Pramod Parajuli
Simulation and Modeling, CS-331

# Markov Chains



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## Markov Chains

A Markov chain is a sequence of random values whose probabilities at a time interval depends upon the value of the number at the previous time

A simple example is the nonreturning random walk, where the walkers are restricted to not go back to the location just previously visited

In this case, each possible position is known as state or condition

The controlling factor in a Markov chain is the ***transition probability***, it is a conditional probability for the system to go to a particular new state, given the current state of the system


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# Markov Chains

Since the probability of moving from one state to another depends on probability of the preceding state, transition probability is a conditional probability

(*blackboard example*)

Markov chain (process) has following properties;

1. The set of all possible states of stochastic (probabilistic) system is finite
2. The variables move from one state to another and the probability of transition from a given state is dependent only on the present state of the system, not in which it was reached
3. The probabilities of reaching to various states from any given state are measurable and remain constant over a time (i.e. throughout the system's operation)



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## Markov Chains

Markov chains are classified by their order

- If the probability occurrence of each state depends only upon the immediate preceding state, then it is known as ***first order Markov chain***
- The zero order Markov chain is memory less chain

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## Matrix of Transition Probabilities

Let $s_j$, $(s_1, s_2, \ldots, s_m;\ j = 1,2,\ldots,m)$ be state of a system and

$p_{ij}$, $(p_{0,0}, p_{0,1}, p_{0,2}, \ldots, p_{m,m})$ be probability of moving from state $s_i$ to state $s_j$

So now, the square matrix of size m×m

Succeeding state

$$P = [p_{ij}]_{m \times m} = \quad \text{Initial state} \quad \begin{array}{c c} & \begin{array}{ccccc} s_1 & s_2 & \cdots & \cdots & s_m \end{array} \\ \begin{array}{c} s_1 \\ s_2 \\ \cdot\cdot \\ \cdot\cdot \\ s_m \end{array} & \begin{bmatrix} p_{11} & p_{12} & \cdot & \cdot & p_{1m} \\ p_{21} & p_{22} & \cdot & \cdot & p_{2m} \\ \cdot\cdot & \cdot\cdot & \cdot & \cdot & \cdot\cdot \\ \cdot\cdot & \cdot\cdot & \cdot & \cdot & \cdot\cdot \\ p_{m1} & p_{m2} & \cdot & \cdot & p_{mm} \end{bmatrix} \end{array}$$



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## Matrix of Transition Probabilities

If there is no transition between $s_i$ and $s_j$, then $p_{ij}= 0$

If only one state is selected while advancing, then $p_{ij} =1$

The probability is distributed over the elements in the row. Therefore;

$$\sum_{j=1}^{m} p_{ij} = 1 \text{ for all } i$$

*and* $$0 \le p_{ij} \le 1$$



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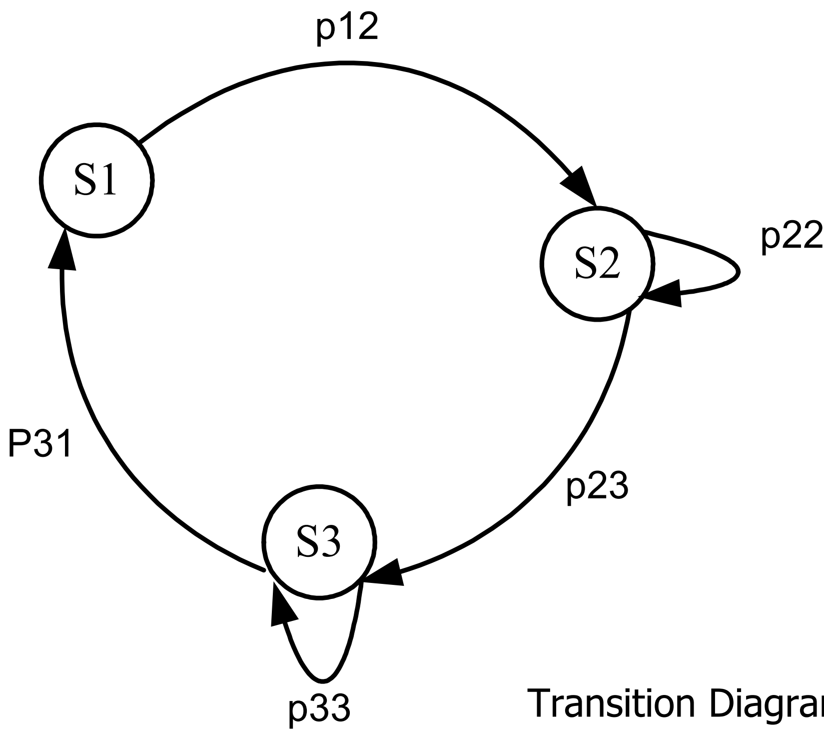
## Diagrams

Two types;

1. *Transition Diagram*
2. *Probability Tree Diagram*




Transition Diagram

| | Succeeding state | | |
|---|---|---|---|
| | ***S1*** | ***S2*** | ***S3*** |
| ***S1*** | 0 | $P_{12}$ | 0 |
| ***S2*** | 0 | $P_{22}$ | $P_{23}$ |
| ***S3*** | $P_{31}$ | 0 | $P_{33}$ |



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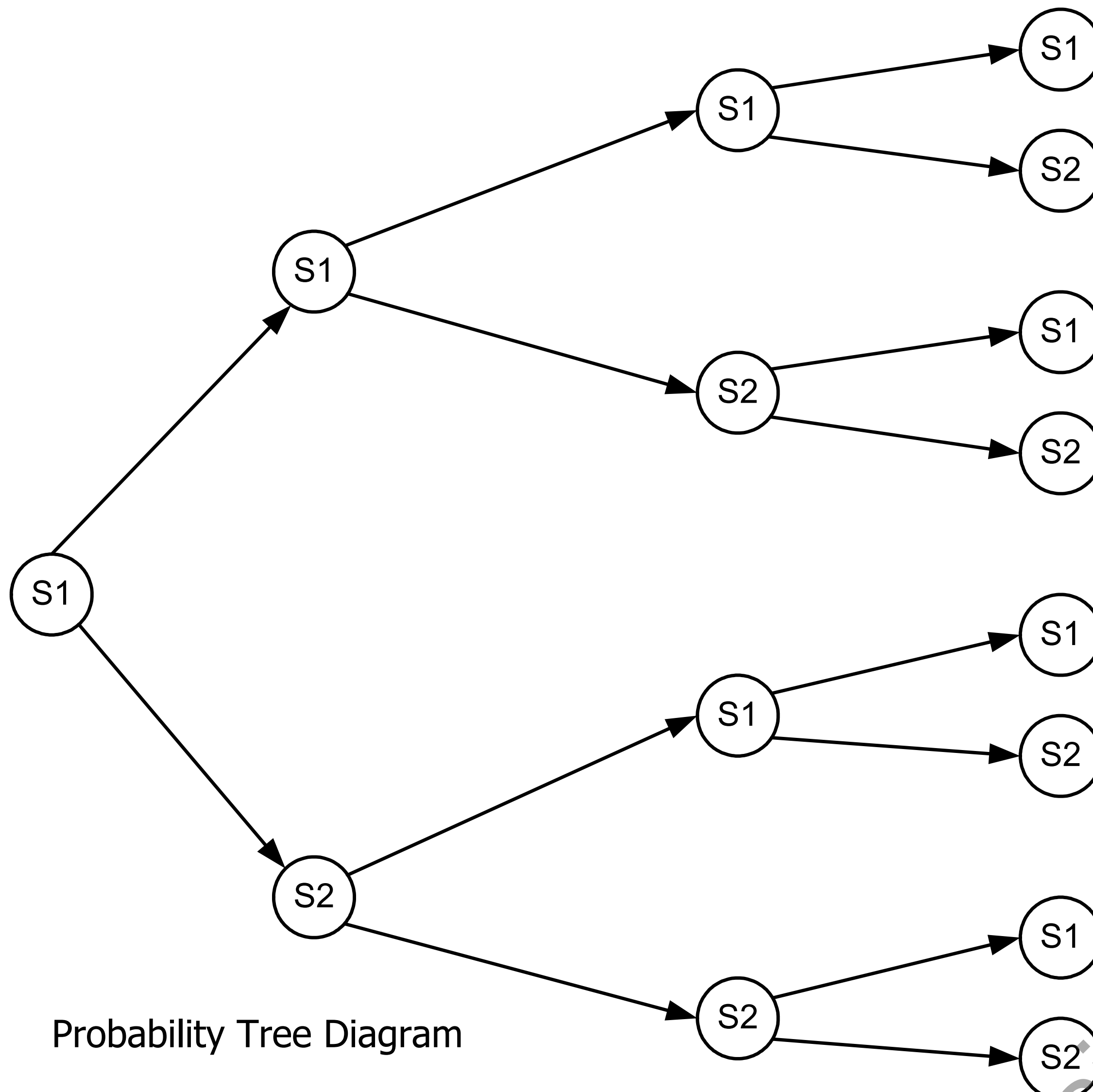
# Diagrams




Probability Tree Diagram



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## Diagrams

The probabilities at various states can be determined by using;

If state1 is the initial state;

P(state 1, shift n+1 | state 1, shift 1)

= 0.7 * P(state 1, shift n | state 1, shift1) +

0.8 * P(state2, shift n | state 1, shift1)

n = 1, 2, 3

Ex;

P(state 1, shift 3 | state 1, shift 1)

= 0.7 (0.7) + 0.8 ( 0.3) = 0.73



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## Diagrams

P(state 1, shift 4 | state 1, shift 1)

= 0.7 P(state 1, shift 3 | state 1, shift 1)

+ 0.8 P(state 1, shift 3 | state 1, shift 1)

= 0.727

P(state 1, shift 4 | state 1, shift 1)

= 0.7 (0.7) (0.7) + 0.7 (0.3)(0.8) + 0.3 (0.8) (0.7) + 0.3 (0.2)(0.8)

= 0.727



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## n-Step Transition Probabilities

Let's represent the initial situation by $R_0$ as;

$$R_0 = [\ p_{11}, p_{12}, p_{13}, \ldots, p_{1m}]$$

And $P = [p_{ij}]_{m \times m}$ be transition probabilities matrix at time period n = 0

Further, let R1 represent the situation after one execution of the experiment i.e. n = 1

$$R_1 = R_0 \text{ x } P$$

Similarly,

$$R_1 = R_0 \text{ x } P = R \text{ x } P^2$$

$$\vdots \qquad\qquad \vdots$$

$$R_n = R_{n-1} \text{ x } P = R_0 \text{ X } P^n$$



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## n-Step Transition Probabilities

$$R_n = R_{n-1} \text{ x } P = R_0 \text{ X } P^n$$

Here, $P^n$ denotes the n-steps transition matrix

Sample calculation – Blackboard demonstration

1. Example 17.1 from page 716 (handouts)
2. Samples from PC Quest



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## Steady-state (equilibrium conditions)

When the number of periods (stage) increases, the probabilities approaches to steady state (equilibrium). At this state, the system becomes independent of time

A Markov chain reaches to equilibrium state if;

1. The transition matrix elements remains positive from one period to the next (regular property of Markov chain)
2. It is possible to go from one state to another in a finite number of steps, regardless of the present state (ergodic property)

Note: All regular Markov chains must be ergodic Markov chains but the converse is not true.



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# Steady-state behavior of Markovian Systems

- Steady state solution solved mathematically

## Inifinite – Population

- Are assumed to follow Poisson process with $\lambda$ arrivals per unit time and inter-arrival time is exponentially distributed with mean $1/\lambda$
- The queue discipline is FIFO
- Mathematically, a system is said to be in steady state, provided the probability that the system is in a given state is not time dependent;

$$P(L(t) = n) = P_n(t) = P_n$$



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# Steady-state behavior of Markovian Systems

- For simple models, steady-state behavior parameter 'L' (time-average of customers in the system) can be computed as;

$$L = \sum_{n=0}^{\infty} n.Pn$$

Ref. Lecture 8, Slide – 46

- If L is given, then other steady-state parameters can be computed by using Little's equation;

$$L = \lambda.w$$

$$w_Q = w - (1/\mu)$$

$$L_Q = \lambda.\ w_Q$$



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# M/G/1

Single server queues with poisson arrivals and unlimited capacity

Mean service time = 1/ μ

Variance = $\sigma^2$

If μ < 1, then M/G/1 queue has a steady-state probability distribution

If λ < μ, then ρ will be server utilization

$$\rho = \frac{\lambda}{\mu}$$

$$L = \rho + \frac{\rho^2(1+\sigma^2\mu^2)}{2(1-\rho)}$$

$$L_Q = \frac{\rho^2(1+\sigma^2\mu^2)}{2(1-\rho)}$$



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## M/G/1

Single server queues with poisson arrivals and unlimited capacity

Let's look at these a bit more closely

Consider first if $\sigma^2$ **= 0**

i.e. the service times are all the same (= mean)

In this case the equations for L and $L_Q$ greatly simplified to

$$L_Q = \frac{\rho^2(1+0^2\mu^2)}{2(1-\rho)} = \frac{\rho^2}{2(1-\rho)}$$

In this case $L_Q$ is dependent solely upon the server utilization, $\rho$

Note as $\rho \rightarrow 0$ (low server utilization) $L_Q \rightarrow 0$

Note as $\rho \rightarrow 1$ (high server utilization) $L_Q \rightarrow \infty$



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# M/G/1

Single server queues with poisson arrivals and unlimited capacity

Again, $\rho = L - L_Q$ is the time average number of customers being served

Example;

Supplements from Banks and Nicol, Page 226, 227



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# M/M/1

M/G/1 (service times are exponential)

- Let's look at the simplest case, an M/M/1 queue with arrival rate $\lambda$ and service rate $\mu$
- Consider the state of the system to be the number of customers in the system
- We can then form a state-transition diagram for this system
  - A transition from state k to state k+1 occurs with probability $\lambda$
  - A transition from state k+1 to state k occurs with probability $\mu$



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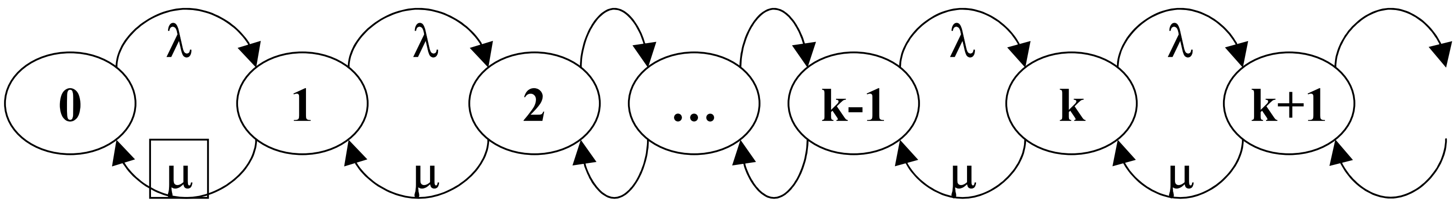
# M/M/1

M/G/1 (service times are exponential)

- From this we can obtain




- We also know that

$$P_k = P_0 \prod_{i=0}^{k-1} \frac{\lambda}{\mu} = P_0 \left( \frac{\lambda}{\mu} \right)^k$$

  - Since the sum of the probabilities in the distribution must equal 1
  - This will allow us to solve for $P_0$

$$\sum_{k=0}^{\infty} P_k = 1$$



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# M/M/1

M/G/1 (service times are exponential)

Before completing the derivation, we must note an important requirement: to be stable, the system utilization $\lambda/\mu < 1$ must be true

$$\sum_{k=0}^{\infty} P_k = 1$$

$$\sum_{k=0}^{\infty} P_0 \left(\frac{\lambda}{\mu}\right)^k = 1$$

$$P_0 \left(1 + \sum_{k=1}^{\infty} \left(\frac{\lambda}{\mu}\right)^k\right) = 1$$

$$P_0 = \frac{1}{1 + \sum_{k=1}^{\infty} \left(\frac{\lambda}{\mu}\right)^k}$$



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# M/M/1

M/G/1 (service times are exponential)

$$P_0 = \frac{1}{\left(1 + \frac{\lambda/\mu}{1-(\lambda/\mu)}\right)} = \frac{1}{\left(\frac{1-(\lambda/\mu)}{1-(\lambda/\mu)} + \frac{\lambda/\mu}{1-(\lambda/\mu)}\right)}$$

$$P_0 = \frac{1}{\left(\frac{1}{1-(\lambda/\mu)}\right)} = 1 - \frac{\lambda}{\mu}$$

- Which is the solution for $P_0$ from the M/G/1 Queue in Table 6.3

- Utilizing these, we can substitute back to get

$$P_k = P_0\left(\frac{\lambda}{\mu}\right)^k = \left(1-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^k = (1-\rho)\rho^k$$

- Which is the formula indicated in Table 6.4
- The other values can also be derived in a similar manner



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## Homework

Read about 'Steady state behavior for finite population model' and write an article about it.

Deadline – February 14, 2005



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