Lab 8 - Kalman Filter

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'KFAS' R package notation

For the linear Gaussian state space model with continuous states and discrete time intervals $t = 1, ..., n$, we have

$$\begin{aligned} \mathbf{y}_t &= (\mathbf{d}_t) + \mathbf{Z}_t\boldsymbol{\alpha}_t + \boldsymbol{\epsilon}_t \quad \boldsymbol{\epsilon}_t \sim \mathcal{N}(\mathbf{0}, \mathbf{H}_t) \quad \text{(observation equation)} \quad (1) \\ \boldsymbol{\alpha}_{t+1} &= (\mathbf{c}_t) + \mathbf{T}_t\boldsymbol{\alpha}_t + \mathbf{R}_t\boldsymbol{\eta}_t \quad \boldsymbol{\eta}_t \sim \mathcal{N}(\mathbf{0}, \mathbf{Q}_t) \quad \text{(state equation)} \quad (2) \end{aligned}$$

where

- $\mathbf{y}_t$ and $\boldsymbol{\epsilon}_t$ are $p \times 1$ vectors
- $\boldsymbol{\alpha}_{t+1}$ is a $m \times 1$ vector
- $\boldsymbol{\eta}_t$ is a $k \times 1$ vector
- $\boldsymbol{\eta}_t$ and $\boldsymbol{\epsilon}_t$ are assumed serially and mutually uncorrelated, i.e., $\mathbb{E}[\boldsymbol{\eta}_t\boldsymbol{\epsilon}'_s] = \mathbf{0} \ \forall t, s$.
- $\boldsymbol{\alpha}_1 \sim \mathcal{N}(\mathbf{a}_1, \mathbf{P}_1)$

See ?KFAS on R for more information.

Example of Gaussian state space model

Our time series consists of yearly alcohol-related deaths per 100,000 persons in Finland for the years 1969–2007 in the age group of 40–49 years. For the observations $y_1, ..., y_n$, we assume that $\frac{\texttt{deaths}_t}{\texttt{population}_t} = y_t \sim \mathcal{N}(\mu_t, \sigma_\epsilon)\ \forall t = 1, ..., n$, where μ_t is a random walk with drift process. We have:

$$\begin{aligned} y_t &\sim \mathcal{N}(\mu_t, \sigma_\epsilon^2) \quad \text{(observation equation)} && (3) \\ \mu_{t+1} &= \mu_t + \nu + \eta_t \quad \eta_t \sim \mathcal{N}(0, \sigma_\eta^2) \quad \text{(state equation)} && (4) \end{aligned}$$

We assume we have no prior information about the initial state μ_1 or the constant slope ν.

The Equations above could be rewritten as

$$\begin{aligned} y_t &= \mu_t + \epsilon_t \quad \epsilon_t \sim \mathcal{N}(0, \sigma_\epsilon^2) \quad \text{(observation equation)} && (5) \\ \mu_{t+1} &= \mu_t + \nu + \eta_t \quad \eta_t \sim \mathcal{N}(0, \sigma_\eta^2) \quad \text{(state equation)} && (6) \end{aligned}$$

We recall that

$$\begin{aligned} \mathbf{y}_t &= (\mathbf{d}_t) + \mathbf{Z}_t\boldsymbol{\alpha}_t + \boldsymbol{\epsilon}_t \quad \boldsymbol{\epsilon}_t \sim \mathcal{N}(\mathbf{0}, \mathbf{H}_t) \quad \text{(observation equation)} && (7) \\ \boldsymbol{\alpha}_{t+1} &= (\mathbf{c}_t) + \mathbf{T}_t\boldsymbol{\alpha}_t + \mathbf{R}_t\boldsymbol{\eta}_t \quad \boldsymbol{\eta}_t \sim \mathcal{N}(\mathbf{0}, \mathbf{Q}_t) \quad \text{(state equation)} && (8) \\ \boldsymbol{\alpha}_1 &\sim \mathcal{N}(\mathbf{a}_1, \mathbf{P}_1) && (9) \end{aligned}$$

So we define

$$\mathbf{Z} = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{H} = \begin{bmatrix} \sigma_\epsilon^2 \end{bmatrix}$$

$$\boldsymbol{\alpha}_t = \begin{bmatrix} \mu_t \\ \nu_t \end{bmatrix}$$

$$\mathbf{T} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{R} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{Q} = \begin{bmatrix} \sigma_\eta^2 \end{bmatrix}$$

$$\mathbf{a}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\mathbf{P}_{*,1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{P}_{\infty,1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

where $\mathbf{P}_1 = k\mathbf{P}_{\infty,1} + \mathbf{P}_{*,1}$, with $k \to \infty$ (*exact diffuse initialization method* for an uninformative diffuse prior).

Example of non-Gaussian state space model

The alcohol-related deaths can also be modeled naturally as a Poisson process. We recall that in this case we have, being $l(\cdot)$ a link function,

$$
\begin{aligned}
\mathbb{E}(\mathbf{y}_t) &= \boldsymbol{\mu}_t \quad \text{with} \quad l(\boldsymbol{\mu}_t) = \mathbf{Z}_t \boldsymbol{\alpha}_t \quad \text{(observation equation)} && (10)\\
\boldsymbol{\alpha}_{t+1} &= (\mathbf{c}_t) + \mathbf{T}_t \boldsymbol{\alpha}_t + \mathbf{R}_t \boldsymbol{\eta}_t \quad \boldsymbol{\eta}_t \sim \mathcal{N}(\mathbf{0}, \mathbf{Q}_t) \quad \text{(state equation)} && (11)\\
\boldsymbol{\alpha}_1 &\sim \mathcal{N}(\mathbf{a}_1, \mathbf{P}_1) && (12)
\end{aligned}
$$

Now our observations $y_t = \texttt{deaths}_t$ are the actual counts of alcohol-related deaths in year t, whereas the varying population size is taken into account by the exposure term u_t.

$$
\begin{aligned}
p(y_t|\mu_t) &= \text{Poisson}(u_t\, e^{\mu_t}) \quad \text{(observation equation)} && (13)\\
\mu_{t+1} &= \mu_t + \nu + \eta_t \quad \eta_t \sim \mathcal{N}(0, \sigma_\eta^2) \quad \text{(state equation)} && (14)
\end{aligned}
$$

$u_t = \texttt{population}_t$ being exposure term. We assume we have no prior information about the initial state μ_1 or the constant slope ν.