1

Vector Spaces

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1.1 INTRODUCTION

Many familiar physical notions, such as forces, velocities,[1] and accelerations, involve both a magnitude (the amount of the force, velocity, or acceleration) and a direction. Any such entity involving both magnitude and direction is called a "vector." A vector is represented by an arrow whose length denotes the magnitude of the vector and whose direction represents the direction of the vector. In most physical situations involving vectors, only the magnitude and direction of the vector are significant; consequently, we regard vectors with the same magnitude and direction as being equal irrespective of their positions. In this section the geometry of vectors is discussed. This geometry is derived from physical experiments that test the manner in which two vectors interact.

Familiar situations suggest that when two like physical quantities act simultaneously at a point, the magnitude of their effect need not equal the sum of the magnitudes of the original quantities. For example, a swimmer swimming upstream at the rate of 2 miles per hour against a current of 1 mile per hour does not progress at the rate of 3 miles per hour. For in this instance the motions of the swimmer and current oppose each other, and the rate of progress of the swimmer is only 1 mile per hour upstream. If, however, the

[1]The word *velocity* is being used here in its scientific sense as an entity having both magnitude and direction. The magnitude of a velocity (without regard for the direction of motion) is called its **speed**.

swimmer is moving downstream (with the current), then his or her rate of progress is 3 miles per hour downstream.

Experiments show that if two like quantities act together, their effect is predictable. In this case, the vectors used to represent these quantities can be combined to form a resultant vector that represents the combined effects of the original quantities. This resultant vector is called the *sum* of the original vectors, and the rule for their combination is called the *parallelogram law.* (See Figure 1.1.)

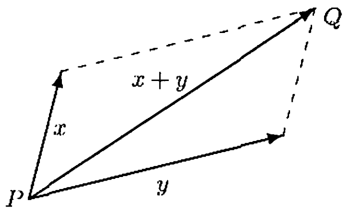

Figure 1.1

Parallelogram Law for Vector Addition. *The sum of two vectors x and y that act at the same point P is the vector beginning at P that is represented by the diagonal of parallelogram having x and y as adjacent sides.*

Since opposite sides of a parallelogram are parallel and of equal length, the endpoint Q of the arrow representing $x + y$ can also be obtained by allowing x to act at P and then allowing y to act at the endpoint of x. Similarly, the endpoint of the vector $x + y$ can be obtained by first permitting y to act at P and then allowing x to act at the endpoint of y. Thus two vectors x and y that both act at the point P may be added "tail-to-head"; that is, either x or y may be applied at P and a vector having the same magnitude and direction as the other may be applied to the endpoint of the first. If this is done, the endpoint of the second vector is the endpoint of $x + y$.

The addition of vectors can be described algebraically with the use of analytic geometry. In the plane containing x and y, introduce a coordinate system with P at the origin. Let (a_1, a_2) denote the endpoint of x and (b_1, b_2) denote the endpoint of y. Then as Figure 1.2(a) shows, the endpoint Q of $x+y$ is $(a_1 + b_1, a_2 + b_2)$. Henceforth, when a reference is made to the coordinates of the endpoint of a vector, the vector should be assumed to emanate from the origin. Moreover, since a vector beginning at the origin is completely determined by its endpoint, we sometimes refer to *the point* x rather than *the endpoint of the vector* x if x is a vector emanating from the origin.

Besides the operation of vector addition, there is another natural operation that can be performed on vectors—the length of a vector may be magnified

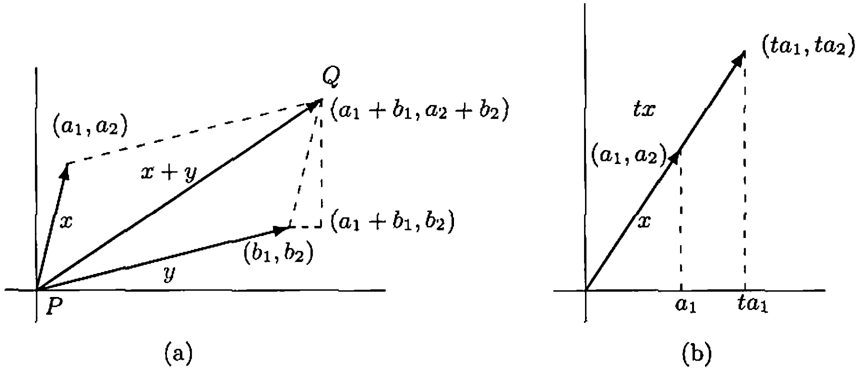

Figure 1.2

or contracted. This operation, called *scalar multiplication*, consists of multiplying the vector by a real number. If the vector x is represented by an arrow, then for any nonzero real number t, the vector tx is represented by an arrow in the same direction if $t > 0$ and in the opposite direction if $t < 0$. *The length of the arrow tx is $|t|$ times the length of the arrow x.* Two nonzero vectors x and y are called **parallel** if $y = tx$ for some nonzero real number t. (Thus nonzero vectors having the same or opposite directions are parallel.)

To describe scalar multiplication algebraically, again introduce a coordinate system into a plane containing the vector x so that x emanates from the origin. If the endpoint of x has coordinates (a_1, a_2), then the coordinates of the endpoint of tx are easily seen to be (ta_1, ta_2). (See Figure 1.2(b).)

The algebraic descriptions of vector addition and scalar multiplication for vectors in a plane yield the following properties:

1. For all vectors x and y, $x + y = y + x$.
2. For all vectors x, y, and z, $(x + y) + z = x + (y + z)$.
3. There exists a vector denoted $\mathit{0}$ such that $x + \mathit{0} = x$ for each vector x.
4. For each vector x, there is a vector y such that $x + y = \mathit{0}$.
5. For each vector x, $1x = x$.
6. For each pair of real numbers a and b and each vector x, $(ab)x = a(bx)$.
7. For each real number a and each pair of vectors x and y, $a(x + y) = ax + ay$.
8. For each pair of real numbers a and b and each vector x, $(a + b)x = ax + bx$.

Arguments similar to the preceding ones show that these eight properties, as well as the geometric interpretations of vector addition and scalar multiplication, are true also for vectors acting in space rather than in a plane. These *results can be used to write equations of lines and planes in space.*

Consider first the equation of a line in space that passes through two distinct points A and B. Let O denote the origin of a coordinate system in space, and let u and v denote the vectors that begin at O and end at A and B, respectively. If w denotes the vector beginning at A and ending at B, then "tail-to-head" addition shows that $u+w=v$, and hence $w=v-u$, where $-u$ denotes the vector $(-1)u$. (See Figure 1.3, in which the quadrilateral $OABC$ is a parallelogram.) Since a scalar multiple of w is parallel to w but possibly of a different length than w, any point on the line joining A and B may be obtained as the endpoint of a vector that begins at A and has the form tw for some real number t. Conversely, the endpoint of every vector of the form tw that begins at A lies on the line joining A and B. Thus an equation of the line through A and B is $x = u + tw = u + t(v - u)$, where t is a real number and x denotes an arbitrary point on the line. Notice also that the endpoint C of the vector $v - u$ in Figure 1.3 has coordinates equal to the difference of the coordinates of B and A.

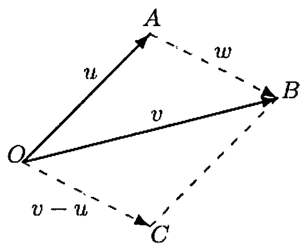

Figure 1.3

Example 1

Let A and B be points having coordinates $(-2, 0, 1)$ and $(4, 5, 3)$, respectively. The endpoint C of the vector emanating from the origin and having the same direction as the vector beginning at A and terminating at B has coordinates $(4, 5, 3) - (-2, 0, 1) = (6, 5, 2)$. Hence the equation of the line through A and B is

$$x = (-2, 0, 1) + t(6, 5, 2). \qquad \blacklozenge$$

Now let A, B, and C denote any three noncollinear points in space. These points determine a unique plane, and its equation can be found by use of our previous observations about vectors. Let u and v denote vectors beginning at A and ending at B and C, respectively. Observe that any point in the plane containing A, B, and C is the endpoint S of a vector x beginning at A and having the form $su + tv$ for some real numbers s and t. The endpoint of su is the point of intersection of the line through A and B with the line through S

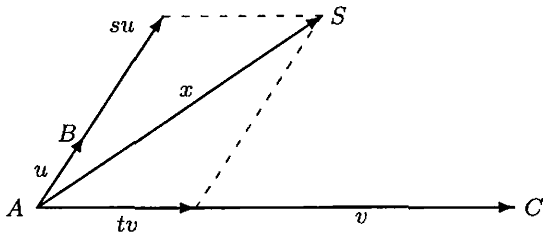

Figure 1.4

parallel to the line through A and C. (See Figure 1.4.) A similar procedure locates the endpoint of tv. Moreover, for any real numbers s and t, the vector $su + tv$ lies in the plane containing A, B, and C. It follows that an equation of the plane containing A, B, and C is

$$x = A + su + tv,$$

where s and t are arbitrary real numbers and x denotes an arbitrary point in the plane.

Example 2

Let A, B, and C be the points having coordinates $(1, 0, 2)$, $(-3, -2, 4)$, and $(1, 8, -5)$, respectively. The endpoint of the vector emanating from the origin and having the same length and direction as the vector beginning at A and terminating at B is

$$(-3, -2, 4) - (1, 0, 2) = (-4, -2, 2).$$

Similarly, the endpoint of a vector emanating from the origin and having the same length and direction as the vector beginning at A and terminating at C is $(1, 8, -5) - (1, 0, 2) = (0, 8, -7)$. Hence the equation of the plane containing the three given points is

$$x = (1, 0, 2) + s(-4, -2, 2) + t(0, 8, -7). \quad \blacklozenge$$

Any mathematical structure possessing the eight properties on page 3 is called a *vector space*. In the next section we formally define a vector space and consider many examples of vector spaces other than the ones mentioned above.

EXERCISES

1. Determine whether the vectors emanating from the origin and terminating at the following pairs of points are parallel.

(a) $(3, 1, 2)$ and $(6, 4, 2)$
(b) $(-3, 1, 7)$ and $(9, -3, -21)$
(c) $(5, -6, 7)$ and $(-5, 6, -7)$
(d) $(2, 0, -5)$ and $(5, 0, -2)$

2. Find the equations of the lines through the following pairs of points in space.

(a) $(3, -2, 4)$ and $(-5, 7, 1)$
(b) $(2, 4, 0)$ and $(-3, -6, 0)$
(c) $(3, 7, 2)$ and $(3, 7, -8)$
(d) $(-2, -1, 5)$ and $(3, 9, 7)$

3. Find the equations of the planes containing the following points in space.

(a) $(2, -5, -1)$, $(0, 4, 6)$, and $(-3, 7, 1)$
(b) $(3, -6, 7)$, $(-2, 0, -4)$, and $(5, -9, -2)$
(c) $(-8, 2, 0)$, $(1, 3, 0)$, and $(6, -5, 0)$
(d) $(1, 1, 1)$, $(5, 5, 5)$, and $(-6, 4, 2)$

4. What are the coordinates of the vector $\mathit{0}$ in the Euclidean plane that satisfies property 3 on page 3? Justify your answer.

5. Prove that if the vector x emanates from the origin of the Euclidean plane and terminates at the point with coordinates (a_1, a_2), then the vector tx that emanates from the origin terminates at the point with coordinates (ta_1, ta_2).

6. Show that the midpoint of the line segment joining the points (a, b) and (c, d) is $((a + c)/2, (b + d)/2)$.

7. Prove that the diagonals of a parallelogram bisect each other.

1.2 VECTOR SPACES

In Section 1.1, we saw that with the natural definitions of vector addition and scalar multiplication, the vectors in a plane satisfy the eight properties listed on page 3. Many other familiar algebraic systems also permit definitions of addition and scalar multiplication that satisfy the same eight properties. In this section, we introduce some of these systems, but first we formally define this type of algebraic structure.

Definitions. *A* ***vector space*** *(or* ***linear space****)* V *over a field*[2] F *consists of a set on which two operations (called* ***addition*** *and* ***scalar multiplication****, respectively) are defined so that for each pair of elements* x, y,

[2]Fields are discussed in Appendix C.

in V *there is a unique element* $x + y$ *in* V, *and for each element* a *in* F *and each element* x *in* V *there is a unique element* ax *in* V, *such that the following conditions hold.*

(VS 1) *For all* x, y *in* V, $x + y = y + x$ *(commutativity of addition).*

(VS 2) *For all* x, y, z *in* V, $(x + y) + z = x + (y + z)$ *(associativity of addition).*

(VS 3) *There exists an element in* V *denoted by* $\mathit{0}$ *such that* $x + \mathit{0} = x$ *for each* x *in* V.

(VS 4) *For each element* x *in* V *there exists an element* y *in* V *such that* $x + y = \mathit{0}$.

(VS 5) *For each element* x *in* V, $1x = x$.

(VS 6) *For each pair of elements* a, b *in* F *and each element* x *in* V, $(ab)x = a(bx)$.

(VS 7) *For each element* a *in* F *and each pair of elements* x, y *in* V, $a(x + y) = ax + ay$.

(VS 8) *For each pair of elements* a, b *in* F *and each element* x *in* V, $(a + b)x = ax + bx$.

The elements $x + y$ *and* ax *are called the* **sum** *of* x *and* y *and the* **product** *of* a *and* x, *respectively.*

The elements of the field F are called **scalars** and the elements of the vector space V are called **vectors**. *The reader should not confuse this use of the word "vector" with the physical entity discussed in Section 1.1: the word "vector" is now being used to describe any element of a vector space.*

A vector space is frequently discussed in the text without explicitly mentioning its field of scalars. The reader is cautioned to remember, however, that *every vector space is regarded as a vector space over a given field, which is denoted by* F. Occasionally we restrict our attention to the fields of real and complex numbers, which are denoted R and C, respectively.

Observe that (VS 2) permits us to define the addition of any finite number of vectors unambiguously (without the use of parentheses).

In the remainder of this section we introduce several important examples of vector spaces that are studied throughout this text. Observe that in describing a vector space, it is necessary to specify not only the vectors but also the operations of addition and scalar multiplication.

An object of the form $(a_1, a_2, \ldots, a_n)$, where the entries $a_1, a_2, \ldots, a_n$ are elements of a field F, is called an ***n*-tuple** with entries from F. The elements

$a_1, a_2, \ldots, a_n$ are called the **entries** or **components** of the n-tuple. Two n-tuples $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ with entries from a field F are called **equal** if $a_i = b_i$ for $i = 1, 2, \ldots, n$.

Example 1

The set of all n-tuples with entries from a field F is denoted by F^n. This set is a vector space over F with the operations of coordinatewise addition and scalar multiplication; that is, if $u = (a_1, a_2, \ldots, a_n) \in \mathsf{F}^n$, $v = (b_1, b_2 \ldots, b_n) \in \mathsf{F}^n$, and $c \in F$, then

$$u + v = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n) \quad \text{and} \quad cu = (ca_1, ca_2, \ldots, ca_n).$$

Thus R^3 is a vector space over R. In this vector space,

$$(3, -2, 0) + (-1, 1, 4) = (2, -1, 4) \quad \text{and} \quad -5(1, -2, 0) = (-5, 10, 0).$$

Similarly, C^2 is a vector space over C. In this vector space,

$$(1 + i, 2) + (2 - 3i, 4i) = (3 - 2i, 2 + 4i) \quad \text{and} \quad i(1 + i, 2) = (-1 + i, 2i).$$

Vectors in F^n may be written as **column vectors**

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

rather than as **row vectors** $(a_1, a_2, \ldots, a_n)$. Since a 1-tuple whose only entry is from F can be regarded as an element of F, we usually write F rather than F^1 for the vector space of 1-tuples with entry from F. ♦

An $m \times n$ **matrix** with entries from a field F is a rectangular array of the form

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix},$$

where each entry a_{ij} $(1 \leq i \leq m,\ 1 \leq j \leq n)$ is an element of F. We call the entries a_{ij} with $i = j$ the **diagonal entries** of the matrix. The entries $a_{i1}, a_{i2}, \ldots, a_{in}$ compose the ***i*th row** of the matrix, and the entries $a_{1j}, a_{2j}, \ldots, a_{mj}$ compose the ***j*th column** of the matrix. The rows of the preceding matrix are regarded as vectors in F^n, and the columns are regarded as vectors in F^m. The $m \times n$ matrix in which each entry equals zero is called the **zero matrix** and is denoted by O.

In this book, we denote matrices by capital italic letters (e.g., A, B, and C), and we denote the entry of a matrix A that lies in row i and column j by A_{ij}. In addition, if the number of rows and columns of a matrix are equal, the matrix is called **square.**

Two $m \times n$ matrices A and B are called **equal** if all their corresponding entries are equal, that is, if $A_{ij} = B_{ij}$ for $1 \le i \le m$ and $1 \le j \le n$.

Example 2

The set of all $m \times n$ matrices with entries from a field F is a vector space, which we denote by $\mathsf{M}_{m\times n}(F)$, with the following operations of **matrix addition** and **scalar multiplication**: For $A, B \in \mathsf{M}_{m\times n}(F)$ and $c \in F$,

$$(A+B)_{ij} = A_{ij} + B_{ij} \quad \text{and} \quad (cA)_{ij} = cA_{ij}$$

for $1 \le i \le m$ and $1 \le j \le n$. For instance,

$$\begin{pmatrix} 2 & 0 & -1 \\ 1 & -3 & 4 \end{pmatrix} + \begin{pmatrix} -5 & -2 & 6 \\ 3 & 4 & -1 \end{pmatrix} = \begin{pmatrix} -3 & -2 & 5 \\ 4 & 1 & 3 \end{pmatrix}$$

and

$$-3\begin{pmatrix} 1 & 0 & -2 \\ -3 & 2 & 3 \end{pmatrix} = \begin{pmatrix} -3 & 0 & 6 \\ 9 & -6 & -9 \end{pmatrix}$$

in $\mathsf{M}_{2\times 3}(R)$. ◆

Example 3

Let S be any nonempty set and F be any field, and let $\mathcal{F}(S, F)$ denote the set of all functions from S to F. Two functions f and g in $\mathcal{F}(S, F)$ are called **equal** if $f(s) = g(s)$ for each $s \in S$. The set $\mathcal{F}(S, F)$ is a vector space with the operations of addition and scalar multiplication defined for $f, g \in \mathcal{F}(S, F)$ and $c \in F$ by

$$(f+g)(s) = f(s) + g(s) \quad \text{and} \quad (cf)(s) = c[f(s)]$$

for each $s \in S$. Note that these are the familiar operations of addition and scalar multiplication for functions used in algebra and calculus. ◆

A **polynomial** with coefficients from a field F is an expression of the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$

where n is a nonnegative integer and each a_k, called the **coefficient** of x^k, is in F. If $f(x) = \mathit{0}$, that is, if $a_n = a_{n-1} = \cdots = a_0 = 0$, then $f(x)$ is called the **zero polynomial** and, for convenience, its degree is defined to be -1;

otherwise, the **degree** of a polynomial is defined to be the largest exponent of x that appears in the representation

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

with a nonzero coefficient. Note that the polynomials of degree zero may be written in the form $f(x) = c$ for some nonzero scalar c. Two polynomials,

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

and

$$g(x) = b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0,$$

are called **equal** if $m = n$ and $a_i = b_i$ for $i = 0, 1, \ldots, n$.

When F is a field containing infinitely many scalars, we usually regard a polynomial with coefficients from F as a function from F into F. (See page 569.) In this case, the value of the function

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

at $c \in F$ is the scalar

$$f(c) = a_n c^n + a_{n-1}c^{n-1} + \cdots + a_1 c + a_0.$$

Here either of the notations f or $f(x)$ is used for the polynomial function

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0.$$

Example 4

Let

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

and

$$g(x) = b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0$$

be polynomials with coefficients from a field F. Suppose that $m \leq n$, and define $b_{m+1} = b_{m+2} = \cdots = b_n = 0$. Then $g(x)$ can be written as

$$g(x) = b_n x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0.$$

Define

$$f(x) + g(x) = (a_n + b_n)x^n + (a_{n-1} + b_{n-1})x^{n-1} + \cdots + (a_1 + b_1)x + (a_0 + b_0)$$

and for any $c \in F$, define

$$cf(x) = ca_n x^n + ca_{n-1}x^{n-1} + \cdots + ca_1 x + ca_0.$$

With these operations of addition and scalar multiplication, the set of all polynomials with coefficients from F is a vector space, which we denote by $\mathsf{P}(F)$. ♦

We will see in Exercise 23 of Section 2.4 that the vector space defined in the next example is essentially the same as $\mathsf{P}(F)$.

Example 5

Let F be any field. A **sequence** in F is a function σ from the positive integers into F. In this book, the sequence σ such that $\sigma(n) = a_n$ for $n = 1, 2, \ldots$ is denoted $\{a_n\}$. Let V consist of all sequences $\{a_n\}$ in F that have only a finite number of nonzero terms a_n. If $\{a_n\}$ and $\{b_n\}$ are in V and $t \in F$, define

$$\{a_n\} + \{b_n\} = \{a_n + b_n\} \quad \text{and} \quad t\{a_n\} = \{ta_n\}.$$

With these operations V is a vector space. ♦

Our next two examples contain sets on which addition and scalar multiplication are defined, but which are *not* vector spaces.

Example 6

Let $S = \{(a_1, a_2) \colon a_1, a_2 \in R\}$. For $(a_1, a_2), (b_1, b_2) \in S$ and $c \in R$, define

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 - b_2) \quad \text{and} \quad c(a_1, a_2) = (ca_1, ca_2).$$

Since (VS 1), (VS 2), and (VS 8) fail to hold, S is not a vector space with these operations. ♦

Example 7

Let S be as in Example 6. For $(a_1, a_2), (b_1, b_2) \in S$ and $c \in R$, define

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, 0) \quad \text{and} \quad c(a_1, a_2) = (ca_1, 0).$$

Then S is not a vector space with these operations because (VS 3) (hence (VS 4)) and (VS 5) fail. ♦

We conclude this section with a few of the elementary consequences of the definition of a vector space.

Theorem 1.1 (Cancellation Law for Vector Addition). *If x, y, and z are vectors in a vector space V such that $x + z = y + z$, then $x = y$.*

Proof. There exists a vector v in V such that $z + v = \mathit{0}$ (VS 4). Thus

$$\begin{aligned} x &= x + \mathit{0} = x + (z + v) = (x + z) + v \\ &= (y + z) + v = y + (z + v) = y + \mathit{0} = y \end{aligned}$$

by (VS 2) and (VS 3). ∎

Corollary 1. *The vector $\mathit{0}$ described in (VS 3) is unique.*

Proof. Exercise. ∎

Corollary 2. *The vector y described in (VS 4) is unique.*

Proof. Exercise. ∎

The vector 0 in (VS 3) is called the **zero vector** of V, and the vector y in (VS 4) (that is, the unique vector such that $x+y=0$) is called the **additive inverse** of x and is denoted by $-x$.

The next result contains some of the elementary properties of scalar multiplication.

Theorem 1.2. *In any vector space* V*, the following statements are true:*
(a) $0x = 0$ *for each* $x \in \mathsf{V}$.
(b) $(-a)x = -(ax) = a(-x)$ *for each* $a \in F$ *and each* $x \in \mathsf{V}$.
(c) $a0 = 0$ *for each* $a \in F$.

Proof. (a) By (VS 8), (VS 3), and (VS 1), it follows that

$$0x + 0x = (0+0)x = 0x = 0x + 0 = 0 + 0x.$$

Hence $0x = 0$ by Theorem 1.1.

(b) The vector $-(ax)$ is the unique element of V such that $ax + [-(ax)] = 0$. Thus if $ax + (-a)x = 0$, Corollary 2 to Theorem 1.1 implies that $(-a)x = -(ax)$. But by (VS 8),

$$ax + (-a)x = [a + (-a)]x = 0x = 0$$

by (a). Consequently $(-a)x = -(ax)$. In particular, $(-1)x = -x$. So, by (VS 6),

$$a(-x) = a[(-1)x] = [a(-1)]x = (-a)x.$$

The proof of (c) is similar to the proof of (a). ∎

EXERCISES

1. Label the following statements as true or false.
   (a) Every vector space contains a zero vector.
   (b) A vector space may have more than one zero vector.
   (c) In any vector space, $ax = bx$ implies that $a = b$.
   (d) In any vector space, $ax = ay$ implies that $x = y$.
   (e) A vector in F^n may be regarded as a matrix in $\mathsf{M}_{n\times 1}(F)$.
   (f) An $m \times n$ matrix has m columns and n rows.
   (g) In $\mathsf{P}(F)$, only polynomials of the same degree may be added.
   (h) If f and g are polynomials of degree n, then $f+g$ is a polynomial of degree n.
   (i) If f is a polynomial of degree n and c is a nonzero scalar, then cf is a polynomial of degree n.

(j) A nonzero scalar of F may be considered to be a polynomial in $\mathsf{P}(F)$ having degree zero.
(k) Two functions in $\mathcal{F}(S, F)$ are equal if and only if they have the same value at each element of S.

2. Write the zero vector of $\mathsf{M}_{3\times 4}(F)$.

3. If

$$M = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix},$$

what are M_{13}, M_{21}, and M_{22}?

4. Perform the indicated operations.

(a) $\begin{pmatrix} 2 & 5 & -3 \\ 1 & 0 & 7 \end{pmatrix} + \begin{pmatrix} 4 & -2 & 5 \\ -5 & 3 & 2 \end{pmatrix}$

(b) $\begin{pmatrix} -6 & 4 \\ 3 & -2 \\ 1 & 8 \end{pmatrix} + \begin{pmatrix} 7 & -5 \\ 0 & -3 \\ 2 & 0 \end{pmatrix}$

(c) $4\begin{pmatrix} 2 & 5 & -3 \\ 1 & 0 & 7 \end{pmatrix}$

(d) $-5\begin{pmatrix} -6 & 4 \\ 3 & -2 \\ 1 & 8 \end{pmatrix}$

(e) $(2x^4 - 7x^3 + 4x + 3) + (8x^3 + 2x^2 - 6x + 7)$
(f) $(-3x^3 + 7x^2 + 8x - 6) + (2x^3 - 8x + 10)$
(g) $5(2x^7 - 6x^4 + 8x^2 - 3x)$
(h) $3(x^5 - 2x^3 + 4x + 2)$

Exercises 5 and 6 show why the definitions of matrix addition and scalar multiplication (as defined in Example 2) are the appropriate ones.

5. Richard Gard ("Effects of Beaver on Trout in Sagehen Creek, California," *J. Wildlife Management*, **25**, 221-242) reports the following number of trout having crossed beaver dams in Sagehen Creek.

Upstream Crossings

	Fall	Spring	Summer
Brook trout	8	3	1
Rainbow trout	3	0	0
Brown trout	3	0	0

Downstream Crossings

	Fall	Spring	Summer
Brook trout	9	1	4
Rainbow trout	3	0	0
Brown trout	1	1	0

Record the upstream and downstream crossings in two 3×3 matrices, and verify that the sum of these matrices gives the total number of crossings (both upstream and downstream) categorized by trout species and season.

6. At the end of May, a furniture store had the following inventory.

	Early American	Spanish	Mediter-ranean	Danish
Living room suites	4	2	1	3
Bedroom suites	5	1	1	4
Dining room suites	3	1	2	6

Record these data as a 3×4 matrix M. To prepare for its June sale, the store decided to double its inventory on each of the items listed in the preceding table. Assuming that none of the present stock is sold until the additional furniture arrives, verify that the inventory on hand after the order is filled is described by the matrix $2M$. If the inventory at the end of June is described by the matrix

$$A = \begin{pmatrix} 5 & 3 & 1 & 2 \\ 6 & 2 & 1 & 5 \\ 1 & 0 & 3 & 3 \end{pmatrix},$$

interpret $2M - A$. How many suites were sold during the June sale?

7. Let $S = \{0, 1\}$ and $F = R$. In $\mathcal{F}(S, R)$, show that $f = g$ and $f + g = h$, where $f(t) = 2t + 1$, $g(t) = 1 + 4t - 2t^2$, and $h(t) = 5^t + 1$.

8. In any vector space V, show that $(a + b)(x + y) = ax + ay + bx + by$ for any $x, y \in \mathsf{V}$ and any $a, b \in F$.

9. Prove Corollaries 1 and 2 of Theorem 1.1 and Theorem 1.2(c).

10. Let V denote the set of all differentiable real-valued functions defined on the real line. Prove that V is a vector space with the operations of addition and scalar multiplication defined in Example 3.

11. Let V $= \{0\}$ consist of a single vector 0 and define $0 + 0 = 0$ and $c0 = 0$ for each scalar c in F. Prove that V is a vector space over F. (V is called the **zero vector space**.)

12. A real-valued function f defined on the real line is called an **even function** if $f(-t) = f(t)$ for each real number t. Prove that the set of even functions defined on the real line with the operations of addition and scalar multiplication defined in Example 3 is a vector space.

13. Let V denote the set of ordered pairs of real numbers. If (a_1, a_2) and (b_1, b_2) are elements of V and $c \in R$, define

$$(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 b_2) \quad \text{and} \quad c(a_1, a_2) = (ca_1, a_2).$$

Is V a vector space over R with these operations? Justify your answer.

14. Let V $= \{(a_1, a_2, \ldots, a_n) \colon a_i \in C \text{ for } i = 1, 2, \ldots n\}$; so V is a vector space over C by Example 1. Is V a vector space over the field of real numbers with the operations of coordinatewise addition and multiplication?

15. Let V $= \{(a_1, a_2, \ldots, a_n) \colon a_i \in R \text{ for } i = 1, 2, \ldots n\}$; so V is a vector space over R by Example 1. Is V a vector space over the field of complex numbers with the operations of coordinatewise addition and multiplication?

16. Let V denote the set of all $m \times n$ matrices with real entries; so V is a vector space over R by Example 2. Let F be the field of rational numbers. Is V a vector space over F with the usual definitions of matrix addition and scalar multiplication?

17. Let V $= \{(a_1, a_2) \colon a_1, a_2 \in F\}$, where F is a field. Define addition of elements of V coordinatewise, and for $c \in F$ and $(a_1, a_2) \in$ V, define

$$c(a_1, a_2) = (a_1, 0).$$

Is V a vector space over F with these operations? Justify your answer.

18. Let V $= \{(a_1, a_2) \colon a_1, a_2 \in R\}$. For $(a_1, a_2), (b_1, b_2) \in$ V and $c \in R$, define

$$(a_1, a_2) + (b_1, b_2) = (a_1 + 2b_1, a_2 + 3b_2) \quad \text{and} \quad c(a_1, a_2) = (ca_1, ca_2).$$

Is V a vector space over R with these operations? Justify your answer.

19. Let $\mathsf{V} = \{(a_1, a_2)\colon a_1, a_2 \in R\}$. Define addition of elements of V coordinatewise, and for (a_1, a_2) in V and $c \in R$, define

$$c(a_1, a_2) = \begin{cases} (0, 0) & \text{if } c = 0 \\ \left(ca_1, \dfrac{a_2}{c}\right) & \text{if } c \neq 0. \end{cases}$$

Is V a vector space over R with these operations? Justify your answer.

20. Let V be the set of sequences $\{a_n\}$ of real numbers. (See Example 5 for the definition of a sequence.) For $\{a_n\}, \{b_n\} \in \mathsf{V}$ and any real number t, define

$$\{a_n\} + \{b_n\} = \{a_n + b_n\} \quad \text{and} \quad t\{a_n\} = \{ta_n\}.$$

Prove that, with these operations, V is a vector space over R.

21. Let V and W be vector spaces over a field F. Let

$$\mathsf{Z} = \{(v, w)\colon v \in \mathsf{V} \text{ and } w \in \mathsf{W}\}.$$

Prove that Z is a vector space over F with the operations

$$(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2) \quad \text{and} \quad c(v_1, w_1) = (cv_1, cw_1).$$

22. How many matrices are there in the vector space $\mathsf{M}_{m\times n}(Z_2)$? (See Appendix C.)

1.3 SUBSPACES

In the study of any algebraic structure, it is of interest to examine subsets that possess the same structure as the set under consideration. The appropriate notion of substructure for vector spaces is introduced in this section.

Definition. *A subset* W *of a vector space* V *over a field* F *is called a* ***subspace*** *of* V *if* W *is a vector space over* F *with the operations of addition and scalar multiplication defined on* V.

In any vector space V, note that V and $\{0\}$ are subspaces. The latter is called the **zero subspace** of V.

Fortunately it is not necessary to verify all of the vector space properties to prove that a subset is a subspace. Because properties (VS 1), (VS 2), (VS 5), (VS 6), (VS 7), and (VS 8) hold for all vectors in the vector space, these properties automatically hold for the vectors in any subset. Thus a subset W of a vector space V is a subspace of V if and only if the following four properties hold.

1. $x+y \in \mathsf{W}$ whenever $x \in \mathsf{W}$ and $y \in \mathsf{W}$. (W is **closed under addition**.)
2. $cx \in \mathsf{W}$ whenever $c \in F$ and $x \in \mathsf{W}$. (W is **closed under scalar multiplication**.)
3. W has a zero vector.
4. Each vector in W has an additive inverse in W.

The next theorem shows that the zero vector of W must be the same as the zero vector of V and that property 4 is redundant.

Theorem 1.3. *Let* V *be a vector space and* W *a subset of* V. *Then* W *is a subspace of* V *if and only if the following three conditions hold for the operations defined in* V.
(a) $0 \in \mathsf{W}$.
(b) $x+y \in \mathsf{W}$ *whenever* $x \in \mathsf{W}$ *and* $y \in \mathsf{W}$.
(c) $cx \in \mathsf{W}$ *whenever* $c \in F$ *and* $x \in \mathsf{W}$.

Proof. If W is a subspace of V, then W is a vector space with the operations of addition and scalar multiplication defined on V. Hence conditions (b) and (c) hold, and there exists a vector $0' \in \mathsf{W}$ such that $x + 0' = x$ for each $x \in \mathsf{W}$. But also $x + 0 = x$, and thus $0' = 0$ by Theorem 1.1 (p. 11). So condition (a) holds.

Conversely, if conditions (a), (b), and (c) hold, the discussion preceding this theorem shows that W is a subspace of V if the additive inverse of each vector in W lies in W. But if $x \in \mathsf{W}$, then $(-1)x \in \mathsf{W}$ by condition (c), and $-x = (-1)x$ by Theorem 1.2 (p. 12). Hence W is a subspace of V. ∎

The preceding theorem provides a simple method for determining whether or not a given subset of a vector space is a subspace. Normally, it is this result that is used to prove that a subset is, in fact, a subspace.

The **transpose** A^t of an $m \times n$ matrix A is the $n \times m$ matrix obtained from A by interchanging the rows with the columns; that is, $(A^t)_{ij} = A_{ji}$. For example,

$$\begin{pmatrix} 1 & -2 & 3 \\ 0 & 5 & -1 \end{pmatrix}^t = \begin{pmatrix} 1 & 0 \\ -2 & 5 \\ 3 & -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}^t = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}.$$

A **symmetric matrix** is a matrix A such that $A^t = A$. For example, the 2×2 matrix displayed above is a symmetric matrix. Clearly, a symmetric matrix must be square. The set W of all symmetric matrices in $\mathsf{M}_{n\times n}(F)$ is a subspace of $\mathsf{M}_{n\times n}(F)$ since the conditions of Theorem 1.3 hold:

1. The zero matrix is equal to its transpose and hence belongs to W.

It is easily proved that for any matrices A and B and any scalars a and b, $(aA+bB)^t = aA^t + bB^t$. (See Exercise 3.) Using this fact, we show that the set of symmetric matrices is closed under addition and scalar multiplication.

2. If $A \in \mathsf{W}$ and $B \in \mathsf{W}$, then $A^t = A$ and $B^t = B$. Thus $(A+B)^t = A^t + B^t = A + B$, so that $A + B \in \mathsf{W}$.

3. If $A \in \mathsf{W}$, then $A^t = A$. So for any $a \in F$, we have $(aA)^t = aA^t = aA$. Thus $aA \in \mathsf{W}$.

The examples that follow provide further illustrations of the concept of a subspace. The first three are particularly important.

Example 1

Let n be a nonnegative integer, and let $\mathsf{P}_n(F)$ consist of all polynomials in $\mathsf{P}(F)$ having degree less than or equal to n. Since the zero polynomial has degree -1, it is in $\mathsf{P}_n(F)$. Moreover, the sum of two polynomials with degrees less than or equal to n is another polynomial of degree less than or equal to n, and the product of a scalar and a polynomial of degree less than or equal to n is a polynomial of degree less than or equal to n. So $\mathsf{P}_n(F)$ is closed under addition and scalar multiplication. It therefore follows from Theorem 1.3 that $\mathsf{P}_n(F)$ is a subspace of $\mathsf{P}(F)$. ♦

Example 2

Let $\mathsf{C}(R)$ denote the set of all continuous real-valued functions defined on R. Clearly $\mathsf{C}(R)$ is a subset of the vector space $\mathcal{F}(R, R)$ defined in Example 3 of Section 1.2. We claim that $\mathsf{C}(R)$ is a subspace of $\mathcal{F}(R, R)$. First note that the zero of $\mathcal{F}(R, R)$ is the constant function defined by $f(t) = 0$ for all $t \in R$. Since constant functions are continuous, we have $f \in \mathsf{C}(R)$. Moreover, the sum of two continuous functions is continuous, and the product of a real number and a continuous function is continuous. So $\mathsf{C}(R)$ is closed under addition and scalar multiplication and hence is a subspace of $\mathcal{F}(R, R)$ by Theorem 1.3. ♦

Example 3

An $n \times n$ matrix M is called a **diagonal matrix** if $M_{ij} = 0$ whenever $i \neq j$, that is, if all its nondiagonal entries are zero. Clearly the zero matrix is a diagonal matrix because all of its entries are 0. Moreover, if A and B are diagonal $n \times n$ matrices, then whenever $i \neq j$,

$$(A+B)_{ij} = A_{ij} + B_{ij} = 0 + 0 = 0 \quad \text{and} \quad (cA)_{ij} = cA_{ij} = c0 = 0$$

for any scalar c. Hence $A + B$ and cA are diagonal matrices for any scalar c. Therefore the set of diagonal matrices is a subspace of $\mathsf{M}_{n \times n}(F)$ by Theorem 1.3. ♦

Example 4

The **trace** of an $n \times n$ matrix M, denoted $\operatorname{tr}(M)$, is the sum of the diagonal entries of M; that is,

$$\operatorname{tr}(M) = M_{11} + M_{22} + \cdots + M_{nn}.$$

It follows from Exercise 6 that the set of $n \times n$ matrices having trace equal to zero is a subspace of $\mathsf{M}_{n\times n}(F)$. ♦

Example 5

The set of matrices in $\mathsf{M}_{m\times n}(R)$ having nonnegative entries is not a subspace of $\mathsf{M}_{m\times n}(R)$ because it is not closed under scalar multiplication (by negative scalars). ♦

The next theorem shows how to form a new subspace from other subspaces.

Theorem 1.4. *Any intersection of subspaces of a vector space* V *is a subspace of* V.

Proof. Let $\mathcal{C}$ be a collection of subspaces of V, and let W denote the intersection of the subspaces in $\mathcal{C}$. Since every subspace contains the zero vector, $\mathit{0} \in$ W. Let $a \in F$ and $x, y \in$ W. Then x and y are contained in each subspace in $\mathcal{C}$. Because each subspace in $\mathcal{C}$ is closed under addition and scalar multiplication, it follows that $x+y$ and ax are contained in each subspace in $\mathcal{C}$. Hence $x+y$ and ax are also contained in W, so that W is a subspace of V by Theorem 1.3. ∎

Having shown that the intersection of subspaces of a vector space V is a subspace of V, it is natural to consider whether or not the union of subspaces of V is a subspace of V. It is easily seen that the union of subspaces must contain the zero vector and be closed under scalar multiplication, but in general the union of subspaces of V need not be closed under addition. In fact, it can be readily shown that the union of two subspaces of V is a subspace of V if and only if one of the subspaces contains the other. (See Exercise 19.) There is, however, a natural way to combine two subspaces W_1 and W_2 to obtain a subspace that contains both W_1 and W_2. As we already have suggested, the key to finding such a subspace is to assure that it must be closed under addition. This idea is explored in Exercise 23.

EXERCISES

1. Label the following statements as true or false.
   (a) If V is a vector space and W is a subset of V that is a vector space, then W is a subspace of V.
   (b) The empty set is a subspace of every vector space.
   (c) If V is a vector space other than the zero vector space, then V contains a subspace W such that W $\neq$ V.
   (d) The intersection of any two subsets of V is a subspace of V.

(e) An $n \times n$ diagonal matrix can never have more than n nonzero entries.
(f) The trace of a square matrix is the product of its diagonal entries.
(g) Let W be the xy-plane in R^3; that is, $\mathsf{W} = \{(a_1, a_2, 0) : a_1, a_2 \in R\}$. Then $\mathsf{W} = \mathsf{R}^2$.

2. Determine the transpose of each of the matrices that follow. In addition, if the matrix is square, compute its trace.

(a) $\begin{pmatrix} -4 & 2 \\ 5 & -1 \end{pmatrix}$ (b) $\begin{pmatrix} 0 & 8 & -6 \\ 3 & 4 & 7 \end{pmatrix}$

(c) $\begin{pmatrix} -3 & 9 \\ 0 & -2 \\ 6 & 1 \end{pmatrix}$ (d) $\begin{pmatrix} 10 & 0 & -8 \\ 2 & -4 & 3 \\ -5 & 7 & 6 \end{pmatrix}$

(e) $\begin{pmatrix} 1 & -1 & 3 & 5 \end{pmatrix}$ (f) $\begin{pmatrix} -2 & 5 & 1 & 4 \\ 7 & 0 & 1 & -6 \end{pmatrix}$

(g) $\begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix}$ (h) $\begin{pmatrix} -4 & 0 & 6 \\ 0 & 1 & -3 \\ 6 & -3 & 5 \end{pmatrix}$

3. Prove that $(aA + bB)^t = aA^t + bB^t$ for any $A, B \in \mathsf{M}_{m \times n}(F)$ and any $a, b \in F$.

4. Prove that $(A^t)^t = A$ for each $A \in \mathsf{M}_{m \times n}(F)$.

5. Prove that $A + A^t$ is symmetric for any square matrix A.

6. Prove that $\operatorname{tr}(aA + bB) = a\operatorname{tr}(A) + b\operatorname{tr}(B)$ for any $A, B \in \mathsf{M}_{n \times n}(F)$.

7. Prove that diagonal matrices are symmetric matrices.

8. Determine whether the following sets are subspaces of R^3 under the operations of addition and scalar multiplication defined on R^3. Justify your answers.

(a) $\mathsf{W}_1 = \{(a_1, a_2, a_3) \in \mathsf{R}^3 : a_1 = 3a_2 \text{ and } a_3 = -a_2\}$
(b) $\mathsf{W}_2 = \{(a_1, a_2, a_3) \in \mathsf{R}^3 : a_1 = a_3 + 2\}$
(c) $\mathsf{W}_3 = \{(a_1, a_2, a_3) \in \mathsf{R}^3 : 2a_1 - 7a_2 + a_3 = 0\}$
(d) $\mathsf{W}_4 = \{(a_1, a_2, a_3) \in \mathsf{R}^3 : a_1 - 4a_2 - a_3 = 0\}$
(e) $\mathsf{W}_5 = \{(a_1, a_2, a_3) \in \mathsf{R}^3 : a_1 + 2a_2 - 3a_3 = 1\}$
(f) $\mathsf{W}_6 = \{(a_1, a_2, a_3) \in \mathsf{R}^3 : 5a_1^2 - 3a_2^2 + 6a_3^2 = 0\}$

9. Let W_1, W_3, and W_4 be as in Exercise 8. Describe $\mathsf{W}_1 \cap \mathsf{W}_3$, $\mathsf{W}_1 \cap \mathsf{W}_4$, and $\mathsf{W}_3 \cap \mathsf{W}_4$, and observe that each is a subspace of R^3.

10. Prove that $\mathsf{W}_1 = \{(a_1, a_2, \ldots, a_n) \in \mathsf{F}^n \colon a_1 + a_2 + \cdots + a_n = 0\}$ is a subspace of F^n, but $\mathsf{W}_2 = \{(a_1, a_2, \ldots, a_n) \in \mathsf{F}^n \colon a_1 + a_2 + \cdots + a_n = 1\}$ is not.

11. Is the set $\mathsf{W} = \{f(x) \in \mathsf{P}(F) \colon f(x) = 0 \text{ or } f(x) \text{ has degree } n\}$ a subspace of $\mathsf{P}(F)$ if $n \geq 1$? Justify your answer.

12. An $m \times n$ matrix A is called **upper triangular** if all entries lying below the diagonal entries are zero, that is, if $A_{ij} = 0$ whenever $i > j$. Prove that the upper triangular matrices form a subspace of $\mathsf{M}_{m \times n}(F)$.

13. Let S be a nonempty set and F a field. Prove that for any $s_0 \in S$, $\{f \in \mathcal{F}(S, F) \colon f(s_0) = 0\}$, is a subspace of $\mathcal{F}(S, F)$.

14. Let S be a nonempty set and F a field. Let $\mathcal{C}(S, F)$ denote the set of all functions $f \in \mathcal{F}(S, F)$ such that $f(s) = 0$ for all but a finite number of elements of S. Prove that $\mathcal{C}(S, F)$ is a subspace of $\mathcal{F}(S, F)$.

15. Is the set of all differentiable real-valued functions defined on R a subspace of $\mathsf{C}(R)$? Justify your answer.

16. Let $\mathsf{C}^n(R)$ denote the set of all real-valued functions defined on the real line that have a continuous nth derivative. Prove that $\mathsf{C}^n(R)$ is a subspace of $\mathcal{F}(R, R)$.

17. Prove that a subset W of a vector space V is a subspace of V if and only if $\mathsf{W} \neq \varnothing$, and, whenever $a \in F$ and $x, y \in \mathsf{W}$, then $ax \in \mathsf{W}$ and $x + y \in \mathsf{W}$.

18. Prove that a subset W of a vector space V is a subspace of V if and only if $\mathit{0} \in \mathsf{W}$ and $ax + y \in \mathsf{W}$ whenever $a \in F$ and $x, y \in W$.

19. Let W_1 and W_2 be subspaces of a vector space V. Prove that $\mathsf{W}_1 \cup \mathsf{W}_2$ is a subspace of V if and only if $\mathsf{W}_1 \subseteq \mathsf{W}_2$ or $\mathsf{W}_2 \subseteq \mathsf{W}_1$.

20.† Prove that if W is a subspace of a vector space V and $w_1, w_2, \ldots, w_n$ are in W, then $a_1 w_1 + a_2 w_2 + \cdots + a_n w_n \in \mathsf{W}$ for any scalars $a_1, a_2, \ldots, a_n$.

21. Show that the set of convergent sequences $\{a_n\}$ (i.e., those for which $\lim_{n \to \infty} a_n$ exists) is a subspace of the vector space V in Exercise 20 of Section 1.2.

22. Let F_1 and F_2 be fields. A function $g \in \mathcal{F}(F_1, F_2)$ is called an **even function** if $g(-t) = g(t)$ for each $t \in F_1$ and is called an **odd function** if $g(-t) = -g(t)$ for each $t \in F_1$. Prove that the set of all even functions in $\mathcal{F}(F_1, F_2)$ and the set of all odd functions in $\mathcal{F}(F_1, F_2)$ are subspaces of $\mathcal{F}(F_1, F_2)$.

†A dagger means that this exercise is essential for a later section.

The following definitions are used in Exercises 23–30.

Definition. *If S_1 and S_2 are nonempty subsets of a vector space V, then the* **sum** *of S_1 and S_2, denoted S_1+S_2, is the set $\{x+y\colon x \in S_1 \text{ and } y \in S_2\}$.*

Definition. *A vector space V is called the* **direct sum** *of W_1 and W_2 if W_1 and W_2 are subspaces of V such that $\mathsf{W}_1 \cap \mathsf{W}_2 = \{0\}$ and $\mathsf{W}_1 + \mathsf{W}_2 = \mathsf{V}$. We denote that V is the direct sum of W_1 and W_2 by writing $\mathsf{V} = \mathsf{W}_1 \oplus \mathsf{W}_2$.*

23. Let W_1 and W_2 be subspaces of a vector space V.
  (a) Prove that $\mathsf{W}_1 + \mathsf{W}_2$ is a subspace of V that contains both W_1 and W_2.
  (b) Prove that any subspace of V that contains both W_1 and W_2 must also contain $\mathsf{W}_1 + \mathsf{W}_2$.

24. Show that F^n is the direct sum of the subspaces

$$\mathsf{W}_1 = \{(a_1, a_2, \ldots, a_n) \in \mathsf{F}^n \colon a_n = 0\}$$

and

$$\mathsf{W}_2 = \{(a_1, a_2, \ldots, a_n) \in \mathsf{F}^n \colon a_1 = a_2 = \cdots = a_{n-1} = 0\}.$$

25. Let W_1 denote the set of all polynomials $f(x)$ in $\mathsf{P}(F)$ such that in the representation

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$

we have $a_i = 0$ whenever i is even. Likewise let W_2 denote the set of all polynomials $g(x)$ in $\mathsf{P}(F)$ such that in the representation

$$g(x) = b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0,$$

we have $b_i = 0$ whenever i is odd. Prove that $\mathsf{P}(F) = \mathsf{W}_1 \oplus \mathsf{W}_2$.

26. In $\mathsf{M}_{m\times n}(F)$ define $\mathsf{W}_1 = \{A \in \mathsf{M}_{m\times n}(F)\colon A_{ij} = 0 \text{ whenever } i > j\}$ and $\mathsf{W}_2 = \{A \in \mathsf{M}_{m\times n}(F)\colon A_{ij} = 0 \text{ whenever } i \leq j\}$. ($\mathsf{W}_1$ is the set of all upper triangular matrices defined in Exercise 12.) Show that $\mathsf{M}_{m\times n}(F) = \mathsf{W}_1 \oplus \mathsf{W}_2$.

27. Let V denote the vector space consisting of all upper triangular $n \times n$ matrices (as defined in Exercise 12), and let W_1 denote the subspace of V consisting of all diagonal matrices. Show that $\mathsf{V} = \mathsf{W}_1 \oplus \mathsf{W}_2$, where $\mathsf{W}_2 = \{A \in \mathsf{V}\colon A_{ij} = 0 \text{ whenever } i \geq j\}$.

28. A matrix M is called **skew-symmetric** if $M^t = -M$. Clearly, a skew-symmetric matrix is square. Let F be a field. Prove that the set W_1 of all skew-symmetric $n \times n$ matrices with entries from F is a subspace of $\mathsf{M}_{n\times n}(F)$. Now assume that F is not of characteristic 2 (see Appendix C), and let W_2 be the subspace of $\mathsf{M}_{n\times n}(F)$ consisting of all symmetric $n \times n$ matrices. Prove that $\mathsf{M}_{n\times n}(F) = \mathsf{W}_1 \oplus \mathsf{W}_2$.

29. Let F be a field that is not of characteristic 2. Define

$$\mathsf{W}_1 = \{A \in \mathsf{M}_{n\times n}(F)\colon A_{ij} = 0 \text{ whenever } i \leq j\}$$

and W_2 to be the set of all symmetric $n \times n$ matrices with entries from F. Both W_1 and W_2 are subspaces of $\mathsf{M}_{n\times n}(F)$. Prove that $\mathsf{M}_{n\times n}(F) = \mathsf{W}_1 \oplus \mathsf{W}_2$. Compare this exercise with Exercise 28.

30. Let W_1 and W_2 be subspaces of a vector space V. Prove that V is the direct sum of W_1 and W_2 if and only if each vector in V can be *uniquely* written as $x_1 + x_2$, where $x_1 \in \mathsf{W}_1$ and $x_2 \in \mathsf{W}_2$.

31. Let W be a subspace of a vector space V over a field F. For any $v \in \mathsf{V}$ the set $\{v\} + \mathsf{W} = \{v + w\colon w \in \mathsf{W}\}$ is called the **coset of W containing** v. It is customary to denote this coset by $v + \mathsf{W}$ rather than $\{v\} + \mathsf{W}$.
   (a) Prove that $v + \mathsf{W}$ is a subspace of V if and only if $v \in \mathsf{W}$.
   (b) Prove that $v_1 + \mathsf{W} = v_2 + \mathsf{W}$ if and only if $v_1 - v_2 \in \mathsf{W}$.

   Addition and scalar multiplication by scalars of F can be defined in the collection $S = \{v + \mathsf{W}\colon v \in \mathsf{V}\}$ of all cosets of W as follows:

$$(v_1 + \mathsf{W}) + (v_2 + \mathsf{W}) = (v_1 + v_2) + \mathsf{W}$$

   for all $v_1, v_2 \in \mathsf{V}$ and

$$a(v + \mathsf{W}) = av + \mathsf{W}$$

   for all $v \in \mathsf{V}$ and $a \in F$.

   (c) Prove that the preceding operations are well defined; that is, show that if $v_1 + \mathsf{W} = v_1' + \mathsf{W}$ and $v_2 + \mathsf{W} = v_2' + \mathsf{W}$, then

$$(v_1 + \mathsf{W}) + (v_2 + \mathsf{W}) = (v_1' + \mathsf{W}) + (v_2' + \mathsf{W})$$

   and

$$a(v_1 + \mathsf{W}) = a(v_1' + \mathsf{W})$$

   for all $a \in F$.
   (d) Prove that the set S is a vector space with the operations defined in (c). This vector space is called the **quotient space of V modulo W** and is denoted by V/W.

1.4 LINEAR COMBINATIONS AND SYSTEMS OF LINEAR EQUATIONS

In Section 1.1, it was shown that the equation of the plane through three noncollinear points A, B, and C in space is $x = A + su + tv$, where u and v denote the vectors beginning at A and ending at B and C, respectively, and s and t denote arbitrary real numbers. An important special case occurs when A is the origin. In this case, the equation of the plane simplifies to $x = su + tv$, and the set of all points in this plane is a subspace of R^3. (This is proved as Theorem 1.5.) Expressions of the form $su + tv$, where s and t are scalars and u and v are vectors, play a central role in the theory of vector spaces. The appropriate generalization of such expressions is presented in the following definitions.

Definitions. *Let* V *be a vector space and* S *a nonempty subset of* V. *A vector* $v \in \mathsf{V}$ *is called a* ***linear combination*** *of vectors of* S *if there exist a finite number of vectors* $u_1, u_2, \ldots, u_n$ *in* S *and scalars* $a_1, a_2, \ldots, a_n$ *in* F *such that* $v = a_1u_1 + a_2u_2 + \cdots + a_nu_n$. *In this case we also say that* v *is a linear combination of* $u_1, u_2, \ldots, u_n$ *and call* $a_1, a_2, \ldots, a_n$ *the* ***coefficients*** *of the linear combination.*

Observe that in any vector space V, $0v = \mathit{0}$ for each $v \in \mathsf{V}$. Thus the zero vector is a linear combination of any nonempty subset of V.

Example 1

TABLE 1.1 Vitamin Content of 100 Grams of Certain Foods

	A (units)	B_1 (mg)	B_2 (mg)	Niacin (mg)	C (mg)
Apple butter	0	0.01	0.02	0.2	2
Raw, unpared apples (freshly harvested)	90	0.03	0.02	0.1	4
Chocolate-coated candy with coconut center	0	0.02	0.07	0.2	0
Clams (meat only)	100	0.10	0.18	1.3	10
Cupcake from mix (dry form)	0	0.05	0.06	0.3	0
Cooked farina (unenriched)	(0)[a]	0.01	0.01	0.1	(0)
Jams and preserves	10	0.01	0.03	0.2	2
Coconut custard pie (baked from mix)	0	0.02	0.02	0.4	0
Raw brown rice	(0)	0.34	0.05	4.7	(0)
Soy sauce	0	0.02	0.25	0.4	0
Cooked spaghetti (unenriched)	0	0.01	0.01	0.3	0
Raw wild rice	(0)	0.45	0.63	6.2	(0)

Source: Bernice K. Watt and Annabel L. Merrill, *Composition of Foods* (Agriculture Handbook Number 8), Consumer and Food Economics Research Division, U.S. Department of Agriculture, Washington, D.C., 1963.

[a]Zeros in parentheses indicate that the amount of a vitamin present is either none or too small to measure.

Table 1.1 shows the vitamin content of 100 grams of 12 foods with respect to vitamins **A**, $\mathbf{B}_1$ (thiamine), $\mathbf{B}_2$ (riboflavin), niacin, and **C** (ascorbic acid).

The vitamin content of 100 grams of each food can be recorded as a column vector in R^5 for example, the vitamin vector for apple butter is

$$\begin{pmatrix} 0.00 \\ 0.01 \\ 0.02 \\ 0.20 \\ 2.00 \end{pmatrix}.$$

Considering the vitamin vectors for cupcake, coconut custard pie, raw brown rice, soy sauce, and wild rice, we see that

$$\begin{pmatrix} 0.00 \\ 0.05 \\ 0.06 \\ 0.30 \\ 0.00 \end{pmatrix} + \begin{pmatrix} 0.00 \\ 0.02 \\ 0.02 \\ 0.40 \\ 0.00 \end{pmatrix} + \begin{pmatrix} 0.00 \\ 0.34 \\ 0.05 \\ 4.70 \\ 0.00 \end{pmatrix} + 2\begin{pmatrix} 0.00 \\ 0.02 \\ 0.25 \\ 0.40 \\ 0.00 \end{pmatrix} = \begin{pmatrix} 0.00 \\ 0.45 \\ 0.63 \\ 6.20 \\ 0.00 \end{pmatrix}.$$

Thus the vitamin vector for wild rice is a linear combination of the vitamin vectors for cupcake, coconut custard pie, raw brown rice, and soy sauce. So 100 grams of cupcake, 100 grams of coconut custard pie, 100 grams of raw brown rice, and 200 grams of soy sauce provide exactly the same amounts of the five vitamins as 100 grams of raw wild rice. Similarly, since

$$2\begin{pmatrix} 0.00 \\ 0.01 \\ 0.02 \\ 0.20 \\ 2.00 \end{pmatrix} + \begin{pmatrix} 90.00 \\ 0.03 \\ 0.02 \\ 0.10 \\ 4.00 \end{pmatrix} + \begin{pmatrix} 0.00 \\ 0.02 \\ 0.07 \\ 0.20 \\ 0.00 \end{pmatrix} + \begin{pmatrix} 0.00 \\ 0.01 \\ 0.01 \\ 0.10 \\ 0.00 \end{pmatrix} + \begin{pmatrix} 10.00 \\ 0.01 \\ 0.03 \\ 0.20 \\ 2.00 \end{pmatrix} + \begin{pmatrix} 0.00 \\ 0.01 \\ 0.01 \\ 0.30 \\ 0.00 \end{pmatrix} = \begin{pmatrix} 100.00 \\ 0.10 \\ 0.18 \\ 1.30 \\ 10.00 \end{pmatrix},$$

200 grams of apple butter, 100 grams of apples, 100 grams of chocolate candy, 100 grams of farina, 100 grams of jam, and 100 grams of spaghetti provide exactly the same amounts of the five vitamins as 100 grams of clams. ♦

Throughout Chapters 1 and 2 we encounter many different situations in which it is necessary to determine whether or not a vector can be expressed as a linear combination of other vectors, and if so, how. This question often reduces to the problem of solving a system of linear equations. In Chapter 3, we discuss a general method for using matrices to solve any system of linear equations. For now, we illustrate how to solve a system of linear equations by showing how to determine if the vector $(2, 6, 8)$ can be expressed as a linear combination of

$$u_1 = (1, 2, 1), \quad u_2 = (-2, -4, -2), \quad u_3 = (0, 2, 3),$$

$$u_4 = (2, 0, -3), \quad \text{and} \quad u_5 = (-3, 8, 16).$$

Thus we must determine if there are scalars a_1, a_2, a_3, a_4, and a_5 such that

$$\begin{aligned}(2,6,8) &= a_1u_1 + a_2u_2 + a_3u_3 + a_4u_4 + a_5u_5 \\ &= a_1(1,2,1) + a_2(-2,-4,-2) + a_3(0,2,3) \\ &\qquad\qquad\qquad\qquad + a_4(2,0,-3) + a_5(-3,8,16) \\ &= (a_1 - 2a_2 + 2a_4 - 3a_5, 2a_1 - 4a_2 + 2a_3 + 8a_5, \\ &\qquad\qquad\qquad\qquad a_1 - 2a_2 + 3a_3 - 3a_4 + 16a_5).\end{aligned}$$

Hence $(2, 6, 8)$ can be expressed as a linear combination of u_1, u_2, u_3, u_4, and u_5 if and only if there is a 5-tuple of scalars $(a_1, a_2, a_3, a_4, a_5)$ satisfying the system of linear equations

$$\begin{aligned} a_1 - 2a_2 \qquad\quad + 2a_4 - \ \ 3a_5 &= 2 \\ 2a_1 - 4a_2 + 2a_3 \qquad\quad + \ \ 8a_5 &= 6 \\ a_1 - 2a_2 + 3a_3 - 3a_4 + 16a_5 &= 8, \end{aligned} \tag{1}$$

which is obtained by equating the corresponding coordinates in the preceding equation.

To solve system (1), *we replace it by another system with the same solutions, but which is easier to solve.* The procedure to be used expresses some of the unknowns in terms of others by eliminating certain unknowns from all the equations except one. To begin, we eliminate a_1 from every equation except the first by adding -2 times the first equation to the second and -1 times the first equation to the third. The result is the following new system:

$$\begin{aligned} a_1 - 2a_2 \qquad + 2a_4 - \ \ 3a_5 &= 2 \\ 2a_3 - 4a_4 + 14a_5 &= 2 \\ 3a_3 - 5a_4 + 19a_5 &= 6. \end{aligned} \tag{2}$$

In this case, it happened that while eliminating a_1 from every equation except the first, we also eliminated a_2 from every equation except the first. This need not happen in general. We now want to make the coefficient of a_3 in the second equation equal to 1, and then eliminate a_3 from the third equation. To do this, we first multiply the second equation by $\frac{1}{2}$, which produces

$$\begin{aligned} a_1 - 2a_2 \qquad + 2a_4 - \ \ 3a_5 &= 2 \\ a_3 - 2a_4 + \ \ 7a_5 &= 1 \\ 3a_3 - 5a_4 + 19a_5 &= 6. \end{aligned}$$

Next we add -3 times the second equation to the third, obtaining

$$\begin{aligned} a_1 - 2a_2 \qquad + 2a_4 - 3a_5 &= 2 \\ a_3 - 2a_4 + 7a_5 &= 1 \\ a_4 - 2a_5 &= 3. \end{aligned} \tag{3}$$

We continue by eliminating a_4 from every equation of (3) except the third. This yields

$$\begin{aligned} a_1 - 2a_2 \qquad\qquad &+ \ a_5 = -4 \\ a_3 \qquad &+ 3a_5 = \ \ 7 \\ a_4 &- 2a_5 = \ \ 3. \end{aligned} \tag{4}$$

System (4) is a system of the desired form: It is easy to solve for the first unknown present in each of the equations (a_1, a_3, and a_4) in terms of the other unknowns (a_2 and a_5). Rewriting system (4) in this form, we find that

$$\begin{aligned} a_1 &= 2a_2 - \ \ a_5 - 4 \\ a_3 &= \qquad - 3a_5 + 7 \\ a_4 &= \qquad\ \ \ 2a_5 + 3. \end{aligned}$$

Thus for any choice of scalars a_2 and a_5, a vector of the form

$$(a_1, a_2, a_3, a_4, a_5) = (2a_2 - a_5 - 4, a_2, -3a_5 + 7, 2a_5 + 3, a_5)$$

is a solution to system (1). In particular, the vector $(-4, 0, 7, 3, 0)$ obtained by setting $a_2 = 0$ and $a_5 = 0$ is a solution to (1). Therefore

$$(2, 6, 8) = -4u_1 + 0u_2 + 7u_3 + 3u_4 + 0u_5,$$

so that $(2, 6, 8)$ is a linear combination of u_1, u_2, u_3, u_4, and u_5.

The procedure just illustrated uses three types of operations to simplify the original system:

1. interchanging the order of any two equations in the system;
2. multiplying any equation in the system by a *nonzero* constant;
3. adding a constant multiple of any equation to another equation in the system.

In Section 3.4, we prove that these operations do not change the set of solutions to the original system. Note that we employed these operations to obtain a system of equations that had the following properties:

1. The first nonzero coefficient in each equation is one.
2. If an unknown is the first unknown with a nonzero coefficient in some equation, then that unknown occurs with a zero coefficient in each of the other equations.
3. The first unknown with a nonzero coefficient in any equation has a larger subscript than the first unknown with a nonzero coefficient in any preceding equation.

To help clarify the meaning of these properties, note that none of the following systems meets these requirements.

$$\begin{aligned} x_1 + 3x_2 \qquad\quad + \ x_4 &= \ \ 7 \\ 2x_3 - 5x_4 &= -1 \end{aligned} \tag{5}$$

$$\begin{aligned} x_1 - 2x_2 + 3x_3 \qquad + \ x_5 &= -5 \\ x_3 \qquad - 2x_5 &= \ \ 9 \\ x_4 + 3x_5 &= \ \ 6 \end{aligned} \tag{6}$$

$$\begin{aligned} x_1 \qquad - 2x_3 \qquad + \ x_5 &= 1 \\ x_4 - 6x_5 &= 0 \\ x_2 + 5x_3 \qquad - 3x_5 &= 2. \end{aligned} \tag{7}$$

Specifically, system (5) does not satisfy property 1 because the first nonzero coefficient in the second equation is 2; system (6) does not satisfy property 2 because x_3, the first unknown with a nonzero coefficient in the second equation, occurs with a nonzero coefficient in the first equation; and system (7) does not satisfy property 3 because x_2, the first unknown with a nonzero coefficient in the third equation, does not have a larger subscript than x_4, the first unknown with a nonzero coefficient in the second equation.

Once a system with properties 1, 2, and 3 has been obtained, it is easy to solve for some of the unknowns in terms of the others (as in the preceding example). *If, however, in the course of using operations* 1, 2, *and* 3 *a system containing an equation of the form* $0 = c$, *where* c *is nonzero, is obtained, then the original system has no solutions.* (See Example 2.)

We return to the study of systems of linear equations in Chapter 3. We discuss there the theoretical basis for this method of solving systems of linear equations and further simplify the procedure by use of matrices.

Example 2

We claim that

$$2x^3 - 2x^2 + 12x - 6$$

is a linear combination of

$$x^3 - 2x^2 - 5x - 3 \quad \text{and} \quad 3x^3 - 5x^2 - 4x - 9$$

in $\mathsf{P}_3(R)$, but that

$$3x^3 - 2x^2 + 7x + 8$$

is not. In the first case we wish to find scalars a and b such that

$$2x^3 - 2x^2 + 12x - 6 = a(x^3 - 2x^2 - 5x - 3) + b(3x^3 - 5x^2 - 4x - 9)$$

$$= (a + 3b)x^3 + (-2a - 5b)x^2 + (-5a - 4b)x + (-3a - 9b).$$

Thus we are led to the following system of linear equations:

$$\begin{aligned} a + 3b &= 2 \\ -2a - 5b &= -2 \\ -5a - 4b &= 12 \\ -3a - 9b &= -6. \end{aligned}$$

Adding appropriate multiples of the first equation to the others in order to eliminate a, we find that

$$\begin{aligned} a + 3b &= 2 \\ b &= 2 \\ 11b &= 22 \\ 0b &= 0. \end{aligned}$$

Now adding the appropriate multiples of the second equation to the others yields

$$\begin{aligned} a &= -4 \\ b &= 2 \\ 0 &= 0 \\ 0 &= 0. \end{aligned}$$

Hence

$$2x^3 - 2x^2 + 12x - 6 = -4(x^3 - 2x^2 - 5x - 3) + 2(3x^3 - 5x^2 - 4x - 9).$$

In the second case, we wish to show that there are no scalars a and b for which

$$3x^3 - 2x^2 + 7x + 8 = a(x^3 - 2x^2 - 5x - 3) + b(3x^3 - 5x^2 - 4x - 9).$$

Using the preceding technique, we obtain a system of linear equations

$$\begin{aligned} a + 3b &= 3 \\ -2a - 5b &= -2 \\ -5a - 4b &= 7 \\ -3a - 9b &= 8. \end{aligned} \tag{8}$$

Eliminating a as before yields

$$\begin{aligned} a + 3b &= 3 \\ b &= 4 \\ 11b &= 22 \\ 0 &= 17. \end{aligned}$$

But the presence of the inconsistent equation $0 = 17$ indicates that (8) has no solutions. Hence $3x^3 - 2x^2 + 7x + 8$ is not a linear combination of $x^3 - 2x^2 - 5x - 3$ and $3x^3 - 5x^2 - 4x - 9$. ♦

Throughout this book, we form the set of all linear combinations of some set of vectors. We now name such a set of linear combinations.

Definition. *Let S be a nonempty subset of a vector space* V. *The* **span** *of S, denoted* $\mathrm{span}(S)$, *is the set consisting of all linear combinations of the vectors in S. For convenience, we define* $\mathrm{span}(\varnothing) = \{0\}$.

In R^3, for instance, the span of the set $\{(1,0,0),(0,1,0)\}$ consists of all vectors in R^3 that have the form $a(1,0,0)+b(0,1,0) = (a,b,0)$ for some scalars a and b. Thus the span of $\{(1,0,0),(0,1,0)\}$ contains all the points in the xy-plane. In this case, the span of the set is a subspace of R^3. This fact is true in general.

Theorem 1.5. *The span of any subset S of a vector space* V *is a subspace of* V. *Moreover, any subspace of* V *that contains S must also contain the span of S.*

Proof. This result is immediate if $S = \varnothing$ because $\mathrm{span}(\varnothing) = \{0\}$, which is a subspace that is contained in any subspace of V.

If $S \neq \varnothing$, then S contains a vector z. So $0z = 0$ is in $\mathrm{span}(S)$. Let $x, y \in \mathrm{span}(S)$. Then there exist vectors $u_1, u_2, \ldots, u_m$, $v_1, v_2, \ldots, v_n$ in S and scalars $a_1, a_2, \ldots, a_m$, $b_1, b_2, \ldots, b_n$ such that

$$x = a_1u_1 + a_2u_2 + \cdots + a_mu_m \quad \text{and} \quad y = b_1v_1 + b_2v_2 + \cdots + b_nv_n.$$

Then

$$x + y = a_1u_1 + a_2u_2 + \cdots + a_mu_m + b_1v_1 + b_2v_2 + \cdots + b_nv_n$$

and, for any scalar c,

$$cx = (ca_1)u_1 + (ca_2)u_2 + \cdots + (ca_m)u_m$$

are clearly linear combinations of the vectors in S; so $x + y$ and cx are in $\mathrm{span}(S)$. Thus $\mathrm{span}(S)$ is a subspace of V.

Now let W denote any subspace of V that contains S. If $w \in \mathrm{span}(S)$, then w has the form $w = c_1w_1 + c_2w_2 + \cdots + c_kw_k$ for some vectors $w_1, w_2, \ldots, w_k$ in S and some scalars $c_1, c_2, \ldots, c_k$. Since $S \subseteq \mathsf{W}$, we have $w_1, w_2, \ldots, w_k \in \mathsf{W}$. Therefore $w = c_1w_1 + c_2w_2 + \cdots + c_kw_k$ is in W by Exercise 20 of Section 1.3. Because w, an arbitrary vector in $\mathrm{span}(S)$, belongs to W, it follows that $\mathrm{span}(S) \subseteq \mathsf{W}$. ∎

Definition. *A subset S of a vector space* V **generates** *(or* **spans***)* V *if* $\mathrm{span}(S) = \mathsf{V}$. *In this case, we also say that the vectors of S generate (or span)* V.

Example 3

The vectors $(1,1,0), (1,0,1)$, and $(0,1,1)$ generate R^3 since an arbitrary vector (a_1, a_2, a_3) in R^3 is a linear combination of the three given vectors; in fact, the scalars r, s, and t for which

$$r(1,1,0) + s(1,0,1) + t(0,1,1) = (a_1, a_2, a_3)$$

are

$$r = \frac{1}{2}(a_1 + a_2 - a_3),\ s = \frac{1}{2}(a_1 - a_2 + a_3), \quad \text{and} \quad t = \frac{1}{2}(-a_1 + a_2 + a_3). \ \blacklozenge$$

Example 4

The polynomials $x^2 + 3x - 2$, $2x^2 + 5x - 3$, and $-x^2 - 4x + 4$ generate $\mathsf{P}_2(R)$ since each of the three given polynomials belongs to $\mathsf{P}_2(R)$ and each polynomial $ax^2 + bx + c$ in $\mathsf{P}_2(R)$ is a linear combination of these three, namely,

$$(-8a + 5b + 3c)(x^2 + 3x - 2) + (4a - 2b - c)(2x^2 + 5x - 3)$$

$$+(-a + b + c)(-x^2 - 4x + 4) = ax^2 + bx + c. \qquad \blacklozenge$$

Example 5

The matrices

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

generate $\mathsf{M}_{2\times 2}(R)$ since an arbitrary matrix A in $\mathsf{M}_{2\times 2}(R)$ can be expressed as a linear combination of the four given matrices as follows:

$$\begin{aligned} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} &= (\frac{1}{3}a_{11} + \frac{1}{3}a_{12} + \frac{1}{3}a_{21} - \frac{2}{3}a_{22}) \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \\ &+ (\frac{1}{3}a_{11} + \frac{1}{3}a_{12} - \frac{2}{3}a_{21} + \frac{1}{3}a_{22}) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \\ &+ (\frac{1}{3}a_{11} - \frac{2}{3}a_{12} + \frac{1}{3}a_{21} + \frac{1}{3}a_{22}) \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \\ &+ (-\frac{2}{3}a_{11} + \frac{1}{3}a_{12} + \frac{1}{3}a_{21} + \frac{1}{3}a_{22}) \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}. \end{aligned}$$

On the other hand, the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

do not generate $M_{2\times 2}(R)$ because each of these matrices has equal diagonal entries. So any linear combination of these matrices has equal diagonal entries. Hence not every 2×2 matrix is a linear combination of these three matrices. ♦

At the beginning of this section we noted that the equation of a plane through three noncollinear points in space, one of which is the origin, is of the form $x = su + tv$, where $u, v \in \mathsf{R}^3$ and s and t are scalars. Thus $x \in \mathsf{R}^3$ is a linear combination of $u, v \in \mathsf{R}^3$ if and only if x lies in the plane containing u and v. (See Figure 1.5.)

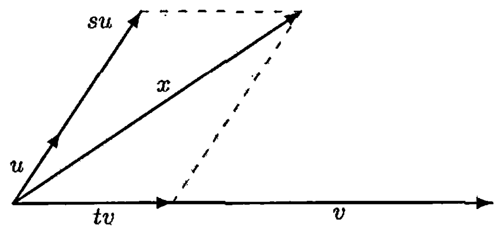

Figure 1.5

Usually there are many different subsets that generate a subspace W. (See Exercise 13.) It is natural to seek a subset of W that generates W and is as small as possible. In the next section we explore the circumstances under which a vector can be removed from a generating set to obtain a smaller generating set.

EXERCISES

1. Label the following statements as true or false.

   (a) The zero vector is a linear combination of any nonempty set of vectors.
   (b) The span of $\varnothing$ is $\varnothing$.
   (c) If S is a subset of a vector space V, then span(S) equals the intersection of all subspaces of V that contain S.
   (d) In solving a system of linear equations, it is permissible to multiply an equation by any constant.
   (e) In solving a system of linear equations, it is permissible to add any multiple of one equation to another.
   (f) Every system of linear equations has a solution.

2. Solve the following systems of linear equations by the method introduced in this section.

(a)
$$\begin{aligned} 2x_1 - 2x_2 - 3x_3 &= -2 \\ 3x_1 - 3x_2 - 2x_3 + 5x_4 &= 7 \\ x_1 - x_2 - 2x_3 - x_4 &= -3 \end{aligned}$$

(b)
$$\begin{aligned} 3x_1 - 7x_2 + 4x_3 &= 10 \\ x_1 - 2x_2 + x_3 &= 3 \\ 2x_1 - x_2 - 2x_3 &= 6 \end{aligned}$$

(c)
$$\begin{aligned} x_1 + 2x_2 - x_3 + x_4 &= 5 \\ x_1 + 4x_2 - 3x_3 - 3x_4 &= 6 \\ 2x_1 + 3x_2 - x_3 + 4x_4 &= 8 \end{aligned}$$

(d)
$$\begin{aligned} x_1 + 2x_2 + 2x_3 &= 2 \\ x_1 + 8x_3 + 5x_4 &= -6 \\ x_1 + x_2 + 5x_3 + 5x_4 &= 3 \end{aligned}$$

(e)
$$\begin{aligned} x_1 + 2x_2 - 4x_3 - x_4 + x_5 &= 7 \\ -x_1 + 10x_3 - 3x_4 - 4x_5 &= -16 \\ 2x_1 + 5x_2 - 5x_3 - 4x_4 - x_5 &= 2 \\ 4x_1 + 11x_2 - 7x_3 - 10x_4 - 2x_5 &= 7 \end{aligned}$$

(f)
$$\begin{aligned} x_1 + 2x_2 + 6x_3 &= -1 \\ 2x_1 + x_2 + x_3 &= 8 \\ 3x_1 + x_2 - x_3 &= 15 \\ x_1 + 3x_2 + 10x_3 &= -5 \end{aligned}$$

3. For each of the following lists of vectors in R^3, determine whether the first vector can be expressed as a linear combination of the other two.

(a) $(-2,0,3), (1,3,0), (2,4,-1)$
(b) $(1,2,-3), (-3,2,1), (2,-1,-1)$
(c) $(3,4,1), (1,-2,1), (-2,-1,1)$
(d) $(2,-1,0), (1,2,-3), (1,-3,2)$
(e) $(5,1,-5), (1,-2,-3), (-2,3,-4)$
(f) $(-2,2,2), (1,2,-1), (-3,-3,3)$

4. For each list of polynomials in $\mathsf{P}_3(R)$, determine whether the first polynomial can be expressed as a linear combination of the other two.

(a) $x^3 - 3x + 5, x^3 + 2x^2 - x + 1, x^3 + 3x^2 - 1$
(b) $4x^3 + 2x^2 - 6, x^3 - 2x^2 + 4x + 1, 3x^3 - 6x^2 + x + 4$
(c) $-2x^3 - 11x^2 + 3x + 2, x^3 - 2x^2 + 3x - 1, 2x^3 + x^2 + 3x - 2$
(d) $x^3 + x^2 + 2x + 13, 2x^3 - 3x^2 + 4x + 1, x^3 - x^2 + 2x + 3$
(e) $x^3 - 8x^2 + 4x, x^3 - 2x^2 + 3x - 1, x^3 - 2x + 3$
(f) $6x^3 - 3x^2 + x + 2, x^3 - x^2 + 2x + 3, 2x^3 - 3x + 1$

5. In each part, determine whether the given vector is in the span of S.

(a) $(2,-1,1)$, $S=\{(1,0,2),(-1,1,1)\}$
(b) $(-1,2,1)$, $S=\{(1,0,2),(-1,1,1)\}$
(c) $(-1,1,1,2)$, $S=\{(1,0,1,-1),(0,1,1,1)\}$
(d) $(2,-1,1,-3)$, $S=\{(1,0,1,-1),(0,1,1,1)\}$
(e) $-x^3+2x^2+3x+3$, $S=\{x^3+x^2+x+1, x^2+x+1, x+1\}$
(f) $2x^3-x^2+x+3$, $S=\{x^3+x^2+x+1, x^2+x+1, x+1\}$
(g) $\begin{pmatrix} 1 & 2 \\ -3 & 4 \end{pmatrix}$, $S=\left\{\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\right\}$
(h) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $S=\left\{\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\right\}$

6. Show that the vectors $(1,1,0)$, $(1,0,1)$, and $(0,1,1)$ generate F^3.

7. In F^n, let e_j denote the vector whose jth coordinate is 1 and whose other coordinates are 0. Prove that $\{e_1, e_2, \ldots, e_n\}$ generates F^n.

8. Show that $\mathsf{P}_n(F)$ is generated by $\{1, x, \ldots, x^n\}$.

9. Show that the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

generate $\mathsf{M}_{2\times 2}(F)$.

10. Show that if

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad M_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

then the span of $\{M_1, M_2, M_3\}$ is the set of all symmetric 2×2 matrices.

11.† Prove that $\operatorname{span}(\{x\}) = \{ax \colon a \in F\}$ for any vector x in a vector space. Interpret this result geometrically in R^3.

12. Show that a subset W of a vector space V is a subspace of V if and only if $\operatorname{span}(\mathsf{W}) = \mathsf{W}$.

13.† Show that if S_1 and S_2 are subsets of a vector space V such that $S_1 \subseteq S_2$, then $\operatorname{span}(S_1) \subseteq \operatorname{span}(S_2)$. In particular, if $S_1 \subseteq S_2$ and $\operatorname{span}(S_1) = \mathsf{V}$, deduce that $\operatorname{span}(S_2) = \mathsf{V}$.

14. Show that if S_1 and S_2 are arbitrary subsets of a vector space V, then $\operatorname{span}(S_1 \cup S_2) = \operatorname{span}(S_1) + \operatorname{span}(S_2)$. (The sum of two subsets is defined in the exercises of Section 1.3.)

15. Let S_1 and S_2 be subsets of a vector space V. Prove that $\text{span}(S_1 \cap S_2) \subseteq \text{span}(S_1) \cap \text{span}(S_2)$. Give an example in which $\text{span}(S_1 \cap S_2)$ and $\text{span}(S_1) \cap \text{span}(S_2)$ are equal and one in which they are unequal.

16. Let V be a vector space and S a subset of V with the property that whenever $v_1, v_2, \ldots, v_n \in S$ and $a_1v_1 + a_2v_2 + \cdots + a_nv_n = 0$, then $a_1 = a_2 = \cdots = a_n = 0$. Prove that every vector in the span of S can be *uniquely* written as a linear combination of vectors of S.

17. Let W be a subspace of a vector space V. Under what conditions are there only a finite number of distinct subsets S of W such that S generates W?

1.5 LINEAR DEPENDENCE AND LINEAR INDEPENDENCE

Suppose that V is a vector space over an infinite field and that W is a subspace of V. Unless W is the zero subspace, W is an infinite set. It is desirable to find a "small" finite subset S that generates W because we can then describe each vector in W as a linear combination of the finite number of vectors in S. Indeed, the smaller that S is, the fewer computations that are required to represent vectors in W. Consider, for example, the subspace W of R^3 generated by $S = \{u_1, u_2, u_3, u_4\}$, where $u_1 = (2, -1, 4)$, $u_2 = (1, -1, 3)$, $u_3 = (1, 1, -1)$, and $u_4 = (1, -2, -1)$. Let us attempt to find a proper subset of S that also generates W. The search for this subset is related to the question of whether or not some vector in S is a linear combination of the other vectors in S. Now u_4 is a linear combination of the other vectors in S if and only if there are scalars a_1, a_2, and a_3 such that

$$u_4 = a_1u_1 + a_2u_2 + a_3u_3,$$

that is, if and only if there are scalars a_1, a_2, and a_3 satisfying

$$(1, -2, -1) = (2a_1 + a_2 + a_3, -a_1 - a_2 + a_3, 4a_1 + 3a_2 - a_3).$$

Thus u_4 is a linear combination of u_1, u_2, and u_3 if and only if the system of linear equations

$$\begin{aligned} 2a_1 + a_2 + a_3 &= 1 \\ -a_1 - a_2 + a_3 &= -2 \\ 4a_1 + 3a_2 - a_3 &= -1 \end{aligned}$$

has a solution. The reader should verify that no such solution exists. This does not, however, answer our question of whether *some* vector in S is a linear combination of the other vectors in S. It can be shown, in fact, that u_3 is a linear combination of u_1, u_2, and u_4, namely, $u_3 = 2u_1 - 3u_2 + 0u_4$.

In the preceding example, checking that some vector in S is a linear combination of the other vectors in S could require that we solve several different systems of linear equations before we determine which, if any, of u_1, u_2, u_3, and u_4 is a linear combination of the others. By formulating our question differently, we can save ourselves some work. Note that since $u_3 = 2u_1 - 3u_2 + 0u_4$, we have

$$-2u_1 + 3u_2 + u_3 - 0u_4 = \mathit{0}.$$

That is, because some vector in S is a linear combination of the others, the zero vector can be expressed as a linear combination of the vectors in S using coefficients that are not all zero. The converse of this statement is also true: If the zero vector can be written as a linear combination of the vectors in S in which not all the coefficients are zero, then some vector in S is a linear combination of the others. For instance, in the example above, the equation $-2u_1 + 3u_2 + u_3 - 0u_4 = \mathit{0}$ can be solved for any vector having a nonzero coefficient; so u_1, u_2, or u_3 (but not u_4) can be written as a linear combination of the other three vectors. Thus, rather than asking whether some vector in S is a linear combination of the other vectors in S, it is more efficient to ask whether the zero vector can be expressed as a linear combination of the vectors in S with coefficients that are not all zero. This observation leads us to the following definition.

Definition. *A subset S of a vector space V is called* ***linearly dependent*** *if there exist a finite number of distinct vectors $u_1, u_2, \ldots, u_n$ in S and scalars $a_1, a_2, \ldots, a_n$, not all zero, such that*

$$a_1u_1 + a_2u_2 + \cdots + a_nu_n = \mathit{0}.$$

In this case we also say that the vectors of S are linearly dependent.

For any vectors $u_1, u_2, \ldots, u_n$, we have $a_1u_1 + a_2u_2 + \cdots + a_nu_n = \mathit{0}$ if $a_1 = a_2 = \cdots = a_n = 0$. We call this the **trivial representation** of $\mathit{0}$ as a linear combination of $u_1, u_2, \ldots, u_n$. Thus, for a set to be linearly dependent, there must exist a nontrivial representation of $\mathit{0}$ as a linear combination of vectors in the set. Consequently, any subset of a vector space that contains the zero vector is linearly dependent, because $\mathit{0} = 1 \cdot \mathit{0}$ is a nontrivial representation of $\mathit{0}$ as a linear combination of vectors in the set.

Example 1

Consider the set

$$S = \{(1, 3, -4, 2), (2, 2, -4, 0), (1, -3, 2, -4), (-1, 0, 1, 0)\}$$

in R^4. We show that S is linearly dependent and then express one of the vectors in S as a linear combination of the other vectors in S. To show that

S is linearly dependent, we must find scalars a_1, a_2, a_3, and a_4, not all zero, such that

$$a_1(1,3,-4,2) + a_2(2,2,-4,0) + a_3(1,-3,2,-4) + a_4(-1,0,1,0) = \mathit{0}.$$

Finding such scalars amounts to finding a nonzero solution to the system of linear equations

$$\begin{aligned} a_1 + 2a_2 + a_3 - a_4 &= 0 \\ 3a_1 + 2a_2 - 3a_3 \phantom{{}+a_4} &= 0 \\ -4a_1 - 4a_2 + 2a_3 + a_4 &= 0 \\ 2a_1 \phantom{{}+2a_2} - 4a_3 \phantom{{}+a_4} &= 0. \end{aligned}$$

One such solution is $a_1 = 4$, $a_2 = -3$, $a_3 = 2$, and $a_4 = 0$. Thus S is a linearly dependent subset of R^4, and

$$4(1,3,-4,2) - 3(2,2,-4,0) + 2(1,-3,2,-4) + 0(-1,0,1,0) = \mathit{0}. \quad \blacklozenge$$

Example 2

In $\mathsf{M}_{2\times 3}(R)$, the set

$$\left\{\begin{pmatrix} 1 & -3 & 2 \\ -4 & 0 & 5 \end{pmatrix}, \begin{pmatrix} -3 & 7 & 4 \\ 6 & -2 & -7 \end{pmatrix}, \begin{pmatrix} -2 & 3 & 11 \\ -1 & -3 & 2 \end{pmatrix}\right\}$$

is linearly dependent because

$$5\begin{pmatrix} 1 & -3 & 2 \\ -4 & 0 & 5 \end{pmatrix} + 3\begin{pmatrix} -3 & 7 & 4 \\ 6 & -2 & -7 \end{pmatrix} - 2\begin{pmatrix} -2 & 3 & 11 \\ -1 & -3 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \blacklozenge$$

Definition. *A subset S of a vector space that is not linearly dependent is called **linearly independent**. As before, we also say that the vectors of S are linearly independent.*

The following facts about linearly independent sets are true in any vector space.

1. The empty set is linearly independent, for linearly dependent sets must be nonempty.
2. A set consisting of a single nonzero vector is linearly independent. For if $\{u\}$ is linearly dependent, then $au = \mathit{0}$ for some nonzero scalar a. Thus

$$u = a^{-1}(au) = a^{-1}\mathit{0} = \mathit{0}.$$

3. A set is linearly independent if and only if the only representations of $\mathit{0}$ as linear combinations of its vectors are trivial representations.

The condition in item 3 provides a useful method for determining whether a finite set is linearly independent. This technique is illustrated in the examples that follow.

Example 3

To prove that the set

$$S = \{(1,0,0,-1),(0,1,0,-1),(0,0,1,-1),(0,0,0,1)\}$$

is linearly independent, we must show that the only linear combination of vectors in S that equals the zero vector is the one in which all the coefficients are zero. Suppose that a_1, a_2, a_3, and a_4 are scalars such that

$$a_1(1,0,0,-1) + a_2(0,1,0,-1) + a_3(0,0,1,-1) + a_4(0,0,0,1) = (0,0,0,0).$$

Equating the corresponding coordinates of the vectors on the left and the right sides of this equation, we obtain the following system of linear equations.

$$\begin{aligned} a_1 \qquad\qquad\qquad\qquad &= 0 \\ a_2 \qquad\qquad\quad &= 0 \\ a_3 \qquad &= 0 \\ -a_1 - a_2 - a_3 + a_4 &= 0 \end{aligned}$$

Clearly the only solution to this system is $a_1 = a_2 = a_3 = a_4 = 0$, and so S is linearly independent. ♦

Example 4

For $k = 0, 1, \ldots, n$ let $p_k(x) = x^k + x^{k+1} + \cdots + x^n$. The set

$$\{p_0(x), p_1(x), \ldots, p_n(x)\}$$

is linearly independent in $\mathsf{P}_n(F)$. For if

$$a_0p_0(x) + a_1p_1(x) + \cdots + a_np_n(x) = \mathit{0}$$

for some scalars $a_0, a_1, \ldots, a_n$, then

$$a_0 + (a_0 + a_1)x + (a_0 + a_1 + a_2)x^2 + \cdots + (a_0 + a_1 + \cdots + a_n)x^n = \mathit{0}.$$

By equating the coefficients of x^k on both sides of this equation for $k = 1, 2, \ldots, n$, we obtain

$$\begin{aligned} a_0 \qquad\qquad\qquad &= 0 \\ a_0 + a_1 \qquad\qquad &= 0 \\ a_0 + a_1 + a_2 \qquad &= 0 \\ &\ \,\vdots \\ a_0 + a_1 + a_2 + \cdots + a_n &= 0. \end{aligned}$$

Clearly the only solution to this system of linear equations is $a_0 = a_1 = \cdots = a_n = 0$. ♦

The following important results are immediate consequences of the definitions of linear dependence and linear independence.

Theorem 1.6. *Let* V *be a vector space, and let* $S_1 \subseteq S_2 \subseteq \mathsf{V}$. *If* S_1 *is linearly dependent, then* S_2 *is linearly dependent.*

Proof. Exercise. ∎

Corollary. *Let* V *be a vector space, and let* $S_1 \subseteq S_2 \subseteq \mathsf{V}$. *If* S_2 *is linearly independent, then* S_1 *is linearly independent.*

Proof. Exercise. ∎

Earlier in this section, we remarked that the issue of whether S is the smallest generating set for its span is related to the question of whether some vector in S is a linear combination of the other vectors in S. Thus the issue of whether S is the smallest generating set for its span is related to the question of whether S is linearly dependent. To see why, consider the subset $S = \{u_1, u_2, u_3, u_4\}$ of R^3, where $u_1 = (2, -1, 4)$, $u_2 = (1, -1, 3)$, $u_3 = (1, 1, -1)$, and $u_4 = (1, -2, -1)$. We have previously noted that S is linearly dependent; in fact,

$$-2u_1 + 3u_2 + u_3 - 0u_4 = 0.$$

This equation implies that u_3 (or alternatively, u_1 or u_2) is a linear combination of the other vectors in S. For example, $u_3 = 2u_1 - 3u_2 + 0u_4$. Therefore every linear combination $a_1u_1 + a_2u_2 + a_3u_3 + a_4u_4$ of vectors in S can be written as a linear combination of u_1, u_2, and u_4:

$$\begin{aligned} a_1u_1 + a_2u_2 + a_3u_3 + a_4u_4 &= a_1u_1 + a_2u_2 + a_3(2u_1 - 3u_2 + 0u_4) + a_4u_4 \\ &= (a_1 + 2a_3)u_1 + (a_2 - 3a_3)u_2 + a_4u_4. \end{aligned}$$

Thus the subset $S' = \{u_1, u_2, u_4\}$ of S has the same span as S!

More generally, suppose that S is any linearly dependent set containing two or more vectors. Then some vector $v \in S$ can be written as a linear combination of the other vectors in S, and the subset obtained by removing v from S has the same span as S. It follows that *if no proper subset of S generates the span of S, then S must be linearly independent.* Another way to view the preceding statement is given in Theorem 1.7.

Theorem 1.7. *Let S be a linearly independent subset of a vector space* V, *and let v be a vector in* V *that is not in S. Then $S \cup \{v\}$ is linearly dependent if and only if $v \in \operatorname{span}(S)$.*

Proof. If $S \cup \{v\}$ is linearly dependent, then there are vectors $u_1, u_2, \ldots, u_n$ in $S \cup \{v\}$ such that $a_1u_1 + a_2u_2 + \cdots + a_nu_n = 0$ for some nonzero scalars $a_1, a_2, \ldots, a_n$. Because S is linearly independent, one of the u_i's, say u_1, equals v. Thus $a_1v + a_2u_2 + \cdots + a_nu_n = 0$, and so

$$v = a_1^{-1}(-a_2u_2 - \cdots - a_nu_n) = -(a_1^{-1}a_2)u_2 - \cdots - (a_1^{-1}a_n)u_n.$$

Since v is a linear combination of $u_2, \ldots, u_n$, which are in S, we have $v \in \operatorname{span}(S)$.

Conversely, let $v \in \operatorname{span}(S)$. Then there exist vectors $v_1, v_2, \ldots, v_m$ in S and scalars $b_1, b_2, \ldots, b_m$ such that $v = b_1v_1 + b_2v_2 + \cdots + b_mv_m$. Hence

$$0 = b_1v_1 + b_2v_2 + \cdots + b_mv_m + (-1)v.$$

Since $v \neq v_i$ for $i = 1, 2, \ldots, m$, the coefficient of v in this linear combination is nonzero, and so the set $\{v_1, v_2, \ldots, v_m, v\}$ is linearly dependent. Therefore $S \cup \{v\}$ is linearly dependent by Theorem 1.6. ∎

Linearly independent generating sets are investigated in detail in Section 1.6.

EXERCISES

1. Label the following statements as true or false.
   - (a) If S is a linearly dependent set, then each vector in S is a linear combination of other vectors in S.
   - (b) Any set containing the zero vector is linearly dependent.
   - (c) The empty set is linearly dependent.
   - (d) Subsets of linearly dependent sets are linearly dependent.
   - (e) Subsets of linearly independent sets are linearly independent.
   - (f) If $a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$ and $x_1, x_2, \ldots, x_n$ are linearly independent, then all the scalars a_i are zero.

2.[3] Determine whether the following sets are linearly dependent or linearly independent.
   - (a) $\left\{\begin{pmatrix} 1 & -3 \\ -2 & 4 \end{pmatrix}, \begin{pmatrix} -2 & 6 \\ 4 & -8 \end{pmatrix}\right\}$ in $\mathsf{M}_{2\times 2}(R)$
   - (b) $\left\{\begin{pmatrix} 1 & -2 \\ -1 & 4 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 2 & -4 \end{pmatrix}\right\}$ in $\mathsf{M}_{2\times 2}(R)$
   - (c) $\{x^3 + 2x^2, -x^2 + 3x + 1, x^3 - x^2 + 2x - 1\}$ in $\mathsf{P}_3(R)$

[3]The computations in Exercise 2(g), (h), (i), and (j) are tedious unless technology is used.

(d) $\{x^3 - x, 2x^2 + 4, -2x^3 + 3x^2 + 2x + 6\}$ in $\mathsf{P}_3(R)$
(e) $\{(1, -1, 2), (1, -2, 1), (1, 1, 4)\}$ in R^3
(f) $\{(1, -1, 2), (2, 0, 1), (-1, 2, -1)\}$ in R^3

(g) $\left\{\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ -4 & 4 \end{pmatrix}\right\}$ in $\mathsf{M}_{2\times 2}(R)$

(h) $\left\{\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 2 & -2 \end{pmatrix}\right\}$ in $\mathsf{M}_{2\times 2}(R)$

(i) $\{x^4 - x^3 + 5x^2 - 8x + 6, -x^4 + x^3 - 5x^2 + 5x - 3,$
$x^4 + 3x^2 - 3x + 5, 2x^4 + 3x^3 + 4x^2 - x + 1, x^3 - x + 2\}$ in $\mathsf{P}_4(R)$
(j) $\{x^4 - x^3 + 5x^2 - 8x + 6, -x^4 + x^3 - 5x^2 + 5x - 3,$
$x^4 + 3x^2 - 3x + 5, 2x^4 + x^3 + 4x^2 + 8x\}$ in $\mathsf{P}_4(R)$

3. In $\mathsf{M}_{3\times 2}(F)$, prove that the set

$$\left\{\begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}\right\}$$

is linearly dependent. /

4. In F^n, let e_j denote the vector whose jth coordinate is 1 and whose other coordinates are 0. Prove that $\{e_1, e_2, \ldots, e_n\}$ is linearly independent.

5. Show that the set $\{1, x, x^2, \ldots, x^n\}$ is linearly independent in $\mathsf{P}_n(F)$.

6. In $\mathsf{M}_{m\times n}(F)$, let E^{ij} denote the matrix whose only nonzero entry is 1 in the ith row and jth column. Prove that $\{E^{ij} \colon 1 \le i \le m,\ 1 \le j \le n\}$ is linearly independent.

7. Recall from Example 3 in Section 1.3 that the set of diagonal matrices in $\mathsf{M}_{2\times 2}(F)$ is a subspace. Find a linearly independent set that generates this subspace.

8. Let $S = \{(1, 1, 0), (1, 0, 1), (0, 1, 1)\}$ be a subset of the vector space F^3.

(a) Prove that if $F = R$, then S is linearly independent.
(b) Prove that if F has characteristic 2, then S is linearly dependent.

9.† Let u and v be distinct vectors in a vector space V. Show that $\{u, v\}$ is linearly dependent if and only if u or v is a multiple of the other.

10. Give an example of three linearly dependent vectors in R^3 such that none of the three is a multiple of another.

11. Let $S = \{u_1, u_2, \ldots, u_n\}$ be a linearly independent subset of a vector space V *over the field Z_2. How many vectors are there in* span(S)? Justify your answer.

12. Prove Theorem 1.6 and its corollary.

13. Let V be a *vector space over a field of characteristic not equal to two.*
    (a) Let u and v be distinct vectors in V. Prove that $\{u, v\}$ is linearly independent if and only if $\{u+v, u-v\}$ is linearly independent.
    (b) Let u, v, and w be distinct vectors in V. Prove that $\{u, v, w\}$ is *linearly independent if and only if* $\{u+v, u+w, v+w\}$ is linearly independent.

14. Prove that a set S is linearly dependent if and only if $S = \{0\}$ or there exist distinct vectors $v, u_1, u_2, \ldots, u_n$ in S such that v is a linear combination of $u_1, u_2, \ldots, u_n$.

15. Let $S = \{u_1, u_2, \ldots, u_n\}$ be a finite set of vectors. Prove that S is linearly dependent if and only if $u_1 = 0$ or $u_{k+1} \in \text{span}(\{u_1, u_2, \ldots, u_k\})$ for some k $(1 \leq k < n)$.

16. Prove that a set S of vectors is linearly independent if and only if each finite subset of S is linearly independent.

17. Let M be a square upper triangular matrix (as defined in Exercise 12 of Section 1.3) with nonzero diagonal entries. Prove that the columns of M are linearly independent.

18. Let S be a set of nonzero polynomials in $\mathsf{P}(F)$ such that no two have the same degree. Prove that S is linearly independent.

19. Prove that if $\{A_1, A_2, \ldots, A_k\}$ is a linearly independent subset of $\mathsf{M}_{n\times n}(F)$, then $\{A_1^t, A_2^t, \ldots, A_k^t\}$ is also linearly independent.

20. Let $f, g, \in \mathcal{F}(R, R)$ be the functions defined by $f(t) = e^{rt}$ and $g(t) = e^{st}$, where $r \neq s$. Prove that f and g are linearly independent in $\mathcal{F}(R, R)$.

1.6 BASES AND DIMENSION

We saw in Section 1.5 that if S is a generating set for a subspace W and no proper subset of S is a generating set for W, then S must be linearly independent. A linearly independent generating set for W possesses a very useful property—every vector in W can be expressed in one and only one way as a linear combination of the vectors in the set. (This property is proved below in Theorem 1.8.) It is this property that makes linearly independent generating sets the building blocks of vector spaces.

Definition. *A* **basis** β *for a vector space* V *is a linearly independent subset of* V *that generates* V. *If* β *is a basis for* V, *we also say that the vectors of* β *form a basis for* V.

Example 1

Recalling that $\text{span}(\varnothing) = \{0\}$ and $\varnothing$ is linearly independent, we see that $\varnothing$ is a basis for the zero vector space. ◆

Example 2

In F^n, let $e_1 = (1, 0, 0, \ldots, 0), e_2 = (0, 1, 0, \ldots, 0), \ldots, e_n = (0, 0, \ldots, 0, 1)$; $\{e_1, e_2, \ldots, e_n\}$ is readily seen to be a basis for F^n and is called the **standard basis** for F^n. ◆

Example 3

In $\mathsf{M}_{m\times n}(F)$, let E^{ij} denote the matrix whose only nonzero entry is a 1 in the ith row and jth column. Then $\{E^{ij} : 1 \le i \le m, 1 \le j \le n\}$ is a basis for $\mathsf{M}_{m\times n}(F)$. ◆

Example 4

In $\mathsf{P}_n(F)$ the set $\{1, x, x^2, \ldots, x^n\}$ is a basis. We call this basis the **standard basis** for $\mathsf{P}_n(F)$. ◆

Example 5

In $\mathsf{P}(F)$ the set $\{1, x, x^2, \ldots\}$ is a basis. ◆

Observe that Example 5 shows that a basis need not be finite. In fact, later in this section it is shown that no basis for $\mathsf{P}(F)$ can be finite. Hence not every vector space has a finite basis.

The next theorem, which is used frequently in Chapter 2, establishes the most significant property of a basis.

Theorem 1.8. *Let* V *be a vector space and* $\beta = \{u_1, u_2, \ldots, u_n\}$ *be a subset of* V. *Then* β *is a basis for* V *if and only if each* $v \in \mathsf{V}$ *can be uniquely expressed as a linear combination of vectors of* β, *that is, can be expressed in the form*

$$v = a_1u_1 + a_2u_2 + \cdots + a_nu_n$$

for unique scalars $a_1, a_2, \ldots, a_n$.

Proof. Let β be a basis for V. If $v \in \mathsf{V}$, then $v \in \text{span}(\beta)$ because $\text{span}(\beta) = \mathsf{V}$. Thus v is a linear combination of the vectors of β. Suppose that

$$v = a_1u_1 + a_2u_2 + \cdots + a_nu_n \quad \text{and} \quad v = b_1u_1 + b_2u_2 + \cdots + b_nu_n$$

are two such representations of v. Subtracting the second equation from the first gives

$$0 = (a_1 - b_1)u_1 + (a_2 - b_2)u_2 + \cdots + (a_n - b_n)u_n.$$

Since β is linearly independent, it follows that $a_1 - b_1 = a_2 - b_2 = \cdots = a_n - b_n = 0$. Hence $a_1 = b_1, a_2 = b_2, \cdots, a_n = b_n$, and so v is uniquely expressible as a linear combination of the vectors of β.

The proof of the converse is an exercise. ∎

Theorem 1.8 shows that if the vectors $u_1, u_2, \ldots, u_n$ form a basis for a vector space V, then every vector in V can be uniquely expressed in the form

$$v = a_1u_1 + a_2u_2 + \cdots + a_nu_n$$

for appropriately chosen scalars $a_1, a_2, \ldots, a_n$. Thus v determines a unique n-tuple of scalars $(a_1, a_2, \ldots, a_n)$ and, conversely, each n-tuple of scalars determines a unique vector $v \in \mathsf{V}$ by using the entries of the n-tuple as the coefficients of a linear combination of $u_1, u_2, \ldots, u_n$. This fact suggests that V is like the vector space F^n, where n is the number of vectors in the basis for V. We see in Section 2.4 that this is indeed the case.

In this book, we are primarily interested in vector spaces having finite bases. Theorem 1.9 identifies a large class of vector spaces of this type.

Theorem 1.9. *If a vector space V is generated by a finite set S, then some subset of S is a basis for V. Hence V has a finite basis.*

Proof. If $S = \varnothing$ or $S = \{0\}$, then $\mathsf{V} = \{0\}$ and $\varnothing$ is a subset of S that is a basis for V. Otherwise S contains a nonzero vector u_1. By item 2 on page 37, $\{u_1\}$ is a linearly independent set. Continue, if possible, choosing vectors $u_2, \ldots, u_k$ in S such that $\{u_1, u_2, \ldots, u_k\}$ is linearly independent. Since S is a finite set, we must eventually reach a stage at which $\beta = \{u_1, u_2, \ldots, u_k\}$ is a linearly independent subset of S, but adjoining to β any vector in S not in β produces a linearly dependent set. We claim that β is a basis for V. Because β is linearly independent by construction, it suffices to show that β spans V. By Theorem 1.5 (p. 30) we need to show that $S \subseteq \text{span}(\beta)$. Let $v \in S$. If $v \in \beta$, then clearly $v \in \text{span}(\beta)$. Otherwise, if $v \notin \beta$, then the preceding construction shows that $\beta \cup \{v\}$ is linearly dependent. So $v \in \text{span}(\beta)$ by Theorem 1.7 (p. 39). Thus $S \subseteq \text{span}(\beta)$. ∎

Because of the method by which the basis β was obtained in the proof of Theorem 1.9, this theorem is often remembered as saying that *a finite spanning set for V can be reduced to a basis for V*. This method is illustrated in the next example.

Example 6

Let

$$S = \{(2, -3, 5), (8, -12, 20), (1, 0, -2), (0, 2, -1), (7, 2, 0)\}.$$

It can be shown that S generates R^3. We can select a basis for R^3 that is a subset of S by the technique used in proving Theorem 1.9. To start, select any nonzero vector in S, say $(2, -3, 5)$, to be a vector in the basis. Since $4(2, -3, 5) = (8, -12, 20)$, the set $\{(2, -3, 5), (8, -12, 20)\}$ is linearly dependent by Exercise 9 of Section 1.5. Hence we do not include $(8, -12, 20)$ in our basis. On the other hand, $(1, 0, -2)$ is not a multiple of $(2, -3, 5)$ and vice versa, so that the set $\{(2, -3, 5), (1, 0, -2)\}$ is linearly independent. Thus we include $(1, 0, -2)$ as part of our basis.

Now we consider the set $\{(2, -3, 5), (1, 0, -2), (0, 2, -1)\}$ obtained by adjoining another vector in S to the two vectors that we have already included in our basis. As before, we include $(0, 2, -1)$ in our basis or exclude it from the basis according to whether $\{(2, -3, 5), (1, 0, -2), (0, 2, -1)\}$ is linearly independent or linearly dependent. An easy calculation shows that this set is linearly independent, and so we include $(0, 2, -1)$ in our basis. In a similar fashion the final vector in S is included or excluded from our basis according to whether the set

$$\{(2, -3, 5), (1, 0, -2), (0, 2, -1), (7, 2, 0)\}$$

is linearly independent or linearly dependent. Because

$$2(2, -3, 5) + 3(1, 0, -2) + 4(0, 2, -1) - (7, 2, 0) = (0, 0, 0),$$

we exclude $(7, 2, 0)$ from our basis. We conclude that

$$\{(2, -3, 5), (1, 0, -2), (0, 2, -1)\}$$

is a subset of S that is a basis for R^3. ◆

The corollaries of the following theorem are perhaps the most significant results in Chapter 1.

Theorem 1.10 (Replacement Theorem). *Let V be a vector space that is generated by a set G containing exactly n vectors, and let L be a linearly independent subset of V containing exactly m vectors. Then $m \leq n$ and there exists a subset H of G containing exactly $n - m$ vectors such that $L \cup H$ generates V.*

Proof. The proof is by mathematical induction on m. The induction begins with $m = 0$; for in this case $L = \varnothing$, and so taking $H = G$ gives the desired result.

Now suppose that the theorem is true for some integer $m \geq 0$. We prove that the theorem is true for $m+1$. Let $L = \{v_1, v_2, \ldots, v_{m+1}\}$ be a linearly independent subset of V consisting of $m+1$ vectors. By the corollary to Theorem 1.6 (p. 39), $\{v_1, v_2, \ldots, v_m\}$ is linearly independent, and so we may apply the induction hypothesis to conclude that $m \leq n$ and that there is a subset $\{u_1, u_2, \ldots, u_{n-m}\}$ of G such that $\{v_1, v_2, \ldots, v_m\} \cup \{u_1, u_2, \ldots, u_{n-m}\}$ generates V. Thus there exist scalars $a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_{n-m}$ such that

$$a_1v_1 + a_2v_2 + \cdots + a_mv_m + b_1u_1 + b_2u_2 + \cdots + b_{n-m}u_{n-m} = v_{m+1}. \qquad (9)$$

Note that $n - m > 0$, lest v_{m+1} be a linear combination of $v_1, v_2, \ldots, v_m$, which by Theorem 1.7 (p. 39) contradicts the assumption that L is linearly independent. Hence $n > m$; that is, $n \geq m+1$. Moreover, some b_i, say b_1, is nonzero, for otherwise we obtain the same contradiction. Solving (9) for u_1 gives

$$\begin{aligned} u_1 = (-b_1^{-1}a_1)v_1 + (-b_1^{-1}a_2)v_2 + \cdots + (-b_1^{-1}a_m)v_m + (b_1^{-1})v_{m+1} \\ + (-b_1^{-1}b_2)u_2 + \cdots + (-b_1^{-1}b_{n-m})u_{n-m}. \end{aligned}$$

Let $H = \{u_2, \ldots, u_{n-m}\}$. Then $u_1 \in \operatorname{span}(L \cup H)$, and because $v_1, v_2, \ldots, v_m$, $u_2, \ldots, u_{n-m}$ are clearly in $\operatorname{span}(L \cup H)$, it follows that

$$\{v_1, v_2, \ldots, v_m, u_1, u_2, \ldots, u_{n-m}\} \subseteq \operatorname{span}(L \cup H).$$

Because $\{v_1, v_2, \ldots, v_m, u_1, u_2, \ldots, u_{n-m}\}$ generates V, Theorem 1.5 (p. 30) implies that $\operatorname{span}(L \cup H) = \mathsf{V}$. Since H is a subset of G that contains $(n-m)-1 = n-(m+1)$ vectors, the theorem is true for $m+1$. This completes the induction. ∎

Corollary 1. *Let V be a vector space having a finite basis. Then every basis for V contains the same number of vectors.*

Proof. Suppose that β is a finite basis for V that contains exactly n vectors, and let γ be any other basis for V. If γ contains more than n vectors, then we can select a subset S of γ containing exactly $n+1$ vectors. Since S is linearly independent and β generates V, the replacement theorem implies that $n+1 \leq n$, a contradiction. Therefore γ is finite, and the number m of vectors in γ satisfies $m \leq n$. Reversing the roles of β and γ and arguing as above, we obtain $n \leq m$. Hence $m = n$. ∎

If a vector space has a finite basis, Corollary 1 asserts that the number of vectors in *any* basis for V is an intrinsic property of V. This fact makes possible the following important definitions.

Definitions. *A vector space is called* **finite-dimensional** *if it has a basis consisting of a finite number of vectors. The unique number of vectors* *in each basis for* V *is called the* **dimension** *of* V *and is denoted by* dim(V). *A vector space that is not finite-dimensional is called* **infinite-dimensional**.

The following results are consequences of Examples 1 through 4.

Example 7

The vector space $\{0\}$ has dimension zero. ♦

Example 8

The vector space F^n has dimension n. ♦

Example 9

The vector space $\mathsf{M}_{m\times n}(F)$ has dimension mn. ♦

Example 10

The vector space $\mathsf{P}_n(F)$ has dimension $n+1$. ♦

The following examples show that the dimension of a vector space depends on its field of scalars.

Example 11

Over the field of complex numbers, the vector space of complex numbers has dimension 1. (A basis is $\{1\}$.) ♦

Example 12

Over the field of real numbers, the vector space of complex numbers has dimension 2. (A basis is $\{1, i\}$.) ♦

In the terminology of dimension, the first conclusion in the replacement theorem states that if V is a finite-dimensional vector space, then no linearly independent subset of V can contain more than dim(V) vectors. From this fact it follows that the vector space $\mathsf{P}(F)$ is infinite-dimensional because it has an infinite linearly independent set, namely $\{1, x, x^2, \ldots\}$. This set is, in fact, a basis for $\mathsf{P}(F)$. Yet nothing that we have proved in this section guarantees an infinite-dimensional vector space must have a basis. In Section 1.7 it is shown, however, that *every vector space has a basis.*

Just as no linearly independent subset of a finite-dimensional vector space V can contain more than dim(V) vectors, a corresponding statement can be made about the size of a generating set.

Corollary 2. *Let* V *be a vector space with dimension* n.

(a) *Any finite generating set for* V *contains at least* n *vectors, and a generating set for* V *that contains exactly* n *vectors is a basis for* V.

(b) *Any linearly independent subset of* V *that contains exactly* n *vectors is a basis for* V.

(c) *Every linearly independent subset of* V *can be extended to a basis for* V.

Proof. Let β be a basis for V.

(a) Let G be a finite generating set for V. By Theorem 1.9 some subset H of G is a basis for V. Corollary 1 implies that H contains exactly n vectors. Since a subset of G contains n vectors, G must contain at least n vectors. Moreover, if G contains exactly n vectors, then we must have $H = G$, so that G is a basis for V.

(b) Let L be a linearly independent subset of V containing exactly n vectors. It follows from the replacement theorem that there is a subset H of β containing $n - n = 0$ vectors such that $L \cup H$ generates V. Thus $H = \varnothing$, and L generates V. Since L is also linearly independent, L is a basis for V.

(c) If L is a linearly independent subset of V containing m vectors, then the replacement theorem asserts that there is a subset H of β containing exactly $n - m$ vectors such that $L \cup H$ generates V. Now $L \cup H$ contains at most n vectors; therefore (a) implies that $L \cup H$ contains exactly n vectors and that $L \cup H$ is a basis for V. ∎

Example 13

It follows from Example 4 of Section 1.4 and (a) of Corollary 2 that

$$\{x^2 + 3x - 2, 2x^2 + 5x - 3, -x^2 - 4x + 4\}$$

is a basis for $\mathsf{P}_2(R)$. ♦

Example 14

It follows from Example 5 of Section 1.4 and (a) of Corollary 2 that

$$\left\{\begin{pmatrix}1 & 1\\1 & 0\end{pmatrix}, \begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}, \begin{pmatrix}1 & 0\\1 & 1\end{pmatrix}, \begin{pmatrix}0 & 1\\1 & 1\end{pmatrix}\right\}$$

is a basis for $\mathsf{M}_{2\times 2}(R)$. ♦

Example 15

It follows from Example 3 of Section 1.5 and (b) of Corollary 2 that

$$\{(1,0,0,-1), (0,1,0,-1), (0,0,1,-1), (0,0,0,1)\}$$

is a basis for R^4. ♦

Example 16

For $k = 0, 1, \ldots, n$, let $p_k(x) = x^k + x^{k+1} + \cdots + x^n$. It follows from Example 4 of Section 1.5 and (b) of Corollary 2 that

$$\{p_0(x), p_1(x), \ldots, p_n(x)\}$$

is a basis for $\mathsf{P}_n(F)$. ♦

A procedure for reducing a generating set to a basis was illustrated in Example 6. In Section 3.4, when we have learned more about solving systems of linear equations, we will discover a simpler method for reducing a generating set to a basis. This procedure also can be used to extend a linearly independent set to a basis, as (c) of Corollary 2 asserts is possible.

An Overview of Dimension and Its Consequences

Theorem 1.9 as well as the replacement theorem and its corollaries contain a wealth of information about the relationships among linearly independent sets, bases, and generating sets. For this reason, we summarize here the main results of this section in order to put them into better perspective.

A basis for a vector space V is a linearly independent subset of V that generates V. If V has a finite basis, then every basis for V contains the same number of vectors. This number is called the dimension of V, and V is said to be finite-dimensional. Thus if the dimension of V is n, every basis for V contains exactly n vectors. Moreover, every linearly independent subset of V contains *no more* than n vectors and can be extended to a basis for V by including appropriately chosen vectors. Also, each generating set for V contains *at least* n vectors and can be reduced to a basis for V by excluding appropriately chosen vectors. The Venn diagram in Figure 1.6 depicts these relationships.

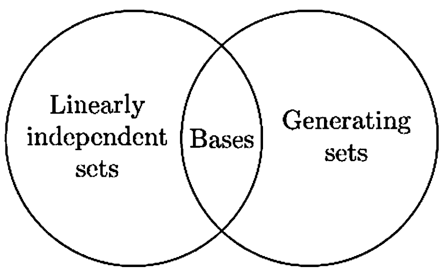

Figure 1.6

The Dimension of Subspaces

Our next result relates the dimension of a subspace to the dimension of the vector space that contains it.

Theorem 1.11. *Let* W *be a subspace of a finite-dimensional vector space* V. *Then* W *is finite-dimensional and* $\dim(\mathsf{W}) \leq \dim(\mathsf{V})$. *Moreover, if* $\dim(\mathsf{W}) = \dim(\mathsf{V})$, *then* $\mathsf{V} = \mathsf{W}$.

Proof. Let $\dim(\mathsf{V}) = n$. If $\mathsf{W} = \{\mathit{0}\}$, then W is finite-dimensional and $\dim(\mathsf{W}) = 0 \leq n$. Otherwise, W contains a nonzero vector x_1; so $\{x_1\}$ is a linearly independent set. Continue choosing vectors, $x_1, x_2, \ldots, x_k$ in W such that $\{x_1, x_2, \ldots, x_k\}$ is linearly independent. Since no linearly independent subset of V can contain more than n vectors, this process must stop at a stage where $k \leq n$ and $\{x_1, x_2, \ldots, x_k\}$ is linearly independent but adjoining any other vector from W produces a linearly dependent set. Theorem 1.7 (p. 39) implies that $\{x_1, x_2, \ldots, x_k\}$ generates W, and hence it is a basis for W. Therefore $\dim(\mathsf{W}) = k \leq n$.

If $\dim(\mathsf{W}) = n$, then a basis for W is a linearly independent subset of V containing n vectors. But Corollary 2 of the replacement theorem implies that this basis for W is also a basis for V; so $\mathsf{W} = \mathsf{V}$. ∎

Example 17

Let

$$\mathsf{W} = \{(a_1, a_2, a_3, a_4, a_5) \in \mathsf{F}^5 \colon a_1 + a_3 + a_5 = 0, a_2 = a_4\}.$$

It is easily shown that W is a subspace of F^5 having

$$\{(-1,0,1,0,0), (-1,0,0,0,1), (0,1,0,1,0)\}$$

as a basis. Thus $\dim(\mathsf{W}) = 3$. ◆

Example 18

The set of diagonal $n \times n$ matrices is a subspace W of $\mathsf{M}_{n\times n}(F)$ (see Example 3 of Section 1.3). A basis for W is

$$\{E^{11}, E^{22}, \ldots, E^{nn}\},$$

where E^{ij} is the matrix in which the only nonzero entry is a 1 in the ith row and jth column. Thus $\dim(\mathsf{W}) = n$. ◆

Example 19

We saw in Section 1.3 that the set of symmetric $n \times n$ matrices is a subspace W of $\mathsf{M}_{n\times n}(F)$. A basis for W is

$$\{A^{ij} \colon 1 \leq i \leq j \leq n\},$$

where A^{ij} is the $n \times n$ matrix having 1 in the ith row and jth column, 1 in the jth row and ith column, and 0 elsewhere. It follows that

$$\dim(\mathsf{W}) = n + (n-1) + \cdots + 1 = \frac{1}{2}n(n+1). \quad \blacklozenge$$

Corollary. *If* W *is a subspace of a finite-dimensional vector space* V, *then any basis for* W *can be extended to a basis for* V.

Proof. Let S be a basis for W. Because S is a linearly independent subset of V, Corollary 2 of the replacement theorem guarantees that S can be extended to a basis for V. ∎

Example 20

The set of all polynomials of the form

$$a_{18}x^{18} + a_{16}x^{16} + \cdots + a_2x^2 + a_0,$$

where $a_{18}, a_{16}, \ldots, a_2, a_0 \in F$, is a subspace W of $\mathsf{P}_{18}(F)$. A basis for W is $\{1, x^2, \ldots, x^{16}, x^{18}\}$, which is a subset of the standard basis for $\mathsf{P}_{18}(F)$. ♦

We can apply Theorem 1.11 to determine the subspaces of R^2 and R^3. Since R^2 has dimension 2, subspaces of R^2 can be of dimensions 0, 1, or 2 only. The only subspaces of dimension 0 or 2 are $\{\mathit{0}\}$ and R^2, respectively. Any subspace of R^2 having dimension 1 consists of all scalar multiples of some nonzero vector in R^2 (Exercise 11 of Section 1.4).

If a point of R^2 is identified in the natural way with a point in the Euclidean plane, then it is possible to describe the subspaces of R^2 geometrically: A subspace of R^2 having dimension 0 consists of the origin of the Euclidean plane, a subspace of R^2 with dimension 1 consists of a line through the origin, and a subspace of R^2 having dimension 2 is the entire Euclidean plane.

Similarly, the subspaces of R^3 must have dimensions 0, 1, 2, or 3. Interpreting these possibilities geometrically, we see that a subspace of dimension zero must be the origin of Euclidean 3-space, a subspace of dimension 1 is a line through the origin, a subspace of dimension 2 is a plane through the origin, and a subspace of dimension 3 is Euclidean 3-space itself.

The Lagrange Interpolation Formula

Corollary 2 of the replacement theorem can be applied to obtain a useful formula. Let $c_0, c_1, \ldots, c_n$ be distinct scalars in an infinite field F. The polynomials $f_0(x), f_1(x), \ldots, f_n(x)$ defined by

$$f_i(x) = \frac{(x-c_0)\cdots(x-c_{i-1})(x-c_{i+1})\cdots(x-c_n)}{(c_i-c_0)\cdots(c_i-c_{i-1})(c_i-c_{i+1})\cdots(c_i-c_n)} = \prod_{\substack{k=0\\k\neq i}}^{n} \frac{x-c_k}{c_i-c_k}$$

are called the **Lagrange polynomials** (associated with $c_0, c_1, \ldots, c_n$). Note that each $f_i(x)$ is a polynomial of degree n and hence is in $\mathsf{P}_n(F)$. By regarding $f_i(x)$ as a polynomial function $f_i\colon F \to F$, we see that

$$f_i(c_j) = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j. \end{cases} \tag{10}$$

This property of Lagrange polynomials can be used to show that $\beta = \{f_0, f_1, \ldots, f_n\}$ is a linearly independent subset of $\mathsf{P}_n(F)$. Suppose that

$$\sum_{i=0}^{n} a_i f_i = \mathit{0} \quad \text{for some scalars } a_0, a_1, \ldots, a_n,$$

where $\mathit{0}$ denotes the zero function. Then

$$\sum_{i=0}^{n} a_i f_i(c_j) = 0 \quad \text{for } j = 0, 1, \ldots, n.$$

But also

$$\sum_{i=0}^{n} a_i f_i(c_j) = a_j$$

by (10). Hence $a_j = 0$ for $j = 0, 1, \ldots, n$; so β is linearly independent. Since the dimension of $\mathsf{P}_n(F)$ is $n+1$, it follows from Corollary 2 of the replacement theorem that β is a basis for $\mathsf{P}_n(F)$.

Because β is a basis for $\mathsf{P}_n(F)$, every polynomial function g in $\mathsf{P}_n(F)$ is a linear combination of polynomial functions of β, say,

$$g = \sum_{i=0}^{n} b_i f_i.$$

It follows that

$$g(c_j) = \sum_{i=0}^{n} b_i f_i(c_j) = b_j;$$

so

$$g = \sum_{i=0}^{n} g(c_i) f_i$$

is the unique representation of g as a linear combination of elements of β. This representation is called the **Lagrange interpolation formula**. Notice that the preceding argument shows that if $b_0, b_1, \ldots, b_n$ are any $n+1$ scalars in F (not necessarily distinct), then the polynomial function

$$g = \sum_{i=0}^{n} b_i f_i$$

is the unique polynomial in $\mathsf{P}_n(F)$ such that $g(c_j) = b_j$. Thus we have found the unique polynomial of degree not exceeding n that has specified values b_j at given points c_j in its domain ($j = 0, 1, \ldots, n$). For example, let us construct the real polynomial g of degree at most 2 whose graph contains the points $(1,8), (2,5)$, and $(3,-4)$. (Thus, in the notation above, $c_0 = 1$, $c_1 = 2$, $c_2 = 3$, $b_0 = 8$, $b_1 = 5$, and $b_2 = -4$.) The Lagrange polynomials associated with c_0, c_1, and c_2 are

$$f_0(x) = \frac{(x-2)(x-3)}{(1-2)(1-3)} = \frac{1}{2}(x^2 - 5x + 6),$$

$$f_1(x) = \frac{(x-1)(x-3)}{(2-1)(2-3)} = -1(x^2 - 4x + 3),$$

and

$$f_2(x) = \frac{(x-1)(x-2)}{(3-1)(3-2)} = \frac{1}{2}(x^2 - 3x + 2).$$

Hence the desired polynomial is

$$\begin{aligned} g(x) &= \sum_{i=0}^{2} b_i f_i(x) = 8f_0(x) + 5f_1(x) - 4f_2(x) \\ &= 4(x^2 - 5x + 6) - 5(x^2 - 4x + 3) - 2(x^2 - 3x + 2) \\ &= -3x^2 + 6x + 5. \end{aligned}$$

An important consequence of the Lagrange interpolation formula is the following result: If $f \in \mathsf{P}_n(F)$ and $f(c_i) = 0$ for $n+1$ distinct scalars $c_0, c_1, \ldots, c_n$ in F, then f is the zero function.

EXERCISES

1. Label the following statements as true or false.

   (a) The zero vector space has no basis.
   (b) Every vector space that is generated by a finite set has a basis.
   (c) Every vector space has a finite basis.
   (d) A vector space cannot have more than one basis.

(e) If a vector space has a finite basis, then the number of vectors in every basis is the same.
(f) The dimension of $\mathsf{P}_n(F)$ is n.
(g) The dimension of $\mathsf{M}_{m\times n}(F)$ is $m+n$.
(h) Suppose that V is a finite-dimensional vector space, that S_1 is a linearly independent subset of V, and that S_2 is a subset of V that generates V. Then S_1 cannot contain more vectors than S_2.
(i) If S generates the vector space V, then every vector in V can be written as a linear combination of vectors in S in only one way.
(j) Every subspace of a finite-dimensional space is finite-dimensional.
(k) If V is a vector space having dimension n, then V has exactly one subspace with dimension 0 and exactly one subspace with dimension n.
(l) If V is a vector space having dimension n, and if S is a subset of V with n vectors, then S is linearly independent if and only if S spans V.

2. Determine which of the following sets are bases for R^3.

(a) $\{(1,0,-1),(2,5,1),(0,-4,3)\}$
(b) $\{(2,-4,1),(0,3,-1),(6,0,-1)\}$
(c) $\{(1,2,-1),(1,0,2),(2,1,1)\}$
(d) $\{(-1,3,1),(2,-4,-3),(-3,8,2)\}$
(e) $\{(1,-3,-2),(-3,1,3),(-2,-10,-2)\}$

3. Determine which of the following sets are bases for $\mathsf{P}_2(R)$.

(a) $\{-1-x+2x^2, 2+x-2x^2, 1-2x+4x^2\}$
(b) $\{1+2x+x^2, 3+x^2, x+x^2\}$
(c) $\{1-2x-2x^2, -2+3x-x^2, 1-x+6x^2\}$
(d) $\{-1+2x+4x^2, 3-4x-10x^2, -2-5x-6x^2\}$
(e) $\{1+2x-x^2, 4-2x+x^2, -1+18x-9x^2\}$

4. Do the polynomials $x^3-2x^2+1, 4x^2-x+3$, and $3x-2$ generate $\mathsf{P}_3(R)$? Justify your answer.

5. Is $\{(1,4,-6),(1,5,8),(2,1,1),(0,1,0)\}$ a linearly independent subset of R^3? Justify your answer.

6. Give three different bases for F^2 and for $\mathsf{M}_{2\times 2}(F)$.

7. The vectors $u_1=(2,-3,1)$, $u_2=(1,4,-2)$, $u_3=(-8,12,-4)$, $u_4=(1,37,-17)$, and $u_5=(-3,-5,8)$ generate R^3. Find a subset of the set $\{u_1,u_2,u_3,u_4,u_5\}$ that is a basis for R^3.

8. Let W denote the subspace of R^5 consisting of all the vectors having coordinates that sum to zero. The vectors

$$\begin{aligned} u_1 &= (2,-3,4,-5,2), & u_2 &= (-6,9,-12,15,-6),\\ u_3 &= (3,-2,7,-9,1), & u_4 &= (2,-8,2,-2,6),\\ u_5 &= (-1,1,2,1,-3), & u_6 &= (0,-3,-18,9,12),\\ u_7 &= (1,0,-2,3,-2), & u_8 &= (2,-1,1,-9,7) \end{aligned}$$

generate W. Find a subset of the set $\{u_1, u_2, \ldots, u_8\}$ that is a basis for W.

9. The vectors $u_1 = (1,1,1,1)$, $u_2 = (0,1,1,1)$, $u_3 = (0,0,1,1)$, and $u_4 = (0,0,0,1)$ form a basis for F^4. Find the unique representation of an arbitrary vector (a_1, a_2, a_3, a_4) in F^4 as a linear combination of u_1, u_2, u_3, and u_4.

10. In each part, use the Lagrange interpolation formula to construct the polynomial of smallest degree whose graph contains the following points.
    (a) $(-2,-6)$, $(-1,5)$, $(1,3)$
    (b) $(-4,24)$, $(1,9)$, $(3,3)$
    (c) $(-2,3)$, $(-1,-6)$, $(1,0)$, $(3,-2)$
    (d) $(-3,-30)$, $(-2,7)$, $(0,15)$, $(1,10)$

11. Let u and v be distinct vectors of a vector space V. Show that if $\{u, v\}$ is a basis for V and a and b are nonzero scalars, then both $\{u+v, au\}$ and $\{au, bv\}$ are also bases for V.

12. Let u, v, and w be distinct vectors of a vector space V. Show that if $\{u, v, w\}$ is a basis for V, then $\{u+v+w, v+w, w\}$ is also a basis for V.

13. The set of solutions to the system of linear equations

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0\\ 2x_1 - 3x_2 + x_3 &= 0 \end{aligned}$$

is a subspace of R^3. Find a basis for this subspace.

14. Find bases for the following subspaces of F^5:

$$\mathsf{W}_1 = \{(a_1, a_2, a_3, a_4, a_5) \in \mathsf{F}^5 \colon a_1 - a_3 - a_4 = 0\}$$

and

$$\mathsf{W}_2 = \{(a_1, a_2, a_3, a_4, a_5) \in \mathsf{F}^5 \colon a_2 = a_3 = a_4 \text{ and } a_1 + a_5 = 0\}.$$

What are the dimensions of W_1 and W_2?

15. The set of all $n \times n$ matrices having trace equal to zero is a subspace W of $\mathsf{M}_{n\times n}(F)$ (see Example 4 of Section 1.3). Find a basis for W. What is the dimension of W?

16. The set of all upper triangular $n \times n$ matrices is a subspace W of $\mathsf{M}_{n\times n}(F)$ (see Exercise 12 of Section 1.3). Find a basis for W. What is the dimension of W?

17. The set of all skew-symmetric $n \times n$ matrices is a subspace W of $\mathsf{M}_{n\times n}(F)$ (see Exercise 28 of Section 1.3). Find a basis for W. What is the dimension of W?

18. Find a basis for the vector space in Example 5 of Section 1.2. Justify your answer.

19. Complete the proof of Theorem 1.8.

20.† Let V be a vector space having dimension n, and let S be a subset of V that generates V.

- **(a)** Prove that there is a subset of S that is a basis for V. (Be careful not to assume that S is finite.)
- **(b)** Prove that S contains at least n vectors.

21. Prove that a vector space is infinite-dimensional if and only if it contains an infinite linearly independent subset.

22. Let W_1 and W_2 be subspaces of a finite-dimensional vector space V. Determine necessary and sufficient conditions on W_1 and W_2 so that $\dim(\mathsf{W}_1 \cap \mathsf{W}_2) = \dim(\mathsf{W}_1)$.

23. Let $v_1, v_2, \ldots, v_k, v$ be vectors in a vector space V, and define $\mathsf{W}_1 = \operatorname{span}(\{v_1, v_2, \ldots, v_k\})$, and $\mathsf{W}_2 = \operatorname{span}(\{v_1, v_2, \ldots, v_k, v\})$.

- **(a)** Find necessary and sufficient conditions on v such that $\dim(\mathsf{W}_1) = \dim(\mathsf{W}_2)$.
- **(b)** State and prove a relationship involving $\dim(\mathsf{W}_1)$ and $\dim(\mathsf{W}_2)$ in the case that $\dim(\mathsf{W}_1) \neq \dim(\mathsf{W}_2)$.

24. Let $f(x)$ be a polynomial of degree n in $\mathsf{P}_n(R)$. Prove that for any $g(x) \in \mathsf{P}_n(R)$ there exist scalars c_0, $c_1, \ldots, c_n$ such that

$$g(x) = c_0 f(x) + c_1 f'(x) + c_2 f''(x) + \cdots + c_n f^{(n)}(x),$$

where $f^{(n)}(x)$ denotes the nth derivative of $f(x)$.

25. Let V, W, and Z be as in Exercise 21 of Section 1.2. If V and W are vector spaces over F of dimensions m and n, determine the dimension of Z.

26. For a fixed $a \in R$, determine the dimension of the subspace of $\mathsf{P}_n(R)$ defined by $\{f \in \mathsf{P}_n(R)\colon f(a) = 0\}$.

27. Let W_1 and W_2 be the subspaces of $\mathsf{P}(F)$ defined in Exercise 25 in Section 1.3. Determine the dimensions of the subspaces $\mathsf{W}_1 \cap \mathsf{P}_n(F)$ and $\mathsf{W}_2 \cap \mathsf{P}_n(F)$.

28. Let V be a finite-dimensional vector space over C with dimension n. Prove that if V is now regarded as a vector space over R, then $\dim \mathsf{V} = 2n$. (See Examples 11 and 12.)

Exercises 29–34 require knowledge of the sum and direct sum of subspaces, as defined in the exercises of Section 1.3.

29. **(a)** Prove that if W_1 and W_2 are finite-dimensional subspaces of a vector space V, then the subspace $\mathsf{W}_1 + \mathsf{W}_2$ is finite-dimensional, and $\dim(\mathsf{W}_1 + \mathsf{W}_2) = \dim(\mathsf{W}_1) + \dim(\mathsf{W}_2) - \dim(\mathsf{W}_1 \cap \mathsf{W}_2)$. *Hint:* Start with a basis $\{u_1, u_2, \ldots, u_k\}$ for $\mathsf{W}_1 \cap \mathsf{W}_2$ and extend this set to a basis $\{u_1, u_2, \ldots, u_k, v_1, v_2, \ldots v_m\}$ for W_1 and to a basis $\{u_1, u_2, \ldots, u_k, w_1, w_2, \ldots w_p\}$ for W_2.

(b) Let W_1 and W_2 be finite-dimensional subspaces of a vector space V, and let $\mathsf{V} = \mathsf{W}_1 + \mathsf{W}_2$. Deduce that V is the direct sum of W_1 and W_2 if and only if $\dim(\mathsf{V}) = \dim(\mathsf{W}_1) + \dim(\mathsf{W}_2)$.

30. Let

$$\mathsf{V} = \mathsf{M}_{2\times 2}(F), \quad \mathsf{W}_1 = \left\{ \begin{pmatrix} a & b \\ c & a \end{pmatrix} \in \mathsf{V}\colon a, b, c \in F \right\},$$

and

$$\mathsf{W}_2 = \left\{ \begin{pmatrix} 0 & a \\ -a & b \end{pmatrix} \in \mathsf{V}\colon a, b \in F \right\}.$$

Prove that W_1 and W_2 are subspaces of V, and find the dimensions of W_1, W_2, $\mathsf{W}_1 + \mathsf{W}_2$, and $\mathsf{W}_1 \cap \mathsf{W}_2$.

31. Let W_1 and W_2 be subspaces of a vector space V having dimensions m and n, respectively, where $m \geq n$.

(a) Prove that $\dim(\mathsf{W}_1 \cap \mathsf{W}_2) \leq n$.

(b) Prove that $\dim(\mathsf{W}_1 + \mathsf{W}_2) \leq m + n$.

32. **(a)** Find an example of subspaces W_1 and W_2 of R^3 with dimensions m and n, where $m > n > 0$, such that $\dim(\mathsf{W}_1 \cap \mathsf{W}_2) = n$.

(b) Find an example of subspaces W_1 and W_2 of R^3 with dimensions m and n, where $m > n > 0$, such that $\dim(\mathsf{W}_1 + \mathsf{W}_2) = m + n$.

(c) Find an example of subspaces W_1 and W_2 of R^3 with dimensions m and n, where $m \geq n$, such that both $\dim(W_1 \cap W_2) < n$ and $\dim(W_1 + W_2) < m + n$.

33. (a) Let W_1 and W_2 be subspaces of a vector space V such that $V = W_1 \oplus W_2$. If β_1 and β_2 are bases for W_1 and W_2, respectively, show that $\beta_1 \cap \beta_2 = \varnothing$ and $\beta_1 \cup \beta_2$ is a basis for V.

(b) Conversely, let β_1 and β_2 be disjoint bases for subspaces W_1 and W_2, respectively, of a vector space V. Prove that if $\beta_1 \cup \beta_2$ is a basis for V, then $V = W_1 \oplus W_2$.

34. (a) Prove that if W_1 is any subspace of a finite-dimensional vector space V, then there exists a subspace W_2 of V such that $V = W_1 \oplus W_2$.

(b) Let $V = R^2$ and $W_1 = \{(a_1, 0) \colon a_1 \in R\}$. Give examples of two different subspaces W_2 and W_2' such that $V = W_1 \oplus W_2$ and $V = W_1 \oplus W_2'$.

The following exercise requires familiarity with Exercise 31 of Section 1.3.

35. Let W be a subspace of a finite-dimensional vector space V, and consider the basis $\{u_1, u_2, \ldots, u_k\}$ for W. Let $\{u_1, u_2, \ldots, u_k, u_{k+1}, \ldots, u_n\}$ be an extension of this basis to a basis for V.

(a) Prove that $\{u_{k+1} + W, u_{k+2} + W, \ldots, u_n + W\}$ is a basis for V/W.

(b) Derive a formula relating dim(V), dim(W), and dim(V/W).

1.7* MAXIMAL LINEARLY INDEPENDENT SUBSETS

In this section, several significant results from Section 1.6 are extended to infinite-dimensional vector spaces. Our principal goal here is to prove that every vector space has a basis. This result is important in the study of infinite-dimensional vector spaces because it is often difficult to construct an explicit basis for such a space. Consider, for example, the vector space of real numbers over the field of rational numbers. There is no obvious way to construct a basis for this space, and yet it follows from the results of this section that such a basis does exist.

The difficulty that arises in extending the theorems of the preceding section to infinite-dimensional vector spaces is that the principle of mathematical induction, which played a crucial role in many of the proofs of Section 1.6, is no longer adequate. Instead, a more general result called the *maximal principle* is needed. Before stating this principle, we need to introduce some terminology.

Definition. *Let $\mathcal{F}$ be a family of sets. A member M of $\mathcal{F}$ is called* **maximal** *(with respect to set inclusion) if M is contained in no member of $\mathcal{F}$ other than M itself.*

Example 1

Let $\mathcal{F}$ be the family of all subsets of a nonempty set S. (This family $\mathcal{F}$ is called the **power set** of S.) The set S is easily seen to be a maximal element of $\mathcal{F}$. ♦

Example 2

Let S and T be disjoint nonempty sets, and let $\mathcal{F}$ be the union of their power sets. Then S and T are both maximal elements of $\mathcal{F}$. ♦

Example 3

Let $\mathcal{F}$ be the family of all finite subsets of an infinite set S. Then $\mathcal{F}$ has no maximal element. For if M is any member of $\mathcal{F}$ and s is any element of S that is not in M, then $M \cup \{s\}$ is a member of $\mathcal{F}$ that contains M as a proper subset. ♦

Definition. *A collection of sets $\mathcal{C}$ is called a* **chain** *(or* **nest** *or* **tower***) if for each pair of sets A and B in $\mathcal{C}$, either $A \subseteq B$ or $B \subseteq A$.*

Example 4

For each positive integer n let $A_n = \{1, 2, \ldots, n\}$. Then the collection of sets $\mathcal{C} = \{A_n : n = 1, 2, 3, \ldots\}$ is a chain. In fact, $A_m \subseteq A_n$ if and only if $m \leq n$. ♦

With this terminology we can now state the maximal principle.

Maximal Principle.[4] *Let $\mathcal{F}$ be a family of sets. If, for each chain $\mathcal{C} \subseteq \mathcal{F}$, there exists a member of $\mathcal{F}$ that contains each member of $\mathcal{C}$, then $\mathcal{F}$ contains a maximal member.*

Because the maximal principle guarantees the existence of maximal elements in a family of sets satisfying the hypothesis above, it is useful to reformulate the definition of a basis in terms of a maximal property. In Theorem 1.12, we show that this is possible; in fact, the concept defined next is equivalent to a basis.

Definition. *Let S be a subset of a vector space V. A* **maximal linearly independent subset** *of S is a subset B of S satisfying both of the following conditions.*

(a) *B is linearly independent.*
(b) *The only linearly independent subset of S that contains B is B itself.*

---

[4]The *Maximal Principle* is logically equivalent to the *Axiom of Choice*, which is an assumption in most axiomatic developments of set theory. For a treatment of set theory using the Maximal Principle, see John L. Kelley, *General Topology*, Graduate Texts in Mathematics Series, Vol. 27, Springer-Verlag, 1991.

Example 5

Example 2 of Section 1.4 shows that

$$\{x^3 - 2x^2 - 5x - 3, 3x^3 - 5x^2 - 4x - 9\}$$

is a maximal linearly independent subset of

$$S = \{2x^3 - 2x^2 + 12x - 6, x^3 - 2x^2 - 5x - 3, 3x^3 - 5x^2 - 4x - 9\}$$

in $\mathsf{P}_2(R)$. In this case, however, any subset of S consisting of two polynomials is easily shown to be a maximal linearly independent subset of S. Thus maximal linearly independent subsets of a set need not be unique. ◆

A basis β for a vector space V is a maximal linearly independent subset of V, because

1. β is linearly independent by definition.
2. If $v \in \mathsf{V}$ and $v \notin \beta$, then $\beta \cup \{v\}$ is linearly dependent by Theorem 1.7 (p. 39) because $\text{span}(\beta) = \mathsf{V}$.

Our next result shows that the converse of this statement is also true.

Theorem 1.12. *Let V be a vector space and S a subset that generates V. If β is a maximal linearly independent subset of S, then β is a basis for V.*

Proof. Let β be a maximal linearly independent subset of S. Because β is linearly independent, it suffices to prove that β generates V. We claim that $S \subseteq \text{span}(\beta)$, for otherwise there exists a $v \in S$ such that $v \notin \text{span}(\beta)$. Since Theorem 1.7 (p. 39) implies that $\beta \cup \{v\}$ is linearly independent, we have contradicted the maximality of β. Therefore $S \subseteq \text{span}(\beta)$. Because $\text{span}(S) = \mathsf{V}$, it follows from Theorem 1.5 (p. 30) that $\text{span}(\beta) = \mathsf{V}$. ∎

Thus a subset of a vector space is a basis if and only if it is a maximal linearly independent subset of the vector space. Therefore we can accomplish our goal of proving that every vector space has a basis by showing that every vector space contains a maximal linearly independent subset. This result follows immediately from the next theorem.

Theorem 1.13. *Let S be a linearly independent subset of a vector space V. There exists a maximal linearly independent subset of V that contains S.*

Proof. Let $\mathcal{F}$ denote the family of all linearly independent subsets of V that contain S. In order to show that $\mathcal{F}$ contains a maximal element, we must show that if $\mathcal{C}$ is a chain in $\mathcal{F}$, then there exists a member U of $\mathcal{F}$ that contains each member of $\mathcal{C}$. We claim that U, the union of the members of $\mathcal{C}$, is the desired set. Clearly U contains each member of $\mathcal{C}$, and so it suffices to prove that $U \in \mathcal{F}$ (i.e., that U is a linearly independent subset of V that contains S). Because each member of $\mathcal{C}$ is a subset of V containing S, we have $S \subseteq U \subseteq \mathsf{V}$. Thus we need only prove that U is linearly independent. Let $u_1, u_2, \ldots, u_n$ be in U and $a_1, a_2, \ldots, a_n$ be scalars such that $a_1u_1 + a_2u_2 + \cdots + a_nu_n = \mathit{0}$. Because $u_i \in U$ for $i = 1, 2, \ldots, n$, there exists a set A_i in $\mathcal{C}$ such that $u_i \in A_i$. But since $\mathcal{C}$ is a chain, one of these sets, say A_k, contains all the others. Thus $u_i \in A_k$ for $i = 1, 2, \ldots, n$. However, A_k is a linearly independent set; so $a_1u_1 + a_2u_2 + \cdots + a_nu_n = \mathit{0}$ implies that $a_1 = a_2 = \cdots = a_n = 0$. It follows that U is linearly independent.

The maximal principle implies that $\mathcal{F}$ has a maximal element. This element is easily seen to be a maximal linearly independent subset of V that contains S. ∎

Corollary. *Every vector space has a basis.*

It can be shown, analogously to Corollary 1 of the replacement theorem (p. 46), that every basis for an infinite-dimensional vector space has the same *cardinality*. (Sets have the same cardinality if there is a one-to-one and onto mapping between them.) (See, for example, N. Jacobson, *Lectures in Abstract Algebra*, vol. 2, Linear Algebra, D. Van Nostrand Company, New York, 1953, p. 240.)

Exercises 4-7 extend other results from Section 1.6 tó infinite-dimensional vector spaces.

EXERCISES

1. *Label the following statements as true or false.*
   (a) Every family of sets contains a maximal element.
   (b) Every chain contains a maximal element.
   (c) If a family of sets has a maximal element, then that maximal element is unique.
   (d) If a chain of sets has a maximal element, then that maximal element is unique.
   (e) A basis for a vector space is a maximal linearly independent subset of that vector space.
   (f) A maximal linearly independent subset of a vector space is a basis for that vector space.

2. Show that the set of convergent sequences is an infinite-dimensional subspace of the vector space of all sequences of real numbers. (See Exercise 21 in Section 1.3.)

3. Let V be the set of real numbers regarded as a vector space over the field of rational numbers. Prove that V is infinite-dimensional. *Hint:*

Use the fact that π is transcendental, that is, π is not a zero of any polynomial with rational coefficients.

4. Let W be a subspace of a (not necessarily finite-dimensional) vector space V. Prove that any basis for W is a subset of a basis for V.

5. Prove the following infinite-dimensional version of Theorem 1.8 (p. 43): Let β be a subset of an infinite-dimensional vector space V. Then β is a basis for V if and only if for each nonzero vector v in V, there exist unique vectors $u_1, u_2, \ldots, u_n$ in β and unique nonzero scalars $c_1, c_2, \ldots, c_n$ such that $v = c_1u_1 + c_2u_2 + \cdots + c_nu_n$.

6. Prove the following generalization of Theorem 1.9 (p. 44): Let S_1 and S_2 be subsets of a vector space V such that $S_1 \subseteq S_2$. If S_1 is linearly independent and S_2 generates V, then there exists a basis β for V such that $S_1 \subseteq \beta \subseteq S_2$. *Hint:* Apply the maximal principle to the family of all linearly independent subsets of S_2 that contain S_1, and proceed as in the proof of Theorem 1.13.

7. Prove the following generalization of the replacement theorem. Let β be a basis for a vector space V, and let S be a linearly independent subset of V. There exists a subset S_1 of β such that $S \cup S_1$ is a basis for V.

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