j=\sigma(i)$ and rearranging the order of factors in each product. To continue, introduce a new summation variable $\beta=(\alpha_{\sigma^{-1}(1)},\ldots,\alpha_{\sigma^{-1}(N)})$. The entries of β are obtained by rearranging the entries of α, so $\mathrm{sort}(\beta)=\mathrm{sort}(\alpha)=\mu$. In fact, the map $\alpha\mapsto\beta$ is a bijection of $M(\mu)$ to itself with inverse $\beta\mapsto(\beta_{\sigma(1)},\ldots,\beta_{\sigma(N)})$. Since addition is commutative, we can continue the calculation by writing

$$\sum_{\alpha\in M(\mu)}\prod_{j=1}^N x_j^{\alpha_{\sigma^{-1}(j)}}=\sum_{\beta\in M(\mu)}\prod_{j=1}^N x_j^{\beta_j}=m_\mu(x_1,\ldots,x_N).$$

10.24. Definition: Λ_N. Let Λ_N be the set of all symmetric polynomials in $K[x_1,\ldots,x_N]$.

If two polynomials f and g are symmetric, so are $f+g$, $-f$, and fg. For example,

$$\begin{aligned}(fg)(x_{\sigma(1)},\ldots,x_{\sigma(N)}) &= f(x_{\sigma(1)},\ldots,x_{\sigma(N)})g(x_{\sigma(1)},\ldots,x_{\sigma(N)})\\ &= f(x_1,\ldots,x_N)g(x_1,\ldots,x_N)\\ &= (fg)(x_1,\ldots,x_N)\qquad\text{(for all }\sigma\in S_N).\end{aligned}$$

Also, any constant polynomial $c\in K$ is certainly symmetric, as is any scalar multiple cf of a symmetric polynomial f. These comments imply that Λ_N is a subring and K-vector subspace of $K[x_1,\ldots,x_N]$. In particular, Λ_N is a commutative ring with identity and a vector space over K.

We have just seen that Λ_N is closed under products. So, we can multiply together polynomials of the form e_k, h_k, or p_k to obtain even more examples of symmetric polynomials. This leads to the following definition.

10.25. Definition: The Symmetric Polynomials e_α, h_α, and p_α. Let $\alpha = (\alpha_1, \ldots, \alpha_s)$ be any sequence of positive integers. Define

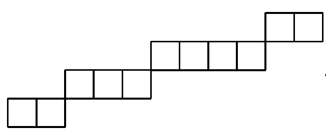
$$\begin{aligned} e_\alpha(x_1, \ldots, x_N) &= \prod_{i=1}^{s} e_{\alpha_i}(x_1, \ldots, x_N); \\ h_\alpha(x_1, \ldots, x_N) &= \prod_{i=1}^{s} h_{\alpha_i}(x_1, \ldots, x_N); \\ p_\alpha(x_1, \ldots, x_N) &= \prod_{i=1}^{s} p_{\alpha_i}(x_1, \ldots, x_N). \end{aligned}$$

We call e_α the *elementary symmetric polynomial* indexed by α; h_α the *complete homogeneous symmetric polynomial* indexed by α; and p_α the *power-sum symmetric polynomial* indexed by α (in N variables).

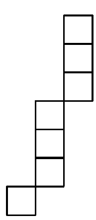
These definitions are most frequently used when α is an *integer partition*. Suppose the sequence α can be sorted to give the partition μ. Then $e_\alpha = e_\mu$, $h_\alpha = h_\mu$, and $p_\alpha = p_\mu$, because multiplication of polynomials is commutative. More generally, if α and β are rearrangements of each other, then $e_\alpha = e_\beta$, $h_\alpha = h_\beta$, and $p_\alpha = p_\beta$.

10.26. Remark. The power-sum polynomials p_α have already appeared in our discussion of Pólya's Formula (§9.19), where they were used to count weighted colorings with symmetries taken into account.

10.27. Remark. The polynomials e_α and h_α are special cases of skew Schur polynomials. For example, consider $h_\alpha = h_{\alpha_1} h_{\alpha_2} \cdots h_{\alpha_s}$. We have seen that each factor h_{α_i} is the generating function for semistandard tableaux of shape (α_i). There exists a skew shape μ/ν consisting of disconnected horizontal rows of lengths $\alpha_1, \ldots, \alpha_s$. When building a semistandard tableau of this shape, each row can be filled with labels independently of the others. So the product rule for weighted sets shows that $h_\alpha(x_1, \ldots, x_N) = s_{\mu/\nu}(x_1, \ldots, x_N)$. For example, given $h_{(2,4,3,2)} = h_2 h_4 h_3 h_2 = h_{(4,3,2,2)}$, we draw the skew shape

 .

Thus we have $h_{(2,4,3,2)} = s_{(11,9,5,2)/(9,5,2)}$. An analogous procedure works for the e_α's, but now we use disconnected vertical columns of lengths given by α. For example, $e_{(3,3,1)} = s_{\mu/\nu}$ if we take

$\mu/\nu =$  $= (3, 3, 3, 2, 2, 2, 1)/(2, 2, 2, 1, 1, 1).$

10.5 Homogeneous Symmetric Polynomials

When studying symmetric polynomials, it is often helpful to focus attention on those polynomials that are *homogeneous* of a given degree.

10.28. Definition: Homogeneous Symmetric Polynomials. For all $k, N \in \mathbb{N}$, let Λ_N^k be the set of symmetric polynomials $p \in \Lambda_N$ such that p is homogeneous of degree k. This means that every monomial x^α appearing in p with nonzero coefficient has degree k (i.e., $\sum_{i=1}^N \alpha_i = k$). In particular, the zero polynomial is homogeneous of every degree.

One can check that for $f, g \in \Lambda_N^k$ and $c \in K$, we have $f + g \in \Lambda_N^k$ and $cf \in \Lambda_N^k$. This means that Λ_N^k *is a K-vector space.* Furthermore, the K-vector space Λ_N is the direct sum of vector spaces

$$\Lambda_N = \bigoplus_{k \geq 0} \Lambda_N^k,$$

since every symmetric polynomial can be uniquely written as a finite sum of its nonzero homogeneous components. Moreover, $p \in \Lambda_N^k$ and $q \in \Lambda_N^j$ imply $pq \in \Lambda_N^{k+j}$, which means that this direct-sum decomposition turns Λ_N into a *graded ring.*

The vector space Λ_N is infinite-dimensional, but each homogeneous piece Λ_N^k is finite-dimensional. A key theme in the theory of symmetric polynomials is the problem of finding different bases of the vector space Λ_N^k and understanding the relations between these bases. We begin in this section by considering the most straightforward basis for this vector space, which consists of suitable monomial symmetric polynomials.

10.29. Theorem: Monomial Basis of Λ_N^k. For every k and N, the indexed set of polynomials

$$\{m_\mu(x_1, \ldots, x_N) : \mu \in \mathrm{Par}_N(k)\} \subseteq K[x_1, \ldots, x_N]$$

is a basis for the K-vector space Λ_N^k.

Proof. For $\mu \in \mathrm{Par}_N(k)$, recall that m_μ is the sum of all distinct monomials x^α such that $\alpha \in \mathbb{N}^N$ can be rearranged to give μ. Each of these monomials has degree $|\mu| = k$, so that each m_μ in the given set is indeed symmetric and homogeneous of degree k. Next, let us prove that the m_μ's are linearly independent over K. Suppose some linear combination of these polynomials is the zero polynomial, say

$$\sum_\mu c_\mu m_\mu(x_1, \ldots, x_N) = 0 \qquad (c_\mu \in K). \tag{10.3}$$

Consider some fixed $\nu \in \mathrm{Par}_N(k)$. Given any partition $\mu \neq \nu$, we cannot rearrange the parts of ν to obtain μ. It follows that m_ν is the only monomial symmetric polynomial in the sum in which x^ν appears with nonzero coefficient. The coefficient of x^ν in m_ν is 1. Extracting the coefficient of x^ν on both sides of (10.3) therefore gives $c_\nu \cdot 1 = 0$. Since ν was arbitrary, all c_ν's are zero, completing the proof of linear independence.

Next, let us prove that the m_μ's span Λ_N^k. Let $f(x_1, \ldots, x_N)$ be any homogeneous symmetric polynomial of degree k. For each $\mu \in \mathrm{Par}_N(k)$, define $d_\mu \in K$ to be the coefficient of x^μ in f. We claim that

$$\sum_\mu d_\mu m_\mu(x_1, \ldots, x_N) = f(x_1, \ldots, x_N). \tag{10.4}$$

It suffices to check that, for every $\alpha \in \mathbb{N}^N$ with $|\alpha| = k$, the coefficient of x^α on both sides of (10.4) is the same. Fix such an α, and note that there is a unique partition $\nu \in \mathrm{Par}_N(k)$ such that $\mathrm{sort}(\alpha) = \nu$. Reasoning as before, we see that the coefficient of x^α in $\sum d_\mu m_\mu$ must be d_ν. On the other hand, since f is symmetric, the coefficient of x^α in f must be the same as the coefficient of x^ν in f, since some permutation of the variables will change x^α into x^ν. But the coefficient of x^ν in f is d_ν by definition, so we are done. $\square$

10.30. Remark. If μ is a partition of k with more than N nonzero parts, then $m_\mu(x_1, \ldots, x_N)$ is not defined. If the number of variables N exceeds k, then the condition $\ell(\mu) \leq N$ automatically holds for all partitions $\mu \vdash k$. Therefore, *when* $N \geq k$, the basis for Λ_N^k reduces to $\{m_\mu(x_1, \ldots, x_N) : \mu \in \mathrm{Par}(k)\}$. So, when $N \geq k$, we have $\dim(\Lambda_N^k) = p(k)$, the number of integer partitions of k.

10.6 Symmetry of Schur Polynomials

Recall the definition of skew Schur polynomials from 10.14:

$$s_{\mu/\nu}(x_1, \ldots, x_N) = \sum_{T \in \mathrm{SSYT}_N(\mu/\nu)} x^{c(T)}.$$

We are about to give a bijective proof that the polynomial appearing in this definition is always symmetric. First, we give names to the coefficients of these polynomials.

10.31. Definition: Kostka Numbers. For each skew shape μ/ν and each $\alpha \in \mathbb{N}^N$, define the *Kostka number* $K_{\mu/\nu,\alpha}$ to be the coefficient of x^α in $s_{\mu/\nu}(x_1, \ldots, x_N)$. Equivalently, $K_{\mu/\nu,\alpha}$ is the number of semistandard tableaux of shape μ/ν and content α.

10.32. Example. The calculations in 10.17 show that

$$K_{(4,3)/(1),(2,2,1,1)} = K_{(4,3)/(1),(1,1,2,2)} = K_{(4,3)/(1),(2,1,1,2)} = 6.$$

Similarly, we see from 10.9 that $K_{(3,2),(1,1,1,1,1)} = 5$.

The following result is the key to proving the symmetry of Schur polynomials.

10.33. Theorem: Symmetry of Kostka Numbers. For all skew shapes μ/ν and all $\alpha, \beta \in \mathbb{N}^N$ such that $\mathrm{sort}(\alpha) = \mathrm{sort}(\beta)$, we have $K_{\mu/\nu,\alpha} = K_{\mu/\nu,\beta}$.

Proof. Fix μ/ν and α, β as in the theorem statement. Since $\mathrm{sort}(\alpha) = \mathrm{sort}(\beta)$, we can pass from α to β by a suitable permutation of the entries of α. This permutation can be achieved in finitely many steps by repeatedly interchanging two consecutive entries of α (cf. 9.29 and 9.179). By induction, it therefore suffices to prove the result when β is obtained from α by switching α_i and α_{i+1} for some $i < N$.

Let Y be the set of all tableaux $T \in \mathrm{SSYT}_N(\mu/\nu)$ such that $c(T) = \alpha$, and let Z be the set of all tableaux $T \in \mathrm{SSYT}_N(\mu/\nu)$ such that $c(T) = \beta$. Since $|Y| = K_{\mu/\nu,\alpha}$ and $|Z| = K_{\mu/\nu,\beta}$, it suffices to define a bijection $f_i : Y \to Z$. The map f_i must take a semistandard tableau of shape μ/ν and create a new semistandard tableau of the same shape in which the number of i's and $(i+1)$'s are switched, while the number of k's (for all $k \neq i, i+1$) is unchanged. We will illustrate the action of f_3 on the following tableau:

$$\begin{array}{ccccccccc}
 & & & 1 & 1 & 1 & 1 & 2 & 3 \\
 & & 2 & 3 & 3 & 3 & 4 & 4 & 4 \\
1 & 3 & 3 & 4 & 4 & 5 & 5 & 6 & \\
4 & 4 & 6 & 6 & 6 & 7 & 9 & & \\
5 & 6 & 7 & 7 & 8 & & & &
\end{array}$$

Observe that certain occurrences of 3 are "matched" with an occurrence of 4 in the cell directly below. Let us underline the 3's and 4's that are *not* part of these matched pairs:

$$\begin{array}{ccccccccc} & & & 1 & 1 & 1 & 1 & 2 & 3 \\ & & 2 & 3 & 3 & \underline{3} & \underline{4} & \underline{4} & 4 \\ 1 & 3 & \underline{3} & 4 & 4 & 5 & 5 & 6 & \\ \underline{4} & 4 & 6 & 6 & 6 & 7 & 9 & & \\ 5 & 6 & 7 & 7 & 8 & & & & \end{array}$$

Notice that each row of the tableau contains a (possibly empty) run of consecutive cells consisting of underlined 3's and 4's. The entries directly above these cells are < 3, while the entries directly below are > 4. So we are free to change the frequency of 3's and 4's within each run without affecting the semistandardness of the tableau. If the run in a given row consists of j threes followed by k fours (where $j, k \geq 0$), we will change this to a run consisting of k threes followed by j fours. Doing this in every row will switch the frequency of 3's and 4's (note that the matched pairs are not touched, and these contribute equally to the frequency counts for 3 and 4). Our example tableau is mapped by f_3 to the following tableau:

$$\begin{array}{ccccccccc} & & & 1 & 1 & 1 & 1 & 2 & 3 \\ & & 2 & 3 & 3 & \underline{3} & \underline{3} & \underline{4} & 4 \\ 1 & 3 & \underline{4} & 4 & 4 & 5 & 5 & 6 & \\ \underline{3} & 4 & 6 & 6 & 6 & 7 & 9 & & \\ 5 & 6 & 7 & 7 & 8 & & & & \end{array}$$

Applying the same run-modification process to this new tableau will restore the original tableau; this means that f_3 is a bijection. As another example of the action of f_3, we have

$$f_3\left(\begin{array}{ccccccccc} 3 & 3 & 3 & 4 & 4 & 4 & 4 & 4 & 6 \\ 4 & & & & & & & & \end{array}\right) = \begin{array}{ccccccccc} 3 & 3 & 3 & 3 & 3 & 3 & 4 & 4 & 6 \\ 4 & & & & & & & & \end{array}.$$

The definition of f_i for general i is exactly the same. We locate and ignore matched pairs consisting of an i directly atop an $i+1$, then underline the remaining i's and $(i+1)$'s, then switch the relative frequencies of the underlined i's and $(i+1)$'s in each row. This action is reversible, maintains semistandardness, and switches the overall frequency of i's and $(i+1)$'s while preserving the frequency of all other letters. So we have found the required bijection. □

10.34. Example. The preceding proof allows us to construct explicit bijections between the collections of tableaux in 10.17, which are counted by various Kostka numbers $K_{(4,3)/(1),\alpha}$ such that $\text{sort}(\alpha) = (2, 2, 1, 1)$. As directed by the proof, we must chain together suitable maps f_i, where the values of i are chosen to rearrange the starting content vector α into the target content vector β. For example, we can go from $(2, 2, 1, 1)$ to $(2, 1, 2, 1)$ to $(2, 1, 1, 2)$ by applying f_2 and then f_3. So, for instance, the first tableau of content $(2, 2, 1, 1)$ in 10.17 is mapped to a tableau of content $(2, 1, 1, 2)$ as follows:

$$\begin{array}{cccc} & 1 & 1 & 2 \\ 2 & 3 & 4 & \end{array} \xrightarrow{f_2} \begin{array}{cccc} & 1 & 1 & 3 \\ 2 & 3 & 4 & \end{array} \xrightarrow{f_3} \begin{array}{cccc} & 1 & 1 & 4 \\ 2 & 3 & 4 & \end{array}.$$

If we continue by applying the maps f_1 and then f_2, we reach a tableau with content $(1, 1, 2, 2)$:

$$\begin{array}{cccc} & 1 & 1 & 4 \\ 2 & 3 & 4 & \end{array} \xrightarrow{f_1} \begin{array}{cccc} & 2 & 2 & 4 \\ 1 & 3 & 4 & \end{array} \xrightarrow{f_2} \begin{array}{cccc} & 2 & 3 & 4 \\ 1 & 3 & 4 & \end{array}.$$

The inverse bijection is computed by applying the maps in the reverse order. For example, the first tableau of content $(1, 1, 2, 2)$ in 10.17 is mapped to a tableau of content $(2, 2, 1, 1)$ via the following steps.

$$\begin{array}{cccc} & 1 & 2 & 3 \\ 3 & 4 & 4 & \end{array} \xrightarrow{f_2} \begin{array}{cccc} & 1 & 2 & 3 \\ 2 & 4 & 4 & \end{array} \xrightarrow{f_1} \begin{array}{cccc} & 1 & 2 & 3 \\ 1 & 4 & 4 & \end{array} \xrightarrow{f_3} \begin{array}{cccc} & 1 & 2 & 4 \\ 1 & 3 & 3 & \end{array} \xrightarrow{f_2} \begin{array}{cccc} & 1 & 2 & 4 \\ 1 & 2 & 3 & \end{array}$$

We can now deduce the symmetry of skew Schur polynomials. In fact, we can even expand these polynomials as linear combinations of monomial symmetric polynomials using the Kostka numbers.

10.35. Theorem: Monomial Expansion of Schur Polynomials. For all skew shapes μ/ν with k boxes and all $N \geq 1$,

$$s_{\mu/\nu}(x_1,\ldots,x_N) = \sum_{\rho\in\mathrm{Par}_N(k)} K_{\mu/\nu,\rho} m_\rho(x_1,\ldots,x_N). \tag{10.5}$$

In particular, $s_{\mu/\nu}(x_1,\ldots,x_N)$ is a *homogeneous, symmetric* polynomial of degree k.

Proof. Consider the following calculation:

$$\begin{aligned} s_{\mu/\nu}(x_1,\ldots,x_N) &= \sum_{\alpha\in\mathbb{N}^N} K_{\mu/\nu,\alpha}x^\alpha = \sum_{\rho\in\mathrm{Par}_N(k)} \sum_{\substack{\alpha\in\mathbb{N}^N:\\ \mathrm{sort}(\alpha)=\rho}} K_{\mu/\nu,\alpha}x^\alpha \\ &= \sum_{\rho\in\mathrm{Par}_N(k)} \sum_{\substack{\alpha\in\mathbb{N}^N:\\ \mathrm{sort}(\alpha)=\rho}} K_{\mu/\nu,\rho}x^\alpha = \sum_{\rho\in\mathrm{Par}_N(k)} K_{\mu/\nu,\rho} \sum_{\substack{\alpha\in\mathbb{N}^N:\\ \mathrm{sort}(\alpha)=\rho}} x^\alpha \\ &= \sum_{\rho\in\mathrm{Par}_N(k)} K_{\mu/\nu,\rho} m_\rho(x_1,\ldots,x_N). \end{aligned}$$

The first step follows from the definition of Kostka numbers. In the second step, we reorder the sum by classifying $\alpha \in \mathbb{N}^N$ based on which partition α sorts to. Only partitions of k occur, since $x^{c(T)}$ is a monomial of degree k for every tableau T on the k-box shape μ/ν. The third step follows from 10.33. The fourth step uses the fact that $K_{\mu/\nu,\rho}$ does not depend on the inner summation index α. The final step follows by definition of m_ρ. □

10.7 Orderings on Partitions

We will use 10.35 to find bases for the vector spaces Λ_N^k consisting of suitable Schur polynomials. First, however, we need to introduce some ordering relations on sets of integer partitions.

10.36. Definition: Lexicographic Ordering of Partitions. Suppose $\mu = (\mu_i : i \geq 1)$ and $\nu = (\nu_i : i \geq 1)$ are partitions of the same integer k. We say that ν is *lexicographically greater* than μ, written $\mu \leq_{\mathrm{lex}} \nu$, iff either $\mu = \nu$ or the first nonzero entry in the vector $\nu - \mu$ is positive. The latter condition means that for some j, $\mu_1 = \nu_1$, $\mu_2 = \nu_2$, …, $\mu_{j-1} = \nu_{j-1}$, and $\mu_j < \nu_j$.

It is routine to check that $\leq_{\mathrm{lex}}$ is a *total order* on $\mathrm{Par}(k)$, for each $k \geq 0$.

10.37. Example. Here is a list of all integer partitions of 6, written in lexicographic order from smallest to largest:

$$\begin{gathered} (1,1,1,1,1,1) \leq_{\mathrm{lex}} (2,1,1,1,1) \leq_{\mathrm{lex}} (2,2,1,1) \leq_{\mathrm{lex}} (2,2,2) \leq_{\mathrm{lex}} (3,1,1,1) \\ \leq_{\mathrm{lex}} (3,2,1) \leq_{\mathrm{lex}} (3,3) \leq_{\mathrm{lex}} (4,1,1) \leq_{\mathrm{lex}} (4,2) \leq_{\mathrm{lex}} (5,1) \leq_{\mathrm{lex}} (6). \end{gathered}$$

For example, $(3,1,1,1) \leq_{\mathrm{lex}} (3,2,1)$ since

$$(3,2,1,0,0,\ldots) - (3,1,1,1,0,\ldots) = (0,1,0,-1,0,\ldots)$$

and the earliest nonzero entry in this vector is positive.

In the coming sections, we will frequently be considering matrices and vectors whose rows and columns are indexed by integer partitions. Unless otherwise specified, we will always use the lexicographic ordering of partitions to determine which partition labels each row and column of the matrix. For instance, when $k=3$, a matrix $A=(c_{\mu,\nu}:\mu,\nu\in\mathrm{Par}(3))$ will be displayed as follows:

$$A=\begin{bmatrix} c_{(1,1,1),(1,1,1)} & c_{(1,1,1),(2,1)} & c_{(1,1,1),(3)} \\ c_{(2,1),(1,1,1)} & c_{(2,1),(2,1)} & c_{(2,1),(3)} \\ c_{(3),(1,1,1)} & c_{(3),(2,1)} & c_{(3),(3)} \end{bmatrix}.$$

Next we consider a partial ordering on partitions that occurs frequently in the theory of symmetric polynomials.

10.38. Definition: Dominance Ordering on Partitions. Let $\mu=(\mu_i:i\geq 1)$ and $\nu=(\nu_i:i\geq 1)$ be two partitions of the same integer k. We say that ν *dominates* μ, written $\mu\trianglelefteq\nu$, iff

$$\mu_1+\mu_2+\cdots+\mu_i\leq\nu_1+\nu_2+\cdots+\nu_i \quad \text{for all } i\geq 1.$$

Note that $\mu\not\trianglelefteq\nu$ iff there exists an $i\geq 1$ with $\mu_1+\cdots+\mu_i>\nu_1+\cdots+\nu_i$.

10.39. Example. We have $(2,2,1,1)\trianglelefteq(4,2)$ since $2\leq 4$, $2+2\leq 4+2$, $2+2+1\leq 4+2+0$, and $2+2+1+1\leq 4+2+0+0$. On the other hand, $(3,1,1,1)\not\trianglelefteq(2,2,2)$ since $3>2$, and $(2,2,2)\not\trianglelefteq(3,1,1,1)$ since $2+2+2>3+1+1$. This example shows that not every pair of partitions is comparable under the dominance relation.

10.40. Theorem: Dominance Partial Order. The dominance relation is a partial ordering on $\mathrm{Par}(k)$, for every $k\geq 0$.

Proof. We will show that $\trianglelefteq$ is reflexive, antisymmetric, and transitive on $\mathrm{Par}(k)$. *Reflexivity:* Given $\mu\vdash k$, we have $\mu_1+\cdots+\mu_i\leq\mu_1+\cdots+\mu_i$ for all $i\geq 1$. So $\mu\trianglelefteq\mu$. *Antisymmetry:* Suppose $\mu,\nu\vdash k$, $\mu\trianglelefteq\nu$, and $\nu\trianglelefteq\mu$. We know $\mu_1+\cdots+\mu_i\leq\nu_1+\cdots+\nu_i$ and also $\nu_1+\cdots+\nu_i\leq\mu_1+\cdots+\mu_i$ for all i, hence $\mu_1+\cdots+\mu_i=\nu_1+\cdots+\nu_i$ for all $i\geq 1$. In particular, taking $i=1$ gives $\mu_1=\nu_1$. For each $i>1$, subtracting the $(i-1)$th equation from the ith equation shows that $\mu_i=\nu_i$. So $\mu=\nu$. *Transitivity:* Fix $\mu,\nu,\rho\vdash k$, and assume $\mu\trianglelefteq\nu\trianglelefteq\rho$; we must prove $\mu\trianglelefteq\rho$. We know $\mu_1+\cdots+\mu_i\leq\nu_1+\cdots+\nu_i$ for all i, and also $\nu_1+\cdots+\nu_i\leq\rho_1+\cdots+\rho_i$ for all i. Combining these inequalities yields $\mu_1+\cdots+\mu_i\leq\rho_1+\cdots+\rho_i$ for all i, so $\mu\trianglelefteq\rho$. □

One can check that $\trianglelefteq$ is a *total* ordering of $\mathrm{Par}(k)$ iff $k\leq 5$.

10.41. Theorem: Lexicographic vs. Dominance Ordering. For all $\mu,\nu\vdash k$, $\mu\trianglelefteq\nu$ implies $\mu\leq_{\mathrm{lex}}\nu$.

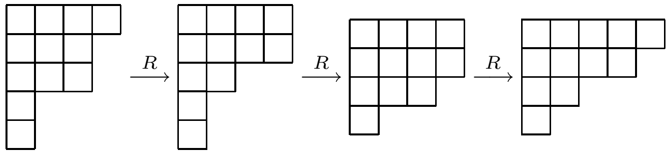
Proof. Fix μ,ν such that $\mu\not\leq_{\mathrm{lex}}\nu$; we will prove that $\mu\not\trianglelefteq\nu$. By definition of the lexicographic order, there must exist an index $j\geq 1$ such that $\mu_i=\nu_i$ for all $i<j$, but $\mu_j>\nu_j$. Adding these relations together, we see that $\mu_1+\cdots+\mu_j>\nu_1+\cdots+\nu_j$, and so $\mu\not\trianglelefteq\nu$. □

The next definition and theorem will allow us to visualize the dominance relation in terms of partition diagrams.

10.42. Definition: Raising Operation. Let μ and ν be two partitions of k. We say that ν is related to μ by a *raising operation*, denoted $\mu R\nu$, iff there exist $i<j$ such that $\nu_i=\mu_i+1$, $\nu_j=\mu_j-1$, and $\nu_s=\mu_s$ for all $s\neq i,j$.

Intuitively, $\mu R\nu$ means that we can go from the diagram for μ to the diagram for ν by taking the last square from some row of μ and moving it to the end of a higher row.

10.43. Example. The following pictures illustrate a sequence of raising operations.



Observe that $(4,3,3,1,1) \trianglelefteq (5,4,2,1)$, so that the last partition in the sequence dominates the first one. The next result shows that this always happens.

10.44. Theorem: Dominance and Raising Operations. Given $\mu, \nu \vdash k$, we have $\mu \trianglelefteq \nu$ iff there exist $m \geq 0$ and partitions $\mu^0, \ldots, \mu^m$ such that $\mu = \mu^0 R \mu^1 R \mu^2 \cdots \mu^{m-1} R \mu^m = \nu$.

Proof. Let us first show that $\mu R \nu$ implies $\mu \trianglelefteq \nu$. Suppose $\nu = (\mu_1, \ldots, \mu_i + 1, \ldots, \mu_j - 1, \ldots)$ as in the definition of dominance ordering. Let us check that $\mu_1 + \cdots + \mu_k \leq \nu_1 + \cdots + \nu_k$ holds for all $k \geq 1$. This is true for $k < i$, since equality holds for these k's. If $k = i$, note that $\nu_1 + \cdots + \nu_k = \mu_1 + \cdots + \mu_{i-1} + (\mu_i + 1)$, so the required inequality does hold. Similarly, for all k with $i \leq k < j$, we have $\nu_1 + \cdots + \nu_k = \mu_1 + \cdots + \mu_k + 1 > \mu_1 + \cdots + \mu_k$. Finally, for all $k \geq j$, we have $\mu_1 + \cdots + \mu_k = \nu_1 + \cdots + \nu_k$ since the $+1$ and -1 adjustments to parts i and j cancel out.

Next, suppose μ and ν are linked by a chain of raising operations as in the theorem statement, say $\mu = \mu^0 R \mu^1 R \mu^2 \cdots R \mu^m = \nu$. By what has just been proved, we have $\mu = \mu^0 \trianglelefteq \mu^1 \trianglelefteq \mu^2 \cdots \trianglelefteq \mu^m = \nu$. Since $\trianglelefteq$ is transitive, we conclude that $\mu \trianglelefteq \nu$, as desired.

Conversely, suppose that $\mu \trianglelefteq \nu$. Consider the vector $(d_1, d_2, \ldots)$ such that $d_s = (\nu_1 + \cdots + \nu_s) - (\mu_1 + \cdots + \mu_s)$. Since $\mu \trianglelefteq \nu$, we have $d_s \geq 0$ for all s. Also, $d_s = 0$ for all large enough s since μ and ν are both partitions of k. We argue by induction on $n = \sum_s d_s$. If $n = 0$, then $\mu = \nu$, and we can take $m = 0$ and $\mu = \mu^0 = \nu$. Otherwise, let i be the least index such that $d_i > 0$, and let j be the least index after i such that $d_j = 0$. The choice of i shows that $\mu_s = \nu_s$ for all $s < i$, but $\mu_i < \nu_i$. If $i > 1$, the inequality $\mu_i < \nu_i \leq \nu_{i-1} = \mu_{i-1}$ shows that it is possible to add one box to the end of row i in $\mathrm{dg}(\mu)$ and still get a partition diagram. If $i = 1$, the addition of this box will certainly give a partition diagram. On the other hand, the relations $d_{j-1} > 0$, $d_j = 0$ mean that $\mu_1 + \cdots + \mu_{j-1} < \nu_1 + \cdots + \nu_{j-1}$ but $\mu_1 + \cdots + \mu_j = \nu_1 + \cdots + \nu_j$, so that $\mu_j > \nu_j$. Furthermore, from $d_j = 0$ and $d_{j+1} \geq 0$ we deduce that $\mu_{j+1} \leq \nu_{j+1}$. So, $\mu_{j+1} \leq \nu_{j+1} \leq \nu_j < \mu_j$, which shows that we can remove a box from row j of $\mathrm{dg}(\mu)$ and still get a partition diagram.

We have just shown that it is permissible to modify μ by a raising operator that moves the box at the end of row j to the end of row i. Let μ^1 be the new partition obtained in this way, so that $\mu R \mu^1$. Consider how the partial sums $\mu_1 + \cdots + \mu_s$ change when we replace μ by μ^1. For $s < i$ or $s \geq j$, the partial sums are the same for μ and μ^1. For $i \leq s < j$, the partial sums increase by 1. Since $d_s > 0$ in the range $i \leq s < j$, it follows that the new differences $d'_s = (\nu_1 + \cdots + \nu_s) - (\mu^1_1 + \cdots + \mu^1_s)$ are all ≥ 0; in other words, $\mu^1 \trianglelefteq \nu$. We have $d'_s = d_s - 1$ for $i \leq s < j$, and $d'_s = d_s$ for all other s; so $\sum d'_s < \sum d_s$. Arguing by induction, we can find a chain of raising operations linking μ^1 to ν. This completes the inductive proof. □

As an application of the previous result, we prove the following fact relating the dominance ordering to the conjugation operation on partitions.

10.45. Theorem: Dominance vs. Conjugation. For all $\mu, \nu \in \mathrm{Par}(k)$, $\mu \trianglelefteq \nu$ iff $\nu' \trianglelefteq \mu'$.

Proof. Fix $\mu, \nu \in \mathrm{Par}(k)$. Note first that $\mu R \nu$ implies $\nu' R \mu'$. This assertion follows from the pictorial description of the raising operation, since the box that moves from a lower row in μ to a higher row in ν necessarily moves from some column to a column strictly to its right. Reversing the direction of motion and transposing the diagrams, we see that we can go from ν' to μ' by moving a box in ν' from a lower row to a higher row.

Next, assuming $\mu \trianglelefteq \nu$, 10.44 shows that there is a chain

$$\mu = \mu^0 R \mu^1 R \mu^2 \cdots \mu^{m-1} R \mu^m = \nu.$$

Applying the remark in the previous paragraph to each link in this chain gives a new chain

$$\nu' = (\mu^m)' R (\mu^{m-1})' R \cdots (\mu^2)' R (\mu^1)' R (\mu^0)' = \mu'.$$

Invoking 10.44 again, we see that $\nu' \trianglelefteq \mu'$.

Conversely, assume that $\nu' \trianglelefteq \mu'$. Applying the result just proved, we get $\mu'' \trianglelefteq \nu''$. Since $\mu'' = \mu$ and $\nu'' = \nu$, we have $\mu \trianglelefteq \nu$. □

10.8 Schur Bases

We now have all the necessary tools to find bases for the vector spaces Λ_N^k consisting of Schur polynomials. First we illustrate the key ideas with an example.

10.46. Example. In 10.15, we computed the Schur polynomials $s_\mu(x_1, x_2, x_3)$ for all partitions $\mu \in \mathrm{Par}(3)$. We can use 10.35 to write these Schur polynomials as linear combinations of monomial symmetric polynomials, where the coefficients are Kostka numbers:

$$\begin{aligned}
s_{(1,1,1)}(x_1,x_2,x_3) &= m_{(1,1,1)}(x_1,x_2,x_3); \\
s_{(2,1)}(x_1,x_2,x_3) &= 2m_{(1,1,1)}(x_1,x_2,x_3) + m_{(2,1)}(x_1,x_2,x_3); \\
s_{(3)}(x_1,x_2,x_3) &= m_{(1,1,1)}(x_1,x_2,x_3) + m_{(2,1)}(x_1,x_2,x_3) + m_{(3)}(x_1,x_2,x_3).
\end{aligned}$$

These equations can be combined to give the following matrix identity:

$$\begin{bmatrix} s_{(1,1,1)} \\ s_{(2,1)} \\ s_{(3)} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} m_{(1,1,1)} \\ m_{(2,1)} \\ m_{(3)} \end{bmatrix}.$$

The 3×3 matrix appearing here is lower-triangular with ones on the main diagonal, hence is invertible. Multiplying by the inverse matrix, we find that

$$\begin{bmatrix} m_{(1,1,1)} \\ m_{(2,1)} \\ m_{(3)} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} s_{(1,1,1)} \\ s_{(2,1)} \\ s_{(3)} \end{bmatrix}.$$

This says that each monomial symmetric polynomial $m_\nu(x_1, x_2, x_3)$ is expressible as a linear combination of the Schur polynomials $s_\mu(x_1, x_2, x_3)$. Since the m_ν's form a basis of the vector space Λ_3^3, the Schur polynomials must span this space. Since $\dim(\Lambda_3^3) = p(3) = 3$, the three-element set $\{s_\mu(x_1, x_2, x_3) : \mu \vdash 3\}$ is in fact a basis of Λ_3^3.

The argument given in the example extends to the general case. The key fact is that the transition matrix from Schur polynomials to monomial symmetric polynomials is always lower-triangular with ones on the main diagonal, as shown next.

10.47. Theorem: Lower Unitriangularity of the Kostka Matrix. For all partitions λ, $K_{\lambda,\lambda} = 1$. For all partitions λ and μ, $K_{\lambda,\mu} \neq 0$ implies $\mu \trianglelefteq \lambda$ (and also $\mu \leq_{\mathrm{lex}} \lambda$, by 10.41).

Proof. The Kostka number $K_{\lambda,\lambda}$ is the number of semistandard tableaux T of shape λ and content λ. Such a tableau must contain λ_i copies of i for each $i \geq 1$. In particular, T contains λ_1 ones. Since T is semistandard, all these ones must occur in the top row, which has λ_1 boxes. So the top row of T contains all ones. For the same reason, the λ_2 twos in T must all occur in the second row, which has λ_2 boxes. Arguing similarly, we see that T must be the tableau whose ith row contains all i's, for $i \geq 1$. Thus, there is exactly one semistandard tableaux of shape λ and content λ. For example, when $\lambda = (4, 2, 2, 1)$, we must have

$$T = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 1 \\ \hline 2 & 2 \\ \cline{1-2} 3 & 3 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array}\ .$$

For the second part of the theorem, we argue by contradiction. Assume $\lambda, \mu \in \mathrm{Par}(k)$ are such that $K_{\lambda,\mu} \neq 0$ and yet $\mu \not\trianglelefteq \lambda$. Since the Kostka number is nonzero, there exists a semistandard tableau T of shape λ and content μ. Since the content of T is μ, the entries of T come from the alphabet $\{1, 2, \ldots, \ell(\mu)\}$. Since the columns of T must strictly increase, we observe that all 1's in T must occur in row 1; all 2's in T must occur in row 1 or row 2; and, in general, all j's in T must occur in the top j rows of $\mathrm{dg}(\lambda)$. Now, the assumption $\mu \not\trianglelefteq \lambda$ means that there is an $i \geq 1$ with $\mu_1 + \cdots + \mu_i > \lambda_1 + \cdots + \lambda_i$. The left side of this inequality is the total number of occurrences of the symbols $1, 2, \ldots, i$ in T. The right side of the inequality is the total number of boxes in the top i rows of T. Our preceding observation now produces the desired contradiction, since there is not enough room in the top i rows of $\mathrm{dg}(\lambda)$ to accommodate the $\mu_1 + \cdots + \mu_i$ occurrences of the symbols $1, 2, \ldots, i$ in T. $\square$

10.48. Example. Let $\lambda = (3, 2, 2)$ and $\mu = (2, 2, 2, 1)$. The Kostka number $K_{\lambda,\mu}$ is 3, as we see by listing the semistandard tableaux of shape λ and content μ:

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 & 3 \\ \cline{1-2} 3 & 4 \\ \cline{1-2}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 1 & 3 \\ \hline 2 & 2 \\ \cline{1-2} 3 & 4 \\ \cline{1-2}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 1 & 4 \\ \hline 2 & 2 \\ \cline{1-2} 3 & 3 \\ \cline{1-2}\end{array}$$

In each tableau, all occurrences of i appear in the top i rows, and we do have $\mu \trianglelefteq \lambda$.

10.49. Theorem: Schur Basis of Λ_N^k. For all $k, N \in \mathbb{N}$, the set of Schur polynomials

$$\{s_\lambda(x_1, \ldots, x_N) : \lambda \in \mathrm{Par}_N(k)\} \subseteq K[x_1, \ldots, x_N]$$

is a basis of the K-vector space Λ_N^k.

Proof. Let $p = |\mathrm{Par}_N(k)|$, and let $\mathbf{S}$ be the $p \times 1$ column vector consisting of the Schur polynomials $\{s_\lambda(x_1, \ldots, x_N) : \lambda \in \mathrm{Par}_N(k)\}$, arranged in lexicographic order. Let $\mathbf{M}$ be the $p \times 1$ column vector consisting of the monomial symmetric polynomials $\{m_\mu(x_1, \ldots, x_N) : \mu \in \mathrm{Par}_N(k)\}$, also arranged in lexicographic order. Finally, let $\mathbf{K}$ be the $p \times p$ matrix, with rows and columns indexed by elements of $\mathrm{Par}_N(k)$ in lexicographic order, such that the entry in row λ and column μ is the Kostka number $K_{\lambda,\mu}$. Now 10.35 says that, for every $\lambda \in \mathrm{Par}_N(k)$,

$$s_\lambda(x_1, \ldots, x_N) = \sum_{\mu \in \mathrm{Par}_N(k)} K_{\lambda,\mu} m_\mu(x_1, \ldots, x_N).$$

These scalar equations are equivalent to the matrix-vector equation $\mathbf{S} = \mathbf{KM}$. Moreover, 10.47 asserts that $\mathbf{K}$ is a lower-triangular matrix of integers with 1's on the main diagonal. So $\mathbf{K}$ has an inverse matrix (whose entries are also integers, since $\det(\mathbf{K}) = 1$).

Multiplying on the left by this inverse matrix, we get $\mathbf{M} = \mathbf{K}^{-1}\mathbf{S}$. This equation means that every m_μ is a linear combination of Schur polynomials. Since the m_μ's generate Λ_N^k, the Schur polynomials must also generate this space. Linear independence follows automatically since the number of Schur polynomials in the proposed basis (namely p) equals the dimension of the vector space, by 10.29. $\square$

10.50. Remark. The matrix $\mathbf{K}$ occurring in this proof is called a *Kostka matrix.* The entries of the *inverse Kostka matrix* $\mathbf{K}^{-1}$ tell us how to expand monomial symmetric polynomials in terms of Schur polynomials. As seen in the 3×3 example, these entries are integers which can be negative. It is natural to ask for a combinatorial interpretation for these matrix entries in terms of suitable collections of *signed* objects. One such interpretation will be discussed in §11.15 below.

10.51. Remark. If $\lambda \in \mathrm{Par}(k)$ has more than N parts, then $s_\lambda(x_1, \ldots, x_N) = 0$. This follows since there are not enough letters available in the alphabet to fill the first column of $\mathrm{dg}(\lambda)$ with a strictly increasing sequence. So there are no semistandard tableaux of this shape on this alphabet.

10.9 Tableau Insertion

We have seen that the Kostka numbers give the coefficients of the monomial expansion of Schur polynomials. Remarkably, the Kostka numbers also relate Schur polynomials to the elementary and complete homogeneous symmetric polynomials. This fact will be a consequence of the *Pieri rules*, which tell us how to rewrite products of the form $s_\mu e_k$ and $s_\mu h_k$ as linear combinations of Schur polynomials.

To develop these results, we need a fundamental combinatorial construction on tableaux called *tableau insertion.* Given a semistandard tableau T of shape μ and a letter x, we wish to build a new semistandard tableau by "inserting x into T." The following recursive procedure allows us to do this.

10.52. Definition: Tableau Insertion Algorithm. Let T be a semistandard tableau of straight shape μ over the ordered alphabet X, and let $x \in X$. We define a new tableau, denoted $T \leftarrow x$, by the following procedure.

1. If $\mu = 0$, so that T is the empty tableau, then $T \leftarrow x$ is the tableau of shape (1) whose sole entry is x.
2. Otherwise, let $y_1 \leq y_2 \leq \cdots \leq y_t$ be the entries in the top row of T.
   - 2a. If $y_t \leq x$, then $T \leftarrow x$ is the tableau of shape $(\mu_1 + 1, \mu_2, \ldots)$ obtained by placing a new box containing x at the right end of the top row of T.
   - 2b. Otherwise, choose the minimal $i \in \{1, 2, \ldots, t\}$ such that $x < y_i$. Let T' be the semistandard tableaux consisting of all rows of T after the first one. To form $T \leftarrow x$, first replace y_i by x in the top row of T. Then replace T' by $T' \leftarrow y_i$, which is computed recursively by the same algorithm.

If step 2b occurs, we say that x has *bumped* y_i out of row 1. In turn, y_i may bump an element from row 2 to row 3, and so on.

This recursively defined insertion algorithm always terminates, since the number of times we execute step 2b is at most $\ell(\mu)$, which is finite. We must also prove that the algorithm always produces a tableau that is *semistandard* and *of partition shape*. We will prove these facts after considering some examples.

10.53. Example. Let us compute $T \leftarrow 3$, where

$$T = \begin{ytableau} 1 & 1 & 2 & 3 & 4 & 4 & 6 \\ 2 & 4 & 5 & 6 & 6 \\ 3 & 5 & 7 & 8 \\ 4 & 6 \end{ytableau}\,.$$

We scan the top row of T from left to right, looking for the first entry *strictly larger* than 3. This entry is the 4 in the fifth box. In step 2b, the 3 bumps the 4 into the second row. The current situation looks like this:

$$\begin{array}{ll} \begin{ytableau} 1 & 1 & 2 & 3 & \underline{3} & 4 & 6 \\ 2 & 4 & 5 & 6 & 6 \\ 3 & 5 & 7 & 8 \\ 4 & 6 \end{ytableau} & \begin{array}{l} \\ \leftarrow 4 \\ \\ \\ \end{array} \end{array}$$

Now we scan the second row from left to right, looking for the first entry strictly larger than 4. It is the 5, so the 4 bumps the 5 into the third row:

$$\begin{array}{ll} \begin{ytableau} 1 & 1 & 2 & 3 & \underline{3} & 4 & 6 \\ 2 & 4 & \underline{4} & 6 & 6 \\ 3 & 5 & 7 & 8 \\ 4 & 6 \end{ytableau} & \begin{array}{l} \\ \\ \leftarrow 5 \\ \\ \end{array} \end{array}$$

Next, the 5 bumps the 7 into the fourth row:

$$\begin{array}{ll} \begin{ytableau} 1 & 1 & 2 & 3 & \underline{3} & 4 & 6 \\ 2 & 4 & \underline{4} & 6 & 6 \\ 3 & 5 & \underline{5} & 8 \\ 4 & 6 \end{ytableau} & \begin{array}{l} \\ \\ \\ \leftarrow 7 \end{array} \end{array}$$

Now, everything in the fourth row is weakly smaller than 7. So, as directed by step 2a, we insert 7 at the end of this row. The final tableau is therefore

$$T \leftarrow 3 \quad = \quad \begin{ytableau} 1 & 1 & 2 & 3 & \underline{3} & 4 & 6 \\ 2 & 4 & \underline{4} & 6 & 6 \\ 3 & 5 & \underline{5} & 8 \\ 4 & 6 & \underline{7} \end{ytableau}\,.$$

We have underlined the entries of $T \leftarrow 3$ that were affected by the insertion process. These entries are the starting value $x = 3$ together with those entries that got bumped during the insertion. Call these entries the *bumping sequence*; in this example, the bumping sequence is $(3, 4, 5, 7)$. The sequence of boxes occupied by the bumping sequence is called the *bumping path*. The lowest box in the bumping path is called the *new box*. It is the only box in $T \leftarrow 3$ that was not present in the original diagram for T.

For a simpler example of tableau insertion, note that

$$T \leftarrow 6 \quad = \quad \begin{ytableau} 1 & 1 & 2 & 3 & 4 & 4 & 6 & \underline{6} \\ 2 & 4 & 5 & 6 & 6 \\ 3 & 5 & 7 & 8 \\ 4 & 6 \end{ytableau}\,.$$

The reader should check that

$$T \leftarrow 1 \quad = \quad \begin{ytableau} 1 & 1 & \underline{1} & 3 & 4 & 4 & 6 \\ 2 & \underline{2} & 5 & 6 & 6 \\ 3 & \underline{4} & 7 & 8 \\ 4 & \underline{5} \\ \underline{6} \end{ytableau}\,, \qquad T \leftarrow 0 \quad = \quad \begin{ytableau} \underline{0} & 1 & 2 & 3 & 4 & 4 & 6 \\ \underline{1} & 4 & 5 & 6 & 6 \\ \underline{2} & 5 & 7 & 8 \\ \underline{3} & 6 \\ \underline{4} \end{ytableau}\,.$$

To prove that $T \leftarrow x$ is always semistandard of partition shape, we need the following result.

10.54. Theorem: Bumping Sequence and Bumping Path. Given a semistandard tableau T and element x, let $(x_1, x_2, \ldots, x_k)$ be the bumping sequence and let $((1, j_1), (2, j_2), \ldots, (k, j_k))$ be the bumping path arising in the computation of $T \leftarrow x$. Then $x = x_1 < x_2 < \cdots < x_k$ and $j_1 \geq j_2 \geq \cdots \geq j_k > 0$. (So the bumping sequence *strictly increases* and the bumping path *moves weakly left* as it goes down.)

Proof. By definition of the bumping sequence, $x = x_1$ and x_i bumps x_{i+1} from row i into row $i+1$, for $1 \leq i < k$. By definition of bumping, x_i bumps an entry strictly larger than itself, so $x_i < x_{i+1}$ for all $i < k$. Next, consider what happens to x_{i+1} when it is bumped out of row i. Before being bumped, x_{i+1} occupied the cell (i, j_i). After being bumped, x_{i+1} will occupy the cell $(i+1, j_{i+1})$, which is either an existing cell in row $i+1$ of T, or a new cell at the end of this row. Consider the cell $(i+1, j_i)$ directly below (i, j_i). If this cell is outside the shape of T, the previous observation shows that $(i+1, j_{i+1})$ must be located weakly left of this cell, so that $j_{i+1} \leq j_i$. On the other hand, if $(i+1, j_i)$ is part of T and contains some value z, then $x_{i+1} < z$ because T is semistandard. Now, x_{i+1} bumps the *leftmost* entry in row $i+1$ that is strictly larger than x_{i+1}. Since z is such an entry, x_{i+1} bumps z or some entry to the left of z. In either case, we again have $j_{i+1} \leq j_i$. □

10.55. Theorem: Output of a Tableau Insertion. If T is a semistandard tableau of shape μ, then $T \leftarrow x$ is a semistandard tableau whose shape is a partition obtained by adding one new box to $\mathrm{dg}(\mu)$.

Proof. Let us first show that the shape of $T \leftarrow x$ is a partition diagram. This shape is obtained from $\mathrm{dg}(\mu)$ by adding one new box (the last box in the bumping path). If this new box is in the top row, then the resulting shape is certainly a partition diagram (namely, $\mathrm{dg}((\mu_1 + 1, \mu_2, \ldots))$). Suppose the new box is in row $i > 1$. Then 10.54 shows that the new box is located weakly left of a box in the previous row that belongs to $\mathrm{dg}(\mu)$. This implies that $\mu_i < \mu_{i-1}$, so adding the new box to row i will still give a partition diagram.

Next we prove that each time an entry of T is bumped during the insertion of x, the resulting tableau is still semistandard. Suppose, at some stage in the insertion process, that an element y bumps z out of the following configuration:

$$\begin{array}{|c|c|c|} \cline{2-2} \multicolumn{1}{c|}{} & a & \multicolumn{1}{c}{} \\ \hline b & z & c \\ \hline \multicolumn{1}{c|}{} & d & \multicolumn{1}{c}{} \\ \cline{2-2} \end{array}.$$

(Some of the boxes containing a, b, c, d may be absent, in which case the following argument should be modified appropriately.) The original configuration is part of a semistandard tableau, so $b \leq z \leq c$ and $a < z < d$. Because y bumps z, z must be the first entry strictly larger than y in its row. This means that $b \leq y < z \leq c$, so replacing z by y will still leave a weakly increasing row. Does the column containing z still strictly increase after the bumping? On one hand, $y < d$, since $y < z < d$. On the other hand, if the box containing a exists (i.e., if z is below the top row), then y was the element bumped out of a's row. Since the bumping path moves weakly left, the original location of y must have been weakly right of z in the row above z. If y was directly above z, then a must have bumped y, and so $a < y$ by definition of bumping. Otherwise, y was located strictly to the right of a before y was bumped, so $a \leq y$. We cannot have $a = y$ in this situation, since otherwise a (or something to its left) would have been bumped instead of y. Thus, $a < y$ in all cases.

Finally, consider what happens at the end of the insertion process, when an element w is inserted in a new box at the end of a (possibly empty) row. This only happens when w

weakly exceeds all entries in its row, so the row containing w is weakly increasing. There is no cell below w in this case. Repeating the argument at the end of the last paragraph, we see that w is strictly greater than the entry directly above it (if any). This completes the proof that $T \leftarrow x$ is semistandard. □

10.10 Reverse Insertion

Given the output $T \leftarrow x$ of a tableau insertion operation, it is generally not possible to determine what T and x were. However, if we also know the location of the new box created by this insertion, then we can recover T and x. More generally, we can start with any semistandard tableau S and any "corner box" of S, and "uninsert" the value in this box to obtain a semistandard tableau T and value x such that $S = T \leftarrow x$. (Here we do not assume in advance that S has the form $T \leftarrow x$.) This process is called *reverse tableau insertion.* Before giving the general definition, we consider some examples.

10.56. Example. Consider the following semistandard tableau:

$$S \quad = \quad \begin{array}{|c|c|c|c|c|}\hline 1&1&2&2&4\\\hline 2&2&3&5\\\cline{1-4} 3&4&4&6\\\cline{1-4} 4&5\\\cline{1-2} 6&6\\\cline{1-2} 7&8\\\cline{1-2}\end{array}$$

There are three corner boxes whose removal from S will still leave a partition diagram; they are the boxes at the end of the first, third, and sixth rows. Removing the corner box in the top row, we evidently will have $S = T_1 \leftarrow 4$, where

$$T_1 \quad = \quad \begin{array}{|c|c|c|c|}\hline 1&1&2&2\\\hline 2&2&3&5\\\hline 3&4&4&6\\\hline 4&5\\\cline{1-2} 6&6\\\cline{1-2} 7&8\\\cline{1-2}\end{array}$$

Suppose instead that we remove the 6 at the end of the third row of S. Reversing the bumping process, we see that 6 must have been bumped into the third row from the second row. What element bumped it? In this case, it is the 5 in the second row. In turn, the 5 must have originally resided in the first row, before being bumped into the second row by the 4. In summary, we have $S = T_2 \leftarrow \underline{4}$, where

$$T_2 \quad = \quad \begin{array}{|c|c|c|c|c|}\hline 1&1&2&2&\underline{5}\\\hline 2&2&3&\underline{6}\\\cline{1-4} 3&4&4\\\cline{1-3} 4&5\\\cline{1-2} 6&6\\\cline{1-2} 7&8\\\cline{1-2}\end{array}$$

(Here we have underlined the entries in the *reverse bumping sequence*, which occupy boxes in the *reverse bumping path.*) Finally, consider what happens when we uninsert the 8 at the end of the last row of S. The 8 was bumped to its current location by one of the 6's in the previous row; it must have been bumped by the rightmost 6, lest semistandardness be violated. Next, the 6 was bumped by the 5 in row 4; the 5 was bumped by the rightmost 4 in row 3; and so on. In general, to determine which element in row i bumped some value z into row $i+1$, we look for the *rightmost* entry in row i that is *strictly less* than z. Continuing in this way, we discover that $S = T_3 \leftarrow \underline{2}$, where

$$T_3 \quad = \quad \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 2 & \underline{3} & 4 \\ \hline 2 & 2 & \underline{4} & 5 \\ \cline{1-4} 3 & 4 & \underline{5} & 6 \\ \cline{1-4} 4 & \underline{6} \\ \cline{1-2} 6 & \underline{8} \\ \cline{1-2} 7 \\ \cline{1-1} \end{array}$$

With these examples in hand, we are ready to give the general definition of reverse tableau insertion.

10.57. Definition: Reverse Tableau Insertion. Suppose S is a semistandard tableau of shape ν. A *corner box* of ν is a box $(i,j) \in \mathrm{dg}(\nu)$ such that $\mathrm{dg}(\nu) \sim \{(i,j)\}$ is still the diagram of some partition μ. Given S and a corner box (i,j) of ν, we define a tableau T and a value x as follows. We will construct a *reverse bumping sequence* $(x_i, x_{i-1}, \ldots, x_1)$ and a *reverse bumping path* $((i,j_i),(i-1,j_{i-1}),\ldots,(1,j_1))$ as follows.

1. Set $j_i = j$ and $x_i = S((i,j))$, which is the value of S in the given corner box.
2. Once x_k and j_k have been found, for some $i \geq k > 1$, scan row $k-1$ of S for the rightmost entry that is strictly less than x_k. Define x_{k-1} to be this entry, and let j_{k-1} be the column in which this entry occurs.
3. At the end, let $x = x_1$, and let T be the tableau obtained by erasing box (i, j_i) from S and replacing the contents of box $(k-1, j_{k-1})$ by x_k for $i \geq k > 1$.

The next results will show that reverse insertion really is the two-sided inverse of ordinary insertion (given knowledge of the location of the new box).

10.58. Theorem: Properties of Reverse Insertion. Suppose we perform reverse tableau insertion on S and (i,j) to obtain T and x as in 10.57. (a) The reverse bumping sequence satisfies $x_i > x_{i-1} > \cdots > x_1 = x$. (b) The reverse bumping path satisfies $j_i \leq j_{i-1} \leq \cdots \leq j_1$. (c) T is a semistandard tableau of shape μ. (d) $(T \leftarrow x) = S$.

Proof. Part (a) follows from the definition of x_{k-1} in 10.57. Note that there does exist an entry in row $k-1$ strictly less than x_k, since the entry directly above x_k (in cell $(k-1, j_k)$ of S) is such an entry. This observation also shows that the rightmost entry strictly less than x_k in row $k-1$ occurs in column j_k or later, proving (b). Part (c) follows from (a) and (b) by an argument similar to that given in 10.55; we let the reader fill in the details. For part (d), consider the bumping sequence $(x'_1, x'_2, \ldots)$ and bumping path $((1,j'_1),(2,j'_2),\ldots)$ for the forward insertion $T \leftarrow x$. We have $x'_1 = x = x_1$ by definition. Recall that $x_1 = S((1,j_1))$ was the rightmost entry in row 1 of S that was strictly less than x_2, and $T((1,j_1)) = x_2$ by definition of T. All other entries in row 1 are the same in S and T. So $T((1,j_1)) = x_2$ will be the leftmost entry of row 1 of T strictly larger than x_1. So, in the insertion $T \leftarrow x$, x_1 bumps x_2 out of cell $(1,j_1)$. In particular, $j'_1 = j_1$ and $x'_2 = x_2$. Repeating this argument in each successive row, we see by induction that $x'_k = x_k$ and $j'_k = j_k$ for all k. At the end of the insertion, we have recovered the starting tableau S. $\square$

10.59. Theorem: Reversing Insertion. Suppose $S = (T \leftarrow x)$ for some semistandard tableau T and value x. Let (i,j) be the new box created by this insertion. If we perform reverse insertion on S starting with box (i,j), we will obtain the original T and x.

Proof. This can be proved by induction, showing step by step that the forward and reverse bumping paths and bumping sequences are the same. The argument is similar to part (d) of 10.58, so we leave it as an exercise for the reader. □

The next theorem summarizes the results of the last two sections.

10.60. Theorem: Invertibility of Tableau Insertion. Let X be an ordered set, and let μ be a fixed partition. Let $P(\mu)$ be the set of all partitions that can be obtained from μ by adding a single box at the end of some row. There exist mutually inverse bijections

$$I : \mathrm{SSYT}_X(\mu) \times X \to \bigcup_{\nu \in P(\mu)} \mathrm{SSYT}_X(\nu), \qquad R : \bigcup_{\nu \in P(\mu)} \mathrm{SSYT}_X(\nu) \to \mathrm{SSYT}_X(\mu) \times X$$

given by $I(T, x) = T \leftarrow x$ and $R(S) =$ the result of applying reverse tableau insertion to S starting at the unique box of S not in μ.

Proof. We have seen that I and R are well-defined functions mapping into the stated codomains. We see from 10.58(d) that $I \circ R$ is the identity map on $\bigcup_{\nu \in P(\mu)} \mathrm{SSYT}_X(\nu)$, while 10.59 says that $R \circ I$ is the identity map on $\mathrm{SSYT}_X(\mu) \times X$. Hence I and R are bijections. □

Let us take $X = \{1, 2, \ldots, N\}$ in 10.60. We can regard X as a weighted set with $\mathrm{wt}(i) = x_i$. The generating function for this weighted set is $x_1 + x_2 + \cdots + x_N = h_1(x_1, \ldots, x_N) = s_{(1)}(x_1, \ldots, x_N) = e_1(x_1, \ldots, x_N)$. Note that the content monomial $x^{c(T \leftarrow j)}$ is $x^{c(T)} x_j$, since $T \leftarrow j$ contains all the entries of T together with one new entry equal to j. This means that $\mathrm{wt}(I(T, j)) = \mathrm{wt}(T)\,\mathrm{wt}(j)$, so that the bijection I in the theorem is *weight-preserving*. Using the product rule for weighted sets and the definition of Schur polynomials, the generating function for the domain of I is $s_\mu(x_1, \ldots, x_N) h_1(x_1, \ldots, x_N)$. Using the sum rule for weighted sets, the generating function for the codomain of I is $\sum_{\nu \in P(\mu)} s_\nu(x_1, \ldots, x_N)$. To summarize, our tableau insertion algorithms have furnished a combinatorial proof of the following *multiplication rule*:

$$s_\mu h_1 = s_\mu e_1 = s_\mu s_{(1)} = \sum_{\nu \in P(\mu)} s_\nu,$$

where we sum over all partitions ν obtained by adding *one* corner box to μ. We have discovered the simplest instance of the *Pieri rules* mentioned at the beginning of §10.9.

10.11 Bumping Comparison Theorem

We now extend the analysis of the previous section to prove the general Pieri rules for expanding $s_\mu h_k$ and $s_\mu e_k$ in terms of Schur polynomials. The key idea is to see what happens when we successively insert k weakly increasing numbers (or k strictly decreasing numbers) into a semistandard tableau by repeated tableau insertion. We begin with some examples to build intuition.

10.61. Example. Consider the semistandard tableau

$$T \quad = \quad \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 2 & 3 & 4 \\ \hline 2 & 3 & 3 & 4 \\ \cline{1-4} 3 & 4 & 5 & 6 \\ \cline{1-4} 5 & 5 & 6 & 7 \\ \cline{1-4} 6 \\ \cline{1-1} \end{array}$$

Let us compute the tableaux that result by successively inserting $2, 3, 3, 5$ into T:

$$T_1 = T \leftarrow 2 \quad = \quad \begin{array}{ccccc} 1 & 1 & 2 & \underline{2} & 4 \\ 2 & 3 & 3 & \underline{3} & \\ 3 & 4 & \underline{4} & 6 & \\ 5 & 5 & \underline{5} & 7 & \\ 6 & \underline{6} & & & \end{array} \quad ; \qquad T_2 = T_1 \leftarrow 3 \quad = \quad \begin{array}{ccccc} 1 & 1 & 2 & 2 & \underline{3} \\ 2 & 3 & 3 & 3 & \underline{4} \\ 3 & 4 & 4 & 6 & \\ 5 & 5 & 5 & 7 & \\ 6 & 6 & & & \end{array} \quad ;$$

$$T_3 = T_2 \leftarrow 3 \quad = \quad \begin{array}{cccccc} 1 & 1 & 2 & 2 & 3 & \underline{3} \\ 2 & 3 & 3 & 3 & 4 & \\ 3 & 4 & 4 & 6 & & \\ 5 & 5 & 5 & 7 & & \\ 6 & 6 & & & & \end{array} \quad ; \qquad T_4 = T_3 \leftarrow 5 \quad = \quad \begin{array}{ccccccc} 1 & 1 & 2 & 2 & 3 & 3 & \underline{5} \\ 2 & 3 & 3 & 3 & 4 & & \\ 3 & 4 & 4 & 6 & & & \\ 5 & 5 & 5 & 7 & & & \\ 6 & 6 & & & & & \end{array} \quad .$$

Consider the skew shape consisting of the four cells in T_4 that are not in T_1, which are marked by asterisks in the following picture:

$$\begin{array}{ccccccc} \square & \square & \square & \square & \square & * & * \\ \square & \square & \square & \square & * & & \\ \square & \square & \square & \square & & & \\ \square & \square & \square & \square & & & \\ \square & * & & & & & \end{array} \quad .$$

Observe that this skew shape is a *horizontal strip* of size 4. Next, compare the bumping paths in the successive insertions of $2, 3, 3, 5$. We see that each path lies *strictly right* of the previous bumping path and ends with a new box in a *weakly higher* row.

Now return to the original tableau T, and consider the insertion of a strictly decreasing sequence $5, 4, 2, 1$. We obtain the following tableaux:

$$S_1 = T \leftarrow 5 \quad = \quad \begin{array}{cccccc} 1 & 1 & 2 & 3 & 4 & \underline{5} \\ 2 & 3 & 3 & 4 & & \\ 3 & 4 & 5 & 6 & & \\ 5 & 5 & 6 & 7 & & \\ 6 & & & & & \end{array} \quad ; \qquad S_2 = S_1 \leftarrow 4 \quad = \quad \begin{array}{cccccc} 1 & 1 & 2 & 3 & 4 & \underline{4} \\ 2 & 3 & 3 & 4 & \underline{5} & \\ 3 & 4 & 5 & 6 & & \\ 5 & 5 & 6 & 7 & & \\ 6 & & & & & \end{array} \quad ;$$

$$S_3 = S_2 \leftarrow 2 \quad = \quad \begin{array}{cccccc} 1 & 1 & 2 & \underline{2} & 4 & 4 \\ 2 & 3 & 3 & \underline{3} & 5 & \\ 3 & 4 & \underline{4} & 6 & & \\ 5 & 5 & \underline{5} & 7 & & \\ 6 & \underline{6} & & & & \end{array} \quad ; \qquad S_4 = S_3 \leftarrow 1 \quad = \quad \begin{array}{cccccc} 1 & 1 & \underline{1} & 2 & 4 & 4 \\ 2 & \underline{2} & 3 & 3 & 5 & \\ 3 & \underline{3} & 4 & 6 & & \\ \underline{4} & 5 & 5 & 7 & & \\ \underline{5} & 6 & & & & \\ \underline{6} & & & & & \end{array} \quad .$$

This time, each successive bumping path is *weakly left* of the previous one and ends in a *strictly lower* row. Accordingly, the new boxes in S_4 form a *vertical strip*:

$$\begin{array}{cccccc} \square & \square & \square & \square & \square & * \\ \square & \square & \square & \square & * & \\ \square & \square & \square & \square & & \\ \square & \square & \square & \square & & \\ \square & * & & & & \\ * & & & & & \end{array} \quad .$$

We now show that the observations in this example hold in general.

10.62. Bumping Comparison Theorem. Let T be a semistandard tableau using letters in X, and let $x, y \in X$. Let the new box in $T \leftarrow x$ be (i, j), and let the new box in $(T \leftarrow x) \leftarrow y$ be (r, s). (a) $x \leq y$ iff $i \geq r$ and $j < s$; (b) $x > y$ iff $i < r$ and $j \geq s$.

Proof. It suffices to prove the forward implications, since exactly one of $x \leq y$ or $x > y$ is true. Let the bumping path for the insertion of x be $((1, j_1), (2, j_2), \ldots, (i, j_i))$ (where $j_i = j$), and let the bumping sequence be $(x = x_1, x_2, \ldots, x_i)$. Let the bumping path for the insertion of y be $((1, s_1), (2, s_2), \ldots, (r, s_r))$ (where $s_r = s$), and let the bumping sequence be $(y = y_1, y_2, \ldots, y_r)$.

Assume $x \leq y$. We prove the following statement by induction: for all k with $1 \leq k \leq r$, we have $i \geq k$ and $x_k \leq y_k$ and $j_k < s_k$. When $k = 1$, we have $i \geq 1$ and $x_1 \leq y_1$ (by assumption). Note that x_1 appears in box $(1, j_1)$ of $T \leftarrow x$. We cannot have $s_1 \leq j_1$, for this would mean that y_1 bumps an entry weakly left of $(1, j_1)$, and this entry is at most $x_1 \leq y_1$, contrary to the definition of bumping. So $j_1 < s_1$. Now consider the induction step. Suppose $k < r$ and the induction hypothesis holds for k; does it hold for $k+1$? Since $k < r$, y_k must have bumped something from position (k, s_k) into the next row. Since $j_k < s_k$, x_k must also have bumped something out of row k, proving that $i \geq k+1$. The object bumped by x_k, namely x_{k+1}, appears to the left of the object bumped by y_k, namely y_{k+1}, in the same row of a semistandard tableau. Therefore, $x_{k+1} \leq y_{k+1}$. Now we can repeat the argument used for the first row to see that $j_{k+1} < s_{k+1}$. Now that the induction is complete, take $k = r$ to see that $i \geq r$ and $j = j_i \leq j_r < s_r = s$ (the first inequality holding since the bumping path for $T \leftarrow x$ moves weakly left as we go down).

Next, assume $x > y$. This time we prove the following by induction: for all k with $1 \leq k \leq i$, we have $r > k$ and $x_k > y_k$ and $j_k \geq s_k$. When $k = 1$, we have $x_1 = x > y = y_1$. Since x appears somewhere in the first row of $T \leftarrow x$, y will necessarily bump something into the second row, so $r > 1$. In fact, the thing bumped by y occurs weakly left of the position $(1, j_1)$ occupied by x, so $s_1 \leq j_1$. For the induction step, assume the induction hypothesis is known for some $k < i$, and try to prove it for $k+1$. Since $k < i$ and $k < r$, both x_k and y_k must bump elements out of row k into row $k+1$. The element y_{k+1} bumped by y_k occurs in column s_k, which is weakly left of the cell (k, j_k) occupied by x_k in $T \leftarrow x$. Therefore, $y_{k+1} \leq x_k$, which is in turn *strictly* less than x_{k+1}, the original occupant of cell (k, j_k) in T. So $x_{k+1} > y_{k+1}$. Repeating the argument used in the first row for row $k+1$, we now see that y_{k+1} must bump something in row $k+1$ into row $k+2$ (so that $r > k+1$), and $s_{k+1} \leq j_{k+1}$. This completes the induction. Taking $k = i$, we finally conclude that $r > i$ and $j = j_i \geq s_i \geq s_r = s$. □

10.12 Pieri Rules

Iteration of the bumping comparison theorem proves the following result.

10.63. Theorem: Inserting a Monotone Sequence into a Tableau. Let T be a semistandard tableau of shape μ, and let S be the semistandard tableau obtained from T by insertion of $z_1, z_2, \ldots, z_k$ (in this order); we write $S = (T \leftarrow z_1 z_2 \cdots z_k)$ in this situation. Let ν be the shape of S.
(a) If $z_1 \leq z_2 \leq \cdots \leq z_k$, then ν/μ is a horizontal strip of size k.
(b) If $z_1 > z_2 > \cdots > z_k$, then ν/μ is a vertical strip of size k.

Since tableau insertion is reversible given the location of the new box, we can also reverse the insertion of a monotone sequence, in the following sense.

10.64. Theorem: Reverse Insertion of a Monotone Sequence. Suppose μ and ν are given partitions, and S is any semistandard tableau of shape ν.
(a) If ν/μ is a horizontal strip of size k, then there exists a unique sequence $z_1 \leq z_2 \leq \cdots \leq$

z_k and a unique semistandard tableau T of shape μ such that $S = (T \leftarrow z_1 z_2 \cdots z_k)$.
(b) If ν/μ is a vertical strip of size k, then there exists a unique sequence $z_1 > z_2 > \cdots > z_k$ and a unique semistandard tableau T of shape μ such that $S = (T \leftarrow z_1 z_2 \cdots z_k)$.

Proof. To prove the existence of T and the z_i's in part (a), we repeatedly perform reverse tableau insertion, erasing each cell in the horizontal strip ν/μ *from right to left.* This produces a sequence of elements $z_k, \ldots, z_2, z_1$ and a semistandard tableau T of shape μ such that $(T \leftarrow z_1 z_2 \cdots z_k) = S$. Keeping in mind the relative locations of the new boxes created by z_i and z_{i+1}, we see from the bumping comparison theorem that $z_i \leq z_{i+1}$ for all i.

As for uniqueness, suppose T' and $z_1' \leq z_2' \leq \cdots \leq z_k'$ also satisfy $S = (T' \leftarrow z_1' z_2' \cdots z_k')$. Since $z_1' \leq z_2' \leq \cdots \leq z_k'$, the bumping comparison theorem shows that the insertion of the z_i''s creates the new boxes of ν/μ in order from left to right, just as the insertion of the z_i's does. Write $T_0 = T$, $T_i = (T \leftarrow z_1 z_2 \cdots z_i)$, $T_0' = T'$, and $T_i' = (T' \leftarrow z_1' z_2' \cdots z_i')$. Since reverse tableau insertion produces a unique answer given the location of the new box, one now sees by reverse induction on i that $T_i = T_i'$ and $z_i = z_i'$ for $k \geq i \geq 0$.

Part (b) is proved similarly. $\square$

10.65. Theorem: Pieri Rules. Given an integer partition μ and positive integer k, let $H_k(\mu)$ consist of all partitions ν such that ν/μ is a horizontal strip of size k, and let $V_k(\mu)$ consist of all partitions ν such that ν/μ is a vertical strip of size k. For every ordered set X, there are weight-preserving bijections

$$F : \mathrm{SSYT}_X(\mu) \times \mathrm{SSYT}_X((k)) \to \bigcup_{\nu \in H_k(\mu)} \mathrm{SSYT}_X(\nu);$$

$$G : \mathrm{SSYT}_X(\mu) \times \mathrm{SSYT}_X((1^k)) \to \bigcup_{\nu \in V_k(\mu)} \mathrm{SSYT}_X(\nu).$$

Consequently, we have the *Pieri rules* in Λ_N:

$$s_\mu h_k = \sum_{\nu \in H_k(\mu)} s_\nu; \qquad s_\mu e_k = \sum_{\nu \in V_k(\mu)} s_\nu.$$

Proof. Recall that a semistandard tableau of shape (k) can be identified with a weakly increasing sequence $z_1 \leq z_2 \leq \cdots \leq z_k$ of elements of X. So, we can define $F(T, z_1 z_2 \ldots z_k) = (T \leftarrow z_1 z_2 \cdots z_k)$. By 10.63, F does map into the stated codomain. Then 10.64 shows that F is a bijection. Moreover, F is weight-preserving, since the content monomial of $(T \leftarrow z_1 z_2 \cdots z_k)$ is $x^{c(T)} x_{z_1} \cdots x_{z_k}$.

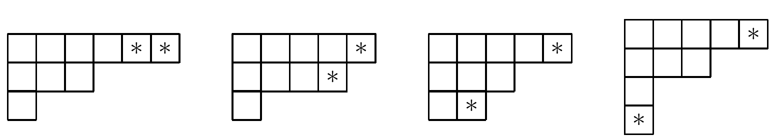
Similarly, a semistandard tableau of shape (1^k) can be identified with a strictly increasing sequence $y_1 < y_2 < \cdots < y_k$. Reversing this gives a strictly decreasing sequence. So we define $G(T, y_1 y_2 \ldots y_k) = (T \leftarrow y_k \cdots y_2 y_1)$. As above, 10.63 and 10.64 show that G is a well-defined bijection.

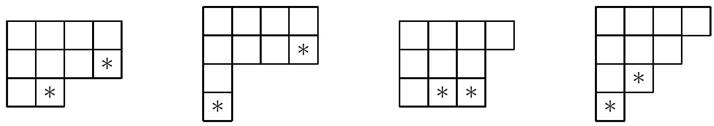
Finally, the Pieri rules follow by passing from weighted sets to generating functions, keeping in mind the sum and product rules for weighted sets, and using $h_k = s_{(k)}$ and $e_k = s_{(1^k)}$. $\square$

10.66. Example. We have

$$s_{(4,3,1)} h_2 = s_{(6,3,1)} + s_{(5,4,1)} + s_{(5,3,2)} + s_{(5,3,1,1)} + s_{(4,4,2)} + s_{(4,4,1,1)} + s_{(4,3,3)} + s_{(4,3,2,1)},$$

as we see by drawing the following pictures:

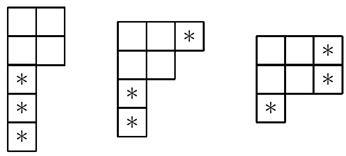




Similarly, we find that

$$s_{(2,2)}e_3 = s_{(2,2,1,1,1)} + s_{(3,2,1,1)} + s_{(3,3,1)}$$

by adding vertical strips to $\mathrm{dg}((2,2))$ as shown here:



10.13 Schur Expansion of h_α

Iteration of the Pieri rules lets us compute the Schur expansions of products of the form $s_\mu h_{\alpha_1} h_{\alpha_2} \cdots h_{\alpha_s}$, or $s_\mu e_{\alpha_1} e_{\alpha_2} \cdots e_{\alpha_s}$, or even "mixed products" involving both h's and e's. Taking $\mu = 0$, so that $s_\mu = 1$, we obtain in particular the expansions of h_α and e_α into sums of Schur polynomials. As we will see, examination of these expansions will lead to another occurrence of the Kostka matrix (cf. 10.50).

10.67. Example. Let us use the Pieri rule to find the Schur expansion of $h_{(2,1,3)} = h_2 h_1 h_3$. To start, recall that $h_2 = s_{(2)}$. Adding one box to $\mathrm{dg}((2))$ in all possible ways gives

$$h_2 h_1 = s_{(3)} + s_{(2,1)}.$$

Now we add a horizontal strip of size 3 in all possible ways to get

$$\begin{aligned} h_2 h_1 h_3 &= s_{(3)} h_3 + s_{(2,1)} h_3 \\ &= [s_{(6)} + s_{(5,1)} + s_{(4,2)} + s_{(3,3)}] + [s_{(5,1)} + s_{(4,2)} + s_{(4,1,1)} + s_{(3,2,1)}] \\ &= s_{(6)} + 2s_{(5,1)} + 2s_{(4,2)} + s_{(4,1,1)} + s_{(3,3)} + s_{(3,2,1)}. \end{aligned}$$

Observe that the Schur polynomials $s_{(5,1)}$ and $s_{(4,2)}$ each occurred twice in the final expansion. Now, consider the computation of $h_{(2,3,1)} = h_2 h_3 h_1$. Since multiplication of polynomials is commutative, this symmetric polynomial must be the same as $h_{(2,1,3)}$. But the computations with the Pieri rule involve different intermediate objects. We initially calculate

$$h_2 h_3 = s_{(2)} h_3 = s_{(5)} + s_{(4,1)} + s_{(3,2)}.$$

Continuing by multiplying by h_1 gives

$$\begin{aligned} h_2 h_3 h_1 &= s_{(5)} h_1 + s_{(4,1)} h_1 + s_{(3,2)} h_1 \\ &= [s_{(6)} + s_{(5,1)}] + [s_{(5,1)} + s_{(4,2)} + s_{(4,1,1)}] + [s_{(4,2)} + s_{(3,3)} + s_{(3,2,1)}], \end{aligned}$$

which is the same as the previous answer after collecting terms. As an exercise, the reader is invited to compute $h_{(3,2,1)} = h_3 h_2 h_1$ and verify that the final answer is again the same.

10.68. Example. We have seen that a given Schur polynomial may appear several times in the Schur expansion of h_α. Is there some way to find the coefficient of a particular Schur polynomial in this expansion, without writing down all the shapes generated by iteration of the Pieri rule? To answer this question, consider the problem of finding the coefficient of $s_{(5,4,3)}$ when $h_{(4,3,3,2)}$ is expanded into a sum of Schur polynomials. Consider the shapes that appear when we repeatedly use the Pieri rule on the product $h_4h_3h_3h_2$. Initially, we have a single shape (4) corresponding to h_4. Next, we add a horizontal strip of size 3 in all possible ways. Then we add another horizontal strip of size 3 in all possible ways. Finally, we add a horizontal strip of size 2 in all possible ways. The coefficient we seek is *the number of ways that the shape* $(5,4,3)$ *can be built by making the ordered sequence of choices just described.* For example, here is one choice sequence that leads to the shape $(5,4,3)$:

$$\begin{ytableau} * & * & * & * \end{ytableau} \;\rightarrow\; \begin{ytableau} \ & \ & \ & \ & * \\ * & * \end{ytableau} \;\rightarrow\; \begin{ytableau} \ & \ & \ & \ & \ \\ \ & \ & * \\ * & * \end{ytableau} \;\rightarrow\; \begin{ytableau} \ & \ & \ & \ & \ \\ \ & \ & \ & * \\ \ & \ & * \end{ytableau}\,.$$

Here is a second choice sequence that leads to the same shape:

$$\begin{ytableau} * & * & * & * \end{ytableau} \;\rightarrow\; \begin{ytableau} \ & \ & \ & \ \\ * & * & * \end{ytableau} \;\rightarrow\; \begin{ytableau} \ & \ & \ & \ & * \\ \ & \ & \ & * \\ * \end{ytableau} \;\rightarrow\; \begin{ytableau} \ & \ & \ & \ & \ \\ \ & \ & \ & \ \\ \ & * & * \end{ytableau}\,.$$

Here is a third choice sequence that leads to the same shape:

$$\begin{ytableau} * & * & * & * \end{ytableau} \;\rightarrow\; \begin{ytableau} \ & \ & \ & \ \\ * & * & * \end{ytableau} \;\rightarrow\; \begin{ytableau} \ & \ & \ & \ \\ \ & \ & \ \\ * & * & * \end{ytableau} \;\rightarrow\; \begin{ytableau} \ & \ & \ & \ & * \\ \ & \ & \ & * \\ \ & \ & \ \end{ytableau}\,.$$

Now comes the key observation. We have exhibited each choice sequence by drawing a succession of shapes showing the sequential addition of each horizontal strip. The same information can be encoded by drawing *one* copy of the final shape $(5,4,3)$ and putting a label in each box to show which horizontal strip caused that box to first appear in the shape. For example, the three choice sequences displayed above are encoded (in order) by the following three objects:

$$\begin{ytableau} 1 & 1 & 1 & 1 & 2 \\ 2 & 2 & 3 & 4 \\ 3 & 3 & 4 \end{ytableau} \qquad \begin{ytableau} 1 & 1 & 1 & 1 & 3 \\ 2 & 2 & 2 & 3 \\ 3 & 4 & 4 \end{ytableau} \qquad \begin{ytableau} 1 & 1 & 1 & 1 & 4 \\ 2 & 2 & 2 & 4 \\ 3 & 3 & 3 \end{ytableau}\,.$$

We have just drawn three semistandard tableaux of shape $(5,4,3)$ and content $(4,3,3,2)$! By definition of the encoding just described, we see that every choice sequence under consideration will be encoded by some tableau of content $(4,3,3,2)$. Since we build the tableau by adding horizontal strips one at a time using increasing labels, it follows that the tableau we get will always be *semistandard.* Finally, we can go backwards in the sense that *any* semistandard tableau of content $(4,3,3,2)$ can be built uniquely by choosing a succession of horizontal strips that tells us where the 1's, 2's, 3's and 4's appear in the tableau. To summarize these remarks, our encoding scheme proves that *the coefficient of* $s_{(5,4,3)}$ *in the Schur expansion of* $h_{(4,3,3,2)}$ *is the number of semistandard tableaux of shape* $(5,4,3)$ *and content* $(4,3,3,2)$. In addition to the three semistandard tableaux already drawn, we have the following tableaux of this shape and content:

$$\begin{ytableau} 1 & 1 & 1 & 1 & 2 \\ 2 & 2 & 3 & 3 \\ 3 & 4 & 4 \end{ytableau} \qquad \begin{ytableau} 1 & 1 & 1 & 1 & 4 \\ 2 & 2 & 2 & 3 \\ 3 & 3 & 4 \end{ytableau} \qquad \begin{ytableau} 1 & 1 & 1 & 1 & 3 \\ 2 & 2 & 2 & 4 \\ 3 & 3 & 4 \end{ytableau}$$

So the desired coefficient in this example is six.

The argument in the last example generalizes to prove the following result.

10.69. Theorem: Schur Expansion of h_α. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_s)$ be any sequence of nonnegative integers with sum k. Then

$$h_\alpha(x_1, \ldots, x_N) = \sum_{\lambda \in \mathrm{Par}_N(k)} K_{\lambda,\alpha} s_\lambda(x_1, \ldots, x_N).$$

(It is also permissible to sum over $\mathrm{Par}(k)$ here.)

Proof. By the Pieri rule, the coefficient of s_λ in h_α is the number of sequences of partitions

$$0 = \mu^0 \subseteq \mu^1 \subseteq \mu^2 \subseteq \cdots \subseteq \mu^s = \lambda \tag{10.6}$$

such that the skew shape μ^i/μ^{i-1} is a horizontal strip of size α_i, for $1 \le i \le s$. (This is a formal way of describing which horizontal strips we choose at each application of the Pieri rule to the product h_α.) On the other hand, $K_{\lambda,\alpha}$ is the number of semistandard tableaux of shape λ and content α. There is a bijection between the sequences (10.6) and these tableaux, defined by filling each strip μ^i/μ^{i-1} with α_i copies of the letter i. The resulting tableau has content α and is semistandard. The inverse map sends a semistandard tableau T to the sequence $(\mu^i : 0 \le i \le s)$, where μ^i consists of the cells of T containing symbols in $\{1, 2, \ldots, i\}$. □

10.70. Remark. Suppose α, β are sequences such that $\mathrm{sort}(\alpha) = \mathrm{sort}(\beta)$. Note that $h_\alpha = h_\beta$ since multiplication of polynomials is commutative. Expanding each side into Schur polynomials gives

$$\sum_{\lambda \vdash k} K_{\lambda,\alpha} s_\lambda(x_1, \ldots, x_N) = \sum_{\lambda \vdash k} K_{\lambda,\beta} s_\lambda(x_1, \ldots, x_N).$$

For $N \ge k$, the Schur polynomials appearing here will be linearly independent by 10.49. So $K_{\lambda,\alpha} = K_{\lambda,\beta}$ for all λ, in confirmation of 10.33. (This remark leads to an algebraic proof of 10.33, provided one first gives an algebraic proof of the linear independence of Schur polynomials.)

10.71. Remark. The previous theorem and remark extend to skew shapes as follows. First,

$$s_\mu h_\alpha = \sum_{\lambda \in \mathrm{Par}_N} K_{\lambda/\mu,\alpha} s_\lambda.$$

One need only change μ^0 from 0 to μ in the proof above. Second, if $\mathrm{sort}(\alpha) = \mathrm{sort}(\beta)$, then $K_{\lambda/\mu,\alpha} = K_{\lambda/\mu,\beta}$.

10.72. Theorem: Complete Homogeneous Basis of Λ_N^k. For all $k, N \in \mathbb{N}$, the set of complete homogeneous polynomials

$$\{h_\mu(x_1, \ldots, x_N) : \mu \in \mathrm{Par}_N(k)\} \subseteq K[x_1, \ldots, x_N]$$

is a basis of the K-vector space Λ_N^k.

Proof. Consider column vectors $\mathbf{S} = (s_\lambda(x_1, \ldots, x_N) : \lambda \in \mathrm{Par}_N(k))$ and $\mathbf{H} = (h_\mu(x_1, \ldots, x_N) : \mu \in \mathrm{Par}_N(k))$, where the entries are listed in lexicographic order. As in the proof of 10.49, let $\mathbf{K} = (K_{\lambda,\mu})$ be the Kostka matrix with rows and columns indexed by partitions in $\mathrm{Par}_N(k)$ in lexicographic order. Recall from 10.47 that $\mathbf{K}$ is a lower-triangular matrix with 1's on the main diagonal. In matrix form, 10.69 asserts that $\mathbf{H} = \mathbf{K}^t\mathbf{S}$, where $\mathbf{K}^t$ is the transpose of the Kostka matrix. This transpose is upper-triangular with 1's on the main diagonal, hence is invertible. Since $\mathbf{H}$ is obtained from $\mathbf{S}$ by application of an invertible matrix of scalars, we see that the elements of $\mathbf{H}$ form a basis by the same reasoning used in the proof of 10.49 (cf. 10.178). □

10.73. Remark. Combining 10.72 with 10.49, we can write $\mathbf{H} = (\mathbf{K}^t\mathbf{K})\mathbf{M}$, where $\mathbf{M}$ is the vector of monomial symmetric polynomials indexed by $\mathrm{Par}_N(k)$. This matrix equation gives the monomial expansion of the complete homogeneous symmetric polynomials h_μ.

10.14 Schur Expansion of e_α

Now we turn to the elementary symmetric polynomials e_α. We can iterate the Pieri rule as we did for h_α, but here we must add vertical strips at each stage.

10.74. Example. Let us compute the Schur expansion of $e_{(2,2,2)} = e_2e_2e_2$. First, $e_2e_2 = s_{(1,1)}e_2 = s_{(2,2)} + s_{(2,1,1)} + s_{(1,1,1,1)}$. Next,

$$\begin{aligned} e_2e_2e_2 &= [s_{(3,3)} + s_{(3,2,1)} + s_{(2,2,1,1)}] \\ &\quad + [s_{(3,2,1)} + s_{(3,1,1,1)} + s_{(2,2,2)} + s_{(2,2,1,1)} + s_{(2,1,1,1,1)}] \\ &\quad + [s_{(2,2,1,1)} + s_{(2,1,1,1,1)} + s_{(1,1,1,1,1,1)}] \\ &= s_{(3,3)} + 2s_{(3,2,1)} + s_{(3,1,1,1)} + s_{(2,2,2)} + 3s_{(2,2,1,1)} + 2s_{(2,1^4)} + s_{(1^6)}. \end{aligned}$$

As in the case of h_α, we can use tableaux to encode the sequence of vertical strips chosen in the repeated application of the Pieri rules. For example, the following tableaux encode the three choice sequences that lead to the shape $(2,2,1,1)$ in the expansion of $e_{(2,2,2)}$:

$$\begin{array}{|c|c|}\hline 1&2\\\hline 1&2\\\hline 3\\\cline{1-1} 3\\\cline{1-1}\end{array}\qquad \begin{array}{|c|c|}\hline 1&2\\\hline 1&3\\\hline 2\\\cline{1-1} 3\\\cline{1-1}\end{array}\qquad \begin{array}{|c|c|}\hline 1&3\\\hline 1&3\\\hline 2\\\cline{1-1} 2\\\cline{1-1}\end{array}$$

Evidently, these tableaux are not semistandard (column-strict). However, transposing the diagrams will produce semistandard tableaux of shape $(2,2,1,1)' = (4,2)$ and content $(2,2,2)$, as shown here:

$$\begin{array}{|c|c|c|c|}\hline 1&1&3&3\\\hline 2&2\\\cline{1-2}\end{array}\qquad \begin{array}{|c|c|c|c|}\hline 1&1&2&3\\\hline 2&3\\\cline{1-2}\end{array}\qquad \begin{array}{|c|c|c|c|}\hline 1&1&2&2\\\hline 3&3\\\cline{1-2}\end{array}$$

This encoding gives a bijection from the relevant choice sequences to the collection of semistandard tableaux of this shape and content. So the coefficient of $s_{(2,2,1,1)}$ in the Schur expansion of $e_{(2,2,2)}$ is the Kostka number $K_{(4,2),(2,2,2)} = 3$. This argument generalizes to prove the following theorem.

10.75. Theorem: Schur Expansion of e_α. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_s)$ be any sequence of nonnegative integers with sum k. Then

$$e_\alpha(x_1, \ldots, x_N) = \sum_{\lambda \in \mathrm{Par}_N(k)} K_{\lambda',\alpha} s_\lambda(x_1, \ldots, x_N) = \sum_{\nu \in \mathrm{Par}_N(k)'} K_{\nu,\alpha} s_{\nu'}(x_1, \ldots, x_N).$$

10.76. Remark. We have written $\mathrm{Par}_N(k)'$ for the set $\{\lambda' : \lambda \in \mathrm{Par}_N(k)\}$. Since conjugation of a partition interchanges the number of parts with the length of the largest part, we have

$$\mathrm{Par}_N(k)' = \{\nu \in \mathrm{Par}(k) : \nu_1 \le N\} = \{\nu \in \mathrm{Par}(k) : \nu_i \le N \text{ for all } i \ge 1\}.$$

It is also permissible to sum over all partitions of k in the theorem, since this will only add zero terms to the sum. If the number of variables is large enough ($N \ge k$), then we will already be summing over all partitions of k.

10.77. Theorem: Elementary Basis of Λ_N^k. For all $k, N \in \mathbb{N}$, the set of elementary symmetric polynomials

$$\{e_\mu(x_1, \ldots, x_N) : \mu \in \mathrm{Par}_N(k)'\} = \{e_{\mu'}(x_1, \ldots, x_N) : \mu \in \mathrm{Par}_N(k)\} \subseteq K[x_1, \ldots, x_N]$$

is a basis of the K-vector space Λ_N^k. Consequently, the set of all polynomials $e_1^{i_1} \cdots e_N^{i_N}$, where the i_j are arbitrary nonnegative integers, is a basis of Λ_N.

Proof. We use the same matrix argument employed earlier, suitably adjusted to account for the shape conjugation. As in the past, let us index the rows and columns of matrices and vectors by the partitions in $\mathrm{Par}_N(k)$, listed in lexicographic order. Introduce column vectors $\mathbf{S} = (s_\lambda(x_1, \ldots, x_N) : \lambda \in \mathrm{Par}_N(k))$ and $\mathbf{E} = (e_{\mu'}(x_1, \ldots, x_N) : \mu \in \mathrm{Par}_N(k))$. Next, consider the modified Kostka matrix $\hat{\mathbf{K}}$ whose entry in row μ and column λ is $K_{\lambda',\mu'}$. Now 10.75 asserts that

$$e_{\mu'}(x_1, \ldots, x_N) = \sum_{\lambda \in \mathrm{Par}_N(k)} K_{\lambda',\mu'} s_\lambda(x_1, \ldots, x_N).$$

By definition of matrix-vector multiplication, the equations just written are equivalent to $\mathbf{E} = \hat{\mathbf{K}}\mathbf{S}$. Since the entries of $\mathbf{S}$ are known to be a basis, it suffices (as in the proofs of 10.49 and 10.72) to argue that $\hat{\mathbf{K}}$ is a triangular matrix with 1's on the diagonal. There are 1's on the diagonal, since $K_{\mu',\mu'} = 1$. On the other hand, by 10.45, we have the implications

$$\hat{\mathbf{K}}(\mu, \lambda) \neq 0 \Rightarrow K_{\lambda',\mu'} \neq 0 \Rightarrow \mu' \trianglelefteq \lambda' \Rightarrow \lambda \trianglelefteq \mu \Rightarrow \lambda \leq_{\mathrm{lex}} \mu.$$

So $\hat{\mathbf{K}}$ is lower-triangular. The final statement about the basis of Λ_N follows by writing partitions in Par'_N in the form $1^{i_1} 2^{i_2} \cdots N^{i_N}$ and noting that the vector space Λ_N is the direct sum of the subspaces Λ_N^k. □

10.78. Remark. Combining this theorem with 10.49, we can write $\mathbf{E} = (\hat{\mathbf{K}}\mathbf{K})\mathbf{M}$, where $\mathbf{M}$ is the vector of monomial symmetric polynomials indexed by $\mathrm{Par}_N(k)$. This matrix equation gives the monomial expansion of elementary symmetric polynomials.

10.15 Algebraic Independence

We will use 10.77 to obtain structural information about the ring Λ_N of symmetric polynomials in N variables. First we need the following definition.

10.79. Definition: Algebraic Independence. Let A be a commutative ring containing K, and let $(z_1, \ldots, z_N)$ be a list of elements of A. We say that $(z_1, \ldots, z_N)$ is *algebraically independent over* K iff the collection of monomials

$$\{z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_N^{\alpha_N} : \alpha \in \mathbb{N}^N\}$$

is linearly independent over K. This means that whenever a finite K-linear combination of the z^α's is zero, say

$$\sum_\alpha c_\alpha z^\alpha = 0 \qquad (c_\alpha \in K),$$

then all c_α's must be zero.

Here is yet another formulation of the definition. Let $K[Z_1, \ldots, Z_N]$ be a polynomial ring in N indeterminates. Given any list $(z_1, \ldots, z_N) \in A^N$, we get an evaluation homomorphism $T : K[Z_1, \ldots, Z_N] \to A$ that sends each Z_i to z_i (see 7.102). One can check that the image of T is the subring B of A generated by K and the z_i's. On the other hand, the kernel of T consists precisely of the polynomials $\sum_\alpha c_\alpha Z^\alpha$ such that $\sum_\alpha c_\alpha z^\alpha = 0$. So, the z_i's are algebraically independent over K iff $\ker(T) = \{0\}$ iff T is *injective*. In this case, T (with codomain restricted to B) is a ring isomorphism, and $K[Z_1, \ldots, Z_N] \cong B$. So we may identify Z_i with z_i and write $B = K[z_1, \ldots, z_N]$.

10.80. Example. Let $K[x_1, \ldots, x_N]$ be a polynomial ring in indeterminates x_i. By the very definition of polynomials, $\sum_\alpha c_\alpha x^\alpha = 0$ implies all c_α are zero. So the indeterminates $x_1, \ldots, x_N$ are algebraically independent over K. The evaluation map T above can be taken to be the identity function on $K[x_1, \ldots, x_N]$. On the other hand, consider the three polynomials $z_1 = x_1 + x_2$, $z_2 = x_1^2 + x_2^2$, and $z_3 = x_1^3 + x_2^3$. The elements z_1, z_2, z_3 are *linearly* independent over K, as one may check. However, they are not *algebraically* independent over K, because of the relation

$$1z_1^3 - 3z_1z_2 + 2z_3 = 0.$$

Later, we will see that z_1 and z_2 are algebraically independent over K.

By 10.77, Λ_N is the subring of $K[x_1, \ldots, x_N]$ generated by K and the elementary symmetric polynomials. Combining the last part of 10.77 with 10.79, we deduce the following structural result.

10.81. Fundamental Theorem of Symmetric Polynomials. The elementary symmetric polynomials

$$\{e_i(x_1, \ldots, x_N) : 1 \leq i \leq N\} \subseteq K[x_1, \ldots, x_N]$$

are algebraically independent over K. Furthermore, if $K[E_1, \ldots, E_N]$ is another polynomial ring, then the evaluation map $T : K[E_1, \ldots, E_N] \to \Lambda_N$ sending E_i to $e_i(x_1, \ldots, x_N)$ is an isomorphism of rings and K-vector spaces. So, for every symmetric polynomial $f(x_1, \ldots, x_N)$, there exists a unique polynomial $g(E_1, \ldots, E_N)$ such that $f = T(g) = g(e_1, \ldots, e_N)$.

10.82. Remark. An algorithmic proof of the existence assertion in the fundamental theorem is sketched in 10.211.

10.16 Power-Sum Symmetric Polynomials

Recall that the *power-sum* symmetric polynomials in N variables are defined by setting $p_k(x_1, \ldots, x_N) = \sum_{i=1}^N x_i^k$ for all $k \geq 1$ and $p_\alpha(x_1, \ldots, x_N) = \prod_{j\geq 1} p_{\alpha_j}(x_1, \ldots, x_N)$. It turns out that the polynomials $(p_1, \ldots, p_N)$ are algebraically independent over K. One way to prove this is to invoke the following determinant criterion for algebraic independence.

10.83. Theorem: Determinant Test for Algebraic Independence. Let $g_1, \ldots, g_N$ be N polynomials in $K[x_1, \ldots, x_N]$. Let $\mathbf{A}$ be the $N \times N$ matrix whose j,k-entry is the formal partial derivative $D_jg_k = \partial g_k/\partial x_j$ (see 7.103), and let $J \in K[x_1, \ldots, x_N]$ be the determinant of $\mathbf{A}$ (see 9.37). If $J \neq 0$, then $g_1, \ldots, g_N$ are algebraically independent over K.

Proof. We prove the contrapositive. Assume $g_1, \ldots, g_N$ are algebraically dependent over K. Then there exist nonzero polynomials $h \in K[Z_1, \ldots, Z_N]$ such that $h(g_1, \ldots, g_N) = 0$. Choose such an h whose total degree in the Z_i's is as small as possible. We can take the partial derivative of $h(g_1, \ldots, g_N)$ with respect to x_j by applying the formal multivariable chain rule (see 7.104), obtaining the relations

$$\sum_{k=1}^{N} (D_k h)(g_1, \ldots, g_N) \frac{\partial g_k}{\partial x_j} = 0 \qquad (1 \leq j \leq N).$$

Let $\mathbf{v}$ be the column vector whose kth entry is $(D_k h)(g_1, \ldots, g_N)$. The preceding relations are equivalent to the matrix identity $\mathbf{Av} = \mathbf{0}$. We will show that $\mathbf{v}$ is not the zero vector. Since $h \in K[Z_1, \ldots, Z_N]$ is nonzero and K contains $\mathbb{Q}$, at least one partial derivative $D_k h \in K[Z_1, \ldots, Z_N]$ must be nonzero. Given such a k with $D_k h \neq 0$, the total degree of $D_k h$ in the Z_i's must be lower than the total degree of h. By choice of h, it follows that $(D_k h)(g_1, \ldots, g_N)$ is nonzero in $K[x_1, \ldots, x_N]$. This polynomial is the kth entry of $\mathbf{v}$, so $\mathbf{v} \neq \mathbf{0}$. Now $\mathbf{Av} = \mathbf{0}$ forces $\mathbf{A}$ to be a singular matrix, so $J = \det(\mathbf{A}) = 0$ by a theorem of linear algebra. □

10.84. Remark. The converse of 10.83 is also true: if $g_1, \ldots, g_N$ are algebraically independent, then the "Jacobian" J will be nonzero. This fact is not needed in the sequel, so we omit the proof.

10.85. Theorem: Algebraic Independence of Power-Sums. Let K be a field containing $\mathbb{Q}$. The power-sum polynomials

$$\{p_k(x_1, \ldots, x_N) : 1 \leq k \leq N\} \subseteq K[x_1, \ldots, x_N]$$

are algebraically independent over K.

Proof. We use the determinant criterion in 10.83. The j, k-entry of the matrix $\mathbf{A}$ is

$$D_j p_k = \frac{\partial}{\partial x_j}(x_1^k + x_2^k + \cdots + x_j^k + \cdots + x_N^k) = k x_j^{k-1}.$$

Accordingly, $J = \det ||k x_j^{k-1}||_{1 \leq j,k \leq N}$. For each column k, we may factor out the scalar k to see that $J = N! \det ||x_j^{k-1}||$. The resulting determinant is called a Vandermonde determinant. This determinant evaluates to $\pm \prod_{1 \leq r < s \leq N} (x_r - x_s)$, which is a nonzero polynomial (see §12.9 for a combinatorial proof of this formula). Since K contains $\mathbb{Q}$, the scalar $N!$ is not zero in K. We conclude that $J \neq 0$, which proves the result. □

Now that we know that the p_k's are algebraically independent, we can obtain power-sum bases for the vector spaces Λ_N^k and Λ_N.

10.86. Theorem: Power-Sum Basis. Let K be a field containing $\mathbb{Q}$. For all $k, N \in \mathbb{N}$, the collection

$$\{p_\mu(x_1, \ldots, x_N) : \mu \in \mathrm{Par}_N(k)'\} \subseteq K[x_1, \ldots, x_N]$$

is a basis of the K-vector space Λ_N^k. The collection $\{p_1^{i_1} \cdots p_N^{i_N} : i_j \geq 0\}$ is a basis of the K-vector space Λ_N. Letting $P_1, \ldots, P_N$ be new indeterminates, there is an isomorphism of rings and K-vector spaces $T : K[P_1, \ldots, P_N] \to \Lambda_N$ such that $T(P_i) = p_i(x_1, \ldots, x_N)$. So, for every symmetric polynomial $f(x_1, \ldots, x_N)$, there exists a unique polynomial g with $f = g(p_1, \ldots, p_N)$.

10.17 Relations between e's and h's

We have seen that, in the polynomial ring $K[x_1,\ldots,x_N]$, the lists $(e_1,\ldots,e_N)$ and $(p_1,\ldots,p_N)$ are each algebraically independent. One might wonder if the polynomials $(h_1,\ldots,h_N)$ are also algebraically independent over K. This would follow (as it did for the e's) if we knew that $\{h_\mu : \mu \in \mathrm{Par}_N(k)'\}$ was a basis of Λ_N^k for all k. However, the basis we found in 10.72 was $\{h_\mu : \mu \in \mathrm{Par}_N(k)\}$, which is indexed by partitions of k with at most N parts, instead of partitions of k with each part at most N. The next result will allow us to overcome this difficulty by providing equations relating $e_1,\ldots,e_N$ to $h_1,\ldots,h_N$.

10.87. Theorem: Recursion for e_i's and h_j's. For all $m,N\in\mathbb{N}$, we have the identity

$$\sum_{i=0}^{m}(-1)^i e_i(x_1,\ldots,x_N)h_{m-i}(x_1,\ldots,x_N)=\chi(m=0). \tag{10.7}$$

Proof. If $m=0$, the identity reads $1=1$, so let us assume $m>0$. We can model the left side of the identity using a collection Z of signed, weighted objects. A typical object in Z is a triple $z=(i,S,T)$, where $0\le i\le m$, $S\in \mathrm{SSYT}_N((1^i))$, and $T\in\mathrm{SSYT}_N((m-i))$. The *weight* of (i,S,T) is $x^{c(S)}x^{c(T)}$, and the *sign* of (i,S,T) is $(-1)^i$. For example, taking $N=9$ and $m=7$, a typical object in Z is

$$z=\left(3,\ \begin{array}{|c|}\hline 2\\\hline 4\\\hline 7\\\hline\end{array},\ \begin{array}{|c|c|c|c|}\hline 3&3&4&6\\\hline\end{array}\right).$$

The signed weight of this object is $(-1)^3(x_2x_4x_7)(x_3^2x_4x_6)=-x_2x_3^2x_4^2x_6x_7$. Recalling that $e_i=s_{(1^i)}$ and $h_{m-i}=s_{(m-i)}$, we see that the left side of (10.7) is precisely

$$\sum_{z\in Z}\mathrm{sgn}(z)\,\mathrm{wt}(z).$$

To prove this expression is zero, we define a sign-reversing involution $I:Z\to Z$ with no fixed points. Given $z=(i,S,T)\in Z$, we compute $I(z)$ as follows. Let $j=S((1,1))$ be the smallest entry in S, and let $k=T((1,1))$ be the leftmost entry in T. If $i=0$, then S is empty and j is undefined; if $i=m$, then T is empty and k is undefined. Since $m>0$, at least one of j or k is defined. If $j\le k$ or k is not defined, move the box containing j from S to T, so that this box is the new leftmost entry in T, and decrement i by 1. Otherwise, if $k<j$ or j is not defined, move the box containing k from T to S, so that this box is the new topmost box in S, and increment i by 1. For example, if z is the object shown above, then

$$I(z)=\left(2,\ \begin{array}{|c|}\hline 4\\\hline 7\\\hline\end{array},\ \begin{array}{|c|c|c|c|c|}\hline 2&3&3&4&6\\\hline\end{array}\right).$$

As another example,

$$I\left((0,\ \emptyset,\ \begin{array}{|c|c|c|c|c|c|c|}\hline 2&2&3&5&5&7&9\\\hline\end{array})\right)=(1,\ \begin{array}{|c|}\hline 2\\\hline\end{array},\ \begin{array}{|c|c|c|c|c|c|}\hline 2&3&5&5&7&9\\\hline\end{array}).$$

From the definition of I, we can check that I does map Z into Z, that $I\circ I=\mathrm{id}_Z$, that I is weight-preserving and sign-reversing, and that I has no fixed points. $\square$

10.88. Theorem: Algebraic Independence of h's. For all $k,N\in\mathbb{N}$, the collection

$$\{h_\mu(x_1,\ldots,x_N):\mu\in\mathrm{Par}_N(k)'\}\subseteq K[x_1,\ldots,x_N]$$

is a basis of the K-vector space Λ_N^k. Consequently, $\{h_1^{i_1}\cdots h_N^{i_N} : i_j \geq 0\}$ is a basis of the K-vector space Λ_N, and $(h_1,\ldots,h_N)$ is algebraically independent over K. Letting $H_1,\ldots,H_N$ be new indeterminates, there is a ring and K-vector space isomorphism $T : K[H_1,\ldots,H_N] \to \Lambda_N$ given by $T(H_i) = h_i(x_1,\ldots,x_N)$. So, for every symmetric polynomial $f(x_1,\ldots,x_N)$, there exists a unique polynomial g with $f = g(h_1,\ldots,h_N)$.

Proof. It suffices to prove the statement about the basis of Λ_N^k, from which the other assertions follow. By 10.72, we know that

$$|\{h_\mu(x_1,\ldots,x_N) : \mu \in \mathrm{Par}_N(k)'\}| \leq |\{h_\mu(x_1,\ldots,x_N) : \mu \in \mathrm{Par}_N(k)\}| = \dim_K(\Lambda_N^k).$$

So, by a theorem of linear algebra, it is enough to prove that $\{h_\mu : \mu \in \mathrm{Par}_N(k)'\}$ is a *spanning set* of Λ_N^k. For each $k \geq 0$, let V_N^k be the vector subspace of Λ_N^k spanned by these h_μ's. We want to prove $V_N^k = \Lambda_N^k$ for all k. It will suffice to show that $e_1^{i_1}\cdots e_N^{i_N} \in V_N^k$ for all $i_1,\ldots,i_N$ that sum to k, since these elementary symmetric polynomials are known to be a basis of Λ_N^k. Now, one can check that $f \in V_N^k$ and $g \in V_N^m$ imply $fg \in V_N^{k+m}$. (This holds when f and g are each products of $h_1,\ldots,h_N$, and the general case follows by linearity and the distributive law.) Using this remark, we can further reduce to proving that $e_j(x_1,\ldots,x_N) \in V_N^j$ for $1 \leq j \leq N$.

We prove this by induction on j. The result is true for $j = 1$, since $e_1 = \sum_{k=1}^N x_k = h_1 \in V_N^1$. Assume $1 < j \leq N$ and the result is known to hold for all smaller values of j. Taking $m = j$ in the recursion (10.7), we have

$$e_j = e_{j-1}h_1 - e_{j-2}h_2 + e_{j-3}h_3 - \cdots \pm e_1 h_{j-1} \mp h_j.$$

Since $e_{j-s} \in V_N^{j-s}$ (by induction) and $h_s \in V_N^s$ (by definition) for $1 \leq s \leq j$, each term on the right side lies in V_N^j. Since V_N^j is a subspace, it follows that $e_j \in V_N^j$, completing the induction. □

10.18 Generating Functions for e's and h's

Another approach to the identities (10.7) involves generating functions.

10.89. Definition: $E_N(t)$ and $H_N(t)$. For each $N \geq 1$, define the polynomial

$$E_N(t) = \prod_{i=1}^N (1 + x_i t) \in F(x_1,\ldots,x_N)[t]$$

and the formal power series

$$H_N(t) = \prod_{i=1}^N \frac{1}{1 - x_i t} \in F(x_1,\ldots,x_N)[[t]].$$

10.90. Theorem: Expansion of $E_N(t)$. For all $N \geq 1$, we have

$$E_N(t) = \sum_{k=0}^N e_k(x_1,\ldots,x_N)t^k.$$

Proof. Let us use the generalized distributive law 2.7 to expand the product in the definition of $E_N(t)$. We obtain

$$E_N(t) = \prod_{i=1}^{N}(1 + x_i t) = \sum_{S \subseteq \{1,2,\ldots,N\}} \prod_{i \in S}(x_i t) \prod_{i \notin S} 1.$$

To get terms involving t^k, we must restrict the sum to subsets S of size k. Such subsets can be identified with increasing sequences $1 \leq i_1 < i_2 < \cdots < i_k \leq N$. Therefore, the coefficient of t^k in $E_N(t)$ is

$$\sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq N} x_{i_1} x_{i_2} \cdots x_{i_k} = e_k(x_1, \ldots, x_N). \qquad \square$$

10.91. Theorem: Relation between Roots and Coefficients of a Polynomial. Suppose a polynomial $p = X^N + a_1 X^{N-1} + \cdots + a_i X^{N-i} + \cdots + a_{N-1}X + a_N \in K[X]$ factors as $p = (X - r_1)(X - r_2)\cdots(X - r_N)$ for some $r_i \in K$. For $1 \leq i \leq N$,

$$a_i = (-1)^i e_i(r_1, r_2, \ldots, r_N).$$

Proof. One can prove this by expanding $\prod_{i=1}^{N}(X - r_i)$ with the generalized distributive law. Alternatively, we can deduce the result from 10.90 as follows. Replacing t by $1/X$ and x_i by $-r_i$ in $E_N(t)$ gives $\prod_{i=1}^{N}(1 - r_i/X) = X^{-N}p$. Using 10.90, we conclude that

$$p = X^N \sum_{k=0}^{n} e_k(-r_1, \ldots, -r_N)X^{-k} = \sum_{k=0}^{N}(-1)^k e_k(r_1, \ldots, r_N)X^{N-k}.$$

Taking the coefficient of X^{N-i} gives the result. $\square$

10.92. Theorem: Expansion of $H_N(t)$. For all $N \geq 1$, we have

$$H_N(t) = \sum_{k=0}^{\infty} h_k(x_1, \ldots, x_N)t^k.$$

Proof. Using the geometric series formula, we have

$$H_N(t) = \prod_{i=1}^{N} \frac{1}{1 - x_i t} = \prod_{i=1}^{N} \sum_{j_i=0}^{\infty} (x_i t)^{j_i}.$$

Next, using the generalized distributive law, we get

$$H_N(t) = \sum_{(j_1,\ldots,j_N)\in\mathbb{N}^N} \prod_{i=1}^{N} x_i^{j_i} t^{j_i} = \sum_{(j_1,\ldots,j_N)\in\mathbb{N}^N} t^{j_1+\cdots+j_N} \prod_{i=1}^{N} x_i^{j_i}.$$

The coefficient of t^k consists of the sum of all possible monomials in $x_1, \ldots, x_N$ of degree k, which is precisely $h_k(x_1, \ldots, x_N)$. $\square$

Now we can give an algebraic proof of 10.87. Note that

$$H_N(t)E_N(-t) = \prod_{i=1}^{N} \frac{1}{(1 - x_i t)} \prod_{i=1}^{N}(1 - x_i t) = 1.$$

Equating the coefficients of t^m on both sides gives the identities (10.7).

10.19 Relations between p's, e's, and h's

In this section, we will study recursions similar to (10.7) that relate the h_n's and e_n's to the p_k's. These recursions can be used to deduce the algebraic independence of the p_k's from the algebraic independence of the h_n's (or e_n's), and vice versa, by adapting the proof of 10.88 to the new recursions.

10.93. Theorem: Recursion for h_i's and p_j's. For all $n, N \geq 1$, the following identity is valid in Λ_N:

$$h_0p_n + h_1p_{n-1} + h_2p_{n-2} + \cdots + h_{n-1}p_1 = nh_n. \tag{10.8}$$

Proof. Let us interpret each side of the desired equation as the generating function for a suitable collection of weighted objects. For the left side, let X be the set of all triples (k, T, U), where: $0 \leq k < n$; $T \in \mathrm{SSYT}_N((k))$; and U consists of a row of $n-k$ boxes all filled with the same integer $i \leq N$. The *weight* of such a triple is $x^{c(T)+c(U)} = x^{c(T)}x_i^{n-k}$. For example, letting $n = 8$ and $N = 9$, here is a typical object in X of weight $x_1^2x_2x_3^3x_4^2$:

$$z_0 = (5,\ \begin{array}{|c|c|c|c|c|}\hline 1&1&2&4&4\\\hline\end{array},\ \begin{array}{|c|c|c|}\hline 3&3&3\\\hline\end{array}).$$

For a fixed value of k, the generating function for the possible T's is $h_k(x_1, \ldots, x_N)$ and the generating function for the possible U's is $p_{n-k}(x_1, \ldots, x_N)$. By the sum and product rules for weighted sets, the left side of (10.8) is the generating function for X.

Now let Y be the set of all pairs (V, j), where $V \in \mathrm{SSYT}_N((n))$ and $1 \leq j \leq n$. We can visualize an object in Y as a semistandard tableau of shape (n) in which the jth cell has been *marked*. For example, here is a typical object in Y of weight $x_1^2x_3^5x_4$:

$$y_0 = \begin{array}{|c|c|c|c|c|c|c|c|}\hline 1&1&3&3^*&3&3&3&4\\\hline\end{array}.$$

The generating function for the weighted set Y is $nh_n(x_1, \ldots, x_N)$.

To prove (10.8), it suffices to define a weight-preserving bijection $f : X \to Y$. Given $(k, T, U) \in X$, note that U consists of a run of $n-k$ copies of some symbol i. To compute $f((k, T, U))$, mark the first box in U and splice the boxes of U into T in the appropriate position to get a weakly increasing sequence. If T already contains one or more i's, the first box of U is inserted immediately after these i's. For example,

$$f(z_0) = \begin{array}{|c|c|c|c|c|c|c|c|}\hline 1&1&2&3^*&3&3&4&4\\\hline\end{array}.$$

This insertion process is reversible, thanks to the marker. More precisely, define $g : Y \to X$ as follows. Given $(V, j) \in Y$, let i be the entry in the jth cell of V. Starting at cell j and scanning right, remove each cell equal to i from V to get a pair of tableaux T and U as in the definition of X. Define $g((V, j)) = (k, T, U)$, where k is the number of boxes in T. For example,

$$g(y_0) = (4,\ \begin{array}{|c|c|c|c|}\hline 1&1&3&4\\\hline\end{array},\ \begin{array}{|c|c|c|c|}\hline 3&3&3&3\\\hline\end{array}).$$

One may check that f and g are weight-preserving and are two-sided inverses of each other. □

10.94. Theorem: Recursion for e_i's and p_j's. For all $n, N \geq 1$, the following identity is valid in Λ_N:

$$e_0p_n - e_1p_{n-1} + e_2p_{n-2} - \cdots + (-1)^{n-1}e_{n-1}p_1 = (-1)^{n-1}ne_n. \tag{10.9}$$

Proof. This time we interpret each side of the equation using suitable *signed*, weighted objects. For the left side, let X be the set of all triples (k,T,U), where: $0 \leq k < n$; $T \in \mathrm{SSYT}_N((1^k))$; and U consists of a row of $n-k$ boxes all filled with the same integer $i \leq N$. The *weight* of this triple is $x^{c(T)+c(U)}$, and the *sign* of this triple is $(-1)^k$. For example, here is a typical object of X whose signed weight is $(-1)^4 x_2 x_4^5 x_5 x_7$:

$$z_0 = \left(4, \begin{array}{|c|}\hline 2\\ \hline 4\\ \hline 5\\ \hline 7\\ \hline\end{array}, \begin{array}{|c|c|c|c|}\hline 4&4&4&4\\ \hline\end{array}\right).$$

Using the sum and product rules for weighted sets, one sees that $\sum_{z\in X}\mathrm{sgn}(z)\,\mathrm{wt}(z)$ is the left side of (10.9).

Now let $Y = \{(T,j) : T \in \mathrm{SSYT}_N((1^n)), 1 \leq j \leq n\}$. We can think of each element of Y as a strictly increasing sequence of n elements of $\{1,2,\ldots,N\}$ in which one of the elements (the jth one) has been *marked*. The generating function for the weighted set Y is $ne_n(x_1,\ldots,x_N)$.

Let us define a weight-preserving, sign-reversing involution $I : X \to X$. Fix $(k,T,U) \in X$. Since $k < n$, U is not empty; let j be the integer appearing in each box of U. The map I acts as follows. On one hand, if $k < n-1$ and j does not appear in T, then increase k by 1, remove one copy of j from U, and insert this number in the proper position in T to get a sorted sequence. On the other hand, if j does appear in T, then decrease k by 1, remove the unique copy of j from T, and place another copy of j in U. If neither of the two preceding cases occurs, (k,T,U) is a fixed point of I. For example,

$$I(z_0) = \left(3, \begin{array}{|c|}\hline 2\\ \hline 5\\ \hline 7\\ \hline\end{array}, \begin{array}{|c|c|c|c|c|}\hline 4&4&4&4&4\\ \hline\end{array}\right).$$

One can check that I is a well-defined, weight-preserving, sign-reversing involution on X.

Let Z be the set of fixed points of I. We see from the description of I that Z consists of all triples $(n-1,T,\boxed{j})$ where j does not appear in T. All of these triples have sign $(-1)^{n-1}$. The proof will be complete if we can find a weight-preserving bijection $g : Z \to Y$. We define g by inserting a marked copy of j into its proper position in the increasing sequence T. The inverse map takes an increasing sequence of size n with one marked element and removes the marked element. For example,

$$g\left(4, \begin{array}{|c|}\hline 2\\ \hline 5\\ \hline 7\\ \hline 8\\ \hline\end{array}, \begin{array}{|c|}\hline 3\\ \hline\end{array}\right) = \begin{array}{|c|}\hline 2\\ \hline 3^*\\ \hline 5\\ \hline 7\\ \hline 8\\ \hline\end{array}.$$

$\square$

10.20 Power-Sum Expansion of h_n and e_n

We can use the recursions in 10.93 and 10.94 to compute expansions for h_n and e_n in terms of the power-sum symmetric polynomials p_μ.

10.95. Example. We know that $h_0 = 1$ and $h_1 = p_1$. Next, since $h_0p_2 + h_1p_1 = 2h_2$, we find that $h_2 = (p_{(2)} + p_{(1,1)})/2$. For $n = 3$, we have

$$h_0 p_3 + h_1 p_2 + h_2 p_1 = 3h_3,$$

so that

$$h_3 = \frac{1}{3}\left(p_3 + p_1 p_2 + \left[\frac{p_2 + p_1^2}{2}\right] p_1\right) = (1/3)p_{(3)} + (1/2)p_{(2,1)} + (1/6)p_{(1,1,1)}.$$

For $n = 4$, we use the relation

$$h_0 p_4 + h_1 p_3 + h_2 p_2 + h_3 p_1 = 4h_4$$

to find, after some calculations,

$$h_4 = (1/4)p_{(4)} + (1/3)p_{(3,1)} + (1/8)p_{(2,2)} + (1/4)p_{(2,1,1)} + (1/24)p_{(1,1,1,1)}.$$

These formulas become nicer if we multiply through by $n!$. For instance,

$$\begin{aligned} 3!h_3 &= 2p_{(3)} + 3p_{(2,1)} + 1p_{(1,1,1)}; \\ 4!h_4 &= 6p_{(4)} + 8p_{(3,1)} + 3p_{(2,2)} + 6p_{(2,1,1)} + 1p_{(1,1,1,1)}. \end{aligned}$$

Similar formulas can be derived for $n!e_n$, but here some signs occur. For instance, calculations with (10.9) lead to the identities

$$\begin{aligned} 3!e_3 &= 2p_{(3)} - 3p_{(2,1)} + 1p_{(1,1,1)}; \\ 4!e_4 &= -6p_{(4)} + 8p_{(3,1)} + 3p_{(2,2)} - 6p_{(2,1,1)} + 1p_{(1,1,1,1)}. \end{aligned}$$

By comparing the coefficients in the power-sum expansion of $4!h_4$ to the entries in Table 9.1, the reader may be led to conjecture the following result.

10.96. Theorem: Power-Sum Expansion of h_n. For all $n, N \geq 1$, the following identity is valid in Λ_N:

$$n!h_n = \sum_{\mu \in \mathrm{Par}(n)} (n!/z_\mu) p_\mu. \tag{10.10}$$

Proof. Recall from 9.134 that $n!/z_\mu$ is the number of permutations $\sigma \in S_n$ with cycle type μ. This suggests the following combinatorial interpretations for the two sides of (10.10). The left side counts all pairs (w, T), where $w = w_1 w_2 \cdots w_n \in S_n$ is a permutation written in *one-line form* and $T = (i_1 \leq i_2 \leq \cdots \leq i_n)$ is an element of $\mathrm{SSYT}_N((n))$. Let X be the set of all such pairs, weighted by the content of T. For example, here is a typical element of X when $n = 8$, written as a two-rowed array:

$$z_0 = \left[\begin{array}{ccccccccc} w: & 4 & 2 & 5 & 8 & 3 & 7 & 1 & 6 \\ T: & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 3 \end{array}\right].$$

The right side of (10.10) counts all triples (μ, σ, C), where $\mu \in \mathrm{Par}(n)$, $\sigma \in S_n$ is a permutation with cycle type μ, and $C : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, N\}$ is a *coloring* of the numbers $1, \ldots, n$ using N available colors such that all elements in the same cycle of σ are assigned the same color (cf. §9.19). Let the *weight* of such a triple be $\prod_{k=1}^{n} x_{C(k)}$, and let Y be the set of all such weighted triples. For example, a typical element of Y is the triple

$$y_0 = \left((3,2,2,1),\ (1,6,3)(2,5)(7,4)(8),\ \left(\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 2 & 3 & 3 & 2 & 3 & 3 & 3 \end{array}\right)\right).$$

To see why the factor $p_\mu(x_1,\ldots,x_N)$ arises, consider how we may choose the coloring function C once μ and σ have been selected. Now, we know σ is a product of cycles of lengths $\mu_1,\mu_2,\ldots,\mu_l$. Choose the common color of the elements in the first cycle in any of N ways. Since μ_1 elements all receive the same color, the generating function for this choice is $x_1^{\mu_1}+x_2^{\mu_1}+\cdots+x_N^{\mu_1}=p_{\mu_1}(x_1,\ldots,x_N)$. Next, choose the common color of the elements in the second cycle, which gives a factor of p_{μ_2}, and so on. Multiplying the generating functions for these choices gives $p_\mu(x_1,\ldots,x_N)$.

To complete the proof, we define weight-preserving maps $f: Y\to X$ and $g: X\to Y$ that are inverses of each other. To understand the definition of f, recall that a given $\sigma\in S_n$ can be written in cycle notation in several different ways, since the cycles can be presented in any order, and elements within each cycle can be cyclically permuted. Given (μ,σ,C), we will specify one particular cycle notation for σ that depends on C, as follows. First, cycles colored with smaller colors are to be written before cycles colored with larger colors. Second, elements within each cycle are cyclically shifted so that the first element in each cycle is the smallest element appearing in that cycle. Third, if there are several cycles that have the same color, these cycles are ordered so that their minimal elements decrease from left to right. For example, starting with the object y_0 above, we obtain the following cycle notation for σ: $(2,5)(8)(4,7)(1,6,3)$. Note that $(2,5)$ is written first because this cycle has color 2. The other cycles, which are all colored 3, are presented in the given order because $8>4>1$. Finally, to compute $f((\mu,\sigma,C))$, we *erase the parentheses* from the chosen cycle notation for σ and write the color $C(i)$ directly beneath each i in the resulting word. For example,

$$f(y_0)=\left[\begin{array}{ccccccccc} w: & 2 & 5 & 8 & 4 & 7 & 1 & 6 & 3 \\ T: & 2 & 2 & 3 & 3 & 3 & 3 & 3 & 3 \end{array}\right].$$

One may check that f is well-defined, maps into X, and preserves weights.

Now consider how to define the inverse map $g: X\to Y$. Given $(w,T)\in X$ with $w=w_1\cdots w_n$ and $T=i_1\leq\cdots\leq i_n$, the coloring map C is defined by setting $C(w_j)=i_j$ for all j. To recover σ from w and T, we need to add parentheses to w to recreate the cycle notation satisfying the rules above. For each color i in turn, look at the substring of w consisting of the symbols located above the i's in T. Scan this substring from left to right, and begin a new cycle each time a number is encountered that is smaller than all of the preceding numbers in this substring. (The numbers that begin new cycles will be called *left-to-right minima relative to color* i.) This procedure defines σ, and finally we set $\mu=\mathrm{type}(\sigma)$. For example,

$$g(z_0)=\left((2,2,1,1,1,1),(4)(2,5)(8)(3,7)(1)(6),\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 1 & 2 & 1 & 1 & 3 & 2 & 2 \end{pmatrix}\right).$$

The reader should check that $g(f(y_0))=y_0$ and $f(g(z_0))=z_0$. One can similarly verify that $g\circ f=\mathrm{id}_Y$ and $f\circ g=\mathrm{id}_X$, so the proof is complete. □

Before considering the analogous theorem for e_n, we introduce the following notation.

10.97. Definition: The Sign Factor ϵ_μ. For every partition $\mu\vdash n$, let

$$\epsilon_\mu=(-1)^{n-\ell(\mu)}=\prod_{i=1}^{\ell(\mu)}(-1)^{\mu_i-1}.$$

We proved in 9.34 that $\epsilon_\mu=\mathrm{sgn}(\sigma)$ for all $\sigma\in S_n$ such that $\mathrm{type}(\sigma)=\mu$.

10.98. Theorem: Power-Sum Expansion of e_n. For all $n,N\geq 1$, the following identity is valid in Λ_N:

$$n!e_n=\sum_{\mu\in\mathrm{Par}(n)}\epsilon_\mu(n!/z_\mu)p_\mu. \tag{10.11}$$

Proof. We use the notation X, Y, f, g, z_0, and y_0 from the proof of 10.96. Recall that $\sum_{\mu \vdash n}(n!/z_\mu)p_\mu$ is the generating function for the weighted set Y. To model the right side of (10.11), we need to get the sign factors ϵ_μ into this sum. We accomplish this by assigning *signs* to objects in Y as follows. Given $(\mu, \sigma, C) \in Y$, write σ in cycle notation as described previously. Attach a $+$ to the first (minimal) element of each cycle, and attach a $-$ to the remaining elements in each cycle. The overall sign of (μ, σ, C) is the product of these signs, which is $\prod_i (-1)^{\mu_i - 1} = \epsilon_\mu$. For example, the object y_0 considered previously is now written

$$y_0 = \left((3,2,2,1),\ (2^+,5^-)(8^+)(4^+,7^-)(1^+,6^-,3^-),\ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 2 & 3 & 3 & 2 & 3 & 3 & 3 \end{pmatrix}\right);$$

the sign of this object is $(-1)^4 = +1$.

The next step is to transfer these signs to the objects in X using the weight-preserving bijection $f : Y \to X$. Given $(w, T) \in X$, find the left-to-right minima relative to each color i (as discussed in the definition of g in the proof of 10.96). Attach a $+$ to these numbers and a $-$ to all other numbers in w. For example, $f(y_0)$ is now written

$$f(y_0) = \left[\begin{array}{ccccccccc} w: & 2^+ & 5^- & 8^+ & 4^+ & 7^- & 1^+ & 6^- & 3^- \\ T: & 2 & 2 & 3 & 3 & 3 & 3 & 3 & 3 \end{array}\right].$$

As another example,

$$z_0 = \left[\begin{array}{ccccccccc} w: & 4^+ & 2^+ & 5^- & 8^+ & 3^+ & 7^- & 1^+ & 6^+ \\ T: & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 3 \end{array}\right].$$

The bijections f and g now preserve both signs and weights, by the way we defined signs of objects in X. It follows that $\sum_{(w,T)\in X} \operatorname{sgn}((w,T)) \operatorname{wt}((w,T))$ is precisely the right side of (10.11).

Now we define a sign-reversing, weight-preserving involution $I : X \to X$. Fix $(w, T) \in X$. If all the entries of T are distinct, then (w, T) will be a fixed point of I. We observe at once that such a fixed point is necessarily positive, and the generating function for such objects is precisely $n! e_n(x_1, \ldots, x_N)$. On the other hand, suppose some color i appears more than one time in T. Choose the smallest color i with this property, and let w_k, w_{k+1} be the first two symbols in the substring of w located above this color. Define $I((w, T))$ by switching w_k and w_{k+1}; one checks that this is a weight-preserving involution. Furthermore, one may verify that switching these two symbols will change the number of left-to-right minima (relative to color i) by exactly 1. For example,

$$I(f(y_0)) = \left[\begin{array}{ccccccccc} w: & 5^+ & 2^+ & 8^+ & 4^+ & 7^- & 1^+ & 6^- & 3^- \\ T: & 2 & 2 & 3 & 3 & 3 & 3 & 3 & 3 \end{array}\right].$$

As another example,

$$I(z_0) = \left[\begin{array}{ccccccccc} w: & 2^+ & 4^- & 5^- & 8^+ & 3^+ & 7^- & 1^+ & 6^+ \\ T: & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 3 \end{array}\right].$$

In general, note that w_k is always labeled by $+$; w_{k+1} is labeled $+$ iff $w_k > w_{k+1}$; and the signs attached to numbers following w_{k+1} do not depend on the order of the two symbols w_k, w_{k+1}. We have now shown that I is sign-reversing, so the proof is complete. $\square$

10.21 The Involution ω

Recall from §10.16 that $p_1, \ldots, p_N \in \Lambda_N$ are algebraically independent over K, so that we can view Λ_N as the polynomial ring $K[p_1, \ldots, p_N]$. By the universal mapping property for polynomial rings (see 7.102), we can define a ring homomorphism with domain Λ_N by sending each p_i to an arbitrary element of a given ring containing K. We now apply this technique to define a certain homomorphism on Λ_N.

10.99. Definition: The Map ω. Let $\omega : \Lambda_N \to \Lambda_N$ be the unique K-linear ring homomorphism such that $\omega(p_i) = (-1)^{i-1} p_i$ for $1 \le i \le N$.

10.100. Theorem: ω is an Involution. We have $\omega^2 = \mathrm{id}_{\Lambda_N}$; in particular, ω is an automorphism of Λ_N.

Proof. Observe that $\omega^2(p_i) = \omega(\omega(p_i)) = \omega((-1)^{i-1}p_i) = (-1)^{i-1}(-1)^{i-1}p_i = p_i = \mathrm{id}(p_i)$ for $1 \le i \le N$. Since ω^2 and id are ring homomorphisms on Λ_N that fix each $c \in K$ and have the same effect on every p_i, the uniqueness part of the UMP for polynomial rings shows that $\omega^2 = \mathrm{id}$. Since ω has a two-sided inverse (namely, itself), ω is a bijection. □

Let us investigate the effect of ω on various bases of Λ_N.

10.101. Theorem: Action of ω on p's, h's, and e's. Suppose $\nu \in \mathrm{Par}_N(k)'$, so $\nu_i \le N$ for all i. The following identities hold in Λ_N: (a) $\omega(p_\nu) = \epsilon_\nu p_\nu$; (b) $\omega(h_\nu) = e_\nu$; (c) $\omega(e_\nu) = h_\nu$.

Proof. (a) Since ω is a ring homomorphism and $\nu_i \le N$ for all i,

$$\omega(p_\nu) = \omega\left(\prod_{i=1}^{\ell(\nu)} p_{\nu_i}\right) = \prod_{i=1}^{\ell(\nu)} \omega(p_{\nu_i}) = \prod_{i=1}^{\ell(\nu)} (-1)^{\nu_i - 1} p_{\nu_i} = \epsilon_\nu p_\nu.$$

(b) First, for any $n \le N$, we have

$$\omega(h_n) = \omega\left(\sum_{\mu \in \mathrm{Par}(n)} z_\mu^{-1} p_\mu\right) = \sum_{\mu \in \mathrm{Par}(n)} z_\mu^{-1} \omega(p_\mu) = \sum_{\mu \in \mathrm{Par}(n)} \epsilon_\mu z_\mu^{-1} p_\mu = e_n$$

by 10.96 and 10.98. Since ω preserves multiplication, $\omega(h_\nu) = e_\nu$ follows.

(c) Part (c) follows by applying ω to both sides of (b), since $\omega^2 = \mathrm{id}$. □

10.102. Theorem: Action of ω on s_λ. If $\lambda \in \mathrm{Par}(n)$ and $n \le N$, then $\omega(s_\lambda) = s_{\lambda'}$ in Λ_N.

Proof. From 10.69, we know that for each $\mu \vdash n$,

$$h_\mu(x_1, \ldots, x_N) = \sum_{\lambda \in \mathrm{Par}(n)} K_{\lambda,\mu} s_\lambda(x_1, \ldots, x_N).$$

We can combine these equations into a single vector equation $\mathbf{H} = \mathbf{K}^t \mathbf{S}$ where $\mathbf{H} = (h_\mu : \mu \in \mathrm{Par}(n))$ and $\mathbf{S} = (s_\lambda : \lambda \in \mathrm{Par}(n))$. Since $\mathbf{K}^t$ (the transpose of the Kostka matrix) is unitriangular and hence invertible, $\mathbf{S} = (\mathbf{K}^t)^{-1}\mathbf{H}$ is the *unique* vector $\mathbf{v}$ satisfying $\mathbf{H} = \mathbf{K}^t \mathbf{v}$.

From 10.75, we know that for each $\mu \vdash n$,

$$e_\mu(x_1, \ldots, x_N) = \sum_{\lambda \in \mathrm{Par}(n)} K_{\lambda,\mu} s_{\lambda'}(x_1, \ldots, x_N).$$

Applying the linear map ω to these equations produces the equations

$$h_\mu = \sum_{\lambda \in \mathrm{Par}(n)} K_{\lambda,\mu} \omega(s_{\lambda'}).$$

But this says that the vector $\mathbf{v} = (\omega(s_{\lambda'}) : \lambda \in \mathrm{Par}(n))$ satisfies $\mathbf{H} = \mathbf{K}^t\mathbf{v}$. By the uniqueness property mentioned above, $\mathbf{v} = \mathbf{S}$. So, for all $\lambda \in \mathrm{Par}(n)$, $s_\lambda = \omega(s_{\lambda'})$. Replacing λ by λ' (or applying ω to both sides) gives the result. □

What happens if we apply ω to the monomial basis of Λ_N? Since ω is a K-linear bijection, we get another basis of Λ_N that is different from those discussed so far. This basis is hard to describe directly, so it is given the following name.

10.103. Definition: Forgotten Basis for Λ_N. For each $\lambda \in \mathrm{Par}_N$, define the *forgotten symmetric polynomial* $\mathrm{fgt}_\lambda = \omega(m_\lambda)$. The set $\{\mathrm{fgt}_\lambda : \lambda \in \mathrm{Par}_N(k)\}$ is a basis of Λ_N^k.

10.22 Permutations and Tableaux

Iteration of the tableau insertion algorithm (§10.9) leads to some remarkable bijections that map permutations, words, and matrices to certain pairs of tableaux. These bijections were studied by Robinson, Schensted, and Knuth, and are therefore called *RSK correspondences.* We begin in this section by showing how permutations can be encoded using pairs of standard tableaux of the same shape.

10.104. Theorem: RSK Correspondence for Permutations. There is a bijection $\mathrm{RSK} : S_n \to \bigcup_{\lambda\in\mathrm{Par}(n)} \mathrm{SYT}(\lambda) \times \mathrm{SYT}(\lambda)$. Given $\mathrm{RSK}(w) = (P(w), Q(w))$, we call $P(w)$ the *insertion tableau for* w and $Q(w)$ the *recording tableau for* w.

Proof. Let $w \in S_n$ have one-line form $w = w_1w_2 \cdots w_n$. We construct a sequence of tableaux $P_0, P_1, \ldots, P_n = P(w)$ and a sequence of tableaux $Q_0, Q_1, \ldots, Q_n = Q(w)$ as follows. Initially, let P_0 and Q_0 be empty tableaux of shape (0). Suppose $1 \le i \le n$ and P_{i-1}, Q_{i-1} have already been constructed. Define $P_i = P_{i-1} \leftarrow w_i$ (the tableau obtained by insertion of w_i into P_i). Let (a, b) be the new cell in P_i created by this insertion. Define Q_i to be the tableau obtained from Q_{i-1} by placing the value i in the new cell (a, b). Informally, we build $P(w)$ by inserting $w_1, \ldots, w_n$ (in this order) into an initially empty tableau. We build $Q(w)$ by placing the numbers $1, 2, \ldots, n$ (in this order) in the new boxes created by each insertion. By construction, $Q(w)$ has the same shape as $P(w)$. Furthermore, since the new box at each stage is a corner box, one sees that $Q(w)$ will be a standard tableau. Finally, set $\mathrm{RSK}(w) = (P(w), Q(w))$.

To see that RSK is a bijection, we present an algorithm for computing the inverse map. Let (P, Q) be any pair of standard tableaux of the same shape $\lambda \in \mathrm{Par}(n)$. The idea is to recover the one-line form $w_1 \cdots w_n$ in reverse by uninserting entries from P, using the entries in Q to decide which box to remove at each stage (cf. §10.10). To begin, note that n occurs in some corner box (a, b) of Q (since Q is standard). Apply reverse insertion to P starting at (a, b) to obtain the unique tableau P_{n-1} and value w_n such that $P_{n-1} \leftarrow w_n$ is P with new box (a, b) (see 10.60). Let Q_{n-1} be the tableau obtained by erasing n from Q. Continue similarly: having computed P_i and Q_i such that Q_i is a standard tableau with i cells, let (a, b) be the corner box of Q_i containing i. Apply reverse insertion to P_i starting at (a, b) to obtain P_{i-1} and w_i. Then delete i from Q_i to obtain a standard tableau Q_{i-1} with $i-1$ cells. The resulting word $w = w_1w_2 \cdots w_n$ is a permutation of $\{1, 2, \ldots, n\}$ (since P contains each of these values exactly once), and our argument has shown that w is the *unique* object satisfying $\mathrm{RSK}(w) = (P, Q)$. So RSK is a bijection. □

10.105. Example. Let $w = 35164872 \in S_8$. Figure 10.1 illustrates the computation of $\mathrm{RSK}(w) = (P(w), Q(w))$. As an example of the inverse computation, let us determine the permutation $v = \mathrm{RSK}^{-1}(Q(w), P(w))$ (note that we have switched the order of the insertion and recording tableaux). Figure 10.2 displays the reverse insertions used to find $v_n, v_{n-1}, \ldots, v_1$. We see that $v = 38152476$.

	Insertion Tableau	Recording Tableau
insert 3:	$\begin{array}{c} 3 \end{array}$	$\begin{array}{c} 1 \end{array}$
insert 5:	$\begin{array}{cc} 3 & 5 \end{array}$	$\begin{array}{cc} 1 & 2 \end{array}$
insert 1:	$\begin{array}{cc} 1 & 5 \\ 3 & \end{array}$	$\begin{array}{cc} 1 & 2 \\ 3 & \end{array}$
insert 6:	$\begin{array}{ccc} 1 & 5 & 6 \\ 3 & & \end{array}$	$\begin{array}{ccc} 1 & 2 & 4 \\ 3 & & \end{array}$
insert 4:	$\begin{array}{ccc} 1 & 4 & 6 \\ 3 & 5 & \end{array}$	$\begin{array}{ccc} 1 & 2 & 4 \\ 3 & 5 & \end{array}$
insert 8:	$\begin{array}{cccc} 1 & 4 & 6 & 8 \\ 3 & 5 & & \end{array}$	$\begin{array}{cccc} 1 & 2 & 4 & 6 \\ 3 & 5 & & \end{array}$
insert 7:	$\begin{array}{cccc} 1 & 4 & 6 & 7 \\ 3 & 5 & 8 & \end{array}$	$\begin{array}{cccc} 1 & 2 & 4 & 6 \\ 3 & 5 & 7 & \end{array}$
insert 2:	$\begin{array}{cccc} 1 & 2 & 6 & 7 \\ 3 & 4 & 8 & \\ 5 & & & \end{array}$	$\begin{array}{cccc} 1 & 2 & 4 & 6 \\ 3 & 5 & 7 & \\ 8 & & & \end{array}$

FIGURE 10.1
Computation of RSK(35164872).

Let us compare the two-line forms of w and v:

$$w = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 5 & 1 & 6 & 4 & 8 & 7 & 2 \end{pmatrix}; \qquad v = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 8 & 1 & 5 & 2 & 4 & 7 & 6 \end{pmatrix}.$$

We see that v and w are inverse permutations!

The phenomenon observed in the last example holds in general: if $w \mapsto (P, Q)$ under the RSK correspondence, then $w^{-1} \mapsto (Q, P)$. To prove this fact, we must introduce a new way of visualizing the construction of the insertion and recording tableaux for w.

10.106. Definition: Cartesian Graph of a Permutation. Given a permutation $w = w_1 w_2 \cdots w_n \in S_n$, the graph of w (in the xy-plane) is the set $G(w) = \{(i, w_i) : 1 \le i \le n\}$.

For example, the graph of $w = 35164872$ is drawn in Figure 10.3.

To analyze the creation of the insertion and recording tableaux for RSK(w), we will annotate the graph of w by drawing lines as described in the following definitions.

10.107. Definition: Shadow Lines. Let $S = \{(x_1, y_1), \ldots, (x_k, y_k)\}$ be a finite set of points in the first quadrant. The *shadow* of S is

$$\mathrm{Shd}(S) = \{(u, v) \in \mathbb{R}^2 : \text{for some } i, u \ge x_i \text{ and } v \ge y_i\}.$$

Informally, the shadow consists of all points northeast of some point in S. The *first shadow line* $L_1(S)$ is the boundary of $\mathrm{Shd}(S)$. This boundary consists of an infinite vertical ray (part of the line $x = a_1$, say), followed by zero or more alternating horizontal and vertical line segments, followed by an infinite horizontal ray (part of the line $y = b_1$, say). Call a_1 and b_1 the *x-coordinate* and *y-coordinate* associated to this shadow line. Next, let S_1 be the set of points in S that lie on the first shadow line of S. The *second shadow line* $L_2(S)$ is

	Insertion Tableau	Recording Tableau	Output Value
initial tableau:	$\begin{array}{llll} 1&2&4&6\\ 3&5&\underline{7}\\ 8 \end{array}$	$\begin{array}{llll} 1&2&6&7\\ 3&4&\underline{8}\\ 5 \end{array}$	
uninsert 7:	$\begin{array}{llll} 1&2&4&\underline{7}\\ 3&5\\ 8 \end{array}$	$\begin{array}{llll} 1&2&6&\underline{7}\\ 3&4\\ 5 \end{array}$	6
uninsert 7:	$\begin{array}{lll} 1&2&\underline{4}\\ 3&5\\ 8 \end{array}$	$\begin{array}{lll} 1&2&\underline{6}\\ 3&4\\ 5 \end{array}$	7
uninsert 4:	$\begin{array}{ll} 1&2\\ 3&5\\ \underline{8} \end{array}$	$\begin{array}{ll} 1&2\\ 3&4\\ \underline{5} \end{array}$	4
uninsert 8:	$\begin{array}{ll} 1&5\\ 3&\underline{8} \end{array}$	$\begin{array}{ll} 1&2\\ 3&\underline{4} \end{array}$	2
uninsert 8:	$\begin{array}{ll} 1&8\\ \underline{3} \end{array}$	$\begin{array}{ll} 1&2\\ \underline{3} \end{array}$	5
uninsert 3:	$\begin{array}{ll} 3&\underline{8} \end{array}$	$\begin{array}{ll} 1&\underline{2} \end{array}$	1
uninsert 8:	$\underline{3}$	$\underline{1}$	8
uninsert 3:	empty	empty	3

FIGURE 10.2
Mapping pairs of standard tableaux to permutations.

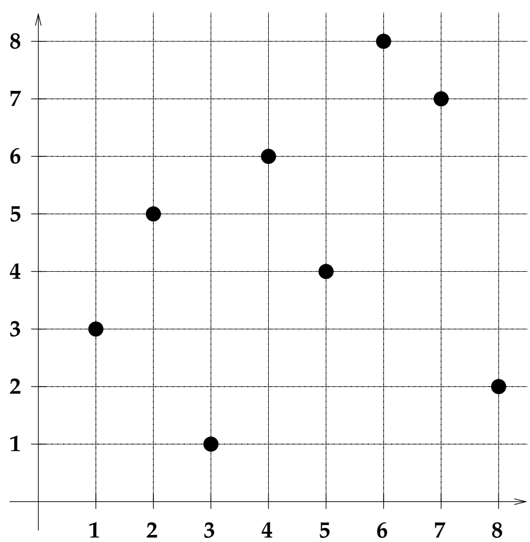


FIGURE 10.3
Cartesian graph of a permutation.

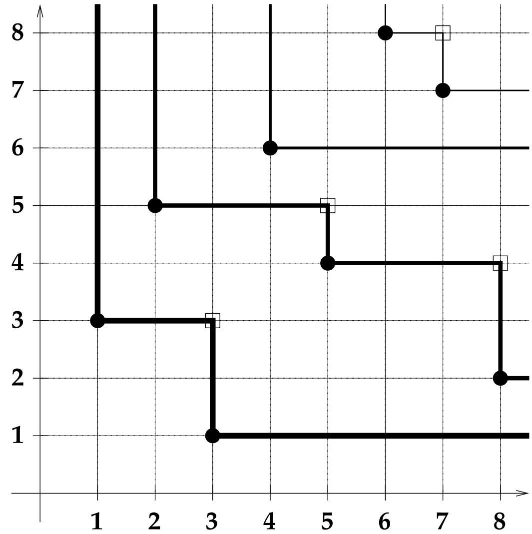


FIGURE 10.4
Shadow lines for a permutation graph.

the boundary of $\mathrm{Shd}(S \sim S_1)$, which has associated coordinates (a_2, b_2). Letting S_2 be the points in S that lie on the second shadow line, the *third shadow line* $L_3(S)$ is the boundary of $\mathrm{Shd}(S \sim (S_1 \cup S_2))$. We continue to generate shadow lines in this way until all points of S lie on some shadow line. Finally, the *first-order shadow diagram of* $w \in S_n$ consists of all shadow lines associated to the graph $G(w)$.

10.108. Example. The first-order shadow diagram of $w = 35164872$ is drawn in Figure 10.4. The x-coordinates associated to the shadow lines of w are $1, 2, 4, 6$. These x-coordinates agree with the entries in the first row of the recording tableau $Q(w)$, which we computed in 10.105. Similarly, the y-coordinates of the shadow lines are $1, 2, 6, 7$, which are precisely the entries in the first row of the insertion tableau $P(w)$. The next result explains why this happens, and shows that the shadow diagram contains complete information about the evolution of the first rows of $P(w)$ and $Q(w)$.

10.109. Theorem: Shadow Lines and RSK. Let $w \in S_n$ have first-order shadow lines $L_1, \ldots, L_k$ with associated coordinates $(x_1, y_1), \ldots, (x_k, y_k)$. Let $P_0, P_1, \ldots, P_n = P(w)$ and $Q_0, Q_1, \ldots, Q_n = Q(w)$ be the sequences of tableaux generated in the computation of $\mathrm{RSK}(w)$. For $0 \le i \le n$, the y-coordinates of the intersections of the shadow lines with the line $x = i + (1/2)$ are the entries in the first row of P_i, whereas the entries in the first row of Q_i consist of all $x_j \le i$. Whenever some shadow line L_r has a vertical segment from (i, a) down to (i, b), then $b = w_i$ and the insertion $P_i = P_{i-1} \leftarrow w_i$ bumps the value a out of the rth cell in the first row of P_{i-1}.

Proof. We proceed by induction on $i \ge 0$. The theorem holds when $i = 0$, since P_0 and Q_0 are empty, and no shadow lines intersect the line $x = 1/2$. Assume the result holds for $i - 1 < n$. Then the first row of P_{i-1} is $a_1 < a_2 < \cdots < a_j$, and the a's are the y-coordinates where the shadow lines hit the line $x = i - 1/2$. Consider the point (i, w_i), which is the unique point in $G(w)$ on the line $x = i$. First consider the case $w_i > a_j$. In this case, the first j shadow lines all pass underneath (i, w_i). It follows that (i, w_i) is the first point of

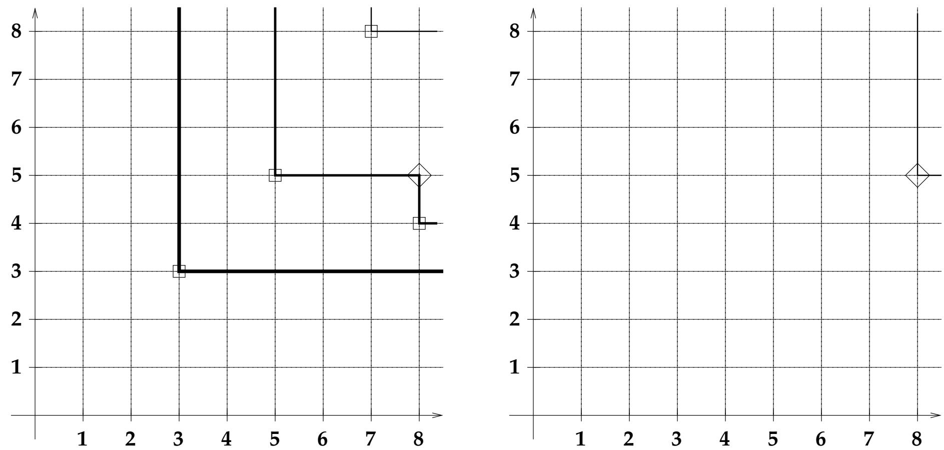


FIGURE 10.5
Higher-order shadow diagrams.

$G(w)$ on shadow line $L_{j+1}(G(w))$, so $x_{j+1} = i$. When we insert w_i into P_{i-1}, w_i goes at the end of the first row of P_{i-1} (since it exceeds the last entry a_j), and we place i in the corresponding cell in the first row of Q_i. The statements in the theorem regarding P_i and Q_i are true in this case. Now consider the case $w_i < a_j$. Suppose a_r is the smallest value in the first row of P_{i-1} exceeding w_i. Then insertion of w_i into P_{i-1} bumps a_r out of the first row. On the other hand, the point (i, w_i) lies between the points (i, a_{r-1}) and (i, a_r) in the shadow diagram (taking $a_0 = 0$). It follows from the way the shadow lines are drawn that shadow line L_r must drop from (i, a_r) to (i, w_i) when it reaches the line $x = i$. The statements of the theorem therefore hold for i in this case as well. □

To analyze the rows of $P(w)$ and $Q(w)$ below the first row, we iterate the shadow diagram construction as follows.

10.110. Definition: Iterated Shadow Diagrams. Let $L_1, \ldots, L_k$ be the shadow lines associated to a given subset S of $\mathbb{R}^2$. An *inner corner* is a point (a, b) at the top of one of the vertical segments of some shadow line. Let S' be the set of inner corners associated to S. The *second-order shadow diagram* of S is the shadow diagram associated to S'. We iterate this process to define all higher-order shadow diagrams of S.

For example, taking $w = 35164872$, Figure 10.5 displays the second-order and third-order shadow diagrams for $G(w)$.

10.111. Theorem: Higher-Order Shadows and RSK. For $w \in S_n$, let $L_1, \ldots, L_k$ be the shadow lines in the rth-order shadow diagram for $w \in S_n$, with associated coordinates $(x_1, y_1), \ldots, (x_k, y_k)$. Let $P_0, P_1, \ldots, P_n = P(w)$ and $Q_0, Q_1, \ldots, Q_n = Q(w)$ be the sequences of tableaux generated in the computation of RSK(w). For $0 \leq i \leq n$, the y-coordinates of the intersections of the shadow lines with the line $x = i + (1/2)$ are the entries in the rth row of P_i, whereas the entries in the rth row of Q_i consist of all $x_j \leq i$. Whenever some shadow line L_c has a vertical segment from (i, a) down to (i, b), then b is the value bumped out of row $r-1$ by the insertion $P_i = P_{i-1} \leftarrow w_i$, and b bumps the value a out of the cth cell in row r of P_{i-1}. (Take $b = w_i$ when $r = 1$.)

Proof. We use induction on $r \geq 1$. The base case $r = 1$ was proved in 10.109. Consider $r = 2$

next. The proof of 10.109 shows that the inner corners of the first-order shadow diagram of w are precisely those points (i, b) such that b is bumped out of the first row of P_{i-1} and inserted into the second row of P_{i-1} when forming P_i. The argument used to prove 10.109 can now be applied to this set of points. Whenever a point (i, b) lies above all second-order shadow lines approaching the line $x = i$ from the left, b gets inserted in a new cell at the end of the second row of P_i and the corresponding cell in Q_i receives the label i. Otherwise, if (i, b) lies between shadow lines L_{c-1} and L_c in the second-order diagram, then b bumps the value in the cth cell of the second row of P_{i-1} into the third row, and shadow line L_c moves down to level b when it reaches $x = i$. The statements in the theorem (for $r = 2$) follow exactly as before by induction on i. Iterating this argument establishes the analogous results for each $r > 2$. □

10.112. Theorem: RSK and Inversion. For all $w \in S_n$, if $\mathrm{RSK}(w) = (P, Q)$, then $\mathrm{RSK}(w^{-1}) = (Q, P)$.

Proof. Consider the picture consisting of $G(w)$ and the first-order shadow diagram. Suppose the shadow lines have associated x-coordinates $(a_1, \ldots, a_k)$ and y-coordinates $(b_1, \ldots, b_k)$. Let us reflect the picture through the line $y = x$ (which interchanges x-coordinates and y-coordinates). This reflection changes $G(w)$ into $G(w^{-1})$, since $(x, y) \in G(w)$ iff $y = w(x)$ iff $x = w^{-1}(y)$ iff $(y, x) \in G(w^{-1})$. We see from the geometric definition that the shadow lines for w get reflected into the shadow lines for w^{-1}. It follows from 10.109 that the first row of both $Q(w)$ and $P(w^{-1})$ is $a_1, \ldots, a_k$, whereas the first row of both $P(w)$ and $Q(w^{-1})$ is $b_1, \ldots, b_k$. The inner corners for w^{-1} are the reflections of the inner corners for w. So, we can apply the same argument to the higher-order shadow diagrams of w and w^{-1}. It follows that all rows of $P(w^{-1})$ (resp. $Q(w^{-1})$) agree with the corresponding rows of $Q(w)$ (resp. $P(w)$). □

10.23 Words and Tableaux

We now generalize the RSK algorithm to operate on arbitrary words, not just permutations.

10.113. Theorem: RSK Correspondence for Words. Let $W = X^n$ be the set of n-letter words over an ordered alphabet X. There is a bijection

$$\mathrm{RSK} : W \to \bigcup_{\lambda \in \mathrm{Par}(n)} \mathrm{SSYT}_X(\lambda) \times \mathrm{SYT}(\lambda).$$

We write $\mathrm{RSK}(w) = (P(w), Q(w))$; $P(w)$ is the *insertion tableau for* w and $Q(w)$ is the *recording tableau for* w. For all $x \in X$, the number of x's in w is the same as the number of x's in $P(w)$.

Proof. Given $w = w_1 w_2 \cdots w_n \in W$, we define sequences of tableaux P_0, P_1, …, P_n and Q_0, Q_1, …, Q_n as follows. P_0 and Q_0 are the empty tableau. If P_{i-1} and Q_{i-1} have been computed for some $i \leq n$, let $P_i = P_{i-1} \leftarrow w_i$. Suppose this insertion creates a new box (c, d); then we form Q_i from Q_{i-1} by placing the value i in the box (c, d). By induction on i, we see that every P_i is semistandard with values in X, every Q_i is standard, and P_i and Q_i have the same shape. We set $\mathrm{RSK}(w) = (P_n, Q_n)$. The letters in P_n (counting repetitions) are exactly the letters in w, so the last statement of the theorem holds.

Next we describe the inverse algorithm. Given (P, Q) with P semistandard and Q standard of the same shape, we construct semistandard tableaux P_n, P_{n-1}, …, P_0, standard

	Insertion Tableau	Recording Tableau
insert 2:	$\begin{array}{c}2\end{array}$	$\begin{array}{c}1\end{array}$
insert 1:	$\begin{array}{c}1\\2\end{array}$	$\begin{array}{c}1\\2\end{array}$
insert 1:	$\begin{array}{cc}1&1\\2&\end{array}$	$\begin{array}{cc}1&3\\2&\end{array}$
insert 3:	$\begin{array}{ccc}1&1&3\\2&&\end{array}$	$\begin{array}{ccc}1&3&4\\2&&\end{array}$
insert 2:	$\begin{array}{ccc}1&1&2\\2&3&\end{array}$	$\begin{array}{ccc}1&3&4\\2&5&\end{array}$
insert 1:	$\begin{array}{ccc}1&1&1\\2&2&\\3&&\end{array}$	$\begin{array}{ccc}1&3&4\\2&5&\\6&&\end{array}$
insert 3:	$\begin{array}{cccc}1&1&1&3\\2&2&&\\3&&&\end{array}$	$\begin{array}{cccc}1&3&4&7\\2&5&&\\6&&&\end{array}$
insert 1:	$\begin{array}{cccc}1&1&1&1\\2&2&3&\\3&&&\end{array}$	$\begin{array}{cccc}1&3&4&7\\2&5&8&\\6&&&\end{array}$

FIGURE 10.6
Computation of RSK(21132131).

tableaux Q_n, Q_{n-1}, $\ldots$, Q_0, and letters $w_n, w_{n-1}, \ldots, w_1$ as follows. Initially, $P_n = P$ and $Q_n = Q$. Suppose, for some $i \leq n$, that we have already constructed tableaux P_i and Q_i such that these tableaux have the same shape and consist of i boxes, P_i is semistandard, and Q_i is standard. The value i lies in a corner cell of Q_i; perform uninsertion starting from the same cell in P_i to get a smaller semistandard tableau P_{i-1} and a letter w_i. Let Q_{i-1} be Q_i with the i erased. At the end, output the word $w_1 w_2 \cdots w_n$. Using 10.60 and induction, one checks that $w = w_1 \cdots w_n$ is the unique word w with $\mathrm{RSK}(w) = (P, Q)$. So the RSK algorithm is a bijection. $\square$

10.114. Example. Let $w = 21132131$. We compute $\mathrm{RSK}(w)$ in Figure 10.6.

Next we investigate how the RSK algorithm is related to certain statistics on words and tableaux.

10.115. Definition: Descents and Major Index for Standard Tableaux. Let Q be a standard tableau with n cells. The *descent set of* Q, denoted $\mathrm{Des}(Q)$, is the set of all $k < n$ such that $k+1$ appears in a lower row of Q than k. The *descent count of* Q, denoted $\mathrm{des}(Q)$, is $|\mathrm{Des}(Q)|$. The *major index of* Q, denoted $\mathrm{maj}(Q)$, is $\sum_{k \in \mathrm{Des}(Q)} k$. (Compare to 6.27, which gives the analogous definitions for words.)

10.116. Example. For the standard tableau $Q = Q(w)$ shown at the bottom of Figure 10.6, we have $\mathrm{Des}(Q) = \{1, 4, 5, 7\}$, $\mathrm{des}(Q) = 4$, and $\mathrm{maj}(Q) = 17$. Here, $w = 21132131$. Note that $\mathrm{Des}(w) = \{1, 4, 5, 7\}$, $\mathrm{des}(w) = 4$, and $\mathrm{maj}(w) = 17$. This is not a coincidence.

10.117. Theorem: RSK Preserves Descents and Major Index. Let $w \in X^n$ be a word with recording tableau $Q = Q(w)$. Then $\mathrm{Des}(w) = \mathrm{Des}(Q)$, $\mathrm{des}(w) = \mathrm{des}(Q)$, and $\mathrm{maj}(w) = \mathrm{maj}(Q)$.

Proof. It suffices to prove $\mathrm{Des}(w) = \mathrm{Des}(Q)$. Let $P_0, P_1, \ldots, P_n$ and $Q_0, Q_1, \ldots, Q_n = Q$ be the sequences of tableaux computed when we apply the RSK algorithm to w. For each $k < n$, note that $k \in \mathrm{Des}(w)$ iff $w_k > w_{k+1}$, whereas $k \in \mathrm{Des}(Q)$ iff $k+1$ appears in a row below k's row in Q. So, for each $k < n$, we must prove $w_k > w_{k+1}$ iff $k+1$ appears in a row below k's row in Q. For this, we use the bumping comparison theorem 10.62. Consider the double insertion $(P_{k-1} \leftarrow w_k) \leftarrow w_{k+1}$. Let the new box in $P_{k-1} \leftarrow w_k$ be (i, j), and let the new box in $(P_{k-1} \leftarrow w_k) \leftarrow w_{k+1}$ be (r, s). By definition of the recording tableau, $Q(i, j) = k$ and $Q(r, s) = k + 1$. Now, if $w_k > w_{k+1}$, part 2 of 10.62 says that $i < r$ (and $j \geq s$). So $k + 1$ appears in a lower row than k in Q. If instead $w_k \leq w_{k+1}$, part 1 of 10.62 says that $i \geq r$ (and $j < s$). So $k + 1$ does not appear in a lower row than k in Q. $\square$

Now suppose the letters $x_1, \ldots, x_N$ in X are variables in some polynomial ring. Then we can view a word $w = w_1 \ldots w_n \in X^n$ as a *monomial* in the x_j's by forming the product of all the letters appearing in w (counting repetitions). Using 2.9, we can write

$$\sum_{w \in X^n} w = (x_1 + \cdots + x_N)^n = p_{(1^n)}(x_1, \ldots, x_N). \tag{10.12}$$

We can use the RSK algorithm and 10.117 to obtain a related identity involving Schur polynomials.

10.118. Theorem: Schur Expansion for Words weighted by Major Index. Let $X = \{x_1, \ldots, x_N\}$ where the x_i's are commuting indeterminates ordered by $x_1 < x_2 < \cdots < x_N$. For every $n \geq 1$,

$$\sum_{w \in X^n} t^{\mathrm{maj}(w)} w = \sum_{\lambda \in \mathrm{Par}(n)} \left(\sum_{Q \in \mathrm{SYT}(\lambda)} t^{\mathrm{maj}(Q)} \right) s_\lambda(x_1, \ldots, x_N). \tag{10.13}$$

Proof. The left side of (10.13) is the generating function for the weighted set X^n, where the weight of a word w is $t^{\mathrm{maj}(w)} w$. On the other hand, let us weight each set $\mathrm{SSYT}_X(\lambda)$ by taking $\mathrm{wt}(P)$ to be the product of the entries in P. Comparing to the definition of Schur polynomials, we see that the generating function for this weighted set is $s_\lambda(x_1, \ldots, x_N)$. Next, weight each set $\mathrm{SYT}(\lambda)$ by taking $\mathrm{wt}(Q) = t^{\mathrm{maj}(Q)}$ for $Q \in \mathrm{SYT}(\lambda)$. Finally, consider the set $\bigcup_{\lambda \in \mathrm{Par}(n)} \mathrm{SSYT}_X(\lambda) \times \mathrm{SYT}(\lambda)$ weighted by setting $\mathrm{wt}(P, Q) = \mathrm{wt}(P)\,\mathrm{wt}(Q)$. By the sum and product rules for weighted sets, the generating function for this last weighted set is precisely the right side of (10.13). To complete the proof, note that the RSK map is a weight-preserving bijection between X^n and $\bigcup_{\lambda \in \mathrm{Par}(n)} \mathrm{SSYT}_X(\lambda) \times \mathrm{SYT}(\lambda)$, because of 10.113 and 10.117. $\square$

Setting $t = 1$ in the previous result and using (10.12) gives the following formula for $p_{(1^n)}$ in terms of Schur polynomials.

10.119. Theorem: Schur Expansion of $p_{(1^n)}$. For all $n, N \geq 1$,

$$p_{(1^n)}(x_1, \ldots, x_N) = \sum_{\lambda \in \mathrm{Par}(n)} |\mathrm{SYT}(\lambda)| s_\lambda(x_1, \ldots, x_N).$$

10.120. Remark. The RSK correspondence can also be used to find the length of the longest weakly increasing or strictly decreasing subsequence of a given word. For details, see §12.11.

10.24 Matrices and Tableaux

Performing the RSK map on a word produces a pair consisting of one *semistandard* tableau and one *standard* tableau. We now define an RSK operation on matrices that will map each matrix to a pair of semistandard tableaux of the same shape. The first step is to encode the matrix as a certain biword.

10.121. Definition: Biword of a Matrix. Let $A = (a_{ij})$ be an $M \times N$ matrix with entries in $\mathbb{N}$. The *biword* of A is a two-row array

$$\mathrm{bw}(A) = \left(\begin{array}{cccc} i_1 & i_2 & \cdots & i_k \\ j_1 & j_2 & \cdots & j_k \end{array}\right)$$

constructed as follows. Start with the empty biword, and scan the rows of A from top to bottom, reading each row from left to right. Whenever a nonzero integer a_{ij} is encountered in the scan, write down a_{ij} copies of $\left(\begin{array}{c} i \\ j \end{array}\right)$ at the end of the current biword. The top row of $\mathrm{bw}(A)$ is called the *row word of* A and denoted $r(A)$. The bottom row of $\mathrm{bw}(A)$ is called the *column word of* A and denoted $c(A)$.

10.122. Example. Suppose A is the matrix

$$A = \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

The biword of A is

$$\mathrm{bw}(A) = \left(\begin{array}{cccccccc} 1 & 1 & 1 & 2 & 2 & 2 & 2 & 3 \\ 1 & 1 & 3 & 2 & 4 & 4 & 4 & 3 \end{array}\right).$$

10.123. Theorem: Matrices vs. Biwords. Let X be the set of all $M \times N$ matrices with entries in $\mathbb{N}$. Let Y be the set of all biwords $w = \left(\begin{array}{cccc} i_1 & i_2 & \cdots & i_k \\ j_1 & j_2 & \cdots & j_k \end{array}\right)$ satisfying the following conditions: (a) $i_1 \leq i_2 \leq \cdots \leq i_k$; (b) if $i_s = i_{s+1}$, then $j_s \leq j_{s+1}$; (c) $i_s \leq M$ for all s; (d) $j_s \leq N$ for all s. The map $\mathrm{bw} : X \to Y$ is a bijection. Suppose $A = (a_{ij})$ has biword $\mathrm{bw}(A)$ as above. Then i appears $\sum_j a_{ij}$ times in $r(A)$, j appears $\sum_i a_{ij}$ times in $c(A)$, and $k = \sum_{i,j} a_{ij}$.

Proof. To show that bw maps X *into* Y, we must show that $\mathrm{bw}(A)$ satisfies conditions (a) through (d). Condition (a) holds since we scan the rows of A from top to bottom. Condition (b) holds since each row is scanned from left to right. Condition (c) holds since A has M rows. Condition (d) holds since A has N columns. We can invert bw as follows. Given a biword $w \in Y$, let A be the $M \times N$ matrix such that, for all $i \leq M$ and all $j \leq N$, a_{ij} is the number of indices s with $i_s = i$ and $j_s = j$. The last statements in the theorem follow from the way we constructed $r(A)$ and $c(A)$. □

10.124. Theorem: RSK Correspondence for Biwords. Let Y be the set of biwords defined in 10.123. Let Z be the set of pairs (P, Q) of semistandard tableaux of the same shape such that P has entries in $\{1, 2, \ldots, N\}$ and Q has entries in $\{1, 2, \ldots, M\}$. There is a bijection $\mathrm{RSK} : Y \to Z$ that "preserves content." This means that if $\binom{v}{w} \in Y$ maps to $(P, Q) \in Z$, then for all $i \leq M$, v and Q contain the same number of i's, and for all $j \leq N$, w and P contain the same number of j's.

Proof. Write $v = i_1 \leq i_2 \leq \cdots \leq i_k$ and $w = j_1, j_2, \ldots, j_k$, where $i_s = i_{s+1}$ implies $j_s \leq j_{s+1}$. As in the previous RSK maps, we build sequences of insertion tableaux $P_0, P_1, \ldots, P_k$ and recording tableaux $Q_0, Q_1, \ldots, Q_k$. Initially, P_0 and Q_0 are empty. Having constructed P_s and Q_s, let $P_{s+1} = P_s \leftarrow j_{s+1}$. If the new box created by this insertion is (a, b), obtain Q_{s+1} from Q_s by setting $Q_{s+1}(a, b) = i_{s+1}$. The final output is $\mathrm{RSK}(\binom{v}{w}) = (P_k, Q_k)$.

By construction, P_k is semistandard with entries consisting of the letters in w, and the entries of Q_k are the letters in v. But, is $Q = Q_k$ a semistandard tableau? To see that it is, note that we obtain Q by successively placing a weakly increasing sequence of numbers $i_1 \leq i_2 \leq \cdots \leq i_k$ into new corner boxes of an initially empty tableau. It follows that the rows and columns of Q weakly increase. To see that columns of Q *strictly* increase, consider what happens during the placement of a run of equal numbers into Q, say $i = i_s = i_{s+1} = \cdots = i_t$. By definition of Y, we have $j_s \leq j_{s+1} \leq \cdots \leq j_t$. When we insert this weakly increasing sequence into the P-tableau, the resulting sequence of new boxes forms a horizontal strip by 10.63. So, the corresponding boxes in Q (which consist of all the boxes labeled i in Q) also form a horizontal strip. This means that there are never two equal numbers in a given column of Q.

The inverse algorithm reconstructs the words v and w in reverse, starting with i_k and j_k. Given (P, Q), look for the rightmost occurrence of the largest letter in Q, which must reside in a corner box. Let i_k be this letter. Erase this cell from Q, and perform reverse insertion on P starting at the same cell to recover j_k. Iterate this process on the resulting smaller tableaux. We have $i_k \geq \cdots \geq i_1$ since we remove the largest letter in Q at each stage. When we remove a string of equal letters from Q, say $i = i_t = i_{t-1} = \cdots = i_s$, the associated letters removed from P must satisfy $j_t \geq j_{t-1} \geq \cdots \geq j_s$. This follows, as above, from the bumping comparison theorem 10.62. For instance, if $j_{t-1} > j_t$, then the new box created at stage t would be weakly left of the new box created at stage $t-1$, which contradicts the requirement of choosing the *rightmost* i in Q when recovering i_t and j_t. It follows that the inverse algorithm does produce a biword in Y, as required. ☐

Composing the two preceding bijections gives the following result.

10.125. Theorem: RSK Correspondence for Matrices. For every $M, N \geq 1$, there is a bijection between the set of $M \times N$ matrices with entries in $\mathbb{N}$ and the set

$$\bigcup_{\lambda \in \mathrm{Par}} \mathrm{SSYT}_N(\lambda) \times \mathrm{SSYT}_M(\lambda)$$

given by $A \mapsto \mathrm{RSK}(\mathrm{bw}(A))$. If $(a_{ij}) \mapsto (P, Q)$ under this bijection, then the number of j's in P is $\sum_i a_{ij}$, and the number of i's in Q is $\sum_j a_{ij}$.

10.126. Example. Let us compute the pair of tableaux associated to the matrix A from 10.122. Looking at the biword of A, we must insert the sequence $c(A) = (1, 1, 3, 2, 4, 4, 4, 3)$ into the P-tableau, recording the entries in $r(A) = (1, 1, 1, 2, 2, 2, 2, 3)$ in the Q-tableau. This computation appears in Figure 10.7.

10.127. Theorem: Cauchy Identity for Schur Polynomials. For all $M, N \geq 1$, we have the formal power series identity in $\mathbb{Q}[[x_1, \ldots, x_M, y_1, \ldots, y_N]]$:

$$\prod_{i=1}^{M} \prod_{j=1}^{N} \frac{1}{1 - x_i y_j} = \sum_{\lambda \in \mathrm{Par}} s_\lambda(y_1, \ldots, y_N) s_\lambda(x_1, \ldots, x_M). \tag{10.14}$$

Proof. We interpret each side as the generating function for a suitable set of weighted

	Insertion Tableau	Recording Tableau
insert 1, record 1:	1	1
insert 1, record 1:	1 1	1 1
insert 3, record 1:	1 1 3	1 1 1
insert 2, record 2:	1 1 2 3	1 1 1 2
insert 4, record 2:	1 1 2 4 3	1 1 1 2 2
insert 4, record 2:	1 1 2 4 4 3	1 1 1 2 2 2
insert 4, record 2:	1 1 2 4 4 4 3	1 1 1 2 2 2 2
insert 3, record 3:	1 1 2 3 4 4 3 3	1 1 1 2 2 2 2 3

FIGURE 10.7
Applying the RSK map to a biword.

objects. For the left side, consider $M \times N$ matrices with entries in $\mathbb{N}$. Let the weight of a matrix $A = (a_{ij})$ be

$$\mathrm{wt}(A) = \prod_{i=1}^{M} \prod_{j=1}^{N} (x_i y_j)^{a_{ij}}.$$

We can build such a matrix by choosing the entries $a_{ij} \in \mathbb{N}$ one at a time. For fixed i and j, the generating function for the choice of a_{ij} is

$$1 + x_i y_j + (x_i y_j)^2 + \cdots + (x_i y_j)^k + \cdots = \frac{1}{1 - x_i y_j}.$$

By the product rule for weighted sets, we see that the left side of (10.14) is the generating function for this set of matrices. On the other hand, the RSK bijection converts each matrix in this set to a pair of semistandard tableaux of the same shape. This bijection $A \mapsto (P, Q)$ will be weight-preserving provided that we weight each occurrence of j in P by y_j and each occurrence of i in Q by x_i. With these weights, the generating function for $\mathrm{SSYT}_N(\lambda)$ is $s_\lambda(y_1, \ldots, y_N)$, and the generating function for $\mathrm{SSYT}_M(\lambda)$ is $s_\lambda(x_1, \ldots, x_M)$. It now follows from the sum and product rules for weighted sets that the right side of (10.14) is the generating function for the weighted set $\bigcup_{\lambda \in \mathrm{Par}} \mathrm{SSYT}_N(\lambda) \times \mathrm{SSYT}_M(\lambda)$. Since RSK is a weight-preserving bijection, the proof is complete. $\square$

10.25 Cauchy Identities

In the last section, we found a formula expressing the product $\prod_{i,j}(1 - x_i y_j)^{-1}$ as a sum of products of Schur polynomials. Next, we derive other formulas for this product that involve other kinds of symmetric polynomials.

10.128. Theorem: Cauchy Identities. For all $M, N \geq 1$, we have the formal power series identities:

$$\begin{aligned}\prod_{i=1}^{M}\prod_{j=1}^{N}\frac{1}{1-x_iy_j} &= \sum_{\lambda\in\mathrm{Par}_N} h_\lambda(x_1,\ldots,x_M)m_\lambda(y_1,\ldots,y_N)\\ &= \sum_{\lambda\in\mathrm{Par}_M} m_\lambda(x_1,\ldots,x_M)h_\lambda(y_1,\ldots,y_N)\\ &= \sum_{\lambda\in\mathrm{Par}} \frac{p_\lambda(x_1,\ldots,x_M)p_\lambda(y_1,\ldots,y_N)}{z_\lambda}.\end{aligned}$$

Proof. Recall from 10.92 the product expansion

$$\prod_{i=1}^{M}\frac{1}{1-x_it} = \sum_{k=0}^{\infty} h_k(x_1,\ldots,x_M)t^k.$$

Replacing t by y_j, where j is a fixed index, we obtain

$$\prod_{i=1}^{M}\frac{1}{1-x_iy_j} = \sum_{k=0}^{\infty} h_k(x_1,\ldots,x_M)y_j^k.$$

Taking the product over j gives

$$\prod_{i=1}^{M}\prod_{j=1}^{N}\frac{1}{1-x_iy_j} = \prod_{j=1}^{N}\sum_{k_j=0}^{\infty} h_{k_j}(x_1,\ldots,x_M)y_j^{k_j}.$$

Let us expand the product on the right side using the generalized distributive law (§2.1), suitably extended to handle infinite series within the product. We obtain

$$\prod_{i=1}^{M}\prod_{j=1}^{N}\frac{1}{1-x_iy_j} = \sum_{k_1=0}^{\infty}\cdots\sum_{k_N=0}^{\infty}\prod_{j=1}^{N} h_{k_j}(x_1,\ldots,x_M)y_j^{k_j}.$$

Let us reorganize the sum on the right side by grouping together summands indexed by sequences $(k_1,\ldots,k_N)$ that can be sorted to give the same partition λ. Since $h_{k_1}h_{k_2}\cdots h_{k_N} = h_\lambda$ for all such sequences, the right side becomes

$$\sum_{\lambda\in\mathrm{Par}_N} h_\lambda(x_1,\ldots,x_M) \sum_{\substack{(k_1,\ldots,k_N)\in\mathbb{N}^N:\\ \mathrm{sort}(k_1,\ldots,k_N)=\lambda}} y_1^{k_1}y_2^{k_2}\cdots y_N^{k_N}.$$

The inner sum is precisely the definition of $m_\lambda(y_1,\ldots,y_N)$. So the first formula of the theorem is proved. The second formula follows from the same argument, interchanging the roles of the x's and y's.

To obtain the formula involving power sums, we again start with 10.92, which can be written

$$\prod_{k=1}^{MN}\frac{1}{1-z_kt} = \sum_{n=0}^{\infty} h_n(z_1,\ldots,z_{MN})t^n.$$

Replace the MN variables z_k by the MN quantities x_iy_j (with $1\leq i\leq M$ and $1\leq j\leq N$). We obtain

$$\prod_{i=1}^{M}\prod_{j=1}^{N}\frac{1}{1-x_iy_jt} = \sum_{n=0}^{\infty} h_n(x_1y_1,x_1y_2,\ldots,x_My_N)t^n.$$

Now use 10.96 to rewrite the right side in terms of power sums:

$$\prod_{i=1}^{M}\prod_{j=1}^{N}\frac{1}{1-x_iy_jt}=\sum_{n=0}^{\infty}t^n\sum_{\lambda\in\mathrm{Par}(n)}p_\lambda(x_1y_1,x_1y_2,\ldots,x_My_N)/z_\lambda.$$

Observe next that, for all $k\geq 1$,

$$\begin{aligned}p_k(x_1y_1,\ldots,x_My_N) &= \sum_{i=1}^{M}\sum_{j=1}^{N}(x_iy_j)^k=\sum_{i=1}^{M}\sum_{j=1}^{N}x_i^ky_j^k\\ &= \left(\sum_{i=1}^{M}x_i^k\right)\cdot\left(\sum_{j=1}^{N}y_j^k\right)=p_k(x_1,x_2,\ldots,x_M)p_k(y_1,y_2,\ldots,y_N).\end{aligned}$$

It follows from this that, for any partition λ,

$$p_\lambda(x_1y_1,\ldots,x_My_N)=p_\lambda(x_1,x_2,\ldots,x_M)p_\lambda(y_1,y_2,\ldots,y_N).$$

We therefore find that

$$\prod_{i=1}^{M}\prod_{j=1}^{N}\frac{1}{1-x_iy_jt}=\sum_{n=0}^{\infty}t^n\sum_{\lambda\in\mathrm{Par}(n)}\frac{p_\lambda(x_1,x_2,\ldots,x_M)p_\lambda(y_1,y_2,\ldots,y_N)}{z_\lambda}. \qquad (10.15)$$

Setting $t=1$ gives the final formula of the theorem. □

10.26 Dual Bases

Now we introduce a scalar product on the vector spaces Λ_N^k. For convenience, we assume that N (the number of variables) is not less than k (the degree of the polynomials in the space), so that the various bases of Λ_N^k are indexed by *all* the integer partitions of k.

10.129. Definition: Hall Scalar Product on Λ_N^k. For $N\geq k$, define the *Hall scalar product* on the vector space Λ_N^k by setting (for all $\mu,\nu\in\mathrm{Par}(k)$)

$$\langle p_\mu,p_\nu\rangle=0 \text{ if } \mu\neq\nu,\qquad \langle p_\mu,p_\mu\rangle=z_\mu,$$

and extending by bilinearity. In more detail, given $f,g\in\Lambda_N^k$, choose scalars $a_\mu,b_\mu\in K$ such that $f=\sum_\mu a_\mu p_\mu$ and $g=\sum_\nu b_\nu p_\nu$. Then $\langle f,g\rangle=\sum_\mu a_\mu b_\mu z_\mu\in K$.

10.130. Definition: Orthonormal Bases and Dual Bases. Suppose $N\geq k$ and $B_1=\{f_\mu:\mu\in\mathrm{Par}(k)\}$ and $B_2=\{g_\mu:\mu\in\mathrm{Par}(k)\}$ are two bases of Λ_N^k. B_1 is called an *orthonormal basis* iff $\langle f_\mu,f_\nu\rangle=\chi(\mu=\nu)$ for all $\mu,\nu\in\mathrm{Par}(k)$. B_1 and B_2 are called *dual bases* iff $\langle f_\mu,g_\nu\rangle=\chi(\mu=\nu)$ for all $\mu,\nu\in\mathrm{Par}(k)$.

For example, taking $F=\mathbb{C}$, $\{p_\mu/\sqrt{z_\mu}:\mu\in\mathrm{Par}(k)\}$ is an orthonormal basis of Λ_N^k. The next theorem allows us to detect dual bases by looking at expansions of the product $\prod_{i,j}(1-x_iy_j)^{-1}$.

10.131. Theorem: Characterization of Dual Bases. Suppose $N\geq k$ and $B_1=\{f_\mu:\mu\in\mathrm{Par}(k)\}$ and $B_2=\{g_\mu:\mu\in\mathrm{Par}(k)\}$ are two bases of Λ_N^k. B_1 and B_2 are dual bases iff

$$\left(\prod_{i=1}^{N}\prod_{j=1}^{N}\frac{1}{1-x_iy_jt}\right)_k=\sum_{\mu\in\mathrm{Par}(k)}f_\mu(x_1,\ldots,x_N)g_\mu(y_1,\ldots,y_N),$$

where the left side is the coefficient of t^k in the indicated product.

Proof. Comparing the displayed equation to (10.15), we must prove that B_1 and B_2 are dual bases iff

$$\sum_{\mu\in\mathrm{Par}(k)} p_\mu(x_1,\ldots,x_N)p_\mu(y_1,\ldots,y_N)/z_\mu = \sum_{\mu\in\mathrm{Par}(k)} f_\mu(x_1,\ldots,x_N)g_\mu(y_1,\ldots,y_N).$$

The idea of the proof is to convert each condition into a statement about matrices. Since $\{p_\mu\}$ and $\{p_\mu/z_\mu\}$ are bases of Λ_N^k, there exist scalars $a_{\mu,\nu}, b_{\mu,\nu} \in K$ satisfying

$$f_\nu = \sum_\mu a_{\mu,\nu}p_\mu, \qquad g_\nu = \sum_\mu b_{\mu,\nu}(p_\mu/z_\mu).$$

Define matrices $\mathbf{A} = (a_{\mu,\nu})$ and $\mathbf{B} = (b_{\mu,\nu})$. By bilinearity, we compute (for all $\lambda,\nu \in \mathrm{Par}(k)$):

$$\begin{aligned}\langle f_\lambda, g_\nu\rangle &= \left\langle \sum_\mu a_{\mu,\lambda}p_\mu, \sum_\rho b_{\rho,\nu}p_\rho/z_\rho \right\rangle\\ &= \sum_{\mu,\rho} a_{\mu,\lambda}b_{\rho,\nu}\langle p_\mu, p_\rho/z_\rho\rangle\\ &= \sum_\mu a_{\mu,\lambda}b_{\mu,\nu} = (\mathbf{A}^t\mathbf{B})_{\lambda,\nu}.\end{aligned}$$

It follows that $\{f_\lambda\}$ and $\{g_\nu\}$ are dual bases iff $\mathbf{A}^t\mathbf{B} = \mathbf{I}$ (the identity matrix of size $|\mathrm{Par}(k)|$).

On the other hand, writing $\vec{x} = (x_1,\ldots,x_N)$ and $\vec{y} = (y_1,\ldots,y_N)$, we have

$$\sum_{\mu\in\mathrm{Par}(k)} f_\mu(\vec{x})g_\mu(\vec{y}) = \sum_{\mu,\alpha,\beta} a_{\alpha,\mu}b_{\beta,\mu}p_\alpha(\vec{x})p_\beta(\vec{y})/z_\beta.$$

Now, one may check that the polynomials

$$\{p_\alpha(\vec{x})p_\beta(\vec{y})/z_\beta : (\alpha,\beta)\in \mathrm{Par}(k)\times\mathrm{Par}(k)\}$$

are linearly independent, using the fact that the power-sum polynomials in one set of variables are linearly independent. It follows that the expression given above for $\sum_{\mu\in\mathrm{Par}(k)} f_\mu(\vec{x})g_\mu(\vec{y})$ will be equal to $\sum_{\alpha\in\mathrm{Par}(k)} p_\alpha(\vec{x})p_\alpha(\vec{y})/z_\alpha$ iff $\sum_\mu a_{\alpha,\mu}b_{\beta,\mu} = 0$ for all $\alpha\neq\beta$ and $\sum_\mu a_{\alpha,\mu}b_{\alpha,\mu} = 1$ for all α. In matrix form, these equations say that $\mathbf{A}\mathbf{B}^t = \mathbf{I}$. This matrix equation is equivalent to $\mathbf{B}^t\mathbf{A} = \mathbf{I}$ (since all the matrices are square), which is equivalent in turn to $\mathbf{A}^t\mathbf{B} = \mathbf{I}$. We saw above that this last condition holds iff B_1 and B_2 are dual bases, so the proof is complete. □

10.132. Theorem: Dual Bases of Λ_N^k. For $N \geq k$, $\{s_\mu(x_1,\ldots,x_N) : \mu\in\mathrm{Par}(k)\}$ is an orthonormal basis of Λ_N^k. Also $\{m_\mu(x_1,\ldots,x_N) : \mu \in \mathrm{Par}(k)\}$ and $\{h_\mu(x_1,\ldots,x_N) : \mu \in \mathrm{Par}(k)\}$ are dual bases of Λ_N^k.

Proof. In 10.127, replace every x_i by tx_i. Since s_λ is homogeneous of degree $|\lambda|$, we obtain

$$\prod_{i=1}^N\prod_{j=1}^N \frac{1}{1-tx_iy_j} = \sum_{\lambda\in\mathrm{Par}} s_\lambda(y_1,\ldots,y_N)s_\lambda(x_1,\ldots,x_N)t^{|\lambda|}.$$

Extracting the coefficient of t^k gives

$$\left(\prod_{i=1}^N\prod_{j=1}^N \frac{1}{1-x_iy_jt}\right)_k = \sum_{\lambda\in\mathrm{Par}(k)} s_\lambda(x_1,\ldots,x_N)s_\lambda(x_1,\ldots,x_N).$$

So 10.131 applies to show that $\{s_\lambda : \lambda \in \text{Par}(k)\}$ is an orthonormal basis. We proceed similarly to see that the m's and h's are dual, starting with 10.128. □

10.133. Theorem: ω is an Isometry. For $N \geq k$, the map $\omega : \Lambda_N^k \to \Lambda_N^k$ is an isometry relative to the Hall scalar product. In other words, for all $f, g \in \Lambda_N^k$, $\langle \omega(f), \omega(g)\rangle = \langle f, g\rangle$. Therefore, ω sends an orthonormal basis (resp. dual bases) of Λ_N^k to an orthonormal basis (resp. dual bases) of Λ_N^k.

Proof. Write $f = \sum_\mu a_\mu p_\mu$ and $g = \sum_\nu b_\nu p_\nu$ for suitable scalars $a_\mu, b_\nu \in K$. By linearity of ω and bilinearity of the Hall scalar product, we compute

$$\begin{aligned}
\langle \omega(f), \omega(g)\rangle &= \left\langle \omega\left(\sum_\mu a_\mu p_\mu\right), \omega\left(\sum_\nu b_\nu p_\nu\right)\right\rangle \\
&= \sum_\mu \sum_\nu a_\mu b_\nu \langle \omega(p_\mu), \omega(p_\nu)\rangle \\
&= \sum_\mu \sum_\nu a_\mu b_\nu \epsilon_\mu \epsilon_\nu \langle p_\mu, p_\nu\rangle \\
&= \sum_\mu a_\mu b_\mu \epsilon_\mu^2 z_\mu .
\end{aligned}$$

The last step follows since we only get a nonzero scalar product when $\nu = \mu$. Now, the last expression is

$$\sum_\mu a_\mu b_\mu z_\mu = \langle f, g\rangle .$$ □

10.134. Theorem: Duality of e_μ's and fgt_ν's. For $N \geq k$, the bases $\{e_\mu : \mu \in \text{Par}(k)\}$ and $\{\text{fgt}_\mu : \mu \in \text{Par}(k)\}$ (forgotten basis) are dual. Moreover,

$$\left(\prod_{i=1}^N \prod_{j=1}^N \frac{1}{1 - x_i y_j t}\right)_k = \sum_{\lambda \in \text{Par}(k)} e_\lambda(x_1, \ldots, x_N)\text{fgt}_\lambda(y_1, \ldots, y_N).$$

Proof. We know that $\{m_\mu\}$ and $\{h_\mu\}$ are dual bases. Since $\text{fgt}_\mu = \omega(m_\mu)$ and $e_\mu = \omega(h_\mu)$, $\{\text{fgt}_\mu\}$ and $\{e_\mu\}$ are dual bases. The indicated product formula now follows from 10.131. □

Summary

Table 10.2 summarizes information about five bases for the vector space Λ_N^k of symmetric polynomials in N variables that are homogeneous of degree k. The statements about dual bases assume $N \geq k$. Recall that $\text{Par}_N(k)$ is the set of integer partitions of k into at most N parts, while $\text{Par}_N(k)'$ is the set of partitions of k where every part is at most N. Table 10.3 gives formulas and recursions for expressing certain symmetric polynomials as linear combinations of other symmetric polynomials. Further identities of a similar kind can be found in the summary of Chapter 11.

- *Skew Shapes and Skew Schur Polynomials.* A skew shape μ/ν is obtained by removing the diagram of ν from the diagram of μ. A semistandard tableau of this shape is a filling

TABLE 10.2
Bases for symmetric polynomials.

Basis of Λ_N^k	Definition	Dual Basis	Action of ω
Monomial $\{m_\mu : \mu \in \mathrm{Par}_N(k)\}$	$m_\mu = \sum_{\substack{\alpha\in\mathbb{N}^N: \\ \mathrm{sort}(\alpha)=\mu}} x_1^{\alpha_1}\cdots x_N^{\alpha_N}$	$\{h_\mu\}$	$\omega(m_\mu) = \mathrm{fgt}_\mu$
Elementary $\{e_\mu : \mu \in \mathrm{Par}_N(k)'\}$	$e_k = \sum_{1\le i_1<i_2<\cdots<i_k\le N} x_{i_1}x_{i_2}\cdots x_{i_k}$, $e_\mu = e_{\mu_1}e_{\mu_2}\cdots e_{\mu_s}$	$\{\mathrm{fgt}_\mu\}$	$\omega(e_\mu) = h_\mu$
Complete $\{h_\mu : \mu \in \mathrm{Par}_N(k)\}$ or $\{h_\mu : \mu \in \mathrm{Par}_N(k)'\}$	$h_k = \sum_{1\le i_1\le i_2\le\cdots\le i_k\le N} x_{i_1}x_{i_2}\cdots x_{i_k}$, $h_\mu = h_{\mu_1}h_{\mu_2}\cdots h_{\mu_s}$	$\{m_\mu\}$	$\omega(h_\mu) = e_\mu$
Power-sum $\{p_\mu : \mu \in \mathrm{Par}_N(k)'\}$	$p_k = \sum_{i=1}^N x_i^k$, $p_\mu = \prod_j p_{\mu_j}$	$\{p_\mu/z_\mu\}$	$\omega(p_\mu) = \epsilon_\mu p_\mu$
Schur $\{s_\mu : \mu \in \mathrm{Par}_N(k)\}$	$s_\mu = \sum_{T\in\mathrm{SSYT}_N(\mu)} x^{c(T)}$	$\{s_\mu\}$	$\omega(s_\mu) = s_{\mu'}$

TABLE 10.3
Expansions and recursions for symmetric polynomials.

Monomial expansion of Schur polys.:	$s_\lambda = \sum_{\mu\in\mathrm{Par}(\lvert\lambda\rvert)} K_{\lambda,\mu} m_\mu$.
Schur expansion of complete polys.:	$h_\alpha = \sum_{\lambda\in\mathrm{Par}(\lvert\alpha\rvert)} K_{\lambda,\alpha} s_\lambda$.
Schur expansion of elementary polys.:	$e_\alpha = \sum_{\lambda\in\mathrm{Par}(\lvert\alpha\rvert)} K_{\lambda',\alpha} s_\lambda$.
Power-sum expansion of h_n:	$h_n = \sum_{\mu\in\mathrm{Par}(n)} z_\mu^{-1} p_\mu$.
Power-sum expansion of e_n:	$e_n = \sum_{\mu\in\mathrm{Par}(n)} \epsilon_\mu z_\mu^{-1} p_\mu$.
Schur expansion of $p_{(1^n)}$:	$p_{(1^n)} = \sum_{\lambda\in\mathrm{Par}(n)} \lvert\mathrm{SYT}(\lambda)\rvert s_\lambda$.
Monomial expansion of skew Schur polys.:	$s_{\mu/\nu} = \sum_{\rho\in\mathrm{Par}(\lvert\mu/\nu\rvert)} K_{\mu/\nu,\rho} m_\rho$.
Schur expansion of $s_\mu h_\alpha$:	$s_\mu h_\alpha = \sum_\lambda K_{\lambda/\mu,\alpha} s_\lambda$.
Schur expansion of $s_\mu e_\alpha$:	$s_\mu e_\alpha = \sum_\lambda K_{\lambda'/\mu',\alpha} s_\lambda$.
Recursion linking e's and h's:	$\sum_{i=0}^m (-1)^i e_i h_{m-i} = \chi(m=0)$.
Recursion linking h's and p's:	$\sum_{i=0}^{n-1} h_i p_{n-i} = n h_n$.
Recursion linking e's and p's:	$\sum_{i=0}^{n-1} (-1)^i e_i p_{n-i} = (-1)^{n-1} n e_n$.

of the cells in μ/ν so that rows weakly increase and columns strictly increase. The skew Schur polynomial in N variables indexed by μ/ν is

$$s_{\mu/\nu}(x_1,\ldots,x_N) = \sum_{T\in\mathrm{SSYT}_N(\mu/\nu)} x^{c(T)},$$

where the power of x_i is the number of i's in T. Skew Schur polynomials are symmetric, since an involution exists that switches the frequencies of i's and $(i+1)$'s in semistandard tableaux of this shape.

- *Orderings on Partitions.* For $\mu,\nu \in \mathrm{Par}(k)$, $\mu \leq_{\mathrm{lex}} \nu$ means that $\mu = \nu$ or the first nonzero entry of $\nu - \mu$ is positive; $\leq_{\mathrm{lex}}$ is a total ordering on $\mathrm{Par}(k)$. We say $\mu \trianglelefteq \nu$ (ν dominates μ) iff $\mu_1 + \cdots + \mu_i \leq \nu_1 + \cdots + \nu_i$ for all $i \geq 1$; $\trianglelefteq$ is a partial ordering on $\mathrm{Par}(k)$. We have $\mu \trianglelefteq \nu$ iff $\nu' \trianglelefteq \mu'$ iff μ can be transformed into ν by a sequence of raising operators (moving one box to a higher row). Also, $\mu \trianglelefteq \nu$ implies $\mu \leq_{\mathrm{lex}} \nu$.

- *Kostka Numbers.* For $\nu \subseteq \mu \in \mathrm{Par}$ and $\alpha \in \mathbb{N}^N$, the Kostka number $K_{\mu/\nu,\alpha}$ is the number of semistandard tableaux of shape μ/ν and content α. We have $K_{\lambda,\lambda} = 1$ for all $\lambda \in \mathrm{Par}$, and $K_{\lambda,\mu} \neq 0$ implies $\mu \trianglelefteq \lambda$ and $\mu \leq_{\mathrm{lex}} \lambda$.

- *Tableau Insertion.* Given a semistandard tableau T and value x, we obtain a new semistandard tableau $T \leftarrow x$ as follows. The element x bumps the leftmost value $y > x$ in the top row into the second row, and this bumping continues recursively until a value is placed in a new box at the end of some row. The bumping path moves weakly left as it goes down. Insertion is invertible if we know which corner box is the new one. If we insert a weakly increasing sequence into T, the new boxes move strictly right and weakly higher, producing a horizontal strip. If we insert a strictly decreasing sequence into T, the new boxes move weakly left and strictly lower, producing a vertical strip.

- *Pieri Rules.* (a) $s_\mu h_k = \sum_\nu s_\nu$ where we sum over all ν such that ν/μ is a horizontal strip of size k. (b) $s_\mu e_k = \sum_\nu s_\nu$ where we sum over all ν such that ν/μ is a vertical strip of size k. If there are N variables, we only use shapes ν with at most N parts.

- *Algebraic Independence.* A list of polynomials $(f_1,\ldots,f_k)$ is algebraically independent over K iff the only polynomial $g \in K[z_1,\ldots,z_k]$ with $g(f_1,\ldots,f_k) = 0$ is the zero polynomial. Equivalently, the monomials $\{f_1^{i_1}\cdots f_k^{i_k} : i_j \in \mathbb{N}\}$ are linearly independent over K. Polynomials $f_1,\ldots,f_k \in K[x_1,\ldots,x_k]$ are algebraically independent over K if (and only if) $\det ||D_i f_j||_{1\leq i,j\leq k} \neq 0$.

- *Algebraically Independent Symmetric Polynomials.* In the ring $K[x_1,\ldots,x_N]$, the lists $(p_1,\ldots,p_N)$, $(h_1,\ldots,h_N)$, and $(e_1,\ldots,e_N)$ are algebraically independent over K. So we can view the ring and K-vector space Λ_N as isomorphic to the polynomial ring $K[z_1,\ldots,z_N]$ in three ways (an isomorphism can send $z_i \in K[z_1,\ldots,z_N]$ to either p_i or h_i or e_i).

- *Generating Functions for e's and h's.* We have

$$E_N(t) = \prod_{i=1}^N (1 + x_i t) = \sum_{k=0}^N e_k(x_1,\ldots,x_N)t^k;$$

$$H_N(t) = \prod_{i=1}^N (1 - x_i t)^{-1} = \sum_{k=0}^N h_k(x_1,\ldots,x_N)t^k;$$

so $H_N(t)E_N(-t) = 1$.

- *Dual Bases and Cauchy Identities.* Assume $N \geq k$. The Hall scalar product on Λ_N^k is defined by setting $\langle p_\mu, p_\nu \rangle = z_\mu \chi(\mu = \nu)$ and extending by bilinearity. Two bases $\{f_\mu : \mu \in \mathrm{Par}(k)\}$ and $\{g_\mu : \mu \in \mathrm{Par}(k)\}$ of Λ_N^k are dual relative to this inner product iff they satisfy the Cauchy identity

$$\text{coefficient of } t^k \text{ in } \prod_{i=1}^{N} \prod_{j=1}^{N} \frac{1}{1 - x_i y_j t} = \sum_{\mu \in \mathrm{Par}(k)} f_\mu(x_1, \ldots, x_N) g_\mu(y_1, \ldots, y_N).$$

- *The Map ω.* The map $\omega : \Lambda_N \to \Lambda_N$ is defined by sending c to c (for $c \in K$) and p_i to $(-1)^{i-1} p_i$, and extending by the universal mapping property of polynomial rings. Note $\omega(p_\mu) = \epsilon_\mu p_\mu$ where $\epsilon_\mu = (-1)^{|\mu| - \ell(\mu)}$. The map ω is an involution ($\omega^2 = \mathrm{id}$), an isomorphism of rings and vector spaces, and (for all $k \leq N$) an isometry of Λ_N^k (so $\langle \omega(f), \omega(g) \rangle = \langle f, g \rangle$ for $f, g \in \Lambda_N^k$).

- *RSK Correspondences.* There are bijections between: (a) permutations in S_n and pairs (P, Q) of standard tableaux of the same shape $\lambda \in \mathrm{Par}(n)$; (b) words in X^n and pairs (P, Q) where $P \in \mathrm{SSYT}_X(\lambda)$ and $Q \in \mathrm{SYT}(\lambda)$ for some $\lambda \in \mathrm{Par}(n)$; (c) $M \times N$ matrices with values in $\mathbb{N}$ and pairs (P, Q) where $P \in \mathrm{SSYT}_N(\lambda)$ and $Q \in \mathrm{SSYT}_M(\lambda)$. In each case, one inserts successive entries into P, using Q to record the locations of new boxes. For (c), one must first encode the matrix as a biword. If $w \in S_n$ maps to (P, Q), then w^{-1} maps to (Q, P). If $w \in X^n$ maps to (P, Q), then $\mathrm{Des}(w) = \mathrm{Des}(Q)$, $\mathrm{des}(w) = \mathrm{des}(Q)$, and $\mathrm{maj}(w) = \mathrm{maj}(Q)$, where $\mathrm{Des}(Q)$ is the set of $k < n$ such that $k+1$ is in a lower row of Q than k, $\mathrm{des}(Q) = |\mathrm{Des}(Q)|$, and $\mathrm{maj}(Q) = \sum_{i \in \mathrm{Des}(Q)} i$.

Exercises

10.135. Draw all skew shapes μ/ν where $\mu \vdash 6$ and $\nu \vdash 3$. Indicate which skew shapes are horizontal (resp. vertical) strips.

10.136. Given a skew shape $S \subseteq \mathbb{N} \times \mathbb{N}$, describe how to calculate the number of different pairs of partitions (μ, ν) such that $S = \mu/\nu$.

10.137. Find necessary and sufficient algebraic conditions on the parts of μ and ν to ensure that the skew shape μ/ν is (a) a horizontal strip; (b) a vertical strip.

10.138. How many horizontal strips are contained in $\{1, 2, \ldots, a\} \times \{1, 2, \ldots, b\}$?

10.139. If $|\mu/\nu| = n$ and $|X| = k$, how many tableaux with values in X have shape μ/ν?

10.140. List all the tableaux in: (a) $\mathrm{SSYT}_5((3, 2))$; (b) $\mathrm{SSYT}_2((3, 2))$; (c) $\mathrm{SYT}((3, 2, 1))$.

10.141. Give a direct counting argument to determine $|\mathrm{SYT}(\mu)|$ when μ is a hook.

10.142. Prove that $|\mathrm{SYT}(\mu/\nu)| = |\mathrm{SYT}(\mu'/\nu')|$ for all skew shapes μ/ν.

10.143. Compute $s_{(2,2)}(x_1, \ldots, x_N)$ for $N = 3, 4, 5$ by enumerating tableaux.

10.144. Compute $s_{(2,2)/(1)}(x_1, \ldots, x_N)$ for $N = 2, 3, 4$ by enumerating tableaux.

10.145. Find the coefficients of the following monomials in $s_{(3,2,1)}(x_1, \ldots, x_6)$ by enumerating tableaux: (a) $x_1 x_2 x_3 x_4 x_5 x_6$; (b) $x_1^2 x_2^2 x_3^2$; (c) $x_1^3 x_2^3$; (d) $x_1^2 x_2 x_3 x_4 x_5$; (e) $x_1 x_2 x_3^2 x_4 x_5$; (f) $x_1 x_2 x_3 x_4 x_5^2$.

10.146. Let $N \geq 4$. Enumerate tableaux to confirm that the coefficients of $x_1^2 x_2 x_3^2 x_4$, $x_1 x_2^2 x_3 x_4^2$, and $x_1 x_2^2 x_3^2 x_4$ in $s_{(4,3)/(1)}(x_1, \ldots, x_N)$ are all equal to 6, as claimed in 10.17. What happens to these coefficients if $N < 4$?

10.147. For which values of N is $s_{\mu/\nu}(x_1, \ldots, x_N) = 0$?

10.148. Compute: (a) $p_4(x_1, x_2, x_3)$; (b) $e_3(x_1, \ldots, x_5)$; (c) $h_3(x_1, x_2, x_3)$; (d) $m_{(3,2,2)}(x_1, \ldots, x_4)$.

10.149. For $\mu = (2, 1)$, compute: (a) $p_\mu(x_1, x_2, x_3)$; (b) $e_\mu(x_1, x_2, x_3)$; (c) $h_\mu(x_1, x_2, x_3)$.

10.150. How many monomials appear with nonzero coefficient in: (a) $e_k(x_1, \ldots, x_n)$; (b) $h_k(x_1, \ldots, x_n)$?

10.151. How many monomials appear with nonzero coefficient in $m_\mu(x_1, \ldots, x_N)$?

10.152. Give a direct proof that the polynomials $e_k(x_1, \ldots, x_N)$ and $h_k(x_1, \ldots, x_N)$ (as defined in 10.21 and 10.22) are symmetric.

10.153. Find a skew shape μ/ν such that $e_3 h_4 h_2 e_5 h_1 = s_{\mu/\nu}$.

10.154. Prove that any finite product of skew Schur polynomials is a skew Schur polynomial.

10.155. Check that the set of homogeneous polynomials of degree k in $R = K[x_1, \ldots, x_N]$ is a vector subspace of R. Conclude that Λ_N^k is a subspace of Λ_N for each $k \geq 0$.

10.156. Write down an explicit basis for the K-vector space Λ_3^7.

10.157. Compute $\dim(\Lambda_N^k)$ for each choice of k and N in the range $1 \leq k \leq 6$ and $1 \leq N \leq 6$.

10.158. Suppose $\{f_i : i \in I\}$ is a collection of nonzero polynomials in $R = K[x_1, \ldots, x_N]$ such that, whenever some x^α appears in some f_i with nonzero coefficient, the coefficient of x^α in every other f_j is zero. Prove that $\{f_i : i \in I\}$ is linearly independent over K.

10.159. Compute the following Kostka numbers:
(a) $K_{(3,3,2),(2,1,2,1,1,1)}$; (b) $K_{(3,2,2,1),(2,2,1,1,1,1)}$; (c) $K_{(5,5),(1^{10})}$; (d) $K_{(3,3,3)/(2,1),(2,2,1,1)}$.

10.160. Compute the image of the first tableau in the proof of 10.33 under the maps f_i, for $i = 1, 2, 4, 5, 6, 7, 8$.

10.161. Use the maps f_i in the proof of 10.33 to compute specific bijections between the three collections of six tableaux in 10.17. (This calculation was begun in 10.34.)

10.162. Express the Schur polynomials $s_\mu(x_1, x_2, x_3, x_4, x_5)$ as explicit linear combinations of monomial symmetric polynomials, for all $\mu \vdash 4$ and $\mu \vdash 5$.

10.163. Express the skew Schur polynomial $s_{(3,3,2)/(1)}(x_1, \ldots, x_8)$ as a linear combination of monomial symmetric polynomials.

10.164. For all partitions $\mu \vdash 3$, express h_μ and e_μ in terms of monomial symmetric polynomials by viewing h_μ and e_μ as instances of skew Schur polynomials.

10.165. (a) Find a recursion characterizing the Kostka numbers $K_{\mu/\nu,\alpha}$. (b) Use (a) to write a computer program for computing Kostka numbers.

10.166. Check that $\leq_{\text{lex}}$ is a total ordering of the set $\text{Par}(k)$, for each $k \geq 0$.

10.167. Prove that $\trianglelefteq$ is a total ordering of $\mathrm{Par}(k)$ iff $k \leq 5$.

10.168. (a) List the integer partitions of 7 in lexicographic order. (b) Find all pairs $\mu, \nu \vdash 7$ such that $\mu \leq_{\mathrm{lex}} \nu$ but $\mu \ntrianglelefteq \nu$.

10.169. (a) Find an ordered sequence of raising operators that changes $\mu = (5, 4, 2, 1, 1)$ to $\nu = (7, 3, 2, 1)$. (b) How many such sequences are there?

10.170. Prove or disprove: for all partitions $\mu, \nu \vdash k$, $\mu \leq_{\mathrm{lex}} \nu$ iff $\nu' \leq_{\mathrm{lex}} \mu'$.

10.171. Let $\mu, \nu \vdash k$. Can you prove that $\mu \trianglelefteq \nu$ implies $\nu' \trianglelefteq \mu'$ by arguing directly from the definitions, without using raising operators?

10.172. Define an ordering $\leq_{\mathrm{lex}}$ on the set $\mathbb{N}^N$ as in 10.36. Show that $\leq_{\mathrm{lex}}$ is a total ordering of $\mathbb{N}^N$ satisfying the following *well-ordering* property: there exists no infinite strictly decreasing sequence

$$\alpha^{(1)} >_{\mathrm{lex}} \alpha^{(2)} >_{\mathrm{lex}} \cdots >_{\mathrm{lex}} \alpha^{(k)} >_{\mathrm{lex}} \cdots \qquad (\alpha^{(j)} \in \mathbb{N}^N).$$

10.173. Define the *lex degree* of a nonzero polynomial $f(x_1, \ldots, x_N) \in R = K[x_1, \ldots, x_N]$, denoted $\deg(f)$, to be the largest $\alpha \in \mathbb{N}^N$ (relative to the lexicographic ordering defined in 10.172) such that x^α occurs with nonzero coefficient in f. Prove that $\deg(gh) = \deg(g) + \deg(h)$ for all nonzero $g, h \in R$, and $\deg(g + h) \leq \max(\deg(g), \deg(h))$ whenever both sides are defined.

10.174. (a) Find the Kostka matrix indexed by all partitions of 4. (b) Invert this matrix, and thereby express the monomial symmetric polynomials $m_\mu(x_1, x_2, x_3, x_4)$ (for $\mu \vdash 4$) as linear combinations of Schur polynomials.

10.175. Find the Kostka matrix indexed by partitions in $\mathrm{Par}_3(7)$, and invert it.

10.176. Let $\mathbf{K}$ be the Kostka matrix indexed by all partitions of 8. How many nonzero entries does this matrix have?

10.177. Suppose A is an $n \times n$ matrix with integer entries such that $\det(A) = \pm 1$. Prove that A^{-1} has all integer entries. (In particular, this applies when A is a Kostka matrix.)

10.178. Suppose $\{v_i : i \in I\}$ is a basis for a finite-dimensional K-vector space V, $\{w_i : i \in I\}$ is an indexed family of vectors in V, and for some total ordering $\leq$ of I and some scalars $a_{ij} \in K$ with $a_{ii} \neq 0$, we have $w_i = \sum_{j \leq i} a_{ij} v_j$ for all $i \in I$. Prove that $\{w_i : i \in I\}$ is a basis of V.

10.179. Let T be the tableau in 10.53. Confirm that $T \leftarrow 1$ and $T \leftarrow 0$ are as stated in that example. Also, compute $T \leftarrow i$ for $i = 2, 4, 5, 7$, and verify that 10.54 holds.

10.180. Let T be the semistandard tableau

$$\begin{array}{|c|c|c|c|c|c|c|}
\hline
2 & 2 & 3 & 5 & 5 & 7 & 7 \\
\hline
3 & 3 & 4 & 6 & 7 & 8 \\
\cline{1-6}
4 & 5 & 5 & 8 & 8 \\
\cline{1-5}
6 & 6 & 6 & 9 \\
\cline{1-4}
7 & 8 & 8 \\
\cline{1-3}
8 \\
\cline{1-1}
\end{array} \; .$$

Compute $T \leftarrow i$ for $1 \leq i \leq 9$.

10.181. Suppose we apply the tableau insertion algorithm 10.52 to a tableau T of skew shape. Are 10.54 and 10.55 still true?

10.182. Give a non-recursive description of $T \leftarrow x$ in the case where: (a) x is larger than every entry of T; (b) x is smaller than every entry of T.

10.183. Let T be the tableau in 10.53. Perform reverse insertion starting at each corner box of T to obtain smaller tableaux T_i and values x_i. Verify that $T_i \leftarrow x_i = T$ for each answer.

10.184. Let T be the tableau in 10.180. Perform reverse insertion starting at each corner box of T, and verify that properties 10.58(a),(b) hold in each case.

10.185. Prove 10.58(c).

10.186. Prove 10.59.

10.187. Express $s_{(4,4,3,1,1)}h_1$ as a sum of Schur polynomials.

10.188. Let T be the tableau in 10.53. Successively insert $1, 2, 2, 3, 5, 5$ into T, and verify that the assertions of the bumping comparison theorem hold.

10.189. Let T be the tableau in 10.53. Successively insert $7, 5, 3, 2, 1$ into T, and verify that the assertions of the bumping comparison theorem hold.

10.190. Let T be the tableau in 10.180. Successively insert $1, 1, 3, 3, 3, 4$ into T, and verify that the assertions of the bumping comparison theorem hold.

10.191. Let T be the tableau in 10.180. Successively insert $7, 6, 5, 3, 2, 1$ into T, and verify that the assertions of the bumping comparison theorem hold.

10.192. Let T be the tableau in 10.61 of shape $\mu = (5, 4, 4, 4, 1)$. For each shape ν such that ν/μ is a horizontal strip of size 3, find a weakly increasing sequence $x_1 \leq x_2 \leq x_3$ such that $(((T \leftarrow x_1) \leftarrow x_2) \leftarrow x_3)$ has shape ν, or prove that no such sequence exists.

10.193. Repeat the previous exercise, replacing horizontal strips by vertical strips and weakly increasing sequences by strictly decreasing sequences.

10.194. Prove 10.64(b).

10.195. Let T be the tableau in 10.180. Find the unique semistandard tableau S of shape $(7, 5, 4, 4, 1, 1)$ and $z_1 \leq z_2 \leq z_3 \leq z_4$ such that $T = S \leftarrow z_1 z_2 z_3 z_4$.

10.196. Let T be the tableau in 10.180. Find the unique semistandard tableau S of shape $(6, 6, 5, 3, 2)$ and $z_1 > z_2 > z_3 > z_4$ such that $T = S \leftarrow z_1 z_2 z_3 z_4$.

10.197. Expand each symmetric polynomial into sums of Schur polynomials: (a) $s_{(4,3,1)}e_2$; (b) $s_{(2,2)}h_3$; (c) $s_{(2,2,1,1,1)}h_4$; (d) $s_{(3,3,2)}e_3$.

10.198. Use the Pieri rule to find the Schur expansions of $h_{(3,2,1)}$, $h_{(3,1,2)}$, $h_{(1,2,3)}$, and $h_{(1,3,2)}$, and verify that the answers agree with those found in 10.67.

10.199. Expand each symmetric polynomial into sums of Schur polynomials: (a) $h_{(2,2,2)}$; (b) $h_{(5,3)}$; (c) $s_{(3,2)}h_{(2,1)}$; (d) $s_{(6,3,2,2)/(3,2)}$.

10.200. Find the coefficients of the following Schur polynomials in the Schur expansion of $h_{(3,2,2,1,1)}$: (a) $s_{(9)}$; (b) $s_{(5,4)}$; (c) $s_{(4,4,1)}$; (d) $s_{(2,2,2,2,1)}$; (e) $s_{(3,3,3)}$; (f) $s_{(3,2,2,1,1)}$.

10.201. Use 10.73 to compute the monomial expansions of $h_\mu(x_1, x_2, x_3, x_4)$ for all partitions μ of size at most four.

10.202. Let $\alpha = (\alpha_1, \ldots, \alpha_s)$. Prove that the coefficient of $m_\lambda(x_1, \ldots, x_N)$ in the monomial expansion of $h_\alpha(x_1, \ldots, x_N)$ is the number of $s \times N$ matrices A with entries in $\mathbb{N}$ such that $\sum_{j=1}^{N} A(i,j) = \alpha_i$ for $1 \le i \le s$ and $\sum_{i=1}^{s} A(i,j) = \lambda_j$ for $1 \le j \le N$.

10.203. Use the Pieri rules to compute the Schur expansions of: (a) $e_{(3,3,1)}$; (b) $e_{(5,3)}$; (c) $s_{(3,2)}e_{(2,1)}$; (d) $s_{(4,3,3,3,1,1)/(3,1,1,1)}$.

10.204. Find the coefficients of the following Schur polynomials in the Schur expansion of $e_{(4,3,2,1)}$: (a) $s_{(4,3,2,1)}$; (b) $s_{(5,5)}$; (c) $s_{(2,2,2,2,2)}$; (d) $s_{(2,2,2,1^4)}$; (e) $s_{(1^{10})}$.

10.205. Use 10.78 to express $e_{(2,2,1)}$ and $e_{(3,2)}$ as linear combinations of monomial symmetric polynomials.

10.206. Prove the formula for $s_\mu e_\alpha$ stated in Table 10.3.

10.207. Let $\alpha = (\alpha_1, \ldots, \alpha_s)$. Find a combinatorial interpretation for the coefficient of $m_\lambda(x_1, \ldots, x_N)$ in the monomial expansion of $e_\alpha(x_1, \ldots, x_N)$ in terms of certain $s \times N$ matrices (cf. 10.202).

10.208. Prove that the following lists of polynomials are algebraically dependent by exhibiting an explicit dependence relation: (a) $h_i(x_1, x_2)$ for $1 \le i \le 3$; (b) $e_i(x_1, x_2, x_3)$ for $1 \le i \le 4$; (c) $p_i(x_1, x_2, x_3)$ for $1 \le i \le 4$.

10.209. Prove that any sublist of an algebraically independent list is algebraically independent.

10.210. Suppose $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{N}^N$ is a partition. Show that $\deg(e_1^{\alpha_1-\alpha_2} e_2^{\alpha_2-\alpha_3} \cdots e_{N-1}^{\alpha_{N-1}-\alpha_N} e_N^{\alpha_N}) = \alpha$ (see 10.173 for the definition of lex degree).

10.211. Algorithmic Proof of the Fundamental Theorem of Symmetric Polynomials. Prove that the following algorithm will express any $f \in \Lambda_N$ as a polynomial in the elementary symmetric polynomials $e_i(x_1, \ldots, x_N)$ (where $1 \le i \le N$) in finitely many steps. If $f = 0$, use the zero polynomial. Otherwise, let the term of largest degree in f (see 10.173) be cx^α where $c \in K$ is nonzero. Use symmetry of f to show that $\alpha_1 \ge \alpha_2 \ge \cdots \ge \alpha_N$, and that $f - ce_1^{\alpha_1-\alpha_2} e_2^{\alpha_2-\alpha_3} \cdots e_{N-1}^{\alpha_{N-1}-\alpha_N} e_N^{\alpha_N}$ is either 0 or has degree $\beta <_{\text{lex}} \alpha$ (see 10.210). Continue similarly to express this new polynomial (and hence f) as a polynomial in the e_i's.

10.212. Use the algorithm in the preceding exercise to express $m_{(2,1)}(x_1, x_2, x_3, x_4)$ and $p_3(x_1, x_2, x_3, x_4)$ as a polynomial in $\{e_i(x_1, x_2, x_3, x_4) : 1 \le i \le 4\}$.

10.213. Use the test in 10.83 to verify that the polynomials $h_i(x_1, x_2, x_3)$ for $1 \le i \le 3$ are algebraically independent. Can you generalize this computation to more than 3 variables?

10.214. Use the test in 10.83 to verify that the polynomials $e_i(x_1, x_2, x_3, x_4)$ for $1 \le i \le 4$ are algebraically independent. Can you generalize this computation to more than 4 variables?

10.215. Compute the images of

$$\left(4, \begin{array}{|c|}\hline 1\\\hline 3\\\hline 4\\\hline 5\\\hline\end{array}, \begin{array}{|c|c|c|c|}\hline 1&2&2&3\\\hline\end{array}\right) \quad \text{and} \quad \left(4, \begin{array}{|c|}\hline 2\\\hline 3\\\hline 4\\\hline 5\\\hline\end{array}, \begin{array}{|c|c|c|c|}\hline 1&1&2&3\\\hline\end{array}\right)$$

under the involution I in the proof of 10.87.

10.216. Write out all the matched pairs $(z, I(z))$ in the proof of 10.87 when: (a) $N = 2$ and $m = 3$; (b) $N = 3$ and $m = 2$.

10.217. Imitate the argument in the proof of 10.88 to show that algebraic independence of $(h_1, \ldots, h_N)$ in $K[x_1, \ldots, x_N]$ implies algebraic independence of $(e_1, \ldots, e_N)$.

10.218. (a) Prove the recursion $e_k(x_1, \ldots, x_N) = e_k(x_1, \ldots, x_{N-1}) + e_{k-1}(x_1, \ldots, x_{N-1})x_N$ for $k, N \geq 1$. What are the initial conditions? (b) Find a similar recursion for $h_k(x_1, \ldots, x_N)$.

10.219. (a) Prove $s'(n, k) = e_{n-k}(1, 2, \ldots, n-1)$. (b) Prove $S(n, k) = h_{n-k}(1, 2, \ldots, k)$.

10.220. Prove 10.91 by expanding $\prod_{i=1}^{N}(X - r_i)$ using the generalized distributive law.

10.221. Consider the polynomial $p = x^5 - 2x^4 + 5x^3 + 7x^2 - x - 4$, which has five roots $r_1, \ldots, r_5 \in \mathbb{C}$. Compute: (a) the sum of the roots; (b) the product of the roots; (c) $e_3(r_1, \ldots, r_5)$; (d) the sum of the squares of the roots; (e) $\sum_{i \neq j} r_i^2 r_j$.

10.222. Use 10.92 to calculate the coefficient of t^4 in the multiplicative inverse of $(1 - 2x)(1 - 3x)(1 - 5x)$.

10.223. Let A be an $n \times n$ complex matrix. What is the relationship between the coefficients of the characteristic polynomial $\det(tI - A)$ and the eigenvalues $r_1, \ldots, r_n$ of A?

10.224. Use (10.8) to show that $p_i(x_1, \ldots, x_N)$ (for $1 \leq i \leq N$) are algebraically independent over K iff $h_i(x_1, \ldots, x_N)$ (for $1 \leq i \leq N$) are algebraically independent over K.

10.225. Use (10.9) to show that $p_i(x_1, \ldots, x_N)$ (for $1 \leq i \leq N$) are algebraically independent iff $e_i(x_1, \ldots, x_N)$ (for $1 \leq i \leq N$) are algebraically independent.

10.226. Consider the maps f and g from the proof of 10.93. Compute

$$f\left(\left(5,\ \begin{array}{|c|c|c|c|c|}\hline 2&4&4&5&5\\\hline\end{array},\ \begin{array}{|c|c|}\hline 4&4\\\hline\end{array}\right)\right) \text{ and } g\left(\begin{array}{|c|c|c|c|c|c|c|c|}\hline 1&1^*&1&2&2&4&6&6\\\hline\end{array}\right).$$

10.227. Consider the maps I and g from the proof of 10.94. Compute

$$I\left(3,\ \begin{array}{|c|}\hline 1\\\hline 3\\\hline 4\\\hline\end{array},\ \begin{array}{|c|c|c|}\hline 3&3&3\\\hline\end{array}\right),\quad I\left(3,\ \begin{array}{|c|}\hline 1\\\hline 3\\\hline 4\\\hline\end{array},\ \begin{array}{|c|c|c|}\hline 2&2&2\\\hline\end{array}\right),\quad I\left(3,\ \begin{array}{|c|}\hline 1\\\hline 3\\\hline 4\\\hline\end{array},\ \begin{array}{|c|}\hline 5\\\hline\end{array}\right),\quad I\left(3,\ \begin{array}{|c|}\hline 1\\\hline 3\\\hline 4\\\hline\end{array},\ \begin{array}{|c|}\hline 4\\\hline\end{array}\right).$$

For any objects that are fixed points of I, compute the images of those objects under g.

10.228. Write $\sum_{i=1}^{N} \frac{x_i}{1 - x_i t}$ in terms of suitable symmetric polynomials.

10.229. Obtain 10.93 and 10.94 algebraically by differentiating the generating functions $H_N(t)$ and $E_N(-t)$.

10.230. Use the recursions 10.93 and 10.94 to verify the formulas for h_4, $3!e_3$, and $4!e_4$ stated in 10.95.

10.231. Complete the proof of 10.96 by checking that $g(f(y_0)) = y_0$ and $f(g(z_0)) = z_0$, and, in general, $g \circ f = \mathrm{id}_Y$ and $f \circ g = \mathrm{id}_X$.

10.232. Let g be the map in the proof of 10.96. Compute $g(z_1)$ and $g(z_2)$, where

$$z_1 = \left[\begin{array}{ccccccccc} w: & 3 & 7 & 2 & 5 & 4 & 6 & 8 & 1 \\ T: & 2 & 4 & 4 & 4 & 4 & 6 & 6 & 6 \end{array}\right]; \quad z_2 = \left[\begin{array}{ccccccccc} w: & 2 & 1 & 4 & 3 & 7 & 5 & 6 & 8 \\ T: & 1 & 1 & 1 & 2 & 3 & 3 & 3 & 3 \end{array}\right].$$

10.233. Let f be the map in the proof of 10.96. Compute $f(y)$, where

$$y = \left((2,2,2,2),(2,5)(3,8)(4,6)(1,7),\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 2 & 3 & 2 & 2 & 2 & 3 & 3 \end{pmatrix}\right).$$

10.234. Let I be the involution in the proof of 10.98. Compute $I(z_1)$, $I(z_2)$, and $I(f(y))$, where z_1, z_2, and y are the objects given in the preceding two exercises.

10.235. Let A be an $n \times n$ complex matrix with eigenvalues $r_1, \ldots, r_n$. (a) Show that the trace of A, defined by $\mathrm{tr}(A) = \sum_{i=1}^{n} A(i,i)$, is $p_1(r_1, \ldots, r_n)$. (b) For $k \geq 1$, express $\mathrm{tr}(A^k)$ as a function of $r_1, \ldots, r_n$. (c) Suppose $n = 5$ and $(\mathrm{tr}(A^k) : k = 1, 2, \ldots, 5) = (3, 41, -93, 693, -2957)$. Find the characteristic polynomial of A.

10.236. Compute: (a) $\omega(h_3)$; (b) $\omega(p_{(3,2,1,1)})$; (c) $\omega(e_{(4,4)})$; (d) $\omega(s_{(5,3,3,1,1,1)})$.

10.237. Show that there exists a unique automorphism of the ring and K-vector space Λ_N mapping each $c \in K$ to itself and sending each p_i to $-p_i$. Compute the image of h_n and of e_n under this automorphism.

10.238. In the proof of 10.101(b), where is the assumption $n \leq N$ needed?

10.239. Compute the polynomials $\mathrm{fgt}_\lambda(x_1, x_2, x_3)$ for all partitions of size at most 3.

10.240. Compute $\mathrm{RSK}(w)$ for all $w \in S_3$.

10.241. Compute $\mathrm{RSK}^{-1}(P,Q)$ for all pairs P, Q of standard tableaux of shape $(2,2)$.

10.242. Let $w = 41572863 \in S_8$. Compute $\mathrm{RSK}(w)$ and $\mathrm{RSK}(w^{-1})$. Verify that 10.112 holds in this case.

10.243. Consider the pair of standard tableaux

$$P = \begin{array}{|c|c|c|}\hline 1 & 3 & 6 \\ \hline 2 & 4 & 8 \\ \hline 5 & 7 \\ \cline{1-2}\end{array} \qquad Q = \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 6 & 7 \\ \hline 4 & 8 \\ \cline{1-2}\end{array}.$$

Compute $w = \mathrm{RSK}^{-1}(P,Q)$ and $v = \mathrm{RSK}^{-1}(Q,P)$, and verify that 10.112 holds in this case.

10.244. (a) Verify that 10.109 holds for the example $w = 35164872$ by comparing the first rows of the tableaux in Figure 10.1 with the shadow diagram in Figure 10.4. (b) Similarly, verify the assertions in 10.111 using Figure 10.5.

10.245. Draw all the shadow diagrams for the permutations w and w^{-1} in 10.242, and use them to verify the assertions in 10.111 for this example.

10.246. Draw all the shadow diagrams for the permutations w and v in 10.243, and use them to verify the assertions in 10.111 for this example.

10.247. (a) Point out why $n! = \sum_{\lambda \vdash n} |\,\mathrm{SYT}(\lambda)|^2$. (b) Verify this identity directly for $n = 5$.

10.248. Show that the number of $w \in S_n$ such that $w^2 = \mathrm{id}$ is given by $\sum_{\lambda \vdash n} |\,\mathrm{SYT}(\lambda)|$.

10.249. Compute $\mathrm{RSK}(w)$ for all words $w \in \{0,1\}^3$.

10.250. Compute $\mathrm{RSK}(313211231)$, and verify that 10.117 holds in this case.

10.251. Compute the word w such that

$$\mathrm{RSK}(w) = \left(\begin{array}{|c|c|c|c|}\hline 1&1&2&2\\ \hline 2&3&4&4\\ \hline 4&5&5\\ \cline{1-3}\end{array}, \begin{array}{|c|c|c|c|}\hline 1&2&4&6\\ \hline 3&5&8&10\\ \hline 7&9&11\\ \cline{1-3}\end{array} \right).$$

Verify that 10.117 holds.

10.252. (a) Compute $\sum_{T \in \mathrm{SYT}((4,1))} q^{\mathrm{maj}(T)}$. (b) Compute $\sum_{T \in \mathrm{SYT}((3,2,1))} q^{\mathrm{maj}(T)}$.

10.253. Express $p_{(1^4)}$ as a linear combination of Schur polynomials.

10.254. (a) Compute the biword and the pair of tableaux associated to the matrix $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 3 & 2 & 0 \end{bmatrix}$. (b) Do the same for the transpose of this matrix.

10.255. (a) Compute the matrix and pair of tableaux associated to the biword

$$\begin{pmatrix} 1 & 1 & 2 & 2 & 2 & 3 & 3 & 5 \\ 2 & 4 & 1 & 1 & 3 & 3 & 3 & 2 \end{pmatrix}.$$

(b) Do the same for the biword obtained by switching the two rows and sorting the new top row into increasing order (using the values in the bottom row to break ties).

10.256. (a) Compute the biword and matrix associated to the pair of tableaux

$$P = \begin{array}{|c|c|c|}\hline 1&1&3\\ \hline 2&3\\ \cline{1-2} 4&4\\ \cline{1-2} 5&5\\ \cline{1-2}\end{array}, \quad Q = \begin{array}{|c|c|c|}\hline 1&2&2\\ \hline 3&3\\ \cline{1-2} 4&5\\ \cline{1-2} 6&6\\ \cline{1-2}\end{array}.$$

(b) Do the same for the pair of tableaux (Q, P).

10.257. Show that if a matrix A maps to (P, Q) under the RSK correspondence, then the transposed matrix A^t maps to (Q, P) under RSK. Do this by generalizing the shadow constructions in §10.22, allowing more than one dot to occupy a given point (i, j) in the graph.

10.258. Give a rigorous justification of the computation

$$\prod_{j=1}^{N} \sum_{k_j=0}^{\infty} h_{k_j}(x_1, \ldots, x_M) y_j^{k_j} = \sum_{k_1=0}^{\infty} \cdots \sum_{k_N=0}^{\infty} \prod_{j=1}^{N} h_{k_j}(x_1, \ldots, x_M) y_j^{k_j},$$

which was used in the proof of 10.128.

10.259. Verify the fact (used in the proof of 10.131) that the polynomials

$$\{p_\alpha(\vec{x}) p_\beta(\vec{y}) / z_\beta : (\alpha, \beta) \in \mathrm{Par}(k) \times \mathrm{Par}(k)\}$$

are linearly independent.

10.260. Suppose A and B are $n \times n$ matrices with entries in K such that $AB = I$. Prove that $BA = I$.

10.261. Suppose $\{f_\mu : \mu \in \mathrm{Par}(k)\}$ is an orthonormal basis of Λ_N^k, and $g \in \Lambda_N^k$. Prove that

$$g = \sum_{\mu \in \mathrm{Par}(k)} \langle g, f_\mu \rangle f_\mu.$$

10.262. Quasisymmetric Polynomials. A polynomial $f \in K[x_1, \ldots, x_N]$ is called *quasisymmetric* iff for all compositions $\alpha = (\alpha_1, \ldots, \alpha_s)$ with $s \leq N$ and all $1 \leq i_1 < i_2 < \cdots < i_s \leq N$, the coefficient of $\prod_{j=1}^{s} x_{i_j}^{\alpha_j}$ in f equals the coefficient of $\prod_{j=1}^{s} x_j^{\alpha_j}$ in f. For each such α, define the *monomial quasisymmetric polynomial* $M_\alpha = \sum_{1 \leq i_1 < i_2 < \cdots < i_s \leq N} \prod_{j=1}^{s} x_{i_j}^{\alpha_j}$. (a) Show that the quasisymmetric polynomials form a subspace of $K[x_1, \ldots, x_N]$ with basis $\{M_\alpha\}$. (b) What is the dimension of the space of homogeneous quasisymmetric polynomials of degree k? (c) Show that every symmetric polynomial is quasisymmetric. More specifically, express each m_λ in terms of the M_α's.

10.263. Fundamental Quasisymmetric Polynomials. For each $n \geq 0$ and each subset S of $\{1, 2, \ldots, n-1\}$, define $Q_{n,S}(x_1, \ldots, x_N) = \sum x_{i_1} x_{i_2} \cdots x_{i_n}$ where we sum over all sequences $1 \leq i_1 \leq i_2 \leq \cdots \leq i_n \leq N$ such that $j \in S$ implies $i_j < i_{j+1}$. $Q_{n,S}$ is called a *fundamental quasisymmetric polynomial.* (a) Show that $Q_{n,\emptyset} = h_n$. What is $Q_{n,\{1,2,\ldots,n-1\}}$? (b) Show that $Q_{n,S}$ is quasisymmetric (as defined in 10.262). (c) Use inclusion-exclusion to express M_α as a linear combination of $Q_{n,S}$'s and vice versa. Use this to find a basis for the space of quasisymmetric polynomials consisting of suitable Q's.

10.264. Q-Expansion of Schur Polynomials. For each integer partition λ of n, prove that $s_\lambda(x_1, \ldots, x_N) = \sum_{U \in \mathrm{SYT}(\lambda)} Q_{n,\mathrm{Des}(U)}(x_1, \ldots, x_N)$, where $Q_{n,S}$ is defined in 10.263.

Notes

Macdonald's book [89] contains a comprehensive treatment of symmetric polynomials, with a heavy emphasis on algebraic methods. A more combinatorial development is given by Stanley [127, Chpt. 7]; see the references to that chapter for an extensive bibliography of the literature in this area. Two other relevant references are Fulton [46], which treats tableaux and their connections to representation theory and geometry, and Sagan [121], which explains the role of symmetric polynomials in the representation theory of symmetric groups.

The bijective proof of 10.33 is due to Bender and Knuth [9]. The algorithmic proof of the existence part of the fundamental theorem of symmetric polynomials (outlined in 10.211) is usually attributed to Waring [130]. Some of the seminal papers by Robinson, Schensted, and Knuth on what is now called the RSK correspondence are [79, 116, 122]. The symmetry property 10.112 was first proved by Schützenberger [124], but the combinatorial proof using shadow lines is due to Viennot [135]. Quasisymmetric polynomials (see 10.262) were introduced by Gessel [51].