Chapter 2

Binomial Coefficients and Multinomial Coefficients

2.1. Introduction

Given $r, n \in \mathbf{Z}$ with $0 \leq r \leq n$, the number $\binom{n}{r}$ or C_r^n was defined in Chapter 1 as the number of r-element subsets of an n-element set. For convenience, we further define that $\binom{n}{r} = 0$ if $r > n$ or $r < 0$. Hence, by the result of (1.4.1), we have:

$$\binom{n}{r} = \begin{cases} \dfrac{n!}{r!(n-r)!} & \text{if } 0 \leq r \leq n, \\ 0 & \text{if } r > n \text{ or } r < 0, \end{cases}$$

for any $r, n \in \mathbf{Z}$ with $n \geq 0$.

In Chapter 1 and Exercise 1, we have learnt some basic identities governing the numbers $\binom{n}{r}$'s. These useful identities are summarized in the following list:

$$\binom{n}{r} = \binom{n}{n-r} \tag{2.1.1}$$

$$\binom{n}{r} = \frac{n}{r}\binom{n-1}{r-1} \quad \text{provided that } r \geq 1 \tag{2.1.2}$$

$$\binom{n}{r} = \frac{n-r+1}{r}\binom{n}{r-1} \quad \text{provided that } r \geq 1 \tag{2.1.3}$$

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r} \tag{2.1.4}$$

$$\binom{n}{m}\binom{m}{r} = \binom{n}{r}\binom{n-r}{m-r} \tag{2.1.5}$$

The numbers $\binom{n}{r}$'s are perhaps the most important and significant numbers in enumeration, and are often called *binomial coefficients* since they appear as the coefficients in the expansion of the binomial expression $(x+y)^n$. In this chapter, we shall derive some more fundamental and useful identities involving the binomial coefficients. Various techniques employed in the derivation of these identities will be discussed. We shall also introduce and study the notion of *multinomial coefficients* that are generalizations of the binomial coefficients.

2.2. The Binomial Theorem

We begin with the following simplest form of the binomial theorem discovered by Issac Newton (1646-1727) in 1676.

Theorem 2.2.1. *For any integer $n \geq 0$,*

$$\begin{aligned}(x+y)^n &= \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \cdots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n \\ &= \sum_{r=0}^{n}\binom{n}{r}x^{n-r}y^r.\end{aligned}$$

First proof – mathematical induction. For $n = 0$, the result is trivial as $(x+y)^0 = 1 = \binom{0}{0}x^0y^0$. Assume that it holds when $n = k$ for some integer $k \geq 0$, that is,

$$(x+y)^k = \sum_{r=0}^{k}\binom{k}{r}x^{k-r}y^r.$$

Consider $n = k+1$. Observe that

$$\begin{aligned}&(x+y)^{k+1} = (x+y)(x+y)^k \\ &= (x+y)\sum_{r=0}^{k}\binom{k}{r}x^{k-r}y^r \quad \text{(by the inductive hypothesis)} \\ &= \sum_{r=0}^{k}\binom{k}{r}x^{k+1-r}y^r + \sum_{r=0}^{k}\binom{k}{r}x^{k-r}y^{r+1} \\ &= \binom{k}{0}x^{k+1} + \binom{k}{1}x^ky + \binom{k}{2}x^{k-1}y^2 + \cdots + \binom{k}{k}xy^k \\ &\qquad + \binom{k}{0}x^ky + \binom{k}{1}x^{k-1}y^2 + \cdots + \binom{k}{k-1}xy^k + \binom{k}{k}y^{k+1}.\end{aligned}$$

Applying (2.1.4) and the trivial results that $\binom{k}{0} = 1 = \binom{k+1}{0}$ and $\binom{k}{k} = 1 = \binom{k+1}{k+1}$, we have

$$(x+y)^{k+1} = \binom{k+1}{0}x^{k+1} + \binom{k+1}{1}x^k y + \cdots + \binom{k+1}{k}xy^k + \binom{k+1}{k+1}y^{k+1}$$

as desired. The result thus follows by induction. ∎

Second proof – combinatorial method. It suffices to prove that the coefficient of $x^{n-r}y^r$ in the expansion of $(x+y)^n$ is $\binom{n}{r}$.

To expand the product $(x+y)^n = \underbrace{(x+y)(x+y)\cdots(x+y)}_{n}$, we choose either x or y from each factor $(x+y)$ and then multiply them together. Thus to form a term $x^{n-r}y^r$, we first select r of the n factors $(x+y)$ and then pick "y" from the r factors chosen (and of course pick "x" from the remaining $(n-r)$ factors). The first step can be done in $\binom{n}{r}$ ways while the second in 1 way. Thus, the number of ways to form the term $x^{n-r}y^r$ is $\binom{n}{r}$ as required. ∎

2.3. Combinatorial Identities

The binomial theorem is a fundamental result in mathematics that has many applications. In this section, we shall witness how Theorem 2.2.1 yields easily a set of interesting identities involving the binomial coefficients. For the sake of comparison, some alternative proofs of these identities will be given.

Example 2.3.1. Show that for all integers $n \geq 0$,

$$\sum_{r=0}^{n}\binom{n}{r} = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n. \qquad (2.3.1)$$

First proof. By letting $x = y = 1$ in Theorem 2.2.1, we obtain immediately

$$\sum_{r=0}^{n}\binom{n}{r} = (1+1)^n = 2^n. \quad ∎$$

Second proof. Let X be an n-element set and $\mathcal{P}(X)$ be the set of all subsets of X. We shall count $|\mathcal{P}(X)|$ in two ways.

For each $r = 0, 1, ..., n$, the number of r-element subsets of X is $\binom{n}{r}$ by definition. Thus

$$|\mathcal{P}(X)| = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}.$$

On the other hand, by the result of Example 1.5.2, $|\mathcal{P}(X)| = 2^n$. The identity thus follows. ∎

Example 2.3.2. Show that for all integers $n \geq 1$,

(i) $\sum_{r=0}^{n}(-1)^r\binom{n}{r} = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n\binom{n}{n} = 0,$ (2.3.2)

(ii) $\binom{n}{0}+\binom{n}{2}+\cdots+\binom{n}{2k}+\cdots = \binom{n}{1}+\binom{n}{3}+\cdots+\binom{n}{2k+1}+\cdots = 2^{n-1}.$(2.3.3)

Proof. By letting $x = 1$ and $y = -1$ in Theorem 2.2.1, we obtain

$$\sum_{r=0}^{n}\binom{n}{r}(-1)^r = (1-1)^n = 0,$$

which is (i). The identity (ii) now follows from (i) and identity (2.3.1). ∎

Remark. A subset A of a non-empty set X is called an *even-element* (resp. *odd-element*) subset of X if $|A|$ is even (resp. odd). Identity (2.3.3) says that given an n-element set X, the number of even-element subsets of X is the same as the number of odd-element subsets of X. The reader is encouraged to establish a bijection between the family of even-element subsets of X and that of odd-element subsets of X (see Problem 2.10).

Example 2.3.3. Show that for all integers $n \in \mathbf{N}$,

$$\sum_{r=1}^{n} r\binom{n}{r} = \binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n} = n \cdot 2^{n-1}. \quad (2.3.4)$$

First proof. Letting $x = 1$ in Theorem 2.2.1 yields

$$(1+y)^n = \sum_{r=0}^{n} \binom{n}{r} y^r.$$

Differentiating both sides of the above identity with respect to y gives

$$n(1+y)^{n-1} = \sum_{r=1}^{n} r\binom{n}{r} y^{r-1}.$$

Finally, by putting $y = 1$, we have

$$\sum_{r=1}^{n} r\binom{n}{r} = n(1+1)^{n-1} = n \cdot 2^{n-1}. \quad \blacksquare$$

Second proof. Identity (2.1.2) can be rewritten as

$$r\binom{n}{r} = n\binom{n-1}{r-1}.$$

It thus follows that

$$\begin{aligned}\sum_{r=1}^{n} r\binom{n}{r} &= \sum_{r=1}^{n} n\binom{n-1}{r-1} = n\sum_{r=1}^{n}\binom{n-1}{r-1}\\ &= n\sum_{s=0}^{n-1}\binom{n-1}{s} \quad \text{(letting } s = r-1)\\ &= n\cdot 2^{n-1} \quad \text{(by (2.3.1))}. \quad \blacksquare\end{aligned}$$

Remark. Extending the techniques used in the two proofs above, one can also show that

$$\sum_{r=1}^{n} r^2\binom{n}{r} = n(n+1)2^{n-2},$$

$$\sum_{r=1}^{n} r^3\binom{n}{r} = n^2(n+3)2^{n-3}$$

for all $n \in \mathbf{N}$ (see Problem 2.47).

In general, what can be said about the summation

$$\sum_{r=1}^{n} r^k\binom{n}{r},$$

where $k \in \mathbf{N}$ and $k \geq 4$ (see Problem 2.48)?

The next result was published by the French mathematician A.T. Vandermonde (1735-1796) in 1772.

Example 2.3.4. (Vandermonde's Identity) Show that for all $m, n, r \in \mathbf{N}$,

$$\sum_{i=0}^{r} \binom{m}{i}\binom{n}{r-i} = \binom{m}{0}\binom{n}{r} + \binom{m}{1}\binom{n}{r-1} + \cdots + \binom{m}{r}\binom{n}{0}$$
$$= \binom{m+n}{r}. \tag{2.3.5}$$

First proof. Expanding the expressions on both sides of the identity

$$(1+x)^{m+n} = (1+x)^m(1+x)^n,$$

we have by Theorem 2.2.1,

$$\sum_{k=0}^{m+n} \binom{m+n}{k} x^k$$
$$= \left(\sum_{i=0}^{m} \binom{m}{i} x^i\right)\left(\sum_{j=0}^{n} \binom{n}{j} x^j\right)$$
$$= \binom{m}{0}\binom{n}{0} + \left\{\binom{m}{0}\binom{n}{1} + \binom{m}{1}\binom{n}{0}\right\} x$$
$$+ \left\{\binom{m}{0}\binom{n}{2} + \binom{m}{1}\binom{n}{1} + \binom{m}{2}\binom{n}{0}\right\} x^2 + \cdots + \binom{m}{m}\binom{n}{n} x^{m+n}.$$

Now, comparing the coefficients of x^r on both sides yields

$$\binom{m+n}{r} = \binom{m}{0}\binom{n}{r} + \binom{m}{1}\binom{n}{r-1} + \cdots + \binom{m}{r}\binom{n}{0}. \blacksquare$$

Second proof. Let $X = \{a_1, a_2, ..., a_m, b_1, b_2, ..., b_n\}$ be a set of $m+n$ objects. We shall count the number of r-combinations A of X.

Assuming that A contains exactly i a's, where $i = 0, 1, ..., r$, then the other $r - i$ elements of A are b's; and in this case, the number of ways to form A is given by $\binom{m}{i}\binom{n}{r-i}$. Thus, by (AP), we have

$$\sum_{i=0}^{r} \binom{m}{i}\binom{n}{r-i} = \binom{m+n}{r}. \blacksquare$$

Remark. If we put $m = n = r$ in identity (2.3.5) and apply identity (2.1.1), we obtain the following

$$\sum_{i=0}^{n}\binom{n}{i}^2 = \binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}. \qquad (2.3.6)$$

We now give an example to show an application of the Vandermonde's identity. In Section 1.6, we showed that H_r^n, the number of r-element multi-subsets of $M = \{\infty \cdot a_1, \infty \cdot a_2, \ldots, \infty \cdot a_n\}$, is given by

$$H_r^n = \binom{r+n-1}{r}.$$

Consider the following 3×3 matrix A whose entries are H_r^n's:

$$A = \begin{pmatrix} H_1^1 & H_2^1 & H_3^1 \\ H_1^2 & H_2^2 & H_3^2 \\ H_1^3 & H_2^3 & H_3^3 \end{pmatrix}.$$

What is the value of the determinant $\det(A)$ of A? We observe that

$$A = \begin{pmatrix} \binom{1}{1} & \binom{2}{2} & \binom{3}{3} \\ \binom{2}{1} & \binom{3}{2} & \binom{4}{3} \\ \binom{3}{1} & \binom{4}{2} & \binom{5}{3} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 6 & 10 \end{pmatrix}$$

and it is easy to check that $\det(A) = 1$.

In general, we have the following interesting result, which can be found in [N] (pp.167 and 256).

Example 2.3.5. Let $A = (H_r^n)$ be the square matrix of order k, where $n, r \in \{1, 2, ..., k\}$, in which the (n, r)-entry is H_r^n. Show that $\det(A) = 1$.

Proof. Let $B = (b_{ij})$ and $C = (c_{ij})$ be the square matrices of order k defined by

$$b_{ij} = \binom{i}{j} \quad \text{and} \quad c_{ij} = \binom{j-1}{j-i};$$

i.e.,

$$B = \begin{pmatrix} \binom{1}{1} & 0 & 0 & \cdots & 0 \\ \binom{2}{1} & \binom{2}{2} & 0 & \cdots & 0 \\ \binom{3}{1} & \binom{3}{2} & \binom{3}{3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \binom{k}{1} & \binom{k}{2} & \binom{k}{3} & \cdots & \binom{k}{k} \end{pmatrix}$$

and

$$C = \begin{pmatrix} \binom{0}{0} & \binom{1}{1} & \binom{2}{2} & \cdots & \binom{k-1}{k-1} \\ 0 & \binom{1}{0} & \binom{2}{1} & \cdots & \binom{k-1}{k-2} \\ 0 & 0 & \binom{2}{0} & \cdots & \binom{k-1}{k-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \binom{k-1}{0} \end{pmatrix}.$$

Claim. $A = BC$.

Indeed, if a_{nr} denotes the (n,r)-entry of the product BC, then

$$\begin{aligned} a_{nr} &= \sum_{i=1}^{k} b_{ni}c_{ir} = \sum_{i=1}^{k} \binom{n}{i}\binom{r-1}{r-i} \\ &= \sum_{i=1}^{r} \binom{n}{i}\binom{r-1}{r-i} \quad \left(\binom{r-1}{r-i} = 0 \text{ if } i > r\right) \\ &= \binom{r+n-1}{r} \quad \text{(by Vandermonde's identity)} \\ &= H_r^n. \end{aligned}$$

Thus, $BC = A$, as claimed.

Now,

$$\begin{aligned} \det(A) &= \det(BC) = \det(B)\det(C) \\ &= \left[\binom{1}{1}\cdot\binom{2}{2}\cdots\binom{k}{k}\right]\cdot\left[\binom{0}{0}\cdot\binom{1}{0}\cdots\binom{k-1}{0}\right] = 1. \quad \blacksquare \end{aligned}$$

2.4. The Pascal's Triangle

The set of binomial coefficients $\binom{n}{r}$'s can be conveniently arranged in a triangular form from top to bottom and left to right in increasing order of the values of n and r respectively, as shown in Figure 2.4.1. This diagram, one of the most influential number patterns in the history of mathematics, is called the *Pascal triangle*, after the renown French mathematician Blaise Pascal (1623-1662) who discovered it and made significant contributions to the understanding of it in 1653. The triangle is also called Yang Hui's triangle in China as it was discovered much earlier by the Chinese mathematician Yang Hui in 1261. The same triangle was also included in the

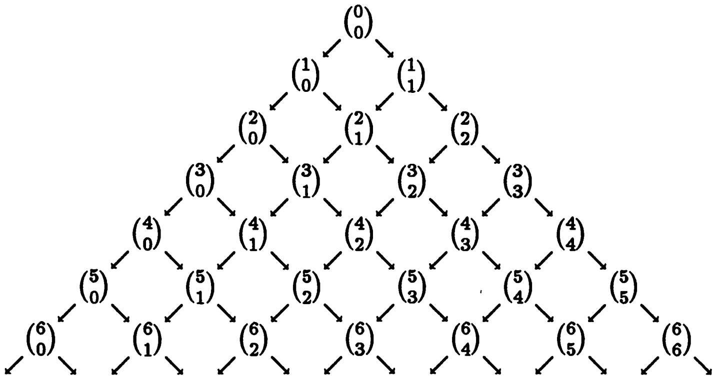


Figure 2.4.1.

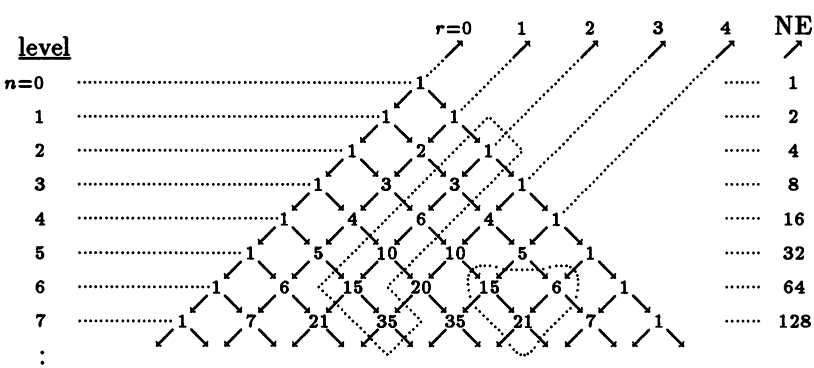


Figure 2.4.2.

book "Precious Mirror of the Four Elements" by another Chinese mathematician Chu Shih-Chieh in 1303. For the history of the Pascal's triangle, the reader may refer to the book [E].

We now make some simple observations with reference to Figure 2.4.2.

(1) The binomial coefficient $\binom{n}{r}$, located at the nth level from the top and rth position from the left, is the number of shortest routes from the top vertex representing $\binom{0}{0}$ to the vertex representing $\binom{n}{r}$. This is identical to what we observed in Example 1.5.1.

(2) As $\binom{n}{r} = \binom{n}{n-r}$, the entries of the triangle are symmetric with respect to the vertical line passing through the vertex $\binom{0}{0}$.

(3) Identity (2.3.1) says that the sum of the binomial coefficients at the n^{th} level is equal to 2^n, and identity (2.3.6) says that the sum of the squares of the binomial coefficients at the n^{th} level is equal to $\binom{2n}{n}$.

(4) Identity (2.1.4), namely $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$, simply says that each binomial coefficient in the interior of the triangle is equal to the sum of the two binomial coefficients on its immediate left and right "shoulders". For instance, $21 = 15 + 6$ as shown in Figure 2.4.2.

2.5. Chu Shih-Chieh's Identity

We proceed with another observation in Figure 2.4.2. Consider the 5 consecutive binomial coefficients:

$$\binom{2}{2} = 1, \ \binom{3}{2} = 3, \ \binom{4}{2} = 6, \ \binom{5}{2} = 10 \text{ and } \binom{6}{2} = 15$$

along the NE line when $r = 2$ from the right side of the triangle as shown. The sum of these 5 number is $1 + 3 + 6 + 10 + 15 = 35$, which is the immediate number we reach after turning left from the route 1-3-6-10-15. Why is this so? Replacing $\binom{2}{2}$ by $\binom{3}{3}$ (they all equal 1) and applying the above observation (4) successively, we have

$$\begin{aligned}
&\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \binom{5}{2} + \binom{6}{2} \\
&= \underbrace{\binom{3}{3} + \binom{3}{2}} + \binom{4}{2} + \binom{5}{2} + \binom{6}{2} \\
&= \underbrace{\binom{4}{3} + \binom{4}{2}} + \binom{5}{2} + \binom{6}{2} \\
&= \underbrace{\binom{5}{3} + \binom{5}{2}} + \binom{6}{2} \\
&= \underbrace{\binom{6}{3} + \binom{6}{2}} \\
&= \binom{7}{3} = 35.
\end{aligned}$$

Evidently, this argument can be used in a general way to obtain the following identity (2.5.1), which was also discovered by Chu Shih-Chieh in 1303.

Example 2.5.1. Show that

(i) $$\binom{r}{r}+\binom{r+1}{r}+\cdots+\binom{n}{r}=\binom{n+1}{r+1} \tag{2.5.1}$$

for all $r, n \in \mathbf{N}$ with $n \geq r$;

(ii) $$\binom{r}{0}+\binom{r+1}{1}+\cdots+\binom{r+k}{k}=\binom{r+k+1}{k} \tag{2.5.2}$$

for all $r, k \in \mathbf{N}$.

Identities (2.5.1) and (2.5.2) can be remembered easily with the help of the patterns as shown in Figure 2.5.1. Due to the symmetry of the Pascal's triangle, identities (2.5.1) and (2.5.2) are equivalent; i.e., (2.5.1) $\Leftrightarrow$ (2.5.2).

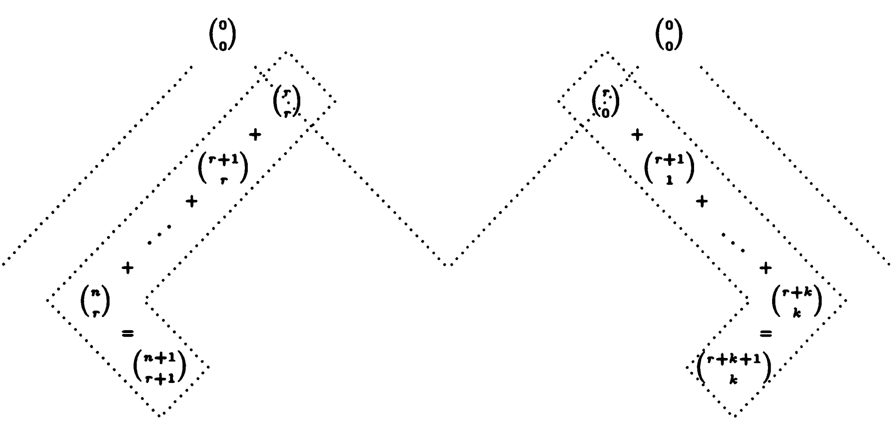


Figure 2.5.1.

We now give a combinatorial proof for identity (2.5.1).

Proof of (i). Let $X = \{a_1, a_2, ..., a_{n+1-r}, ..., a_{n+1}\}$ be a set of $n+1$ elements. We shall count the number of $(r+1)$-element subsets A of X. Consider the following $n+1-r$ cases:

(1): $a_1 \in A$. We need another r elements from $X \backslash \{a_1\}$ to form A. There are $\binom{n}{r}$ ways that A can be formed.

(2): $a_1 \notin A$ and $a_2 \in A$. We need another r elements from $X \backslash \{a_1, a_2\}$ to form A. There are $\binom{n-1}{r}$ ways that A can be formed.

$\cdots$

$(n-r)$: $a_1, a_2, ..., a_{n-1-r} \notin A$ and $a_{n-r} \in A$. We need another r elements from $X \backslash \{a_1, a_2, ..., a_{n-r}\}$ to form A. There are $\binom{(n+1)-(n-r)}{r} = \binom{r+1}{r}$ ways that A can be formed.

$(n+1-r)$: $a_1, a_2, ..., a_{n-r} \notin A$. In this case, there is $\binom{r}{r} = 1$ way to form A, namely, $A = \{a_{n-r+1}, a_{n-r+2}, ..., a_{n+1}\}$.

We note that all the above $n+1-r$ cases are pairwise disjoint and exhaustive. Thus we have by (AP),

$$\binom{n}{r} + \binom{n-1}{r} + \cdots + \binom{r+1}{r} + \binom{r}{r} = \binom{n+1}{r+1},$$

proving identity (2.5.1). ∎

We shall present two examples showing some applications of identity (2.5.1).

Example 2.5.2. **(IMO, 1981/2)** Let $1 \leq r \leq n$ and consider all r-element subsets of the set $\{1, 2, ..., n\}$. Each of these subsets has a smallest member. Let $F(n, r)$ denote the arithmetic mean of these smallest numbers. Prove that

$$F(n, r) = \frac{n+1}{r+1}.$$

Before proving the result, we give an illustrating example to help us understand the problem better. Take $n = 5$ and $r = 3$. All the 3-element subsets of $\{1, 2, 3, 4, 5\}$ and their respective smallest members are shown in Table 2.5.1.

3-element subsets of $\{1,2,3,4,5\}$	Smallest members
$\{1,2,3\}$	1
$\{1,2,4\}$	1
$\{1,2,5\}$	1
$\{1,3,4\}$	1
$\{1,3,5\}$	1
$\{1,4,5\}$	1
$\{2,3,4\}$	2
$\{2,3,5\}$	2
$\{2,4,5\}$	2
$\{3,4,5\}$	3

Table 2.5.1.

Thus $F(5,3) = \frac{1}{10}(6 \cdot 1 + 3 \cdot 2 + 1 \cdot 3) = \frac{3}{2}$, while $\frac{n+1}{r+1} = \frac{6}{4} = \frac{3}{2}$, and they are equal.

Two questions are in order. First, which numbers in the set $\{1, 2, ..., n\}$ could be the smallest members of some r-element subsets of $\{1, 2, ..., n\}$? How many times does each of these smallest members contribute to the sum? Observe that $\{n-r+1, n-r+2, ..., n\}$ consists of $n-(n-r+1)+1 = r$ elements; and it is the r-element subset of $\{1, 2, ..., n\}$ consisting of the largest possible members. Thus the numbers $1, 2, ..., n - r + 1$ are all the possible smallest members of the r-element subsets of $\{1, 2, ..., n\}$. This answers the first question. Let $m = 1, 2, ..., n - r + 1$. The number of times that m contributes to the sum is the number of r-element subsets of $\{1, 2, ..., n\}$ which contain m as the smallest member. This, however, is equal to the number of ways to form $(r - 1)$-element subsets from the set $\{m + 1, m + 2, ..., n\}$. Thus the desired number of times that m contributes to the sum is $\binom{n-m}{r-1}$, which answers the second question. We are now ready to see how identity (2.5.1) can be applied to prove the statement of Example 2.5.2.

Proof. For $m = 1, 2, \ldots, n-r+1$, the number of r-element subsets of $\{1, 2, \ldots, n\}$ which contain m as the smallest member is given by $\binom{n-m}{r-1}$. Thus the sum S of the smallest members of all the r-element subsets of $\{1, 2, \ldots, n\}$ is given by

$$S = 1\binom{n-1}{r-1} + 2\binom{n-2}{r-1} + 3\binom{n-3}{r-1} + \cdots + (n-r+1)\binom{r-1}{r-1}$$

$$\begin{array}{cccccc} = \binom{n-1}{r-1} & +\binom{n-2}{r-1} & +\binom{n-3}{r-1} & +\cdots & +\binom{r-1}{r-1} & \\ & +\binom{n-2}{r-1} & +\binom{n-3}{r-1} & +\cdots & +\binom{r-1}{r-1} & \\ & & +\binom{n-3}{r-1} & +\cdots & +\binom{r-1}{r-1} & \\ & & & +\cdots & +\binom{r-1}{r-1} & \\ & & & & +\binom{r-1}{r-1} & \end{array} \left.\vphantom{\begin{array}{c} 1\\1\\1\\1\\1 \end{array}}\right\} (n-r+1) \text{ rows}$$

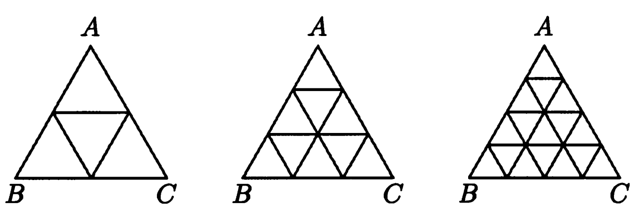
Applying identity (2.5.1) to each row yields

$$S = \binom{n}{r} + \binom{n-1}{r} + \binom{n-2}{r} + \cdots + \binom{r}{r},$$

which is equal to $\binom{n+1}{r+1}$, applying (2.5.1) once more. Since the number of r-element subsets of $\{1, 2, \ldots, n\}$ is $\binom{n}{r}$, it follows that

$$F(n, r) = \frac{\binom{n+1}{r+1}}{\binom{n}{r}} = \frac{(n+1)!}{(r+1)!(n-r)!} \cdot \frac{r!(n-r)!}{n!} = \frac{n+1}{r+1}. \quad \blacksquare$$

We now consider another example. For $n \in \mathbf{N}$, the n^{th} *subdivision* of an equilateral triangle ABC is the configuration obtained by (i) dividing each side of $\triangle ABC$ into $n+1$ equal parts by n points; and (ii) adding $3n$ line segments to join the $3n$ pairs of such points on adjacent sides so that the line segments are parallel to the third side. The configurations for $n = 1, 2, 3$ are shown in Figure 2.5.2.



$n = 1$ $\qquad$ $n = 2$ $\qquad$ $n = 3$

Figure 2.5.2.

Let $g(n)$ denote the number of parallelograms contained in the nth subdivision of an equilateral triangle. It can be checked from Figure 2.5.2 that

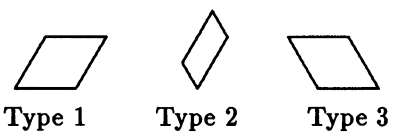
$$g(1) = 3, \quad g(2) = 15, \quad \text{and} \quad g(3) = 45.$$

The general case is discussed below.

Example 2.5.3. For each $n \in \mathbf{N}$, evaluate $g(n)$.

Remark. This problem, which can be found, for instance, in [MM], was given to the trainees in the 1990 Singapore Mathematical Olympiad Team as a test problem on 15 June, 1989. Lin Ziwei, a member in the team, of age 14 then, was able to solve the problem within an hour. His solution is presented below.

Solution. There are 3 types of parallelograms:



By symmetry, the number of parallelograms of each type is the same. Thus we need only to count the number of parallelograms of one type, say Type 1.

Any parallelogram of Type 1 is formed by 4 line segments ℓ_1, ℓ_2, ℓ_3 and ℓ_4 as shown in Figure 2.5.3.

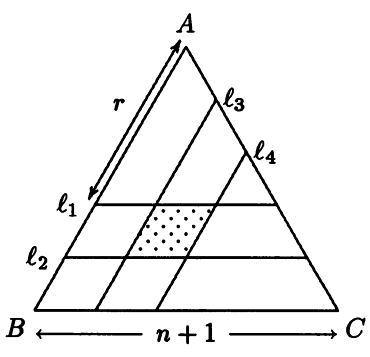


Figure 2.5.3

Each side of ΔABC is of length $n+1$ units. When ℓ_1 is r units from the vertex A, the number of choices for ℓ_2 is $n+1-r$, and the number of ways of choosing the pair $\{\ell_3, \ell_4\}$ is $\binom{r+1}{2}$. Thus the total number of parallelograms of Type 1 is given by

$$\begin{aligned}
&\sum_{r=1}^{n}(n+1-r)\binom{r+1}{2} \\
&= \binom{2}{2}n + \binom{3}{2}(n-1) + \binom{4}{2}(n-2) + \cdots + \binom{n+1}{2}1 \\
&= \left.\begin{matrix}
\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \cdots + \binom{n}{2} + \binom{n+1}{2} \\
+ \binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \cdots + \binom{n}{2} \\
\vdots \\
+ \binom{2}{2} + \binom{3}{2} \\
+ \binom{2}{2}
\end{matrix}\right\} n \text{ rows} \\
&= \binom{n+2}{3} + \binom{n+1}{3} + \cdots + \binom{4}{3} + \binom{3}{3} \quad \text{(by identity (2.5.1))} \\
&= \binom{n+3}{4} \quad \text{(by identity (2.5.1) again).}
\end{aligned}$$

Hence $g(n) = 3\binom{n+3}{4}$. ∎

Remark. Since the above answer for $g(n)$ is a simple binomial coefficient, one may wonder whether there is any shorter or more direct combinatorial argument proving the result. We present one below.

First, extend the sides AB and AC of the equilateral triangle ABC to AB' and AC' respectively such that $\frac{AB}{AB'} = \frac{AC}{AC'} = \frac{n+1}{n+2}$. Thus the nth subdivision of ΔABC is part of the $(n+1)$th subdivision of $\Delta AB'C'$ (see Figure 2.5.4(a)). Note that including B' and C', there are exactly $n+3$ subdivision points on $B'C'$ with respect to the $(n+1)$th subdivision of $\Delta AB'C'$. Now, observe that any parallelogram of Type 2 in the nth subdivision of ΔABC corresponds to a unique set of four subdivision points of $B'C'$ as shown in Figure 2.5.4(b). It can be shown that this correspondence

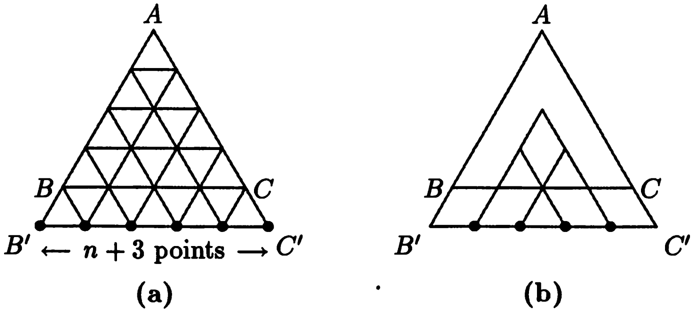


Figure 2.5.4.

is a bijection. Thus by (BP), the number of parallelograms of Type 2 in the nth subdivision of ΔABC is equal to the number of 4-element subsets of the set of $n+3$ subdivision points on $B'C'$. Since the latter is equal to $\binom{n+3}{4}$, we have $g(n) = 3\binom{n+3}{4}$. ∎

2.6. Shortest Routes in a Rectangular Grid

A point (a, b) in the x-y plane is called a *lattice point* if both a and b are integers. Figure 2.6.1 shows a rectangular grid in the x-y plane consisting of $(m+1) \times (n+1)$ lattice points, and a shortest route from the lattice point $(0,0)$ to the lattice point (m,n), where $m, n \in \mathbf{N}$. It follows from Example 1.5.1 that the number of shortest routes from $(0,0)$ to (m,n) is given by $\binom{m+n}{m}$ or $\binom{m+n}{n}$.

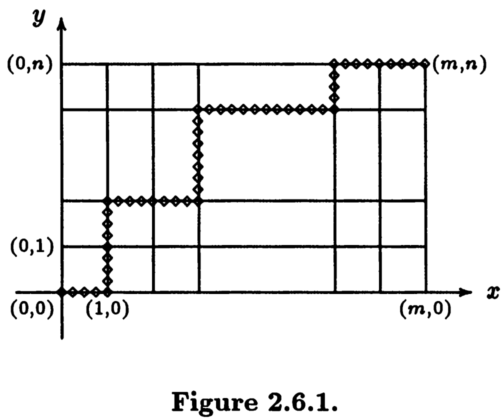
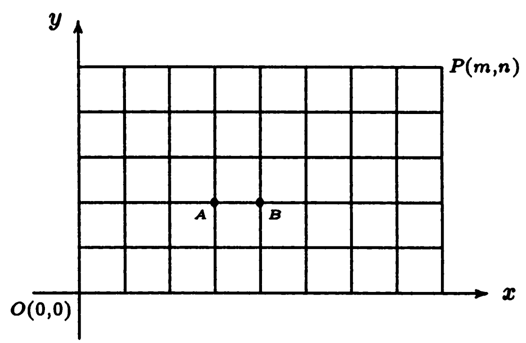


Figure 2.6.1.

In this section, we shall see that the technique of counting shortest routes between two lattice points in a rectangular grid can serve as a way of deriving combinatorial identities involving binomial coefficients. To begin with, we state the following two useful observations, which are related to Problem 1.25.

1° In Figure 2.6.2, the number of shortest routes from $O(0,0)$ to $A(x,y)$ is $\binom{x+y}{x}$, and the number of shortest routes from $A(x,y)$ to $P(m,n)$ is $\binom{(m-x)+(n-y)}{m-x}$. Thus the number of shortest routes from $O(0,0)$ to $P(m,n)$ that pass through $A(x,y)$ is given by

$$\binom{x+y}{x}\binom{(m-x)+(n-y)}{m-x}.$$



$A=(x,y)$, $B=(x+1,y)$

Figure 2.6.2.

2° In Figure 2.6.2, the number of shortest routes from $O(0,0)$ to $A(x,y)$ is $\binom{x+y}{x}$, and the number of shortest routes from $B(x+1,y)$ to $P(m,n)$ is $\binom{(m-x-1)+(n-y)}{m-x-1}$. Thus the number of shortest routes from $O(0,0)$ to $P(m,n)$ that pass through the line segment AB is given by

$$\binom{x+y}{x}\binom{(m-x-1)+(n-y)}{m-x-1}.$$

As the first example, we derive the Vandermonde's identity using the technique of counting shortest routes.

Example 2.6.1. Show that

$$\binom{m}{0}\binom{n}{r}+\binom{m}{1}\binom{n}{r-1}+\cdots+\binom{m}{r}\binom{n}{0}=\binom{m+n}{r}$$

where $m, n, r \in \mathbf{N}$ with $m, n \geq r$.

Proof. Consider the rectangular grid shown in Figure 2.6.3.

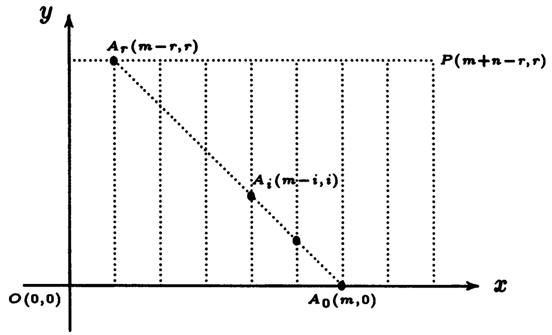


Figure 2.6.3.

The number of shortest routes from $O(0,0)$ to $P(m+n-r,r)$ is equal to

$$\binom{m+n-r+r}{r}=\binom{m+n}{r}.$$

We now count this number in a different way. Consider the line segment joining $A_0(m,0), A_1(m-1,1), \ldots$, and $A_r(m-r,r)$ as shown in Figure 2.6.3. We note that each shortest route from O to P passes through one and only one A_i $(i = 0, 1, ..., r)$ on the line segment. The number of such shortest routes passing through $A_i(m-i,i)$ is, by observation 1°, given by

$$\binom{m-i+i}{i}\binom{(m+n-r-m+i)+(r-i)}{r-i}=\binom{m}{i}\binom{n}{r-i}.$$

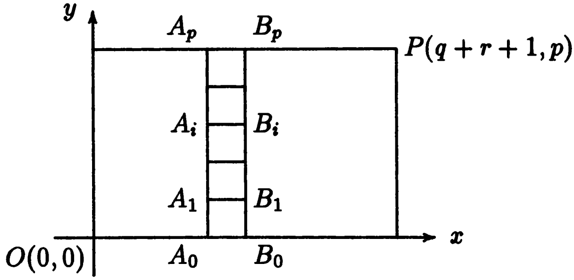
The identity now follows by (AP). ∎

The next example makes use of observation 2°.

Example 2.6.2. Show that for all integers $p, q, r \geq 0$,

$$\binom{p+q}{q}\binom{r}{r} + \binom{p+q-1}{q}\binom{r+1}{r} + \cdots + \binom{q}{q}\binom{r+p}{r}$$
$$= \binom{p+q+r+1}{q+r+1} = \binom{p+q+r+1}{p}. \tag{2.6.1}$$

Proof. Consider the grid shown in Figure 2.6.4.



$A_i = (r, i),\ B_i = (r+1, i),\ i = 0, 1, ..., p$

Figure 2.6.4.

The number of shortest routes from $O(0,0)$ to $P(q+r+1, p)$ is given by

$$\binom{q+r+1+p}{p}.$$

Another way of counting this number is as follows: Consider the sequence of unit line segments $A_0B_0, A_1B_1, \ldots, A_pB_p$, where $A_i = (r, i)$ and $B_i = (r+1, i)$, $i = 0, 1, ..., p$, as shown in Figure 2.6.4. We note that each shortest route from O to P must pass through one and only one unit line segment A_iB_i. The number of such shortest routes passing through A_iB_i is, by observation 2°, given by

$$\binom{r+i}{r}\binom{(q+r+1-r-1)+(p-i)}{q} = \binom{r+i}{r}\binom{p+q-i}{q}.$$

Identity (2.6.1) thus follows by (AP). ∎

From now on, let $\mathbf{N}_k = \{1, 2, ..., k\}$, where $k \in \mathbf{N}$. In Problem 1.81, we counted the number of mappings $\alpha : \mathbf{N}_n \to \mathbf{N}_m$ $(n, m \in \mathbf{N})$ that may satisfy some additional conditions. These results are summarized in the following table.

$\alpha : \mathbf{N}_n \to \mathbf{N}_m$	Number of α
α is a mapping	m^n
α is injective ($n \leq m$)	$P_n^m = m(m-1)\cdots(m-n+1)$
α is surjective ($n \geq m$)	$m!S(n,m)$
α is strictly increasing	$\binom{m}{n}$

Our next aim is to apply the technique of counting shortest routes to enumerate two special types of mappings.

A mapping $\alpha : \mathbf{N}_n \to \mathbf{N}_m$ is said to be *increasing* if $\alpha(a) \leq \alpha(b)$ whenever $a \leq b$ in $\mathbf{N}_n$. The first problem we shall study is the enumeration of increasing mappings $\alpha : \mathbf{N}_n \to \mathbf{N}_m$.

Given an increasing mapping $\alpha : \mathbf{N}_n \to \mathbf{N}_m$, we construct a shortest route $R(\alpha)$ from $(1,1)$ to $(n+1,m)$ as follows:

(i) Join $(1,1)$ to $(1,\alpha(1))$ if $\alpha(1) > 1$;

(ii) For each $i = 1,2,\ldots,n-1$,
join $(i,\alpha(i))$ to $(i+1,\alpha(i+1))$ if $\alpha(i) = \alpha(i+1)$,
join $(i,\alpha(i))$ to $(i+1,\alpha(i))$ and $(i+1,\alpha(i))$ to $(i+1,\alpha(i+1))$ if $\alpha(i) < \alpha(i+1)$;

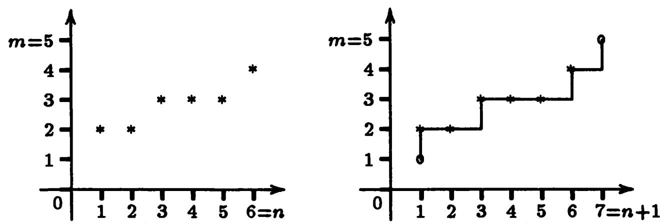
(iii) Join $(n,\alpha(n))$ to $(n+1,m)$ if $\alpha(n) = m$;
Join $(n,\alpha(n))$ to $(n+1,\alpha(n))$ and $(n+1,\alpha(n))$ to $(n+1,m)$ if $\alpha(n) < m$.

For instance, if $\alpha : \mathbf{N}_6 \to \mathbf{N}_5$ is the increasing mapping defined in Figure 2.6.5(a), then $R(\alpha)$ is the shortest route shown in Figure 2.6.5(b).

On the other hand, if R is the shortest route form $(1,1)$ to $(7,5)$ as shown in Figure 2.6.6(a), then $R = R(\alpha)$, where $\alpha : \mathbf{N}_6 \to \mathbf{N}_5$ is the increasing mapping shown in Figure 2.6.6(b), by reversing the above procedure.

We are now ready to deal with the following:

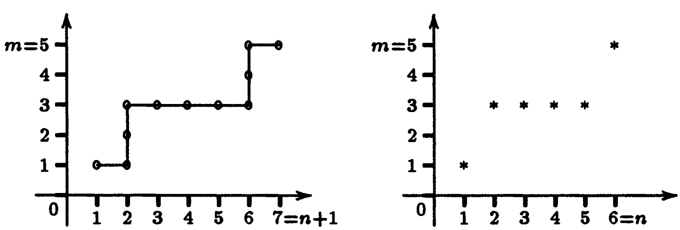
Example 2.6.3. Show that the number of increasing mappings from $\mathbf{N}_n$ to $\mathbf{N}_m$ ($m,n \in \mathbf{N}$) equals the number of shortest routes from the lattice point $(1,1)$ to the lattice point $(n+1,m)$, which is $\binom{m+n-1}{n}$.



(a) α (b) $R(\alpha)$

i	$\alpha(i)$
1	2
2	2
3	3
4	3
5	3
6	4

Figure 2.6.5.



(a) R (b) α

i	$\alpha(i)$
1	1
2	3
3	3
4	3
5	3
6	5

Figure 2.6.6.

Proof. Let X be the set of increasing mappings $\alpha : \mathbf{N}_n \to \mathbf{N}_m$, and Y be the set of shortest routes from $(1,1)$ to $(n+1,m)$. Define a mapping $f : X \to Y$ by putting for each $\alpha \in X$, $f(\alpha) = R(\alpha)$, which is the shortest route associated with α as defined above. It can be checked that f is indeed a bijection between X and Y. Hence we have by (BP),

$$|X| = |Y| = \binom{(n+1-1)+(m-1)}{n} = \binom{n+m-1}{n}. \quad \blacksquare$$

Remarks. (1) The reader should understand why we choose the lattice point $(n+1,m)$ but not (n,m) as a terminus in the above argument.

(2) The answer in Example 2.6.3, which is $H_n^m = \binom{m+n-1}{n}$, suggests another way to enumerate $|X|$. Indeed, given an increasing mapping $\alpha : \mathbf{N}_n \to \mathbf{N}_m$, there corresponds a unique n-element multi-subset $\{\alpha(1), \alpha(2), ..., \alpha(n)\}$ of the multi-set $M = \{\infty \cdot 1, \infty \cdot 2, ..., \infty \cdot m\}$ (thus the mapping of Figure 2.6.5(a) is associated with the 6-element multi-subset $\{2 \cdot 2, 3 \cdot 3, 1 \cdot 4\}$ of $M = \{\infty \cdot 1, \infty \cdot 2, ..., \infty \cdot 5\}$ and the mapping of Figure 2.6.6(b) is associated with $\{1 \cdot 1, 4 \cdot 3, 1 \cdot 5\}$ of M), and conversely, every n-element multi-subset of M corresponds to a unique increasing mapping from $\mathbf{N}_n$ to $\mathbf{N}_m$. The existence of this one-to-one correspondence shows that $|X| = H_n^m$.

Let $\mathbf{N}_k^* = \{0\} \cup \mathbf{N}_k$. Our next problem is to enumerate the number of increasing mappings $\alpha : \mathbf{N}_n \to \mathbf{N}_{n-1}^*$ such that $\alpha(a) < a$ for each $a \in \mathbf{N}_n$. First of all, we establish a useful principle about shortest routes in a grid, called the *reflection principle.*

Let $L : y = x + k$ $(k \in \mathbf{Z})$ be a line of slope 1 on the x-y plane. Suppose P and Q are two lattice points on one side of L and P' is the reflection of P with respect to L as shown in Figure 2.6.7. Then we have:

Reflection Principle (RP). The number of shortest routes from P to Q that meet the line L is equal to the number of shortest routes from P' to Q.

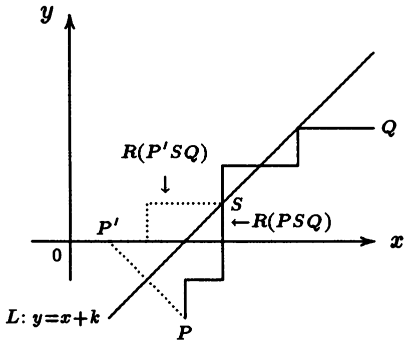


Figure 2.6.7.

Proof. Let X be the set of shortest routes from P to Q which meet L and Y be the set of shortest routes from P' to Q. The equality $|X| = |Y|$ will be proved by establishing a bijection between X and Y.

Let $R(PSQ)$ be a member in X where S is the first lattice point that $R(PSQ)$ meets L as it traverses from P to Q (see Figure 2.6.7). Let $R(P'SQ)$ be the union of (1) the shortest route from P' to S, which is the reflection of the portion from P to S in $R(PSQ)$ with respect to L, and (2) the shortest route from S to Q contained in $R(PSQ)$. Obviously, $R(P'SQ)$ is a member in Y. It can be checked that the correspondence

$$R(PSQ) \to R(P'SQ)$$

is indeed a bijection between X and Y. Thus $|X| = |Y|$ and so (RP) follows. ∎

Example 2.6.4. Show that the number of increasing mappings $\alpha : \mathbf{N}_n \to \mathbf{N}^*_{n-1}$, $n \in \mathbf{N}$, such that $\alpha(a) < a$ is given by $\frac{1}{n+1}\binom{2n}{n}$.

For instance, if $n = 3$, there are $\frac{1}{3+1}\binom{6}{3} = 5$ such mappings from $\mathbf{N}_3$ to $\mathbf{N}^*_2$. They are exhibited in the table below.

i	$\alpha_1(i)$	$\alpha_2(i)$	$\alpha_3(i)$	$\alpha_4(i)$	$\alpha_5(i)$
1	0	0	0	0	0
2	0	0	0	1	1
3	0	1	2	1	2

Proof. It follows from the discussion in the proof of Example 2.6.3 that the number of increasing mappings $\alpha : \mathbf{N}_n \to \mathbf{N}^*_{n-1}$ such that $\alpha(a) < a$ is equal to the number of shortest routes from $(1,0)$ to $(n+1, n-1)$ which do not meet the line $y = x$. For convenience, let δ_1 denote this number; and further let δ_2 denote the number of shortest routes from $(1,0)$ to $(n+1, n-1)$ and δ_3 denote the number of shortest routes from $(1,0)$ to $(n+1, n-1)$ which meet the line $y = x$.

It is clear that $\delta_1 = \delta_2 - \delta_3$ and $\delta_2 = \binom{(n+1-1)+(n-1)}{n} = \binom{2n-1}{n}$.

It remains to evaluate δ_3. First we note that the mirror image of the lattice point $(1,0)$ with respect to the line $y = x$ is $(0,1)$. Thus by (RP),

δ_3 is equal to the number of shortest routes from $(0,1)$ to $(n+1, n-1)$, which is given by

$$\delta_3 = \binom{(n+1)+(n-1-1)}{n+1} = \binom{2n-1}{n+1}.$$

Thus,

$$\delta_1 = \delta_2 - \delta_3 = \binom{2n-1}{n} - \binom{2n-1}{n+1} = \frac{1}{n+1}\binom{2n}{n}. \quad \blacksquare$$

Remarks. (1) Both the ideas of (RP) and the above argument are essentially due to the French combinatorist Désiré André (1840-1917), who made use of them to solve a problem in 1887, called the *ballot problem*, which was posed and also solved by Joseph Louis Francois Bertrand (1822-1900) in the same year. Readers may refer to the interesting survey article [BM] for the history and some generalizations of the problem.

(2) The first few terms of the numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$ just obtained are $1, 2, 5, 14, 42, 132, 429, 1430, \ldots$. They are called *Catalan numbers* after the Belgium mathematician Eugene Charles Catalan (1814-1894) who found the sequence of numbers in 1838 when he enumerated the number of ways of putting brackets in the product $x_1 x_2 \cdots x_n$ of n numbers. (Thus 1 way for $n = 2$: (x_1x_2); 2 ways for $n = 3$: $((x_1x_2)x_3), (x_1(x_2x_3))$; 5 ways for $n = 4$: $(((x_1x_2)x_3)x_4)$, $((x_1(x_2x_3))x_4)$, $((x_1x_2)(x_3x_4))$, $(x_1((x_2x_3)x_4))$, $(x_1(x_2(x_3x_4)))$ and so on). Some other problems related to Catalan numbers will be discussed in Chapter 6.

2.7. Some Properties of Binomial Coefficients

In the previous sections, we studied several identities involving binomial coefficients and introduced different techniques used to derive them. In this section, we shall state without proof some useful and striking properties about the binomial coefficients.

First of all, we have the following *unimodal* property:

1° For even $n \in \mathbf{N}$,

$$\binom{n}{0} < \binom{n}{1} < \cdots < \binom{n}{\frac{n}{2}} > \cdots > \binom{n}{n-1} > \binom{n}{n}. \tag{2.7.1}$$

and for odd $n \in \mathbf{N}$,

$$\binom{n}{0} < \binom{n}{1} < \cdots < \binom{n}{\frac{n-1}{2}} = \binom{n}{\frac{n+1}{2}} > \cdots > \binom{n}{n-1} > \binom{n}{n}. \quad (2.7.2)$$

2° Let $n \geq 2$ be an integer. Mann and Shanks [MS] showed that

$$n \text{ is a prime} \quad \text{iff} \quad n \mid \binom{n}{r} \text{ for all } r = 1, 2, \ldots, n-1.$$

Recently, this result has been improved by Z. Hao who showed (via private communication) that an integer $n > 2$ is prime iff

$$n \,\Bigg|\, \binom{n}{6k \pm 1},$$

for all k with $1 \leq k \leq \lfloor \frac{\sqrt{n}}{6} \rfloor$, where $\lfloor x \rfloor$ denotes the greatest integer not exceeding the real number x.

3° For $a, b, c \in \mathbf{Z}$, we write $a \equiv b \pmod{c}$ iff $c \mid (a-b)$. The following results are due to the 19th-century French mathematician E. Lucas (1842-1891).

Let p be a prime. Then

(i) $\binom{n}{p} \equiv \lfloor \frac{n}{p} \rfloor \pmod{p}$ for every $n \in \mathbf{N}$,

(ii) $\binom{p}{r} \equiv 0 \pmod{p}$ for every r such that $1 \leq r \leq p-1$,

(iii) $\binom{p+1}{r} \equiv 0 \pmod{p}$ for every r such that $2 \leq r \leq p-1$,

(iv) $\binom{p-1}{r} \equiv (-1)^r \pmod{p}$ for every r such that $0 \leq r \leq p-1$,

(v) $\binom{p-2}{r} \equiv (-1)^r (r+1) \pmod{p}$ for every r such that $0 \leq r \leq p-2$,

(vi) $\binom{p-3}{r} \equiv (-1)^r \binom{r+2}{2} \pmod{p}$ for every r such that $0 \leq r \leq p-3$.

4° Given a prime p, one can always find an $n \in \mathbf{N}^* = \mathbf{N} \cup \{0\}$ such that

$$p \nmid \binom{n}{r} \quad \text{for every } r = 0, 1, \ldots, n.$$

For instance, take $n = 0, 1, 2, \ldots, p-1$ (see properties (iv)–(vi) above). Besides these, there are other such numbers n, and so the problem is:

Given a prime p, determine the set

$$A = \{n \in \mathbf{N} \mid p \nmid \binom{n}{r}, \text{ for every } r = 0, 1, \ldots, n\}.$$

According to Honsberger [H2], this problem was posed and also solved by two Indian mathematicians M.R. Railkar and M.R. Modak in 1976. They proved that

$$n \in A \quad \text{iff} \quad n = kp^m - 1$$

where m is a nonnegative integer and $k = 1, 2, ..., p-1$.

5° Let $n, r \in \mathbf{N}$ and p be a prime. Write n and r with base p as follows:

$$n = n_0 + n_1 p + n_2 p^2 + \cdots + n_k p^k,$$
$$r = r_0 + r_1 p + r_2 p^2 + \cdots + r_k p^k,$$

where k is a nonnegative integer and $n_i, r_i \in \{0, 1, ..., p-1\}$ for every $i = 0, 1, ..., k$. In 1878, Lucas proved the following important result:

$$\binom{n}{r} \equiv \binom{n_0}{r_0}\binom{n_1}{r_1} \cdots \binom{n_k}{r_k} \pmod{p}.$$

In particular, if we take $p = 2$ and write n and r in binary system:

$$n = (n_k n_{k-1} \cdots n_1 n_0)_2$$
$$r = (r_k r_{k-1} \cdots r_1 r_0)_2$$

where $n_i, r_i \in \{0, 1\}$ for every $i = 0, 1, ..., k$, then we have the following interesting result:

$$\binom{n}{r} \text{ is odd} \quad \text{iff} \quad n_i \geq r_i \quad \text{for every } i = 0, 1, ..., k. \tag{2.7.3}$$

For instance, take $a = 11 = (a_3 a_2 a_1 a_0)_2 = (1011)_2$, $b = 9 = (b_3 b_2 b_1 b_0)_2 = (1001)_2$, and $c = 6 = (c_3 c_2 c_1 c_0)_2 = (0110)_2$. Since $a_i \geq b_i$ for every $i = 0, 1, 2, 3$, $\binom{a}{b} = \binom{11}{9}$ is odd; and since $a_2 \not\geq c_2$, $\binom{a}{c} = \binom{11}{6}$ is even.

6° According to Honsberger [H1], the following problem had been studied and solved by Fine [F]: Fix $n \in \mathbf{N}$, how many odd binomial coefficients $\binom{n}{r}$ are there at the n^{th} level of the Pascal's triangle? We shall apply result (2.7.3) to answer this question.

Write $n = (n_k n_{k-1} \cdots n_1 n_0)_2$ in binary system and let $w(n) = \sum_{i=0}^{k} n_i$, which is equal to the number of 1's in the multi-set $\{n_0, n_1, ..., n_k\}$. Given $r \in \mathbf{Z}$ such that $0 \leq r \leq n$, write $r = (r_k r_{k-1} \cdots r_1 r_0)_2$. By result (2.7.3), $\binom{n}{r}$ is odd iff $r_i \leq n_i$. In order that $r_i \leq n_i$, we have $r_i = 0$ if $n_i = 0$, and $r_i \in \{0, 1\}$ if $n_i = 1$. Thus the number of choices for r is $2^{w(n)}$. We therefore conclude that given $n \in \mathbf{N}$, the number of odd binomial coefficients $\binom{n}{r}$'s at the n^{th} level is given by $2^{w(n)}$. For instance, if $n = 11 = (1011)_2$, then $w(11) = 3$, and among the 12 binomial coefficients $\binom{11}{0}, \binom{11}{1}, \ldots, \binom{11}{11}$ at 11^{th} level, the following $8 (= 2^3)$ are odd:

$$\binom{11}{0} = \binom{11}{11} = 1, \quad \binom{11}{1} = \binom{11}{10} = 11, \quad \binom{11}{2} = \binom{11}{9} = 55, \quad \binom{11}{3} = \binom{11}{8} = 165.$$

2.8. Multinomial Coefficients and the Multinomial Theorem

By changing the symbols x, y to x_1, x_2 respectively, the binomial expansion may be written as

$$(x_1 + x_2)^n = \sum_{r=0}^{n} \binom{n}{r} x_1^r x_2^{n-r},$$

where $n \in \mathbf{N}$. Naturally, one may wish to find the coefficients in the expansion of the following more general product:

$$(x_1 + x_2 + \cdots + x_m)^n \tag{2.8.1}$$

where $n, m \in \mathbf{N}$ and $m \geq 2$.

To do this, let us first introduce a family of numbers, that can be regarded as extensions of binomial coefficients. Let

$$\binom{n}{n_1, n_2, \ldots, n_m} \tag{2.8.2}$$

denote the number of ways to distribute n distinct objects into m distinct boxes such that n_1 of them are in box 1, n_2 in box 2, ..., and n_m in box m, where $n, m, n_1, n_2, \ldots, n_m$ are nonnegative integers with

$$n_1 + n_2 + \cdots + n_m = n. \tag{2.8.3}$$

What can be said about the number (2.8.2) when $m = 2$? Since there are $\binom{n}{n_1}$ ways to select n_1 of n distinct objects and put them in box 1, and 1 way to put the remaining $n_2 = n - n_1$ objects in box 2, we see that $\binom{n}{n_1, n_2} = \binom{n}{n_1}$, which is the usual binomial coefficient. Actually, in general, numbers of the form (2.8.2) can be expressed as a product of a sequence of binomial coefficients as follows: From the given n distinct objects, there are

$\binom{n}{n_1}$ ways to select n_1 objects and put them in box 1,

$\binom{n-n_1}{n_2}$ ways to select n_2 objects from the remaining objects and put them in box 2,

$\vdots$

$\binom{n-(n_1+\cdots+n_{m-2})}{n_{m-1}}$ ways to select n_{m-1} objects from the remaining objects and put them in box $(m-1)$, and

$\binom{n-(n_1+\cdots+n_{m-1})}{n_m} = 1$ way to put the remaining objects in box m.

Thus we have

$$\binom{n}{n_1, n_2, \ldots, n_m} = \binom{n}{n_1}\binom{n-n_1}{n_2}\cdots\binom{n-(n_1+n_2+\cdots n_{m-1})}{n_m}. \tag{2.8.4}$$

Note that, as proved in Section 1.6, the right-hand product of (2.8.4) is equal to $\frac{n!}{n_1!n_2!\cdots n_m!}$. Thus we have:

$$\binom{n}{n_1, n_2, \ldots, n_m} = \frac{n!}{n_1!n_2!\cdots n_m!}. \tag{2.8.5}$$

We shall see the role played by the family of numbers (2.8.2) in the expansion of the product (2.8.1).

In expanding the product,

$$(x_1 + x_2 + \cdots + x_m)^n = \underbrace{(x_1 + x_2 + \cdots + x_m)\cdots(x_1 + x_2 + \cdots + x_m)}_{n},$$

we choose, for each of the above n factors, a symbol x_i from $\{x_1, x_2, \ldots, x_m\}$ and then multiply them together. Thus each term in the expansion is of the form:

$$x_1^{n_1}x_2^{n_2}\cdots x_m^{n_m} \tag{2.8.6}$$

for some nonnegative integers $n_1, n_2, ..., n_m$ with $\sum_{i=1}^m n_i = n$. If the like terms are grouped together, then the coefficient of (2.8.6) can be found. Let A be the set of ways that (2.8.6) can be formed, and B the set of ways of distributing n distinct objects into m distinct boxes such that n_i objects are put in box i for each $i = 1, 2, ..., m$, where $\sum_{i=1}^m n_i = n$. We claim that $|A| = |B|$. Define a mapping $f : A \to B$ as follows. For each member of the form $a = x_1^{n_1} x_2^{n_2} \cdots x_m^{n_m}$ in A, let $f(a)$ be the way of putting n_i objects in box i (corresponding to x_i). It is evident that f is a bijection between A and B and so $|A| = |B|$, as claimed.

We thus conclude that the coefficient of (2.8.6) in the expansion is given by

$$|A| = |B| = \binom{n}{n_1, n_2, ..., n_m}.$$

Combining this with identity (2.8.5), we arrive at the following generalization of the binomial theorem, that was first formulated by G.W. Leibniz (1646-1716) and later on proved by Johann Bernoulli (1667-1748).

Theorem 2.8.1 (The Multinomial Theorem). *For $n, m \in \mathbf{N}$,*

$$(x_1 + x_2 + \cdots + x_m)^n = \sum \binom{n}{n_1, n_2, ..., n_m} x_1^{n_1} x_2^{n_2} \cdots x_m^{n_m}$$

where the sum is taken over all m-ary sequences $(n_1, n_2, ..., n_m)$ of nonnegative integers with $\sum_{i=1}^m n_i = n$, and

$$\binom{n}{n_1, n_2, ..., n_m} = \frac{n!}{n_1! n_2! \cdots n_m!}.$$

Example 2.8.1. For $n = 4$ and $m = 3$, we have by Theorem 2.8.1,

$$\begin{aligned}(x_1+x_2+x_3)^4 &= \binom{4}{4,0,0}x_1^4 + \binom{4}{3,1,0}x_1^3x_2 + \binom{4}{3,0,1}x_1^3x_3 \\ &+ \binom{4}{2,2,0}x_1^2x_2^2 + \binom{4}{2,1,1}x_1^2x_2x_3 + \binom{4}{2,0,2}x_1^2x_3^2 \\ &+ \binom{4}{1,3,0}x_1x_2^3 + \binom{4}{1,2,1}x_1x_2^2x_3 + \binom{4}{1,1,2}x_1x_2x_3^2 \\ &+ \binom{4}{1,0,3}x_1x_3^3 + \binom{4}{0,4,0}x_2^4 + \binom{4}{0,3,1}x_2^3x_3 \\ &+ \binom{4}{0,2,2}x_2^2x_3^2 + \binom{4}{0,1,3}x_2x_3^3 + \binom{4}{0,0,4}x_3^4 \\ &= x_1^4 + 4x_1^3x_2 + 4x_1^3x_3 + 6x_1^2x_2^2 + 12x_1^2x_2x_3 + 6x_1^2x_3^2 \\ &+ 4x_1x_2^3 + 12x_1x_2^2x_3 + 12x_1x_2x_3^2 + 4x_1x_3^3 + x_2^4 \\ &+ 4x_2^3x_3 + 6x_2^2x_3^2 + 4x_2x_3^3 + x_3^4.\end{aligned}$$

Because of Theorem 2.8.1, the numbers of the form (2.8.2) are usually called the *multinomial coefficients*. Since multinomial coefficients are generalizations of binomial coefficients, it is natural to ask whether some results about binomial coefficients can be generalized to multinomial coefficients. We end this chapter with a short discussion on this.

1° The identity $\binom{n}{n_1} = \binom{n}{n-n_1}$ for binomial coefficients may be written as $\binom{n}{n_1,n_2} = \binom{n}{n_2,n_1}$ (here of course $n_1 + n_2 = n$). By identity (2.8.5), it is easy to see in general that

$$\binom{n}{n_1, n_2, \ldots, n_m} = \binom{n}{n_{\alpha(1)}, n_{\alpha(2)}, \ldots n_{\alpha(m)}} \tag{2.8.7}$$

where $\{\alpha(1), \alpha(2), \ldots, \alpha(m)\} = \{1, 2, \ldots, m\}$.

2° The identity $\binom{n}{n_1} = \binom{n-1}{n_1-1} + \binom{n-1}{n_1}$ for binomial coefficients may be written:

$$\binom{n}{n_1, n_2} = \binom{n-1}{n_1-1, n_2} + \binom{n-1}{n_1, n_2-1}.$$

In general, we have:

$$\begin{aligned}\binom{n}{n_1, n_2, \ldots, n_m} &= \binom{n-1}{n_1-1, n_2, \ldots, n_m} + \binom{n-1}{n_1, n_2-1, \ldots, n_m} + \cdots \\ &+ \binom{n-1}{n_1, n_2, \ldots, n_m - 1}.\end{aligned} \tag{2.8.8}$$

3° For binomial coefficients, we have the identity $\sum_{r=0}^{n}\binom{n}{r} = 2^n$. By letting $x_1 = x_2 = \cdots = x_m = 1$ in the multinomial theorem, we have

$$\sum \binom{n}{n_1, n_2, \ldots, n_m} = m^n \tag{2.8.9}$$

where the sum is taken over all m-ary sequences $(n_1, n_2, \ldots, n_m)$ of nonnegative integers with $\sum_{i=1}^{m} n_i = n$.

Identity (2.8.9) simply says that the sum of the coefficients in the expansion of $(x_1 + x_2 + \cdots + x_m)^n$ is given by m^n. Thus, in Example 2.8.1, the sum of the coefficients in the expansion of $(x_1 + x_2 + x_3)^4$ is 81, which is 3^4.

4° In the binomial expansion $(x_1 + x_2)^n = \sum_{r=0}^{n}\binom{n}{r}x_1^r x_2^{n-r}$, the number of distinct terms is $n + 1$. How many distinct terms are there in the expansion of $(x_1 + x_2 + \cdots + x_m)^n$? To answer this question, let us first look at Example 2.8.1. The distinct terms obtained in the expansion of $(x_1 + x_2 + x_3)^4$ are shown on the left column below:

$$\begin{aligned}
x_1^4 &\rightarrow \{4 \cdot x_1\} \\
x_1^3 x_2 &\rightarrow \{3 \cdot x_1, x_2\} \\
x_1^3 x_3 &\rightarrow \{3 \cdot x_1, x_3\} \\
x_1^2 x_2^2 &\rightarrow \{2 \cdot x_1, 2 \cdot x_2\} \\
x_1^2 x_2 x_3 &\rightarrow \{2 \cdot x_1, x_2, x_3\} \\
x_1^2 x_3^2 &\rightarrow \{2 \cdot x_1, 2 \cdot x_3\} \\
x_1 x_2^3 &\rightarrow \{x_1, 3 \cdot x_2\} \\
x_1 x_2^2 x_3 &\rightarrow \{x_1, 2 \cdot x_2, x_3\} \\
x_1 x_2 x_3^2 &\rightarrow \{x_1, x_2, 2 \cdot x_3\} \\
x_1 x_3^3 &\rightarrow \{x_1, 3 \cdot x_3\} \\
x_2^4 &\rightarrow \{4 \cdot x_2\} \\
x_2^3 x_3 &\rightarrow \{3 \cdot x_2, x_3\} \\
x_2^2 x_3^2 &\rightarrow \{2 \cdot x_2, 2 \cdot x_3\} \\
x_2 x_3^3 &\rightarrow \{x_2, 3 \cdot x_3\} \\
x_3^4 &\rightarrow \{4 \cdot x_3\}.
\end{aligned}$$

Observe that each of them corresponds to a unique 4-element multi-subset of $M = \{\infty \cdot x_1, \infty \cdot x_2, \infty \cdot x_3\}$, and vice versa, as shown on the right column above. Thus by (BP), the number of distinct terms in the expansion of $(x_1+x_2+x_3)^4$ is equal to the number of 4-element multi-subsets of M, which is $H_4^3 = \binom{4+3-1}{4} = \binom{6}{4} = 15$. In general, one can prove that (see Problem 2.62)

the number of distinct terms in the expansion of $(x_1+x_2+\cdots+x_m)^n$ is given by $H_n^m = \binom{n+m-1}{n}$.

In particular, for binomial expansion, we have $H_n^2 = \binom{2+n-1}{n} = n+1$, which agrees with what we mentioned before.

5° It follows from (2.7.1) and (2.7.2) that for a given positive integer n, the maximum value of the binomial coefficients $\binom{n}{r}$, $r = 0, 1, ..., n$, is equal to

$$\begin{cases} \dbinom{n}{\frac{n}{2}} & \text{if } n \text{ is even,} \\ \dbinom{n}{\frac{n-1}{2}} = \dbinom{n}{\frac{n+1}{2}} & \text{if } n \text{ is odd.} \end{cases}$$

What can we say about the maximum value of multinomial coefficients $\binom{n}{n_1,n_2,\dots,n_m}$? This problem has recently been solved by Wu [W]. For $n, m \geq 2$, let

$$M(n,m) = \max\left\{ \binom{n}{n_1, n_2, \dots, n_m} \middle| n_i \in \mathbf{N}^* \text{ and } \sum_{i=1}^m n_i = n \right\}.$$

Case 1. $m|n$.

Let $n = mr$ for some $r \in \mathbf{N}$. Then

$$M(n,m) = \binom{n}{\underbrace{r, r, \dots, r}_{m}} = \frac{n!}{(r!)^m},$$

and $\binom{n}{\underbrace{r, r, \dots, r}_{m}}$ is the only term attaining this maximum value.

Case 2. $m \nmid n$.

Suppose that $n = mr + k$ for some $r, k \in \mathbf{N}$ with $1 \leq k \leq m-1$. Then

$$\begin{aligned} M(n,m) &= \binom{n}{\underbrace{r, r, \dots, r,}_{m-k} \underbrace{(r+1), (r+1), \dots, (r+1)}_{k}} \\ &= \frac{n!}{(r!)^{m-k}((r+1)!)^k} = \frac{n!}{(r+1)^k (r!)^m}, \end{aligned}$$

and $\binom{n}{n_1, n_2, \ldots, n_m}$, where $\{n_1, n_2, \ldots, n_m\} = \{(m-k)\cdot r, k\cdot(r+1)\}$ as multi-sets, are the $\binom{m}{k}$ terms attaining this maximum value.

For instance, in Example 2.8.1, we have

$$n = 4, \quad m = 3, \quad r = 1 \text{ and } k = 1.$$

Thus the maximum coefficient is

$$M(4,3) = \frac{4!}{2!(1!)^3} = 12,$$

which is attained by the following $\binom{m}{k} = 3$ terms:

$$\binom{4}{1,1,2}, \quad \binom{4}{1,2,1} \text{ and } \binom{4}{2,1,1}.$$

Exercise 2

1. The number 4 can be expressed as a sum of one or more positive integers, taking order into account, in 8 ways:

$$\begin{aligned} 4 &= 1+3 = 3+1 = 2+2 = 1+1+2 \\ &= 1+2+1 = 2+1+1 = 1+1+1+1. \end{aligned}$$

   In general, given $n \in \mathbf{N}$, in how many ways can n be so expressed?

2. Find the number of $2n$-digit binary sequences in which the number of 0's in the first n digits is equal to the number of 1's in the last n digits.

3. Let $m, n, r \in \mathbf{N}$. Find the number of r-element multi-subsets of the multi-set

$$M = \{a_1, a_2, \ldots, a_n, m \cdot b\}$$

   in each of the following cases:

   (i) $r \le m$, $r \le n$;

   (ii) $n \le r \le m$;

   (ii) $m \le r \le n$.

4. Ten points are marked on a circle. How many distinct convex polygons of three or more sides can be drawn using some (or all) of the ten points as vertices? (Polygons are distinct unless they have exactly the same vertices.) (AIME, 1989/2)

5. Find the coefficient of x^5 in the expansion of $(1+x+x^2)^8$.

6. Find the coefficient of x^6 in the expansion of $(1+x+x^2)^9$.

7. Find the coefficient of x^{18} in the expansion of

$$(1+x^3+x^5+x^7)^{100}.$$

8. Find the coefficient of x^{29} in the expansion of

$$(1+x^5+x^7+x^9)^{1000}.$$

9. In the expansion of

$$(1+x+x^2+\cdots+x^{10})^3,$$

what is the coefficient of

(i) x^5?   (ii) x^8?

10. Given an n-element set X, where $n \in \mathbf{N}$, let $\mathcal{O} = \{A \subseteq X \mid |A| \text{ is odd}\}$ and $\mathcal{E} = \{A \subseteq X \mid |A| \text{ is even}\}$. Show that $|\mathcal{O}| = |\mathcal{E}|$ by establishing a bijection between $\mathcal{O}$ and $\mathcal{E}$.

11. Find the number of permutations of the multi-set $\{m \cdot 1, n \cdot 2\}$, where $m, n \in \mathbf{N}$, which must contain the m 1's.

12. Let $1 \leq r \leq n$ and consider all r-element subsets of the set $\{1, 2, \ldots, n\}$. Each of these subsets has a *largest* member. Let $H(n,r)$ denote the arithmetic mean of these largest members. Find $H(n,r)$ and simplify your result (see Example 2.5.2).

13. For $n \in \mathbf{N}$, let $\Delta(n)$ denote the number of triangles XYZ in the nth subdivision of an equilateral triangle ABC (see Figure 2.5.2) such that $YZ//BC$, and X and A are on the same side of YZ. Evaluate $\Delta(n)$. (For other enumeration problems relating to this, see M.E. Larsen, The eternal triangle – A history of a counting problem, *The College Math. J.* **20** (1989), 370-384.)

14. Find the coefficients of x^n and x^{n+r} $(1 \leq r \leq n)$ in the expansion of

$$(1+x)^{2n} + x(1+x)^{2n-1} + x^2(1+x)^{2n-2} + \cdots + x^n(1+x)^n.$$

15. A polynomial in x is defined by

$$a_0 + a_1x + a_2x^2 + \cdots + a_{2n}x^{2n} = (x + 2x^2 + \cdots + nx^n)^2.$$

Show that

$$\sum_{i=n+1}^{2n} a_i = \frac{n(n+1)(5n^2+5n+2)}{24}.$$

16. Show that

$$P_r^r + P_r^{r+1} + \cdots + P_r^{2r} = P_r^{2r+1},$$

where r is a nonnegative integer.

17. Given $r, n, m \in \mathbf{N}^*$ with $r < n$, show that

$$P_r^n + P_r^{n+1} + \cdots + P_r^{n+m} = \frac{1}{r+1}(P_{r+1}^{n+m+1} - P_{r+1}^n).$$

(See Problem 2.35.)

18. Show that

(i) for even $n \in \mathbf{N}$,

$$\binom{n}{i} < \binom{n}{j} \quad \text{if } 0 \le i < j \le \frac{n}{2};$$

and

$$\binom{n}{i} > \binom{n}{j} \quad \text{if } \frac{n}{2} \le i < j \le n.$$

(ii) for odd $n \in \mathbf{N}$,

$$\binom{n}{i} < \binom{n}{j} \quad \text{if } 0 \le i < j \le \frac{1}{2}(n-1);$$

and

$$\binom{n}{i} > \binom{n}{j} \quad \text{if } \frac{1}{2}(n+1) \le i < j \le n.$$

19. Give three different proofs for each of the following identities:

(i) $\binom{2(n+1)}{n+1} = \binom{2n}{n+1} + 2\binom{2n}{n} + \binom{2n}{n-1}$;

(ii) $\binom{n+1}{m} = \binom{n}{m-1} + \binom{n-1}{m} + \binom{n-1}{m-1}$.

20. Give a combinatorial proof for the identity

$$\binom{n}{m}\binom{m}{r} = \binom{n}{r}\binom{n-r}{m-r}.$$

21. Show that for $n \in \mathbf{N}^*$,

$$\sum_{r=0}^{n} \frac{(2n)!}{(r!)^2((n-r)!)^2} = \binom{2n}{n}^2.$$

22. By using the identity $(1-x^2)^n = (1+x)^n(1-x)^n$, show that for each $m \in \mathbf{N}^*$ with $m \leq n$,

$$\sum_{i=0}^{2m}(-1)^i\binom{n}{i}\binom{n}{2m-i} = (-1)^m\binom{n}{m},$$

and

$$\sum_{i=0}^{2m+1}(-1)^i\binom{n}{i}\binom{n}{2m+1-i} = 0.$$

Deduce that

$$\sum_{i=0}^{n}(-1)^i\binom{n}{i}^2 = \begin{cases} (-1)^{\frac{n}{2}}\binom{n}{\frac{n}{2}} & \text{if n is even} \\ 0 & \text{if n is odd.} \end{cases}$$

23. What is the value of the sum

$$S = m! + \frac{(m+1)!}{1!} + \frac{(m+2)!}{2!} + \cdots + \frac{(m+n)!}{n!} \ ?$$

(Beijing Math. Contest (1962))

Prove each of the following identities in Problems 24–43, where $m, n \in \mathbf{N}^*$:

24. $\sum\limits_{r=0}^{n} 3^r\binom{n}{r} = 4^n$,

25. $\sum\limits_{r=0}^{n} (r+1)\binom{n}{r} = (n+2)2^{n-1}$,

26. $\sum\limits_{r=0}^{n} \frac{1}{r+1}\binom{n}{r} = \frac{1}{n+1}(2^{n+1}-1)$,

27. $\sum\limits_{r=0}^{n} \frac{(-1)^r}{r+1}\binom{n}{r} = \frac{1}{n+1}$,

28. $\sum_{r=m}^{n}\binom{n}{r}\binom{r}{m}=2^{n-m}\binom{n}{m}$ for $m\le n$,

29. $\sum_{r=0}^{m}(-1)^r\binom{n}{r}=\begin{cases}(-1)^m\binom{n-1}{m} & \text{if } m<n\\ 0 & \text{if } m=n,\end{cases}$

30. $\sum_{r=0}^{m}(-1)^{m-r}\binom{n}{r}=\binom{n-1}{m}$ for $m\le n-1$,

31. $\sum_{r=0}^{n}(-1)^r r\binom{n}{r}=0$,

32. $\sum_{r=0}^{n-1}\binom{2n-1}{r}=2^{2n-2}$,

33. $\sum_{r=0}^{n}\binom{2n}{r}=2^{2n-1}+\frac{1}{2}\binom{2n}{n}$,

34. $\sum_{r=0}^{n}r\binom{2n}{r}=n2^{2n-1}$,

35. $\sum_{r=k}^{m}\binom{n+r}{n}=\binom{n+m+1}{n+1}-\binom{n+k}{n+1}$ for $k\in\mathbf{N}^*$ and $k\le m$,

36. $\sum_{r=1}^{n}\binom{n}{r}\binom{n-1}{r-1}=\binom{2n-1}{n-1}$,

37. $\sum_{r=m}^{n}(-1)^r\binom{n}{r}\binom{r}{m}=\begin{cases}(-1)^m & \text{if } m=n\\ 0 & \text{if } m<n,\end{cases}$

38. $\sum_{r=1}^{n-1}(n-r)^2\binom{n-1}{n-r}=n(n-1)2^{n-3}$,
(See Problem 2.47(i).)

39. $\sum_{r=0}^{n}\binom{2n}{r}^2=\frac{1}{2}\left\{\binom{4n}{2n}+\binom{2n}{n}^2\right\}$,

40. $\sum_{r=0}^{n}\binom{n}{r}^2\binom{r}{n-k}=\binom{n}{k}\binom{n+k}{k}$ for $k\in\mathbf{N}^*$, $0\le k\le n$,

41. $\sum_{r=0}^{n}\binom{n}{r}\binom{m+r}{n}=\sum_{r=0}^{n}\binom{n}{r}\binom{m}{r}2^r$,

42. $\sum_{r=0}^{m}\binom{m}{r}\binom{n}{r}\binom{p+r}{m+n}=\binom{p}{m}\binom{p}{n}$, for $p\in\mathbf{N}$, $p\ge m,n$;
(Li Shanlan, 1811-1882)

43. $\sum_{r=0}^{m}\binom{m}{r}\binom{n}{r}\binom{p+m+n-r}{m+n}=\binom{p+m}{m}\binom{p+n}{n}$, for $p\in\mathbf{N}$.

(Li Shanlan)

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44. Prove the following identities using the technique of counting shortest routes in a grid:

(i) $\sum\limits_{r=0}^{n} \binom{n}{r} = 2^n$,

(ii) $\sum\limits_{k=0}^{n} \binom{r+k}{r} = \binom{r+n+1}{r+1}$.

45. Use the technique of finding the number of shortest routes in rectangular grid to prove the following identity:

$$\binom{p}{q}\binom{r}{0} + \binom{p-1}{q-1}\binom{r+1}{1} + \cdots + \binom{p-q}{0}\binom{r+q}{q} = \binom{p+r+1}{q}.$$

46. Give a combinatorial proof for the following identity:

$$\sum_{r=1}^{n} r\binom{n}{r} = n \cdot 2^{n-1}.$$

47. Given $n \in \mathbf{N}$, show that

(i) $\sum\limits_{r=1}^{n} r^2\binom{n}{r} = n(n+1)2^{n-2}$;

(Putnam, 1962)

(ii) $\sum\limits_{r=1}^{n} r^3\binom{n}{r} = n^2(n+3)2^{n-3}$;

(iii) $\sum\limits_{r=1}^{n} r^4\binom{n}{r} = n(n+1)(n^2+5n-2)2^{n-4}$.

48. (i) Prove that for $r, k \in \mathbf{N}$,

$$r^k = \sum_{i=0}^{k} \binom{k}{i}(r-1)^{k-i}.$$

(ii) For $n, k \in \mathbf{N}$, let

$$R(n,k) = \sum_{r=1}^{n} r^k \binom{n}{r}.$$

Show that

$$R(n,k) = n \cdot \sum_{j=0}^{k-1} \binom{k-1}{j} R(n-1,j).$$

Remark. Two Chinese teachers, Wei Guozhen and Wang Kai (1988) showed that

$$\sum_{r=1}^{n} r^k \binom{n}{r} = \sum_{i=1}^{k} S(k,i) \cdot P_i^n \cdot 2^{n-i},$$

where $k \leq n$ and $S(k,i)$'s are the Stirling numbers of the second kind.

49. Prove that

$$\sum_{r=1}^{n} \frac{1}{r}\binom{n}{r} = \sum_{r=1}^{n} \frac{1}{r}(2^r - 1).$$

50. Give two different proofs for the following identity:

$$\sum_{r=1}^{n} r\binom{n}{r}^2 = n\binom{2n-1}{n-1},$$

where $n \in \mathbf{N}$.

51. Let p be a prime. Show that

$$\binom{p}{r} \equiv 0 \pmod{p}$$

for all r such that $1 \leq r \leq p-1$. Deduce that $(1+x)^p \equiv (1+x^p) \pmod{p}$.

52. Let p be an odd prime. Show that

$$\binom{2p}{p} \equiv 2 \pmod{p}.$$

53. Let n, m, p be integers such that $1 \leq p \leq m \leq n$.
    (i) Express, in terms of $S(n,p)$, the number of mappings $f : \mathbf{N}_n \to \mathbf{N}_m$ such that $|f(\mathbf{N}_n)| = p$.
    (ii) Express, in terms of $S(n,k)$'s, where $p \leq k \leq m$, the number of mappings $f : \mathbf{N}_n \to \mathbf{N}_m$ such that $\mathbf{N}_p \subseteq f(\mathbf{N}_n)$.

54. Recall that for nonnegative integers n, r, $H_r^n = \binom{r+n-1}{r}$. Prove each of the following identities:
    (a) $H_r^n = \frac{n}{r} H_{r-1}^{n+1}$;
    (b) $H_r^n = \frac{n+r-1}{r} H_{r-1}^n$;
    (c) $H_r^n = H_{r-1}^n + H_r^{n-1}$;

(d) $\sum_{k=0}^{r} H_k^n = H_r^{n+1}$;

(e) $\sum_{k=1}^{r} kH_k^n = nH_{r-1}^{n+2}$;

(f) $\sum_{k=0}^{r} H_k^m H_{r-k}^n = H_r^{m+n}$.

55. For $n, k \in \mathbf{N}$ with $n \geq 2$ and $1 \leq k \leq n$, let

$$d_k(n) = \left|\binom{n}{k} - \binom{n}{k-1}\right|,$$

$$d_{\min}(n) = \min\{d_k(n) \mid 1 \leq k \leq n\}.$$

Show that

(i) $d_{\min}(n) = 0$ iff n is odd;

(ii) For odd n, $d_k(n) = 0$ iff $k = \frac{1}{2}(n+1)$.

Let $d^*_{\min}(n) = \min\{d_k(n) \mid 1 \leq k \leq n, \ \ k \neq \frac{1}{2}(n+1)\}$. Show that

(iii) For $n \neq 4$, $d^*_{\min}(n) = n - 1$;

(iv) For $n \neq 4$ and $n \neq 6$,

$$d_k(n) = n - 1 \ \text{ iff } \ k = 1 \ \text{ or } \ k = n;$$

(v) For $n = 6, d_k(6) = 5$ iff $k = 1, 3, 4$ or 6.

(See Z. Shan and E.T.H. Wang, The gaps between consecutive binomial coefficients, *Math. Magazine*, **63** (1990), 122-124.)

56. Prove that

(i) $\binom{\binom{n}{2}}{2} = 3\binom{n+1}{4}$ for $n \in \mathbf{N}$;

(ii) $\binom{\binom{n}{2}}{3} > \binom{\binom{n}{3}}{2}$ for $n \in \mathbf{N}, n \geq 3$;

(iii) $\binom{\binom{n}{r}}{2} = \sum_{j=1}^{r} \binom{\binom{r}{j}+\epsilon_j}{2}\binom{n+r-j}{2r}$,

where $\epsilon_j = \begin{cases} 1 & \text{if j is odd} \\ 0 & \text{if j is even,} \end{cases}$ and $n, r \in \mathbf{N}$ with $r \leq n$;

(iv) $\binom{\binom{n}{r}}{2} = \sum_{j=1}^{r} \binom{2j-1}{j}\binom{r+1}{2j}\binom{n}{r+j}$,

for $n, r \in \mathbf{N}$ with $r \leq n$.

(For more results on these iterated binomial coefficients, see S.W. Golomb, Iterated binomial coefficients, *Amer. Math. Monthly*, **87** (1980), 719-727.)

57. Let $a_n = 6^n + 8^n$. Determine the remainder on dividing a_{83} by 49. (AIME, 1983/6)

58. The increasing sequence 1, 3, 4, 9, 10, 12, 13, ... consists of all those positive integers which are powers of 3 or sums of distinct powers of 3. Find the 100th term of this sequence (where 1 is the 1st term, 3 is the 2nd term, and so on). (AIME, 1986/7)

59. The polynomial $1 - x + x^2 - x^3 + \cdots + x^{16} - x^{17}$ may be written in the form $a_0 + a_1y + a_2y^2 + a_3y^3 + \cdots + a_{16}y^{16} + a_{17}y^{17}$, where $y = x + 1$ and the a_i's are constants. Find the value of a_2. (AIME, 1986/11)

60. In an office, at various times during the day, the boss gives the secretary a letter to type, each time putting the letter on top of the pile in the secretary's in-box. When there is time, the secretary takes the top letter off the pile and types it. There are nine letters to be typed during the day, and the boss delivers them in the order 1, 2, 3, 4, 5, 6, 7, 8, 9.

    While leaving for lunch, the secretary tells a colleague that letter 8 has already been typed, but says nothing else about the morning's typing. The colleague wonders which of the nine letters remain to be typed after lunch and in what order they will be typed. Based upon the above information, how many such *after-lunch typing orders* are possible? (That there are no letters left to be typed is one of the possibilities.) (AIME, 1988/15)

61. Expanding $(1 + 0.2)^{1000}$ by the binomial theorem and doing no further manipulation gives

$$\begin{aligned}&\tbinom{1000}{0}(0.2)^0+\tbinom{1000}{1}(0.2)^1+\tbinom{1000}{2}(0.2)^2+\cdots+\tbinom{1000}{1000}(0.2)^{1000}\\ &= A_0 + A_1 + A_2 + \cdots + A_{1000},\end{aligned}$$

    where $A_k = \binom{1000}{k}(0.2)^k$ for $k = 0, 1, 2, \ldots, 1000$. For which k is A_k the largest? (AIME, 1991/3)

62. Prove that the number of distinct terms in the expansion of

$$(x_1 + x_2 + \cdots + x_m)^n$$

    is given by $H_n^m = \binom{n+m-1}{n}$.

63. Show by two different methods that

$$\binom{n}{n_1, n_2, \ldots, n_m} = \sum_{i=1}^{m} \binom{n-1}{n_1, \ldots, n_i - 1, n_{i+1}, \ldots, n_m}.$$

64. For $n, m \in \mathbf{N}$, show that

$$\sum \binom{n}{n_1, n_2, \ldots, n_m} = m!S(n, m),$$

where the sum is taken over all m-ary sequences $(n_1, n_2, \ldots, n_m)$ such that $n_i \neq 0$ for all i, and $S(n, m)$ is a Stirling number of the second kind.

65. Prove that

$$\sum \binom{n}{n_1, n_2, \ldots, n_m} (-1)^{n_2+n_4+n_6+\cdots} = \begin{cases} 1 & \text{if m is odd} \\ 0 & \text{if m is even,} \end{cases}$$

where the sum is taken over all m-ary sequences $(n_1, n_2, \ldots, n_m)$ of nonnegative integers with $\sum_{i=1}^{m} n_i = n$.

66. Prove the following generalized Vandermonde's identity for multinomial coefficients: for $p, q \in \mathbf{N}$,

$$\binom{p+q}{k_1, k_2, \ldots, k_m} = \sum \binom{p}{j_1, j_2, \ldots, j_m} \binom{q}{k_1 - j_1, k_2 - j_2, \ldots, k_m - j_m},$$

where the sum is taken over all m-ary sequences $(j_1, j_2, \ldots, j_m)$ of non-negative integers with $j_1 + j_2 + \cdots + j_m = p$.

67. Given any prime p and $m \in \mathbf{N}$, show that

$$\binom{p}{n_1, n_2, \cdots, n_m} \equiv 0 \pmod{p},$$

if $p \neq n_i$ for any $i = 1, 2, \ldots, m$.

Deduce that

$$\left(\sum_{i=1}^{m} x_i\right)^p \equiv \sum_{i=1}^{m} x_i^p \pmod{p}.$$

68. Let p be a prime, and $n \in \mathbf{N}$. Write n in base p as follows:

$$n = n_0 + n_1 p + n_2 p^2 + \cdots + n_k p^k,$$

where $n_i \in \{0, 1, \ldots, p-1\}$ for each $i = 1, 2, \ldots, k$.
Given $m \in \mathbf{N}$, show that the number of terms in the expansion of $(x_1 + x_2 + \cdots + x_m)^n$ whose coefficients are not divisible by p is

$$\prod_{i=0}^{k} \binom{n_i + m - 1}{m - 1}.$$

(See F.T. Howard, The number of multinomial coefficients divisible by a fixed power of a prime, *Pacific J. Math.*, **50** (1974), 99-108.)

69. Show that

$$\sum_{r=0}^{n} \binom{n}{r}^2 \binom{2n + m - r}{2n} = \binom{m + n}{n}^2.$$

(Li Jen Shu)

70. Show that

$$\sum_{r=1}^{n} \frac{(-1)^{r-1}}{r} \binom{n}{r} = \sum_{k=1}^{n} \frac{1}{k},$$

where $n \in \mathbf{N}$.

71. Given $r \in \mathbf{N}$ with $r \geq 2$, show that

$$\sum_{n=r}^{\infty} \frac{1}{\binom{n}{r}} = \frac{r}{r-1}.$$

(see H.W. Gould, *Combinatorial Identities*, Morgantown, W.V. (1972), 18-19)

72. Let $S = \{1, 2, \ldots, n\}$. For each $A \subseteq S$ with $A \neq \emptyset$, let $M(A) = \max\{x \mid x \in A\}$, $m(A) = \min\{x \mid x \in A\}$ and $\alpha(A) = M(A) + m(A)$. Evaluate the arithmetic mean of all the $\alpha(A)$'s when A runs through all nonempty subsets of S.

73. Given $a_n = \sum\limits_{k=0}^{n} \binom{n}{k}^{-1}$, $n \in \mathbf{N}$,
show that

$$\lim_{n \to \infty} a_n = 2.$$

(Putnam, November 1958)

74. Let $(z)_0 = 1$ and for $n \in \mathbf{N}$, let

$$(z)_n = z(z-1)(z-2)\cdots(z-n+1).$$

Show that

$$(x+y)_n = \sum_{i=0}^{n} \binom{n}{i}(x)_i(y)_{n-i},$$

for all $n \in \mathbf{N}^*$. (Putnam, 1962)

75. In how many ways can the integers from 1 to n be ordered subject to the condition that, except for the first integer on the left, every integer differs by 1 from some integer to the left of it? (Putnam, 1965)

76. Show that, for any positive integer n,

$$\sum_{r=0}^{\lfloor\frac{n-1}{2}\rfloor} \left\{\frac{n-2r}{n}\binom{n}{r}\right\}^2 = \frac{1}{n}\binom{2n-2}{n-1}.$$

(Putnam, 1965)

77. Show that for $n \in \mathbf{N}$ with $n \geq 2$,

$$\sum_{r=1}^{n} r\sqrt{\binom{n}{r}} < \sqrt{2^{n-1}n^3}.$$

(Spanish MO, 1988)

78. Let $n, r \in \mathbf{N}$ with $r \leq n$ and let k be the HCF of the following numbers:

$$\binom{n}{r}, \binom{n+1}{r}, \ldots, \binom{n+r}{r}.$$

Show that $k = 1$.

79. Show that there are no four consecutive binomial coefficients $\binom{n}{r}$, $\binom{n}{r+1}$, $\binom{n}{r+2}$, $\binom{n}{r+3}$ ($n, r \in \mathbf{N}$ with $r+3 \leq n$) which are in arithmetic progression. (Putnam, 1972)

80. Find the greatest common divisor (i.e., HCF) of

$$\binom{2n}{1}, \binom{2n}{3}, \ldots, \binom{2n}{2n-1}.$$

(Proposed by N.S. Mendelsohn, see *Amer. Math. Monthly*, **78** (1971), 201.)

81. Let $n \in \mathbf{N}$. Show that $\binom{n}{r}$ is odd for each $r \in \{0, 1, 2, \ldots, n\}$ iff $n = 2^k - 1$ for some $k \in \mathbf{N}$.

82. An unbiased coin is tossed n times. What is the expected value of $|H - T|$, where H is the number of heads and T is the number of tails? In other words, evaluate in *closed form*:

$$\frac{1}{2^{n-1}} \sum_{k < \frac{n}{2}} (n - 2k) \binom{n}{k}.$$

("closed form" means a form not involving a series.) (Putnam, 1974)

83. Prove that

$$\binom{pa}{pb} \equiv \binom{a}{b} \pmod{p}$$

for all integers p, a and b with p a prime, and $a \geq b \geq 0$. (Putnam, 1977)

84. The geometric mean (G.M.) of k positive numbers $a_1, a_2, \ldots, a_k$ is defined to be the (positive) kth root of their product. For example, the G.M. of 3, 4, 18 is 6. Show that the G.M. of a set S of n positive numbers is equal to the G.M. of the G.M.'s of all nonempty subsets of S. (Canadian MO, 1983)

85. For $n, k \in \mathbf{N}$, let $S_k(n) = 1^k + 2^k + \cdots + n^k$. Show that

(i)

$$\sum_{k=0}^{m-1} \binom{m}{k} S_k(n) = (n+1)^m - 1,$$

(ii)

$$S_m(n) - \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} S_k(n) = n^m,$$

where $m \in \mathbf{N}$.

86. Let $P(x)$ be a polynomial of degree $n, n \in \mathbf{N}$, such that $P(k) = 2^k$ for each $k = 1, 2, \ldots, n+1$. Determine $P(n+2)$. (Proposed by M. Klamkin, see *Pi Mu Epsilon*, **4** (1964), 77, Problem 158.)

87. Let $X = \{1, 2, \ldots, 10\}$, $\mathcal{A} = \{A \subset X \mid |A| = 4\}$, and $f : \mathcal{A} \to X$ be an arbitrary mapping. Show that there exists $S \subset X$, $|S| = 5$ such that

$$f(S - \{r\}) \neq r$$

for each $r \in S$.

88. (i) Applying the arithmetic-geometric mean inequality on

$$\binom{n+1}{1}, \binom{n+1}{2}, \ldots, \binom{n+1}{n},$$

show that

$$(2^{n+1}-2)^n \geq n^n \prod_{r=1}^{n} \binom{n+1}{r},$$

where $n \in \mathbf{N}$.

(ii) Show that

$$(n!)^{n+1} = \left(\prod_{r=1}^{n} r^r\right) \left(\prod_{r=1}^{n} (r!)\right).$$

(iii) Deduce from (i) and (ii) or otherwise, that

$$\left(\frac{n(n+1)!}{2^{n+1}-2}\right)^{\frac{n}{2}} \leq \frac{(n!)^{n+1}}{\prod_{r=1}^{n} r^r}.$$

(iv) Show that the equality in (iii) holds iff $n = 1$ or $n = 2$. (See *The College Math. J.* **20** (1989), 344.)

89. Find, with proof, the number of positive integers whose base-n representation consists of distinct digits with the property that, except for the leftmost digit, every digit differs by ± 1 from some digit further to the left. (Your answer should be an explicit function of n in simplest form.) (USA MO, 1990/4)

90. Let $S_n = \sum_{k=0}^{n} \binom{3n}{3k}$. Prove that

$$\lim_{n \to \infty} (S_n)^{\frac{1}{3n}} = 2.$$

(Bulgarian Spring Competition, 1985)

91. (i) If $f(n)$ denotes the number of 0's in the decimal representation of the positive integer n, what is the value of the sum

$$S = 2^{f(1)} + 2^{f(2)} + \cdots + 2^{f(\underbrace{9999999999}_{10})}?$$

(ii) Let a be a nonzero real number, and $b, k, m \in \mathbf{N}$. Denote by $f(k)$ the number of zeros in the base $b+1$ representation of k. Compute

$$S_n = \sum_{k=1}^{n} a^{f(k)},$$

where $n = (b+1)^m - 1$.

Remark. Part (i) was a 1981 Hungarian Mathematical Competition problem. Part (ii) is a generalization of part (i), and was formulated by M.S. Klamkin (see *Crux Mathematicorum*, **9** (1983), 17-18).

92. Prove that the number of binary sequences of length n which contain exactly m occurrences of "01" is $\binom{n+1}{2m+1}$. (Great Britain MO, 1982/6)

93. There are n people in a gathering, some being acquaintances, some strangers. It is given that every 2 strangers have exactly 2 common friends, and every 2 acquaintances have no common friends. Show that everyone has the same number of friends in the gathering. (23rd Moscow MO)

94. Let $n \in \mathbf{N}^*$. For $p = 1, 2, \ldots$, define

$$A_p(n) = \sum_{0 \le k \le \frac{n}{2}} (-1)^k \left\{ \binom{n}{k} - \binom{n}{k-1} \right\}^p.$$

Prove that, whenever n is odd, $A_2(n) = nA_1(n)$. (Proposed by H.W. Gould, see *Amer. Math. Monthly*, **80** (1973), 1146.)

95. Let $n \in \mathbf{N}^*$. For $p = 1, 2, \ldots$, define

$$B_p(n) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \left\{ \binom{n}{k} - \binom{n}{k-1} \right\}^p.$$

Evaluate $B_2(n)$. (Proposed by E.T. Ordman, see *Amer. Math. Monthly*, **80** (1973), 1066.)

96. Show that

(i) $\binom{2n-1}{r}^{-1} = \frac{2n}{2n+1} \left\{ \binom{2n}{r}^{-1} + \binom{2n}{r+1}^{-1} \right\}$, where $r, n \in \mathbf{N}$;

(ii) $\sum\limits_{r=1}^{2n-1} (-1)^{r-1} \binom{2n-1}{r}^{-1} \sum\limits_{j=1}^{r} \frac{1}{j} = \frac{2n}{2n+1} \sum\limits_{r=1}^{2n} \frac{1}{r}$.

(Proposed by I. Kaucký, see *Amer. Math. Monthly*, **78** (1971), 908.)

97. Given $\ell, m, n \in \mathbf{N}^*$ with $\ell, n \le m$, evaluate the double sum

$$\sum_{i=0}^{\ell}\sum_{j=0}^{i}(-1)^j\binom{m-i}{m-\ell}\binom{n}{j}\binom{m-n}{i-j}.$$

(Proposed by D.B. West, see *Amer. Math. Monthly*, **97** (1990), 428-429.)

98. Show that

$$\sum_{r=0}^{n}\binom{n}{r}\binom{p}{r+s}\binom{q+r}{m+n}=\sum_{r=0}^{n}\binom{n}{r}\binom{q}{m+r}\binom{p+r}{n+s},$$

where $m, n, p, q, s \in \mathbf{N}^*$. (See R.C. Lyness, The mystery of the double sevens, *Crux Mathematicorum*, **9** (1983), 194-198.)

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