Quality Data Analysis

Control charts for small persistent shifts: CUSUM – EWMA – MA

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[Course slides of Prof. Bianca Maria Colosimo]

Reference: Montgomery – Introduction to Statistical Quality Control: Chapter 7

Remark

The control chart scheme (Shewhart W.A. – 1931) consists of the (repeated) application of a test on the last sample (i.e., it exploits only the last available information).

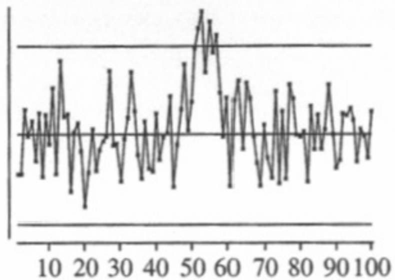
Why not use ALL the available information?

The following control chart are based on this idea:

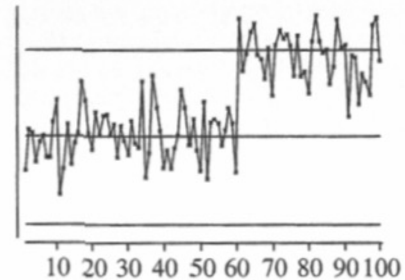
- CUmulative SUMmation – CUSUM
- Exponentially Weighted Moving Average – EWMA
- Moving Average – MA

Suitable to detect small persistent shifts of the process mean.

MA, CUSUM and EWMA are suitable procedures to detect (non-random) small shifts of the process mean

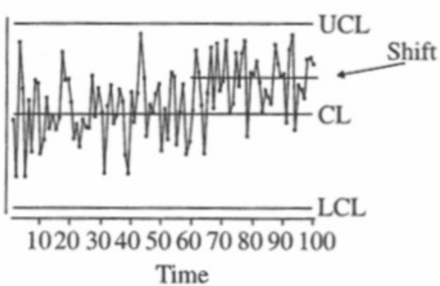


(a) **Shewhart control limits detecting a temporary shift**

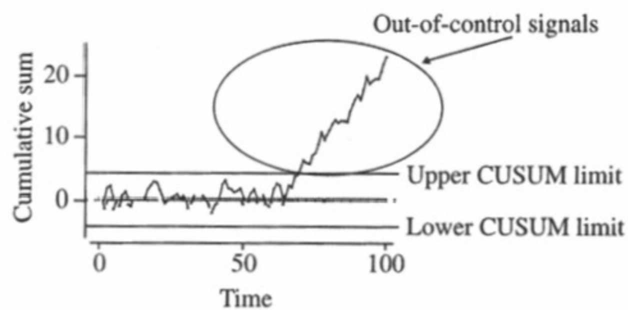


(b) **Shewhart control limits detecting a sustaining shift**

Shewhart:
Large shifts of the process mean – short or long duration.



(a) **Shewhart control chart fails to detect small shift**



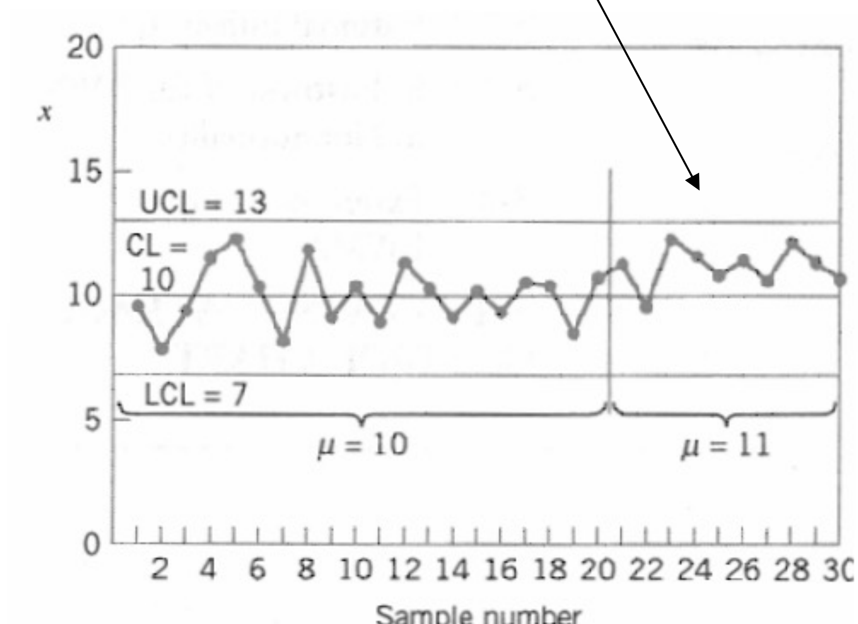
(b) **CUSUM control chart detects small shift**

MA, CUSUM, EWMA:
Small shifts of the process mean – long duration.

Example

	Sample, i	(a) x_i
	1	9.45
	2	7.99
	3	9.29
	4	11.66
	5	12.16
	6	10.18
	7	8.04
	8	11.46
	9	9.20
$N(10,1)$	10	10.34
	11	9.03
	12	11.47
	13	10.51
	14	9.40
	15	10.08
	16	9.37
	17	10.62
	18	10.31
	19	8.52
	20	10.84
	21	10.90
	22	9.33
	23	12.29
	24	11.50
	25	10.60
$N(11,1)$	26	11.08
	27	10.38
	28	11.62
	29	11.31
	30	10.52

In-control?



The MA control chart

$$M_i = \frac{\bar{x}_i + \bar{x}_{i-1} + \ldots + \bar{x}_{i-w+1}}{w}$$

$$V(M_i) = \frac{1}{w^2} \sum_{j=i-w+1}^{i} V(\bar{x}_j) = \frac{1}{w^2} \sum_{j=i-w+1}^{i} \frac{\sigma^2}{n} = \frac{\sigma^2}{nw}$$

$$\begin{cases} UCL = \mu + 3\dfrac{\sigma}{\sqrt{nw}} \\ LCL = \mu - 3\dfrac{\sigma}{\sqrt{nw}} \end{cases}$$

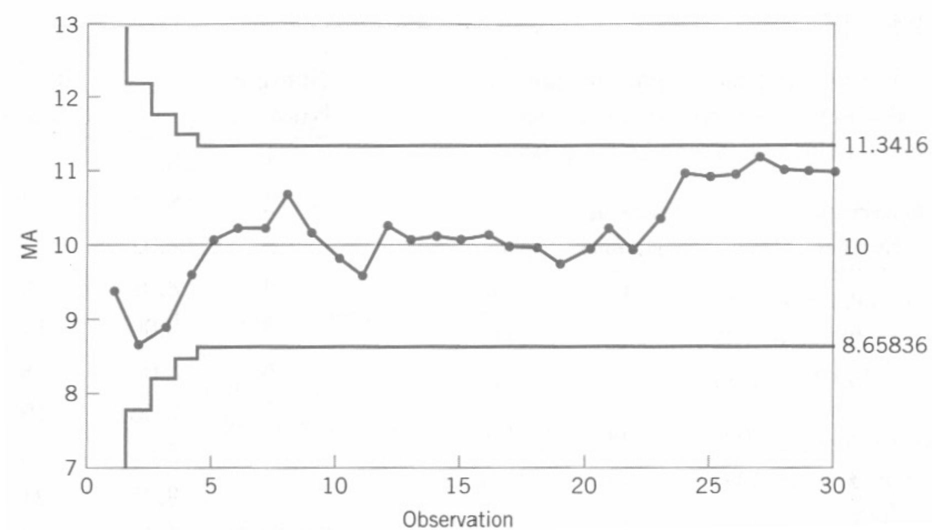
-If estimated from data: $\hat{\mu} = \bar{\bar{x}}$

-If individual measures: $n = 1 \quad x \rightarrow \bar{x} \quad \hat{\mu} = \bar{x}$

Table 8-13 Moving Average Chart for Example 8-3

Observation, i	x_i	M_i
1	9.45	9.45
2	7.99	8.72
3	9.29	8.91
4	11.66	9.5975
5	12.16	10.11
6	10.18	10.256
7	8.04	10.266
8	11.46	10.7
9	9.2	10.208
10	10.34	9.844
11	9.03	9.614
12	11.47	10.3
13	10.51	10.11
14	9.4	10.15
15	10.08	10.098
16	9.37	10.166
17	10.62	9.996
18	10.31	9.956
19	8.52	9.78
20	10.84	9.932
21	10.9	10.238
22	9.33	9.98
23	12.29	10.376
24	11.5	10.972
25	10.6	10.924
26	11.08	10.96
27	10.38	11.17
28	11.62	11.036
29	11.31	10.998
30	10.52	10.982

The MA control chart



$$M_i = \frac{x_i + x_{i-1} + \cdots + x_{i-4}}{5}$$

$$\begin{cases} \text{UCL} = \mu_0 + \dfrac{3\sigma}{\sqrt{w}} = 10 + \dfrac{3(1.0)}{\sqrt{5}} = 11.34 \\ \\ \text{LCL} = \mu_0 - \dfrac{3\sigma}{\sqrt{w}} = 10 - \dfrac{3(1.0)}{\sqrt{5}} = 8.66 \end{cases}$$

A comparison of control charting schemes

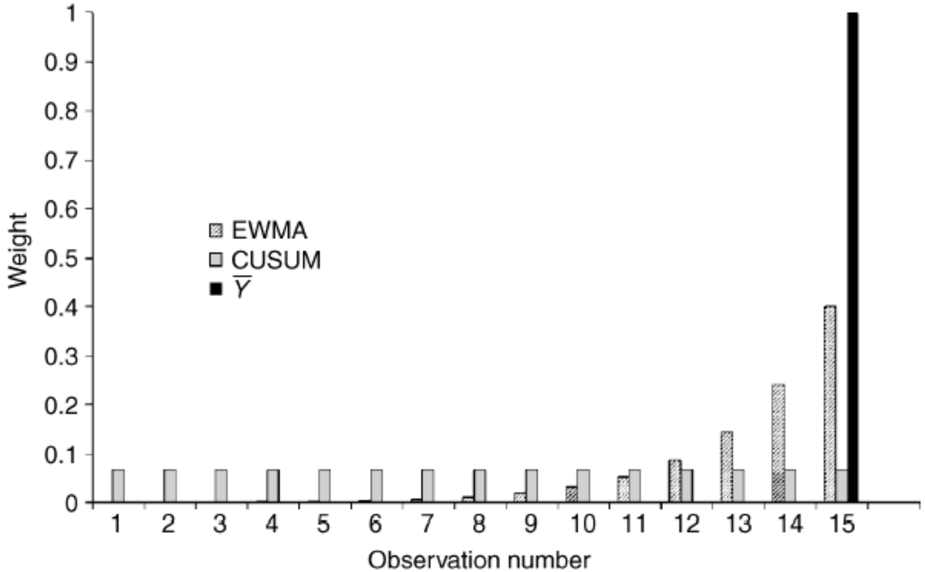
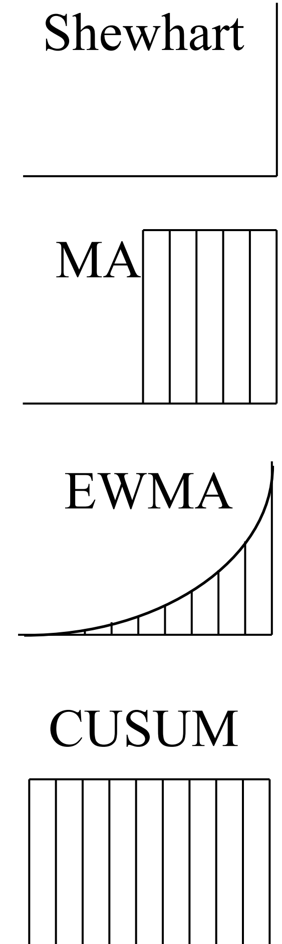


Figure 1.7 Different weights given to the last 15 observations of a process. The weights for the EWMA chart correspond to a value $\lambda = 0.4$.



Control charts for small shifts: the EWMA control chart

Exponentially Weighted Moving Average (EWMA)

- For samples of size $n \geq 1$, the EWMA for process mean is defined as:

$$\boxed{z_i = \lambda \bar{x}_i + (1-\lambda) z_{i-1}}$$

where $0 < \lambda \leq 1$ is a constant and $z_0 = \mu_0 =$ target (or $z_0 = \hat{\mu}_0 = \bar{\bar{x}}$)

- The control limits are computed as follows:

$$V_i = z_i \qquad \begin{cases} LCL = \mu_{V_i} - L\sigma_{V_i} \\ CL = \mu_{V_i} \\ UCL = \mu_{V_i} + L\sigma_{V_i} \end{cases}$$

Where L is a constant (typically L=3)

The EWMA control chart

$$X \sim \mathrm{NID}(\mu_0, \sigma^2) \Rightarrow \bar{X} \sim \mathrm{NID}(\mu_0, \frac{\sigma^2}{\mathrm{n}}) \Rightarrow z_i \sim \mathrm{N}(?,?)$$

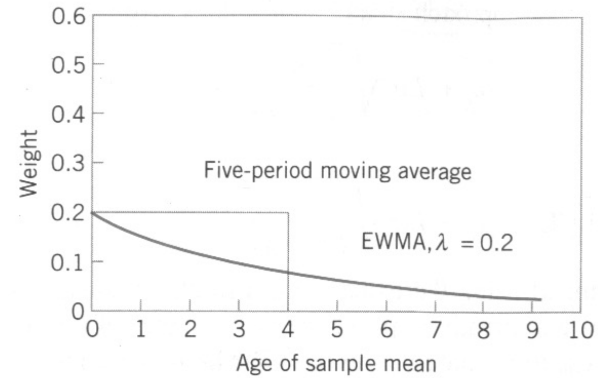
$$z_i = \lambda \bar{x}_i + (1-\lambda) z_{i-1} = \lambda \bar{x}_i + (1-\lambda)[\lambda \bar{x}_{i-1} + (1-\lambda) z_{i-2}] =$$

$$= \lambda \bar{x}_i + \lambda(1-\lambda)\bar{x}_{i-1} + (1-\lambda)^2 z_{i-2}$$

$$= \lambda \bar{x}_i + \lambda(1-\lambda)\bar{x}_{i-1} + (1-\lambda)^2 [\lambda \bar{x}_{i-2} + (1-\lambda) z_{i-3}] =$$

$$= \lambda \bar{x}_i + \lambda(1-\lambda)\bar{x}_{i-1} + \lambda(1-\lambda)^2 \bar{x}_{i-2} + (1-\lambda)^3 z_{i-3}$$

$$\ldots$$



$$\boxed{z_i = \lambda \sum_{j=0}^{i-1} (1-\lambda)^j \bar{x}_{i-j} + (1-\lambda)^i z_0}$$

Reminder: (finite) sum of terms in a geometric progression: $\sum_{j=0}^{i-1} q^j = \frac{1-q^i}{1-q}$

$$E(z_i) = \lambda \sum_{j=0}^{i-1} (1-\lambda)^j E(\bar{x}_{i-j}) + (1-\lambda)^i \mu_0 = \frac{\lambda[1-(1-\lambda)^i]}{1-(1-\lambda)} \mu_0 + (1-\lambda)^i \mu_0 =$$

$$= [1-(1-\lambda)^i]\mu_0 + (1-\lambda)^i \mu_0 = \mu_0$$

The EWMA control chart

$$\boxed{z_i = \lambda \sum_{j=0}^{i-1} (1-\lambda)^j \bar{x}_{i-j} + (1-\lambda)^i z_0} \qquad \sum_{j=0}^{i-1} q^j = \frac{1-q^i}{1-q}$$

$V(z_0)$=0 or negligible

$$V(z_i) = \lambda^2 \sum_{j=0}^{i-1} (1-\lambda)^{2j} V(\bar{x}_{i-j}) = \lambda^2 \sum_{j=0}^{i-1} (1-\lambda)^{2j} \frac{\sigma^2}{n} = \frac{\lambda^{\cancel{2}}[1-(1-\lambda)^{2i}]}{\cancel{1}-\cancel{1}-\lambda^{\cancel{2}}+2\cancel{\lambda}} \frac{\sigma^2}{n}$$

$$= \frac{\lambda}{2-\lambda}[1-(1-\lambda)^{2i}]\frac{\sigma^2}{n} \quad \overset{i \to \infty}{\Rightarrow} \frac{\lambda}{2-\lambda}\frac{\sigma^2}{n} \le \frac{\sigma^2}{n}$$

Note: $E(z_i)$ and $V(z_i)$ estimated under IID hypothesis (not NID)

$$\begin{cases} LCL = \mu_{V_i} - L\sigma_{V_i} \\ CL = \mu_{V_i} \\ UCL = \mu_{V_i} + L\sigma_{V_i} \end{cases} \qquad \begin{cases} LCL = \mu_0 - L\frac{\sigma}{\sqrt{n}}\sqrt{\frac{\lambda}{2-\lambda}\left[1-(1-\lambda)^{2i}\right]} \\ CL = \mu_0 \\ UCL = \mu_0 + L\frac{\sigma}{\sqrt{n}}\sqrt{\frac{\lambda}{2-\lambda}\left[1-(1-\lambda)^{2i}\right]} \end{cases}$$

$$\begin{cases} LCL = \mu_0 - L\frac{\sigma}{\sqrt{n}}\sqrt{\frac{\lambda}{2-\lambda}\left[1-(1-\lambda)^{2i}\right]} \\ CL = \mu_0 \\ UCL = \mu_0 + L\frac{\sigma}{\sqrt{n}}\sqrt{\frac{\lambda}{2-\lambda}\left[1-(1-\lambda)^{2i}\right]} \end{cases}$$

for $i \to \infty$

$1-(1-\lambda)^{2i} \to 1$

$$\begin{cases} LCL = \mu_0 - L\frac{\sigma}{\sqrt{n}}\boxed{\sqrt{\frac{\lambda}{2-\lambda}}} \\ CL = \mu_0 \\ UCL = \mu_0 + L\frac{\sigma}{\sqrt{n}}\boxed{\sqrt{\frac{\lambda}{2-\lambda}}} \end{cases}$$

For single measurements (n=1): $\boxed{z_i = \lambda x_i + (1-\lambda)z_{i-1}}$

$$\begin{cases} LCL = \mu_0 - L\sigma\sqrt{\frac{\lambda}{2-\lambda}\left[1-(1-\lambda)^{2i}\right]} \\ CL = \mu_0 \\ UCL = \mu_0 + L\sigma\sqrt{\frac{\lambda}{2-\lambda}\left[1-(1-\lambda)^{2i}\right]} \end{cases}$$

for $i \to \infty$

$1-(1-\lambda)^{2i} \to 1$

$$\begin{cases} LCL = \mu_0 - \boxed{L\sigma}\sqrt{\frac{\lambda}{2-\lambda}} \\ CL = \mu_0 \\ UCL = \mu_0 + \boxed{L\sigma}\sqrt{\frac{\lambda}{2-\lambda}} \end{cases}$$

If estimated from data: $\hat{\mu}_0 = \bar{\bar{x}}$

The EWMA control chart

$Note:$

$\lambda \to 1 : z_i \to \bar{x}_i$: Shewhart chart

$\lambda \to 0$: Slowly decaying weights : $CUSUM$

$$\boxed{z_i = \lambda \sum_{j=0}^{i-1} (1-\lambda)^j \bar{x}_{i-j} + (1-\lambda)^i z_0}$$

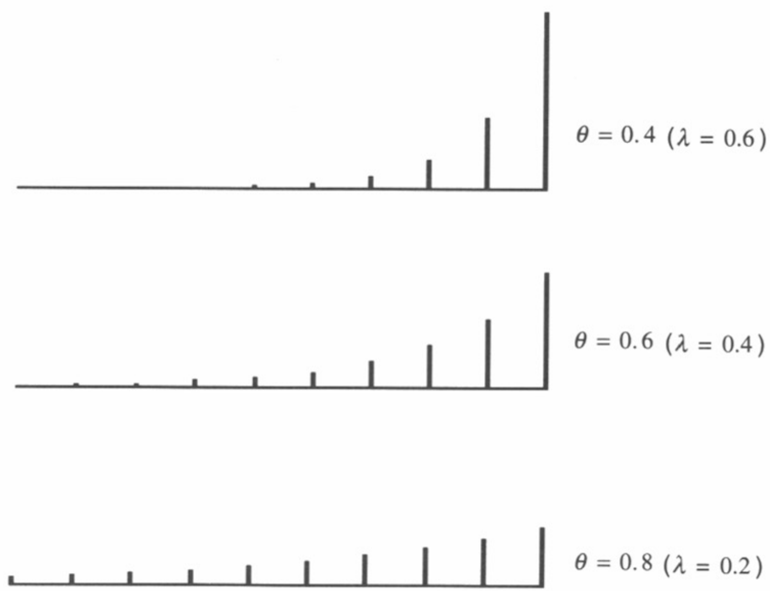
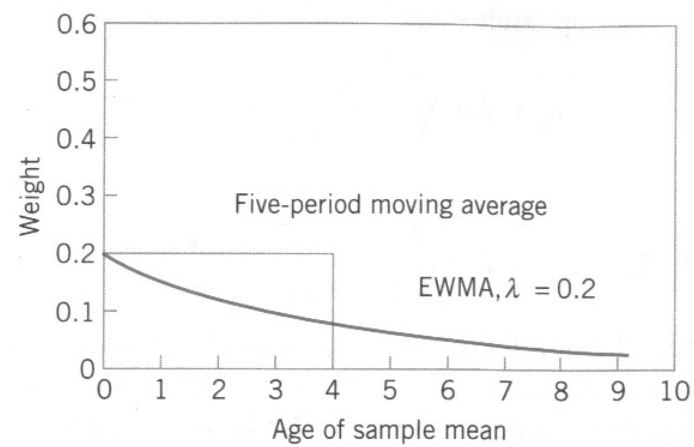


Figure 3.5 Exponential weights for $\theta = 0.4, 0.6$, and 0.8.

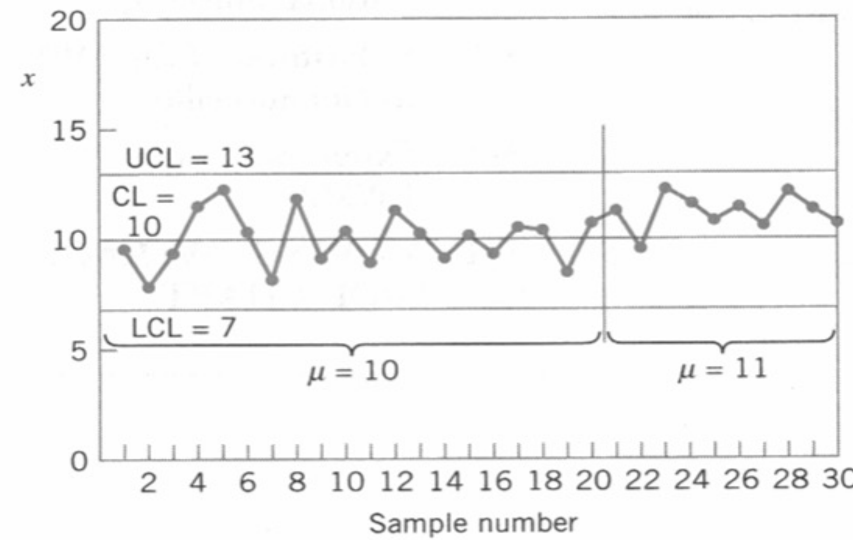


Example

Sample, i	(a) x_i
1	9.45
2	7.99
3	9.29
4	11.66
5	12.16
6	10.18
7	8.04
8	11.46
9	9.20
10	10.34
11	9.03
12	11.47
13	10.51
14	9.40
15	10.08
16	9.37
17	10.62
18	10.31
19	8.52
20	10.84
21	10.90
22	9.33
23	12.29
24	11.50
25	10.60
26	11.08
27	10.38
28	11.62
29	11.31
30	10.52

Samples 1–20: $N(10,1)$

Samples 21–30: $N(11,1)$



$$\lambda = 0.1$$
$$L = 2.7$$
$$\mu_0 = 10$$

$$z_1 = \lambda x_1 + (1-\lambda)z_0 = 0.1(9.45) + 0.9(10) = 9.945$$

$$z_2 = \lambda x_2 + (1-\lambda)z_1 = 0.1(7.99) + 0.9(9.945) = 9.7495$$

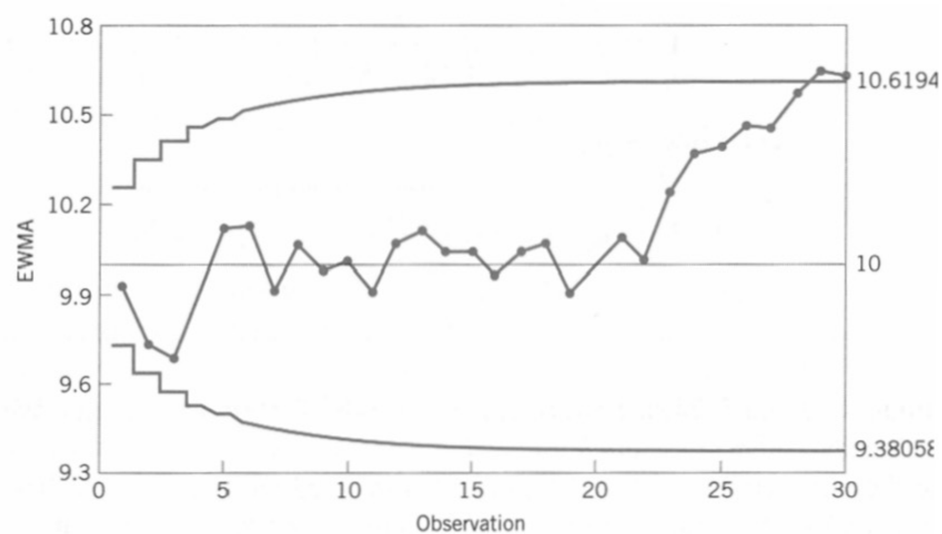
$$
\left\{
\begin{aligned}
\mathrm{UCL} &= \mu_0 + L\sigma\sqrt{\frac{\lambda}{(2-\lambda)}\left[1-(1-\lambda)^{2i}\right]} \\
&= 10 + 2.7(1)\sqrt{\frac{0.1}{(2-0.1)}\left[1-(1-0.1)^{2(1)}\right]} \\
&= 10.27 \\[1em]
\mathrm{LCL} &= \mu_0 - L\sigma\sqrt{\frac{\lambda}{(2-\lambda)}\left[1-(1-\lambda)^{2i}\right]} \\
&= 10 - 2.7(1)\sqrt{\frac{0.1}{(2-0.1)}\left[1-(1-0.1)^{2(1)}\right]} \\
&= 9.73
\end{aligned}
\right.
$$

$$
\left\{
\begin{aligned}
\mathrm{UCL} &= \mu_0 + L\sigma\sqrt{\frac{\lambda}{(2-\lambda)}\left[1-(1-\lambda)^{2i}\right]} \\
&= 10 + 2.7(1)\sqrt{\frac{0.1}{(2-0.1)}\left[1-(1-0.1)^{2(2)}\right]} \\
&= 10.36 \\[1em]
\mathrm{LCL} &= \mu_0 - L\sigma\sqrt{\frac{\lambda}{(2-\lambda)}\left[1-(1-\lambda)^{2i}\right]} \\
&= 10 - 2.7(1)\sqrt{\frac{0.1}{(2-0.1)}\left[1-(1-0.1)^{2(2)}\right]} \\
&= 9.64
\end{aligned}
\right.
$$

$$
\left\{
\begin{aligned}
\mathrm{LCL} &= \mu_0 + L\sigma\sqrt{\frac{\lambda}{(2-\lambda)}} \\
&= 10 + 2.7(1)\sqrt{\frac{0.1}{(2-0.1)}} \\
&= 10.62 \\[1em]
\mathrm{LCL} &= \mu_0 - L\sigma\sqrt{\frac{\lambda}{(2-\lambda)}} \\
&= 10 - 2.7(1)\sqrt{\frac{0.1}{(2-0.1)}} \\
&= 9.38
\end{aligned}
\right.
$$

Example

Subgroup, i	* = Beyond Limits x_i	EWMA, z_i
1	9.45	9.945
2	7.99	9.7495
3	9.29	9.70355
4	11.66	9.8992
5	12.16	10.1253
6	10.18	10.1307
7	8.04	9.92167
8	11.46	10.0755
9	9.2	9.98796
10	10.34	10.0232
11	9.03	9.92384
12	11.47	10.0785
13	10.51	10.1216
14	9.4	10.0495
15	10.08	10.0525
16	9.37	9.98426
17	10.62	10.0478
18	10.31	10.074
19	8.52	9.91864
20	10.84	10.0108
21	10.9	10.0997
22	9.33	10.0227
23	12.29	10.2495
24	11.5	10.3745
25	10.6	10.3971
26	11.08	10.4654
27	10.38	10.4568
28	11.62	10.5731
29	11.31	10.6468*
30	10.52	10.6341*



Design of an EWMA control chart

The design parameters of the EWMA control chart are L and λ.

They can be chosen to give ARL performances that approximate the desired ones

Generally speaking, $0.05 \leq \lambda \leq 0.25$ is the range of typically applied values ($\lambda = 0.20$ is the most common choice)

$L = 3$ is the value to be used in combination with large values of λ.

L ranging between 2.6 and 2.8 is suitable for small values of λ (i.e., $\lambda \leq 0.1$).

Table 8-10 Average Run Lengths for Several EWMA Control Schemes [Adapted from Lucas and Saccucci (1990)]

Shift in Mean (multiple of σ)	$L = 3.054$ $\lambda = 0.40$	2.998 0.25	2.962 0.20	2.814 0.10	2.615 0.05
0	500	500	500	500	500
0.25	224	170	150	106	84.1
0.50	71.2	48.2	41.8	31.3	28.8
0.75	28.4	20.1	18.2	15.9	16.4
1.00	14.3	11.1	10.5	10.3	11.4
1.50	5.9	5.5	5.5	6.1	7.1
2.00	3.5	3.6	3.7	4.4	5.2
2.50	2.5	2.7	2.9	3.4	4.2
3.00	2.0	2.3	2.4	2.9	3.5
4.00	1.4	1.7	1.9	2.2	2.7

	EWMA			Shewhart
λ	0.05	0.1	0.2	
L	2.492	2.703	2.86	3.00
Normal	370.4	370.8	370.5	370.4

Robustness of the EWMA to nonnormality

Analogously to the CUSUM chart, the EWMA chart is suitable to detect small shift of the process mean, but it not able to rapidly detect large shifts (contrary to the Shewhart chart).

The EWMA chart often outperforms the CUSUM for large shifts, especially when $\lambda > 0.1$

The EWMA control chart is robust against departures from normality

Table 8-11 In-Control ARLs for the EWMA and the Individuals Control Charts for Various Gamma Distributions

	EWMA			Shewhart
λ	0.05	0.1	0.2	
L	2.492	2.703	2.86	3.00
Normal	370.4	370.8	370.5	370.4
Gam(4, 1)	372	341	259	97
Gam(3, 1)	372	332	238	85
Gam(2, 1)	372	315	208	71
Gam(1, 1)	369	274	163	55
Gam(0.5, 1)	357	229	131	45

Table 8-12 In-Control ARLs for the EWMA and the Individuals Control Charts for Various *t* Distributions

	EWMA			Shewhart
λ	0.05	0.1	0.2	1
L	2.492	2.703	2.86	3.00
Normal	370.4	370.8	370.5	370.4
t_{50}	369	365	353	283
t_{40}	369	363	348	266
t_{30}	368	361	341	242
t_{20}	367	355	325	204
t_{15}	365	349	310	176
t_{10}	361	335	280	137
t_{8}	358	324	259	117
t_{6}	351	305	229	96
t_{4}	343	274	188	76

The CUSUM control chart (for process mean)

- The CUSUM control chart directly incorporates all the information in the sequence of sample values by plotting the cumulative sums of the deviations of the sample values from a target value (μ_0)

- $\bar{x}_j$ is the mean of the j[th] sample:

$$C_1 = \bar{x}_1 - \mu_0$$
$$C_2 = \bar{x}_2 - \mu_0 + \bar{x}_1 - \mu_0 = C_1 + \bar{x}_2 - \mu_0$$
$$C_3 = \bar{x}_3 - \mu_0 + \bar{x}_2 - \mu_0 + \bar{x}_1 - \mu_0 = C_2 + \bar{x}_3 - \mu_0$$
$$\ldots$$

- The cumulative sum is:

$$\boxed{C_i = \sum_{j=1}^{i} (\bar{x}_j - \mu_0) = C_{i-1} + \bar{x}_i - \mu_0}$$

- If (generally) $n = 1$

$$\boxed{C_i = \sum_{j=1}^{i} (x_j - \mu_0) = C_{i-1} + x_i - \mu_0}$$

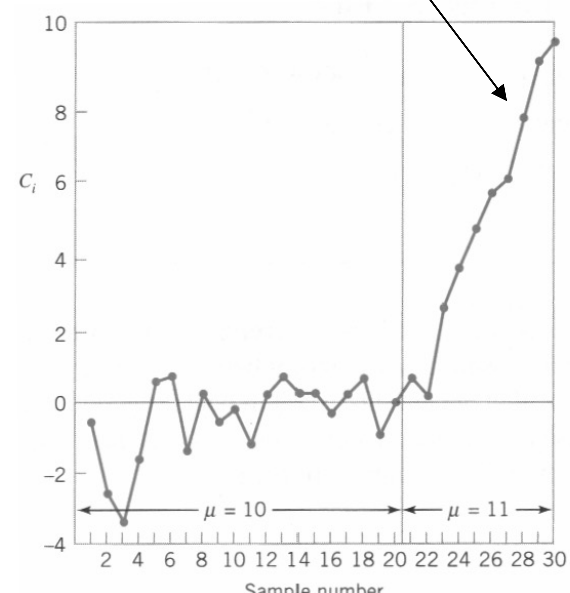
- It can be used to monitor the process mean, the process variability and it can be adapted to particular processes (Poisson and binomial variables)

The CUSUM control chart (for process mean)

	Sample, i	(a) x_i	(b) $x_i - 10$	(c) $C_i = (x_i - 10) + C_{i-1}$
$N(10,1)$	1	9.45	−0.55	−0.55
	2	7.99	−2.01	−2.56
	3	9.29	−0.71	−3.27
	4	11.66	1.66	−1.61
	5	12.16	2.16	0.55
	6	10.18	0.18	0.73
	7	8.04	−1.96	−1.23
	8	11.46	1.46	0.23
	9	9.20	−0.80	−0.57
	10	10.34	0.34	−0.23
	11	9.03	−0.97	−1.20
	12	11.47	1.47	0.27
	13	10.51	0.51	0.78
	14	9.40	−0.60	0.18
	15	10.08	0.08	0.26
	16	9.37	−0.63	−0.37
	17	10.62	0.62	0.25
	18	10.31	0.31	0.56
	19	8.52	−1.48	−0.92
	20	10.84	0.84	−0.08
$N(11,1)$	21	10.90	0.90	0.82
	22	9.33	−0.67	0.15
	23	12.29	2.29	2.44
	24	11.50	1.50	3.94
	25	10.60	0.60	4.54
	26	11.08	1.08	5.62
	27	10.38	0.38	6.00
	28	11.62	1.62	7.62
	29	11.31	1.31	8.93
	30	10.52	0.52	9.45

$$
\begin{aligned}
C_i &= \sum_{j=1}^{i}(x_j - 10) \\
&= (x_i - 10) + \sum_{j=1}^{i-1}(x_j - 10) \\
&= (x_i - 10) + C_{i-1}
\end{aligned}
$$

Trend



The CUSUM control chart (for process mean)

- $n = 1$: let x_i be the i^{th} observation of the monitored process, by assuming σ to be known (or to be estimated from data)
- We define the reference value, K:
  - It is often chosen about halfway between the target, μ_0, and the out-of-control value of the mean, μ_1, that we are interested in detecting quickly.
  - If the shift is expressed in standard deviation units as $\mu_1 = \mu_0 + \Delta = \mu_0 + \delta\sigma$, then K is one-half the magnitude of the shift, i.e.:

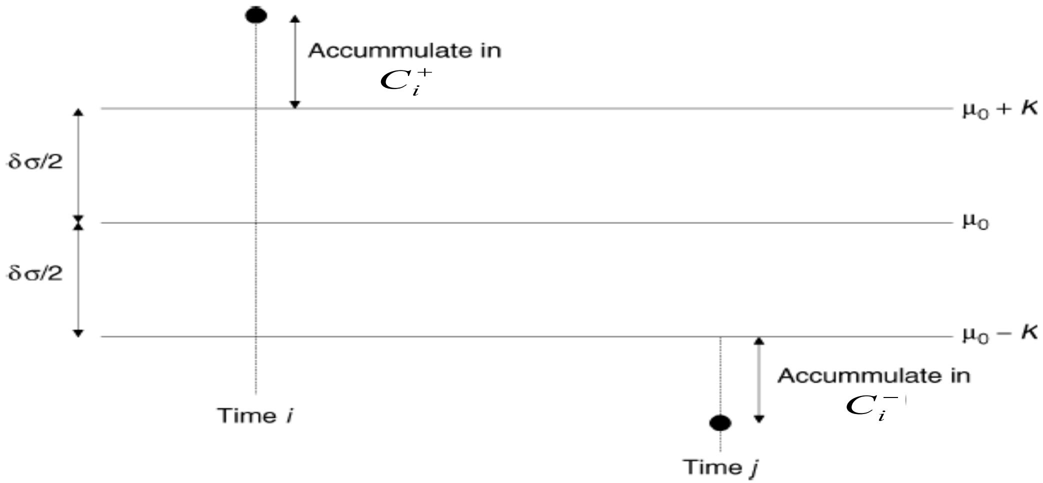
$$K = \frac{\Delta}{2} = \frac{\delta}{2}\sigma = \frac{|\mu_1 - \mu_0|}{2}$$

$$K = k\sigma \quad \text{where } k = \frac{\delta}{2}$$

The CUSUM control chart (for process mean)

The tabular CUSUM works by accumulating deviations from μ_0:

- Deviations that are **above** target μ_0 (larger than) – called ***upper cusum*** (accumulator) C^+
- Deviations that are **below** target μ_0 (smaller than) – called ***lower cusum*** (accumulator) C^-



$$\boxed{\begin{aligned} C_i^+ &= \max\left[0, x_i - (\mu_0 + K) + C_{i-1}^+\right] \\ C_i^- &= \max\left[0, (\mu_0 - K) - x_i + C_{i-1}^-\right] \end{aligned}}$$

$$\boxed{C_0^+ = C_0^- = 0}$$

The CUSUM control chart (for process mean)

Out-of-control decision rule:

If at least one of the two statistics violates a control limit, H, the process is considered out-of-control:

$H = h\sigma$ where h is often set to 5: $H=5\sigma$

$$x_i \leftarrow \bar{x}_i \qquad \begin{cases} C_i^+ = \max[0, \bar{x}_i - (\mu_0 + K) + C_{i-1}^+] \\ C_i^- = \max[0, (\mu_0 - K) - \bar{x}_i + C_{i-1}^-] \end{cases}$$

Note:

If $n > 1$ (control chart for process mean):

$$\sigma \leftarrow \sigma_{\bar{X}} = \sigma / \sqrt{n} \qquad \begin{cases} \mu_1 = \mu_0 + \Delta \rightarrow \Delta = \delta\sigma_{\bar{X}} \\ \mathrm{K} = k\sigma_{\bar{X}} = \dfrac{\Delta}{2} = \dfrac{\delta}{2}\sigma_{\bar{X}} \\ \mathrm{H} = h\sigma_{\bar{X}} \end{cases}$$

The CUSUM control chart (for process mean)

- μ_0 = 10, n = 1, σ = 1
- The magnitude of the shift we want to detect is: 1.0 σ = 1.0 (1.0) = 1.0
- The out-of-control value of the process mean is: μ_1= 10 + 1 = 11
- K = k σ = ½ σ = ½ we will use: H = 5σ = 5
- The equations for the upper e lower cusums are:

$$\mu_0 + K = 10.5 \rightarrow C_i^+ = \max\left[0, x_i - 10.5 + C_{i-1}^+\right]$$
$$\mu_0 - K = 9.5 \rightarrow C_i^- = \max\left[0, 9.5 - x_i + C_{i-1}^-\right]$$

The quantities N^+ and N^- indicate the number of consecutive periods that the susums C_i^+ and C_i^- have been non-zero

$$C_0^+ = C_0^- = 0,$$

Period i	x_i	$x_i - 10.5$	C_i^+	N^+	$9.5 - x_i$	C_i^-	N^-
		a			**b**		
1	9.45	−1.05	0	0	0.05	0.05	1
2	7.99	−2.51	0	0	1.51	1.56	2
3	9.29	−1.21	0	0	0.21	1.77	3
4	11.66	1.16	1.16	1	−2.16	0	0
5	12.16	1.66	2.82	2	−2.66	0	0
6	10.18	−0.32	2.50	3	−0.68	0	0
7	8.04	−2.46	0.04	4	1.46	1.46	1
8	11.46	0.96	1.00	5	−1.96	0	0
9	9.20	−1.3	0	0	0.30	0.30	1
10	10.34	−0.16	0	0	−0.84	0	0
11	9.03	−1.47	0	0	0.47	0.47	1
12	11.47	0.97	0.97	1	−1.97	0	0
13	10.51	0.01	0.98	2	−1.01	0	0
14	9.40	−1.10	0	0	0.10	0.10	1
15	10.08	−0.42	0	0	−0.58	0	0
16	9.37	−1.13	0	0	0.13	0.13	1
17	10.62	0.12	0.12	1	−1.12	0	0
18	10.31	−0.19	0	0	−0.81	0	0
19	8.52	−1.98	0	0	0.98	0.98	1
20	10.84	0.34	0.34	1	−1.34	0	0
21	10.90	0.40	0.74	2	−1.40	0	0
22	9.33	−1.17	0	0	0.17	0.17	1
23	12.29	1.79	1.79	1	−2.79	0	0
24	11.50	1.00	2.79	2	−2.00	0	0
25	10.60	0.10	2.89	3	−1.10	0	0
26	11.08	0.58	3.47	4	−1.58	0	0
27	10.38	−0.12	3.35	5	−0.88	0	0
28	11.62	1.12	4.47	6	−2.12	0	0
29	11.31	0.81	5.28	7	−1.81	0	0
30	10.52	0.02	5.30	8	−1.02	0	0

$$C_1^+ = \max[0, x_1 - 10.5 + C_0^+]$$

$$C_1^- = \max[0, 9.5 - x_1 + C_0^-]$$

$$C_1^+ = \max[0, 9.45 - 10.5 + 0] = 0$$

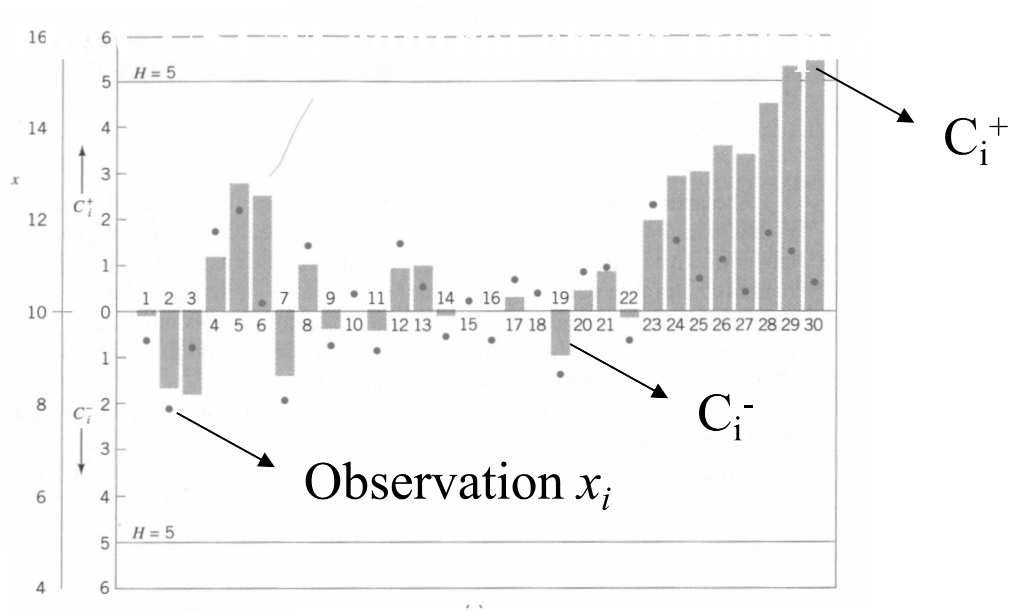
$$C_1^- = \max[0, 9.5 - 9.45 + 0] = 0.05$$

$$C_2^+ = \max[0, x_2 - 10.5 + C_1^+]$$
$$= \max[0, x_2 - 10.5 + 0]$$

$$C_2^- = \max[0, 9.5 - x_2 + C_1^-]$$
$$= \max[0, 9.5 - x_2 + 0.05]$$

$$C_2^+ = \max[0, 7.99 - 10.5 + 0] = 0$$

$$C_2^- = \max[0, 9.5 - 7.99 + 0.05] = 1.56$$

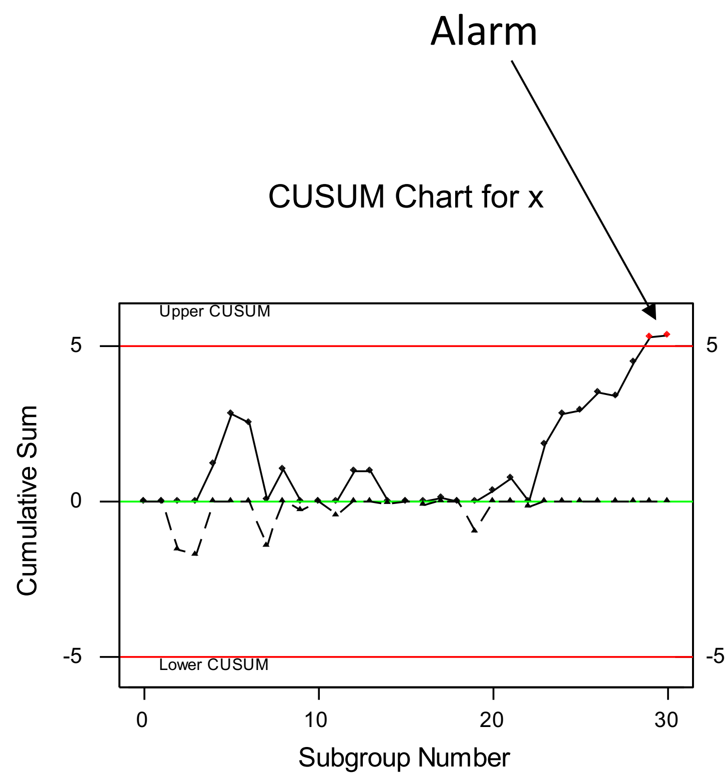


The process is out-of-control

We need to:

- Search for an assignable cause and possible corrective actions are required
- Re-initialize the cusums at zero.

We need to know the magnitude of the shift



Remarks

It may be helpful to standardize the process variables x_i before applying the CUSUM control chart. Then:

$$y_i = \frac{x_i - \mu_0}{\sigma}$$

$$C_i^+ = \max\left[0, x_i - (\mu_0 + K) + C_{i-1}^+\right] \qquad H = h \qquad C_i^+ = \max\left[0, y_i - k + C_{i-1}^+\right]$$

$$C_i^- = \max\left[0, (\mu_0 - K) - x_i + C_{i-1}^-\right] \qquad K = k \qquad C_i^- = \max\left[0, -k - y_i + C_{i-1}^-\right]$$

Alarm if:

$$C_i^+ > h$$

$$C_i^- > h$$

- The Shewhart control chart is generally enhanced by using large sample sizes, but this is not necessarily true for the CUSUM control chart
- Generally speaking, the CUSUM control chart for the process mean is applied on single measures
- It is possible to use one-sided charts (one-sided CUSUM).

$$H = h\sigma \qquad K = k\sigma$$

Table 8-3 ARL Performance of the Tabular Cusum with $k = \frac{1}{2}$ and $h = 4$ or $h = 5$

Shift in Mean (multiple of σ)	$h = 4$	$h = 5$	ARL (Shewhart chart wih K=3)
0	168	465	370.38
0.25	74.2	139	281.14
0.50	26.6	38.0	155.22
0.75	13.3	17.0	81.22
1.00	8.38	10.4	43.89
1.50	4.75	5.75	14.97
2.00	3.34	4.01	6.30
2.50	2.62	3.11	3.24
3.00	2.19	2.57	2.00
4.00	1.71	2.01	1.19

Table 8-4 Values of k and the Corresponding Values of h That Give $ARL_0 = 370$ for the Two-Sided Tabular Cusum [from Hawkins (1993a)]

k	0.25	0.5	0.75	1.0	1.25	1.5
h	8.01	4.77	3.34	2.52	1.99	1.61

The CUSUM control chart

The CUSUM control chart is not as effective as the Shewhart chart in detecting large shifts of the process mean.

An alternative solution is to use a combined CUSUM-Shewhart procedure to enhance the performances of both charts.

Table 8-5 ARL Values for Some Modifications of the Basic Cusum with $k = \frac{1}{2}$ and $h = 5$ (If subgroups of size $n > 1$ are used, then $\sigma = \sigma_{\bar{x}} = \sigma/\sqrt{n}$)

Shift in Mean (multiple of σ)	(a) Basic Cusum	(b) Cusum-Shewhart (Shewhart limits at 3.5σ)	(c) Cusum with FIR	(d) FIR Cusum-Shewhart (Shewhart limits at 3.5σ)
0	465	391	430	360
0.25	139	130.9	122	113.9
0.50	38.0	37.20	28.7	28.1
0.75	17.0	16.80	11.2	11.2
1.00	10.4	10.20	6.35	6.32
1.50	5.75	5.58	3.37	3.37
2.00	4.01	3.77	2.36	2.36
2.50	3.11	2.77	1.86	1.86
3.00	2.57	2.10	1.54	1.54
4.00	2.01	1.34	1.16	1.16