CHAPTER

STACKS

4

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4.1 What is a Stack?

A stack is a simple data structure used for storing data (similar to Linked Lists). In a stack, the order in which the data arrives is important. A pile of plates in a cafeteria is a good example of a stack. The plates are added to the stack as they are cleaned and they are placed on the top. When a plate, is required it is taken from the top of the stack. The first plate placed on the stack is the last one to be used.

Definition: A *stack* is an ordered list in which insertion and deletion are done at one end, called *top*. The last element inserted is the first one to be deleted. Hence, it is called the Last in First out (LIFO) or First in Last out (FILO) list.

Special names are given to the two changes that can be made to a stack. When an element is inserted in a stack, the concept is called *push*, and when an element is removed from the stack, the concept is called *pop*. Trying to pop out an empty stack is called *underflow* and trying to push an element in a full stack is called *overflow*. Generally, we treat them as exceptions. As an example, consider the snapshots of the stack.
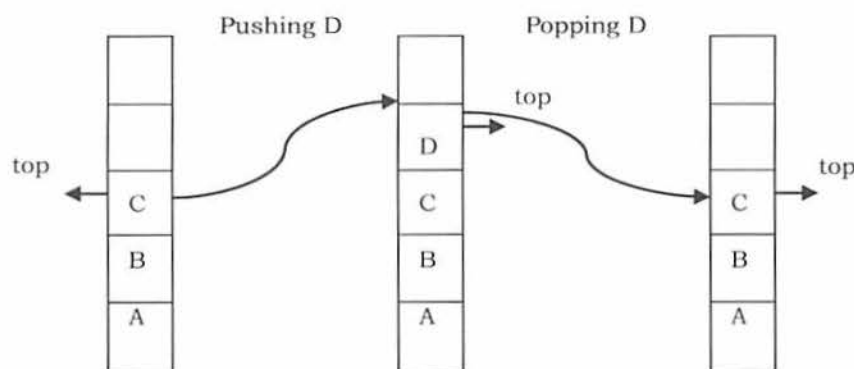
4.2 How Stacks are Used

Consider a working day in the office. Let us assume a developer is working on a long-term project. The manager then gives the developer a new task which is more important. The developer puts the long-term project aside and begins work on the new task. The phone rings, and this is the highest priority as it must be answered immediately. The developer pushes the present task into the pending tray and answers the phone.

When the call is complete the task that was abandoned to answer the phone is retrieved from the pending tray and work progresses. To take another call, it may have to be handled in the same manner, but eventually the new task will be finished, and the developer can draw the long-term project from the pending tray and continue with that.

4.3 Stack ADT

The following operations make a stack an ADT. For simplicity, assume the data is an integer type.

Main stack operations

- Push (int data): Inserts *data* onto stack.
- int Pop(): Removes and returns the last inserted element from the stack.

Auxiliary stack operations

- int Top(): Returns the last inserted element without removing it.
- int Size(): Returns the number of elements stored in the stack.
- int IsEmptyStack(): Indicates whether any elements are stored in the stack or not.
- int IsFullStack(): Indicates whether the stack is full or not.

Exceptions

Attempting the execution of an operation may sometimes cause an error condition, called an exception. Exceptions are said to be "thrown" by an operation that cannot be executed. In the Stack ADT, operations pop and top cannot be performed if the stack is empty. Attempting the execution of pop (top) on an empty stack throws an exception. Trying to push an element in a full stack throws an exception.

4.4 Applications

Following are some of the applications in which stacks play an important role.

Direct applications

- Balancing of symbols
- Infix-to-postfix conversion
- Evaluation of postfix expression
- Implementing function calls (including recursion)
- Finding of spans (finding spans in stock markets, refer to *Problems* section)
- Page-visited history in a Web browser [Back Buttons]
- Undo sequence in a text editor
- Matching Tags in HTML and XML

Indirect applications

- Auxiliary data structure for other algorithms (Example: Tree traversal algorithms)
- Component of other data structures (Example: Simulating queues, refer *Queues* chapter)

4.5 Implementation

There are many ways of implementing stack ADT; below are the commonly used methods.
- Simple array based implementation
- Dynamic array based implementation
- Linked lists implementation

Simple Array Implementation

This implementation of stack ADT uses an array. In the array, we add elements from left to right and use a variable to keep track of the index of the top element.

The array storing the stack elements may become full. A push operation will then throw a *full stack exception*. Similarly, if we try deleting an element from an empty stack it will throw *stack empty exception*.

```python
class Stack(object):
  def __init__(self, limit = 10):
        self.stk = []
        self.limit = limit

  def isEmpty(self):
        return len(self.stk) <= 0

  def push(self, item):
        if len(self.stk) >= self.limit:
                print 'Stack Overflow!'
        else:
                self.stk.append(item)
        print 'Stack after Push',self.stk

  def pop(self):
        if len(self.stk) <= 0:
                print 'Stack Underflow!'
                return 0
        else:
                return self.stk.pop()

  def peek(self):
        if len(self.stk) <= 0:
                print 'Stack Underflow!'
                return 0
        else:
                return self.stk[-1]

  def size(self):
        return len(self.stk)

our_stack = Stack(5)
our_stack.push("1")
our_stack.push("21")
our_stack.push("14")
our_stack.push("31")
our_stack.push("19")
our_stack.push("3")
our_stack.push("99")
our_stack.push("9")
print our_stack.peek()
print our_stack.pop()
print our_stack.peek()
print our_stack.pop()
```

Performance & Limitations

Performance

Let n be the number of elements in the stack. The complexities of stack operations with this representation can be given as:

Space Complexity (for n push operations)	$O(n)$
Time Complexity of Push()	$O(1)$
Time Complexity of Pop()	$O(1)$
Time Complexity of Size()	$O(1)$
Time Complexity of IsEmptyStack()	$O(1)$
Time Complexity of IsFullStack()	$O(1)$
Time Complexity of DeleteStack()	$O(1)$

Limitations

The maximum size of the stack must first be defined and it cannot be changed. Trying to push a new element into a full stack causes an implementation-specific exception.

Dynamic Array Implementation

First, let's consider how we implemented a simple array based stack. We took one index variable *top* which points to the index of the most recently inserted element in the stack. To insert (or push) an element, we increment *top* index and then place the new element at that index.

Similarly, to delete (or pop) an element we take the element at *top* index and then decrement the *top* index. We represent an empty queue with *top* value equal to -1. The issue that still needs to be resolved is what we do when all the slots in the fixed size array stack are occupied?

First try: What if we increment the size of the array by 1 every time the stack is full?

- Push(): increase size of S[] by 1
- Pop(): decrease size of S[] by 1

Problems with this approach?

This way of incrementing the array size is too expensive. Let us see the reason for this. For example, at $n = 1$, to push an element create a new array of size 2 and copy all the old array elements to the new array, and at the end add the new element. At $n = 2$, to push an element create a new array of size 3 and copy all the old array elements to the new array, and at the end add the new element.

Similarly, at $n = n - 1$, if we want to push an element create a new array of size n and copy all the old array elements to the new array and at the end add the new element. After n push operations the total time $T(n)$ (number of copy operations) is proportional to $1 + 2 + ... + n \approx O(n^2)$.

Alternative Approach: Repeated Doubling

Let us improve the complexity by using the array *doubling* technique. If the array is full, create a new array of twice the size, and copy the items. With this approach, pushing n items takes time proportional to n (not n^2).

For simplicity, let us assume that initially we started with $n = 1$ and moved up to $n = 32$. That means, we do the doubling at $1, 2, 4, 8, 16$. The other way of analyzing the same approach is: at $n = 1$, if we want to add (push) an element, double the current size of the array and copy all the elements of the old array to the new array.

At $n = 1$, we do 1 copy operation, at $n = 2$, we do 2 copy operations, and at $n = 4$, we do 4 copy operations and so on. By the time we reach $n = 32$, the total number of copy operations is $1 + 2 + 4 + 8 + 16 = 31$ which is approximately equal to $2n$ value (32). If we observe carefully, we are doing the doubling operation $\log n$ times.

Now, let us generalize the discussion. For n push operations we double the array size $\log n$ times. That means, we will have $\log n$ terms in the expression below. The total time $T(n)$ of a series of n push operations is proportional to

$$1 + 2 + 4 + 8 ... + \frac{n}{4} + \frac{n}{2} + n = n + \frac{n}{2} + \frac{n}{4} + \frac{n}{8} ... + 4 + 2 + 1$$
$$= n\left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} ... + \frac{4}{n} + \frac{2}{n} + \frac{1}{n}\right)$$
$$= n(2) \approx 2n = O(n)$$

$T(n)$ is $O(n)$ and the amortized time of a push operation is $O(1)$.

```python
class Stack(object):
    def __init__(self, limit = 10):
        self.stk = limit*[None]
        self.limit = limit

    def isEmpty(self):
        return len(self.stk) <= 0

    def push(self, item):
        if len(self.stk) >= self.limit:
            self.resize()
        self.stk.append(item)
        print 'Stack after Push',self.stk

    def pop(self):
        if len(self.stk) <= 0:
            print 'Stack Underflow!'
            return 0
        else:
            return self.stk.pop()
```

```
        def peek(self):
                if len(self.stk) <= 0:
                        print 'Stack Underflow!'
                        return 0
                else:
                        return self.stk[-1]
        def size(self):
                return len(self.stk)
        def resize(self):
                newStk = list(self.stk)
                self.limit = 2*self.limit
                self.stk = newStk

    our_stack = Stack(5)
    our_stack.push("1")
    our_stack.push("21")
    our_stack.push("14")
    our_stack.push("11")
    our_stack.push("31")
    our_stack.push("14")
    our_stack.push("15")
    our_stack.push("19")
    our_stack.push("3")
    our_stack.push("99")
    our_stack.push("9")
    print our_stack.peek()
    print our_stack.pop()
    print our_stack.peek()
    print our_stack.pop()
```

Performance

Let n be the number of elements in the stack. The complexities for operations with this representation can be given as:

Space Complexity (for n push operations)	$O(n)$
Time Complexity of CreateStack()	$O(1)$
Time Complexity of Push()	$O(1)$ (Average)
Time Complexity of Pop()	$O(1)$
Time Complexity of Top()	$O(1)$
Time Complexity of IsEmptyStack()	$O(1)$)
Time Complexity of IsFullStack()	$O(1)$
Time Complexity of DeleteStack()	$O(1)$

Note: Too many doublings may cause memory overflow exception.

Linked List Implementation

The other way of implementing stacks is by using Linked lists. Push operation is implemented by inserting element at the beginning of the list. Pop operation is implemented by deleting the node from the beginning (the header/top node).

top
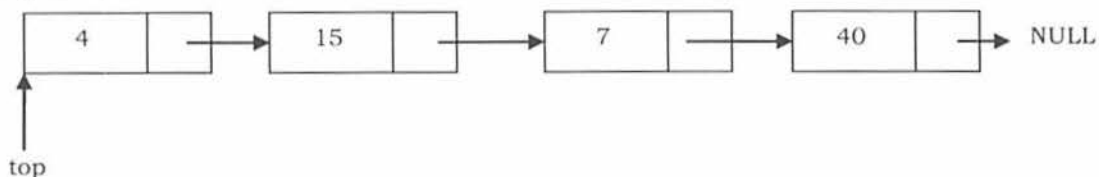
```
    #Node of a Singly Linked List
    class Node:
        #constructor
        def __init__(self):
            self.data = None
            self.next = None
```

```
                    #method for setting the data field of the node
            def setData(self,data):
                self.data = data
            #method for getting the data field of the node
            def getData(self):
                return self.data
             #method for setting the next field of the node
            def setNext(self,next):
                self.next = next
             #method for getting the next field of the node
            def getNext(self):
                return self.next
            #returns true if the node points to another node
            def hasNext(self):
                    return self.next != None

        class Stack(object):
            def __init__(self, data=None):
                self.head = None
                if data:
                    for data in data:
                        self.push(data)

            def push(self, data):
                temp = Node()
                temp.setData(data)
                temp.setNext(self.head)
                self.head = temp

            def pop(self):
                if self.head is None:
                    raise IndexError
                temp = self.head.getData()
                self.head = self.head.getNext()
                return temp

            def peek(self):
                if self.head is None:
                    raise IndexError
                return self.head.getData()

        our_list = ["first", "second", "third", "fourth"]
        our_stack = Stack(our_list)
        print our_stack.pop()
        print our_stack.pop()
```

Performance

Let n be the number of elements in the stack. The complexities for operations with this representation can be given as:

Space Complexity (for n push operations)	$O(n)$
Time Complexity of CreateStack()	$O(1)$
Time Complexity of Push()	$O(1)$ (Average)
Time Complexity of Pop()	$O(1)$
Time Complexity of Top()	$O(1)$
Time Complexity of IsEmptyStack()	$O(1)$
Time Complexity of DeleteStack()	$O(n)$

4.6 Comparison of Implementations

Comparing Incremental Strategy and Doubling Strategy

We compare the incremental strategy and doubling strategy by analyzing the total time $T(n)$ needed to perform a series of n push operations. We start with an empty stack represented by an array of size 1.

We call *amortized* time of a push operation is the average time taken by a push over the series of operations, that is, $T(n)/n$.

Incremental Strategy

The amortized time (average time per operation) of a push operation is $O(n)$ $[O(n^2)/n]$.

Doubling Strategy

In this method, the amortized time of a push operation is $O(1)$ $[O(n)/n]$.

Note: For analysis, refer to the *Implementation* section.

Comparing Array Implementation and Linked List Implementation

Array Implementation

- Operations take constant time.
- Expensive doubling operation every once in a while.
- Any sequence of n operations (starting from empty stack) – "*amortized*" bound takes time proportional to n.

Linked List Implementation

- Grows and shrinks gracefully.
- Every operation takes constant time $O(1)$.
- Every operation uses extra space and time to deal with references.

4.7 Stacks: Problems & Solutions

Problem-1   Discuss how stacks can be used for checking balancing of symbols.

Solution: Stacks can be used to check whether the given expression has balanced symbols. This algorithm is very useful in compilers. Each time the parser reads one character at a time. If the character is an opening delimiter such as (, {, or [- then it is written to the stack. When a closing delimiter is encountered like), }, or]- the stack is popped.

The opening and closing delimiters are then compared. If they match, the parsing of the string continues. If they do not match, the parser indicates that there is an error on the line. A linear-time $O(n)$ algorithm based on stack can be given as:

Algorithm:
  a)   Create a stack.
  b)   while (end of input is not reached) {
        1)   If the character read is not a symbol to be balanced, ignore it.
        2)   If the character is an opening symbol like (, [, {, push it onto the stack
        3)   If it is a closing symbol like),],}, then if the stack is empty report an error. Otherwise pop the stack.
        4)   If the symbol popped is not the corresponding opening symbol, report an error.
       }
  c)   At end of input, if the stack is not empty report an error

Examples:

Example	Valid?	Description
(A+B)+(C-D)	Yes	The expression has a balanced symbol
((A+B)+(C-D)	No	One closing brace is missing
((A+B)+[C-D])	Yes	Opening and immediate closing braces correspond
((A+B)+[C-D]}	No	The last closing brace does not correspond with the first opening parenthesis

For tracing the algorithm let us assume that the input is: () (() [()])

Input Symbol, A[i]	Operation	Stack	Output
(	Push (	(	
)	Pop (Test if (and A[i] match? YES		
(	Push (	(	

(	Push (	((	
)	Pop (Test if (and A[i] match? YES	(	
[	Push [	([	
(	Push (	([(	
)	Pop (Test if[and A[i] match? YES	([	
]	Pop [Test if [and A[i] match? YES	(	
)	Pop (Test if(and A[i] match? YES		
	Test if stack is Empty? YES		TRUE

Time Complexity: $O(n)$. Since we are scanning the input only once. Space Complexity: $O(n)$ [for stack].

```
def checkSymbolBalance(input):
    symbolstack = Stack()
    balanced = 0
    for symbols in input:
        if symbols in ["(","{","["]:
            symbolstack.push(symbols)
        else:
            if symbolstack.isEmpty():
                balanced = 0
            else:
                topSymbol = symbolstack.pop()
            if not matches(topSymbol,symbols):
                    balanced = 0
                else:
                    balanced = 1
    return balanced
print checkSymbolBalance("([)]")
'''Output: 0'''
print checkSymbolBalance("{{([][])}()}")
'''Output: 1'''
```

Problem-2   Discuss infix to postfix conversion algorithm using stack.

Solution: Before discussing the algorithm, first let us see the definitions of infix, prefix and postfix expressions.

Infix: An infix expression is a single letter, or an operator, proceeded by one infix string and followed by another Infix string.

      A
      A+B
      (A+B)+ (C-D)

Prefix: A prefix expression is a single letter, or an operator, followed by two prefix strings. Every prefix string longer than a single variable contains an operator, first operand and second operand.

      A
      +AB
      ++AB-CD

Postfix: A postfix expression (also called Reverse Polish Notation) is a single letter or an operator, preceded by two postfix strings. Every postfix string longer than a single variable contains first and second operands followed by an operator.

      A
      AB+
      AB+CD-+

Prefix and postfix notions are methods of writing mathematical expressions without parenthesis. Time to evaluate a postfix and prefix expression is $O(n)$, where n is the number of elements in the array.

Infix	Prefix	Postfix
A+B	+AB	AB+
A+B-C	-+ABC	AB+C-
(A+B)*C-D	-*+ABCD	AB+C*D-

Now, let us focus on the algorithm. In infix expressions, the operator precedence is implicit unless we use parentheses. Therefore, for the infix to postfix conversion algorithm we have to define the operator precedence (or priority) inside the algorithm.

The table shows the precedence and their associativity (order of evaluation) among operators.

Token	Operator	Precedence	Associativity
( ) [ ] → .	function call array element struct or union member	17	left-to-right
-- ++	increment, decrement	16	left-to-right
-- ++ ! ~ - + & * sizeof	decrement, increment logical not one's complement unary minus or plus address or indirection size (in bytes)	15	right-to-left
(type)	type cast	14	right-to-left
* / %	multiplicative	13	Left-to-right
+ -	binary add or subtract	12	left-to-right
<< >>	shift	11	left-to-right
> >= < <=	relational	10	left-to-right
== !=	equality	9	left-to-right
&	bitwise and	8	left-to-right
^	bitwise exclusive or	7	left-to-right
|	bitwise or	6	left-to-right
&&	logical and	5	left-to-right
||	logical or	4	left-to-right
?:	conditional	3	right-to-left
= += -= /= *= %= <<= >>= &= ^=	assignment	2	right-to-left
,	comma	1	left-to-right

Important Properties

- Let us consider the infix expression 2 + 3 * 4 and its postfix equivalent 2 3 4 * +. Notice that between infix and postfix the order of the numbers (or operands) is unchanged. It is 2 3 4 in both cases. But the order of the operators * and + is affected in the two expressions.
- Only one stack is enough to convert an infix expression to postfix expression. The stack that we use in the algorithm will be used to change the order of operators from infix to postfix. The stack we use will only contain operators and the open parentheses symbol '('.
  Postfix expressions do not contain parentheses. We shall not output the parentheses in the postfix output.

Algorithm:

a) Create a stack
b) for each character t in the input stream{
       if(t is an operand)
            output t
       else if(t is a right parenthesis){
            Pop and output tokens until a left parenthesis is popped (but not output)
       }
       else // t is an operator or left parenthesis{
            pop and output tokens until one of lower priority than t is encountered or a left parenthesis is encountered or the stack is empty
            Push t
       }

    }
    c)   pop and output tokens until the stack is empty

For better understanding let us trace out an example: A * B- (C + D) + E

Input Character	Operation on Stack	Stack	Postfix Expression
A		Empty	A
*	Push	*	A
B		*	AB
-	Check and Push	-	AB*
(	Push	-(	AB*
C		-(	AB*C
+	Check and Push	-(+	AB*C
D			AB*CD
)	Pop and append to postfix till '('	-	AB*CD+
+	Check and Push	+	AB*CD+-
E		+	AB*CD+-E
End of input	Pop till empty		AB*CD+-E+

```python
class Stack:
    def __init__(self):
        self.items = []

    #method for pushing an item on a stack
    def push(self,item):
        self.items.append(item)

    #method for popping an item from a stack
    def pop(self):
        return self.items.pop()

    #method to check whether the stack is empty or not
    def isEmpty(self):
        return (self.items == [])

    #method to get the top of the stack
    def peek(self):
        return self.items[-1]

    def __str__(self):
        return str(self.items)

def infixToPostfix(infixexpr):
    prec = {}
    prec["*"] = 3
    prec["/"] = 3
    prec["+"] = 2
    prec["-"] = 2
    prec["("] = 1
    opStack = Stack()
    postfixList = []
    tokenList = infixexpr.split()

    for token in tokenList:
        if token in "ABCDEFGHIJKLMNOPQRSTUVWXYZ" or token in "0123456789":
            postfixList.append(token)
        elif token == '(':
            opStack.push(token)
        elif token == ')':
            topToken = opStack.pop()

            while topToken != '(':
                postfixList.append(topToken)
                topToken = opStack.pop()
        else:
            while (not opStack.isEmpty()) and \
               (prec[opStack.peek()] >= prec[token]):
                  postfixList.append(opStack.pop())
            opStack.push(token)

    while not opStack.isEmpty():
```

```
    postfixList.append(opStack.pop())
  return " ".join(postfixList)
print(infixToPostfix("A * B + C * D"))
print(infixToPostfix("( A + B ) * C - ( D - E ) * ( F + G )"))
```

**Problem-3**   Discuss postfix evaluation using stacks?

Solution:

Algorithm:

1 Scan the Postfix string from left to right.
2 Initialize an empty stack.
3 Repeat steps 4 and 5 till all the characters are scanned.
4 If the scanned character is an operand, push it onto the stack.
5 If the scanned character is an operator, and if the operator is a unary operator, then pop an element from the stack. If the operator is a binary operator, then pop two elements from the stack. After popping the elements, apply the operator to those popped elements. Let the result of this operation be retVal onto the stack.
6 After all characters are scanned, we will have only one element in the stack.
7 Return top of the stack as result.

Example: Let us see how the above-mentioned algorithm works using an example. Assume that the postfix string is 123*+5-.
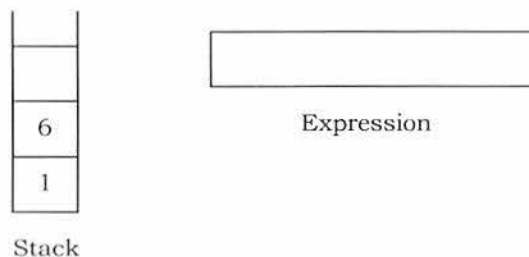
Initially the stack is empty. Now, the first three characters scanned are 1, 2 and 3, which are operands. They will be pushed into the stack in that order.
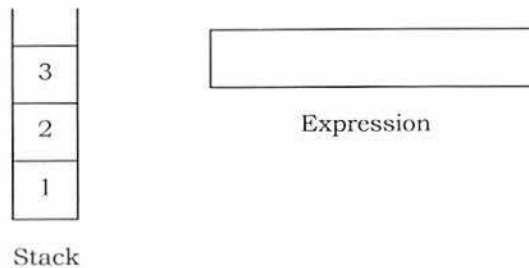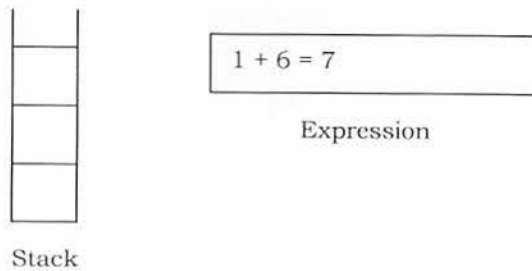

The next character scanned is "*", which is an operator. Thus, we pop the top two elements from the stack and perform the "*" operation with the two operands. The second operand will be the first element that is popped.
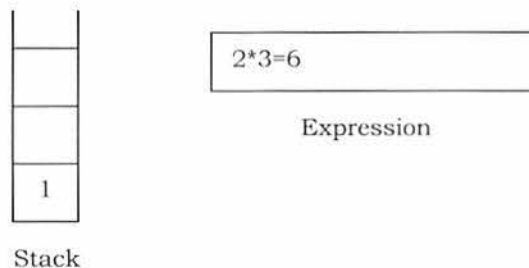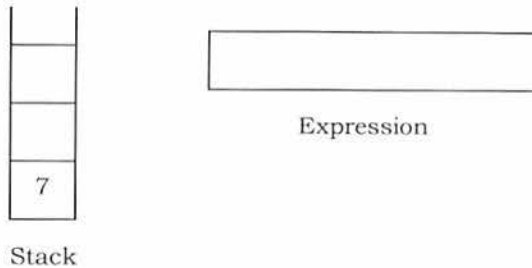

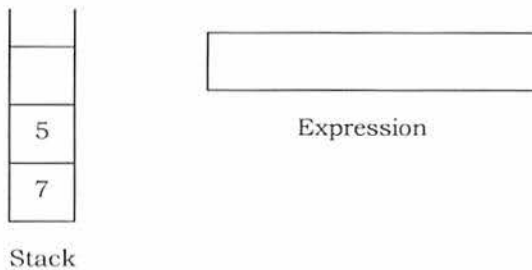The value of the expression (2*3) that has been evaluated (6) is pushed into the stack.

The next character scanned is "+", which is an operator. Thus, we pop the top two elements from the stack and perform the "+" operation with the two operands. The second operand will be the first element that is popped.
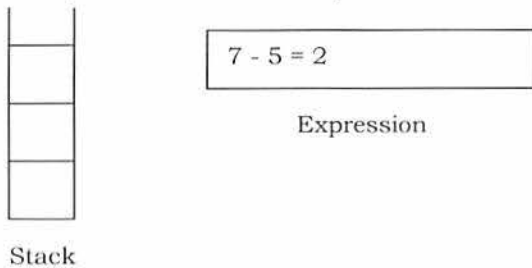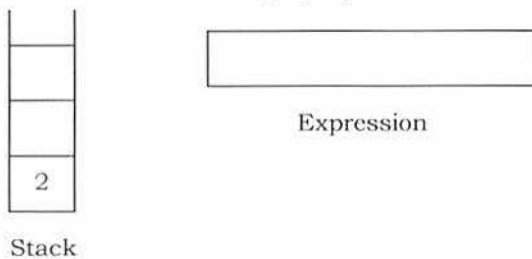
```
         ┌───┐
         │   │                    ┌────────────────────────┐
         │   │                    │  1 + 6 = 7             │
         ├───┤                    └────────────────────────┘
         │   │                           Expression
         ├───┤
         │   │
         └───┘
          Stack
```

The value of the expression (1+6) that has been evaluated (7) is pushed into the stack.

```
         ┌───┐
         │   │                    ┌────────────────────────┐
         │   │                    │                        │
         ├───┤                    └────────────────────────┘
         │   │                           Expression
         ├───┤
         │ 7 │
         └───┘
          Stack
```

The next character scanned is "5", which is added to the stack.

```
         ┌───┐
         │   │                    ┌────────────────────────┐
         │   │                    │                        │
         ├───┤                    └────────────────────────┘
         │ 5 │                           Expression
         ├───┤
         │ 7 │
         └───┘
          Stack
```

The next character scanned is "-", which is an operator. Thus, we pop the top two elements from the stack and perform the "-" operation with the two operands. The second operand will be the first element that is popped.

```
         ┌───┐
         │   │                    ┌────────────────────────┐
         │   │                    │  7 - 5 = 2             │
         ├───┤                    └────────────────────────┘
         │   │                           Expression
         ├───┤
         │   │
         └───┘
          Stack
```

The value of the expression(7-5) that has been evaluated(23) is pushed into the stack.

```
         ┌───┐
         │   │                    ┌────────────────────────┐
         │   │                    │                        │
         ├───┤                    └────────────────────────┘
         │   │                           Expression
         ├───┤
         │ 2 │
         └───┘
          Stack
```

Now, since all the characters are scanned, the remaining element in the stack (there will be only one element in the stack) will be returned. End result:

- Postfix String : 123*+5-
- Result : 2

```
class Stack:
    def __init__(self):
```

```
        self.items = []
    #method for pushing an item on a stack
    def push(self,item):
        self.items.append(item)

    #method for popping an item from a stack
    def pop(self):
        return self.items.pop()

    #method to check whether the stack is empty or not
    def isEmpty(self):
        return (self.items == [])

    def __str__(self):
        return str(self.items)
def postfixEval(postfixExpr):
    operandStack = Stack()
    tokenList = postfixExpr.split()

    for token in tokenList:
        if token in "0123456789":
            operandStack.push(int(token))
        else:
            operand2 = operandStack.pop()
            operand1 = operandStack.pop()
            result = doMath(token,operand1,operand2)
            operandStack.push(result)
    return operandStack.pop()

def doMath(op, op1, op2):
    if op == "*":
        return op1 * op2
    elif op == "/":
        return op1 / op2
    elif op == "+":
        return op1 + op2
    else:
        return op1 - op2
print(postfixEval('1 2 3 * + 5 -'))
```

Problem-4   Can we evaluate the infix expression with stacks in one pass?

Solution: Using 2 stacks we can evaluate an infix expression in 1 pass without converting to postfix.

Algorithm:
1) Create an empty operator stack
2) Create an empty operand stack
3) For each token in the input string
   a. Get the next token in the infix string
   b. If next token is an operand, place it on the operand stack
   c. If next token is an operator
       i. Evaluate the operator (next op)
4) While operator stack is not empty, pop operator and operands (left and right), evaluate left operator right and push result onto operand stack
5) Pop result from operator stack

Problem-5   How to design a stack such that GetMinimum( ) should be O(1)?

Solution: Take an auxiliary stack that maintains the minimum of all values in the stack. Also, assume that each element of the stack is less than its below elements. For simplicity let us call the auxiliary stack *min stack*.

When we *pop* the main stack, *pop* the min stack too. When we push the main stack, push either the new element or the current minimum, whichever is lower. At any point, if we want to get the minimum, then we just need to return the top element from the min stack. Let us take an example and trace it out. Initially let us assume that we have pushed 2, 6, 4, 1 and 5. Based on the above-mentioned algorithm the *min stack* will look like:

Main stack	Min stack
5 → top	1 → top
1	1
4	2
6	2
2	2

After popping twice we get:

Main stack	Min stack
4 --→ top	2 → top
6	2
2	2

Based on the discussion above, now let us code the push, pop and GetMinimum() operations.

```python
class SmartStack:
    def __init__(self):
        self.stack= []
        self.min = []

    def stack_push(self,x):
      self.stack.append(x)
      if not self.min or x <= self.stack_min():
            self.min.append(x)
      else:
            self.min.append(self.min[-1])

    def stack_pop(self):
      x = self.stack.pop()
      self.min.pop()
      return x

    def stack_min(self):
      return self.min[-1]
```

Time complexity: O(1). Space complexity: O(n) [for Min stack]. This algorithm has much better space usage if we rarely get a "new minimum or equal".

Problem-6   For Problem-5 is it possible to improve the space complexity?

Solution: Yes. The main problem of the previous approach is, for each push operation we are pushing the element on to min stack also (either the new element or existing minimum element). That means, we are pushing the duplicate minimum elements on to the stack.

Now, let us change the algorithm to improve the space complexity. We still have the min stack, but we only pop from it when the value we pop from the main stack is equal to the one on the min stack. We only *push* to the min stack when the value being pushed onto the main stack is less than *or equal* to the current min value. In this modified algorithm also, if we want to get the minimum then we just need to return the top element from the min stack. For example, taking the original version and pushing 1 again, we'd get:

Main stack	Min stack
1 → top	
5	
1	
4	1 → top
6	1
2	2

Popping from the above pops from both stacks because 1 == 1, leaving:

Main stack	Min stack
5 → top	
1	
4	
6	1 → top
2	2

Popping again *only* pops from the main stack, because 5 > 1:

Main stack	Min stack
1 → top	
4	
6	1 → top
2	2

Popping again pops both stacks because 1 == 1:

Main stack	Min stack
4 → top	
6	
2	2 → top

Note: The difference is only in push & pop operations.

```python
class SmartStack:
    def __init__(self):
        self.stack= []
        self.min = []
    def stack_push(self,x):
        self.stack.append(x)
        if not self.min or x <= self.stack_min():
            self.min.append(x)

    def stack_pop(self):
        x = self.stack.pop()
        if x == self.stack_min():
            self.min.pop()
        return x

    def stack_min(self):
        return self.min[-1]
```

Time complexity: $O(1)$. Space complexity: $O(n)$ [for Min stack]. But this algorithm has much better space usage if we rarely get a "new minimum or equal".

Problem-7   For a given array with n symbols how many stack permutations are possible?

Solution: The number of stack permutations with n symbols is represented by Catalan number and we will discuss this in the *Dynamic Programming* chapter.

Problem-8   Given an array of characters formed with a's and b's. The string is marked with special character X which represents the middle of the list (for example: ababa...ababXbabab.....baaa). Check whether the string is palindrome.

Solution: This is one of the simplest algorithms. What we do is, start two indexes, one at the beginning of the string and the other at the end of the string. Each time compare whether the values at both the indexes are the same or not. If the values are not the same then we say that the given string is not a palindrome.

If the values are the same then increment the left index and decrement the right index. Continue this process until both the indexes meet at the middle (at X) or if the string is not palindrome.

```python
def isPalindrome(A):
    i=0
    j = len(A)-1
    while (i < j and A[i] == A[j]):
        i += 1
        j -= 1
    if (i < j ):
        print("Not a Palindrome")
        return 0
    else:
        print("Palindrome")
        return 1

isPalindrome(['m', 'a', 'd','a', 'm'])
```

Time Complexity: $O(n)$. Space Complexity: $O(1)$.

Problem-9   For Problem-8, if the input is in singly linked list then how do we check whether the list elements form a palindrome (That means, moving backward is not possible).

Solution: Refer Linked Lists chapter.

Problem-10   Can we solve Problem-8 using stacks?

Solution: Yes.

Algorithm:
- Traverse the list till we encounter X as input element.
- During the traversal push all the elements (until X) on to the stack.
- For the second half of the list, compare each element's content with top of the stack. If they are the same then pop the stack and go to the next element in the input list.
- If they are not the same then the given string is not a palindrome.
- Continue this process until the stack is empty or the string is not a palindrome.

```python
def isPalindrome(str):
    strStack = Stack()
    palindrome = False
    for char in str:
        strStack.push(char)
    for char in str:
        if char == strStack.pop():
            palindrome = True
        else:
            palindrome = False
    return palindrome
print isPalindrome("smadams")
```

Time Complexity: $O(n)$. Space Complexity: $O(n/2) \approx O(n)$.

Problem-11   Given a stack, how to reverse the elements of the stack using only stack operations (push & pop)?

Solution:

Algorithm:
- First pop all the elements of the stack till it becomes empty.
- For each upward step in recursion, insert the element at the bottom of the stack.

```python
class Stack(object):
    def __init__(self,items=[]):
        self.stack = items
    def is_empty(self):
        return not self.stack
    def pop(self):
        return self.stack.pop()
    def push(self,data):
        self.stack.append(data)
    def __repr__(self):
        return "Stack {0}".format(self.stack)
def reverseStack(stack):
    def reverseStackRecursive(stack,newStack=Stack()):
        if not stack.is_empty():
            newStack.push(stack.pop())
            reverseStackRecursive(stack,newStack)
        return newStack
    return reverseStackRecursive(stack)

stk = Stack(range(10))
print stk
print reverseStack(stk)
```

Time Complexity: $O(n^2)$. Space Complexity: $O(n)$, for recursive stack.

Problem-12   Show how to implement one queue efficiently using two stacks. Analyze the running time of the queue operations.
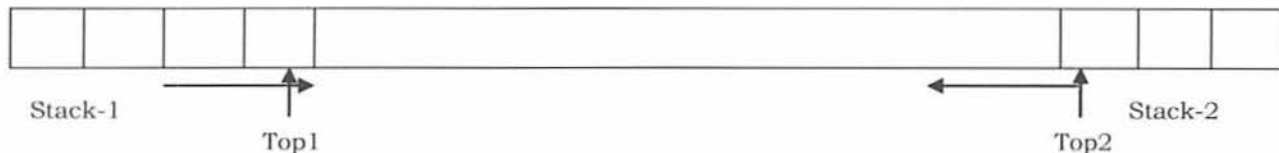
Solution: Refer Queues chapter.

Problem-13   Show how to implement one stack efficiently using two queues. Analyze the running time of the stack operations.

Solution: Refer Queues chapter.

Problem-14   How do we implement *two* stacks using only one array? Our stack routines should not indicate an exception unless every slot in the array is used?

Solution:

Algorithm:
- Start two indexes one at the left end and the other at the right end.
- The left index simulates the first stack and the right index simulates the second stack.
- If we want to push an element into the first stack then put the element at the left index.
- Similarly, if we want to push an element into the second stack then put the element at the right index.
- The first stack grows towards the right, and the second stack grows towards the left.

Time Complexity of push and pop for both stacks is $O(1)$. Space Complexity is $O(1)$.

Problem-15   3 stacks in one array: How to implement 3 stacks in one array?

Solution: For this problem, there could be other ways of solving it. Given below is one possibility and it works as long as there is an empty space in the array.
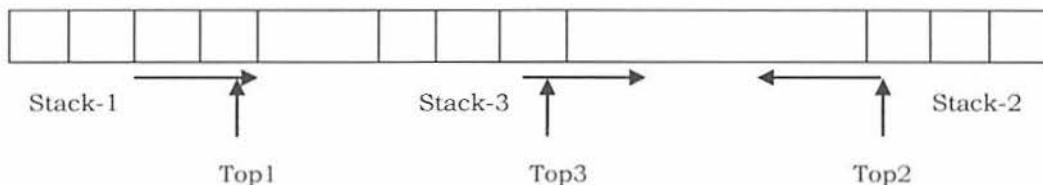
To implement 3 stacks we keep the following information.
- The index of the first stack (Top1): this indicates the size of the first stack.
- The index of the second stack (Top2): this indicates the size of the second stack.
- Starting index of the third stack (base address of third stack).
- Top index of the third stack.

Now, let us define the push and pop operations for this implementation.

Pushing:
- For pushing on to the first stack, we need to see if adding a new element causes it to bump into the third stack. If so, try to shift the third stack upwards. Insert the new element at (start1 + Top1).
- For pushing to the second stack, we need to see if adding a new element causes it to bump into the third stack. If so, try to shift the third stack downward. Insert the new element at (start2 – Top2).
- When pushing to the third stack, see if it bumps into the second stack. If so, try to shift the third stack downward and try pushing again. Insert the new element at (start3 + Top3).

Time Complexity: $O(n)$. Since, we may need to adjust the third stack. Space Complexity: $O(1)$.

Popping: For popping, we don't need to shift, just decrement the size of the appropriate stack.

Time Complexity: $O(1)$. Space Complexity: $O(1)$.

**Problem-16**   For Problem-15, is there any other way of implementing the middle stack?

Solution: Yes. When either the left stack (which grows to the right) or the right stack (which grows to the left) bumps into the middle stack, we need to shift the entire middle stack to make room. The same happens if a push on the middle stack causes it to bump into the right stack.

To solve the above-mentioned problem (number of shifts) what we can do is: alternating pushes can be added at alternating sides of the middle list (For example, even elements are pushed to the left, odd elements are pushed to the right). This would keep the middle stack balanced in the center of the array but it would still need to be shifted when it bumps into the left or right stack, whether by growing on its own or by the growth of a neighboring stack.

We can optimize the initial locations of the three stacks if they grow/shrink at different rates and if they have different average sizes. For example, suppose one stack doesn't change much. If we put it at the left, then the middle stack will eventually get pushed against it and leave a gap between the middle and right stacks, which grow toward each other. If they collide, then it's likely we've run out of space in the array. There is no change in the time complexity but the average number of shifts will get reduced.

Problem-17   Multiple (m) stacks in one array: Similar to Problem-15, what if we want to implement m stacks in one array?

Solution: Let us assume that array indexes are from 1 to n. Similar to the discussion in Problem-15, to implement m stacks in one array, we divide the array into m parts (as shown below). The size of each part is $\frac{n}{m}$.
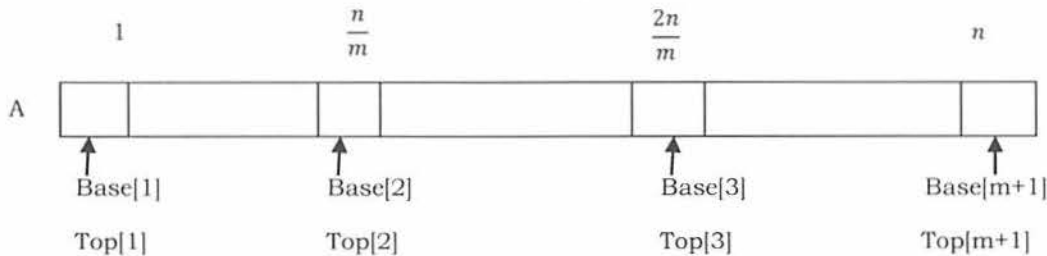
From the above representation we can see that, first stack is starting at index 1 (starting index is stored in Base[1]), second stack is starting at index $\frac{n}{m}$ (starting index is stored in Base[2]), third stack is starting at index $\frac{2n}{m}$ (starting index is stored in Base[3]), and so on. Similar to *Base* array, let us assume that *Top* array stores the top indexes for each of the stack. Consider the following terminology for the discussion.

- Top[i], for $1 \leq i \leq m$ will point to the topmost element of the stack i.
- If Base[i] == Top[i], then we can say the stack i is empty.
- If Top[i] == Base[i+1], then we can say the stack i is full.
  Initially Base[i] = Top[i] = $\frac{n}{m}(i-1)$, for $1 \leq i \leq m$.
- The ith stack grows from Base[i]+1 to Base[i+1].

Pushing on to i^{th} stack:

1) For pushing on to the ith stack, we check whether the top of ith stack is pointing to Base[i+1] (this case defines that i^{th} stack is full). That means, we need to see if adding a new element causes it to bump into the $i+1^{th}$ stack. If so, try to shift the stacks from $i+1^{th}$ stack to m^{th} stack toward the right. Insert the new element at (Base[i] + Top[i]).
2) If right shifting is not possible then try shifting the stacks from 1 to $i-1^{th}$ stack toward the left.
3) If both of them are not possible then we can say that all stacks are full.

```
def push(StackID, data):
    if Top[i] == Base[i+1]:
        print (i^th Stack is full and does the necessary action (shifting))
    Top[i] = Top[i]+1
    A[Top[i]] = data
```

Time Complexity: $O(n)$. Since we may need to adjust the stacks. Space Complexity: $O(1)$.

Popping from i^{th} stack: For popping, we don't need to shift, just decrement the size of the appropriate stack. The only case to check is stack empty case.

```
def Pop(StackID):
    if(Top[i] == Base[i])
        print (i^th Stack is empty)
    return  A[Top[i]--]
```

Time Complexity: $O(1)$. Space Complexity: $O(1)$.

Problem-18   Consider an empty stack of integers. Let the numbers $1,2,3,4,5,6$ be pushed on to this stack in the order they appear from left to right. Let S indicate a push and X indicate a pop operation. Can they be permuted in to the order 325641(output) and order 154623?

Solution: SSSXXSSXSXXX outputs 325641. 154623 cannot be output as 2 is pushed much before 3 so can appear only after 3 is output.

Problem-19   Earlier in this chapter, we discussed that for dynamic array implementation of stacks, the 'repeated doubling' approach is used. For the same problem, what is the complexity if we create a new array whose size is $n + K$ instead of doubling?

Solution: Let us assume that the initial stack size is 0. For simplicity let us assume that $K = 10$. For inserting the element we create a new array whose size is $0 + 10 = 10$. Similarly, after 10 elements we again create a new array whose size is $10 + 10 = 20$ and this process continues at values: $30, 40 \dots$ That means, for a given n value, we are creating the new arrays at: $\frac{n}{10}, \frac{n}{20}, \frac{n}{30}, \frac{n}{40} \dots$ The total number of copy operations is:

$$= \frac{n}{10} + \frac{n}{20} + \frac{n}{30} + \cdots 1 = \frac{n}{10}\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots \frac{1}{n}\right) = \frac{n}{10}logn \approx O(nlogn)$$

If we are performing n push operations, the cost per operation is $O(logn)$.

Problem-20   Given a string containing $n\ S's$ and $n\ X's$ where S indicates a push operation and X indicates a pop operation, and with the stack initially empty, formulate a rule to check whether a given string S of operations is admissible or not?

Solution: Given a string of length $2n$, we wish to check whether the given string of operations is permissible or not with respect to its functioning on a stack. The only restricted operation is pop whose prior requirement is that the stack should not be empty. So while traversing the string from left to right, prior to any pop the stack shouldn't be empty, which means the number of $S's$ is always greater than or equal to that of $X's$. Hence the condition is at any stage of processing of the string, the number of push operations (S) should be greater than the number of pop operations (X).

Problem-21   Suppose there are two singly linked lists which intersect at some point and become a single linked list. The head or start pointers of both the lists are known, but the intersecting node is not known. Also, the number of nodes in each of the lists before they intersect are unknown and both lists may have a different number. $List1$ may have n nodes before it reaches the intersection point and $List2$ may have m nodes before it reaches the intersection point where m and n may be $m = n, m < n$ or $m > n$. Can we find the merging point using stacks?
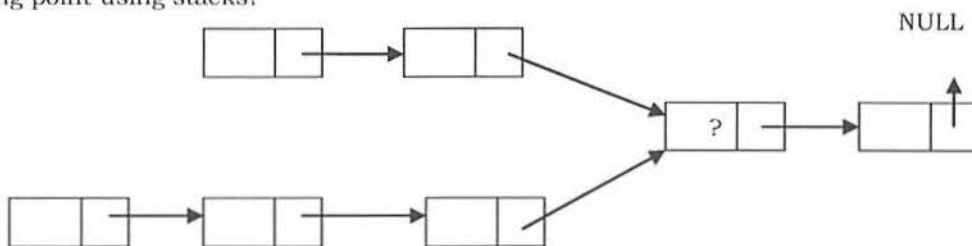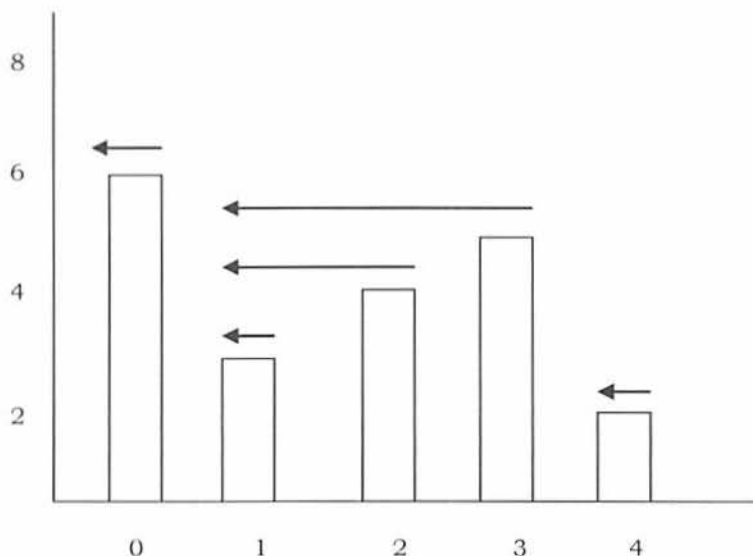

Solution: Yes. For algorithm refer to *Linked Lists* chapter.

Problem-22   **Finding Spans:** Given an array A, the span $S[i]$ of $A[i]$ is the maximum number of consecutive elements $A[j]$ immediately preceding $A[i]$ and such that $A[j] \leq A[i]$?
Another way of asking: Given an array A of integers, find the maximum of $j - i$ subjected to the constraint of $A[i] < A[j]$.

Solution:

Day: Index i	Input Array A[i]	S[i]: Span of A[i]
0	6	1
1	3	1
2	4	2
3	5	3
4	2	1

This is a very common problem in stock markets to find the peaks. Spans are used in financial analysis (E.g., stock at 52-week high). The span of a stock price on a certain day, i, is the maximum number of consecutive days (up to the current day) the price of the stock has been less than or equal to its price on i.

As an example, let us consider the table and the corresponding spans diagram. In the figure the arrows indicate the length of the spans. Now, let us concentrate on the algorithm for finding the spans. One simple way is, each day, check how many contiguous days have a stock price that is less than the current price.

```python
class Stack:
    def __init__(self):
        self.items = []

    #method for pushing an item on a stack
    def push(self,item):
        self.items.append(item)

    #method for popping an item from a stack
    def pop(self):
        return self.items.pop()

    #method to check whether the stack is empty or not
    def isEmpty(self):
        return (self.items == [])

    #method to get the top of the stack
    def peek(self):
        return self.items[-1]

    def __str__(self):
        return str(self.items)

def findingSpans(A):
    s = [None]*len(A)
    for i in range(0,len(A)):
        j = 1
        while j <= i and A[i] > A[i-j]:
            j = j + 1
            s[i] = j
        print s

findingSpans(['6', '3', '4', '5', '2'])
```

Time Complexity: $O(n^2)$. Space Complexity: $O(1)$.

Problem-23   Can we improve the complexity of Problem-22?

Solution: From the example above, we can see that span $S[i]$ on day i can be easily calculated if we know the closest day preceding i, such that the price is greater on that day than the price on day i. Let us call such a day as P. If such a day exists then the span is now defined as $S[i] = i - P$.

```python
class Stack:
    def __init__(self):
        self.items = []

    #method for pushing an item on a stack
    def push(self,item):
        self.items.append(item)

    #method for popping an item from a stack
    def pop(self):
        return self.items.pop()

    #method to check whether the stack is empty or not
    def isEmpty(self):
        return (self.items == [])
```

```
        #method to get the top of the stack
     def peek(self):
        return self.items[-1]

     def __str__(self):
        return str(self.items)
  def findingSpans(A):
     D = Stack()
     S = [None]*len(A)
     for i in range (0,len(A)):
              while not D.isEmpty() and A[i] > A[D.peek()]:
                    D.pop()
              if D.isEmpty():
                 P = -1
              else:
                       P = D.peek()
              S[i] = i-P
              D.push(i)
     print S
  findingSpans(['6', '3', '4', '5', '2'])
```

Time Complexity: Each index of the array is pushed into the stack exactly once and also popped from the stack at most once. The statements in the while loop are executed at most n times. Even though the algorithm has nested loops, the complexity is $O(n)$ as the inner loop is executing only n times during the course of the algorithm (trace out an example and see how many times the inner loop becomes successful). Space Complexity: $O(n)$ [for stack].

Problem-24   **Largest rectangle under histogram:** A histogram is a polygon composed of a sequence of rectangles aligned at a common base line. For simplicity, assume that the rectangles have equal widths but may have different heights. For example, the figure on the left shows a histogram that consists of rectangles with the heights 3, 2, 5, 6, 1, 4, 4, measured in units where 1 is the width of the rectangles. Here our problem is: given an array with heights of rectangles (assuming width is 1), we need to find the largest rectangle possible. For the given example, the largest rectangle is the shared part.
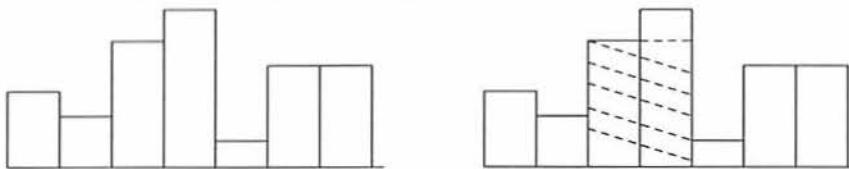
Solution: A straightforward answer is to go to each bar in the histogram and find the maximum possible area in the histogram for it. Finally, find the maximum of these values. This will require $O(n^2)$.

Problem-25   For Problem-24, can we improve the time complexity?

Solution: Linear search using a stack of incomplete sub problems: There are many ways of solving this problem. *Judge* has given a nice algorithm for this problem which is based on stack. Process the elements in left-to-right order and maintain a stack of information about started but yet unfinished sub histograms.

If the stack is empty, open a new sub problem by pushing the element onto the stack. Otherwise compare it to the element on top of the stack. If the new one is greater we again push it. If the new one is equal we skip it. In all these cases, we continue with the next new element. If the new one is less, we finish the topmost sub problem by updating the maximum area with respect to the element at the top of the stack. Then, we discard the element at the top, and repeat the procedure keeping the current new element.

This way, all sub problems are finished when the stack becomes empty, or its top element is less than or equal to the new element, leading to the actions described above. If all elements have been processed, and the stack is not yet empty, we finish the remaining sub problems by updating the maximum area with respect to the elements at the top.

```
  def largestRectangleArea(self, height):
       stack=[]; i=0; maxArea=0
       while i<len(height):
          if stack==[] or height[i]>height[stack[len(stack)-1]]:
                stack.append(i)
          else:
                curr=stack.pop()
```

```
                    width=i if stack==[] else i-stack[len(stack)-1]-1
                    maxArea=max(maxArea,width*height[curr])
                    i-=1
            i+=1
        while stack!=[]:
            curr=stack.pop()
            width=i if stack==[] else len(height)-stack[len(stack)-1]-1
            maxArea=max(maxArea,width*height[curr])
        return maxArea
```

At the first impression, this solution seems to be having $O(n^2)$ complexity. But if we look carefully, every element is pushed and popped at most once, and in every step of the function at least one element is pushed or popped. Since the amount of work for the decisions and the update is constant, the complexity of the algorithm is $O(n)$ by amortized analysis.

Space Complexity: $O(n)$ [for stack].

Problem-26   Given a stack of integers, how do you check whether each successive pair of numbers in the stack is consecutive or not. The pairs can be increasing or decreasing, and if the stack has an odd number of elements, the element at the top is left out of a pair. For example, if the stack of elements are [4, 5, -2, -3, 11, 10, 5, 6, 20], then the output should be true because each of the pairs (4, 5), (-2, -3), (11, 10), and (5, 6) consists of consecutive numbers.

Solution: Refer *Queues* chapter.

Problem-27   Recursively remove all adjacent duplicates: Given a string of characters, recursively remove adjacent duplicate characters from string. The output string should not have any adjacent duplicates.

Input: careermonk	*Input*: mississippi
Output: camonk	*Output*: m

Solution: This solution runs with the concept of in-place stack. When element on stack doesn't match the current character, we add it to stack. When it matches to stack top, we skip characters until the element matches the top of stack and remove the element from stack.

```
def removeAdjacentDuplicates(str):
    stkptr = -1
    i = 0
    size=len(str)
    while i<size:
        if (stkptr == -1 or str[stkptr]!=str[i]):
            stkptr += 1
            str[stkptr]=str[i]
            i += 1
        else:
            while i < size and str[stkptr]==str[i]:
                i += 1
            stkptr -= 1
    stkptr += 1
    str = str[0:stkptr]
    print str
removeAdjacentDuplicates(['6', '2', '4', '1', '2', '1', '2', '2', '1'])
```

Time Complexity: $O(n)$. Space Complexity: $O(1)$ as the stack simulation is done inplace.

Problem-28   Given an array of elements, replace every element with nearest greater element on the right of that element.

Solution: One simple approach would involve scanning the array elements and for each of the elements, scan the remaining elements and find the nearest greater element.

```
def replaceWithNearestGreaterElement(A):
    nextNearestGreater = float("-inf")
    i = j = 0
    for i in range(0,len(A)-1):
            nextNearestGreater = float("-inf")
            for j in range(i+1,len(A)):
                    if A[i] < A[j]:
                            nextNearestGreater = A[j]
```

```
                              break
                print("For "+ str(A[i]) +", " + str(nextNearestGreater) +" is the nearest greater element")
```

Time Complexity: $O(n^2)$. Space Complexity: $O(1)$.

Problem-29   For Problem-28, can we improve the complexity?

Solution: The approach is pretty much similar to Problem-22. Create a stack and push the first element. For rest of the elements, mark the current element as *nextNearestGreater*. If stack is not empty, then pop an element from stack and compare it with *nextNearestGreater*. If *nextNearestGreater* is greater than the popped element, then *nextNearestGreater* is the next greater element for the popped element. Keep popping from the stack while the popped element is smaller than *nextNearestGreater*. *nextNearestGreater* becomes the next greater element for all such popped elements. If *nextNearestGreater* is smaller than the popped element, then push the popped element back.

```python
def replaceWithNearestGreaterElementWithStack(A):
    i = 0
    S = Stack()
    S.push(A[0])
    for i in range(0,len(A)):
            nextNearestGreater = A[i]
            if not S.isEmpty():
                    element = S.pop()
                    while (element < nextNearestGreater):
                        print(str(element)+"--->"+str(nextNearestGreater)
                        if S.isEmpty():
                            break
                        element = S.pop()
                    if element > nextNearestGreater:
                            S.push(element)
            S.push(nextNearestGreater)

    while (not S.isEmpty()):
            element = S.pop()
            nextNearestGreater = float("-inf")
            print(str(element)+"→"+str(nextNearestGreater)
replaceWithNearestGreaterElementWithStack([6, 12, 4, 1, 2, 111, 2, 2, 10])
```

Time Complexity: $O(n)$. Space Complexity: $O(n)$.

Problem-30   Given a singly linked list L: $L_1 -> L_2 -> L_3 ... -> L_{n-1} -> L_n$, reorder it to: $L_1 -> L_n -> L_2 -> L_{n-1} ......$

Solution:

```python
def reorderList(self, head):
    if head == None:
        return head
    stack = []
    temp = head
    while temp != None:
        stack.append(temp)
        temp = temp.next
    list = head
    fromHead = head
    fromStack = True
    while (fromStack and list != stack[-1]) or ( not fromStack and list != fromHead):
        if fromStack:
                fromHead = fromHead.next
                list.next = stack.pop()
                fromStack = False
        else:
                list.next = fromHead
                fromStack = True
        list = list.next
    list.next = None
```

Time Complexity: $O(n)$. Space Complexity: $O(n)$.