Chapter 5

Generating Functions

5.1. Ordinary Generating Functions

As seen in the previous chapters, one of the main tasks in combinatorics is to develop tools for counting. Perhaps, one of the most powerful tools frequently used in counting is the notion of generating functions. This notion has its roots in the work of de Moivre around 1720 and was developed by Euler in 1748 in connection with his study on the partitions of integers. It was later on extensively and systematically treated by Laplace in the late 18th century. In fact, the name "generating functions" owes its origin to Lapalce in his great work "Théorie Analytique des Probabilities" (Paris, 1812).

Let $(a_r) = (a_0, a_1, ..., a_r, ...)$ be a sequence of numbers. The (ordinary) *generating function* for the sequence (a_r) is defined to be the power series

$$A(x) = \sum_{r=0}^{\infty} a_r x^r = a_0 + a_1 x + a_2 x^2 + \cdots .$$

Two generating functions $A(x)$ and $B(x)$ for the sequences (a_r) and (b_r), respectively are considered *equal* (written $A(x) = B(x)$) if and only if $a_i = b_i$ for each $i \in \mathbf{N}^*$.

In considering the summation in a generating function, we may assume that x has been chosen such that the series converges. In fact, we do not have to concern ourselves so much with questions of convergence of the series, since we are only interested in the coefficients. Ivan Niven [N] gave an excellent account of the theory of formal power series, that allows us to ignore questions of convergence, so that we can add and multiply formal power series term by term like polynomials, as given below:

Let $A(x)$ and $B(x)$ be, respectively, the generating functions for the sequences (a_r) and (b_r). That is,

$$A(x) = a_0 + a_1x + a_2x^2 + \cdots ,$$
$$B(x) = b_0 + b_1x + b_2x^2 + \cdots .$$

The *sum* $A(x) + B(x)$ and the *product* $A(x)B(x)$ of $A(x)$ and $B(x)$ are defined by:

$$A(x) + B(x) = c_0 + c_1x + c_2x^2 + \cdots ,$$

and

$$A(x)B(x) = d_0 + d_1x + d_2x^2 + \cdots ,$$

where

$$c_r = a_r + b_r, \quad \text{for } r = 0, 1, 2, \ldots, \quad \text{and}$$
$$d_r = a_0b_r + a_1b_{r-1} + a_2b_{r-2} + \cdots + a_{r-1}b_1 + a_rb_0, \quad \text{for } r = 0, 1, 2, \ldots .$$

Writing explicitly, we have:

$$A(x) + B(x) = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \cdots ,$$

and

$$A(x)B(x) = (a_0b_0) + (a_0b_1 + a_1b_0)x + (a_0b_2 + a_1b_1 + a_2b_0)x^2 + \cdots .$$

Also, for each constant α, we put

$$\alpha A(x) = (\alpha a_0) + (\alpha a_1)x + (\alpha a_2)x^2 + \cdots .$$

Remark. The sequence (c_r) above is defined "componentwise", whereas, the sequence (d_r) is called the *Cauchy product* or the *convolution* of the sequences (a_r) and (b_r). When the sequences are finite, both operations are exactly the same as those for polynomials. In this chapter, we shall see how combinatorial considerations can be converted into algebraic manipulations using generating functions. This, in fact, is the main advantage of the theory of generating functions.

Now, for each $\alpha \in \mathbf{R}$ and each $r \in \mathbf{N}$, we introduce the "generalized binomial coefficient" $\binom{\alpha}{r}$ by putting:

$$\binom{\alpha}{r} = \frac{P_r^\alpha}{r!},$$

where $P_r^\alpha = \alpha(\alpha-1)(\alpha-2)\cdots(\alpha-r+1)$. We further set $\binom{\alpha}{0} = 1$ for each $\alpha \in \mathbf{R}$.

We now state the following generalized binomial expansion due to Newton: For every $\alpha \in \mathbf{R}$,

$$\begin{aligned}(1 \pm x)^\alpha &= \sum_{r=0}^{\infty} \binom{\alpha}{r} (\pm x)^r \\ &= 1 \pm \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 \pm \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \cdots \\ &\quad + (-1)^r \frac{\alpha(\alpha-1)\cdots(\alpha-r+1)}{r!}x^r + \cdots \end{aligned} \tag{5.1.1}$$

The proof of this expansion can be found in many books on advanced calculus. Note that the series in (5.1.1) is infinite if α is not a positive integer. The generalized notion $\binom{\alpha}{r}$ has some properties similar to those of the usual binomial coefficients. For instance, we have:

$$\binom{\alpha}{r-1} + \binom{\alpha}{r} = \binom{\alpha+1}{r},$$

for each $\alpha \in \mathbf{R}$ and $r \in \mathbf{N}$.

By (5.1.1), we have

$$\frac{1}{1-x} = (1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots,$$

and

$$\frac{1}{(1-x)^2} = (1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \cdots.$$

In general,

$$\boxed{\begin{aligned}\frac{1}{(1-x)^n} &= (1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \cdots \\ &= 1 + \binom{1+n-1}{1}x + \binom{2+n-1}{2}x^2 + \cdots + \binom{r+n-1}{r}x^r + \cdots,\end{aligned}}$$

for each $n \in \mathbf{N}$.

Example 5.1.1. **(a)** For each $n \in \mathbf{N}^*$, let (a_r) be the sequence where

$$a_r = \begin{cases} 1 & \text{if } r = n, \\ 0 & \text{otherwise.} \end{cases}$$

That is,

$$(a_r) = (\underset{\substack{\uparrow \\ 0}}{0}, \underset{\substack{\uparrow \\ 1}}{0}, \ldots, 0, \underset{\substack{\uparrow \\ n}}{1}, 0, 0, \ldots).$$

Then the generating function for (a_r) is x^n.

(b) The generating function for the sequence $(\binom{n}{0}, \binom{n}{1}, ..., \binom{n}{n}, 0, 0, \ldots)$ is

$$\sum_{r=0}^{n} \binom{n}{r} x^r = (1+x)^n. \tag{5.1.2}$$

(c) The generating function for the sequence $(1, 1, 1, ...)$ is

$$1 + x + x^2 + \cdots = \frac{1}{1-x}. \tag{5.1.3}$$

More generally, the generating function for the sequence $(1, k, k^2, ...)$, where k is an arbitrary constant, is

$$1 + kx + k^2x^2 + k^3x^3 + \cdots = \frac{1}{1-kx}. \tag{5.1.4}$$

(d) The generating function for the sequence $(1, 2, 3, ...)$ is

$$1 + 2x + 3x^2 + \cdots = \frac{1}{(1-x)^2}. \tag{5.1.5}$$

(e) The generating function for the sequence

$$\left(\binom{n-1}{0}, \binom{1+n-1}{1}, \ldots, \binom{r+n-1}{r}, \ldots\right)$$

is

$$\sum_{r=0}^{\infty} \binom{r+n-1}{r} x^r = \frac{1}{(1-x)^n}. \quad \blacksquare \tag{5.1.6}$$

Formulae (5.1.2) – (5.1.6) are very useful in finding the coefficients of generating functions, as illustrated in the following example.

Example 5.1.2. Find the coefficient of x^k, $k \geq 18$, in the expansion of

$$(x^3 + x^4 + x^5 + \cdots)^6.$$

Solution. Observe that

$$\begin{aligned}
&(x^3 + x^4 + x^5 + \cdots)^6 \\
&= \{x^3(1 + x + x^2 + \cdots)\}^6 \\
&= x^{18}(1 + x + x^2 + \cdots)^6 \\
&= x^{18}\left(\frac{1}{1-x}\right)^6 \quad \text{(by (5.1.3))} \\
&= x^{18}\sum_{r=0}^{\infty}\binom{r+6-1}{r}x^r \quad \text{(by (5.1.6))} \\
&= x^{18}\sum_{r=0}^{\infty}\binom{r+5}{5}x^r.
\end{aligned}$$

Thus the coefficient of x^k, $k \geq 18$, in the expansion of $(x^3 + x^4 + x^5 + \cdots)^6$ is the coefficient of x^{k-18} in $\sum\binom{r+5}{5}x^r$, which is $\binom{k-18+5}{5} = \binom{k-13}{5}$.

In particular, the coefficient of x^{30} in $(x^3 + x^4 + x^5 + \cdots)^6$ is $\binom{17}{5}$. ∎

To facilitate algebraic manipulations of generating functions, we have the following results.

Theorem 5.1.1. (Operations on Generating Functions) *Let $A(x)$ and $B(x)$ be, respectively, the generating functions for the sequences (a_r) and (b_r). Then*

(i) *for any numbers α and β, $\alpha A(x) + \beta B(x)$ is the generating function for the sequence (c_r), where*

$$c_r = \alpha a_r + \beta b_r, \quad \textit{for all } r;$$

(ii) *$A(x)B(x)$ is the generating function for the sequence (c_r), where*

$$c_r = a_0b_r + a_1b_{r-1} + a_2b_{r-2} + \cdots + a_{r-1}b_1 + a_rb_0, \quad \textit{for all } r;$$

(iii) *$A^2(x)$ is the generating function for the sequence (c_r), where*

$$c_r = a_0a_r + a_1a_{r-1} + a_2a_{r-2} + \cdots + a_{r-1}a_1 + a_ra_0, \quad \textit{for all } r;$$

(iv) $x^m A(x)$, $m \in \mathbf{N}$, *is the generating function for the sequence* (c_r), *where*

$$c_r = \begin{cases} 0 & \text{if } 0 \le r \le m-1 \\ a_{r-m} & \text{if } r \ge m; \end{cases}$$

(v) $A(kx)$, *where* k *is a constant, is the generating function for the sequence* (c_r), *where*

$$c_r = k^r a_r, \quad \text{for all } r;$$

(vi) $(1-x)A(x)$ *is the generating function for the sequence* (c_r), *where*

$$c_0 = a_0 \quad \text{and} \quad c_r = a_r - a_{r-1}, \quad \text{for all } r \ge 1;$$

i.e.,
$$(c_r) = (a_0, a_1 - a_0, a_2 - a_1, \ldots);$$

(vii) $\dfrac{A(x)}{1-x}$ *is the generating function for the sequence* (c_r), *where*

$$c_r = a_0 + a_1 + \cdots + a_r, \quad \text{for all } r;$$

i.e.,
$$(c_r) = (a_0, a_0 + a_1, a_0 + a_1 + a_2, \ldots);$$

(viii) $A'(x)$ *is the generating function for the sequence* (c_r), *where*

$$c_r = (r+1)a_{r+1}, \quad \text{for all } r;$$

i.e.,
$$(c_r) = (a_1, 2a_2, 3a_3, \ldots);$$

(ix) $xA'(x)$ *is the generating function for the sequence* (c_r), *where*

$$c_r = ra_r, \quad \text{for all } r;$$

i.e.,
$$(c_r) = (0, a_1, 2a_2, 3a_3, \ldots);$$

(x) $\int_0^x A(t)dt$ *is the generating function for the sequence* (c_r), *where*

$$c_0 = 0 \quad \text{and} \quad c_r = \frac{a_{r-1}}{r}, \quad \text{for all } r \ge 1;$$

i.e.,
$$(c_r) = (0, a_0, \frac{a_1}{2}, \frac{a_2}{3}, \ldots).$$

Proof. (i), (ii) and (v) follows directly from the definition, whereas (iii), (iv) and (vi) are special cases of (ii). Also, (viii), (ix) and (x) are straightfoward. We shall prove (vii) only.

(vii) By (5.1.3), $\frac{1}{1-x} = 1 + x + x^2 + \cdots$. Thus

$$\begin{aligned}\frac{A(x)}{1-x} &= (a_0 + a_1 x + a_2 x^2 + \cdots)(1 + x + x^2 + \cdots) \\ &= a_0 + (a_0 + a_1)x + (a_0 + a_1 + a_2)x^2 + \cdots .\end{aligned}$$

Hence $\frac{A(x)}{1-x}$ is the generating function for the sequence (c_r), where $c_r = a_0 + a_1 + \cdots + a_r$. ∎

We see from Theorem 5.1.1 that operations on the terms of sequences correspond to simpler operations on their generating functions. Thus the generating function becomes a useful tool in the algebraic manipulations of sequences.

Example 5.1.3. Express the generating function for each of the following sequences (c_r) in closed form (i.e., a form not involving any series):

(i) $c_r = 3r + 5$ for each $r \in \mathbf{N}^*$;

(ii) $c_r = r^2$ for each $r \in \mathbf{N}^*$.

Solution. (i) Let $a_r = r$ and $b_r = 1$ for all r. The generating function for the sequence (a_r) is $\frac{x}{(1-x)^2}$, by (5.1.5) and Theorem 5.1.1(iv); while the generating function for (b_r) is $\frac{1}{1-x}$, by (5.1.3). Thus, by Theorem 5.1.1(i), the generating function for the sequence (c_r), where $c_r = 3r+5 = 3a_r + 5b_r$, is given by $\frac{3x}{(1-x)^2} + \frac{5}{1-x}$.

(ii) Let $a_r = r$ for all r. As in (i), the generating function for the sequence (a_r) is $A(x) = \frac{x}{(1-x)^2}$. Since $c_r = r^2 = ra_r$, by Theorem 5.1.1(ix), the generating function for the sequence (c_r) is

$$xA'(x) = x \cdot \frac{(1-x)^2 + x \cdot 2(1-x)}{(1-x)^4} = \frac{x(1+x)}{(1-x)^3}. \quad \blacksquare$$

5.2. Some Modelling Problems

In this section, we shall discuss how the notion of generating functions, as introduced in the preceding section, can be used to solve some combinatorial problems. Through the examples provided, the reader will be able to see the applicability of the technique studied here.

To begin with, let $S = \{a, b, c\}$. Consider the various ways of selecting objects from S.

To select one object from S, we have:

$$\{a\} \text{ or } \{b\} \text{ or } \{c\} \text{ (denoted by } a + b + c\text{).}$$

To select two objects from S, we have:

$$\{a, b\} \text{ or } \{b, c\} \text{ or } \{c, a\} \text{ (denoted by } ab + bc + ca\text{).}$$

To select three objects from S, we have:

$$\{a, b, c\} \text{ (denoted by } abc\text{).}$$

These symbols can be found in the following expression:

$$\begin{aligned}&(1 + ax)(1 + bx)(1 + cx)\\ &= 1x^0 + (a + b + c)x^1 + (ab + bc + ca)x^2 + (abc)x^3. \qquad (*)\end{aligned}$$

We may write $1 + ax = x^0 + ax^1$, which may be interpreted as "a is not selected or a is selected once" (see the figure below).

$$\begin{array}{ccccc} 1+ax & = & x^0 & + & ax^1 \\ & & \uparrow & \uparrow & \uparrow \\ & & \boxed{\text{“}a\text{” is not selected}} & \boxed{\text{or}} & \boxed{\text{“}a\text{” is selected once}} \end{array}$$

Similarly, $1 + bx$ and $1 + cx$ may be interpreted likewise. Now, expanding the product on the LHS of the equality $(*)$, we obtain the expression on the RHS, from which we see that the exponent of x in a term indicates the number of objects in a selection and the corresponding coefficient shows all the possible ways of selections.

Since we are only interested in the number of ways of selection, we may simply let $a = b = c = 1$ and obtain the following:

$$(1 + x)(1 + x)(1 + x) = 1 + 3x + 3x^2 + 1x^3,$$

which is the generating function for the sequence $(1, 3, 3, 1, 0, 0, \ldots)$ (or simply $(1, 3, 3, 1)$ after truncating the 0's at the end of the sequence). Hence the generating function for the number of ways to select r objects from 3 distinct objects is $(1 + x)^3$.

Example 5.2.1. Let $S = \{s_1, s_2, \ldots, s_n\}$, and let a_r denote the number of ways of selecting r elements from S. Then the generating function for the sequence (a_r) is given by

$$\underset{(s_1)}{(1+x)}\underset{(s_2)}{(1+x)}\cdots\underset{(s_n)}{(1+x)} = (1+x)^n = \sum_{r=0}^{n} \binom{n}{r} x^r.$$

Thus $\sum_{r=0}^{\infty} a_r x^r = \sum_{r=0}^{n} \binom{n}{r} x^r$, which implies that

$$a_r = \begin{cases} \binom{n}{r} & \text{if } 0 \le r \le n, \\ 0 & \text{if } r \ge n+1. \end{cases} \blacksquare$$

Now, let S be the multi-set $\{2 \cdot a, 1 \cdot b\}$. Consider the various ways of selecting objects from S.

To select one object from S, we have:

$$\{a\} \text{ or } \{b\} \text{ (denoted by } a + b).$$

To select two objects from S, we have:

$$\{a, a\} \text{ or } \{a, b\} \text{ (denoted by } a^2 + ab).$$

To select three objects from S, we have:

$$\{a, a, b\} \text{ (denoted by } a^2 b).$$

These symbols can be found in the following expression:

$$(1 + ax + a^2x^2)(1 + bx) = 1x^0 + (a + b)x^1 + (a^2 + ab)x^2 + (a^2b)x^3.$$

As before, the exponent of x in the equality indicates the number of objects in a selection and the corresponding coefficients show all the possible ways of selections.

Again, since we are only interested in the number of ways of selection, we may simply let $a = b = 1$ and obtain the following

$$(1 + x + x^2)(1 + x) = 1 + 2x + 2x^2 + 1x^3,$$

which is the generating function for the sequence $(1, 2, 2, 1, 0, 0, \ldots)$ (or simpy $(1, 2, 2, 1)$). Hence the generating function for the number of ways to select r objects from the multi-set $\{2 \cdot a, 1 \cdot b\}$ is $(1 + x + x^2)(1 + x)$.

Example 5.2.2. Find the number of ways to select 4 members from the multi-set $M = \{2 \cdot b, 1 \cdot c, 2 \cdot d, 1 \cdot e\}$.

Solution. Let a_r be the number of ways of selecting r members from M. Then the generating function for the sequence (a_r) is given by

$$\begin{aligned}&\underset{(b)}{(1+x+x^2)}\,\underset{(c)}{(1+x)}\,\underset{(d)}{(1+x+x^2)}\,\underset{(e)}{(1+x)}\\ &= (1+2x+2x^2+x^3)(1+2x+2x^2+x^3).\end{aligned}$$

The required answer is a_4, which is the coefficient of x^4. Thus $a_4 = 2+4+2 = 8$. ∎

More generally, we have:

> Let a_r be the number of ways of selecting r members from the multi-set $M = \{n_1 \cdot b_1, n_2 \cdot b_2, \ldots, n_k \cdot b_k\}$. Then the generating function for the sequence (a_r) is given by
>
> $$\underset{(b_1)}{(1+x+\cdots+x^{n_1})}\,\underset{(b_2)}{(1+x+\cdots+x^{n_2})}\cdots\underset{(b_k)}{(1+x+\cdots+x^{n_k})}.$$
>
> That is, a_r is the coefficient of x^r in the expansion of the above product.

Example 5.2.3. Let a_r be the number of ways of selecting r members from the multi-set $M = \{\infty \cdot b_1, \infty \cdot b_2, \cdots, \infty \cdot b_k\}$. Then the generating function for the sequence (a_r) is given by

$$\begin{aligned}&\underset{(b_1)}{(1+x+x^2+\cdots)}\,\underset{(b_2)}{(1+x+x^2+\cdots)}\cdots\underset{(b_k)}{(1+x+x^2+\cdots)}\\ &= \left(\frac{1}{1-x}\right)^k = \sum_{r=0}^{\infty}\binom{r+k-1}{r}x^r.\end{aligned}$$

Thus $a_r = \binom{r+k-1}{r}$. ∎

Remark. The answer a_r in Example 5.2.3 can also be obtained by enumerating the coefficient of x^r in the expansion of the following generating function:

$$(1+x+x^2+\cdots+x^r)^k.$$

Though now the number of terms in each factor is finite, it actually does not simplify the computation, as

$$(1+x+\cdots+x^r)^k = \left(\frac{1-x^{r+1}}{1-x}\right)^k,$$

which leads to a more complicated expansion than the one given in the example above.

Example 5.2.4. Let a_r be the number of ways of distributing r identical objects into n distinct boxes. Then the generating function for (a_r) is:

$$\underset{\text{(box 1)}}{(1+x+x^2+\cdots)}\,\underset{\text{(box 2)}}{(1+x+x^2+\cdots)}\cdots\underset{\text{(box n)}}{(1+x+x^2+\cdots)}$$

$$= \left(\frac{1}{1-x}\right)^n = \sum_{r=0}^{\infty}\binom{r+n-1}{r}x^r.$$

Thus $a_r = \binom{r+n-1}{r}$. ∎

Example 5.2.5. Let a_r be the number of ways of distributing r identical objects into n distinct boxes such that no box is empty. Since a box must hold at least one object, the corresponding generating function for each box is $(x+x^2+x^3+\cdots)$. Hence, the generating function for (a_r) is

$$\begin{aligned}(x+x^2+\cdots)^n &= x^n(1+x+x^2+\cdots)^n\\ &= x^n\left(\frac{1}{1-x}\right)^n = x^n\sum_{i=0}^{\infty}\binom{i+n-1}{i}x^i.\end{aligned}$$

Thus

$$a_r = \begin{cases} 0 & \text{if } r < n, \\ \binom{r-n+n-1}{n-1} = \binom{r-1}{n-1} & \text{if } r \geq n. \end{cases}$$ ∎

Example 5.2.6. Each of the 3 boys tosses a die once. Find the number of ways for them to get a total of 14.

Solution. Let a_r be the number of ways to get a total of r. Since the outcomes of tossing a die are $1, 2, 3, 4, 5$ and 6, the generating function for (a_r) is

$$\begin{aligned}&(x+x^2+\cdots+x^6)^3\\&=x^3(1+x+\cdots+x^5)^3=x^3\left(\frac{1-x^6}{1-x}\right)^3\\&=x^3(1-3x^6+3x^{12}-x^{18})\sum_{i=0}^{\infty}\binom{i+2}{2}x^i.\end{aligned}$$

The required answer is a_{14}, which is the coefficient of x^{14}. Thus

$$a_{14}=\binom{11+2}{2}-3\binom{5+2}{2}=\binom{13}{2}-3\binom{7}{2}. \quad \blacksquare$$

5.3. Partitions of Integers

A *partition* of a positive integer n is a collection of positive integers whose sum is n (or a way of expressing n as a sum of positive integers, ordering not taken into account). Since the ordering is immaterial, we may regard a partition of n as a finite nonincreasing sequence $n_1 \geq n_2 \geq \cdots \geq n_k$ of positive integers such that $\sum_{i=1}^k n_i = n$. The number of different partitions of n is denoted by $p(n)$.

Example 5.3.1. The following table shows the partitions of 1, 2, 3, 4 and 5.

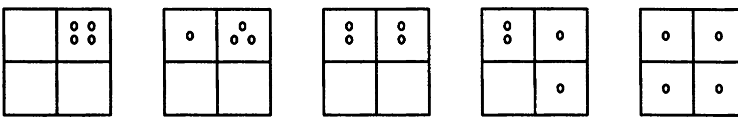
n	partitions of n	$p(n)$
1	1	1
2	$2=1+1$	2
3	$3=2+1=1+1+1$	3
4	$4=3+1=2+2=2+1+1=1+1+1+1$	5
5	$5=4+1=3+2=3+1+1=2+2+1$ $=2+1+1+1=1+1+1+1+1$	7

$\blacksquare$

Notes. **(1)** If $n = n_1 + n_2 + \cdots n_k$ is a partition of n, we say that n is partitioned into k parts of sizes $n_1, n_2, \ldots, n_k$ respectively. Thus, in the partition $9 = 3 + 3 + 2 + 1$, there are 4 parts of sizes 3, 3, 2 and 1 respectively.

(2) A partition of n is equivalent to a way of distributing n identical objects into n identical boxes (with empty boxes allowed), as illustrated below:

4 =4 =3+1 =2+2 =2+1+1 =1+1+1+1



Example 5.3.2. Let a_r be the number of partitions of an integer r into parts of sizes 1, 2 or 3. The generating function for (a_r) is

$$\underset{\text{(size 1)}}{(1 + x + x^2 + \cdots)}\,\underset{\text{(size 2)}}{(1 + x^2 + x^4 + \cdots)}\,\underset{\text{(size 3)}}{(1 + x^3 + x^6 + \cdots)}$$

$$= \frac{1}{(1-x)(1-x^2)(1-x^3)}.$$

Note that the three factors in the above generating function are of the form

$$(x^k)^0 + (x^k)^1 + (x^k)^2 + \cdots,$$

where $k = 1, 2, 3$ and a term $(x^k)^j$ indicates that, in the partition, there are j parts of size k.

Now, consider the term containing x^3 and its coefficient. We see that the coefficient of x^3 is 3 since there are 3 ways of getting x^3 (namely, $x^3 = (x^1)^0(x^2)^0(x^3)^1 = (x^1)^1(x^2)^1(x^3)^0 = (x^1)^3(x^2)^0(x^3)^0$) in the above generating function, as illustrated in the following table.

$$\begin{array}{cccccl}
 & \text{Size 1} & \text{Size 2} & \text{Size 3} & & \\
\hline
 & x^0 & x^0 & x^3 & & 3 = 3 \\
3x^3 \left\{ \right. & x^1 & x^2 & x^0 & & 3 = 1 + 2 \\
 & x^3 & x^0 & x^0 & & 3 = 1 + 1 + 1
\end{array}$$

Similarly, consider the term containing x^4 and its coefficient. We see that the coefficient of x^4 is 4 since there are 4 ways of getting x^4 in the above generating function, as illustrated in the following table.

$$
\begin{array}{cccccc}
 & \text{Size 1} & \text{Size 2} & \text{Size 3} & \\
\hline
\multirow{4}{*}{$4x^4$} \left\{ \right. & x^0 & x^4 & x^0 & 4=2+2 \\
 & x^1 & x^0 & x^3 & 4=1+3 \\
 & x^2 & x^2 & x^0 & 4=1+1+2 \\
 & x^4 & x^0 & x^0 & 4=1+1+1+1. \quad \blacksquare
\end{array}
$$

Example 5.3.3. Let a_r be the number of partitions of r into *distinct* parts of sizes 1, 2, 3 or 4. The generating function for (a_r) is

$$(1+x)(1+x^2)(1+x^3)(1+x^4).$$

We note that, in this partition, no repetition is allowed. So a part of size k is used at most once and thus the corresponding generating function is $(x^k)^0 + (x^k)^1 = 1 + x^k$. There are two ways to form x^6, namely:

$$
\begin{aligned}
x^6 &= 1 \cdot x^2 \cdot 1 \cdot x^4 &&\leftrightarrow && 6 = 2+4 \\
x^6 &= x \cdot x^2 \cdot x^3 \cdot 1 &&\leftrightarrow && 6 = 1+2+3.
\end{aligned}
$$

Thus $a_6 = 2$. ∎

Example 5.3.4. Let a_r denote the number of partitions of r into *distinct* parts (of arbitrary sizes). For instance,

$$
\begin{aligned}
&6 = 5+1 = 4+2 = 3+2+1; \\
&7 = 6+1 = 5+2 = 4+3 = 4+2+1; \\
&8 = 7+1 = 6+2 = 5+3 = 5+2+1 = 4+3+1.
\end{aligned}
$$

Thus, $a_6 = 4$, $a_7 = 5$ and $a_8 = 6$.

It is easy to see that the generating function for (a_r) is

$$\underset{(1)}{(1+x)}\,\underset{(2)}{(1+x^2)}\,\underset{(3)}{(1+x^3)}\cdots = \prod_{i=1}^{\infty}(1+x^i).$$

We note that, since the size of each part is arbitrary, the number of terms on the LHS is infinite.

For example, in the above product, there are 4 ways to form x^6, namely:

$$\begin{array}{lcl} x^6 = x^6 & \leftrightarrow & 6 = 6 \\ x^6 = x^1x^5 & \leftrightarrow & 6 = 5+1. \\ x^6 = x^2x^4 & \leftrightarrow & 6 = 4+2. \\ x^6 = x^1x^2x^3 & \leftrightarrow & 6 = 3+2+1. \end{array}$$

Thus $a_6 = 4$. ∎

Example 5.3.5. A part in a partition is said to be *odd* if its size is odd. Let b_r denote the number of partitions of r into odd parts. For instance,

$$\begin{aligned} 6 &= 5+1 = 3+3 = 3+1+1+1 = 1+1+1+1+1+1; \\ 7 &= 7 = 5+1+1 = 3+3+1 = 3+1+1+1+1 \\ &= 1+1+1+1+1+1+1; \\ 8 &= 7+1 = 5+3 = 5+1+1+1 = 3+3+1+1 \\ &= 3+1+1+1+1+1 = 1+1+1+1+1+1+1+1. \end{aligned}$$

Thus, $b_6 = 4$, $b_7 = 5$ and $b_8 = 6$.

The generating function for (b_r) is

$$\underset{(1)}{(1+x+x^2+\cdots)}\underset{(3)}{(1+x^3+x^6+\cdots)}\underset{(5)}{(1+x^5+x^{10}+\cdots)}\cdots$$
$$= \frac{1}{(1-x)(1-x^3)(1-x^5)\cdots}.$$

For example, in the above product, there are 4 ways to form x^6, namely:

$$\begin{array}{cccc} (x^1)^6, & (x^1)^3x^3, & x^1x^5, & (x^3)^2. \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 1+1+1+1+1+1 & 1+1+1+3 & 1+5 & 3+3 \end{array}$$

Thus $b_6 = 4$. ∎

From the two examples above, we notice that $a_6 = 4 = b_6$, $a_7 = 5 = b_7$ and $a_8 = 6 = b_8$. This is by no means a coincidence. In fact, these equalities are just special cases of the following result due to Euler, who laid the foundation of the theory of partitions, around 1748, by proving many beautiful theorems about partitions.

Theorem 5.3.1. (Euler) *The number of partitions of r into distinct parts is equal to the number of partitions of r into odd parts.*

Proof. Let a_r (resp., b_r) denote the number of partitions of r into distinct (resp., odd) parts. Then the generating function for (a_r) is

$$
\begin{aligned}
&(1+x)(1+x^2)(1+x^3)(1+x^4)\cdots \\
&= \frac{1-x^2}{1-x}\,\frac{1-x^4}{1-x^2}\,\frac{1-x^6}{1-x^3}\,\frac{1-x^8}{1-x^4}\cdots \\
&= \frac{1}{(1-x)(1-x^3)(1-x^5)\cdots},
\end{aligned}
$$

which is exactly equal to the generating function for (b_r). Hence $a_r = b_r$ for each $r = 1, 2, \ldots$. ∎

The technique used in the proof of Theorem 5.3.1 can be utilized to prove many other results of the "Euler type". For instance, we have:

Theorem 5.3.2. *For each $n \in \mathbf{N}$, the number of partitions of n into parts each of which appears at most twice, is equal to the number of partitions of n into parts the sizes of which are not divisible by 3.*

Before proving the result, let us examine it by taking $n = 6$. We see that there are 7 ways of partitioning 6 into parts, each of which appears at most twice, as shown below:

$$
\begin{aligned}
6 &= 5+1 = 4+2 = 4+1+1 = 3+3 \\
&= 3+2+1 = 2+2+1+1.
\end{aligned}
$$

There are also 7 ways of partitioning 6 into parts, the sizes of which are not divisible by 3, namely:

$$
\begin{aligned}
&5+1 = 4+2 = 4+1+1 = 2+2+2 = 2+2+1+1 \\
&= 2+1+1+1+1 = 1+1+1+1+1+1.
\end{aligned}
$$

Proof of Theorem 5.3.2. The generating function for the number of partitions of n into parts, each of which appears at most twice, is

$$(1+x+x^2)(1+x^2+x^4)(1+x^3+x^6)(1+x^4+x^8)\cdots.$$

However, we have

$$\begin{aligned}
&(1+x+x^2)(1+x^2+x^4)(1+x^3+x^6)(1+x^4+x^8)\cdots\\
&=\frac{(1-x)(1+x+x^2)}{(1-x)}\,\frac{(1-x^2)(1+x^2+x^4)}{(1-x^2)}\,\frac{(1-x^3)(1+x^3+x^6)}{(1-x^3)}\\
&\quad\cdot\frac{(1-x^4)(1+x^4+x^8)}{(1-x^4)}\cdots\\
&=\frac{1-x^3}{1-x}\,\frac{1-x^6}{1-x^2}\,\frac{1-x^9}{1-x^3}\,\frac{1-x^{12}}{1-x^4}\cdots\\
&=\frac{1}{1-x}\,\frac{1}{1-x^2}\,\frac{1}{1-x^4}\,\frac{1}{1-x^5}\,\frac{1}{1-x^7}\cdots\\
&=\prod\left(\frac{1}{1-x^k}\,\middle|\, k\in\mathbf{N}, 3\nmid k\right),
\end{aligned}$$

which is exactly the generating function for the number of partitions of n into parts the sizes of which are not divisible by 3. ∎

Theorem 5.3.2 has the following generalization that was discovered by J.W.L. Glaisher in 1883.

Theorem 5.3.3. [G] *For any $n, k \in \mathbf{N}$, the number of partitions of n into parts, each of which appears at most k times, is equal to the number of partitions of n into parts the sizes of which are not divisible by $k+1$.* ∎

We leave the proofs of this and the following result as exercises for the reader (see Problems 5.59).

Theorem 5.3.4. *For any $n \in \mathbf{N}$, the number of partitions of n into parts each of which appears at least twice is equal to the number of partitions of n into parts the sizes of which are not congruent to 1 or $-1 \pmod 6$.* ∎

Theorem 5.3.4 first appeared in the literature as an exercise in the book [An1] by Andrew. Interested readers may read the paper by H.L. Alder [Al] which gives a very comprehensive account of results and problems of the Euler type.

Ferrers Diagram

A convenient tool, in the form of a diagram, to study partitions of integers is due to Norman M. Ferrers (1829-1903). The *Ferrers diagram* for a partition $n = n_1 + n_2 + \cdots + n_k$ of a positive integer n $(n_1 \geq n_2 \geq \cdots \geq n_k)$ is an array of asterisks in left-justified rows with n_i asterisks in the ith row. For each partition P of an integer, we shall denote by $\mathcal{F}(P)$ the Ferrers diagram for P.

Example 5.3.6. Let P be the partition of the number 15 as shown below:

$$P : 15 = 6 + 3 + 3 + 2 + 1.$$

Then, the Ferrers diagram $\mathcal{F}(P)$ of P is:

$$\mathcal{F}(P) : \quad \left\{ \begin{array}{llllll} * & * & * & * & * & * \\ * & * & * & & & \\ * & * & * & & & \\ * & * & & & & \\ * & & & & & \end{array} \right.$$

The transpose $\mathcal{F}^t$ of a Ferrers diagram is the Ferrers diagram whose rows are the columns of $\mathcal{F}$. Thus the transpose of the Ferrers diagram above is:

$$\mathcal{F}(P)^t : \quad \left\{ \begin{array}{lllll} * & * & * & * & * \\ * & * & * & * & \\ * & * & * & & \\ * & & & & \\ * & & & & \\ * & & & & \end{array} \right.$$

which gives another partition of 15:

$$Q : 15 = 5 + 4 + 3 + 1 + 1 + 1. \quad \blacksquare$$

Two partitions of n whose Ferrers diagrams are transpose of each other are called *conjugate partitions.* Thus, P and Q in the above example are conjugate partitions. It follows readily from the definition that the number of parts in P (which is 5, the number of rows in $\mathcal{F}(P)$) is equal to the largest size in Q (which is the number of columns in $\mathcal{F}(P)^t$). This observation enables us to have the following simple proof of another result, also due to Euler.

Theorem 5.3.5. (Euler) *Let $k, n \in \mathbf{N}$ with $k \leq n$. Then the number of partitions of n into k parts is equal to the number of partitions of n into parts the largest size of which is k.*

Proof. Let $\mathcal{P}$ be the family of all partitions of n into k parts and $\mathcal{Q}$ be the family of all partitions of n into parts the largest size of which is k. Define a mapping $f : \mathcal{P} \to \mathcal{Q}$ as follows: For each $P \in \mathcal{P}$, we put $f(P)$ to be the partition of n whose Ferrers diagram is $\mathcal{F}(P)^t$ (i.e., $f(P)$ is just the conjugate of P). It is easy to see that f establishes a bijection between $\mathcal{P}$ and $\mathcal{Q}$. Thus $|\mathcal{P}| = |\mathcal{Q}|$, by (BP). ∎

We illustrate the above proof for $n = 8$ and $k = 3$ by the following table:

partitions of 8 into 3 parts		partitions of 8 into parts the largest size of which is 3
P		$f(P)$
$6+1+1$	$\longleftrightarrow$	$3+1+1+1+1+1$
$5+2+1$	$\longleftrightarrow$	$3+2+1+1+1$
$4+3+1$	$\longleftrightarrow$	$3+2+2+1$
$4+2+2$	$\longleftrightarrow$	$3+3+1+1$
$3+3+2$	$\longleftrightarrow$	$3+3+2$

An application of Theorem 5.3.5 is given in the following example.

Example 5.3.7. Let a_r denote the number of ways of distributing r identical objects into 3 identical boxes such that no box is empty. Find the generating function for (a_r).

Solution. First, note that a_r is equal to the number of partition of r into 3 parts. Thus by Theorem 5.3.5, a_r is equal to the number of partitions of r into parts the largest size of which is 3. With this observation, we can now obtain the generating function for (a_r), as shown below:

$$\begin{aligned} & \underset{\text{(size 1)}}{(1+x+x^2+\cdots)}\,\underset{\text{(size 2)}}{(1+x^2+x^4+\cdots)}\,\underset{\text{(size 3)}}{(x^3+x^6+\cdots)} \\ = \quad & \frac{x^3}{(1-x)(1-x^2)(1-x^3)}. \end{aligned}$$ ∎

Remark. The reader should notice that the third factor on the LHS of the above equality is $(x^3+x^6+\cdots)$ rather than $(1+x^3+x^6+\cdots)$ (why?).

As an immediate consequence of Theorem 5.3.5, we have:

Corollary. *Let $m, n \in \mathbf{N}$ with $m \leq n$. Then the number of partitions of n into at most m parts is equal to the number of partitions of n into parts with sizes not exceeding m.* ∎

For a detailed and advanced treatment of the theory of partitions of numbers, we refer the reader to the book [An2] by Andrew.

5.4. Exponential Generating Functions

From the problems as discussed in the previous two sections, we see that (ordinary) generating functions are applicable in distribution problems or arrangement problems, in which the ordering of the objects involved is immaterial. In this section, we shall study the so-called "exponential generating functions" that will be useful in the counting of arrangements of objects where the ordering is taken into consideration.

The *exponential generating function* for the sequence of numbers (a_r) is defined to be the power series

$$a_0 + a_1\frac{x}{1!} + a_2\frac{x^2}{2!} + a_3\frac{x^3}{3!} + \cdots + a_r\frac{x^r}{r!} + \cdots = \sum_{r=0}^{\infty} a_r\frac{x^r}{r!}.$$

Example 5.4.1. **(1)** The exponential generating function for $(1, 1, ..., 1, ...)$ is

$$\sum_{r=0}^{\infty} \frac{x^r}{r!} = e^x.$$

(This explains why the name "exponential generating function" is used, as it includes the exponential function e^x as a special case.)

(2) The exponential generating function for $(0!, 1!, 2!, ..., r!, ...)$ is

$$\sum_{r=0}^{\infty} r!\frac{x^r}{r!} = 1 + x + x^2 + \cdots = \frac{1}{1-x}.$$

(3) The exponential generating function for $(1, k, k^2, ..., k^r, ...)$, where k is a nonzero constant, is

$$1 + \frac{kx}{1!} + \frac{k^2x^2}{2!} + \cdots = \sum_{r=0}^{\infty} \frac{(kx)^r}{r!} = e^{kx}. \quad \blacksquare$$

Example 5.4.2. Show that the exponential generating function for the sequence

$$(1,\ 1 \cdot 3,\ 1 \cdot 3 \cdot 5,\ 1 \cdot 3 \cdot 5 \cdot 7,\ ...)$$

is $(1 - 2x)^{-\frac{3}{2}}$.

Proof. It suffices to show that the coefficient of x^r in $(1 - 2x)^{-\frac{3}{2}}$ is $\frac{1 \cdot 3 \cdot 5 \cdots (2r+1)}{r!}$.

Indeed, the coefficient of x^r in $(1 - 2x)^{-\frac{3}{2}} = \sum_{i=0}^{\infty} \binom{-\frac{3}{2}}{i}(-2x)^i$ is

$$\begin{aligned} (-2)^r \binom{-\frac{3}{2}}{r} &= (-2)^r \frac{(-\frac{3}{2})(-\frac{3}{2} - 1) \cdots (-\frac{3}{2} - r + 1)}{r!} \\ &= (-2)^r (-\frac{1}{2})^r \frac{3 \cdot 5 \cdot 7 \cdots (2r+1)}{r!} = \frac{1 \cdot 3 \cdot 5 \cdots (2r+1)}{r!}, \end{aligned}$$

as required. $\blacksquare$

Exponential generating functions for permutations

Recall that P_r^n denotes the number of r-permutations of n distinct objects, and

$$P_r^n = \binom{n}{r} \cdot r!\,.$$

Then

$$\sum_{r=0}^{n} P_r^n \frac{x^r}{r!} = \sum_{r=0}^{n} \binom{n}{r} x^r = (1 + x)^n.$$

Thus, by definition, the exponential generating function for the sequence $(P_r^n)_{r=0,1,2,...}$ is $(1 + x)^n$.

Note that

$$(1 + x)^n = \underset{(1)}{(1 + \frac{x^1}{1!})}\underset{(2)}{(1 + \frac{x^1}{1!})} \cdots \underset{(n)}{(1 + \frac{x^1}{1!})},$$

where, as before, each bracket on the RHS corresponds to a distinct object in the arrangement.

Example 5.4.3. Let a_r denote the number of r-permutations of p identical objects. The exponential generating function for (a_r) is

$$1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots+\frac{x^p}{p!},$$

since $a_r = 1$ for each $r = 0, 1, 2, \ldots, p$ and $a_r = 0$ for each $r > p$. ∎

Example 5.4.4. Let a_r denote the number of r-permutations of p identical blue balls and q identical red balls. The exponential generating function for (a_r) is

$$\underset{\text{(B)}}{\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots+\frac{x^p}{p!}\right)}\underset{\text{(R)}}{\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots+\frac{x^q}{q!}\right)}. \quad ∎$$

In general, we have

> Let a_r denote the number of r-permutations of the multi-set
>
> $$\{n_1 \cdot b_1, n_2 \cdot b_2, \ldots, n_k \cdot b_k\}.$$
>
> Then the exponential generating function for (a_r) is
>
> $$\underset{(b_1)}{\left(1+\frac{x}{1!}+\cdots+\frac{x^{n_1}}{n_1!}\right)}\underset{(b_2)}{\left(1+\frac{x}{1!}+\cdots+\frac{x^{n_2}}{n_2!}\right)}\cdots\underset{(b_k)}{\left(1+\frac{x}{1!}+\cdots+\frac{x^{n_k}}{n_k!}\right)},$$
>
> and a_r is the coefficient of $\frac{x^r}{r!}$ in the expansion of the above product.

Example 5.4.5. In how many ways can 4 of the letters from PAPAYA be arranged?

Solution. Let a_r be the number of r-permutations of the multi-set

$$\{3 \cdot A, 2 \cdot P, 1 \cdot Y\}$$

formed by all the letters from PAPAYA. Then the exponential generating function for (a_r) is

$$\underset{\text{(A)}}{\left(1+\frac{x^1}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}\right)}\underset{\text{(P)}}{\left(1+\frac{x^1}{1!}+\frac{x^2}{2!}\right)}\underset{\text{(Y)}}{\left(1+\frac{x^1}{1!}\right)}.$$

Grouping the like terms x^4 in the product, we have

$$x \cdot \frac{x^2}{2!} \cdot x + \frac{x^2}{2!} \cdot x \cdot x + \frac{x^2}{2!} \cdot \frac{x^2}{2!} \cdot 1 + \frac{x^3}{3!} \cdot x \cdot 1 + \frac{x^3}{3!} \cdot 1 \cdot x$$
$$= a_4 \frac{x^4}{4!},$$

where

$$a_4 = \underset{\{A,2\cdot P,Y\}}{\frac{4!}{2!}} + \underset{\{2\cdot A,P,Y\}}{\frac{4!}{2!}} + \underset{\{2\cdot A,2\cdot P\}}{\frac{4!}{2!2!}} + \underset{\{3\cdot A,P\}}{\frac{4!}{3!}} + \underset{\{3\cdot A,Y\}}{\frac{4!}{3!}},$$

which is the required answer. ∎

Example 5.4.6. Let (a_r) denote the number of r-permutations of the multi-set $\{\infty \cdot b_1, \infty \cdot b_2, \ldots, \infty \cdot b_k\}$. Then the exponential generating function for (a_r) is

$$(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots)^k = (e^x)^k = e^{kx} = \sum_{r=0}^{\infty} \frac{(kx)^r}{r!} = \sum_{r=0}^{\infty} k^r \frac{x^r}{r!}.$$

Thus, $a_r = k^r$ for each $r \in \mathbf{N}^*$. ∎

Example 5.4.7. For each $r \in \mathbf{N}^*$, let a_r denote the number of r-digit quaternary sequences (whose digits are 0, 1, 2, 3) in which each of the digits 2 and 3 appears at least once. Find a_r.

Solution. The exponential generating function for (a_r) is

$$\begin{aligned}\underset{(0)\ (1)}{(1 + x + \frac{x^2}{2!} + \cdots)^2} \underset{(2)\ (3)}{(x + \frac{x^2}{2!} + \cdots)^2} &= (e^x)^2(e^x - 1)^2 \\ &= e^{2x}(e^{2x} - 2e^x + 1) \\ &= e^{4x} - 2e^{3x} + e^{2x} \\ &= \sum_{r=0}^{\infty} (4^r - 2 \cdot 3^r + 2^r) \frac{x^r}{r!}.\end{aligned}$$

Thus, $a_r = 4^r - 2 \cdot 3^r + 2^r$ for each $r \in \mathbf{N}^*$. ∎

Remark. Since

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

and

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots + (-1)^n \frac{x^n}{n!} + \cdots,$$

we have:

$$\boxed{\begin{aligned} \frac{e^x + e^{-x}}{2} &= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots, \\ \frac{e^x - e^{-x}}{2} &= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots. \end{aligned}}$$

We note that, in the above expansions, $\frac{e^x+e^{-x}}{2}$ involves only even powers of x whereas, $\frac{e^x-e^{-x}}{2}$ involves only odd powers of x. This observation is useful in solving the following problem.

Example 5.4.8. For each $r \in \mathbf{N}^*$, let a_r denote the number of r-digit ternary sequences that contain an odd number of 0's and an even number of 1's. Find a_r.

Solution. The exponential generating function for (a_r) is

$$\underset{(0)}{\left(\frac{e^x - e^{-x}}{2}\right)} \underset{(1)}{\left(\frac{e^x + e^{-x}}{2}\right)} \underset{(2)}{e^x}$$

$$= \frac{1}{4} e^x (e^{2x} - e^{-2x})$$

$$= \frac{1}{4} (e^{3x} - e^{-x})$$

$$= \frac{1}{4} \sum_{r=0}^{\infty} \{3^r - (-1)^r\} \frac{x^r}{r!}.$$

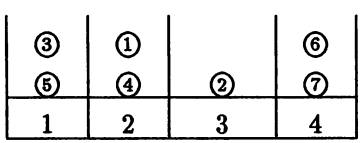
Thus, $a_r = \frac{1}{4}\{3^r - (-1)^r\}$. ∎

Distribution Problems

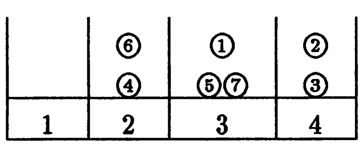
We have seen in Sections 2 and 3 that the notion of (ordinary) generating functions can be used to tackle distribution problems where the objects to be distributed are *identical*. On the other hand, if we are to distribute *distinct* objects into *distinct* boxes, then the notion of exponential generating functions turns out to be very helpful, as seen in the following two examples.

Example 5.4.9. For each $r \in \mathbf{N}^*$, find a_r, the number of ways of distributing r distinct objects into 4 distinct boxes such that boxes 1 and 2 must hold an even number of objects and box 3 must hold an odd number of objects.

Before giving the solution, we shall see how the given distribution problem can be transformed into the problem of finding the number of certain r-digit quaternary sequences. Assume here that the 4 digits used are $1, 2, 3$ and 4. Then a_r is the number of r-digit quaternary sequences that contain an even number of 1's, an even number of 2's and an odd number of 3's. For instance, when $r = 7$, the correspondence between such distributions and quaternary sequences is illustrated in the following figure.



$\leftrightarrow$ 2312144



$\leftrightarrow$ 3442323

Note that a ball labelled i is placed in a box labelled j if and only if j occurs in the ith position in the corresponding quaternary sequence. For instance, in the sequence 3442323, there are two 4's in the 2nd and 3rd positions; and thus, in this distribution, balls labelled 2 and 3 are placed in box 4.

In view of the correspondence mentioned above, we can now used the notion of exponential generating function to solve the given distribution problem.

Solution. The exponential generating function for (a_r) is

$$\begin{aligned}
&\underset{(1)(2)}{\left(\frac{e^x+e^{-x}}{2}\right)^2}\ \underset{(3)}{\left(\frac{e^x-e^{-x}}{2}\right)}\ \underset{(4)}{e^x}\\
&=\frac{1}{8}(e^{2x}-e^{-2x})(e^{2x}+1)\\
&=\frac{1}{8}(e^{4x}-1+e^{2x}-e^{-2x})\\
&=\frac{1}{8}\left\{-1+\sum_{r=0}^{\infty}(4^r+2^r-(-2)^r)\frac{x^r}{r!}\right\}.
\end{aligned}$$

Thus, $a_r=\frac{1}{8}\{4^r+2^r-(-2)^r\}$, for each $r\in\mathbf{N}$. ∎

Example 5.4.10. For each $r\in\mathbf{N}^*$, find a_r, the number of ways of distributing r distinct objects into n distinct boxes such that no box is empty.

Solution. The exponential generating function for (a_r) is

$$\begin{aligned}
&(x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots)^n\\
&=(e^x-1)^n\\
&=\sum_{i=0}^{n}\binom{n}{i}(e^x)^{n-i}(-1)^i\\
&=\sum_{i=0}^{n}\binom{n}{i}\left(\sum_{r=0}^{\infty}\frac{(n-i)^r x^r}{r!}\right)(-1)^i\\
&=\sum_{r=0}^{\infty}\left(\sum_{i=0}^{n}(-1)^i\binom{n}{i}(n-i)^r\right)\frac{x^r}{r!}.
\end{aligned}$$

Thus, $a_r=\sum_{i=0}^{n}(-1)^i\binom{n}{i}(n-i)^r$, which is the number $F(r,n)$ of surjective mappings from $\mathbf{N}_r$ to $\mathbf{N}_n$, as shown in Theorem 4.5.1. ∎

Generally speaking, distribution problems can be classified into 4 types, according to whether the objects to be distributed are identical or distinct and the boxes containing these objects are identical or distinct. We end this chapter by summarizing the results for these 4 types of problems that we have obtained so far in the following table, where # denotes the number of ways to distibute r objects into n boxes subject to the conditions in columns 2 and 3 of the table.

Type	r objects	n boxes	#
I	distinct	distinct	n^r
II	identical	distinct	$\binom{r+n-1}{r}$
III	distinct	identical	$\sum_{i=1}^{n} S(r,i)$
IV	identical	identical	number of partitions of r into n or fewer parts

In this chapter, we have seen how generating functions can be used to solve distribution problems of Types I, II and IV. Briefly speaking, in a distribution problem, when identical objects are distributed to distinct boxes (cf., Examples 5.2.4 and 5.2.5), the corresponding generating function for each box is just an ordinary generating function. When distinct objects are distributed to distinct boxes (cf., Examples 5.4.9 and 5.4.10), we introduce an exponential generating function for each box. The case when identical objects are distributed to identical boxes (cf., Example 5.3.7) is just a partition problem and we introduce an ordinary generating function for the size of each part in the partition.

Exercise 5

1. Find the coefficient of x^{20} in the expansion of $(x^3 + x^4 + x^5 + \cdots)^3$.

2. Find the coefficients of x^9 and x^{14} in the expansion of $(1 + x + x^2 + \cdots + x^5)^4$.

3. Prove Theorem 5.1.1 (iv), (vi), (viii), (ix) and (x).

4. Find the generating function for the sequence (c_r), where $c_0 = 0$ and $c_r = \sum_{i=1}^{r} i^2$ for $r \in \mathbf{N}$. Hence show that

$$\sum_{i=1}^{r} i^2 = \binom{r+1}{3} + \binom{r+2}{3}.$$

5. Find the generating function for the sequence (c_r), where $c_r = \sum_{i=0}^{r} i2^i$ with $r \in \mathbf{N}^*$. Hence show that

$$\sum_{i=0}^{r} i2^i = 2 + (r-1)2^{r+1}.$$

6. (i) For $r \in \mathbf{N}^*$, let $a_r = \frac{1}{4^r}\binom{2r}{r}$. Show that the generating function for the sequence (a_r) is given by $(1-x)^{-\frac{1}{2}}$.

   (ii) Using the identity

$$(1-x)^{-1} = (1-x)^{-\frac{1}{2}}(1-x)^{-\frac{1}{2}},$$

   show that

$$\sum_{k=0}^{n} \binom{2k}{k}\binom{2(n-k)}{n-k} = 4^n$$

   for each $n \in \mathbf{N}^*$.

7. Show that

$$\sum_{r=1}^{n} r\binom{n}{r}\binom{m}{r} = n\binom{n+m-1}{n}.$$

8. Find the number of ways to distribute 10 identical pieces of candy to 3 children so that no child gets more than 4 pieces.

9. Find the number of ways to distribute 40 identical balls to 7 distinct boxes if box 1 must hold at least 3, and at most 10, of the balls.

10. Find the number of ways to select $2n$ balls from n identical blue balls, n identical red balls and n identical white balls, where $n \in \mathbf{N}$.

11. In how many ways can 100 identical chairs be divided among 4 different rooms so that each room will have 10, 20, 30, 40 or 50 chairs?

12. Let a_r be the number of ways of distributing r identical objects into 5 distinct boxes so that boxes 1, 3 and 5 are not empty. Let b_r be the number of ways of distributing r identical objects into 5 distinct boxes so that each of the boxes 2 and 4 contains at least two objects.

    (i) Find the generating function for the sequence (a_r).

    (ii) Find the generating function for the sequence (b_r).

    (iii) Show that $a_r = b_{r+1}$ for each $r = 1, 2, \ldots$.

13. For $r \in \mathbf{N}^*$, let a_r denote the number of integer solutions to the equation

$$x_1 + x_2 + x_3 = r$$

where $3 \le x_1 \le 9$, $0 \le x_2 \le 8$ and $7 \le x_3 \le 17$. Find the generating function for (a_r), and determine the value of a_{28}.

14. In how many ways can 3000 identical pencils be divided up, in packages of 25, among four student groups so that each group gets at least 150, but not more than 1000, of the pencils?

15. Find the number of selections of 10 letters from "F, U, N, C, T, I, O" that contain at most three U's and at least one O.

16. Find the generating function for the sequence (a_r) in each of the following cases: a_r is
   (i) the number of selections of r letters (not necessarily distinct) from the set $\{D, R, A, S, T, I, C\}$ that contain at most 3 D's and at least 2 T's;
   (ii) the number of partitions of r into parts of sizes 1, 2, 3, 5, and 8;
   (iii) the number of partitions of r into distinct parts of sizes 5, 10, and 15;
   (iv) the number of partitions of r into distinct odd parts;
   (v) the number of partitions of r into distinct even parts;
   (vi) the number of integer solutions to the inequality

$$x_1 + x_2 + x_3 + x_4 + x_5 \le r$$

   with $1 \le x_i \le 6$ for each $i = 1, 2, \ldots, 5$.

17. Find the number of $4n$-element multi-subsets of the multi-set

$$\{(3n) \cdot x, (3n) \cdot y, (3n) \cdot z\},$$

where $n \in \mathbf{N}$.

18. Find the number of $3n$-element multi-subsets of the multi-set

$$M = \{n \cdot z_1, n \cdot z_2, \ldots, n \cdot z_m\},$$

where $n, m \in \mathbf{N}$ and $n, m \ge 3$.

19. What is the probability that a roll of 5 distinct dice yields a sum of 17?

20. Find the generating function for the sequence (a_r), where a_r is the number of ways to obtain a sum of r by a roll of *any* number of distinct dice.

21. For $k, m \in \mathbf{N}$ and $r \in \mathbf{N}^*$, let a_r denote the number of ways of distributing r identical objects into $2k+1$ distinct boxes such that the first $k+1$ boxes are non-empty, and b_r denote the number of ways of distributing r identical objects into $2k+1$ distinct boxes such that each of the last k boxes contains at least m objects.

    (i) Find the generating function for the sequence (a_r);

    (ii) Find the generating function for the sequence (b_r);

    (iii) Show that $a_r = b_{r+(m-1)k-1}$.

22. Find the generating function for the sequence (a_r), where a_r is the number of integer solutions to the equation

$$x_1 + 2x_2 + 3x_3 + 4x_4 = r$$

with $x_i \geq 0$ for each $i = 1, 2, 3, 4$.

23. For $r \in \mathbf{N}^*$, let a_r denote the number of ways of selecting 4 distinct integers from $\{1, 2, \ldots, r\}$ such that no two are consecutive. Find the generating function for (a_r) and deduce that $a_r = \binom{r-3}{4}$.

24. For $r \in \mathbf{N}^*$, and $m, t \in \mathbf{N}$, let a_r denote the number of m-element subsets $\{n_1, n_2, \ldots, n_m\}$ of the set $\{1, 2, \ldots, r\}$, where $n_1 < n_2 < \cdots < n_m$ and $n_{i+1} - n_i \geq t$ for each $i = 1, 2, \ldots, m-1$. Find the generating function for (a_r) and deduce that

$$a_r = \binom{r - (m-1)(t-1)}{m}.$$

(See Problem 1.91.)

25. For $r \in \mathbf{N}^*$, let a_r be the number of integer solutions to the inequality

$$x_1 + x_2 + x_3 + x_4 \leq r,$$

where $3 \leq x_1 \leq 9$, $1 \leq x_2 \leq 10$, $x_3 \geq 2$ and $x_4 \geq 0$. Find the generating function for the sequence (a_r) and the value of a_{20}.

26. Prove that if $-1 < \alpha < 0$ and $n \in \mathbf{N}^*$, then

$$\binom{2\alpha}{2n} \geq (2n+1)\binom{\alpha}{n}^2;$$

while if $\alpha < -1$, the inequality is reversed. (Proposed by S. I. Rosencrans, see *Amer. Math. Monthly*, **79** (1972), 1136.)

27. For $n \in \mathbf{N}$, let

$$a_{n-1} = \sum_{k=0}^{n-1} \left\{\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{k}\right\} \cdot \left\{\binom{n}{k+1} + \binom{n}{k+2} + \cdots + \binom{n}{n}\right\}.$$

Let $B(x)$ be the generating function for the sequence (b_k), where $b_k = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{k}$.

(i) Show that

$$B(x) = \frac{(1+x)^n}{1-x}.$$

(ii) Find the generating function for the sequence (a_n), and deduce that

$$a_{n-1} = \sum_{r=0}^{n-1} \binom{2n}{r}(n-r).$$

(iii) Show that

$$a_{n-1} = \frac{n}{2}\binom{2n}{n}.$$

(G. Chang and Z. Shan, 1984.)

28. For $m, n \in \mathbf{N}$ and $r \in \mathbf{N}^*$, a generalized quantity $\binom{n}{r}_m$ of binomial coefficients is defined as follows:

$$\binom{1}{r}_m = \begin{cases} 1 & \text{if } 0 \leq r \leq m-1 \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\binom{n}{r}_m = \sum_{i=0}^{m-1} \binom{n-1}{r-i}_m \quad \text{for } n \geq 2.$$

Note that $\binom{n}{r}_2 = \binom{n}{r}$. Show that

(i) $\binom{n}{r}_m$ is the number of integer solutions to the equation

$$x_1 + x_2 + \cdots + x_n = r$$

with $0 \le x_i \le m-1$ for each $i = 1, 2, \ldots, n$;

(ii) $\binom{n}{0}_m = 1$;

(iii) $\binom{n}{1}_m = n$, where $m \ge 2$;

(iv) $\binom{n}{r}_m = \binom{n}{s}_m$, where $r + s = n(m-1)$;

(v) $\sum_{r=0}^{n(m-1)} \binom{n}{r}_m = m^n$;

(vi) the generating function for $\left(\binom{n}{r}_m\right)_{r=0,1,2,\ldots}$ is $(1 + x + \cdots + x^{m-1})^n$;

(vii) $\sum_{r=0}^{n(m-1)} (-1)^r \binom{n}{r}_m = \begin{cases} 0 & \text{if } m \text{ is even} \\ 1 & \text{if } m \text{ is odd} \end{cases}$

(viii) $\sum_{r=1}^{n(m-1)} r\binom{n}{r}_m = \frac{n(m-1)m^n}{2}$;

(ix) $\sum_{r=1}^{n(m-1)} (-1)^{r-1} r\binom{n}{r}_m = \begin{cases} 0 & \text{if } m \text{ is even} \\ \frac{n(1-m)}{2} & \text{if } m \text{ is odd} \end{cases}$

(x) $\sum_{i=0}^{r} \binom{p}{i}_m \binom{q}{r-i}_m = \binom{p+q}{r}_m$, where $p, q \in \mathbf{N}$;

(xi) $\binom{n}{r}_m = \sum_{i=0}^{n} (-1)^i \binom{n}{i} \binom{n-1+r-mi}{n-1}$.

(See C. Cooper and R. E. Kennedy, A dice-tossing problem, *Crux Mathematicorum*, **10** (1984), 134-138.)

29. Given $n \in \mathbf{N}$, evaluate the sum

$$S_n = \sum_{r=0}^{n} 2^{r-2n} \binom{2n-r}{n}.$$

(Proposed by the Israeli Team at the 31st IMO.)

30. For each $r \in \mathbf{N}^*$, let

$$a_r = 1 \cdot 4 \cdot 7 \cdots (3r+1).$$

Show that the exponential generating function for the sequence (a_r) is given by $(1-3x)^{-\frac{4}{3}}$.

31. For $n \in \mathbf{N}$, find the number of ways to colour the n squares of a $1 \times n$ chessboard using the colours: blue, red and white, if each square is coloured by a colour and an even number of squares are to be coloured red.

32. Find the number of n-digit quaternary sequences that contain an odd number of 0's, an even number of 1's and at least one 3.

33. For $n \in \mathbf{N}$, find the number of words of length n formed by the symbols: $\alpha, \beta, \gamma, \delta, \epsilon, \lambda$ in which the total number of α's and β's is (i) even, (ii) odd.

34. For $r \in \mathbf{N}^*$, find the number of ways of distributing r distinct objects into 5 distinct boxes such that each of the boxes 1, 3, and 5 must hold an odd number of objects while each of the remaining boxes must hold an even number of objects.

35. Prove the following summations for all real z :
(i) $\sum_{k=0}^{n} \binom{z}{2k}\binom{z-2k}{n-k}2^{2k} = \binom{2z}{2n}$,
(ii) $\sum_{k=0}^{n} \binom{z+1}{2k+1}\binom{z-2k}{n-k}2^{2k+1} = \binom{2z+2}{2n+1}$.
(Proposed by M. Machover and H. W. Gould, see *Amer. Math. Monthly*, **75** (1968), 682.)

36. Prove that
$$\sum_{r=1}^{n}\sum_{k=0}^{r}(-1)^{k+1}\frac{k}{r}\binom{r}{k}k^{n-1} = 0,$$
where $n = 2, 3, 4, \ldots$. (Proposed by G. M. Lee, see *Amer. Math. Monthly*, **77** (1970), 308-309.)

37. Prove that
$$\sum \frac{1}{k_1!k_2!\cdots k_n!} = \frac{1}{r!}\binom{n-1}{r-1},$$
where the sum is taken over all $k_1, k_2, \ldots, k_n \in \mathbf{N}^*$ with $\sum_{i=1}^{n} k_i = r$ and $\sum_{i=1}^{n} ik_i = n$.
(Proposed by D. Ž. Djoković, see *Amer. Math. Monthly*, **77** (1970), 659.)

38. Ten female workers and eight male workers are to be assigned to work in one of four different departments of a company. In how many ways can this be done if

(i) each department gets at least one worker?

(ii) each department gets at least one female worker?

(iii) each department gets at least one female worker and at least one male worker?

39. For $r \in \mathbf{N}^*$, find the number of r-permutations of the multi-set

$$\{\infty \cdot \alpha,\ \infty \cdot \beta,\ \infty \cdot \gamma,\ \infty \cdot \lambda\}$$

in which the number of α's is odd while the number of λ's is even.

40. For $r \in \mathbf{N}^*$ and $n \in \mathbf{N}$, let $a_r = F(r,n)$, which is the number of ways to distribute r distinct objects into n distinct boxes so that no box is empty (see Theorem 4.5.1). Thus $a_r = n!S(r,n)$, where $S(r,n)$ is a Stirling number of the second kind. Find the exponential generating function for the sequence (a_r), and show that for $r \geq 2$,

$$\sum_{m=0}^{\infty}(-1)^m m!S(r,m+1) = 0.$$

41. For $n \in \mathbf{N}$, let $A_n(x)$ be the exponential generating function for the sequence $(S(0,n),\ S(1,n),\ldots,\ S(r,n),\ldots)$. Find $A_n(x)$ and show that

$$\frac{d}{dx}A_n(x) = nA_n(x) + A_{n-1}(x),$$

where $n \geq 2$.

42. Let $B_0 = 1$ and for $r \in \mathbf{N}$, let $B_r = \sum_{k=1}^{r} S(r,k)$. The numbers B_r's are called the Bell numbers (see Section 1.7). Show that the exponential generating function for the sequence (B_r) is given by e^{e^x-1}.

43. Let $n \in \mathbf{N}$ and $r \in \mathbf{N}^*$.

(a) Find the number of ways of distributing r distinct objects into n distinct boxes such that the objects in each box are ordered.

(b) Let a_r denote the number of ways to select at most r objects from r distinct objects and to distribute them into n distinct boxes such that the objects in each box are ordered. Show that

(i) $a_r = \sum_{i=0}^{r} \binom{r}{i} n^{(i)}$, where $n^{(i)} = n(n+1)\cdots(n+i-1)$ with $n^{(0)} = 1$;

(ii) the exponential generating function for the sequence (a_r) is given by

$$e^x(1-x)^{-n}.$$

44. Find the generating function for the sequence (a_r) in each of the following cases: a_r is the number of ways of distributing r identical objects into

(i) 4 distinct boxes;

(ii) 4 distinct boxes so that no box is empty;

(iii) 4 identical boxes so that no box is empty;

(iv) 4 identical boxes.

45. For $n \in \mathbf{N}$, show that the number of partitions of n into parts where no even part occurs more than once is equal to the number of partitions of n in which parts of each size occur at most three times.

46. For $r \in \mathbf{N}^*$ and $n \in \mathbf{N}$, let a_r be the number of integer solutions to the equation

$$x_1 + x_2 + \cdots + x_n = r,$$

where $x_1 \geq x_2 \geq \cdots \geq x_n \geq 1$. Find the generating function for the sequence (a_r).

47. For $r \in \mathbf{N}^*$ and $n \in \mathbf{N}$, let b_r be the number of integer solutions to the equation

$$x_1 + x_2 + \cdots + x_n = r,$$

where $x_1 \geq x_2 \geq \cdots \geq x_n \geq 0$. Find the generating function for the sequence (b_r).

48. For $r \in \mathbf{N}^*$ and $n \in \mathbf{N}$, let a_r denote the number of ways to distribute r identical objects into n identical boxes, and b_r denote the number of integer solutions to the equation

$$\sum_{k=1}^{n} kx_k = r$$

with $x_k \geq 0$ for each $k = 1, 2, \ldots, n$. Show that $a_r = b_r$ for each $r \in \mathbf{N}^*$.

49. For $r \in \mathbf{N}^*$, let a_r denote the number of partitions of r into distinct powers of 2.
   (i) Find the generating function for (a_r);
   (ii) Show that $a_r = 1$ for all $r \geq 1$;
   (iii) Give an interpretation of the result in (ii).

50. Show that for $n \in \mathbf{N}$, the number of partitions of $2n$ into *distinct even* parts is equal to the number of partitions of n into *odd* parts.

51. Let $k, n \in \mathbf{N}$. Show that the number of partitions of n into odd parts is equal to the number of partitions of kn into distinct parts whose sizes are multiples of k.

52. Let $p(n)$ be the number of partitions of n. Show that

$$p(n) \leq \frac{1}{2}(p(n+1) + p(n-1)),$$

where $n \in \mathbf{N}$ with $n \geq 2$.

53. For $n, k \in \mathbf{N}$ with $k \leq n$, let $p(n,k)$ denote the number of partitions of n into exactly k parts.
   (i) Determine the values of $p(5,1)$, $p(5,2)$, $p(5,3)$ and $p(8,3)$.
   (ii) Show that

$$\sum_{k=1}^{m} p(n,k) = p(n+m, m),$$

where $m \in \mathbf{N}$ and $m \leq n$.

54. (i) With $p(n,k)$ as defined in the preceding problem, determine the values of $p(5,3)$, $p(7,2)$ and $p(8,3)$.
   (ii) Show that

$$p(n-1, k-1) + p(n-k, k) = p(n,k).$$

55. Given $n, k \in \mathbf{N}$ with $n \leq k$, show that

$$p(n+k, k) = p(2n, n) = p(n).$$

56. For $n, k \in \mathbf{N}$ with $k \leq n$, show that

$$p(n,k) \geq \frac{1}{k!}\binom{n-1}{k-1}.$$

57. Given $n, k \in \mathbf{N}$, show that the number of partitions of n into k *distinct* parts is equal to $p\left(n - \binom{k}{2}, k\right)$.

58. Prove the corollary to Theorem 5.3.2.

59. (i) Prove Theorem 5.3.3

    (ii) Prove Theorem 5.3.4.

60. For positive integers n, let $C(n)$ be the number of representations of n as a sum of nonincreasing powers of 2, where no power can be used more than three times. For example, $C(8) = 5$ since the representations for 8 are:

$$8,\ 4+4,\ 4+2+2,\ 4+2+1+1,\ \text{and}\ 2+2+2+1+1.$$

    Prove or disprove that there is a polynoimal $Q(x)$ such that $C(n) = \lfloor Q(n) \rfloor$ for all positive integers n.

    (Putnam, 1983.)

61. For $n \in \mathbf{N}$, let $C(n)$ be the number defined in the preceding problem. Show that the generating function for the sequence $(C(n))$ is given by

$$\frac{1}{(1+x)(1-x)^2}.$$

    Deduce that $C(n) = \lfloor \frac{n+2}{2} \rfloor$ for each $n \in \mathbf{N}$.

62. (a) (i) List all partitions of 8 into 3 parts.

    (ii) List all noncongruent triangles whose sides are of integer length $a,\ b,\ c$ such that $a+b+c=16$.

    (iii) Is the number of partitions obtained in (i) equal to the number of noncongruent triangles obtained in (ii)?

    (b) For $r \in \mathbf{N}^*$, let a_r denote the number of noncongruent triangles whose sides are of integer length $a,\ b,\ c$ such that $a+b+c=2r$, and let b_r denote the number of partitions of r into 3 parts.

    (i) Show by (BP) that $a_r = b_r$ for each $r \in \mathbf{N}^*$.

    (ii) Find the generating function for (a_r).

63. A partition P of a positive integer n is said to be *self-conjugate* if P and its conjugate have the same Ferrers diagram.

    (i) Find all the self-conjugate partitions of 15.

    (ii) Find all the partitions of 15 into distinct odd parts.

    (iii) Show that the number of the self-conjugate partitions of n is equal to the number of partitions of n into distinct odd parts.

64. Show that the number of self-conjugate partitions of n with largest size equal to m is equal to the number of self-conjugate partitions of $n-2m+1$ with largest size not exceeding $m-1$.

65. (i) The largest square of asterisks in the upper left-hand corner of the Ferrers diagram is called the *Durfee* square of the diagram. Find the generating function for the number of self-conjugate partitions of r whose Durfee square is an $m \times m$ square, where $m \in \mathbf{N}$.

    (ii) Deduce that

$$\prod_{k=0}^{\infty}(1+x^{2k+1}) = 1 + \sum_{m=1}^{\infty} \frac{x^{m^2}}{\prod_{k=1}^{m}(1-x^{2k})}.$$

66. Let $A(x)$ be the generating function for the sequence $(p(r))$ where $p(r)$ is the number of partitions of r.

    (i) Find $A(x)$;

    (ii) Use the notion of Durfee square to prove that

$$\left[\prod_{k=1}^{\infty}(1-x^k)\right]^{-1} = 1 + \sum_{m=1}^{\infty} \frac{x^{m^2}}{\prod_{k=1}^{m}(1-x^k)^2}.$$

67. By considering isosceles right triangles of asterisks in the upper left-hand corner of a Ferrers diagram, show that

$$\prod_{k=1}^{\infty}(1+x^{2k}) = 1 + \sum_{m=1}^{\infty} \frac{x^{m(m+1)}}{\prod_{k=1}^{m}(1-x^{2k})}.$$

68. Let $p, q, r \in \mathbf{N}$ with $p < r$ and $q < r$. Show that the number of partitions of $r-p$ into $q-1$ parts with sizes not exceeding p, is equal to the number of partitions of $r-q$ into $p-1$ parts with sizes not exceeding q.

69. For $n \in \mathbf{N}$, let $p_e(n)$ (resp., $p_o(n)$) denote the number of partitions of n into an even (resp., odd) number of *distinct* parts. Show that

$$p_e(n) - p_o(n) = \begin{cases} (-1)^k & \text{if } n = \frac{k(3k\pm 1)}{2} \\ 0 & \text{otherwise.} \end{cases}$$

70. Prove the following Euler's pentagonal number theorem:

$$\prod_{k=1}^{\infty}(1-x^k) = \sum_{m=-\infty}^{\infty} (-1)^m x^{\frac{1}{2}m(3m-1)}.$$

71. For $n \in \mathbf{N}$, show that

$$\begin{aligned}&p(n)-p(n-1)-p(n-2)+p(n-5)+p(n-7)\\&+\cdots+(-1)^m p(n-\frac{1}{2}m(3m-1))+\cdots\\&+(-1)^m p(n-\frac{1}{2}m(3m+1))+\cdots=0.\end{aligned}$$

72. For $j \in \mathbf{N}^*$, let $\beta(j)=\frac{3j^2+j}{2}$. Prove the following Euler identity:

$$\sum_{j \text{ even}} p(n-\beta(j)) = \sum_{j \text{ odd}} p(n-\beta(j))$$

by (BP), where $n \in \mathbf{N}$.

(See D. M. Bressoud and D. Zeilberger, Bijecting Euler's partitions-recurrence, *Amer. Math. Monthly*, **92** (1985), 54-55.)

73. For $r, n \in \mathbf{N}$, let $f(r,n)$ denote the number of partitions of n of the form

$$n = n_1 + n_2 + \cdots + n_s,$$

where, for $i = 1, 2, \ldots, s-1$, $n_i \geq r n_{i+1}$, and let $g(r,n)$ denote the number of partitions of n, where each part is of the form $1 + r + r^2 + \cdots + r^k$ for some $k \in \mathbf{N}^*$. Show that

$$f(r,n) = g(r,n).$$

(See D. R. Hickerson, A partition identity of the Euler type, *Amer. Math. Monthly*, **81** (1974), 627-629.)

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