vector - Matrix equation

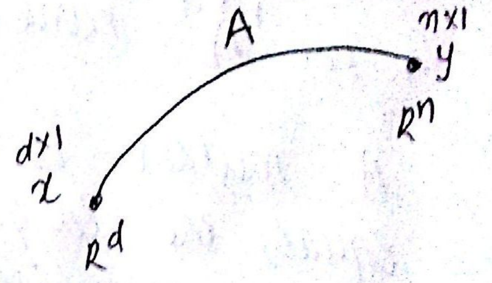
$$\underset{n\times d}{A} \cdot \underset{d\times 1}{x} = \underset{n\times 1}{y} \quad \text{— (A)}$$

we also have

$$A = [\underline{a}_1, \underline{a}_2 \ldots\ldots a_d] \quad \text{— (B)}$$ a_i is a $n\times 1$ column vector

and,

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{pmatrix} \quad \text{— (C)}$$



with the help of (B) & (C), we can express (A) as

$$y = \sum_{k=1}^{d} \underline{a}_i x_i$$

Here,

y is formed as a linear combination of the columns.

⇒ The set of all linear combination (using any x) of the column A is the range of A → R(A)

- For $y \notin R(A)$, trying to find an $\underline{x}$ that satisfies matrix equation $A\underline{x} = \underline{y}$ is trivial

For $n = d$,

Equation (A) can be interpreted as that of change of co-ordinate system (basis vector). $\underline{x}$ is the representation of a point in R^d w.r.t the original co-ordinate system, $\underline{y}$ is the representation of a point w.r.t a new - co-ordinate system. And Here matrix A related original co- ordinate system to new. co-ordinate system

Matrix rank

Let a matrix as in previous case defined in terms of column vectors

- Rank of A, denoted by rank(A) is defined as the number of linearly independent columns of A

+ For $\overset{n \times d}{A}$, rank(A) $\leq d$ [provided $n \geq d$]

- A matrix is said to have full rank if its rank equals the largest possible for a matrix of the same dimension, which is the lesser of the number of rows and columns

- A matrix is said to be rank deficient if it doesn't have full rank)

e.g $\begin{bmatrix} 1 & 2 & 1 \\ -2 & -3 & 1 \\ 3 & 5 & 0 \end{bmatrix}$ Rank = ?

2.1.4 Inner and outer products

* Inner product

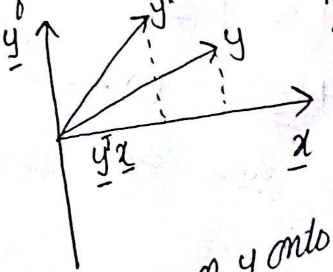
If $\underline{x}$ and $\underline{y}$ are real $d \times 1$ vectors, their vector inner product is denoted using braces ($\langle \rangle$) and defined to be the scalar given by

$$(\underline{x}, \underline{y}) = (\underline{x})^T \underline{y} = \underline{y}^T \underline{x}$$

The inner product since it is clear, is symmetric

- Geometrically $(\underline{x}, \underline{y})$ is visualized as the projection of $\underline{y}$ onto x (or vice-versa) as in figure below
- The inner product provides a measure of the closeness of two vectors

$$\langle \underset{n\times 1}{\underline{x}}, \underset{n\times 1}{\underline{y}} \rangle = \underline{x}^T y = \underline{y}^T \underline{x}$$



Projection of $\underline{y}$ onto $\underline{x}$ or $\underline{x}$ onto $\underline{y}$.

Also,
$$\langle x, x \rangle = \underline{x}^T \underline{x} = \sum_{k=1}^{d} x_k^2$$

Here, the Euclidean length or norm of vector $\underline{x}$ denoted by

$$\|x\| = \sqrt{\sum_{k=1}^{d} x_k^2}$$

$$= \left[\langle \underline{x}, \underline{x} \rangle\right]^{1/2}$$

$$\Rightarrow \|x\|^2 = \langle \underline{x}, \underline{x} \rangle$$

with respect to matrix inner product can be formed as

$$\langle \underline{x}, R\underline{x} \rangle = \underline{x}^T R \underline{x} = \|X\|_R^2 \quad — \text{(C)}$$

and

$$\langle \underline{x}, R\underline{y} \rangle = \underline{x}^T R \underline{y} = \underline{y}^T R \underline{x} \quad — \text{D}$$

Equation (C) is used to denote a non-Euclidean vector norm

- A particularly useful form of equation (C) involves the difference (vector) of two vectors and provides a scalar measure of the closeness $\underline{x}$ & $\underline{y}$ i.e.

$$\|\underline{x}-\underline{y}\|_R = \langle (\underline{x}-\underline{y}), R(\underline{x}-\underline{y}) \rangle$$

$$= (\underline{x}-\underline{y})^T R (\underline{x}-\underline{y})$$

$$= \|\underline{x}\|_R^2 + \|\underline{y}\|_R^2 - (\langle \underline{y}, R\underline{x} \rangle + \langle \underline{x}, R\underline{y} \rangle)$$

Solution of this equation is possible for symmetric R,

$$\|\underline{x}-\underline{y}\|_R = \langle (\underline{x}-\underline{y}), R(\underline{x}-\underline{y}) \rangle$$

$$= \|\underline{x}\|_R^2 + \|\underline{y}\|_R^2 - 2\langle \underline{x}, R\underline{y} \rangle$$

Also,

The inner product is linear

$$\langle a\underline{x}_1 + b\underline{x}_2, \underline{y} \rangle = a\langle x_1, y \rangle + b\langle x_2, y \rangle$$

* If $\langle \underline{x}, \underline{y} \rangle = 0$, vectors $\underline{x}$ and y are said to be orthogonal