

**FOR MORE MATERIAL EE,  
ME, CIVIL,CSE, ECE, EEE, GS  
VISIT  
NOTESGURU.INFO**

**ALL IN ONE PLATFORM FOR ENGINEERING  
STUDENTS AND GATE ASPIRANTS**

**SEE ALSO :**

---

**COLLEGE NOTES**

---

---

**UNIVERSITY NOTES**

---

---

**COMPETITION NOTES**

---

---

**STUDY MATERIAL**

---

---

**NON TECH NOTES**

---

Thanks for choosing  
notesguru



*Do not copy this material. If  
you find any website or blog  
publishing this material with  
'notesguru.info' logo or  
watermark kindly report us to  
notesguru01@gmail.com*

## Linear Algebra [Matrices]

### \* Properties of Determinants :-

i) If two rows or columns of a matrix are identical then the determinant = 0

$$\Delta = \begin{vmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & 1 & 2 \end{vmatrix} = 0$$

ii) If two rows or columns of a matrix are interchanged, then the sign of determinant changes.

$$-\Delta = \begin{vmatrix} 1 & 0 & 2 \\ 2 & 1 & 3 \\ 1 & 0 & 2 \end{vmatrix}$$

iii) If three rows or columns of matrix are interchanged, then the sign of determinant is unaltered.

$$\Delta = \begin{vmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 1 & 1 \\ 0 & 1 & 2 \end{vmatrix}$$

iv) In the determinant of a matrix, if any column containing the sum or difference of two elements then it can be split into sum or difference of two determinants.

$$\begin{vmatrix} a & a^2 & a^3+1 \\ b & b^2 & b^3+1 \\ c & c^2 & c^3+1 \end{vmatrix} = \begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix} + \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix}$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\Delta = ad - bc$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\Delta = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

\* Lower Triangular Matrix! - If all elements above the principal diagonal are zeroes.

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 1 & 2 & 5 \end{bmatrix}$$

$$\Delta = 15$$

\* Upper Triangular Matrix! - If all elements below the principal diagonal are zeroes, then it is said to be an upper triangular matrix.

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & 2 \\ 0 & 0 & 7 \end{bmatrix}$$

$$\Delta = -28$$

NOTE:-

If a matrix is either lower triangular or upper triangular then the determinant is the product of the principle diagonal elements.

Q. Find the determinant of Matrix

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$\rightarrow$

$$\begin{array}{|ccc|} \hline ① & a & a^2 \\ ② & b & b^2 \\ ③ & c & c^2 \\ \hline \end{array}$$

$$R_1 \rightarrow R_1 - R_2$$

$$R_2 \rightarrow R_2 - R_3$$

$$= \begin{vmatrix} 0 & a-b & a^2-b^2 \\ 0 & b-c & b^2-c^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$= (a-b)(b-c) \begin{vmatrix} 0 & 1 & a+b \\ 0 & 1 & b+c \end{vmatrix} = (a-b)(b-c) [bc - a^2 - b^2 + ab] \\ = (a-b)(b-c)(c-a)$$

\*

$$|A| = |A^T|$$

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$|A| = |A^T| = (a-b)(b-c)(c-a)$$

Q.

Imp  
Result

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+a & 1 \\ 1 & 1 & 1+b \end{vmatrix} \quad R_2 - R_1, R_3 - R_1$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 0 & a & 0 \\ 0 & 0 & b \end{vmatrix} = ab$$

Q.

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 5 & 1 \\ 1 & 1 & 6 \end{vmatrix} = ?$$

Ⓐ 20 Ⓑ 30 Ⓒ 40 Ⓓ 0

Q.

Imp  
Result

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1+a & 1 & 1 \\ 1 & 1 & 1+b & 1 \\ 1 & 1 & 1 & 1+c \end{vmatrix} = abc$$

a.

$$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix}$$

Imp  
Result

$$= abc \begin{vmatrix} 1+\frac{1}{a} & \frac{1}{a} & \frac{1}{a} \\ \frac{1}{b} & 1+\frac{1}{b} & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & 1+\frac{1}{c} \end{vmatrix} \quad R_1 \rightarrow R_1 + (R_2 + R_3)$$

$$= abc \left( 1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \begin{vmatrix} 1 & 1 & 1 \\ \frac{1}{b} & 1+\frac{1}{b} & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & 1+\frac{1}{c} \end{vmatrix} \quad C_2 - C_1, C_3 - C_1$$

$$= abc \left( 1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \begin{vmatrix} 1 & 0 & 0 \\ \frac{1}{b} & 1 & 0 \\ \frac{1}{c} & 0 & 1 \end{vmatrix}$$

$$= abc \left( 1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right).$$

a..

$$\begin{vmatrix} 1+a & 1 & 1 & 1 \\ 1 & 1+b & 1 & 1 \\ 1 & 1 & 1+c & 1 \\ 1 & 1 & 1 & 1+d \end{vmatrix}$$

Imp  
Result

$$= abcd \left( 1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \right)$$

Q.

$$\begin{vmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{vmatrix} = ?$$

- Ⓐ 4 Ⓛ 5 Ⓜ 6 Ⓝ 7

Q.

$$\begin{vmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{vmatrix} = 0( ) - 1(1-9) + 2(1-6) = 8 - 10 = -2$$

\* SHORTCUT to determinant of Matrix.

~~W~~ [Applicable only for  $3 \times 3$  Matrix].

Q.

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$$

$$0 + 9 + 2 - 12 - 0 - 1$$

$$= -2.$$

Q.

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

$$\Delta = 1 + 8 + 8 - 4 - 4 - 4$$

$$\Delta = 17 - 12 = 5$$

Q.

$$\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

$$\Delta = 3 + 0 - 4 - 0 - 0 + 2$$

$$\Delta = 1$$

Q.

$$\begin{bmatrix} 1 & 2 & 5 \\ 3 & 1 & 4 \\ 1 & 1 & 2 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\Delta = 2 + 8 + 15 - 5 - 4 - 12$$

$$\Delta = 25 - 21$$

$$\Delta = 4$$

## # INVERSE OF MATRIX! -

$$\boxed{A^{-1} = \frac{\text{adj } A}{\Delta}}$$

If  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

- Principle
- Interchange diagonal element
  - 2 sign change in second diagonal element.

$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Q.  $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$$A^{-1} = \frac{1}{4-6} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

Q.  $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$

$$A^{-1} = \frac{1}{10-4} \begin{bmatrix} 2 & -4 \\ -1 & 5 \end{bmatrix}$$

$$Q. A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\begin{aligned} A^{-1} &= \frac{1}{\cos^2\theta + \sin^2\theta} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \end{aligned}$$

# For  $3 \times 3$  Inverse matrix:-

$$A^{-1} = \frac{\text{adj } A}{\Delta}$$

$\text{adj } A$  = Transpose of cofactors Matrix

Cofactor of elements =  $(-1)^{i+j}$  minor

↳ Just to check sign for even or odd  
(row+column) no

Minor of an Element = determinant of sq. sub matrix in which the row & column of the particular element lines to be deleted.

ex:-

0	1	2
①	2	3
3	1	1

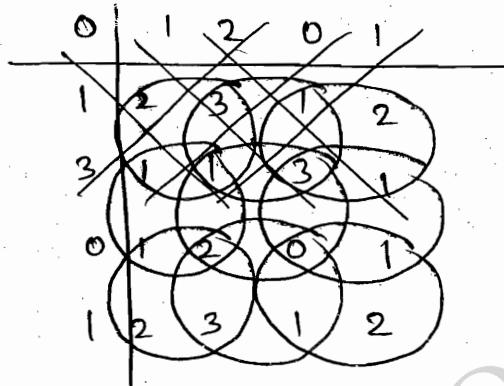
$$\text{minor of } 1 \rightarrow \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = 1-2 = -1$$

$$\text{Cofactor of } 1 = (-1)^{2+1}(-1) = 1$$

\* Shortcut to find Cofactor of  $3 \times 3$  Matrix!-

Q.  $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$

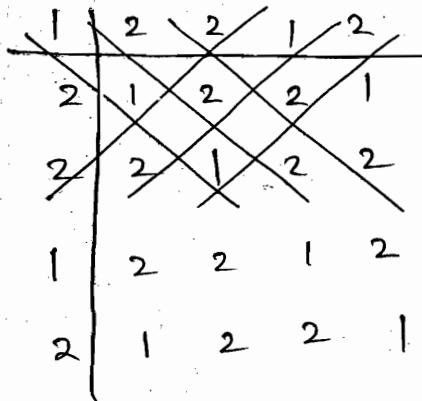
$$\Delta = 0 + 9 + 2 - 12 - 0 - 1 = -2$$



$$A^{-1} = \frac{1}{-2} \begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ 3 & -1 & 1 \end{bmatrix}$$

Q.  $\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$

$$\Delta = 1 + 8 + 8 - 4 - 4 - 4 = 17 - 12 = 5$$



$$A^{-1} = \frac{1}{5} \begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix}$$

Q.

$$\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

$$\Delta = 1(3 \times 1) - (-1)(-2 \times 1) + 0 = 3 + 0 - 4 - 0 - 0 + 2 = 1.$$

$$\left[ \begin{array}{ccccc} 1 & 2 & -2 & 1 & 2 \\ -1 & 3 & 0 & -1 & 3 \\ 0 & -2 & 1 & 0 & -2 \\ 1 & 2 & -2 & 1 & 2 \\ -1 & 3 & 0 & -1 & 3 \end{array} \right]$$

$$A^{-1} = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$

Q.

$$\begin{bmatrix} 1 & 2 & 5 \\ 3 & 1 & 4 \\ 1 & 1 & 2 \end{bmatrix}$$

$$\Delta = 2 + 8 + 15 - 5 - 4 - 12 = 25 - 21 = 4$$

$$\left[ \begin{array}{ccccc} 1 & 2 & 5 & 1 & 2 \\ 3 & 1 & 4 & 3 & 1 \\ 1 & 1 & 2 & 1 & 1 \\ 1 & 2 & 5 & 1 & 2 \\ 3 & 1 & 4 & 3 & 1 \end{array} \right]$$

$$A^{-1} = \frac{1}{4} \begin{bmatrix} -2 & 1 & 3 \\ -2 & -3 & 11 \\ 2 & 1 & -5 \end{bmatrix}$$

Q.

$$Q \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$Q \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Delta = \cos^2\theta + \sin^2\theta = 1$$

$$A^{-1} = \begin{array}{c|ccccc} \cos\theta & -\sin\theta & 0 & \cos\theta & -\sin\theta \\ \hline \sin\theta & \cos\theta & 0 & \sin\theta & \cos\theta \\ 0 & 0 & 1 & 0 & 0 \\ \cos\theta & -\sin\theta & 0 & \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta & 0 & \sin\theta & \cos\theta \end{array}$$

$$Q \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{array}{c|cccc} 0 & 1 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{array}$$

$$A^{-1} = \frac{1}{-1} \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

## # RANK OF MATRIX!-

If all minors of order  $(r+1)$  are zeroes, but there is atleast one non zero minor of order  $\bullet$  'r'. if exists, It is called the rank of matrix and is denoted by  $R(A) = r$

### \* Properties of Rank:-

i) If A is a null matrix or zero matrix then rank of A = 0

ex:-  $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$  etc.

ii) If A is a non zero matrix, then rank of A  $\geq 1$

iii) If 'I' be the unit matrix or identity matrix of order  $m \times n$ , then rank of  $I_n = n$

ex:-  $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$  rank = 2

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
 rank = 3

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
 rank = 4.

iv) If A is a matrix of order  $m \times n$ , then rank of A  $\leq \min\{m, n\}$

ex:-  $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}_{2 \times 3}$  rank  $\leq 2$ .

\* If all determinants of  $2 \times 2 = 0$  rank  $< 2$

\* If at least one determinant of  $2 \times 2$  is non zero, then rank = 2.

Q. Find rank of

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ -2 & 0 & 5 & 7 \end{bmatrix}_{2 \times 4}$$

$$P(A) \leq 2$$

By checking determinant of any  $2 \times 2$  <sup>one of</sup>  $\neq 0$  here

$$P(A) = 2$$

\* \*

Determinant checking

1<sup>st</sup> D check = 0

2<sup>nd</sup> D check = 0

3<sup>rd</sup> D check = 0

4<sup>th</sup> D check (no need)

of course There must be rows identical with some common factors.

$\therefore$  rank will  $< \min\{m, n\}$   
 $\neq \min\{m, n\}$

Q.

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$$

\* purpose  $\rightarrow$  make at least one row = 0

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$$

$$P(A) \leq 3$$

$$R_3 - (R_1 + R_2)$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow P(A) = 2$$

Q.

$$\left[ \begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{array} \right] \xrightarrow{\text{Row operations}} \left[ \begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & -2 & 2 & -2 \\ 0 & 0 & 2 & 2 \end{array} \right] \xrightarrow{\text{Row operations}} \left[ \begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 2 & 2 \end{array} \right] \xrightarrow{\text{Row operations}} \left[ \begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 \end{array} \right]$$

$\neq 0 \Rightarrow P(A) = 2.$

$P(A) \leq 3$

Can be made '0'

Q.

$$\left[ \begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & -2 & 2 & -2 \\ 2 & 2 & 0 & 0 \end{array} \right] \xrightarrow{\text{Row operations}} \left[ \begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 \\ 2 & 2 & 0 & 0 \end{array} \right] \xrightarrow{\text{Row operations}} \left[ \begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 2 & 2 \end{array} \right] \xrightarrow{\text{Row operations}} \left[ \begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 \end{array} \right]$$

$\neq 0 \Rightarrow P(A) = 2.$

$P(A) \leq 3$

Q.

$$\left[ \begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ -1 & 2 & 2 & -1 \\ 0 & 5 & 6 & 1 \end{array} \right] \xrightarrow{\text{Row operations}} \left[ \begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 5 & 6 & 1 \end{array} \right] \xrightarrow{\text{Row operations}} \left[ \begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 5 & 6 & 1 \end{array} \right]$$

$P(A) \leq 3$

$P(A) = 3.$

Q.

$$\left[ \begin{array}{cccc|c} 2 & 3 & 4 & -1 & 1 \\ 5 & 2 & 0 & -1 & 1 \\ -4 & 5 & 12 & -1 & 1 \end{array} \right] \xrightarrow{\text{Row operations}} \left[ \begin{array}{cccc|c} 2 & 3 & 4 & -1 & 1 \\ 0 & -1 & -4 & 0 & 0 \\ -4 & 5 & 12 & -1 & 1 \end{array} \right] \xrightarrow{\text{Row operations}} \left[ \begin{array}{cccc|c} 2 & 3 & 4 & -1 & 1 \\ 0 & -1 & -4 & 0 & 0 \\ 0 & 2 & 8 & -1 & 1 \end{array} \right] \xrightarrow{\text{Row operations}} \left[ \begin{array}{cccc|c} 2 & 3 & 4 & -1 & 1 \\ 0 & 1 & 4 & 0 & 0 \\ 0 & 0 & 4 & -1 & 1 \end{array} \right] \xrightarrow{\text{Row operations}} \left[ \begin{array}{cccc|c} 2 & 3 & 4 & -1 & 1 \\ 0 & 1 & 4 & 0 & 0 \\ 0 & 0 & 1 & -\frac{1}{4} & \frac{1}{4} \end{array} \right]$$

$P(A) \leq 3.$

$R_3 - 4R_2$

$$\left[ \begin{array}{ccc|c} 2 & 3 & 4 & -1 \\ 5 & 2 & 0 & -1 \\ -10 & -4 & 0 & 2 \end{array} \right] \xrightarrow{\text{Row operations}} \left[ \begin{array}{ccc|c} 2 & 3 & 4 & -1 \\ 0 & -1 & -4 & 0 \\ 0 & 0 & 4 & 2 \end{array} \right] \xrightarrow{\text{Row operations}} \left[ \begin{array}{ccc|c} 2 & 3 & 4 & -1 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 1 & \frac{1}{2} \end{array} \right]$$

$= 0 \Rightarrow P(A) \leq 3$

$P(A) = 2.$

Q.

$$\left[ \begin{array}{cccc|c} 1 & -1 & 3 & 6 & 1 \\ 1 & 8 & -3 & -4 & 1 \\ 5 & 3 & 3 & 11 & 1 \end{array} \right] \xrightarrow{\text{Row operations}} \left[ \begin{array}{cccc|c} 1 & -1 & 3 & 6 & 1 \\ 0 & 9 & 0 & 2 & 0 \\ 0 & 8 & 0 & 1 & 0 \end{array} \right] \xrightarrow{\text{Row operations}} \left[ \begin{array}{cccc|c} 1 & -1 & 3 & 6 & 1 \\ 0 & 1 & 0 & \frac{2}{9} & 0 \\ 0 & 8 & 0 & 1 & 0 \end{array} \right] \xrightarrow{\text{Row operations}} \left[ \begin{array}{cccc|c} 1 & -1 & 3 & 6 & 1 \\ 0 & 1 & 0 & \frac{2}{9} & 0 \\ 0 & 0 & 0 & \frac{1}{9} & 0 \end{array} \right]$$

$P(A) \leq 3.$

$R_2 + R_1$

$R_3 - R_1$

$$\left[ \begin{array}{ccc|c} 1 & -1 & 3 & 6 \\ 2 & 7 & 0 & 2 \\ 4 & 4 & 0 & 5 \end{array} \right] \xrightarrow{\text{Row operations}} \left[ \begin{array}{ccc|c} 1 & -1 & 3 & 6 \\ 0 & 8 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{array} \right] \xrightarrow{\text{Row operations}} \left[ \begin{array}{ccc|c} 1 & -1 & 3 & 6 \\ 0 & 1 & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 1 \end{array} \right]$$

$\neq 0 \Rightarrow$

Q.

$$\left[ \begin{array}{cccc} 1 & 3 & 4 & 3 \\ 3 & 9 & 12 & 3 \\ \hline 1 & 3 & 4 & 1 \end{array} \right] \quad P(A) = 2$$

$\Rightarrow P(A) \leq 3$

$P(A) < 3 \quad \left\{ \begin{array}{l} \therefore R_2, R_1 \text{ identical} \end{array} \right\}$

Q.

$$\left[ \begin{array}{cccc} 1 & 2 & 3 & 8 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 4 & 3 \\ 6 & 8 & 7 & 5 \end{array} \right] \quad 4 \times 4$$

$P(A) \leq 4$

$R_4 - (R_1 + R_2 + R_3)$

$$\left[ \begin{array}{cccc} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$P(A) < 4$

$R_2 - 2R_1$

$$\left[ \begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 3 & 2 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$\neq 0 \Rightarrow P(A) = 3.$

Q.

$$\left[ \begin{array}{cccc} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{array} \right] \quad P(A) \leq 4$$

$R_{12}$

$$\left[ \begin{array}{cccc} 1 & -1 & -2 & 4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{array} \right] \quad R_2 - 2R_1, \quad R_3 - 3R_1$$

$R_4 - (R_1 + R_2 + R_3)$

$$= \left[ \begin{array}{ccc|c} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ \hline 0 & 0 & 0 & 0 \end{array} \right] \quad P(A) < 4$$

$\cancel{\neq 0} \Rightarrow P(A) = 3$

Q.

$$\left[ \begin{array}{cccc} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 10 & 3 & 9 & 7 \\ \hline 16 & 4 & 12 & 15 \end{array} \right] \quad P(A) \leq 4$$

$R_4 - (R_1 + R_3)$   
 $R_3 - (R_1 + R_2)$

$$\left[ \begin{array}{cccc} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$P(A) < 4$   
 $P(A) < 3$   
 $P(A) = 2$

→ have sol<sup>n</sup>      → have no sol<sup>n</sup>

## # The consistency & inconsistency of the system of eqn:-

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$

$$a_3x + b_3y + c_3z = d_3$$

Coefficient Matrix       $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$  ,      Constant Matrix       $B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$  ,      Variable Matrix       $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$$AB = \begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{bmatrix}$$

Augmented Matrix.

Eq<sup>n</sup> form  $\rightarrow$  
$$\boxed{AX = B}$$

- \* Any System is said to be a consistent system, if it has a solution.
- \* Inconsistent System has no solution.

$\rightarrow$  If  $P(AB) = P(A) = \text{No. of Unknowns}$ , then the system is said to be consistent and having unique solution.

$\rightarrow$  If  $P(AB) = P(A) < \text{No. of Unknowns}$ , then the system is said to be consistent and having more than one solution, or  $\infty$  no. of solutions.

\* Remember  $\rightarrow$  either 1 sol<sup>n</sup> or  $\infty$  sol<sup>n</sup> exist for System  
To confuse, in exam option will be 2 sol<sup>n</sup> or 3 sol<sup>n</sup>  $\times$

$\rightarrow$  If  $P(AB) \neq P(A)$ , then the System is said to be inconsistent and have no solution.

$$\begin{aligned}
 0. \quad & x + y + z = 3 \\
 & x + 2y + 3z = 4 \\
 & x + 4y + 9z = 6
 \end{aligned}$$

$\textcircled{0} \ 0 \ \textcircled{1} \ \textcircled{X} \ \textcircled{0} \ \infty$

### NOTE!-

- If AB is rectangle matrix, then last column must be excluded.
- If AB is a square matrix, then last column must be included
- If last column is excluded, then it represents both the matrices' rank.
- If last column included, then it represents only AB Matrix rank.

Now,

$$[AB] = \left[ \begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 6 \end{array} \right] \quad R_2 - R_1, \quad R_3 - R_1$$

$3 \times 4$ .

$$= \left[ \begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 3 & 8 & 3 \end{array} \right]$$

$\rightarrow \neq 0 \Rightarrow P(AB) = P(A) = 3 = 3 \quad (\text{no. of Unknown})$

$\therefore$  System has unique soln.

Q. Check the consistency of system -

$$x - 2y + 3z = 2$$

$$2x - 3z = 3$$

$$x + y + z = 0$$

$\rightarrow$

$$\left[ \begin{array}{ccc|c} 1 & -2 & 3 & 2 \\ 2 & 0 & -3 & 3 \\ 1 & 1 & 1 & 0 \end{array} \right] \quad 3 \times 4$$

$$R_1 + 2R_3$$

$$\left[ \begin{array}{ccc|c} 3 & 0 & 5 & 2 \\ 2 & 0 & -3 & 3 \\ 1 & 1 & 1 & 0 \end{array} \right]$$

$$\Rightarrow \neq 0 \Rightarrow \rho(AB) = \rho(A) = 3 = \text{no. of unknowns}$$

$\therefore$  System showing consistent and having unique soln.

Q.

$$3x + 3y + 2z = 1$$

$$x + 2y = 4$$

$$10y + 3z = -2$$

$$2x - 3y - z = 5$$

$$[AB] = \left[ \begin{array}{cccc} 1 & 2 & 0 & 4 \\ 0 & 3 & 2 & 1 \\ 0 & 10 & 3 & -2 \\ 2 & -3 & -1 & 5 \end{array} \right] \quad 4 \times 4$$

$$R_2 - 3R_1, R_4 - 2R_1$$

$$= \left[ \begin{array}{cccc} 1 & 2 & 0 & 4 \\ 0 & -3 & 2 & -1 \\ 0 & 10 & 3 & -2 \\ 0 & 7 & -1 & 3 \end{array} \right]$$

$$R_2 + 2R_4$$

$$R_3 + 3R_4$$

Note:-

If '0' present, make adjacent '0'

If no '0' present, then check if 1 or -1  $\rightarrow$  make adjacent of 1 or -1 = 0

$$\left[ \begin{array}{cccc|c} 1 & 2 & 0 & 4 \\ 0 & -17 & 0 & -17 \\ 0 & -11 & 0 & -11 \\ 0 & -7 & -1 & -3 \end{array} \right]$$

take common

take common

then  $R_3 - R_2$  and interchange  $R_4 \& R_3$

$$\left[ \begin{array}{cccc|c} 1 & 2 & 0 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & -7 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$\neq 0 \Rightarrow \rho(A) = \rho(AB) = 3 = \text{no. of unknowns}$

∴ System is consistent & having unique solution.

Q. 
$$\begin{aligned} 4x - 2y + 6z &= 8 \\ x + y - 3z &= -1 \\ 15x - 3y + 9z &= 21 \end{aligned}$$

$$[AB] = \left[ \begin{array}{ccc|c} 1 & 1 & -3 & -1 \\ 2 & -1 & 3 & 4 \\ 5 & -1 & 3 & 7 \end{array} \right] \quad \begin{matrix} R_2 + R_1 \\ R_3 + R_1 \end{matrix}$$

$3 \times 4$

$$\left[ \begin{array}{ccc|c} 1 & 1 & -3 & -1 \\ 3 & 0 & 0 & 3 \\ 6 & 0 & 0 & 6 \end{array} \right]$$

$\rightarrow \neq 0 \Rightarrow$

$\therefore \rho(AB) \neq \rho(A) = 2 < 3$

System - consistent  
But  $\infty$  solution.

We can write system again as—

$$\begin{aligned}x + y - 3z &= -1 \\3x &= 3 \Rightarrow x = 1\end{aligned}$$

$$1 + y - 3z = -1 \\ \Rightarrow y - 3z = -2 \quad \text{--- } \textcircled{1}$$

Since, we are discussing the consistency of only real system of equations, therefore solutions always real numbers.

Let  $z = k$  where  $k \in \mathbb{R}$

$$\begin{aligned}y - 3k &= -2 \\ \Rightarrow \boxed{\begin{aligned}y &= 3k - 2 \\z &= k \\x &= 1\end{aligned}}\end{aligned}$$

For different values of  $k$ , we will have diff. solutions and these solutions are infinite.

A.  $2x - y + z = 4$

$$3x - y + z = 6$$

$$4x - y + 2z = 7$$

$$-x + y - z = 9$$

Check the consistency of System.

$$\rightarrow [AB] = \left[ \begin{array}{cccc} -1 & 1 & -1 & 9 \\ 2 & -1 & 1 & 4 \\ 3 & -1 & 1 & 6 \\ 4 & -1 & 2 & 7 \end{array} \right] \quad 4 \times 4$$

$$\begin{aligned}R_2 + R_1 \\ R_3 + R_1 \\ R_4 + R_1\end{aligned}$$

$$\therefore \left[ \begin{array}{cccc} -1 & 1 & -1 & 9 \\ 1 & 0 & 0 & 13 \\ 2 & 0 & 0 & 15 \\ 3 & 1 & 1 & 14 \end{array} \right]$$

C<sub>12</sub>

$$\left[ \begin{array}{c|cccc} 1 & -1 & -1 & 9 \\ \hline 0 & 1 & 0 & 13 \\ 0 & 2 & 0 & 15 \\ 0 & 3 & 1 & 16 \end{array} \right]$$

$$\rightarrow \neq 0 \Rightarrow \rho(AB) = 4$$

$$\text{But } \rho(A) = 3$$

$$\therefore \rho(AB) \neq \rho(A)$$

Hence, System is inconsistent  
and having no solution.

- Q. for what values of  $\lambda$  and  $\mu$ , does the system of equations

$$x + y + z = 6$$

$$x + 2y + 3z = 10$$

$$x + 2y + \lambda z = \mu$$

has —

- (A) no solution
- (B) Unique soln
- (C) More than one solution

→

$$[AB] = \left[ \begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{array} \right] \quad 3 \times 4$$

$$R_3 - R_2$$

$$\left[ \begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 0 & 0 & \lambda-3 & \mu-10 \end{array} \right]$$

$$\rightarrow \neq 0 \Rightarrow \rho(AB) = \rho(A) = 3 \quad \text{if } \lambda-3 \neq 0$$

$$< 3 \quad \text{if } \lambda-3 = 0$$

Case - I.

Case - II

Case - III.

A. 
$$\begin{aligned} 2x + 3y + 5z &= 9 \\ 7x + 3y - 2z &= 8 \\ 2x + 3y + \lambda z &= 4 \end{aligned}$$



for what values of  $\lambda$  &  $\mu$  system  
has —  
 A no sol<sup>n</sup>.  
 B unique sol<sup>n</sup>.  
 C more than one sol<sup>n</sup>.

## # Eigen Values and Eigen Vectors :-

# Characteristic equation:- Let A be the sq. matrix of order  $n \times n$  and I be the unit matrix of order  $n \times n$  then  $|A - \lambda I| = 0$  is called the characteristic eq<sup>n</sup> where  $\lambda$  is a parameter.

The roots of characteristic eq<sup>n</sup> are called characteristic roots / latent / eigen / proper values.

$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$  or  $[x_1 \ x_2 \ \dots \ x_n]^T$  which satisfies the matrix eq<sup>n</sup>  $[A - \lambda I]X = 0$  is called the corresponding eigen vector of the matrix.

### Note:-

i). The sum of eigen values of any matrix is equal to sum of the elements of its principal diagonals.

Ex:-

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

- ① 1 ② 3 ③ 5 ④ 7

$$\text{Trace of } A = 1+5+1 = 7.$$

ii). The Product of eigen values of any matrix is equal to its determinant.

Ex:- The characteristic roots of matrix  $\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$

① 2, 5 ✓

② 3, 4 ✗

③ 1, 6 ✓

④ 2, 3 ✗

→ Sum = sum of principle diagonal

$6+1=7$

Product = |determinant value|

iii). The eigen values of a symmetric matrix ( $A^T = A$ ) are purely real.

Ex:-  $A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} = A^T$  ∵ Eigen values are 1 & 5 Which are purely real

Cross

$1+5=6$   $1 \times 5 = 5$  equal to  
to sum of principle diagonals  $\downarrow$  determinant value

v) The eigen value of skew symmetric matrix are ( $A^T = -A$ ) either purely imaginary or zeroes

Ex:-  $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

Eigen Values?  
 ① 1, -1    ②  $i, -i$     ③ -1, 0    ④ 0, 1  
 X            ✓            X            X

$$A^T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = - \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = -A$$

v) If a matrix is either lower triangular or upper triangular, then principle diagonal elements are called eigen values.

\* If Given Matrix is L.T.M or U.T.M (without operation only original)

Ex:-

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & 2 \\ 0 & 0 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{bmatrix}$$

Don't Use this Property to find eigen value when L.T.M or U.T.M is made by some operation.

Then only follow this rule.

$$\lambda = 1, -4, 7$$

$$\lambda = 1, 3, 6$$

v) If  $\lambda$  is an eigen value of A

$$\text{then } \lambda^2 \text{ _____ } A^2$$

$$\lambda^3 \text{ _____ } A^3$$

$$\lambda^n \text{ _____ } A^n$$

$$\frac{1}{\lambda} \text{ _____ } A^{-1}$$

Ex:- If  $S = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$  and  $\lambda = 1, 5$

then for  $S^2$  what will be eigen value  $\rightarrow$  After getting result  $\lambda$ , verify it by sum & product rule

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

then eigen values of  $A^3$  are -

a) 5, 4

b) 9, 1

c) 16, -1

d) 27, -1

D. find the eigen values and corresponding eigen vectors of

$$\text{Matrix } A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$$

$$\rightarrow |A - \lambda I| = 0$$

$$\left| \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0$$

$$\left| \begin{array}{cc} 5-\lambda & 4 \\ 1 & 2-\lambda \end{array} \right| = 0$$

$$\Rightarrow 10 - 7\lambda + \lambda^2 - 4 = 0$$

$$\Rightarrow \lambda^2 - 7\lambda + 6 = 0$$

$$\Rightarrow (\lambda - 1)(\lambda - 6) = 0$$

$$\lambda = 1, 6.$$

Eigen Vector —

$$[A - \lambda I] X = 0.$$

Subtract  $\lambda$  from matrix and multiply ~~vector~~ X Matrix

$$\text{for } \lambda = 1 \quad \begin{bmatrix} 4 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\Rightarrow 4x_1 + 4x_2 = 0$$

$$x_1 + x_2 = 0$$

$$\Rightarrow x_1 = -x_2$$

$$\Rightarrow \frac{x_1}{1} = \frac{x_2}{-1} \quad \text{eigen vector} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & -1 \end{bmatrix}^T$$

for  $\lambda = 6$  —

$$\begin{bmatrix} -1 & 4 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0.$$

$$-x_1 + 4x_2 = 0$$

$$\frac{x_1}{4} = \frac{x_2}{1}$$

$$\text{eigen vector} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \text{ or } \begin{bmatrix} 4 & 1 \end{bmatrix}^T$$

Q. find the ch. roots of matrix  
or Eigen Value.

1st Step

- (A) 3, 7, 8 ✓
- (B) 2, 2, 14 ✓
- (C) 0, 3, 15 ✓
- (D) 1, 4, 9 ✗

By sum of diagonal rule  
= sum of eigen value.

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

2nd step

check determinant.

$$\Delta = 0.$$

So, one root must be 0

so that to make product rule of eigen values correct

Q. find the latent root of matrix -

$$\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$$

- |             |                 |                 |
|-------------|-----------------|-----------------|
| (A) 1, 2, 3 | <u>1st step</u> | <u>2nd step</u> |
| (B) 0, 2, 4 | ✓ X             | 4 + 2 = 6.      |
| (C) 1, 1, 4 | ✓ X             |                 |
| (D) 2, 2, 2 | ✓ X             |                 |

Q. find Eigen Values and Corresponding eigen vectors of

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & 2 \\ 0 & 0 & 7 \end{bmatrix}$$

$$\lambda = 1, -4, 7. \quad \{ \text{it is } U.T.M \}$$

for  $\lambda = 1$

$$\begin{bmatrix} 0 & 2 & 3 \\ 0 & -5 & 2 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0.$$

$$2x_2 + 3x_3 = 0.$$

$$-5x_2 + 2x_3 = 0.$$

Remember

\* for any eigen values if ~~two~~ <sup>in</sup> eigen vectors = 0 then third must be only non zero

not able to identify any <sup>infn</sup> abt  $x_1$ .  
set, two values are

$\therefore$  eigen vector set =  $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

for  $\lambda = -4$

$$\begin{bmatrix} 5 & 2 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 11 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0.$$

$$5x_1 + 2x_2 + 3x_3 = 0 \Rightarrow 5x_1 + 2x_2 = 0.$$

$$2x_3 = 0 \Rightarrow x_3 = 0 \Rightarrow 5x_1 = -2x_2$$

$$11x_3 = 0 \Rightarrow x_3 = 0 \Rightarrow \frac{x_1}{2} = \frac{x_2}{-5}$$

Eigen Vector =  $\begin{bmatrix} 2 \\ -5 \\ 0 \end{bmatrix}$ .

for  $\lambda = 7$  -

$$\begin{bmatrix} -6 & 2 & 3 \\ 0 & -11 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0.$$

$$-6x_1 + 2x_2 + 3x_3 = 0,$$

$$-11x_2 + 2x_3 = 0 \Rightarrow 6x_1 = 37 \Rightarrow x_1 = \frac{37}{6},$$

$$\Rightarrow \frac{x_2}{2} = \frac{x_3}{11}$$

Eigen vector =  $\begin{bmatrix} \frac{37}{6} \\ 2 \\ 11 \end{bmatrix} \propto \begin{bmatrix} 37 \\ 12 \\ 66 \end{bmatrix}$

Q. For the matrix P:

$$\begin{bmatrix} 3 & 2 & 2 \\ 0 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

One of the eigen value is

-2 : Which of the following is an eigen vector.

- Ⓐ  $\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$  Ⓑ  $\begin{bmatrix} -3 \\ 2 \\ -1 \end{bmatrix}$  Ⓒ  $\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$  Ⓓ  $\begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix}$

→ for  $\lambda = -2$

$$\begin{bmatrix} 5 & -2 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0.$$

$$3x_1 - 2x_2 + 2x_3 = 0 \Rightarrow 5x_1 - 2x_2 = 0.$$

$$\left. \begin{array}{l} x_3 = 0 \\ 3x_3 = 0 \end{array} \right\} x_3 = 0$$

$$\frac{x_1}{2} = \frac{x_2}{5}$$

eigen vector  
↙

$$= \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix}$$

Q. for the matrix

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

which one of the following is an eigen vector.

a)  $[1 \ -1 \ 1]^T$

Give 1<sup>st</sup> pref to least eigen value.

b)  $[1 \ 2 \ 1]^T$

check only by subtracting -2 in 1<sup>st</sup> row  
check eigen value.

c)  $[2 \ -1 \ 1]^T$

If option not found Again 2<sup>nd</sup> pref to 2<sup>nd</sup> least eigen value

d)  $[1 \ 1 \ -1]^T$

Again if option not found same subtract  $\lambda$  from 2<sup>nd</sup> row  
check eigen value

Again 3<sup>rd</sup> pref to 3<sup>rd</sup> least eigen value

but do not subtract to 3<sup>rd</sup> row  
subtract from either 1<sup>st</sup> or 2<sup>nd</sup>

row (no effect) —

check eigen value

## # Cayley - Hamilton Theorem :-

Every sq. matrix satisfies its own characteristic eqn.

Ex:-  $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$

$$\begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow 10 - 7\lambda + \lambda^2 - 4 = 0$$

$$\Rightarrow \lambda^2 - 7\lambda + 6 = 0$$

$$\Rightarrow \lambda =$$

Means —

$$A^2 - 7A + 6I = 0 \rightarrow \text{A/c to Cayle Hamilton Theorem}$$

$\downarrow$   
For Inverse finding  
without knowing  
adj of A & determinant.

$$\Rightarrow 6I = 7A - A^2$$

$$\Rightarrow 6A^{-1} = 7A^{-1} - A^{-1}$$

$$\Rightarrow 6A^{-1} = 7I - A$$

$$= \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} - \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$$

$$= \frac{1}{6} \begin{bmatrix} 2 & -4 \\ -1 & 5 \end{bmatrix}$$

- Q. Let M be  $3 \times 3$  matrix with ch. eqn  $\alpha^3 + 1 = 0$  then  
the inverse of M = —

$\rightarrow$  A/c to Cayle Hamilton theorem —

$$M^3 + I = 0$$

$$I = -M^3$$

$$IM^{-1} = -M^3 M^{-1}$$

~~$M^{-1} = -M^2$~~

Q. Let  $P$  be a  $3 \times 3$  matrix with ch. eqn  $\lambda^3 + \lambda^2 + 2\lambda + 1 = 0$   
then inverse of  $P =$  —

- (A)  $P^2 + P + I$
- (B)  $P^2 + P + 2I$
- (C)  $-(P^2 + P + 2I)$
- (D)  $-(P^2 + P + I)$ .

$$\rightarrow P^3 + P^2 + 2P + I = 0$$

$$\Rightarrow I = -(P^3 + P^2 + 2P)$$

$$\Rightarrow P^{-1} = -(P^2 + P + 2I)$$

Q. find the value of  $\lambda$  for which the following system of eqns

$$\begin{aligned} \cancel{\lambda x + 3y + 5z = 1} \\ 2x - 4\lambda y + \lambda z = 0 \\ -4x + 18y + 7z = 0 \end{aligned}$$

have a non trivial solution.

Note:-

for the system of Eqn  $AX = 0$

$X = 0$  will always a solution

and this soln. is called a trivial soln.

For a non trivial solution determinant of coefficient matrix = 0 (always)

Note:-

$AX = 0$  is called Homogeneous.

$X = 0$  will always be a solution.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

This is called Trivial equation.

$$\left[ \begin{array}{ccc|cc} \lambda & 3 & 5 & \lambda & 3 \\ 2 & -4\lambda & 1 & 2 & -4\lambda \\ -4 & 18 & 7 & -4 & 18 \end{array} \right]$$

$$-28\lambda^2 - 12\lambda + 180 - 80\lambda - 18\lambda^2 - 42 = 0$$

$$\Rightarrow -46\lambda^2 - 92\lambda + 138 = 0$$

$$\Rightarrow \lambda^2 + 2\lambda - 3 = 0$$

$$\lambda = -3 \text{ or } 1$$

Q. Rank of

$$\begin{bmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 1 \end{bmatrix}$$

① 1 ② 2 ③ 3 ④ 4.

→

$$\begin{bmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 1 \end{bmatrix}$$

$$r(A) \leq 3$$

$$R_2 - 4R_1$$

3x4

$$\left( \begin{array}{ccc|c} 4 & 2 & 1 & 3 \\ -10 & -5 & 0 & +5 \\ 2 & 1 & 0 & 1 \end{array} \right)$$

$$\rightarrow = 0 \Rightarrow r(A) < 3$$

$$r(A) = 2$$

Q. Rank and nullity of matrix

$$\begin{bmatrix} 6 & 1 & 8 & 3 \\ 2 & 3 & 0 & 2 \\ 4 & -1 & -8 & -3 \end{bmatrix}$$

\* \* \* \* \*  
Nullity of A = no. of columns - Rank

$$\left( \begin{array}{ccc|cc} 10 & 0 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 & 2 \\ 4 & -1 & -8 & -3 & \end{array} \right)$$

$$r(A) \leq 3$$

$$\rightarrow \neq 0 \Rightarrow r(A) = 3$$

$$\therefore \text{nullity of } A = 4 - 3$$

$$= 1 \text{ Ans}$$

Q. The rank of  $\begin{bmatrix} 1 & 2 & 3 \\ 3 & -1 & 2 \\ 4 & 1 & K \end{bmatrix}$  is 2

then  $K = \underline{5}$

Q. The inverse of

$$\begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0.2 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} \cancel{\frac{1}{5}}^2 & 0 & 0 \\ 0 & \cancel{\frac{1}{1}}^1 & 0 \\ 0 & 0 & \cancel{\frac{1}{0.2}}^5 \end{bmatrix}$$

\* \* \* \* \*  
Inverse of Diagonal matrix is inverse of each element in that diagonal.

Q. The rank of

$$\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix}$$

$9 \times 9$

\* \* \* \* \*  
Rank of diagonal matrix  
= no. of non zero elements in that diagonal.

rank = 4

Q. Let  $A, B, C, D$  be  $n \times n$  matrices each with non zero determinant and if  $ABCD = I$  then  $B^{-1} = \underline{\quad}$

$$(ABCD)^{-1} = I^{-1}$$

$$D^{-1} C^{-1} B^{-1} A^{-1} = I$$

$$\boxed{D^{-1} C^{-1} B^{-1} A^{-1} = I}$$

$$DD^{-1} C^{-1} B^{-1} A^{-1} = ID$$

$$DC^{-1} B^{-1} A^{-1} = IDC$$

$$B^{-1} A^{-1} / A = IDC A = DCA$$

Q. The eigen vector pair of matrix

$$\begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$$

a)  $\begin{bmatrix} 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

b)  $\begin{bmatrix} 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

c)  $\begin{bmatrix} -2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

d)  $\begin{bmatrix} -2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$$\rightarrow \begin{bmatrix} 3-\lambda & 4 \\ 4 & -3-\lambda \end{bmatrix} \Rightarrow -9 - 3\lambda + 3\lambda + \lambda^2 - 16 = 0$$

$$\Rightarrow \lambda^2 = 25$$

$$\lambda = 5, -5$$

for  $\lambda = 5$

$$\begin{bmatrix} -2 & 4 \\ 4 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0.$$

$$-2x_1 + 4x_2 = 0$$

$$4x_1 - 8x_2 = 0.$$

$$\frac{x_1}{2} = \frac{x_2}{4}$$

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

for  $\lambda = -5$

$$\begin{bmatrix} 8 & 4 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0.$$

$$8x_1 + 4x_2 = 0$$

$$\frac{x_1}{2} = \frac{x_2}{-8}$$

$$\begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Q. How many of the following matrices have an eigen value = 1

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \quad \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}$$

$\lambda = 1, 0$        $\lambda = 0, 0$        $(1-\lambda)^2 + 1 = 0$        $(-1-\lambda)^2 = 0$

Ⓐ 1    Ⓑ 2    Ⓒ 3    Ⓓ 4.

Q. If the following represents eq<sup>n</sup> of line then line passes through  
the point

$$\begin{bmatrix} x & 2 & 4 \\ y & 8 & 0 \\ 1 & 1 & 1 \end{bmatrix} = 0$$

- a) (0, 0)
- ✓ (3, 4)
- c) (4, 3)
- d) (4, 4)

→

$$\begin{array}{ccccc} x & 2 & 4 & x & 2 \\ y & 8 & 0 & y & 8 \\ 1 & 1 & 1 & 1 & 0 \end{array}$$

$$8x + 4y - 32 - 2y = 0$$

$$\Rightarrow 8x + 2y = 32$$

$$\boxed{\begin{array}{l} -3x + 0 \rightarrow 2y - 8x - 0 + 0 = 0 \\ \Rightarrow -8x + 2y + 32 = 0 \\ \Rightarrow 8x + 2y = 32 \end{array}}$$

- Q. A is  $3 \times 4$  real matrix and  $Ax = B$  is an inconsistent system of eqn's then the highest possible rank of A —  
 @ 1    2    3    4 ..

$$A = 3 \times 4$$

$$[AB] = 3 \times 5 \Rightarrow r(A) \leq 3$$

for inconsistent System  $r(A) \neq r(AB)$   $\left\{ \begin{array}{l} r(A) \text{ can't be} \\ \text{greater than} \\ r(AB) \end{array} \right\}$   
 $\therefore r(A) < 3$

$r(A) = 2$

- Q. Given an orthogonal matrix

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{then } (AA^T)^{-1} = I$$

NOTE! -

A matrix is said to be orthogonal matrix if  $AA^T = I$ .

Q.  $A = \begin{bmatrix} 2 & -0.1 \\ 0 & 3 \end{bmatrix}, A^{-1} = \begin{bmatrix} \frac{1}{2} & a \\ 0 & b \end{bmatrix}$

then  $a+b =$  —

$$A^T = \frac{1}{6} \begin{bmatrix} 3 & 0.1 \\ 0 & 2 \end{bmatrix}$$

$$\therefore a = \frac{0.1}{6}$$

$$b = \frac{2}{6}$$

$$\therefore a+b = \frac{2.1}{6} = \frac{21}{60} = \frac{7}{20} \quad \underline{\text{Ans}}$$

Q. Let  $A = \begin{bmatrix} -2 & 0 & 0 \\ -1 & -2 & 0 \\ 4 & 2 & -1 \end{bmatrix}$  and  $B = 2A^2$  then  $|B| =$  \_\_\_\_\_

- Ⓐ 16 Ⓑ 32 Ⓒ 64 Ⓓ 128.

$$B = 2A^2$$

$$|B| = |2A^2| \rightarrow \text{order } \hat{=} \underline{\underline{3 \times 3}}$$

$$= 2^3 |A^2|$$

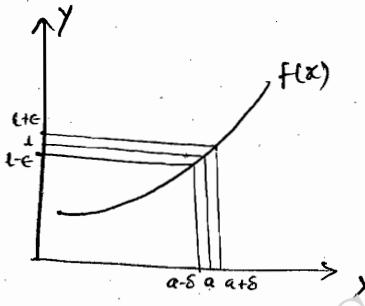
$$= 8(16)$$

$$= 128.$$

# DIFFERENTIAL CALCULUS

Limit :- A no.  $l$  is said to be the limit of the fn  $f(x)$  as  $x \rightarrow a$  if  $\forall \epsilon > 0$  (however small)  $\exists$  a  $\delta > 0$  such that  $|f(x) - l| < \epsilon$  whenever  $|x - a| < \delta$

$$\boxed{\lim_{x \rightarrow a} f(x) = l}$$



$$\therefore l - \epsilon < f(x) < l + \epsilon$$

$$\forall a - \delta < x < a + \delta$$

\* Left limit :-  $\lim_{x \rightarrow a^-} f(x) = \lim_{h \rightarrow 0} f(a-h)$ , ( $a-\delta < x < a$ )

\* Right limit :-  $\lim_{x \rightarrow a^+} f(x) = \lim_{h \rightarrow 0} f(a+h)$ , ( $a < x < a+\delta$ )

\* Existence of Limit :- The limit of a function exists if both the left and right limits are existed and are equal.

ex:-  $\lim_{x \rightarrow a} \frac{1}{x-a}$

$$\therefore \lim_{x \rightarrow a^-} f(x) = \lim_{h \rightarrow 0} f(a-h) = \frac{1}{a-h-a} = \frac{1}{h}$$

$$\lim_{x \rightarrow a^+} f(x) = \lim_{h \rightarrow 0} f(a+h) = \frac{1}{a+h-a} = \frac{1}{h}$$

L.H limit  $\neq$  R.H limit

$\therefore$  limit not exists

## # Formulae :-

$$\lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = n a^{n-1}$$

$$\lim_{x \rightarrow a} \frac{x^m - a^m}{x^n - a^n} = \frac{m}{n} a^{m-n}$$

$$\lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1$$

$$\lim_{x \rightarrow 0} \frac{e^{mx} - 1}{x} = m$$

$$\lim_{x \rightarrow 0} \frac{a^x - 1}{x} = \log a$$

$$\lim_{x \rightarrow 0} \frac{a^x - b^x}{x} = \log\left(\frac{a}{b}\right)$$

•  $\lim_{x \rightarrow 0} (1+x)^{1/x} = e$

•  $\lim_{x \rightarrow 0} (1+ax)^{1/x} = e^a$

$$\lim_{x \rightarrow 0} (1+ax)^{b/x} = e^{ab}$$

•  $\lim_{x \rightarrow 0} (1+ax)^{1/bx} = e^{a/b}$

$$\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = e$$

$$\lim_{x \rightarrow \infty} \left(1 - \frac{1}{x}\right)^x = e^{-1}$$

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

$$\lim_{x \rightarrow 0} \frac{\tan x}{x} = 1$$

$$\lim_{x \rightarrow 0} \frac{\sin^{-1} x}{x} = 1$$

$$\lim_{x \rightarrow 0} \frac{\tan^{-1} x}{x} = 1$$

$$\lim_{x \rightarrow 0} \frac{1 - \cos mx}{1 - \cos nx} = \frac{m^2}{n^2}$$

$$\lim_{x \rightarrow 0} \frac{\cos ax - \cos bx}{x^2} = \frac{b^2 - a^2}{2}$$

$\frac{0}{0}, \frac{\infty}{\infty}, \frac{\infty}{0}, 0 \times \infty, 1^\infty$  etc  
are indeterminate form.

If  $\frac{0}{0}, \frac{\infty}{\infty}$  then apply

L.H. rule

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

$$= \lim_{x \rightarrow a} \frac{f''(x)}{g''(x)} = \dots$$

$$\lim_{x \rightarrow a} [f(x)]^{g(x)} = e^{\lim_{x \rightarrow a} g(x)[f(x)-1]}$$

$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)}$ , if  $f(x)$  &  
 $g(x)$  are algebraic fn.

Case-1 if degree of  $f(x) > g(x)$  degree.  
then result =  $\infty$

$$\text{ex!- } \lim_{x \rightarrow \infty} \frac{2x^3 + 3x + 5}{5x^2 + 6x + 7} = \infty$$

Case-2 if degree of  $f(x) < g(x)$  degree  
then result = 0

$$\text{ex!- } \lim_{x \rightarrow \infty} \frac{2x^2 + 3x + 4}{x^3 + 8} = 0$$

Case-3 If  $f(x)$  and  $g(x)$  have equal degrees then -

$$\text{result} = \frac{\text{co-eff of } N^r}{\text{co-eff of } D^r}$$

$$\text{ex!- } \lim_{x \rightarrow \infty} \frac{2x^2 + 3x + 4}{5x^2 + 6x + 7} = \frac{2}{5}$$

\*  $|x| = x$  if  $x > 0$

\*  $|x| = -x$  if  $x < 0$

\*  $[x] = \text{greatest integer not greater than } x$ .

\* Examples :-

$$\textcircled{1} \quad \underset{x \rightarrow 0}{\text{Lt}} \frac{1 - \cos x}{x^2} = \frac{1}{2}$$

$$\textcircled{2} \quad \underset{x \rightarrow 0}{\text{Lt}} \frac{1 - \cos x}{x \sin x} = \frac{1 - \cos x}{x^2} = \frac{\sin x}{2x} = \frac{\cos x}{2} = \frac{1}{2}$$

In Trigonometric  
fn., avoid  
LH rule  
Try alternates

$$\textcircled{3} \quad \underset{x \rightarrow 0}{\text{Lt}} \frac{\tan x - \sin x}{\sin x \cos x} = \frac{1 - \cos x}{\sin x(1 + \cos x)} = \frac{1}{1 + 1} = \frac{1}{2}$$

$$\textcircled{4} \quad \underset{x \rightarrow 1}{\text{Lt}} \frac{\cos(\frac{\pi x}{2})}{1 - \sqrt{x}} = \frac{\frac{\pi}{2} \sin(\frac{\pi x}{2})}{-\frac{1}{2\sqrt{x}}} = \frac{\pi/2}{1/2} = \pi$$

$$\textcircled{5} \quad \underset{x \rightarrow 0}{\text{Lt}} \frac{2 \cos x}{8} \left( \frac{\sin^8(\frac{\pi}{6} + x) - \sin^8 \frac{\pi}{6}}{x} \right)$$

$$= \frac{2}{8} \left[ \frac{8 \sin^7(\frac{\pi}{6} + x) \cos(\frac{\pi}{6} + x)}{1} - 0 \right]$$

$$= \frac{2}{8} \left[ 8 \times \frac{1}{128} \cdot \frac{\sqrt{3}}{2} \right]$$

$$= \underline{\underline{\sqrt{3}}}$$

$$\textcircled{6} \quad \underset{x \rightarrow 0}{\text{Lt}} \frac{\sqrt{1-x^2} - \sqrt{1+x^2}}{x^2} = -1$$

$$\textcircled{7} \quad \underset{x \rightarrow 0}{\text{Lt}} \frac{\sqrt[3]{1+x} - \sqrt[3]{1-x}}{x} = \frac{2}{3}$$

$$\textcircled{8} \quad \underset{x \rightarrow 0}{\text{Lt}} \frac{\log(1+x)}{x} = 1.$$

$$\textcircled{9} \quad \text{Lt}_{x \rightarrow 0} (1 + \sin x) \csc x = e^{\text{Lt}_{x \rightarrow 0} \csc x [1 + \sin x - x]} = 1$$

$$\textcircled{10} \quad \text{Lt}_{x \rightarrow 0} (1 + 2x)^{1/3x} = e^{\text{Lt}_{x \rightarrow 0} \frac{1}{3x} [1 + 2x - x]} = e^{2/3}$$

$$\textcircled{11} \quad \text{Lt}_{n \rightarrow \infty} \left(1 - \frac{1}{n}\right)^n = e^{\text{Lt}_{n \rightarrow \infty} n \left[1 - \frac{1}{n} - x\right]} = e^{-1/2}$$

$$\textcircled{12} \quad \text{Lt}_{x \rightarrow \infty} \left(\frac{x-1}{x-2}\right)^x = e^{\text{Lt}_{x \rightarrow \infty} x \left[\frac{x-1-x+2}{x-2}\right]} = e$$

$$\textcircled{13} \quad \text{Lt}_{n \rightarrow \infty} \frac{1+2+3+\dots+n}{n^2} = \frac{n(n+1)}{2n^2} = \frac{1}{2}$$

$$\textcircled{14} \quad \text{Lt}_{n \rightarrow \infty} \frac{1+4+9+\dots+n^2}{n^3} = \frac{n(n+1)(2n+1)}{6n^3} = \frac{2n^2+3n+1}{6n^2}, \frac{2}{3}$$

$$\textcircled{15} \quad \text{Lt}_{n \rightarrow \infty} \frac{n [1^3 + 2^3 + \dots + n^3]^2}{(1^2 + 2^2 + \dots + n^2)^3} = n \left[ \frac{n^2}{n(n+1)^4} \times 6^3 \right] = \frac{6^3}{4^2} \frac{n^3(n+1)^3}{(2n+1)^3} = \frac{6^3}{4^2 \times 8} = \frac{27}{16}$$

For trick for  
partial fraction  
 $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \dots + \frac{1}{n(n+1)}$

for two terms  
उदाहरण -  $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \dots + \frac{1}{n(n+1)}$

$$\text{Lt}_{n \rightarrow \infty} \left[ \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \dots + \frac{1}{n(n+1)} \right] = \sum \frac{1}{n(n+1)} = \sum \frac{1}{n} - \frac{1}{n+1} = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \dots - \frac{1}{n+1} = 1$$

$$\textcircled{16} \quad \text{Lt}_{x \rightarrow 0} \frac{1 - \cos 2x}{1 - \cos 3x} = \frac{+2 \sin 2x}{3 \sin 3x} = \frac{4 \cos 2x}{9 \cos 3x} = \frac{4}{9}$$

$$\textcircled{18} \quad \text{Lt}_{x \rightarrow 0} \frac{\cos 2x - \cos 3x}{x^2} = \frac{-2 \sin 2x + 3 \sin 3x}{2x} = \frac{-4 \cos 2x + 9 \cos 3x}{2} = \frac{5}{2}$$

## # Continuity of function:-

A function  $f$  is said to be continuous at  $x=a$ , if  
 $\lim_{x \rightarrow a} f(x) = f(a)$ , otherwise it is said to be a discontinuous function.

## \* Types of Discontinuous fn:-

### i) Discontinuity of 1<sup>st</sup> Type (Jumped Discontinuity)

A fn.  $f$  is said to have discontinuity of 1<sup>st</sup> type if

$$\lim_{x \rightarrow a^-} f(x) \neq \lim_{x \rightarrow a^+} f(x)$$

### ii) Discontinuity of 2<sup>nd</sup> Type

A fn.  $f$  is said to have discontinuity of 2<sup>nd</sup> type if either the left limit or the right limit are both does not exist.

### iii) Discontinuity of 3<sup>rd</sup> Type (Removal Discontinuity)

A fn.  $f$  is said to have discontinuity of 3<sup>rd</sup> type

$$\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x) \neq f(a)$$

## \* Certain standard functions:-

### \* Standard Continuous function:-

①  $x^n$  is a continuous fn.  $\forall x$  when  $n > 0$

$$(x, x^2, x^3, -x^{1/3}, x^{3/2}, \dots)$$

②  $x^n$  is a continuous fn.  $\forall x$  except  $x=0$  When  $n < 0$

$$(x^{-2} = \frac{1}{x^2}, x^{-3} = \frac{1}{x^3}, \dots)$$

③  $|x|$  is a continuous fn.  $\forall x$ .

- iv)  $\log x$  is a continuous fn  $\forall x > 0$ .
- v) Every exponential function ( $e^x$  or  $a^x$ ) is continuous.
- vi)  $\sin x$  and  $\cos x$  are continuous fn  $\forall x$
- vii)  $\tan x$  and  $\sec x$  are continuous fn  $\forall x$  except  $x = (2n+1)\frac{\pi}{2}$
- viii)  $\cot x$ ,  $\operatorname{cosec} x$  are continuous fn  $\forall x$  except  $x = n\pi$

Q.  $f(x) = \frac{1}{1+2^{1/x}}$  at  $x=0$

$$\lim_{h \rightarrow 0^-} f(0-h) = \frac{1}{1+2^{\frac{1}{0-h}}} = 1$$

$$\lim_{h \rightarrow 0^+} f(0+h) = \frac{1}{1+2^{\frac{1}{0+h}}} = 0.$$

$\therefore$  L.H limit  $\neq$  R.H limit

$\therefore$  discontinuous at  $x=0$ .

Q.  $f(x) = \frac{x(e^{1/x}-1)}{e^{\frac{1}{x}}+1}, x \neq 0$   
 $= 0, x=0.$

$$\begin{aligned} \lim_{h \rightarrow 0^-} f(0-h) &= \frac{-h(e^{\frac{1}{-h}}-1)}{e^{\frac{1}{-h}}+1} \\ &= \frac{-h \times -1}{1} = 0 \end{aligned}$$

$$\lim_{h \rightarrow 0^+} f(0+h) = \frac{h(e^{\frac{1}{h}}-1)}{e^{\frac{1}{h}}+1} = \frac{h\left(1-\frac{1}{e^{1/h}}\right)}{\left(1+\frac{1}{e^{1/h}}\right)} = h = 0$$

$$f(0) = 0$$

$\therefore$  L.H limit = R.H limit  $\Rightarrow f(0)$   $\therefore$  Continuous at  $x=0$

Q. A fn  $f(x)$  is defined by —

$$f(x) = \begin{cases} 0, & x \leq 0 \\ \frac{1}{2} - x, & 0 < x < \frac{1}{2} \\ \frac{1}{2}, & x = \frac{1}{2} \\ \frac{3}{2} - x, & \frac{1}{2} < x < 1 \\ 1, & x \geq 1 \end{cases}$$

Which of the following is true —

- f is continuous at  $x=0$
- f is discontinuous at  $x=\frac{1}{2}$
- continuous at  $x=1$
- All are true

Q.

25.9.14:

Q. A fn  $f(x)$  is given by -

$$f(x) = \begin{cases} 0, & x \leq 0 \\ 5x-4, & 0 < x \leq 1 \\ 4x^2-3x, & 1 \leq x \leq 2 \\ 3x+4, & x > 2 \end{cases}$$

Check continuity @  $x = 0, 1, 2$  -

- 
- At  $x=0$  → discontinuous
  - At  $x=1$  → continuous.
  - At  $x=2$  → continuous.

### DIFFERENTIABILITY

- A fn  $f$  is said to be differentiable at  $x=a$  if both  
L.H.D

$$\text{L.H.D} = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{-h}$$

$$\text{R.H.D} = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

exists and are equal. They are equal to  $f'(a)$ .

$$d(x^n) = nx^{n-1}$$

$$d\sqrt{x} = \frac{1}{2\sqrt{x}}$$

$$d\left(\frac{1}{\sqrt{x}}\right) = -\frac{1}{2x\sqrt{x}}$$

$$d\left(\frac{1}{x}\right) = -\frac{1}{x^2}$$

$$d\left(\frac{1}{x^2}\right) = -\frac{2}{x^3}$$

$$d(e^{ax}) = ae^{ax}$$

$$d(a^x) = a^x \log a$$

$$d(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$$

$$d(\sin^{-1}hx) = \frac{1}{\sqrt{1+x^2}}$$

$$d(\tan^{-1}x) = \frac{1}{1+x^2}$$

$$*\frac{d}{dx}(x^x) = x^x(1+\log x)$$

$$*\frac{d}{dx}(x^{\frac{1}{x}}) = x^{\frac{1}{x}} \cdot \frac{1}{x^2}(1-\log x)$$

$$\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$$

$$\frac{d}{dx}(u \cdot v) = \frac{udv}{dx} + v \frac{du}{dx}$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$$

$$\text{If } y = \sin^{-1} \left( \frac{2x}{1+x^2} \right)$$

$$= \cos^{-1} \left( \frac{1-x^2}{1+x^2} \right)$$

$$= \tan^{-1} \left( \frac{2x}{1-x^2} \right)$$

$$\text{then } \frac{dy}{dx} = \frac{2}{1+x^2}$$


---

Q. If  $y = x^{x^x}$  then  $\frac{dy}{dx} = ?$

$$y = x^y$$

$$\log y = y \log x.$$

$$\frac{dy}{y} = y \left( \frac{1}{x} dx \right) + \log x dy$$

$$dy \left( \frac{1}{y} - \log x \right) = \frac{y}{x} dx$$

$$\Rightarrow \frac{dy}{dx} = \frac{(y/x)}{(1/y - \log x)} = \frac{y^2}{x - xy \log x}$$

Q.  $x^y = e^{(x-y)}$  in terms of  $x \frac{dy}{dx} = ?$

~~2 marks~~  $y \log x = x-y$

$$y(\log x + 1) = x$$

$$\frac{dy}{dx} = \frac{(\log x + 1) - x \left( \frac{1}{x} \right)}{(\log x + 1)^2} dx$$

$$\frac{dy}{dx} = \frac{\log x}{(\log x + 1)^2}$$

Q. If  $x^m \cdot y^n = a^{m+n}$  then  $\frac{dy}{dx}$

$$m \log x + n \log y = m+n \log a.$$

$$\frac{m}{x} dx + \frac{n}{y} dy = 0$$

$$\cancel{\frac{dx}{dx}} = \cancel{x}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{m}{x^n} y = -\frac{m}{n} \cdot \frac{y}{x}$$

Q.  $x^m \cdot y^n = (x+y)^{m+n} \Rightarrow \frac{dy}{dx} = ?$

$$m \log x + n \log y = m+n \log(x+y)$$

$$\frac{m}{x} dx + \frac{n}{y} dy = \frac{m+n}{x+y} \cdot (dx+dy)$$

$$\Rightarrow \left( \frac{m}{x} - \frac{m+n}{x+y} \right) dx = \left( \frac{m+n}{x+y} - \frac{n}{y} \right) dy$$

Imp  
Result.

$$\boxed{\frac{dy}{dx} = \frac{y}{x}}$$

Q. If  $y = \sqrt{x+\sqrt{x+\sqrt{x+\dots}}}$  then  $\frac{dy}{dx} = ?$

$$y = \sqrt{x+y}$$

$$\Rightarrow y^2 = x+y$$

$$\Rightarrow 2y \cdot dy = dx + dy$$

$$\Rightarrow dy(dy-1) = dx$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{dy-1}$$

Top  
Result

$$\text{If } y = \sqrt{f(x) + \sqrt{f(x) + \sqrt{f(x) + \dots}}} \text{ then } \frac{dy}{dx} = \frac{f'(x)}{2y-1}$$

Q.  $y = \sqrt{\tan x + \sqrt{\tan x + \sqrt{\tan x + \dots}}}$  then  $\frac{dy}{dx} = ?$   $\frac{\sec^2 x}{2y-1}$

Q.  $y = \sqrt{\log x + \sqrt{\log x + \sqrt{\log x + \dots}}}$  then  $\frac{dy}{dx} = ?$   $\frac{1/x}{2y-1}$

Q.  $y = \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x + \dots}}}$  then  $\frac{dy}{dx} = ?$   $\frac{\cos x}{2y-1}$

Q.  $y = \frac{\sin^{-1} x}{\sqrt{1-x^2}}$  then  $(1-x^2)y_1 - xy = ?$

Ⓐ Ⓛ Ⓜ Ⓝ Ⓞ Ⓟ Ⓠ Ⓡ

*Leave term*  
*uv form*  $\rightarrow$   $(y \sqrt{1-x^2})^2 = (\sin^{-1} x)^2$   
*is easy than  $y^2$*   
 $\Rightarrow y^2(1-x^2) = (\sin^{-1} x)^2$

$\Rightarrow 2y \frac{dy}{dx} x$

$$\Rightarrow (1-x^2) 2y \frac{dy}{dx} + y^2(-2x) = 2 \sin^{-1} x \cdot \frac{1}{\sqrt{1-x^2}} dx$$

$$\Rightarrow (1-x^2) 2y \frac{dy}{dx} = dx \left( \frac{2 \sin^{-1} x}{\sqrt{1-x^2}} + 2y^2 x \right)$$

$$\Rightarrow 2y [(1-x^2)y_1 - xy] = 2y$$

$$\Rightarrow (1-x^2)y_1 - xy = 1.$$

Q.  $y = a \cos(\log x) + b \sin(\log x)$  then  $x^2 y_2 + xy_1 = ?$

$$y_1 = \frac{-a \sin(\log x) + b \cos(\log x)}{x}$$

$$xy_1 = -a \sin(\log x) + b \cos(\log x)$$

$$xy_2 + y_1 \cdot 1 = \frac{-a \cos(\log x) - b \sin(\log x)}{x}$$

$$x^2 y_2 + xy_1 = y.$$

Q. If  $y = \tan^{-1} \left[ \frac{(3-x)\sqrt{x}}{1-3x} \right] \Rightarrow \frac{dy}{dx} \Big|_{x=1} = ?$

General differentiation  
concerns time  
∴ replacement  
Substitute.

$$\begin{aligned}
 & \text{Let } x = \tan^2 \theta \Rightarrow \theta = \tan^{-1} \sqrt{x}. \\
 & \therefore y = \tan^{-1} \left( \frac{(3 - \tan^2 \theta) \tan \theta}{1 - 3 \tan^2 \theta} \right) \\
 & = \tan^{-1} \left( \frac{3 - \tan^3 \theta}{1 - 3 \tan^2 \theta} \right) \\
 & = \tan^{-1} (\tan 3\theta) \\
 & = 3\theta \\
 & y = 3 \tan^{-1} \sqrt{x} \\
 & \Rightarrow \frac{dy}{dx} = 3 \cdot \frac{1}{1 + (\sqrt{x})^2} \times \frac{1}{2\sqrt{x}} \\
 & \Rightarrow \frac{dy}{dx} \Big|_{x=1} = \frac{3}{1+1} \times \frac{1}{2} = \frac{3}{4} \text{ Ans}
 \end{aligned}$$

Q.  $y = \tan^{-1} \left( \frac{\cos x}{1 + \sin x} \right) \Rightarrow \frac{dy}{dx} = ?$

$1 + \sin x$   
we don't have formula  
But  $1 + \cos x$   
we have formula  
That's why this approach

$$\begin{aligned}
 & = \tan^{-1} \left( \frac{\sin(90-x)}{1 + \cos(90-x)} \right) \\
 & = \tan^{-1} \left( \frac{\sin(90-x)}{2 \cos^2(\frac{45-x}{2})} \right) \\
 & = \tan^{-1} \left( \frac{2 \sin(\frac{45-x}{2}) \cdot \cos(\frac{45-x}{2})}{2 \cos^2(\frac{45-x}{2})} \right) \\
 & = 45 - \frac{x}{2}.
 \end{aligned}$$

$$y = \frac{\pi}{4} - \frac{x}{2}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{2} \text{ Ans}$$

Q. If  $x = a \cos^3 \theta$ ,  $y = a \sin^3 \theta \Rightarrow \frac{dy}{dx} = ?$

$$\begin{aligned} dx &= -3a \cos^2 \theta \cdot \cancel{\sin^2 \theta} \cdot \cancel{\cos \theta} \cdot \sin \theta \\ dy &= 3a \sin^2 \theta \cdot \cancel{\cos^2 \theta} \cdot \cancel{\sin \theta} \cdot \cos \theta \\ \Rightarrow \frac{dy}{dx} &= -\frac{\sin^2 \theta}{\cos^2 \theta} = -\tan^2 \theta \end{aligned}$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{3a \sin^2 \theta \cdot \cos \theta}{-3a \cos^2 \theta \cdot \sin \theta} = -\tan \theta. \end{aligned}$$

Q.  $x = 3 \cos \theta - \cos^3 \theta$

$$y = 3 \sin \theta - \sin^3 \theta$$

$$\frac{dy}{dx} = 3 \cos \theta - 3 \sin^2 \theta \cos \theta = 3 \cos \theta (1 - \sin^2 \theta)$$

$$\frac{dx}{dy} = -3 \sin \theta + 3 \cos^2 \theta \cdot \sin \theta = -3 \sin \theta (1 - \cos^2 \theta)$$

$$\therefore \frac{dy}{dx} = \frac{3 \cos \theta \cdot \cos^2 \theta}{-3 \sin \theta \cdot \sin^2 \theta} = -\cot^3 \theta \text{ Ans}$$

Q.  $x = a(\theta + \sin \theta)$

$$y = a(1 - \cos \theta)$$

$$\frac{dy}{dx} = \frac{a(\theta + \sin \theta)}{a(1 - \cos \theta)} = \frac{a \sin \theta / 2 \cdot \cos \theta / 2}{a \cos^2 \theta / 2} = \tan \frac{\theta}{2} \text{ Ans}$$

Q.  $x = a[\theta \sin \theta + \cos \theta]$

$$y = a[\sin \theta - \theta \cos \theta]$$

$$\frac{dy}{dx} = \tan \theta \text{ Ans}$$

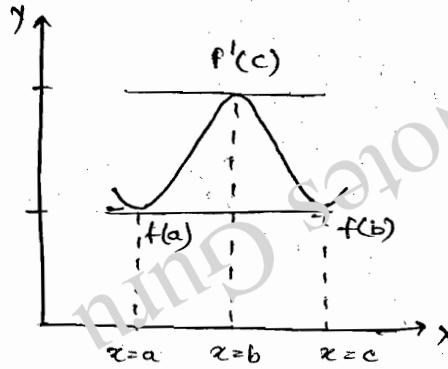
## # MEAN VALUE THEOREM:-

### (I). Rolle's mean value theorem:-

Let  $f(x)$  be a function defined such that -

- i)  $f(x)$  is continuous in  $[a, b]$
- ii)  $f(x)$  is differentiable in  $(a, b)$
- iii)  $f(a) = f(b)$

then  $\exists$  at least one value  $c \in (a, b)$  such that  $f'(c)=0$



Proof:-

$f'(c) = \text{slope of tangent line}$ ..

$= \text{slope of } \overline{AB} \quad \left\{ \because \text{two parallel line have equal slopes} \right\}$

$$= \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{f(b) - f(a)}{b - a}$$

$$= \frac{0}{b-a} \quad \left\{ \because f(a) = f(b) \right\}$$

$f'(c) = 0$

Q. find  $c$  for the Rolle's mean value theorem for -

$$f(x) = x^3 - 4x \text{ in } [-2, 2].$$

- Ⓐ  $\frac{2}{\sqrt{3}}$  Ⓑ  $-\frac{2}{\sqrt{3}}$  Ⓒ  $\pm \frac{2}{\sqrt{3}}$  Ⓓ none

→  $f(-2) = -8 + 8 = 0.$

$$f(2) = 8 - 8 = 0$$

$$\therefore f'(x) = 3x^2 - 4$$

$$f'(c) = 3c^2 - 4 = 0$$

$$\Rightarrow c^2 = \pm \sqrt{\frac{4}{3}} = \pm \frac{2}{\sqrt{3}} \text{ Ans.}$$

But Problem is not over.

Cross-verify the points in case of  $\pm \sqrt{ }$

→ are points inside interval or not?

Here  $\pm \frac{2}{\sqrt{3}}$  both are inside so ans is  $\pm \frac{2}{\sqrt{3}}$  Ans.

Q.  $f(x) = \frac{\sin x}{e^x}$  in  $[0, \pi]$ .

$$f(0) = \frac{0}{1} = 0$$

$$f(\pi) = \frac{0}{e^\pi} = 0$$

$$\therefore f'(x) = \frac{e^x \cos x - \sin x e^x}{e^{2x}}$$

$$f'(c) = e^c \cos c - \sin c \cdot e^c = 0$$

$$\Rightarrow e^c \cos c = e^c \sin c$$

$$\Rightarrow \tan c = 1$$

$$\therefore c = \frac{\pi}{4} \text{ Ans}$$

$c \in [0, \pi]$  must satisfy this

Q.  $f(x) = \frac{x^2 + ab}{x(a+b)}$ ,  $[a, b]$ ,  $a > 0, b > 0$ .

- (A)  $-\sqrt{ab}$  (B)  $\sqrt{ab}$  (C)  $\pm\sqrt{ab}$  (D) none.

$$f'(x) = x(a+b)[2x] - [x^2 + ab][a+b] = 0$$

$$\Rightarrow 2x^2(a+b) = (x^2 + ab)(a+b)$$

$$\Rightarrow x^2 = ab$$

$$\Rightarrow x = \pm\sqrt{ab}$$

But  $x = -\sqrt{ab}$  should not  $\in [a, b]$

where  $a > 0, b > 0$

$$\therefore x = \sqrt{ab}$$

$$\Rightarrow c = \boxed{\sqrt{ab}}$$

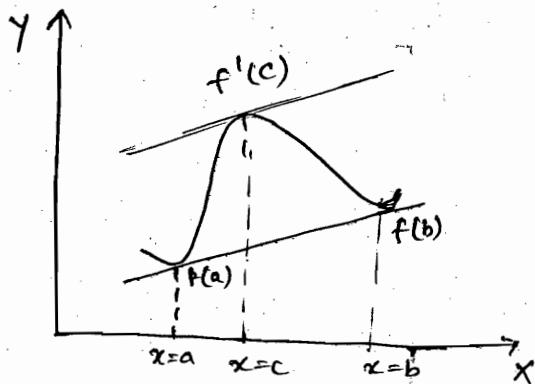
## # Lagrange's mean value theorem :-

Let  $f(x)$  be a function defined such that -

- i).  $f(x)$  is continuous in  $[a, b]$
- ii).  $f(x)$  is differentiable in  $(a, b)$
- iii).  $f(a) \neq f(b)$

Then  $\exists$  at least one value  $c \in (a, b)$  such that

$$\boxed{f'(c) = \frac{f(b) - f(a)}{b-a}}$$



$$Q. f(x) = \log x \text{ in } [1, e].$$

$$f(e) = \log e$$

$$f(1) = \log 1 = 0.$$

$$\therefore f'(c) = \frac{\log e - \log 1}{e-1} = \frac{\log e}{e-1}$$

$$\Rightarrow \frac{1}{c} = \frac{\log e}{e-1}$$

$$\Rightarrow c = \frac{e-1}{\cancel{\log e}} \quad \underline{\text{Ans}}$$

$$\Rightarrow c = e-1 \in (1, e) \quad \left\{ \because e \approx 2.71 \right\}$$

$$Q. f(x) = x^3 - 6x^2 + 11x - 6 \text{ in } [0, 4].$$

$$3x^2 - 12x + 11 = \frac{4^3 - 6(4)^2 + 11(4) - 6 - (\cancel{6})}{4-0}$$

$$\Rightarrow 3x^2 - 12x + 11 = \frac{64 - 96 + 44}{4} = \frac{108 - 96}{4} = \frac{12}{4} = 3$$

$$\Rightarrow 3x^2 - 12x + 9 = 0$$

$$\Rightarrow 3c^2 - 12c + 9 = 0$$

$$\Rightarrow c = \frac{12 \pm \sqrt{144 - 96}}{6}$$

$$= \frac{12 \pm \sqrt{48}}{6}$$

$$= 2 \pm \frac{4\sqrt{3}}{6}$$

$$= 2 \pm \frac{2}{\sqrt{3}} \in (0, 4)$$

$$\therefore c = 2 \pm \frac{2}{\sqrt{3}} \quad \underline{\text{Ans}}$$

## # CAUCHY'S MEAN VALUE THEOREM :-

Let  $f(x)$  and  $g(x)$  be two functions defined such that

i)  $f$  and  $g$  are continuous in the closed interval  $[a,b]$

ii)  $f$  and  $g$  are differentiable in  $(a,b)$

iii)  $g'(x) \neq 0 \quad \forall x \in (a,b)$  then there exists at least one value  $c \in (a,b)$  such that —

$$\boxed{\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}}$$

Q.  $f(x) = \sqrt{x}$ ,  $g(x) = \frac{1}{\sqrt{x}}$  in  $[a,b]$

Ⓐ  $\frac{ab}{2}$  Ⓑ  $\frac{a+b}{2}$  Ⓒ  $\sqrt{ab}$  Ⓓ  $\frac{2ab}{a+b}$

$$\frac{\frac{1}{2}x^{-\frac{1}{2}}}{-\frac{1}{2}x^{-\frac{3}{2}}} = \frac{\sqrt{b} - \sqrt{a}}{\frac{1}{\sqrt{b}} - \frac{1}{\sqrt{a}}}$$

$$\Rightarrow -x^{-\frac{1}{2} + \frac{3}{2}} = \frac{\sqrt{b} - \sqrt{a}}{\frac{\sqrt{b} - \sqrt{a}}{\sqrt{ab}}}$$

$$\Rightarrow f(c) = f(\sqrt{ab})$$

$$\therefore c = \sqrt{ab} \quad \text{Ans}$$

which  $\in (a,b)$

Q.  $f(x) = e^x$ ,  $g(x) = e^{-x}$  in  $[a,b]$ .

$$\frac{e^x}{-e^{-c}} = \frac{e^b - e^a}{e^{-b} - e^{-a}}$$

$$\Rightarrow -e^{2c} = \frac{e^b - e^a}{e^a - e^b} \Rightarrow e^{2c} = e^{b+a}$$

$$\Rightarrow c = \frac{b+a}{2} \quad \text{Ans}$$

$$Q. \quad f(x) = \frac{1}{x^2} \quad g(x) = \frac{1}{x} \quad \text{in } [a, b]$$

$$\rightarrow \frac{f(2c)^3}{f(c)^{-2}} = \frac{\frac{1}{b^2} - \frac{1}{a^2}}{\frac{1}{b} - \frac{1}{a}}$$

$$\Rightarrow 2c^{-1} = b^2 \cdot \frac{a^2/b^2}{a+b} \times \frac{ab}{a^2b^2}$$

$$\Rightarrow c = \frac{2ab}{a+b}$$

$$Q. \quad f(x) = \sin x, \quad g(x) = \cos x \quad \text{in } [0, \frac{\pi}{2}]$$

$$\Rightarrow \frac{\cos x}{-\sin x} = \frac{1-c}{0-1}$$

$$\Rightarrow -\cot x = -1$$

$$\Rightarrow \tan x = 1$$

$$\therefore c = \frac{\pi}{4}$$

#

## # Taylor's mean value Theorem :-

Let  $f(x)$  be a function defined such that -

i)  $f(x)$  is continuous in  $[a, a+h]$

ii)  $f^{n-1}(x)$  is differentiable in  $[a, a+h]$

then there exists at least one value  $\theta \in (0, 1)$  such that -

$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a) + \dots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{h^n}{n!} f^n(\theta)$$

Let  $a+h = x \Rightarrow h = (x-a)$ .

\*\*\*

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} + f''(a) + \dots$$

## # Maclaurin's formula :- [a=0]

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} + f''(0) + \frac{x^3}{3!} f'''(0) + \dots$$

①  $f(x) = \sin x$  about  $x=0$

$$f(0) = 0$$

$$f'(0) = \cos 0 = 1$$

$$f''(0) = -\sin 0 = 0$$

$$f'''(0) = -\cos 0 = -1$$

\*\*\*

$$f(x) = \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

Imp Result.

Q.  $f(x) = e^x$  about  $x=0$

$$f(0) = 1$$

$$f'(0) = e^0 = 1$$

$$f''(0) = e^0 = 1$$

$$f'''(0) = e^0 = 1$$

Temp  
Result

$$\boxed{f(x) = e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots}$$

Q.  $f(x) = \log(1+x)$  about  $x=0$ .

$$f(0) = 0$$

$$f'(0) = \frac{1}{1+x} = 1$$

$$f''(0) = -(1+x)^{-2} = -\frac{1}{2}$$

$$f'''(0) = +2(1+x)^{-3} = 2$$

$$f''''(0) = -6(1+x)^{-4} = -6$$

$\therefore f(x) = \log(1+x) = x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} (-6)$

Temp  
Result

$$\boxed{\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots}$$

Q.  $f(x) = \frac{x}{1+x}$  about  $x=0$ .

$$f(x) = \frac{x}{1+x} = 1 - \frac{1}{1+x} = 0 \quad \frac{(1+x)x}{1+x} \left( 1 - \frac{1}{x+1} \right)$$

$$f'(x) = \frac{1}{(1+x)^2} = 1$$

$$f''(x) = \frac{-2}{(1+x)^3} = -2$$

$$f'''(x) = \frac{6}{(1+x)^4}$$

Top Result

$$\therefore \frac{x}{1+x} = 0 + x(1) + \frac{x^2(-2)}{2!} + \frac{x^3(6)}{3!} + \dots$$

$$\boxed{\frac{x}{1+x} = x - x^2 + x^3 - x^4 + \dots}$$

### Alternate method

$$\begin{aligned} f(x) &= \frac{x}{1+x} \\ &= x(1+x)^{-1} \\ &= x(1-x+x^2-x^3+\dots) \quad \left\{ \text{Binomial Expansion} \right\} \\ &= x - x^2 + x^3 - x^4 + \dots \end{aligned}$$

- Q. find the coeff. of  $x^2$  in the expansion of  $f(x) = \cos^2 x$  about  $x=0$ .

$$\begin{aligned} \rightarrow f(x) &= \cos^2 x \\ &= \cos x \cdot \cos x \\ &= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right) \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right) \\ &= -\frac{x^2}{2!} - \frac{x^2}{2!} \\ &= -\frac{2x^2}{2!} \\ &= -x^2 = -1 \end{aligned}$$

### Alternate method

$\because$  coeff. of  $x^2$  asked.

$$\hookrightarrow = \frac{f''(0)}{2!}$$

$$\therefore f(x) = \cos^2 x =$$

$$f'(x) = 2 \cos x (-\sin x) = -2 \sin 2x$$

$$f''(x) = -2 \cos 2x$$

$$\therefore \frac{f''(0)}{2!} = \frac{-2}{2!} = -1$$

Q. find the coeff of  $(x-2)^4$  in the expansion of  $e^x$  about

$$x=2$$

$$\rightarrow \text{coeff of } (x-2)^4 = \frac{f''(a)}{4!}$$

$$f(x) = e^x$$

$$f'''(x) = e^{x^2} = e^2$$

$$\therefore \frac{f''(a)}{4!} = \frac{e^2}{4!} \quad \underline{\text{Ans}}$$

Q. find expansion of  $x^2$  about  $a=1$

④  $1-x+x^2$

⑤  $1+x^2$

⑥  $x^2$        $\left. \begin{array}{l} \because f(x) = x^2 \text{ is pure algebraic fn} \\ \therefore \text{expansion not possible} \end{array} \right\}$

⑦  $1+x+\frac{x^2}{2}+\frac{x^3}{3}$

Q.  $f(x) = \log\left(\frac{1+x}{1-x}\right)$  about  $x=0$ .

$$f(x) = \log(1+x) - \log(1-x)$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \dots$$

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} \dots$$

$$\log\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \dots\right)$$

## # PARTIAL AND TOTAL DERIVATIVES :-

O.D	P.D
$y = f(x)$	$z = f(x, y)$
$z = g(y)$	$u = f(x, y, z)$

Q.  $z = x^2 - xy + y^2$

$$\begin{array}{|l|l|} \hline \frac{\partial z}{\partial x} = 2x - y & \frac{\partial z}{\partial y} = -x + 2y \\ \frac{\partial^2 z}{\partial y \partial x} = -1 & \frac{\partial^2 z}{\partial z \cdot \partial y} = -1 \\ \hline \end{array}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$$

### \* HOMOGENEOUS FUNCTION :-

A function  $f(x, y)$  is said to be homogeneous of degree  $n$  in  $x$  and  $y$  if —

$$f(kx, ky) = k^n f(x, y)$$

or else check overall degree of each term must be same.

ex:-  $f(x, y) = x^2 - xy + y^2 \rightarrow$  Homogeneous (Directly by degree)  
 $f(kx, ky) = k^2 x^2 - k^2 xy + k^2 y^2$   
 $= k^2 (x^2 - xy + y^2)$

Note:-

The product of two homogeneous fn is again a homogeneous function.

ex:-  $f(x, y) = (x^3 + y^3)(x^2 - y^2)$

$$n = 3+2 = 5$$

If a fn is in rational form, if both numerator and Denominator are homogeneous function, therefore the given function is also homogeneous.

$$\text{ex:- } f(x,y) = \frac{x^3+y^3}{x^2-y^2}$$

$$n = 3-2=1$$

$$f(x,y) = \frac{x^{1/4}+y^{1/4}}{x^{1/5}-y^{1/5}}$$

$$n = \frac{1}{4} - \frac{1}{5} = \frac{1}{20}$$

### # Euler's Theorem:-

If  $z$  is a homogeneous fn of degree  $n$  in  $x$  and  $y$

then  $x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = n \cdot z$

Note:-

This formula is applicable directly if  $z$  is an algebraic fn

$$\text{ex:- } \underline{0.} \quad z = (x^2+y^2)^{1/3} \quad \text{then } x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = \frac{2}{3}z$$

$$\underline{0.} \quad z = \frac{xy}{x+y} \Rightarrow xz_x + yz_y = 1 \cdot z$$

$$Q. \text{ If } z = a_0 x^n + a_1 x^{n-1} y + a_2 x^{n-2} y^2 + \dots + a_n y^n$$

$$\text{then } x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = nz$$

$$Q. \text{ If } z = \log(x^2+y^2) \quad \text{then } x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = -$$

Note:-

If  $z$  is not algebraic and let  $\phi(z)$  is algebraic and homogeneous of degree  $n$  in  $x$  and  $y$  then

$$\frac{x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y}}{\phi'(z)} = n \frac{\phi(z)}{\phi'(z)}$$

Shortcut

Q. given  $z = \log(x^2+y^2)$

$\Rightarrow e^z = x^2+y^2 \rightarrow$  homogeneous of degree 2

$$\therefore x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y}$$

$$x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = 2 \cdot \frac{e^z}{e^z} = 2 \text{ Ans}$$

Q. If  $z = \sin^{-1}\left(\frac{x^2+y^2}{x-y}\right)$

$$\Rightarrow x z_x + y z_y = 1 \cdot \frac{\sin z}{\cos z} = \tan z \text{ Ans}$$

Q. If  $z = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$  then  $x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = ?$

$$\Rightarrow \tan z = \frac{x^3+y^3}{x-y} \text{ homogeneous of order 2.}$$

$$\therefore x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = 2 \cdot \frac{\tan z}{\sec^2 z} = \sin 2z \text{ Ans}$$

Q. If  $z = x^2 \sin^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right)$

then  $x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = 2z$

$\left. \begin{array}{l} \text{at } \frac{y}{x} \neq 0 \text{ i.e. } y=0 \text{ & } x \neq 0 \\ \text{inverse fn not exists} \\ \text{so it becomes } z = x^2 - y^2 \end{array} \right\}$

Q. If  $z = (x^3+y^3) e^{-x/y} \Rightarrow x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = 3z$

$\left. \begin{array}{l} \text{when } x=y=0 \\ \text{the } e^{-x/y} \text{ does not} \\ \text{exist} \\ \therefore z = x^3+y^3 \\ \text{homogeneous of} \\ \text{order 3} \end{array} \right\}$

## Application of Euler's Theorem:-

If  $z$  is a homogeneous fn. of degree  $n$  in  $x$  and  $y$   
then —

$$x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = n(n-1)z$$

Note:-

This formula is applicable directly if  $z$  is an algebraic fn.

Ex:-

Q. If  $z = (x^2 + y^2)^{1/3}$  then  $x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = \frac{2}{3}(\frac{2}{3}-1)z$   
 $= -\frac{2}{9}z$

Q. If  $z = \frac{x^3y^3}{x^2+y^2}$  then  $x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = 4(4-1)z$   
 $\hookrightarrow 6-2=4$

Q. If  $z = \tan^{-1} \left( \frac{x^3+y^3}{x-y} \right)$  then  $x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} =$

Shortcut

Note:-

If  $z$  is not algebraic then

$$x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = \frac{n \phi(z)}{\phi'(z)} = F(z)$$

and  $x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = F(z) [F'(z) - 1]$

$$\therefore F(z) = z \cdot \frac{\tan z}{\sec^2 z} = \sin 2z$$

$$\begin{aligned} \therefore F(z) [F'(z) - 1] &= \sin 2z [2 \cos 2z - 1] \\ &= \underline{\sin 4z - \sin 2z} \end{aligned}$$

$$Q. \text{ If } \log\left(\frac{x^4+y^4}{x+y}\right) \text{ then } x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = ?$$

$$\rightarrow F(z) = 3 \cdot \frac{e^z}{e^2} = 3$$

$$F(z)[F(z)-1] = 3[0-1] = -3 \quad \underline{\text{Ans}}$$

### # Explicit Function :-

Any function which we can express in the form of either  $y = f(x)$  or  $x = g(y)$  etc is called an explicit fn.

means  $y = \text{purely fn of } x$  }  
 $x = \text{purely fn of } y$  }  $\Rightarrow$  Explicit fn.

$$\text{ex:- } y = ax^2 + bx + c$$

Ordinary Derivatives upto  $n^{\text{th}}$  order can be easily evaluated.

### # Implicit function:-

Any function which is not explicit is said to be an implicit function.

$$\text{ex:- } y = ax^2 + bxy + c$$

Ordinary Derivatives upto  $n^{\text{th}}$  order evaluation becomes complex.

∴ Partial Derivative is used

Shortcut

$$\frac{dy}{dx} = - \frac{\partial f / \partial x}{\partial f / \partial y}$$

$$\frac{d^2y}{dx^2} = - \frac{[q^2r - 2pqrs + p^2t]}{q^3}$$

$$P = \frac{\partial z}{\partial x}$$

$$Q = \frac{\partial z}{\partial y}$$

$$R = \frac{\partial^2 z}{\partial x^2}$$

$$S = \frac{\partial^2 z}{\partial x \partial y} \quad \& \quad T = \frac{\partial^2 z}{\partial y^2}$$

\* If  $ax^2 + 2hxy + by^2 = 1$ .

Shortcut

$$\boxed{\frac{d^2y}{dx^2} = \frac{h^2 - ab}{(hx + by)^3}}$$

Ex:-

Q.  $2x^2 + 4xy + 3y^2 = 1$

$$a = 2$$

$$b = 3$$

$$h = 2$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{2^2 - 2 \times 3}{(2x + 3y)^3} = \frac{-2}{(2x + 3y)^3} \quad \text{Ans}$$

Q.  $x^2 + xy + 2y^2 = 1$

$$\frac{d^2y}{dx^2} = \frac{\left(\frac{1}{2}\right)^2 - 2}{\left(\frac{1}{2}x + 2y\right)^3} = \frac{-\frac{15}{4}}{(x + 4y)^3} \quad \text{Ans}$$

\* If  $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

Shortcut

$$\frac{d^2y}{dx^2} = \frac{abc + 2fgh - af^2 - bg^2 - ch^2}{(hx + by + f)^3}$$

Ex:-

Q.  $2x^2 + 4xy + 3y^2 + 2x + 4y + 1 = 0$

$$a = 2$$

$$b = 3$$

$$c = 1$$

$$h = 2$$

$$g = 1$$

$$f = 2$$

$$\therefore \frac{d^2y}{dx^2} = \frac{6 + 8' - 8 - 3 - 4}{(2x + 3y + 2)^3} = \frac{-1}{(2x + 3y + 2)^3}$$

$$0. \quad y^2 - 5x + 4x^2 = 8$$

$$a = 1$$

$$b = 1$$

$$c = -8$$

$$h = 0$$

$$g = -\frac{5}{2}$$

$$f = 0.$$

$$\frac{d^2y}{dx^2} = \frac{-32 + \frac{25}{4}}{(1y)^3} = -\frac{153}{4y^3}$$

Alternate method (By Explicit)

$$y^2 = 8 + 5x - 4x^2$$

$$y = \sqrt{8 + 5x - 4x^2}$$

$\frac{dy}{dx^2}$  will become complex.

$\therefore$  Implicit shortcut is very Easy to use.

## # COMPOSITE FUNCTION

$$\rightarrow (g \circ f)x = g[f(x)].$$

Any function in function is said to be a composite fn.

or

If  $z$  is a fn in  $x$  and  $y$  and  $x, y$  are functions in  $t$  then  $z$  is called a composite fn in  $t$ .

$$z = f(x, y)$$

### \* Total Derivative of a composite fn :-

If  $z = f(x, y)$  and  $x = g(t)$ ,  $y = h(t)$  then the total derivative of  $z$  w.r.t.  $t$  is denoted by —

$$\boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}}$$

Total Derivative means combination of ordinary + partial derivative.

$$\boxed{\frac{du}{dt} = \frac{\partial u}{\partial x} \frac{dx}{dt} + \frac{\partial u}{\partial y} \frac{dy}{dt} + \frac{\partial u}{\partial z} \frac{dz}{dt}}$$

Q:  $z = e^x \sin y$

$$x = \log t$$

$$y = t^2$$

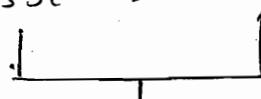
$$\frac{dz}{dt} = ?$$

$$\rightarrow \left. \begin{array}{l} \frac{\partial z}{\partial x} = e^x \sin y \\ \frac{\partial z}{\partial y} = e^x \cos y \\ \frac{\partial z}{\partial t} = \frac{1}{t} \\ \frac{dy}{dt} = 2t \end{array} \right\} \therefore \frac{dz}{dt} = e^x \sin y \left( \frac{1}{t} \right) + e^x \cos y (2t) \\ = \frac{e^x}{t} [ \sin y + 2t^2 \cos y ]$$

Q.  $u = x^2 + y^2 + z^2$

$$x = e^{2t}, y = e^{2t} \cos 3t, z = e^{2t} \sin 3t$$

then  $\frac{du}{dt} = ?$



$\frac{dy}{dt}$  &  $\frac{dz}{dt}$  will take time as in  $uv$  form.

Alternate  
Method:

$$u = (e^{2t})^2 + (e^{2t} \cos 3t)^2 + (e^{2t} \sin 3t)^2 \\ = e^{4t} [ 1 + \cos^2 3t + \sin^2 3t ]$$

$$u = 2e^{4t}$$

$$\frac{du}{dt} = 8e^{4t}$$

$$Q. \quad u = x^2 - y^2$$

$$x = e^t \cos t, \quad y = e^t \sin t$$

$$\frac{du}{dt} \Big|_{t=0} = ?$$

$$\rightarrow u = e^{2t} \cos^2 t - e^{2t} \sin^2 t$$

$$= e^{2t} (\cos^2 t - \sin^2 t)$$

$$u = e^{2t} \cdot \cos 2t$$

$$\Rightarrow \frac{du}{dt} \Big|_{t=0} = 2e^{2t} (-\sin 2t) + 2 \cos 2t \cdot e^{2t}$$

$$= 2 \cdot 1 =$$

$$= 2 \text{ Ans}$$

$$Q. \quad u = x^3 y e^z$$

$$x = t, \quad y = t^2, \quad z = \log t$$

$$\frac{du}{dt} \Big|_{t=2} = ?$$

$$u = t^3 t^2 e^{\log t}$$

$$u = t^6$$

$$\frac{du}{dt} \Big|_{t=2} = 6t^5 = 6 \times 2^5 = 192 \text{ Ans}$$

$$Q. \quad \text{If } x = r \cos \theta, \quad y = r \sin \theta.$$

$$\text{then } ① \frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = \underline{\hspace{2cm}}$$

$$② \frac{\partial^2 \theta}{\partial x \partial y} = \underline{\hspace{2cm}}$$

$$\frac{y \sin \theta}{r \cos \theta} = \frac{y}{x}$$

$$\Rightarrow \tan \theta = \frac{y}{x}$$

$$\theta = \tan^{-1} \frac{y}{x}$$

$$\frac{\partial \theta}{\partial x} = \frac{1}{1 + \frac{y^2}{x^2}} \times -\frac{y}{x^2}$$

$$\frac{\partial \theta}{\partial x} = \frac{-y}{x^2 + y^2}$$

$$\begin{aligned}\frac{\partial^2 \theta}{\partial x^2} &= \frac{-y \times -1}{(x^2 + y^2)^2} \times 2x \\ &= \frac{2xy}{(x^2 + y^2)^2}\end{aligned}$$

Similarly  $\frac{\partial^2 \theta}{\partial y^2} = \frac{-2xy}{(x^2 + y^2)^2}$

$$\therefore \boxed{\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = 0.}$$

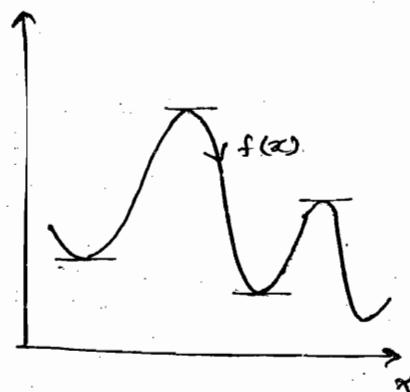
$$\therefore \frac{\partial \theta}{\partial x} = -\frac{y}{x^2 + y^2}$$

$$\frac{\partial \theta}{\partial x \partial y}$$

## Maxima and Minima.

### Maximum Value:-

If a continuous fn  $f(x)$  increases to a certain value and then decreases that value is called maximum value of the function.



### Minimum Value:-

If a continuous fn  $f(x)$  decreases to a certain value and then increases, that value is called minimum value of the fn.

#### Note:-

- i) The maxima and minima occurs alternatively.
- ii) A fn can have several maximum and several minimum values.
- iii) The min<sup>m</sup> value may be greater than the max<sup>m</sup> value.
- iv) The least min<sup>m</sup> value is called the global minimum or universal minimum and the highest max<sup>m</sup> value is called the global max<sup>m</sup> or universal max<sup>m</sup>.

→ For a maxima or minima —

$$f'(x) = 0$$

$$x = \alpha, \beta, \lambda \dots$$

$x = \alpha$ ,  $f''(\alpha) = 0$ ,  $f$  is minimum at  $x = \alpha$ . Min. value =  $f(\alpha)$

$x = \beta$ ,  $f''(\beta) < 0$ ,  $f$  is maximum at  $x = \beta$ . Max<sup>m</sup> value =  $f(\beta)$

$x = \lambda$ ,  $f''(\lambda) = 0$ ,  $f$  is stationary.

if  $f'''(\lambda) \neq 0$  then ' $\lambda$ ' is said to be inflection point.

Q.  $f(x) = x^x$

$$f'(x) = x^x(1 + \log x)$$

$$f''(x) = x^x\left(\frac{1}{x}\right) + (1 + \log x)x^x(1 + \log x)$$

$$f''(x) = x^{x-1} + x^x(1 + \log x)^2$$

$$\Rightarrow f'(x) = 0$$

$$\Rightarrow x^x(1 + \log x) = 0$$

↳ can't be never 0

$$\therefore 1 + \log x = 0$$

$$\Rightarrow \log x = -1$$

$$\Rightarrow x = e^{-1} = \frac{1}{e}$$

$$f''(x) = f''\left(\frac{1}{e}\right) = \left(\frac{-1}{e}\right)^{\frac{1}{e}-1} > 0 \quad \left. \begin{array}{l} 2^2 = 4 > 0 \\ 2^{-2} = \frac{1}{4} > 0 \end{array} \right\}$$

for  $f(x) = x^x$ , F is min<sup>m</sup> at  $x = \frac{1}{e}$ .

∴ Min<sup>m</sup> value =  $f\left(\frac{1}{e}\right) = \left(\frac{1}{e}\right)^{\frac{1}{e}} = e^{-\frac{1}{e}}$  Ans.

Top Result

Q.  $f(x) = x^{\frac{1}{x}}$  has its max<sup>m</sup> at  $x = e$ .

Max<sup>m</sup> value =  $f(e) = e^{\frac{1}{e}}$ .

Top Result

\*\*\*\*\*

$f_n$	Min value	Max value	At
$x^x$	$e^{-\frac{1}{e}}$	-	$x = \frac{1}{e}$
$x^{\frac{1}{x}}$	-	$e^{\frac{1}{e}}$	$x = e$

f

$$0. \quad f(x) = \frac{\log x}{x} = \frac{1}{x} \log x$$

$$f'(x) = \frac{1}{x} \left( \frac{1}{x} \right) + \log x \left( -\frac{1}{x^2} \right)$$

$$f'(x) = \frac{1}{x^2} (1 - \log x)$$

$$f''(x) = -\frac{2}{x^3} (1 - \log x) + \frac{1}{x^2} \left( -\frac{1}{x} \right)$$

$$f'(x) = 0.$$

$$\Rightarrow \frac{1}{x^2} (1 - \log x) = 0.$$

$$\Rightarrow \log x = 1$$

$$\therefore x = e.$$

$$f''(x) = f''(e) = 0 - \frac{1}{e^3} < 0$$

$\therefore f$  is  $\max^m$  at  $x=e$ .

$$\text{Max}^m \text{ value} = f(e) = \frac{\log e}{e} = \frac{1}{e}.$$

$$a. \quad f(x) = x^3 + \frac{3}{x}$$

$$f'(x) = 3x^2 - \frac{3}{x^2}$$

$$f''(x) = 6x + \frac{3 \times 2 x^{-3}}{x^2} = 6x + \frac{6}{x^3}$$

$$\Rightarrow 3x^2 - \frac{3}{x^2} = 0$$

$$\Rightarrow x^2 = \frac{1}{x^2}$$

$$\Rightarrow x^4 - 1 = 0$$

$$\Rightarrow (x^2 - 1)(x^2 + 1) = 0$$

$$\Rightarrow x^2 - 1 = 0$$

$$\Rightarrow x = \pm 1$$

$$f''(1) = 6 + 6 = 12 > 0 \rightarrow \min^m \text{ at } x=1$$

$$f''(-1) = -6 - 6 = -12 < 0 \rightarrow \max^m \text{ at } x=-1$$

Min. value = 4.  
 $\therefore \text{Max}^m \text{ value} = -4$

Q  $f(x) = x^2 e^{-x}$

$$f'(x) = e^{-x} \times (2x) + x^2 (-e^{-x}) = 0$$

$$= 2xe^{-x} - x^2 e^{-x} = 0$$

$$\Rightarrow e^{-x} (2x - x^2) = 0$$

$$\Rightarrow \cancel{e^{-x}} x \cancel{x} (2-x) = 0$$

$$x = 0, 2.$$

$$f''(x) = -2x e^{-x} + e^{-x} \times 2 - [e^{-x} (2x) - x^2 (-e^{-x})]$$

$$= e^{-x} (2 - 4x + x^2)$$

at  $x=0$   $f''(x) = 2 > 0 \rightarrow \text{min}^m$  at  $x=0$

at  $x=2$   $f''(x) = \frac{-2}{e^2} < 0 \rightarrow \text{max}^m$  at  $x=2$

$$\text{Max value} = \frac{4e^{-2}}{4}$$

Q.  $f(x) = x^1 e^{-x}$  has its max at  $x = \underline{1}$ .

Foilek ~~\*\*\*\*~~ Remember: in this type max<sup>m</sup> occurs at degree ..

Q.

$$f(x) = a \cos x + b \sin x + c$$

$$\text{Min value} = c - \sqrt{a^2 + b^2}$$

$$\text{Max value} = c + \sqrt{a^2 + b^2}$$

Shortcut

Q.

$$f(x) = 3 \cos x + 4 \sin x + 2.$$

$$\text{Min value} = 2 - \sqrt{3^2 + 4^2} = 2 - 5 = -3$$

$$\text{Max value} = 2 + \sqrt{3^2 + 4^2} = 2 + 5 = 7$$

$$\begin{aligned}
 Q. \quad f(x) &= 5 \cos x + 3 \cos\left(x + \frac{\pi}{3}\right) + 3 \\
 &= 5 \cos x + 3 \cos x \cdot \cos \frac{\pi}{3} - 3 \sin x \cdot \sin \frac{\pi}{3} + 3 \\
 &= \cos x \left(5 + \frac{3}{2}\right) - \frac{3\sqrt{3}}{2} \sin x + 3
 \end{aligned}$$

~~Min value~~

$$\begin{aligned}
 &= c - \sqrt{a^2 + b^2} \\
 &= 3 - \sqrt{\left(\frac{13}{2}\right)^2 + \left(-\frac{3\sqrt{3}}{2}\right)^2} \\
 &= 3 - \sqrt{\frac{169}{4} + \frac{27}{4}} \\
 &= 3 - \sqrt{\frac{196}{4}} \\
 &= 3 - \frac{14}{2} \\
 &= -4
 \end{aligned}$$

~~Max value~~ =  $3 + 7 = 10$

Q. Find the height of the cone of maxm volume that can be inscribed in a sphere of radius 1 unit.

$$r^2 = 1 - h^2$$

$$\therefore \text{volume of cone} = \frac{1}{3} \pi r^2 h$$

$$= \frac{1}{3} \pi (1-h^2)(1+h)$$

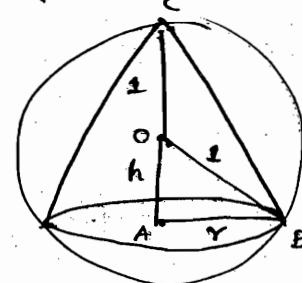
$$V = \frac{\pi}{3} (1+h-h^2-h^3)$$

$$\frac{dV}{dh} = \frac{\pi}{3} (1-2h-3h^2) = 0$$

$$\Rightarrow 1-2h-3h^2 = 0$$

$$\Rightarrow 3h^2+2h-1 = 0$$

$$h = \frac{-2 \pm \sqrt{4+12}}{6} = \frac{-2 \pm 4}{6} = \frac{-84}{58}, \frac{2}{58} \cdot \frac{1}{3}$$



$$\therefore \text{height} = 1 + h = 1 + \frac{1}{3} = \frac{4}{3}$$

Q. find the height of the cylinder of maximum volume that can be inscribed in a sphere of radius  $a$  units.

$$\rightarrow r^2 = a^2 - \frac{h^2}{4}$$

$$\text{volume of cylinder} = \pi r^2 h$$

$$V = \pi r^2 h$$

$$= \pi \left( a^2 - \frac{h^2}{4} \right) h$$

$$V = \pi \left( a^2 h - \frac{h^3}{4} \right)$$

$$\frac{dV}{dh} = \pi \left( a^2 - \frac{3h^2}{4} \right) = 0$$

$$\Rightarrow a^2 - \frac{3h^2}{4} = 0$$

$$\Rightarrow 3h^2 = 4a^2$$

$$\Rightarrow h^2 = \frac{4a^2}{3}$$

$$h = \frac{2a}{\sqrt{3}}$$

height for which max<sup>th</sup> volume of cylinder occurs in sphere

Imp Result:

If  $Z = f(x, y)$

$$P = \frac{\partial z}{\partial x}$$

$$Q = \frac{\partial z}{\partial y}$$

$$R = \frac{\partial^2 z}{\partial x^2}$$

$$S = \frac{\partial^2 z}{\partial x \partial y}$$

$$T = \frac{\partial^2 z}{\partial y^2}$$

→ for a maxima and minima, Let  $P = 0$  (since 1<sup>st</sup> order derivative are only  $P$  &  $Q$ )

→ Since,  $P$  and  $Q$  are fn. for  $x$  and  $y$

so, we have two eqn in  $x$  &  $y$ .

→ solve these eqns and get a relation bet<sup>n</sup>  $x$  and  $y$ .

Ex:-  $x = y, x = -y, y = 2x, x = 2y$  etc type

→ Put this relation either in  $P = 0$  or in  $Q = 0$ ; then the eqn transformed to either purely in  $x$  or in  $y$

Solve the Eqn and get the roots

$x_1, x_2, \dots$  and  $y_1, y_2, \dots$  etc  
 $x_1, y_1, \dots$  are critical points.

→ Take a point  $x_0, y_0$ , find the values of  $r, s$  and  $t$

- \* **Case-1** → If  $rt - s^2 > 0$  and  $r > 0$ , then the fn attains its minimum at that point.  
→ Min<sup>m</sup> value =  $f(x_0, y_0)$ .

- \* **Case-2** → If  $rt - s^2 > 0$  and  $r < 0$ , then the fn attains its maximum at that point.  
→ Max<sup>m</sup> value =  $f(x_1, y_1)$ .

- \* **Case-3** → If  $rt - s^2 < 0$  then the fn has no extremum.

- Case-4** → If  $rt - s^2 = 0$  then the case is doubtful and needs further investigation.

Q.  $f(x, y) = x^2 + y^2 + 6x + 2$  has

a) min<sup>m</sup> at  $(-3, 0)$

b) max<sup>m</sup> at  $(-3, 0)$

c) no extremum

d) none.

$$\begin{aligned} \rightarrow P &= 2x + 6 \\ Q &= \frac{\partial f}{\partial y} = 2y \\ R &= \frac{\partial^2 f}{\partial x^2} = 2 \\ S &= \frac{\partial^2 f}{\partial x \partial y} = 0 \\ t &= \frac{\partial^2 f}{\partial y^2} = 2. \end{aligned} \quad \left. \begin{array}{l} rt - s^2 = 2(-4 - 0^2) > 0. \text{ and } r > 0. \\ \text{minimum} \end{array} \right\}$$

a.  $f = 1 - x^2 - y^2$  has.

- (A) min at  $(0,0)$
- (B) max at  $(0,0)$
- (C) no extremum
- (D) none

$$\rightarrow \left. \begin{array}{l} P = -2x \\ Q = -2y \\ R = -2 \\ S = 0 \\ T = -2 \end{array} \right\} \quad rt - s^2 = 4 > 0 \quad \text{and} \quad -2 < 0.$$

maximum.

a.  $f(x,y) = x^3 + y^3 - 3xy$

- (A) max at  $(1,-1)$
- (B) min at  $(1,1)$
- (C) max at  $(-1,-1)$
- (D) min at  $(1,1)$

$$\rightarrow \left. \begin{array}{l} P = 3x^2 - 3y \\ Q = 3y^2 - 3x \\ R = 6x \\ S = -3 \\ T = 6y \end{array} \right\}$$

at  $1, -1$

$$6 \times (-6) - 9 < 0 \quad \text{and} \quad r = 6 > 0.$$

max.  
no extremum

or for multiple points check  
 $P$  and  $Q$  must be 0 at that point  
then consider max & min

$$6 \times 6 - 9 > 0 \quad \text{and} \quad r = 6 > 0.$$

min

Q.  $f(x, y) = x^2 + y^2 + xy + x - 4y + 5$  has \_\_\_\_\_

- (A) min at  $(2, -3)$
- (B) min at  $(-2, 3)$
- (C) max at  $(2, -3)$
- (D) max at  $(-2, 3)$

$$\rightarrow \left. \begin{array}{l} P = 2x + y + 1 \\ Q = 2y + x - 4 \\ R = 2 \\ S = 1 \\ T = 2 \end{array} \right\} \quad \begin{aligned} RT - S^2 &= 4 - 1 > 0 \quad \& \quad R = 2 > 0. \\ &\text{minimum} \end{aligned}$$

Q.  $f(x, y) = 4x^2 + 5y^2 - 8x - 4y + 8$  has \_\_\_\_\_

- (A) min  $\frac{8}{3}$
- (B) max  $\frac{8}{3}$
- (C) min  $\frac{10}{3}$
- (D) max  $\frac{10}{3}$ .

$$\rightarrow \left. \begin{array}{l} P = 8x - 8 = 0 \Rightarrow x = 1 \\ Q = 12y - 4 = 0 \Rightarrow y = \frac{1}{3} \\ R = 8 \\ S = 0 \\ T = 12 \end{array} \right\} \quad \begin{aligned} RT - S^2 &= 96 - 0 > 0 \quad \& \quad R = 8 > 0 \\ &\text{minimum} \end{aligned}$$

$$\therefore f\left(1, \frac{1}{3}\right) = 4 + 6\left(\frac{1}{3^2}\right) - 8 - \frac{4}{3} + 8$$

$$= 4 + \frac{2}{3} - 8 - \frac{4}{3} + 8$$

$$= 4 - \frac{2}{3}$$

$$= \frac{10}{3}$$

Q.  $f(x, y) = \sin x + \sin y + \sin(x+y)$  has \_\_\_\_\_

- Ⓐ  $\min \frac{\sqrt{3}}{2}$  Ⓑ  $\max \frac{\sqrt{3}}{2}$  Ⓒ  $\min \frac{3\sqrt{3}}{2}$  Ⓓ  $\max \frac{3\sqrt{3}}{2}$

$$\rightarrow P = \cos x + \cos(x+y)$$

$$q = \cos y + \cos(x+y)$$

$$r = \frac{\partial^2 f}{\partial x^2} = -\sin x - \sin(x+y) \Big|_{\frac{\pi}{3}, \frac{\pi}{3}} = -\sqrt{3}$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = -\sin(x+y) \Big|_{\frac{\pi}{3}, \frac{\pi}{3}} = -\frac{\sqrt{3}}{2}$$

$$t = \frac{\partial^2 f}{\partial y^2} = -\sin y - \sin(x+y) \Big|_{\frac{\pi}{3}, \frac{\pi}{3}} = -\sqrt{3}$$

$\rightarrow$  Make  $P=0$  or  $q=0$ .

$$1. \cos x + \cos(x+y) = 0$$

$$\cos y + \cos(x+y) = 0$$

$$\cos x - \cos y = 0.$$

$$\Rightarrow x = y$$

$$\therefore \cos x + \cos 2x = 0$$

$$\Rightarrow \cos 2x = -\cos x$$

$$\Rightarrow \cos 2x = \cos(\pi - x)$$

$$2x = \pi - x$$

$$\Rightarrow x = \frac{\pi}{3} = y$$

$$\therefore rt - s^2 = 3 - \frac{3}{4} > 0 \quad \text{and } r = -\sqrt{3} < 0$$

$$\underbrace{\max}_{\text{maxm}} \cdot \frac{3\sqrt{3}}{2}$$

## # Definite and Indefinite Integrals:-

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$$

$$\int_a^b f(x) dx = \int_a^b f(a+b-x) dx$$

$$\int_a^b \frac{f(x)}{f(x) + f(a+b-x)} dx = \frac{b-a}{2}$$

$$\int_0^a f(x) dx = \int_0^{a-x} f(a-x) dx$$

$$\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx, \text{ if } f(x) \text{ is even.}$$

$$= 0, \text{ if } f(x) \text{ is odd.}$$

$$\int_0^{2a} f(x) dx = 2 \int_0^a f(x) dx, \text{ if } f(2a-x) = f(x)$$

$$= 0, \text{ if } f(2a-x) = -f(x)$$

$$\int_0^{\pi/2} \frac{f(\sin x)}{f(\sin x) + f(\cos x)} dx = \int_0^{\pi/2} \frac{f(\tan x)}{f(\tan x) + f(\cot x)} dx = \frac{\pi}{4}.$$

$$\int_0^{\pi/2} \sin^n x dx = \int_0^{\pi/2} \cos^n x dx = \frac{(n-1)(n-3)(n-5)\dots}{n(n-2)(n-4)\dots} \times K$$

$$\left\{ \begin{array}{l} K=1, \text{ if } n \text{ is odd} \\ K=\frac{\pi}{2}, \text{ if } n \text{ is even} \end{array} \right\}$$

$$\int_0^{\pi/2} \sin^m x \cdot \cos^n x \, dx = \frac{(m-1)(m-3)\dots(n-1)(n-3)\dots}{(m+n)(m+n-2)(m+n-4)} \times k$$

$\left. \begin{array}{l} \left\{ \begin{array}{l} k=1, \text{ if either } m \text{ or } n \text{ or} \\ \text{both are odd} \end{array} \right. \\ \left. \begin{array}{l} k=\frac{\pi}{2}, \text{ if both } m \text{ and } n \text{ are} \\ \text{even} \end{array} \right. \end{array} \right\}$

$$\int \cos x \, dx = \sin x$$

$$\int \sec^2 x \, dx = \tan x$$

$$\int \sec x \tan x \, dx = \sec x$$

$$\int \tan x \, dx = \log \sec x$$

$$\int \cot x \, dx = \log \sin x$$

$$\int \sin h x \, dx = \cosh x$$

$$\int \cosh x \, dx = \sin h x$$

$$\int \frac{1}{\sqrt{a^2-x^2}} \, dx = \sin^{-1}\left(\frac{x}{a}\right)$$

$$\int \frac{1}{\sqrt{a^2+x^2}} \, dx = \sinh^{-1}\left(\frac{x}{a}\right)$$

$$\int \frac{1}{\sqrt{x^2-a^2}} \, dx = \cosh^{-1}\left(\frac{x}{a}\right)$$

$$\int \frac{1}{a^2+x^2} \, dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right)$$

$$\int \frac{1}{a^2-x^2} \, dx = \frac{1}{2a} \log\left(\frac{a+x}{a-x}\right)$$

$$\int \frac{1}{x^2-a^2} \, dx = \frac{1}{2a} \log\left(\frac{x-a}{x+a}\right)$$

$$\int \frac{f'(x)}{f(x)} dx = \log f(x)$$

$$\textcircled{2} \quad \int \frac{f'(x)}{\sqrt{f(x)}} dx = 2\sqrt{f(x)}$$

$$\int \frac{f'(x)}{1+(f(x))^2} dx = \tan^{-1}[f(x)]$$

$$\int [f(x)]^n f'(x) dx = \frac{[f(x)]^{n+1}}{n+1}$$

$$\int e^{f(x)} \cdot f'(x) dx = e^{f(x)}$$

$$\int e^x [f(x) + f'(x)] dx = e^x \cdot f(x)$$

$$\int f(x) g(x) dx = f(x) \int g(x) dx - \int f'(x) \int g(x) dx dx$$

To choose  $f(x)$ , remember I L A T E

Shortcut  $\rightarrow \int \text{algebraic} \times e^x dx = e^x (\text{algebraic with derivate of } f^n \text{ alternate sign})$

$$\int x e^x dx = e^x (x-1)$$

$$\int x^2 e^x dx = e^x (x^2 - 2x + 2)$$

$$\int x^3 e^x dx = e^x (x^3 - 3x^2 + 6x - 6)$$

$$\int (x^2 + x) e^x dx = e^x [(x^2+x) - (2x+1) + 2]$$

Shortcut when  $e^{ax}$  is present

$$\int x e^{ax} dx = e^{ax} \left( \frac{x}{a} - \frac{1}{a^2} \right)$$

$$\int x^2 e^{ax} dx = e^{ax} \left[ \frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3} \right]$$

$$\int \log x \, dx = x \log x - x$$

$$\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2} [a \sin bx - b \cos bx]$$

$$\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2} [a \cos bx + b \sin bx]$$

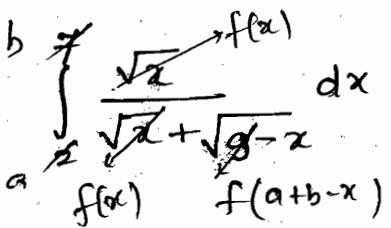
$$\text{Q. } \textcircled{1} \quad \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1-x^2}} \, dx = (\sin^{-1} x) \Big|_0^{\frac{\pi}{2}} = \sin^{-1} \frac{\pi}{2} - \sin^{-1} 0 = \frac{\pi}{6}$$

$$\text{Q. } \textcircled{2} \quad \int_{\log 2}^{\log 3} \frac{e^x}{1+e^x} \, dx = \left. \log(1+e^x) \right|_{\log 2}^{\log 3} = \log(1+e^{\log 3}) - \log(1+e^{\log 2}) \\ = \log 4 - \log 3 \\ = \log \frac{4}{3} \quad \underline{\text{Ans}}$$

$$\text{Q. } \textcircled{3} \quad \int_0^1 \frac{1}{e^x + e^{-x}} \, dx \\ = \int_0^1 \frac{e^x}{e^{2x} + 1} \, dx \\ = \int_0^1 \frac{e^x}{1+(e^x)^2} \, dx = \left. \tan^{-1}(e^x) \right|_0^1 = \tan^{-1} e - \tan^{-1} 1 \\ = \tan^{-1} e - \frac{\pi}{4} \quad \underline{\text{Ans.}}$$

$$\text{Q. } \textcircled{4} \quad \int_0^1 x e^x \, dx = \left. e^x(x-1) \right|_0^1 = e^1(1-1) - e^0(0-1) \\ = 1. \quad \underline{\text{Ans}}$$

$$\text{Q. } \int_a^b \frac{\sqrt{x}}{\sqrt{x} + \sqrt{b-x}} \, dx = \frac{b-a}{2} = \frac{7-2}{2} = \frac{5}{2}$$



$$\text{Q. } \int_0^{20} \frac{f(x)}{f(x)+f(20-x)} \, dx = \frac{20-0}{2} = 10$$

Q.  $\int_0^2 |1-x| dx$  observe fn within limits when modulus present

$$= \int_0^2 |1-x| dx = \cancel{x} \cancel{|x^2 - 1|} \cancel{\frac{1}{2}} \cancel{x^2} \cancel{|x|}$$

$$= \int_0^1 |1-x| dx + \int_1^2 |x-1| dx$$

$$= \left[ x - \frac{x^2}{2} \right]_0^1 + \left[ \frac{x^2}{2} - x \right]_1^2$$

$$\Rightarrow 1 - \frac{1}{2} + (2 - \cancel{\frac{1}{2}}) - (\frac{1}{2} - 1)$$

$$= \frac{1}{2} + \frac{1}{2}$$

= 1. Ans. When Greatest Integer fn present

Q.  $\int_{-1.5}^1 [x+1] dx$  observe limit bet<sup>n</sup> limits if integer lies  $\rightarrow$  split if no integer lie bet<sup>n</sup> limits  $\rightarrow$  no need to split

$$\int_{-1.5}^{-1} [x+1] dx + \int_{-1}^0 [x+1] dx + \int_0^1 [x+1] dx$$

$$[x+1] = \begin{cases} -1, & -1.5 < x < -1 \\ 0, & -1 < x < 0 \\ 1, & 0 < x < 1 \end{cases}$$

$$\int_{-1.5}^{-1} -1 dx + \int_{-1}^0 0 dx + \int_0^1 1 dx$$

$$= -x \Big|_{-1.5}^{-1} + x \Big|_0^1$$

$$= -(-1 + 1.5) + (1 - 0) = -0.5 + 1 = 0.5 \quad \underline{\text{Ans}}$$

$$Q. \int^2 x[x] dx$$

$$-\frac{1}{7}\pi$$

$$\mathcal{X}[x] \leftarrow$$

$$\int_1^2 x \, dx = \frac{x^2}{2} \Big|_1^2 = \left( \frac{2^2 - 1^2}{2} \right) = \frac{3}{2} \quad \underline{\text{Ans}}$$

a.  $\int_{-1}^1 \frac{1}{1+x^2} dx$   \* \* \* When limits  $-a$  to  $a$  check even or odd

$$= 2 \int_0^1 \frac{1}{1+x^2} dx$$

$$= 2 \tan^{-1} x \Big|_0^1$$

$$= 2 \left( \tan^{-1} 1 - \tan^{-1} 0 \right)$$

$$= 2 \left( \frac{\pi}{4} - 0 \right)$$

$$= \frac{\pi}{2} \text{ Ans}$$

no even no odd

$$a. \quad \int_{-a}^a \sqrt{a+x} \sqrt{a-x}$$

Always try to eliminate sq. root from numerator not from denominator

WHEN  $\sqrt{f(x)}$  IS PRESENT.

$$= \int_{-a}^a \frac{a+x}{\sqrt{a^2-x^2}}$$

$$\int_0^a \frac{a}{\sqrt{a^2 - x^2}} dx = \left[ a \sin^{-1} \left( \frac{x}{a} \right) \right]_0^a = a \left( \sin^{-1} 1 - \sin^{-1} 0 \right) = a \left( \frac{\pi}{2} - 0 \right)$$

<https://notesguru.info>

Q.  $\int_0^\pi \frac{x \sin x}{1 + \cos^2 x} dx$

Q.  $\int_{-a}^a \sqrt{\frac{a-x}{a+x}} = a\pi$  (also)

- ②  $a\pi$  ④  $\frac{a}{\pi}$  ⑤  $\frac{\pi}{a}$  ⑥  $\frac{1}{\pi a}$

When sign changes if  $x$   
even though answer is same  
so forget the sign  
but remember the answer  
always.  
=

\* \* \*  
Q.  $I = \int_0^\pi \frac{x \sin x}{1 + \cos^2 x} dx = \frac{\pi^2}{4}$

Top result.

For Removal of <sup>algebraic</sup> function always use —

$$\int_0^a f(x) dx = \int_0^a f(a-x) dx$$

$$I = \int_0^\pi \frac{(\pi-x) \sin(\pi-x)}{1 + \cos^2(\pi-x)}$$

$$I = \int_0^\pi \frac{(\pi-x) \sin x}{1 + \cos^2 x}$$

$$2I = \int_0^\pi \frac{(\pi+x-\pi-x) \sin x}{1 + \cos^2 x} dx$$

$$2I = -\pi \int_0^\pi -\frac{\sin x}{1 + (\cos x)^2} dx$$

$$\int \frac{f'(x)}{1 + [f(x)]^2} dx = \tan^{-1}[f(x)]$$

$$I = -\frac{\pi}{2} \cdot [\tan^{-1}(\cos x)]_0^\pi = -\frac{\pi}{2} [\tan^{-1}(-1) - \tan^{-1}(1)] \\ = -\frac{\pi}{2} \left( -\frac{\pi}{4} - (-\frac{\pi}{2}) \right) = \frac{\pi^2}{4} \text{ Ans}$$

$$I = \int_0^{\pi/2} \log(\tan x) dx = 0$$

~~Imp Results~~

$$I = \int_0^{\pi/2} \log(\cot x) dx = 0$$

Q.  $I = \int_0^{\pi/2} \log(\tan x) dx$

$$= \int_0^{\pi/2} \log(\tan(\frac{\pi}{2}-x)) dx = \int_0^{\pi/2} \log \cot x dx.$$

$$Q.I = \int_0^{\pi/2} [\log(\tan x) + \log(\cot x)] dx$$

$$= \int_0^{\pi/2} \log(\tan x / \cot x) dx$$

$$= 0 \text{ Ans}$$

Whenever limit  $-a$  to  $a$  given

Check even or odd

Q.  $\int_{-\pi}^{\pi} x^4 \sin^5 x dx$

↓ even × odd  
↓  
= odd

$$= 0.$$

Q.  $\int_0^{\pi/2} \sin^5 x dx$

for Numerator } If formula forgot  
if power even → write odd no  
if power odd → write even no

Q. ~~Ans.~~

$$= \frac{4 \times 2}{5 \times 3 \times 1} = \frac{8}{15}$$

for Denominator } if power even → write even  
if power odd → write odd

} If power = odd → keep result as it is

} If power = even → multiply result by  $\frac{1}{2}$ .

$$Q. \int_0^{\pi/2} \cos^7 x dx = \frac{6 \times 4 \times 2}{7 \times 5 \times 3} = \frac{16}{35}$$

$$Q. \int_0^{\pi/2} \sin^8 x dx = \frac{7 \times 5 \times 3 \times 1}{8 \times 6 \times 4 \times 2} \times \frac{\pi}{2} = \frac{35}{256} \pi$$

$$Q. \int_0^1 \frac{x^6 dx}{\sqrt{1-x^2}}$$

$$\text{Let } x = \sin \theta$$

$$dx = \cos \theta d\theta$$

$$\text{if } x \rightarrow 0 \quad \theta \rightarrow 0$$

$$\text{if } x \rightarrow 1 \quad \theta \rightarrow \frac{\pi}{2}$$

$$1. \int_0^{\pi/2} \frac{\sin^6 \theta}{\cos^4 \theta} \cos \theta d\theta$$

$$= \frac{5 \times 3 \times 1}{8 \times 4 \times 2} \times \frac{\pi}{2} = \frac{5\pi}{32}$$

$$Q. \int_0^{\pi/2} \sin^4 x \cos^5 x dx \rightarrow \text{both are even then only multiply by } \frac{1}{2}$$

$$= \frac{3 \times 1 \times 4 \times 2}{9 \times 7 \times 5 \times 3 \times 1} \rightarrow \text{do individually as before in numerator}$$

$$= \frac{8}{315} \rightarrow \text{In Denominator - start from sum of powers and continue.}$$

Q.  $\int_0^{\pi/2} \sin^6 x \cos^4 x dx$

$$= \frac{5 \times 3 \times 1 \times 3 \times 1}{10 \times 8 \times 6 \times 4 \times 2} \times \frac{\pi}{2} = \frac{3\pi}{512}$$

Q.  $\int_0^{\pi/2} \sin^3 x \cos^5 x dx$

$$= \frac{2 \times 4 \times 2}{8 \times 6 \times 4 \times 2} = \frac{1}{24}$$

Q.  $\int_0^x \sin^3 x dx$

$$\therefore \int_0^{2a} f(x) dx = 2 \int_0^a f(x) dx \text{ if } f(2a-x) = f(x).$$

$$\therefore f(\pi-x) = \sin^3(\pi-x) = \sin^3 \pi = f(x)$$

→ Converted in  $\frac{\pi}{2}$ .

$\therefore \int_0^{\pi/2} \sin^3 x dx$

$$= 2 \times \frac{2}{3 \times 1} = \frac{4}{3}$$

To use formula

## # Definite and Improper Integrals :-

An integral  $\int_a^b f(x) dx$  is said to be an improper integral

if ①  $f$  becomes infinite in the interval of integration.

② one or both of the limits are infinite.

$$\text{Ex:- } \int_0^1 \frac{1}{x} dx, \int_0^1 \sqrt{\frac{1+x}{1-x}} dx, \int_0^3 \frac{1}{(x-1)^{2/3}} dx.$$

$$\int_1^\infty \frac{1}{x^3} dx, \int_{-\infty}^0 \sin bx dx, \int_{-\infty}^\infty \frac{1}{1+x^2} dx.$$

$$0. \int_0^1 \frac{1}{x} dx = \log|x| \Big|_0^1 = \log 1 - \log 0 = \log \frac{1}{0}, \log \infty = \infty$$

Divergent

$$a. \int_0^1 \sqrt{\frac{1+x}{1-x}} dx. \quad \text{A/c to prev. discussion}$$

when limit was -1 to 1 ans was  $\pi$

Here limit is 0 to 1 ans will be  $\frac{\pi}{2}$

$$= \int_0^1 \frac{1+\cancel{x}}{\sqrt{1-x^2}} + \frac{x}{\sqrt{1-x^2}} dx.$$

But some extra term would be present also

$$= \int_0^1 \frac{1}{\sqrt{1-x^2}} dx + \int_{\frac{1}{2}}^1 \frac{-2x}{\sqrt{1-x^2}} dx$$

$$= \left( \sin^{-1} x - \frac{1}{2} \times 2 \sqrt{1-x^2} \right) \Big|_0^1$$

$$= (\sin^{-1} 1 - \cancel{\sqrt{1-1}}) - (\sin^{-1} 0 - \sqrt{1-0}).$$

$$= \frac{\pi}{2} + 1$$

$$Q. \int_0^3 \frac{1}{(x-1)^{2/3}} dx$$

$$= \int_0^3 (x-1)^{-2/3} dx$$

$$= \left[ \frac{(x-1)^{-2/3+1}}{-\frac{2}{3}+1} \right]_0^3$$

$$= \left. \frac{(x-1)^{1/3}}{\frac{1}{3}} \right|_0^3$$

$$= 3 \left[ x^{1/3} - \left. \frac{1}{3} \right|_0^3 \right]$$

$$= 3 \left[ \sqrt[3]{2} + 1 \right] \text{ Ans}$$

$$Q. \int_1^\infty \frac{1}{x^3} dx$$

$$= \left( \frac{-1}{2x^2} \right)_1^\infty$$

$$= -\frac{1}{2} \left[ \frac{1}{\infty} - \frac{1}{1} \right]$$

$$= -\frac{1}{2} [0 - 1]$$

$$= \frac{1}{2} \text{ Ans}$$

$$Q. \int_{-\infty}^0 \sin hx dx$$

$$= \left. \cos hx dx \right|_{-\infty}^0$$

$$= \left. \frac{e^x + e^{-x}}{2} \right|_{-\infty}^0$$

$$= \cancel{\frac{e^0 + e^0}{2}} - e^{-\infty} + e^{\infty} = -\infty \text{ Ans}$$

$$\begin{aligned}
 Q. \int_{-\infty}^{\infty} \frac{1}{1+x^2} dx &= 2 \int_0^{\infty} \frac{1}{1+x^2} dx \\
 &= 2 \left[ \tan^{-1} x \right]_0^{\infty} \\
 &= 2 \left( \tan^{-1}(\infty) - \tan^{-1}(0) \right) \\
 &= 2 \times \frac{\pi}{2} \\
 &= \pi \quad \text{Ans}
 \end{aligned}$$

$$\begin{aligned}
 Q. \int_0^{\infty} x e^{-x^2} dx &= \frac{1}{2} \int_0^{\infty} e^{-x^2} (-2x) dx \quad \left\{ \int e^{f(x)} f'(x) = e^{f(x)} \right\} \\
 &= -\frac{1}{2} \left( e^{-x^2} \right)_0^{\infty} \\
 &= -\frac{1}{2} \left[ e^0 - e^{-0} \right] \\
 &= -\frac{1}{2} [0 - 1] = \frac{1}{2}
 \end{aligned}$$

~~\*\*\*~~ ~~Q.~~

$$\begin{aligned}
 \int_0^{\infty} e^{-x^2} dx &\quad \left\{ \begin{array}{l} \text{Let } t = x^2 \\ dt = 2x dx \\ dx = \frac{dt}{2\sqrt{t}} \\ x \rightarrow 0 \rightarrow t \rightarrow 0 \\ x \rightarrow \infty \rightarrow t \rightarrow \infty \end{array} \right\} \quad \begin{array}{l} \text{donot take } t = -x^2 \\ \Rightarrow x^2 = -t \\ \text{sq never becomes} \\ \rightarrow \text{ve} \\ \text{Wrong Idea} \end{array} \\
 &\text{Definite Improper integral} \\
 &\text{When , becomes complex to evaluate} \\
 &\text{Take Help of } \Gamma \text{ function.} \\
 &= \frac{1}{2} \int_0^{\infty} \frac{e^{-t}}{\sqrt{t}} dt \\
 &= \frac{1}{2} \int_0^{\infty} e^{-t} \cdot t^{-\frac{1}{2}} dt \\
 &= \frac{1}{2} \sqrt{\frac{1}{2}} = \frac{\sqrt{\pi}}{2} \quad \left| \begin{array}{l} \sqrt{1} = 1 \\ \sqrt{\frac{1}{2}} = \sqrt{\pi} \end{array} \right. \quad \left| \begin{array}{l} \sqrt{n} = \int_0^{\infty} e^{-xt} x^{n-1} dx \\ \sqrt{n+1} = n\sqrt{n}, \text{ if } n \geq 0 \\ \Gamma(n) = \int_0^{\infty} e^{-xt} t^{n-1} dt \end{array} \right.
 \end{aligned}$$

Do clearly Prev. problem here.

$$Q. \int_0^{\infty} \int_0^{\infty} e^{-(x^2+y^2)} dx dy.$$

$$= \int_0^{\infty} \int_0^{\infty} e^{-x^2} \cdot e^{-y^2} dx dy.$$

$$= \int_0^{\infty} e^{-x^2} dx \int_0^{\infty} e^{-y^2} dy$$

$$= \frac{\sqrt{\pi}}{2} \times \frac{\sqrt{\pi}}{2} = \frac{\pi}{4}$$

$$Q. \int_0^{\infty} \int_0^{\infty} \int_0^{\infty} e^{-(x^2+y^2+z^2)} dx dy dz$$

$$= \frac{\sqrt{\pi}}{2} \times \frac{\sqrt{\pi}}{2} \times \frac{\sqrt{\pi}}{2} = \frac{\sqrt{\pi}}{8}$$

Q.  $\int_0^\infty x^3 e^{-x} dx = 3!$  Power will become factorial

Shortcut  $= \int_0^\infty x^3 e^{-x} \left[ \frac{x^3}{1} - \frac{3x^2}{1^2} + \frac{6x}{1^3} - \frac{6}{1^4} \right]_0^\infty = 0 \times \infty$  (indeterminate form)

Q.  $\int_0^\infty \frac{x}{(x^2+9)^2} dx$  Let  $t = x^2 + 9$   
 $\Rightarrow dt = 2x dx \Rightarrow dx = \frac{dt}{2x}$

$$= \int_9^\infty \frac{x}{t^2} \times \frac{dt}{2x} \quad x \rightarrow 0, t \rightarrow 9 \\ x \rightarrow \infty, t \rightarrow \infty$$

$$= \frac{1}{2} \left[ -\frac{1}{t} \right]_9^\infty$$

$$= -\frac{1}{2} \left[ \frac{1}{\infty} - \frac{1}{9} \right]$$

$$= -\frac{1}{2} \left[ 0 - \frac{1}{9} \right]$$

$$= \frac{1}{18} \text{ Ans}$$

Q.  $\int_{-\infty}^\infty \frac{x}{(x^2+9)^2} dx \Rightarrow$  result = 0 fn is odd

## Vector Calculus

### # Gradient of a scalar fn:-

Let  $\phi(x, y, z) = c$  be any scalar fn, then the grad of  $\phi$  is denoted by "grad  $\phi$ " or " $\nabla \phi$ " and is defined as -

$$\nabla \phi = i \frac{\partial \phi}{\partial x} + j \frac{\partial \phi}{\partial y} + k \frac{\partial \phi}{\partial z}$$

we know that -

$$\vec{r} = xi + yj + zk$$

$$|\vec{r}| = \sqrt{x^2 + y^2 + z^2}$$

$$r^2 = x^2 + y^2 + z^2$$

$$\Rightarrow \cancel{fr} \frac{\partial r}{\partial x} = \cancel{fx}, \cancel{fr} \frac{\partial r}{\partial y} = \cancel{fy}, \cancel{fr} \frac{\partial r}{\partial z} = \cancel{fz}$$

$$\Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

$$\frac{\partial r}{\partial y} = \frac{y}{r}$$

$$\frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\therefore \nabla r = i \frac{\partial r}{\partial x} + j \frac{\partial r}{\partial y} + k \frac{\partial r}{\partial z}$$

$$= i \left( \frac{x}{r} \right) + j \left( \frac{y}{r} \right) + k \left( \frac{z}{r} \right)$$

$$\nabla r = \frac{xi + yj + zk}{r}$$

$\therefore \text{grad } r = \nabla r = \frac{\vec{r}}{r}$

$$\nabla(\log r) = \frac{1}{r} \nabla r = \frac{1}{r} \cdot \frac{\vec{r}}{r} = \frac{\vec{r}}{r^2}$$

$$\nabla\left(\frac{1}{r}\right) = -\frac{1}{r^2} \nabla r = -\frac{1}{r^2} \cdot \frac{\vec{r}}{r} = -\frac{\vec{r}}{r^3}$$

$$\nabla(r^n) = n r^{n-1} \nabla r = n r^{n-1} \frac{\vec{r}}{r} = n r^{n-2} \vec{r}$$

## # Tangent vector to a curve:-

Let  $\vec{r}(t)$  be the given vector curve, then  $\frac{d\vec{r}}{dt}$  is called the tangent vector to the given curve

$$\vec{r} = xi + yj + zk \\ (\text{fn of } x)$$

$$\boxed{\frac{d\vec{r}}{dt} = \frac{dx}{dt} i + \frac{dy}{dt} j + \frac{dz}{dt} k}$$

## # Normal to a surface:-

Let  $\phi(x, y, z) = c$  be any surface then

$\nabla\phi$  is called the normal to the surface  $\phi$ .

and  $\frac{\nabla\phi}{|\nabla\phi|}$  is called the unit normal vector to the

surface ' $\phi$ ' and is denoted by ' $N$ '.

$$\text{i.e. } \phi(x, y, z) = c$$

$$\text{then } N = \frac{\nabla\phi}{|\nabla\phi|}$$

## # Directional Derivative :-

Let  $\phi(x, y, z) = c$  be any surface, then

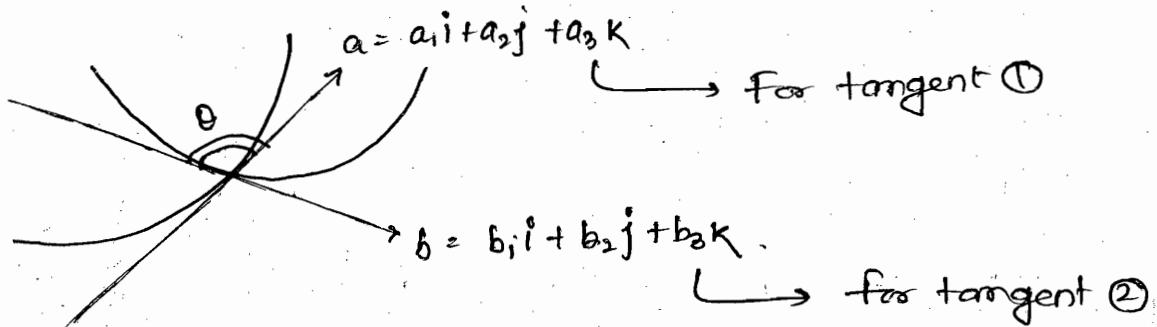
$|\nabla\phi \cdot e|$  is called the directional derivative to the surface  $\phi$ . Where  $e$  is the unit ~~normal~~ vector in the dir<sup>n</sup> of given vector.

$$\text{i.e. } e = \frac{\vec{a}}{|\vec{a}|}$$

The max<sup>m</sup> value of the directional derivative or the greatest value of directional derivative or the magnitude of a gradient of a scalar fn  $\phi$  is defined by  $|\nabla\phi|$

## # The angle between the curves :-

The angle bet<sup>n</sup> the two curves is the angle bet<sup>n</sup> tangents at their point of intersection.



Note :-

If the two curves cuts each orthogonally then

$$a \cdot b = 0 \quad \left\{ \begin{array}{l} \because \theta \text{ will become } 90^\circ \\ \& \cos \theta = \cos 90^\circ = 0 \end{array} \right.$$

## # The angle bet<sup>n</sup> two surfaces:-

The angle bet<sup>n</sup> two surfaces is the angle bet<sup>n</sup> normals drawn at their surfaces.

If  $f$  and  $g$  are any two surfaces and  $\theta$  is the angle bet<sup>n</sup> them then

$$\cos \theta = \frac{\nabla f \cdot \nabla g}{|\nabla f| |\nabla g|}$$

Note!:-

If the two surfaces cuts each orthogonally then  
 $\nabla f \cdot \nabla g = 0$ .

- a. find the unit normal vector to surface at the point  $(1, 2, -1)$  where  $\phi = x^3 + y^3 + 3xyz = 3$   
 (Surface)

$$\rightarrow \phi = x^3 + y^3 + 3xyz = 3.$$

$$\nabla \phi = (3x^2 + 3yz) \mathbf{i} + (3y^2 + 3xz) \mathbf{j} + (3xy) \mathbf{k}$$

$$\nabla \phi \Big|_{(1,2,-1)} = -3\mathbf{i} + 9\mathbf{j} + 6\mathbf{k}.$$

∴  $N \doteq \frac{\nabla \phi}{|\nabla \phi|}$  is unit normal to surface

$$\therefore N = \frac{(-3\mathbf{i} + 9\mathbf{j} + 6\mathbf{k})}{\sqrt{1+81+36}} = \frac{-3\mathbf{i} + 9\mathbf{j} + 6\mathbf{k}}{\sqrt{14}}$$

Q. Unit normal to  $x^2y + 2xz = 4$  at  $(2, -2, 3)$

Ⓐ  $\frac{i + 2j - 2k}{3}$

Ⓑ  $\frac{i - 2j + 2k}{3}$

Ⓒ  $\frac{-i + 2j + 2k}{3}$

Ⓓ  $\frac{i + 2j + 2k}{3}$

$$\rightarrow \nabla \phi \Big|_{(2, -2, 3)} = (2xy + 2z)i + (x^2)j + (2x)k \\ = -2i + 4j + 4k$$

$$\therefore N = \frac{\nabla \phi}{|\nabla \phi|} = \frac{\phi'(-i + j + 2k)}{2\sqrt{1+4+4}} = \frac{-i + j + 2k}{3}$$

Q.  $0 \cdot N$  to  $x^2 + y^2 + z^2 = 1$  at  $(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$

2011

Ⓐ  $\frac{i + j}{\sqrt{2}}$

Ⓑ  $\frac{j + k}{\sqrt{2}}$

Ⓒ  $\frac{i + k}{\sqrt{2}}$

Ⓓ  $\frac{i + j + k}{\sqrt{3}}$

Type-1

### DIRECTIONAL DERIVATIVE (4 types total)

Q. find the directional derivative of  $f = xy + yz + zx$  at  $(1, 2, 0)$  in the dir<sup>n</sup> of  $\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ .

$$\rightarrow \nabla f = i \frac{\partial f}{\partial x} + j \frac{\partial f}{\partial y} + k \frac{\partial f}{\partial z}$$

$$\nabla f = i(y+z) + j(x+z) + k(y+x)$$

$$\left. \nabla f \right|_{(1,2,0)} = 2i + j + 3k$$

$$a = i + 2j + 2k$$

$$e = \frac{a}{|a|} = \frac{i + 2j + 2k}{\sqrt{1+4+4}}$$

$$e = \frac{i + 2j + 2k}{3}$$

$$D \cdot D = \nabla f \cdot e$$

$$= 2\left(\frac{1}{3}\right) + 1\left(\frac{2}{3}\right) + 3\left(\frac{2}{3}\right)$$

$$= \frac{10}{3}$$

Q. find D·D of  $f = 2xy + z^2$  at  $(1, -1, 3)$  in the dir<sup>n</sup> of  $\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ .

$$\rightarrow \nabla f \Big|_{(1,-1,3)} = (2y)i + (2x)j + (2z)k$$

$$= -2i + 2j + 6k$$

$$a = i + 2j + 3k$$

$$e = \frac{i + 2j + 3k}{\sqrt{1+4+9}}$$

$$e = \frac{i + 2j + 3k}{\sqrt{14}}$$

$$D \cdot D = \nabla f \cdot e$$

$$= -2\left(\frac{1}{\sqrt{14}}\right) + 2\left(\frac{2}{\sqrt{14}}\right) + 6\left(\frac{3}{\sqrt{14}}\right)$$

$$= \frac{20}{\sqrt{14}}$$

### Type-II:

- Q. find the directional derivative of  $f = x^2 - y^2 + 2z^2$  at  $P(1, 2, 3)$  in the dir<sup>n</sup> of  $\overrightarrow{PQ}$  where  $Q(5, 0, 4)$

$$\rightarrow a = \overrightarrow{PQ} = Q-P = 4i - 2j + k$$

$$e = \frac{a}{|a|} = \frac{4i - 2j + k}{\sqrt{16+4+1}} = \frac{4i - 2j + k}{\sqrt{21}}$$

dot product

$$\nabla f \Big|_{1,2,3} = (2x)i - (2y)j + (4z)k \\ = 2i - 4j + 12k$$

$$\therefore D \cdot D = \frac{8+8+12}{\sqrt{21}} = \frac{28}{\sqrt{21}} \text{ Ans.}$$

- Q. find the D.D of  $f = xy + yz + zx$  at  $P(1, 2, -1)$  in the dir<sup>n</sup> of  $\overrightarrow{PQ}$  where  $Q = 1, 2, 3$ .

$$\rightarrow \nabla f \Big|_{1,2,-1} = (y+z)i + (x+z)j + (x+y)k \\ = i + 3k$$

$$a = \overrightarrow{PQ} = Q-P = 4k$$

$$\therefore e = \frac{a}{|a|} = \frac{4k}{4} = k$$

$$e = k$$

$$\boxed{\therefore D \cdot D = 3.}$$

### Type-III

- a. find the directional derivative of  $f = xy^2 + yz^2 + zx^2$  at  $(1,1,1)$  along the tangent to the curve  
 $x=t, y=t^2, z=t^3$

$$\nabla f \Big|_{(1,1,1)} = i(y^2 + 2xz) + j(2xy + z^2) + k(2yz + x^2)$$

$$\nabla f \Big|_{(1,1,1)} = (3i + 3j + 3k)$$

$$\therefore D \cdot D = \nabla f \cdot e$$

$$= \frac{3+6+9}{\sqrt{14}}$$

$$= \frac{18}{\sqrt{14}}$$

$$a = \frac{dr}{dt} = \frac{dx}{dt} i + \frac{dy}{dt} j + \frac{dz}{dt} k$$

$$x = t \quad | \quad y = t^2 \quad | \quad z = t^3$$

$$\frac{dx}{dt} = 1 \quad | \quad \frac{dy}{dt} = 2t \quad | \quad \frac{dz}{dt} = 3t^2$$

→ To take  $t$  value  
 take easy relation.

→ If you choose complicated relation to find ' $t$ ' then after verifying we know that for other ~~points~~ our point varies (Ultimately time loss)

$$\therefore a = i + 2tj + 3t^2k$$

$$= i + 2j + 3k \quad \{ \text{as } t=1 \}$$

$$\therefore e = \frac{a}{|a|} = \frac{i + 2j + 3k}{\sqrt{14}}$$

Type-IV

Q. find the directional derivative of surface  $f = xy^2 + z^2$  at  $(1,1,1)$  in the dir<sup>n</sup> of normal to surface  $3x^2y^2 + y = z$  at  $(0,1,1)$

$$\nabla f = i(yz^2 + z) + j(xz^2) + k(2xyz + x)$$

$$\nabla f \Big|_{(1,1,1)} = 2i + j + 3k$$

$$a = \nabla g = i(3y^2) + j(6xy + 1) + k(-1)$$

$\nabla g \Big|_{(0,1,1)}$

$$= 3i + j - k$$

$$e = \frac{a}{|a|} = \frac{3i + j - k}{\sqrt{11}}$$

$$\therefore D \circ D = \nabla f \cdot e$$

$$= \frac{6+1-3}{\sqrt{11}}$$

$$= \frac{4}{\sqrt{11}}$$

Q. find the maximum value of the directional derivative to the surface  $\phi = x^2yz^3$  at  $(2,1,-1)$

$$\rightarrow \nabla \phi = i(2xyz^3) + j(x^2z^3) + k(3x^2yz^2)$$

$$\nabla \phi \Big|_{(2,1,-1)} = -4i - 4j + 12k$$

$$|\nabla \phi| = \sqrt{16 + 16 + 144}$$

$$= \sqrt{176}$$

$$= \sqrt{16 \times 11}$$

$$= 4\sqrt{11}$$

✓

a. find the greatest value of dir<sup>n</sup> derivative of  
 $\phi = x^2yz$  at  $(1, 4, 1)$

$$\rightarrow \nabla \phi \Big|_{(1,4,1)} = (2xyz)i + (x^2z)j + (x^2y)k \\ = 8i + j + 4k$$

$$|\nabla \phi| = \sqrt{64+16+1} \\ = \sqrt{81} \\ = 9$$

b. find the magnitude of gradient  $u = \frac{x^2}{2} + \frac{y^2}{3}$   
at  $(1, 3)$

④  $\frac{1}{\sqrt{10}}$  ⑤  $\frac{1}{\sqrt{5}}$  ⑥  $\sqrt{5}$  ⑦  $\sqrt{10}$

$$\rightarrow \nabla u \Big|_{(1,3)} = \frac{2x}{2}i + \frac{2y}{3}j \\ = i + 2j$$

$$|\nabla u| = \sqrt{1+4} = \sqrt{5}$$

c. find the angle bet<sup>n</sup> surfaces  $x^2+y^2+z^2=9$  and  
 $x^2+y^2-z=3$  at the point  $(2, -1, 2)$

$$\rightarrow \nabla f = \frac{2x}{2}i + \frac{2y}{2}j + \frac{2z}{2}k = 2i - 2j + 4k$$

$$\nabla g = \frac{2x}{2}i + \frac{2y}{2}j - k = 2i - 2j - k$$

$$\cos \theta = \frac{\nabla f \cdot \nabla g}{|\nabla f| |\nabla g|} = \frac{16+4-4}{\sqrt{36} \sqrt{21}} = \frac{16}{36\sqrt{21}}$$

$$\therefore \theta = \cos$$

Q. find acute angle betw the surfaces  $xy^2z = 3x + z^2$   
and  $3x^2 - y^2 + 2z = 1$  at the point  $(1, -2, 1)$ .

$$\rightarrow \nabla f \Big|_{(1, -2, 1)} = i(y^2z - 3) + j(2xyz) + k(xy^2 - 2z)$$

$$= i(+7) + j(-4) + k(2)$$

$$\nabla g = 6xi - 2yj + 2k$$

$$(\nabla g)_{(1, -2, 1)} = 6i + 4j + 2k$$

$$\therefore \cos \theta = \frac{|6 - 16 + 4|}{\sqrt{1+16+4} \sqrt{36+16+4}}$$

$$= \frac{6}{\sqrt{7 \times 3} \sqrt{7 \times 4 \times 2}}$$

$$= \frac{6}{2 \times 7 \times \sqrt{6}}$$

$$\theta = \cos^{-1} \left( \frac{3}{7\sqrt{6}} \right)$$

Q. find the const. a, b so that the surfaces  $ax^2 - byz = (a+2)x$   
and  $4x^2y + z^3 = 4$  may intersect orthogonally at the  
point  $(1, 1, 2)$ .

$$\rightarrow \nabla f \Big|_{(1, 1, 2)} = (2az - a - 2)i - bzj - byk$$

$$= (a-2)i - 2bj + bk$$

$$\nabla g \Big|_{(1, 1, 2)} = (8xy)i + (4x^2)j + (3z^2)k$$

$$= -8i + 4j + 12k$$

A/c to Q-

$$\nabla f \cdot \nabla g = 0 \Rightarrow -8(a-2) - 8b + 12b = 0$$

A/c to Q - 1, -1, 2 lies also on surface ④ -

$$\therefore a(1)^2 - b(-1)(2) = (a+2) \quad \text{--- } ①$$

$$\Rightarrow a + 2b = a + 2$$

$$\Rightarrow 2b = 2$$

$$\Rightarrow b = 1 \quad \underline{\text{Ans}}$$

∴ from ① -

$$a = \frac{5}{2} \quad \underline{\text{Ans}}$$

## # Divergence of a vector function :-

Let  $\mathbf{F} = F_1 \mathbf{i} + F_2 \mathbf{j} + F_3 \mathbf{k}$  be any differentiable vector fn then the divergence of  $\mathbf{f}$  is denoted by  $\text{div } \mathbf{F}$  or  $\nabla \cdot \mathbf{F}$  and is defined as -

$$\text{div } \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

Ex:-  $\overline{\mathbf{r}} = xi + yj + zk$

$$\text{div } \overline{\mathbf{r}} = 1 + 1 + 1$$

$$\therefore \text{div } \overline{\mathbf{r}} = \nabla \cdot \overline{\mathbf{r}} = 3$$

## # Solenoidal Vector:-

The necessary and sufficient cond<sup>n</sup> for a vector fn  $\mathbf{f}$  to be solenoidal is -

$$\text{div } \mathbf{F} = 0$$

$$\boxed{\nabla \cdot \mathbf{F} = 0}$$

Note:-

Let  $\phi$  is a scalar and  $a$  is vector then -

$$\operatorname{div}(\phi A) = \nabla \cdot (\phi A) = (\operatorname{grad} \phi) \cdot A + \phi (\operatorname{div} A)$$

$$\boxed{\nabla \cdot (\phi A) = (\nabla \phi) \cdot A + \phi (\nabla \cdot A)}$$

Q.  $\nabla^2 \left( \frac{1}{r} \right) = 0$  where  $r = |\vec{r}|$  and  $\vec{r} = xi + yj + zk$

From  
Result

$$\rightarrow \nabla \cdot \nabla \left( \frac{1}{r} \right) = \nabla \cdot \left( -\frac{1}{r^2} \nabla r \right) = \nabla \cdot \left( -\frac{1}{r^2} \frac{\vec{r}}{r} \right) = \nabla \cdot \left( -\frac{1}{r^3} \vec{r} \right) \quad \text{A}$$

$$= \nabla \left( -\frac{1}{r^3} \right) \cdot \vec{r} + \left( -\frac{1}{r^3} \right) (\nabla \cdot \vec{r})$$

$$= -\frac{3}{r^4} \nabla r \cdot \vec{r} - \frac{3}{r^3}$$

$$= \frac{3}{r^4} \frac{\vec{r}}{r} \cdot \vec{r} - \frac{3}{r^3}$$

$$= \frac{3}{r^4} \times r^2 - \frac{3}{r^3}$$

$$= \frac{3}{r^3} - \frac{3}{r^3}$$

$$= 0$$

\*\*\*\*\*

$$\text{SHORTCUT} - \nabla^2 [f(r)] = f''(r) + \frac{2}{r} f'(r)$$

①  $\nabla^2 (\log r) = -\frac{1}{r^2} + \frac{2}{r} \cdot \frac{1}{r} = \frac{1}{r^2}$

②  $\nabla^2 (r^n) = n(n-1)r^{n-2} + \frac{2}{r} n \cdot r^{n-1}$   
 $= r^{n-2} [n(n-1) + 2n]$

## # Curl of a vector function :-

Let  $\mathbf{F} = F_1 \mathbf{i} + F_2 \mathbf{j} + F_3 \mathbf{k}$  be any differentiable vector function then the curl of  $\mathbf{F}$  is denoted by  $\text{curl } \mathbf{F}$  or  $\nabla \times \mathbf{F}$  and is defined as -

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

Ex:-  $\overline{\mathbf{r}} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

$$\begin{aligned} \text{Curl } \overline{\mathbf{r}} &= \nabla \times \overline{\mathbf{r}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} \\ &= \mathbf{i}(0-0) - \mathbf{j}(0-0) + \mathbf{k}(0-0) \\ &= \mathbf{0} \end{aligned}$$

$\therefore \boxed{\text{curl } \overline{\mathbf{r}} = \mathbf{0}}$

## # Irrotational Vector :-

The necessary and sufficient cond" for a vector fn  $\mathbf{F}$  to be irrotational is -

$\boxed{\nabla \times \mathbf{F} = \mathbf{0}}$

Note:-

Let  $\mathbf{F}$  be any non zero vector function then - ~~div~~

$$\text{div}(\text{curl } \mathbf{F}) = \mathbf{0}$$

$$\nabla \cdot (\nabla \times \mathbf{F}) = \mathbf{0}$$

Q. If  $\mathbf{F} = (x+2y)\mathbf{i} + (y+2z)\mathbf{j} + (x+\lambda z)\mathbf{k}$  is solenoidal then value of  $\lambda$  = ?

$$\rightarrow \nabla \cdot \mathbf{F} = 0 \quad (\text{for solenoidal})$$

$$\Rightarrow \nabla \cdot [(x+2y)\mathbf{i} + (y+2z)\mathbf{j} + (x+\lambda z)\mathbf{k}] = 0$$

$$\Rightarrow 1 + 1 + \lambda = 0$$

$$\Rightarrow \lambda = -2$$

Q. Find the value of  $\lambda$  so that —

$$\mathbf{F} = (\lambda x^2 y + yz)\mathbf{i} + (4xy^2 + xz)\mathbf{j} + 2xyz\mathbf{k} \quad \text{is}$$

zero divergence.

$$\rightarrow \nabla \cdot \mathbf{F} = 0$$

$$\Rightarrow 2\lambda xy + 8xy + 2xy = 0$$

$$\Rightarrow xy(2\lambda + 8 + 2) = 0$$

$$\Rightarrow \lambda + 4 + 1 = 0$$

$$\Rightarrow \lambda = -5 \quad \underline{\text{Ans}}$$

Q. If  $f = x^3 + y^3 + z^3 - 3xyz$  then  $\operatorname{div}(\operatorname{grad} f) =$  —

$$\rightarrow \nabla \cdot (\nabla f) = \nabla^2 f$$

$$= 6x + 6y + 6z$$

$$= 6(x+y+z)$$

Q. If  $\mathbf{F} = 2xy\mathbf{i} - x^2z\mathbf{j}$  then  $(\operatorname{curl} \mathbf{F})_{(1,1,1)} =$  —

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy & -x^2z & 0 \end{vmatrix} = \mathbf{i}(0+x^2) - \mathbf{j}(0-0) + \mathbf{k}(-2xz-2x^2) \\ = \mathbf{i} - 4\mathbf{k} \quad @ \text{point } (1,1,1)$$

Q. If  $F = 3x^2\mathbf{i} + 5xy^2\mathbf{j} + xyz^3\mathbf{k}$   
then  $(\operatorname{div} F) = \underline{\quad}$

$$\rightarrow \nabla \cdot F = 6x + 10xy + 3xyz^2 \\ = 6 + 20 + 54 \\ = 80 \text{ Ans}$$

Q. If  $\nabla \cdot (\gamma^n \bar{r}) = 0$  then  $n = \underline{\quad}$

$$\rightarrow \because \nabla \cdot (\phi A) = (\nabla \phi) \cdot A + \phi (\nabla \cdot A)$$

$$\Rightarrow (\nabla \gamma^n) \cdot \bar{r} + \gamma^n (\cancel{\nabla \cdot \bar{r}}) = 0$$

$$\Rightarrow n \gamma^{n-1} \nabla \gamma \cdot \bar{r} + 3\gamma^n = 0$$

$$\Rightarrow n \gamma^{n-2} \frac{\bar{r} \cdot \bar{r}}{r} + 3\gamma^n = 0$$

$$\Rightarrow n \gamma^{n-2} r^2 + 3\gamma^n = 0$$

$$\Rightarrow n \gamma^n + 3\gamma^n = 0$$

$$\Rightarrow n + 3 = 0$$

$$\Rightarrow n = -3 \text{ not}$$

Q. Find the constants  $a, b, c$  so that the vector field  $F = (x+2y+az)\mathbf{i} + (bx-3y-z)\mathbf{j} + (4x+cy+2z)\mathbf{k}$  is irrotational.

$$\rightarrow \nabla \times F = 0 \Rightarrow \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix}$$

$$\Rightarrow i(c+1) - j(4-a) + k(b-2) = 0$$

$$\Rightarrow c+1=0 \Rightarrow c=-1$$

$$\text{and } 4-a=0 \Rightarrow a=4$$

$$\text{and } b-2=0 \Rightarrow b=2$$

Ans

## # Vector Integration :-

- ① Line integral  $\int$
- ② Surface integral  $\iint$
- ③ Volume integral  $\iiint$

### (1). Line Integral :-

Any integral which is evaluated along the curve is called the line integral.

Let  $F$  be any differentiable vector function defined along 'curve 'C' , then the line integral of  $F$  is defined by —  $\int_C F \cdot d\vec{r}$

If  $C$  is a closed curve , it is denoted by —  $\oint_C F \cdot d\vec{r}$

#### Cartesian form :-

$$\text{Let } F = F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k}$$

$$\vec{r} = x \hat{i} + y \hat{j} + z \hat{k}$$

$$d\vec{r} = dx \hat{i} + dy \hat{j} + dz \hat{k}$$

$$\int_C F \cdot d\vec{r} = \int_C (F_1 dx + F_2 dy + F_3 dz)$$

is called  
Cartesian form  
of line integral

Note:- If  $F(x, y, z)$  are fn. of  $t$  , then

$$\int_C F \cdot d\vec{r} = \int_C \left[ F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt} + F_3 \frac{dz}{dt} \right] dt$$

a. Evaluate  $\int_C \mathbf{F} \cdot d\mathbf{r}$  where  $\mathbf{F} = (2y+3)\mathbf{i} + xz\mathbf{j} + (yz-x)\mathbf{k}$

① Where  $C$  is the curve  $x = 2t^2$ ,  $y = t$ ,  $z = t^3$   
joining  $(0,0,0)$  to  $(2,1,1)$

② Where  $C$  is the curve of line joining  $(0,0,0) \rightarrow (0,0,1)$   
 $\rightarrow (0,1,1) \rightarrow (2,1,1)$

$$\rightarrow (1). \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \left[ F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt} + F_3 \frac{dz}{dt} \right] dt$$

$$= \begin{array}{|c|c|c|} \hline x & y & z \\ \hline 2t^2 & t & t^3 \\ \frac{dx}{dt} = 4t & \frac{dy}{dt} = 1 & \frac{dz}{dt} = 3t^2 \\ \hline \end{array}$$

$$\begin{aligned} &\because y = t \\ &\therefore y \rightarrow 0, t \rightarrow 0 \\ &y \rightarrow 1, t \rightarrow 1 \end{aligned}$$

$$= \int_{t=0}^1 \left[ (2y+3) \frac{dx}{dt} + yz \frac{dy}{dt} + (yz-x) \frac{dz}{dt} \right] dt$$

$$= \int_{t=0}^1 \left[ 8t^2 + 12t + 2t^5 + 3t^6 - 6t^4 \right] dt$$

$$= \left[ \frac{8t^3}{3} + \frac{12t^2}{2} + \frac{2t^6}{6} + \frac{3t^7}{7} - \frac{6t^5}{5} \right]_0^1$$

$$= \frac{8}{3} + 6 + \frac{1}{3} + \frac{3}{7} - \frac{6}{5}$$

$$= 9 + \frac{3}{7} - \frac{6}{5}$$

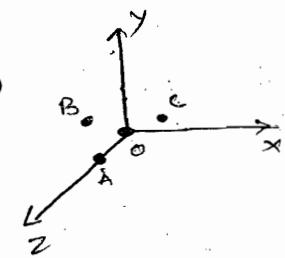
$$= \frac{288}{35}$$

$$\textcircled{2} \quad \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C F_1 dx + F_2 dy + F_3 dz.$$

① along  $\overline{OA}$   
 choose  $dx, dy, dz$   
 such that derivative  
 only that derivative  
 point varies  
 other remains const.  
 $\rightarrow$  remains const. & others  
 two varies  
 as  $x=0$  to  $x=1$

$$x=0, y=0 \\ dx=0, dy=0 \\ \text{as } z=0 \text{ to } z=1$$

$$= \int_{OA} (yz - x) dz = 0 \quad \left\{ \begin{array}{l} x=0 \\ y=0 \end{array} \right\}$$



② along  $\overline{AB}$   
 $x=0, z=1$   
 $dx=0, dz=0$   
 $\text{as } y=0 \text{ to } 1$

$$= \int_{AB} xz dy = 0 \quad \left\{ \begin{array}{l} x=0 \\ y=0 \end{array} \right\}$$

③ along  $\overline{AC}$   $\overline{BC}$  -  $\overline{AB}$   
 $y=1, z=1$   
 $dy=0, dz=0$   
 $\text{as } x=0 \text{ to } 2$

$$= \int_{x=0}^2 (2y+3) dz = \int_{x=0}^2 [2(1)+3] dx = 5[x]_0^2 = 10$$

$$\therefore \int_C \mathbf{F} \cdot d\mathbf{r} = 0 + 0 + 10 = \underline{\underline{10}}$$

Q.  $\int \mathbf{F} \cdot d\mathbf{r}$  where  $\mathbf{F} = 3x^2 \mathbf{i} + (2xz - y) \mathbf{j} + z \mathbf{k}$  where  $C$  is the st. line joining  $(0, 0, 0)$  to  $(2, 1, 3)$

$$\rightarrow \text{eqn of } OA = \frac{x-0}{2-0} = \frac{y-0}{1-0} = \frac{z-0}{3-0} = t$$

$$\Rightarrow \frac{x}{2} = \frac{y}{1} = \frac{z}{3} = t$$

$$\Rightarrow x = 2t, y = t, z = 3t$$

The eqn of a st. line joining  $(x_1, y_1, z_1)$  and  $(x_2, y_2, z_2)$  in the symmetric form is  $\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1} = t$

Where  $t$  is some scalar.

$$\therefore \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \left[ F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt} + F_3 \frac{dz}{dt} \right] dt$$

$$= \int_{t=0}^1 [3(4t^2)(2) + (2(2t)(3t) - t)(1) + 3t(3)] dt$$

$$= \int_{t=0}^1 (24t^2 + 12t^2 - t + 9t) dt$$

$$= \left[ \frac{36t^3}{3} + \frac{8t^2}{2} \right]_0^1 = 12 + 4 = 16$$

Q.  $\int_C \mathbf{F} \cdot d\mathbf{r} = ?$  where  $\mathbf{F} = 2xyz \mathbf{i} + x^2y \mathbf{j} + x^2z \mathbf{k}$  where  $C$  is the st. line joining  $(0, 0, 0)$  to  $(1, 1, 1)$

$\rightarrow$  eqn of st line —

$$\frac{x}{1} = \frac{y}{1} = \frac{z}{1} = t$$

$$x = t, y = t, z = t$$

$$\frac{dx}{dt} = \frac{dy}{dt} = \frac{dz}{dt} = 1$$

$$\begin{aligned}\therefore \int \mathbf{F} \cdot d\mathbf{r} &= \int_C F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt} + F_3 \frac{dz}{dt} \\ &= \int_{t=0}^1 [2t^3(1) + t^3(1) + t^3(1)] dt \\ &= \int_{t=0}^1 4t^3 dt = 4 \frac{t^4}{4} \Big|_0^1 = 1 - 0 = 1\end{aligned}$$

Q.  $\int_C \mathbf{F} \cdot d\mathbf{r}$ , where  $\mathbf{F} = 3xy \mathbf{i} - y^2 \mathbf{j}$  and  $C$  is the curve  $y = 2x^2$  in the  $xy$  plane joining  $(0,0)$  to  $(1,2)$

$$\begin{aligned}\rightarrow \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_C [3xy dx - y^2 dy] \\ &\quad \text{y = } 2x^2 \\ &\quad dy = 4x dx \\ &\quad x \rightarrow 0 \text{ to } 1 \\ &= \int_{x=0}^1 3x(2x^2) dx - (2x^2)^2 4x dx \\ &= \left| \frac{6x^4}{4} - \frac{16x^6}{6} \right|_0^1 = \frac{3}{2} - \frac{8}{3} = -\frac{7}{6}.\end{aligned}$$

Q.  $\int_C \mathbf{F} \cdot d\mathbf{r} = ?$  Where  $\mathbf{F} = (5xy - 6x^2) \mathbf{i} + (2y - 4x) \mathbf{j}$  and  $C$  is the curve  $y = x^3$  in the  $xy$  plane joining  $(1,1)$  to  $(2,8)$

$$\begin{aligned}\rightarrow \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_C (5xy - 6x^2) dx + (2y - 4x) dy \\ &\quad y = x^3 \\ &\quad dy = 3x^2 dx \\ &\quad x \rightarrow 1 \text{ to } 2 \\ &= \int_1^2 (5x^4 - 6x^2) dx + (2x^3 - 4x) 3x^2 dx\end{aligned}$$

$$= \int_{x=0}^2 [5x^4 - 6x^2 + 6x^5 - 12x^3] dx$$

$$= \left[ \frac{5x^5}{5} - \frac{6x^3}{3} + \frac{6x^6}{6} - \frac{12x^4}{4} \right]_0^2$$

$$= \cancel{[10x^5 - 18x^3 + 6x^6 - 12x^4]}_0^2$$

$$= -28$$

$$= (32 - 16 + 64 - 48) - (1 - 2 + 1 - 3)$$

$$= 32 + 3 = 35 \quad \text{Ans}$$

Q.  $\oint_C F \cdot d\vec{r} = ?$  where  $F = (2x-y+z) \hat{i} + (x+y-z^2) \hat{j} + (3x+2y-4z) \hat{k}$

where  $C$  is the circle in  $xy$  plane having the centre at origin and radius 3 units.

$$\rightarrow \oint F \cdot d\vec{r} = \int (2x-y+\cancel{z}) dx + (x+y-\cancel{z^2}) dy$$

{ Circle is in  $xy$  plane  
 $\therefore$  do not consider  $dz$  }

Our Relation is —

$$x^2 + y^2 = 9$$

$$x = r \cos \theta = 3 \cos \theta$$

$$y = r \sin \theta = 3 \sin \theta$$

$$\Rightarrow dx = -3 \sin \theta d\theta$$

$$dy = 3 \cos \theta d\theta$$

as  $\theta \Rightarrow 0$  to  $2\pi$  to complete a circle

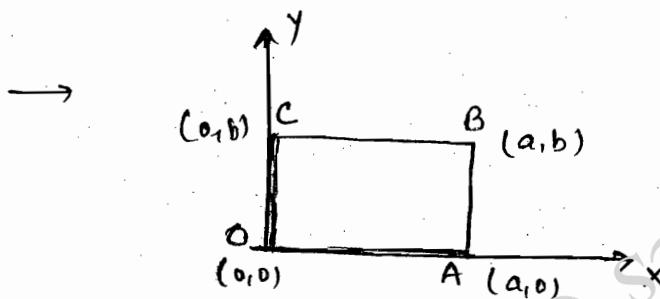
$$= \int_{\theta=0}^{2\pi} 2(3 \cos \theta - 3 \sin \theta) (-3 \sin \theta) + (3 \cos \theta + 3 \sin \theta) \times (3 \cos \theta) d\theta$$

$$= \int_{\theta=0}^{2\pi} (-9(\sin \theta) \cos \theta + 9) d\theta$$

$$= \left[ -q \frac{\sin^2 \theta}{2} + q\theta \right]_0^{2\pi}$$

$$= 18\pi$$

- Q.  $\oint_C \mathbf{F} \cdot d\mathbf{r}$  where  $\mathbf{F} = (x^2+y^2) \mathbf{i} - 2xy \mathbf{j}$  where C is the rectangle bounded by  $x=0, x=a$   
 $y=0, y=b$ .



$$\int \mathbf{F} \cdot d\mathbf{r} = \oint (x^2+y^2) dx - 2xy dy$$

① along OA —  $y=0 \rightarrow dy=0$   
 $x \rightarrow 0 \text{ to } a$

$$\int \mathbf{F} \cdot d\mathbf{r} = \int_{x=0}^a (x^2+0) dx = \left( \frac{x^3}{3} \right)_0^a = \frac{a^3}{3}$$

② along AB —  $x=a \rightarrow dx=0$   
 $y \rightarrow 0 \text{ to } b$

$$\int \mathbf{F} \cdot d\mathbf{r} = \int_{y=0}^b -2ay dy = -2a \left( \frac{y^2}{2} \right)_0^b = -ab^2$$

③ along BC —  $y=b \rightarrow dy=0$   
 $x \rightarrow a \text{ to } 0$

$$\int \mathbf{F} \cdot d\mathbf{r} = \int_{x=a}^0 (x^2+b^2) dx = 0 - \frac{a^3}{3} - ab^2$$

④ along CO —  $x=0 \rightarrow dx=0$

$$\int \mathbf{F} \cdot d\mathbf{r} = \int 0 - 0 = 0 \therefore \text{Ans}$$

∴  $\int \mathbf{F} \cdot d\mathbf{r} = 0 = -2ab^2$

## # Surface Integrals—

Any integral which is evaluated over surface is called a surface integral.

Let  $\mathbf{F}$  be any differentiable vector field defined over a surface  $S$  then the surface integral is defined by

$$\iint_S \mathbf{F} \cdot \mathbf{N} d\mathbf{s} \quad \text{where } \mathbf{N} \text{ is the outer unit}$$

normal vector to given surface.

$$\iint_S \mathbf{F} \cdot \mathbf{N} d\mathbf{s} = \iint_{R_1} \mathbf{F} \cdot \mathbf{N} \frac{dx dy}{|\mathbf{N} \cdot \mathbf{k}|} \quad \text{where } R_1 \text{ is projection}$$

in  $xy$  plane.

$$= \iint_{R_2} \mathbf{F} \cdot \mathbf{N} \frac{dy dz}{|\mathbf{N} \cdot \mathbf{i}|} \quad R_2 \text{ --- } yz \text{ plane}$$

$$= \iint_{R_3} \mathbf{F} \cdot \mathbf{N} \frac{dz dx}{|\mathbf{N} \cdot \mathbf{j}|} \quad R_3 \text{ --- } zx \text{ plane}$$

Q. Evaluate  $\iint_S \mathbf{F} \cdot \mathbf{N} d\mathbf{s}$  where  $\mathbf{F} = z\mathbf{i} + x\mathbf{j} - 3y^2z\mathbf{k}$  and  $S$  is.

the surface of cylinder  $x^2 + y^2 = 16$  included in the 1<sup>st</sup>.

octant bounded by  $z=0$  &  $z=5$

$\therefore z$  is given  $\therefore$  either  $xz$  plane or  
 $yz$  plane will be chosen for projection

Q. Evaluate  $\iint_S \mathbf{F} \cdot \mathbf{N} d\mathbf{s}$  where  $\mathbf{F} = 4xz\mathbf{i} - y^2\mathbf{j} + yz\mathbf{k}$  and  $S$  is the cube bounding  $0 \leq x, y, z \leq 1$ .

→ for POAR —

$$\mathbf{N} = \mathbf{i}, \quad x=1 \text{ (const)}$$

$$\mathbf{F} \cdot \mathbf{N} = 4x^2z^2 = 4z$$

$$\iint_S \mathbf{F} \cdot \mathbf{N} d\mathbf{s} = \iint_{y=0, z=0}^1 4z \, dy \, dz$$

$$= 4 \cdot \frac{z^2}{2} \Big|_0^1 = 4 \cdot \frac{1}{2} = 2$$

$$= 4 \times \frac{1}{2} \times 1$$

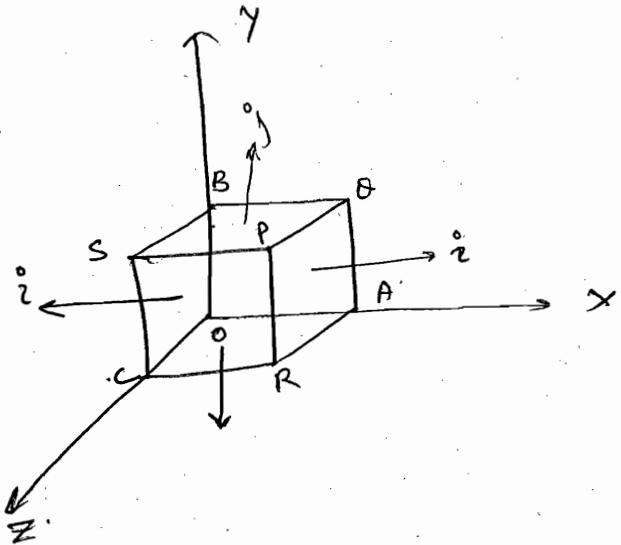
$$= 2$$

For SBOC —

$$\mathbf{N} = -\mathbf{i}, \quad x=0$$

$$\mathbf{F} \cdot \mathbf{N} = -4xz = 0$$

$$\therefore \iint_S \mathbf{F} \cdot \mathbf{N} d\mathbf{s} = 0$$



For PQBS —

$$N = \vec{y}, \quad y = 1.$$
$$\int F \cdot N ds = \iint_{S \setminus \{(x=0, z=0)\}} -y^2 dx dz = -\int_0^1 \int_0^1 z dx dz = -1$$

For OCRA —

$$N = \vec{y}, \quad y = 0.$$

$$\int F \cdot N ds = 0.$$

For SPRC —

$$N = \vec{z}, \quad z = 1.$$

$$\begin{aligned} \int_S F \cdot N ds &= \int_0^1 \int_0^1 y z^2 dx dy \\ &= \frac{y^2}{2} \Big|_0^1 \Big|_0^1 \\ &= \frac{1}{2} \end{aligned}$$

for BQOA —  $z = 0, \quad N = -\vec{z}$

$$\int F \cdot N ds = \text{edge} = 0$$

$$\begin{aligned} \iint_S F \cdot N ds &= 2 + 0 + (-1) + 0 + \frac{1}{2} \\ &= 1 + \frac{1}{2} = \frac{3}{2} \text{ Ans.} \end{aligned}$$

## # Volume Integral :-

Any integral which is evaluated over a volume is called the volume integral.

Q.  $\iiint_V \nabla \cdot F \, dv$  where  $F = (2x^2 - 3z)i - 2xyj + 2zk$  where  $V$  is

the region bounded by  $x=0, y \geq 0, z=0$  and  $2x+2y+z=4$ .

$$\nabla \cdot F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$\rightarrow \nabla \cdot F = 4x - 2z$$

$$\begin{aligned} z &\rightarrow \text{in terms of } xy \rightarrow 0 \text{ to } 4 - 2x - 2y \\ y &\rightarrow \text{from } 0 \text{ to } 2-x \\ x &\rightarrow \text{from } 0 \text{ to } 2. \end{aligned}$$

$$\begin{aligned} \iiint_V \nabla \cdot F \, dv &= \int_0^2 \int_0^{2-x} \int_{2x}^{4-2x-2y} 2x \, dz \, dy \, dx \\ &= 2 \int_0^2 \int_0^{2-x} x(4-2x-2y) \, dy \, dx \\ &= 2 \int_0^2 \left[ 4x^2 - \frac{2x^3}{3} - 2xy^2 \right]_0^{2-x} \, dx \\ &= 2 \int_0^2 (8 - 4x^2 - 4 + x^3 - 4x^2 - 4y^2 + 2y^3) \, dx \\ &= 2 \left( \frac{8}{3}x^3 - 8x^2 - 4x + \frac{x^4}{4} - 4y^2 + 2y^3 \right) \Big|_0^2 \\ &= 2 \int_0^2 x(2-x)^2 \, dx \\ &= 2 \left[ \frac{4x^2}{2} + \frac{x^4}{4} - \frac{4x^3}{3} \right]_0^2 \\ &= 2 \left[ \frac{16}{2} + \frac{16}{4} - \frac{32}{3} \right] = \frac{8}{3} \end{aligned}$$

## Vector Transformation

### # Gauss Divergence Theorem :-

→ The gauss divergence theorem use the connection betn surface to volume integrals.

→ Let  $S$  be the closed surface enclosed by a volume  $V$  and Let  $F$  be any differentiable vector fn then

$$\iint_S F \cdot N \, ds = \iiint_V (\operatorname{div} F) \, dv = \iiint_V (\nabla \cdot F) \, dv.$$

### \* Cartesian form of Divergence Theorem:-

$$\iint_S (F_1 dy dz + F_2 dz dx + F_3 dx dy) = \iiint_V \left( \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) dx dy dz$$

- Q. The value of  $\iint_S (ax^i + by^j + cz^k) \cdot N \, ds$ , where  $S$  is the surface of sphere  $x^2 + y^2 + z^2 = 1$ .

$$\rightarrow \nabla \cdot F = a + b + c$$

$$\begin{aligned} \iint_S F \cdot N \, ds &= \iiint_V (\operatorname{div} F) \, dv \\ &= (a+b+c) \frac{4}{3} \pi r^3 \\ &= \frac{4}{3} \pi (a+b+c) \underline{\text{Ans}} \end{aligned}$$

a. Find  $\int_S (ax\mathbf{i} + by\mathbf{j} + cz\mathbf{k}) \cdot \mathbf{N} ds$  where  $S$  is sphere

$$x^2 + y^2 + z^2 = 1$$

Same Problem

or  $\int_S (ax^2 + by^2 + cz^2) ds$  where  $S$  is sphere  $x^2 + y^2 + z^2 = 1$

Top Result

$$= (a+b+c) \frac{4\pi}{3}$$

b.  $\int_S \bar{r} \cdot \mathbf{N} ds$ , where ' $S$ ' is the closed region enclosed by a volume ' $V$ '.

$$\rightarrow \nabla \cdot \bar{r} = 3$$

$$\therefore \int_S \bar{r} \cdot \mathbf{N} ds = \int_V (\nabla \cdot \bar{r}) dv = 3V \quad \underline{\text{Ans}}$$

c.  $\int_S \bar{r} \cdot \mathbf{N} ds$  where  $S$  is sphere  $x^2 + y^2 + z^2 = a^2$

$$\rightarrow \nabla \cdot \bar{r} = 3$$

$$\therefore \int_S \bar{r} \cdot \mathbf{N} ds = \int_V (\nabla \cdot \bar{r}) dv = \frac{4}{3}\pi a^3 \times 3 = 4\pi a^3 \quad \underline{\text{Ans}}$$

d.  $\int_S (x dy dz + y dz dx + z dx dy)$  where  $S$  is the cube of unit length.

$$\nabla \cdot F = 1 + 1 + 1 = 3$$

$$\int_S F \cdot \mathbf{N} ds = \int_V (\operatorname{div} F) dv = 3V = 3 \cancel{a^3} = 3 \quad \underline{\text{Ans}}$$

Q.  $\int_S (x+z) dy dz + (y+z) dz dx + (x+y) dx dy$  where

$S$  is sphere  $x^2 + y^2 + z^2 = 4$ .

$$\Rightarrow \nabla \cdot F = 1 + 1 + 0 = 2$$

$$\therefore \int_S F \cdot N ds = \int_V (\nabla \cdot v) dv = 2V = 2 \times \frac{4}{3} \pi \times 2^3 = 3 \cdot \frac{64\pi}{3} \text{ Ans}$$

Q.  $\int_S (4xz\mathbf{i} - y^2\mathbf{j} + yz\mathbf{k}) \cdot N ds$  where  $S$  is cube bounded by  $0 \leq x, y, z \leq 1$ .

$$\rightarrow \nabla \cdot F = 4z - 2y + y = 4z - y$$

$$\begin{aligned} \int_S F \cdot N ds &= \int_V (\operatorname{div} F) dv = \int_{x=0}^1 \int_{y=0}^1 \int_{z=0}^1 [4z - y] dx dy dz \\ &= (x)_0^1 \left[ 4 \left( \frac{z^2}{2} \right)_0^1 (y)_0^1 - \left( \frac{y^2}{2} \right)_0^1 (z)_0^1 \right] \\ &= (1-0) \left[ 2 - \frac{1}{2} \right] \\ &= \frac{3}{2} \text{ Ans} \end{aligned}$$

Q.  $\int_S [(x^2 - yz)\mathbf{i} + (y^2 - zx)\mathbf{j} + (z^2 - xy)\mathbf{k}] \cdot N ds$  where

$S$  is the cuboid bounded by  $0 \leq x \leq a$

$0 \leq y \leq b$

$0 \leq z \leq c$

$$\rightarrow \nabla \cdot F = 2x - 2y + 2z$$

$$\therefore \int_S F \cdot N ds = \int_V (\nabla \cdot F) dv = \int_{x=0}^a \int_{y=0}^b \int_{z=0}^c (2x - 2y + 2z) dx dy dz$$

$$\begin{aligned}
 &= [x^2yz + y^2zx + z^2xy] \\
 &\quad \begin{matrix} x=0 \\ y=0 \\ z=0 \end{matrix} \\
 &= abc(a+b+c) \quad \underline{\text{Ans}}
 \end{aligned}$$

Q. A region of volume 10 cubic units is bounded by a closed surface 'S'. and  $\mathbf{N}$  is the outer unit normal vector to  $S$ , then -

$$\int_S (4x\mathbf{i} + 2y\mathbf{j} - z\mathbf{k}) \cdot \mathbf{N} \, ds = \underline{\hspace{2cm}}$$

$$\rightarrow \quad \nabla \cdot F = 4 + 2 - 1 = 5$$

$$\int v^5 dv = \frac{5v^6}{6} \Big|_1^{10} = 50.$$

~~(ONLY APPLICABLE WHEN R IS ALSO IN ONLY X-Y PLANE, NO Z PLANE)~~

# GREEN'S THEOREM in a plane :-  $(\int \rightarrow \iint)$

Let 'R' be the closed region in the xy plane bounded by a [closed and non intersecting curve] 'c'

and Let  $P$  and  $Q$  are continuous fn of  $x$  and  $y$   
 possessing the first order partial derivatives , then

$$\oint_C P dx + Q dy = \iint_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy.$$

Q. The value of  $\oint_C (2x-y)dx + (x+y)dy$  where  $C$  is the circle  $x^2+y^2=9$

$$\rightarrow \frac{\partial P}{\partial x} = -1, \quad \frac{\partial Q}{\partial x} = 1.$$

$\rightarrow$  Const., so  
no need to consider limit.

} only for  $\square$  and  $\square$   
we will take const. limit  
of  $x$  and  $y$ .

for other curves except rectangle & square } else where, we will take one limit const and other as a fn of former one.

$$\therefore \iint_R (1 - (-1)) dx dy$$

$$= \int_R 2 dR = 2R \Big|_3^9 = 18\pi \text{ Ans}$$

Q.  $\oint_C (x^2+y^2)dx - 2xydy$ , where  $C$  is the rectangle bounded by  $x=0, x=a$   
 $y=0, y=b$ .

$$\rightarrow \frac{\partial P}{\partial y} = 2y, \quad \frac{\partial Q}{\partial x} = -2y$$

$$\iint_R (-2y - 2y) dx dy$$

$$= \int_{x=0}^a \int_{y=0}^b -4y dx dy$$

$$= (x)_0^a \left( -4 \frac{y^2}{2} \right)_0^b = ax - 2b^2 = -2ab^2$$

a.  $\oint_C (x^2 - xy^3) dx + (y - 2xy) dy$  where 'C' is the square bounded by the points  $(0,0), (2,0), (2,2), (0,2)$ .

$$\rightarrow \frac{\partial P}{\partial y} = -3xy^2, \quad \frac{\partial Q}{\partial x} = -2y$$

$$\therefore \iint_R -2y - (-3xy^2) dx dy$$

$$= \iint_{x=0, y=0}^{2,2} -2y + 3xy^2 dx dy$$

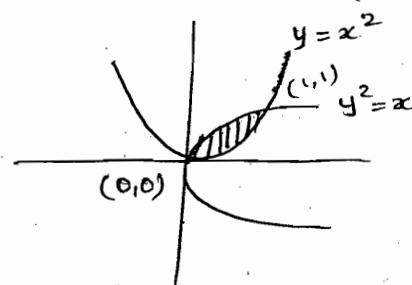
$$= -2\left(\frac{y^2}{2}\right)_0^2 (x)_0^2 + 3\left(\frac{x^2}{2}\right)_0^2 \left(\frac{y^3}{3}\right)_0^2$$

$$= -8 + 16 = 8 \quad \underline{\text{Ans}}$$

a.  $\oint_C (3x^2 - 8y^2) dx + (4y - 6xy) dy$  where 'C' is the region bounded by  $y = \sqrt{x}$  and  $y = x^2$

$$\rightarrow \frac{\partial P}{\partial y} = -16y, \quad \frac{\partial Q}{\partial x} = -6y$$

$$\iint_{x=0, y=x^2}^{\sqrt{x}, 16y} (-6y + 16y) dx dy$$



$$= \int_0^1 16 \left( \frac{y^2}{2} \right)_{x^2}^{\sqrt{x}} dx$$

$$= 5 \int_0^1 (x - x^4) dx$$

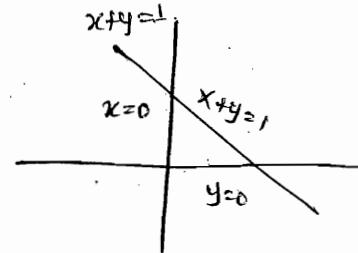
$$= 5 \left[ \frac{x^2}{2} - \frac{x^5}{5} \right]_0^1 = 5 \left( \frac{3}{10} \right) = \frac{3}{2} \quad \underline{\text{Ans}}$$

$$\begin{aligned} x^2 &= \sqrt{x} \\ x^4 &= x \\ x(x^3 - 1) &= 0 \\ x = 0, 1 & \\ y = 0, 1 & \end{aligned}$$

Q.  $\oint_C (3x^2 - 8y^2) dx + (4y - 6xy) dy$  where C is the region bounded by  $x=0, y=0, x+y=1$

$$\rightarrow \frac{\partial P}{\partial y} = -16y, \quad \frac{\partial Q}{\partial x} = -6y$$

$$\int_{x=0}^1 \int_{y=0}^{1-x} (-6y + 16y) dx dy$$



$$= \int_0^1 \int_{y=0}^{1-x} 10y dx dy$$

$$= \int_0^1 10 \left( \frac{y^2}{2} \right) \Big|_0^{1-x} dx$$

$$= 5 \int_0^1 (1-x)^2 dx$$

$$= 5 \left| \frac{(1-x)^3}{-3} \right|_0^1$$

$$= -\frac{5}{3}(0-1)$$

$$= \frac{5}{3} \quad \underline{\text{Ans}}$$

# STOKE'S THEOREM :-  $\int \rightarrow \iint$

Let  $S$  be an open surface bounded by a closed and non-intersecting curve ' $C$ ' and  $F$  be any differentiable vector function then —

$$\oint F \cdot d\vec{r} = \iint_S (\text{curl } F) \cdot N \, ds = \iint_S (\nabla \times F) \cdot N \, ds$$

Q. Evaluate by the Stoke's theorem

$\oint_C F \cdot d\vec{r}$  where  $F = -y^3 i + x^3 j$  and  $S$  is the circular disc  $x^2 + y^2 \leq 1$ ,  $z=0$ .

$$\rightarrow \nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^3 & x^3 & 0 \end{vmatrix}$$

$$\begin{aligned} &= i(0-0) - j(0-0) + k(3x^2 + 3y^2) \\ &= 3(x^2 + y^2)k \\ &= 3k \quad \left\{ \because x^2 + y^2 \leq 1 \right\} \end{aligned}$$

Directly we can say  $N = \vec{R}$   $\left. \begin{array}{l} z=0 \text{ given} \\ \text{means } xy \text{ plane} \\ \text{and normal vector} \\ \text{will be only } k \end{array} \right\}$

$$\therefore \int \partial k \cdot k \, ds$$

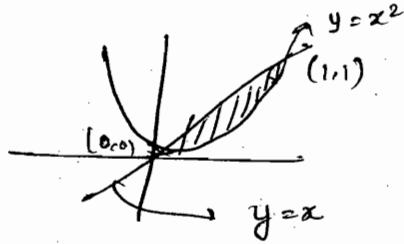
$$= 3 \int ds$$

$$= 3s$$

$$= 3\pi(1)^2$$

$$= 3\pi \cancel{\int ds}$$

Q. The area bounded by  $y = x^2$  and  $y = x$  -



$$\begin{aligned}x^2 &= x \\x(x-1) &= 0 \\x &= 0, 1 \\y &= 0, 1\end{aligned}$$

Point Trick

$$\begin{aligned}\text{The area} &= \int_0^1 (x - x^2) dx \quad \xrightarrow{\text{from Graph}} \left[ \text{Area will be always upper part - lower part.} \right] \\&= \left[ \frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 \\&= \frac{1}{2} - \frac{1}{3} \\&= \frac{1}{6} \quad \underline{\text{Ans}}\end{aligned}$$

Q. The area bounded by  $y^2 = 4ax$  &  $x^2 = 4ay$  is  $\frac{16a^3}{3}$

The area bounded by  $y^2 = 8x$  &  $x^2 = 8y$  is  $\frac{64}{3}$

$$\begin{array}{lll}\xrightarrow{\hspace{1cm}} y^2 = 4x \text{ & } x^2 = 4y \text{ is } & \frac{16}{3} \\ \xrightarrow{\hspace{1cm}} y^2 = x \text{ & } x^2 = y \text{ is } & \frac{1}{3}\end{array}$$

\*\* Q. The arc length of the curve  $y = f(x)$  between  $x=a$  &  $x=b$  is  $\int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$

Important Result

Q.  $y = \frac{2}{3}x^{3/2}$  between  $x=0$  and  $x=1$ .

$$\frac{dy}{dx} = \frac{2}{3} \times \frac{3}{2} x^{\frac{3}{2}-1} = \sqrt{x}$$

$$\begin{aligned}\therefore \int_a^b \sqrt{1 + (\sqrt{x})^2} dx &= \int_a^b \sqrt{1+x} dx = \int_a^b (1+x)^{\frac{1}{2}} dx \\&= (1+x)^{3/2} \Big|_a^b = \frac{2}{3} [2\sqrt{2} - 1]\end{aligned}$$

Q. A parabolic arc  $y = \sqrt{x}$ ,  $1 \leq x \leq 2$  is revolving around  $x$ -axis then the volume of the solid revolution.

$$\rightarrow \boxed{\begin{aligned} \text{volume} &= \int_a^b \pi y^2 dx \quad (\text{$x$-axis}) \\ &= \int_a^b \pi x^2 dy \quad (\text{$y$-axis}) \end{aligned}}$$

$$\therefore \text{volume} = \int_1^2 \pi (\sqrt{x})^2 dx \\ = \pi \left( \frac{x^2}{2} \right) \Big|_1^2 = \frac{\pi}{2} (4-1) = \frac{3\pi}{2} \text{ Ans}$$

WB  
97

Pg-19

$$\text{volume} = \int_a^b \pi x^2 dy.$$

$$y^2 = 8x, x=2$$

$$y^2 = 8(2)$$

$$y^2 = 16$$

$$\Rightarrow y = \pm 4$$

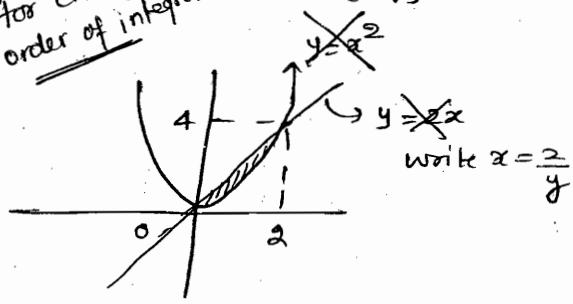
$$\therefore \text{volume} = \int_{-4}^4 \pi x^2 dy \\ = \int_{-4}^4 \pi \left( \frac{y^2}{8} \right) dy \\ = \frac{\pi}{8^2} \int_{-4}^4 (y^2)^2 dy \\ = \frac{\pi}{64} \cdot 2 \int_0^4 y^4 dy \\ = \frac{\pi}{32} \times \frac{y^5}{5} \Big|_0^4 \\ = \frac{\pi}{32} \times \frac{4^5 - 0^5}{5} = \frac{32\pi}{5} \text{ Ans}$$

(96)

$$\int_0^2 \int_{y=x^2}^{x=\sqrt{y}} f(x,y) dy dx$$

$\Rightarrow x = \sqrt{y}$

for changing  
order of integration write  
 $x = \sqrt{y}$

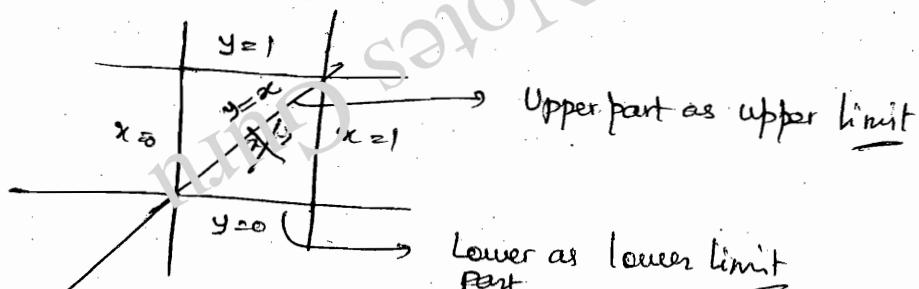


write  $x = \frac{y}{2}$

$$= \int_0^4 \int_{y/2}^{\sqrt{y}} f(x,y) dx dy \quad \underline{\text{Ans}}$$

(97)

$$\int_0^1 \int_{x=y}^{x=1} xy \sin(xy) dx dy = \int_0^1 \int_a^b xy \sin(xy) dy dx$$

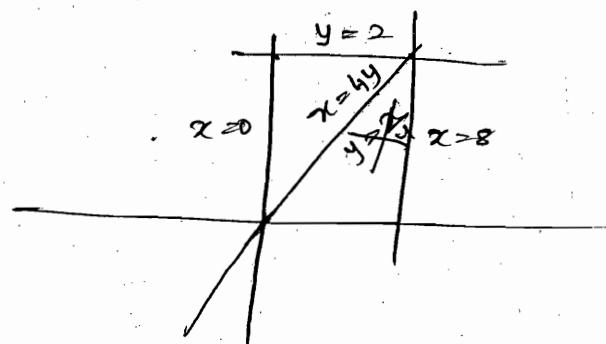


By changing  
order of integration.

$$= \int_0^1 \int_0^x xy \sin(xy) dy dx$$

(98)

$$\int_0^8 \int_{y=2/x}^2 f(x,y) dy dx \approx \int_8^9 \int_{y=0}^4 f(x,y) dy dx$$

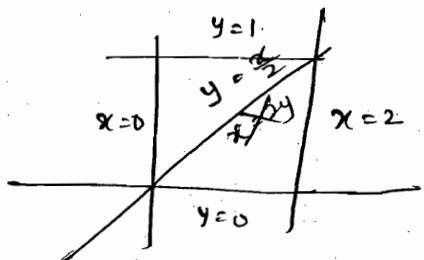


$$\approx \int_{y=0}^4 \int_{x=0}^{4y} f(x,y) dx dy$$

(85)

$$\int_0^1 \int_{x=2y}^{x=2} e^{x^2} dx dy$$

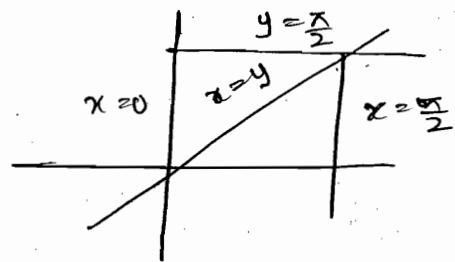
By changing order of integration —



$$\begin{aligned} &= \int_0^2 \int_0^{x/2} e^{x^2} dy dx \\ &= \int_0^2 e^{x^2} (y) \Big|_0^{x/2} dx \\ &= \int_0^2 e^{x^2} \frac{x}{2} dx \\ &= \frac{1}{2x^2} \int_0^1 e^{x^2} 2x dx \\ &= \frac{1}{4} (e^{x^2}) \Big|_0^1 = \frac{e^4 - e^0}{4} = \frac{e^4 - 1}{4} \underline{\Delta y} \end{aligned}$$

(8)

$$\int_0^{\pi/2} \int_{y/x}^{y=\pi/2} \frac{\cos y}{y} dy dx$$

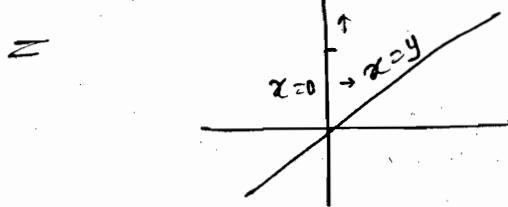


By changing order of integration —

$$\begin{aligned}& \approx \int_0^{\pi/2} \int_0^y \frac{\cos y}{y} dz dy \\& = \int_0^{\pi/2} \frac{\cos y}{y} (z)_0^y dy \\& = \int_0^{\pi/2} \cos y dy \\& = (\sin y)_0^{\pi/2} \\& = \sin \frac{\pi}{2} - \sin 0 \\& = 1 - 0 \\& = 1 \text{ Ans}\end{aligned}$$

~~77~~  
78

$$\int_0^{\infty} \int_y^{\infty} \frac{1}{y} e^{-y/2} dy dx$$



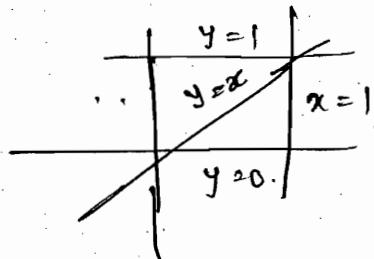
$$= \int_0^{\infty} \int_0^y \frac{1}{y} e^{-y/2} dx dy$$

$$= \int_0^{\infty} \int_0^y \frac{1}{y} e^{-y/2} (x)_0^y dy$$

$$= -2 \left[ \frac{e^{-\infty}}{0^{-\frac{1}{2}}} \right]_0^{\infty} = -2 [e^{-\infty} - e^0] = -2(0-1) = 2$$

~~77~~  
79

$$\int_0^1 \int_{x-y}^{x+1} y \sqrt{1+x^3} dx dy$$



$$= \int_0^1 \int_{y^2}^x y \sqrt{1+x^3} dy dx$$

$$= \int_0^1 \sqrt{1+x^3} \left( \frac{y^2}{2} \right)_0^x dx$$

$$= \int_0^1 \sqrt{1+x^3} \frac{x^2}{2} dx$$

$$= \frac{1}{2} x \frac{1}{3} \int_0^1 \sqrt{1+x^3} \frac{1}{2} \cdot 3x^2 dx$$

$$= \frac{1}{\frac{3}{8}} \left| \frac{(1+x^3)^{3/2}}{x} \right|_0^1$$

$$= \frac{1}{9} [2\sqrt{2} - 1]$$

Q1  $f(n) = \int_0^{\pi/4} \tan^n x \, dx$  then  $f(3) + f(1) = ?$

$$\rightarrow f(3) + f(1) = \int_0^{\pi/4} (\tan^3 x + \tan x) \, dx.$$

$$= \int_0^{\pi/4} (\tan x) (\sec^2 x) \, dx$$

$$= \left( \frac{\tan^2 x}{2} \right)_0^{\pi/4} = \frac{1}{2} \text{ Ans}$$

Q2  $\int_0^{\pi/2} \frac{\cos x - \sin x}{1 + \sin x \cos x} \, dx = ?$

$$\therefore \int_0^{\pi/2} f(x) \, dx = \int_0^a f(a-x) \, dx$$

$$I = \int_0^{\pi/2} \frac{\sin x - \cos x}{1 + \sin x \cos x} \, dx$$

$$I = \int_0^{\pi/2} \frac{\sin x - \sin x + \sin x - \cos x}{1 + \sin x \cos x} \, dx$$

$$= 0$$

$$\Rightarrow I = 0 \quad \text{Ans}$$

(56)

$$P(x, y, z) \\ z^2 = 1 + xy \\ (0, 0, 0)$$

$$OP = \sqrt{x^2 + y^2 + z^2} \\ = \sqrt{x^2 + y^2 + 1 + xy}$$

$$f = x^2 + y^2 + xy + 1$$

$$P = \frac{\partial f}{\partial x} = 2x + y = 0 \leftarrow$$

$$Q = \frac{\partial f}{\partial y} = 2y + x = 0$$

$$x - y = 0$$

$$x = y$$

$$\begin{array}{l} 3x^2 = 0 \\ x = 0 \end{array} \quad \begin{array}{l} 3x = 0 \\ x = 0 \\ y = 0 \end{array}$$

$$\therefore OP = \sqrt{0+0+0+1} \\ = 1$$

(57)

$$x = 2 \text{ after making } f''(x) = 0$$

} Since maximum of slope needed  
not maximum of curve demanding

(45)

$$x = uv, y = \frac{u}{v}$$

$$f(x, y) dx dy = \iint f(uv) \cdot f(uv, \frac{u}{v}) \phi(u, v) du dv$$

$$\phi(u, v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ -\frac{u}{v^2} & \frac{1}{u} \end{vmatrix}$$

A. A curve  $C$  is defined as  $x = a \cos^3 \theta$

$$y = a \sin^2 \theta \text{ in } [0, \frac{\pi}{2}]$$

What will be point  $P$  on curve  $C$  where tangent to curve is parallel to chord joining points  $(a, 0) \in (0, a)$ .

$$\rightarrow x = a \cos^3 \theta, y = a \sin^2 \theta.$$

$$\frac{dy/d\theta \sin^2 \theta \times \cos \theta}{dx/d\theta \cos^2 \theta \times -\sin \theta} = \frac{a-0}{0-a}$$

$$\Rightarrow \tan \theta = -1$$

$$\Rightarrow \theta = \frac{\pi}{4}$$

∴ putting values —

we get —

$$\left( \frac{a}{2\sqrt{2}}, \frac{a}{2\sqrt{2}} \right)$$

Q. 19 — correct  $y = a(1 - \cos \theta)$ .

Q. ⑧. Let  $\lim_{n \rightarrow \infty} \left( \frac{n!}{e^n n^n} \right)^{\frac{1}{n}}$

$$l = \left( \frac{n!}{e^n n^n} \right)^{\frac{1}{n}}$$

$$\log l = \frac{1}{n} [\log n! - n \log n]$$

$$\cancel{\log l} = \cancel{\frac{1}{n} \log n!} + \cancel{\log n}$$

$$= \frac{1}{n} \left[ \log \frac{1 \times 2 \times 3 \times \dots \times n}{n \times n \times n \dots \times n} \right]$$

$$= \frac{1}{n} \left[ \log\left(\frac{1}{n}\right) + \log\left(\frac{2}{n}\right) + \dots \right]$$

Doubt formula

$$\text{Let } \sum_{n=1}^{\infty} \frac{1}{n} \log\left(\frac{x}{n}\right) = \int_0^1 \log x \, dx \\ = [x \log x - x]_0^1$$

$$\log 2 = (1 \log 1 - 1)$$

$$e = e^{-1} = \frac{1}{e} \quad \underline{\text{Ans}}$$

$$Q6 = \lim_{n \rightarrow \infty} \left[ \frac{1}{1+n} + \frac{1}{2+n} + \dots + \frac{1}{n+n} \right]$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n} \left[ \frac{1}{1+\frac{1}{n}} + \frac{1}{1+\frac{2}{n}} + \dots \right]$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^n \frac{1}{1 + \left(\frac{r}{n}\right)}$$

$$= \int_0^1 \log \frac{1}{1+x} \, dx$$

$$= \left. \log(1+x) \right|_0^1$$

$$= \log 2 - \log 1$$

$$= \log 2 \quad \underline{\text{Ans}}$$

$$⑤ \quad \lim_{n \rightarrow \infty} \left[ \frac{n}{n^2} + \frac{n}{n^2+1^2} + \dots + \frac{n}{n^2+(n-1)^2} \right]$$

$$\lim_{n \rightarrow \infty} \frac{n}{n^2} \left[ \sum_{x=0}^{n-1} \frac{1}{1+(\frac{x}{n})^2} \right]$$

$$= \int_0^1 \frac{1}{1+x^2} dx$$

$$= \left( \tan^{-1} x \right)_0^1$$

$$= \tan^{-1} 1 - \tan^{-1} 0$$

$$= \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

$$④ \quad \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1}{3n+k}$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \frac{1}{3 + (\frac{k}{n})}$$

$$= \int_0^1 \frac{1}{3+x} dx = \left[ \log(3+x) \right]_0^1 = \log 4 - \log 3 = \log \frac{4}{3}$$

## DIFFERENTIAL EQ<sup>n</sup>

---

An eqn consisting of differential coeff.  $\frac{dy}{dx}$ ,  $\frac{d^2y}{dx^2}$ ,  $\frac{d^3y}{dx^3}$ , ...,  $\frac{dy^n}{dx^n}$

is called a differential eqn:-

\* Order -  $\frac{d^n y}{dx^n}$  is order order

\* Degree - The degree of diff eqn is the degree of the highest order

$$\frac{d^2y}{dx^2} + 5 \left( \frac{dy}{dx} \right)^3 + 6y = 0 \quad D \rightarrow 2 \quad O \rightarrow 1$$

$$\left( \frac{d^2y}{dx^2} \right)^3 + 5 \frac{dy}{dx} + 6y = k \left( \frac{d^3y}{dx^3} \right)^2 \quad D \rightarrow 3 \quad O \rightarrow 2$$

$$\left( \frac{d^2y}{dx^2} \right)^2 + \sqrt{1 + \left( \frac{dy}{dx} \right)^2} = 5y \quad D \rightarrow 4 \quad O \rightarrow 2$$

$$\left( \frac{d^2y}{dx^2} \right)^3 + 5 \cos \left( \frac{dy}{dx} \right) + 6y = 0 \quad D \rightarrow \text{not defined determined} \quad O \rightarrow 2$$

### # Formation of a Differential Eqn :-

---

A diff eqn can be formed by eliminating arbitrary const. in the solution.

Ex:-  $y^2 = 4ax$

$$2y \frac{dy}{dx} = 4a$$

$$y' = \cancel{x} \cdot \cancel{\frac{2dy}{dx}} \cdot x \cdot \frac{2dy}{dx}$$

$$2x \frac{dy}{dx} - y = 0$$

Q.  $x^2 + y^2 = a^2$   
 $\cancel{x} + \cancel{y} \frac{dy}{dx} = 0$

$$\boxed{x + y \frac{dy}{dx} = 0}$$

Note :-

Differentiate only  
 that much time  
 upto there <sup>No. of</sup> arbitrary  
 constants.

Q.  $y = ae^x + be^{-x}$

$$\frac{dy}{dx} = ae^x - be^{-x}$$

$$\frac{d^2y}{dx^2} = ae^x + be^{-x}$$

$$\boxed{\frac{d^2y}{dx^2} = y}$$

Q.  $y = e^x (a \cos x + b \sin x)$

~~$$\frac{dy}{dx} =$$~~ 
$$y_1 = e^x (a \cos x + b \sin x) + e^x (-a \sin x + b \cos x)$$

$$y_1 - y = e^x (-a \sin x + b \cos x)$$

$$y_2 - y_1 = \underbrace{e^x [-a \sin x + b \cos x]}_{y_1 - y} + \underbrace{e^x [-a \cos x + b \sin x]}_{-y}$$

$$y_2 - y_1 = y_1 - y - y$$

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$$

## # Solution of a Differential Eq<sup>n</sup> :-

### \* Variable Separable Method :-

$$\textcircled{1} \quad \frac{dy}{dx} + \sqrt{\frac{1+y^2}{1+x^2}} = 0$$

$$\frac{dy}{dx} = -\frac{\sqrt{1+y^2}}{\sqrt{1+x^2}}$$

$$\int \frac{dy}{\sqrt{1+y^2}} = \int -\frac{dx}{\sqrt{1+x^2}}$$

$$\Rightarrow \boxed{\sinh^{-1} y = -\sinh^{-1} x + C}$$

$$\textcircled{2} \quad \frac{dy}{dx} = e^{x+y}$$

$$\frac{dy}{dx} = e^x \cdot e^y$$

$$\int dy e^{-y} = \int e^x dx$$

$$-e^{-y} = e^x + C$$

$$\Rightarrow e^x + e^{-y} + C = 0$$

$$\textcircled{3} \quad \frac{dy}{dx} = x^2 e^{x-y}$$

$$\int \frac{dy}{e^y} = \int x^2 e^x dx$$

$$e^y = e^x [x^2 - 2x + 2] + C$$

∴

$$\textcircled{4} \quad \frac{dy}{dx} + y^2 = 0$$

$$\int dy y^{-2} = - \int dx$$

$$\frac{y^{-2+1}}{-1} = -x + c$$

$$\Rightarrow -y^{-1} = -x + c$$

$$\Rightarrow y^{-1} + x + c = 0 \quad \text{or} \quad \boxed{y = \frac{1}{x+c}}$$

$$\textcircled{5} \quad e^x \tan y dx + (1+e^x) \sec^2 y dy = 0$$

$$e^x \tan y dx = -(1+e^x) \sec^2 y dy$$

$$\frac{e^x dx}{1+e^x} = -\frac{\sec^2 y dy}{\tan y}$$

$$\int \frac{e^x}{1+e^x} dx + \int \frac{\sec^2 y}{\tan y} dy = \int 0$$

$$\log(1+e^x) + \log \tan y = \log C$$

$$\Rightarrow \boxed{(1+e^x) \tan y = C}$$

\textcircled{6}

## II<sup>nd</sup> Method

\* Eq<sup>n</sup> Reducable to variable - separable form:-

If the given diff. eq<sup>n</sup> is not directly variable separable form, by making substitution, again we can transform and under variable separable form.

Q.  $\frac{dy}{dx} = (4x+y+1)^2$        $\hookrightarrow$  Let  $z = 4x+y+1$   
 $\frac{dz}{dx} = 4 + \frac{dy}{dx}$

$$4 + \frac{dy}{dx} = 4 + (4x+y+1)^2$$

$$\Rightarrow 4 \frac{dz}{dx} = 4 + z^2$$

$$\Rightarrow \int \frac{dz}{z^2+4} = \int dx$$

$$\Rightarrow \frac{1}{2} \tan^{-1} \frac{z}{2} = x + C$$

$$\Rightarrow \boxed{\frac{1}{2} \tan^{-1} \left( \frac{4x+y+1}{2} \right) = x + C}$$

Q.  $1 + \frac{dy}{dx} = 1 + \sin(x+y) + \cos(x+y)$

$$\frac{dz}{dx} = 1 + \sin z + \cos z$$

$$x+y = z$$

$$1 + \frac{dy}{dx} = \frac{dz}{dx}$$

$$\Rightarrow \int \frac{dz}{1 + \sin z + \cos z} = \int dx$$

Remember

WHENEVER INTEGRAL IS -  
In form of  $\frac{1}{a+b \sin z} dz$  or  $\int \frac{1}{a+bx+cx^2} dz$ ,  $\int \frac{1}{a \cos z + b \sin z} dz$

$$\text{Put } t = \tan \frac{z}{2} \Rightarrow dz = \frac{2dt}{1+t^2}$$

$$\sin z = \frac{2t}{1+t^2}$$

$$x = \frac{1-t^2}{1+t^2}$$

$$\int \frac{1}{1 + \frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2}} \times \frac{2dt}{1+t^2} = x+c$$

$$\int \frac{2dt}{(1+t^2+2t+1-t^2)} = x+c$$

$$\int \frac{2}{2(1+t)} dt = x+c$$

$$\log(1+t) = x+c$$

$$\log\left(1+\tan\frac{x+y}{2}\right) = x+c$$

## # HOMOGENEOUS DIFFERENTIABLE EQUATION :-

A diff. eq<sup>n</sup> of the form -

$\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$  is said to be homogeneous differential eq<sup>n</sup>, if both  $f(x,y)$  and  $g(x,y)$  are homogeneous function of the same degree.

$$y = vx \Rightarrow v = \frac{y}{x}$$

$$\boxed{\frac{dy}{dx} = v + x \frac{dv}{dx}}$$

$\checkmark$  when  $\frac{dx}{dy} = \frac{f(x,y)}{g(x,y)}$

$$x = vy \Rightarrow v = \frac{x}{y}$$

$$\boxed{\frac{dx}{dy} = v + y \frac{dv}{dy}}$$

ILATE

Ex:-

$$\frac{dy}{dx} = \frac{x^2+y^2}{2xy}$$

$$\frac{dx}{dy} = \frac{2xy}{x^2+y^2}$$

$$\therefore \frac{dy}{dx} = \frac{x^2+y^2}{2xy}$$

→ which form should be selected for substitution?

→ ~~for~~ numerator ~~or~~ no. of terms ~~more~~

$$v + x \frac{dv}{dx} = \frac{x^2 + v^2}{2x \cdot vx}$$

$$x \frac{dv}{dx} = \frac{1+v^2}{2v} - v$$

$$x \frac{dv}{dx} = \frac{1-v^2}{2v}$$

$$\int \frac{-2v}{1-v^2} dv = \int -\frac{dx}{x}$$

$$\Rightarrow \log(1-v^2) = \log \frac{c}{x}$$

$$\Rightarrow 1 - \frac{y^2}{x^2} = \frac{c}{x}$$

$$\Rightarrow \frac{x^2-y^2}{x^2} = \frac{c}{x}$$

$$\Rightarrow \boxed{x^2-y^2=cx}$$

$$\frac{1}{x} \rightarrow \log cx$$

$$-\frac{1}{x} \rightarrow \log \frac{c}{x}$$

$$\frac{2}{x} \rightarrow \log cx^2$$

$$-\frac{2}{x} \rightarrow \log \frac{c}{x^2}$$

$$\frac{3}{x} \rightarrow \log cx^3$$

$$-\frac{3}{x} \rightarrow \log \frac{c}{x^3}$$

Q.  $x^2y dx - (x^3+y^3)dy = 0.$

$$\frac{dy}{dx} = \frac{x^2y}{x^3+y^3}$$

$$\Rightarrow \frac{dx}{dy} = \frac{x^3+y^3}{x^2y}$$

$$v + y \frac{dv}{dy} = \frac{v^3y^3 + y^3}{v^2y^3}$$

$$v + y \frac{dv}{dy} = \frac{v^3+1}{v^2}$$

$$y \frac{dv}{dy} = \frac{1}{v^2}$$

$$\Rightarrow \int \frac{dy}{y} = \int v^2 dv$$

$$\Rightarrow \log y = \frac{v^3}{3}.$$

$$\Rightarrow \log y = \frac{x^3}{3y^3} \underline{\Delta x}.$$

$$Q. x \frac{dy}{dx} = y + x \tan\left(\frac{y}{x}\right)$$

$$\frac{dy}{dx} = \frac{y}{x} + \tan \frac{y}{x}$$

$$y + x \frac{dy}{dx} = x + \tan y$$

$$\int \frac{dy}{\tan y} = \int \frac{dx}{x}$$

$$\log \sin y \cos x = \log cx$$

$$\boxed{\sin\left(\frac{y}{x}\right) = cx} \quad \text{ie}$$

$$Q. (1+e^{xy})dx + e^{xy}(1-\frac{x}{y})dy = 0$$

$$\frac{dx}{dy} = \frac{e^{xy}(x/y - 1)}{(1+e^{xy})}$$

$$v + y \frac{dv}{dy} = \frac{e^v(v-1)}{1+e^v}$$

$$y \frac{dv}{dy} = \frac{e^v(v-1)}{(1+e^v)} - v$$

$$y \frac{dv}{dy} = \frac{v/e^v - e^v - v - v/e^v}{(1+e^v)}$$

$$y \frac{dv}{dy} = -\frac{(v+e^v)}{(1+e^v)}$$

$$\int \frac{(1+e^v)}{v+e^v} dv = \int -\frac{dy}{y}$$

$$\log(v+e^v) = \log \frac{c}{y}$$

$$\Rightarrow \frac{x}{y} + e^{-y} = \frac{c}{x+y \cdot e^{-y}} \quad \text{ie}$$

Tellig  $\rightarrow$  Angle  $\omega$   
 Algebraic  $\rightarrow$  power numerator & denominator

$$\text{exponents } \frac{x}{y} - \frac{dx}{dy}$$

$$\frac{y}{x} - \frac{dy}{dx}$$

MD: 9660127702  
 NEW DELHI-110009  
 WEST END MARKS STUDY LABS  
 ADD: MEER MADE EASY LINE NO. 4  
 100% PHOTOGRAPHIC BOOK CENTRE

## # NON HOMOGENEOUS DIFFERENTIAL FUNCTIONS:-

A diff. eq<sup>n</sup> of the form —

$$\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$$

~~$a_1x + b_1y + c_1$~~       is said to be a non

homogeneous differential eq<sup>n</sup>

Case-1:-

If  $a_1b_2 - a_2b_1 \neq 0$ , then the substitution is —

$$x = X + h$$

$$y = Y + K$$

where  $h, K$  are constants to be determined.

Case-2:-

If  $a_1b_2 - a_2b_1 = 0$ , then ~~the~~ some part of numerator and denominator are equal and that part is considered as substitution.

Q. Find the substitution that transforms a non homogeneous diff eq<sup>n</sup>

$$\frac{dy}{dx} = \frac{2x+2y-2}{3x+y-5}$$

~~$2x+2y-2$~~       to homogeneous form.  
 $\downarrow \neq 0 \rightarrow 1^{\text{st}}$  form

→

$$\frac{dy}{dx} = \frac{2(x+h) + 2(y+k) - 2}{3(x+h) + y - 5}$$

~~$x$~~      ~~$y$~~      ~~$x+h$~~      ~~$y+k$~~      ~~$x$~~      ~~$y$~~

h	K	1	h	K	
2	2	-2	2	2	
3	1	-5	3	1	

→ write like this only

→ Numerator

→ Denominator

$$\frac{h}{-10+2} = \frac{K}{-6+10} = \frac{1}{2-6}$$

$$\frac{h}{-8} = \frac{K}{4} = \frac{1}{-4}$$

$$h = 2 \cdot K = -1 \quad \Rightarrow \quad \left. \begin{array}{l} x = X + 2 \\ y = Y + 1 \end{array} \right\}$$

Q. Find Substitution —

$$\frac{dy}{dx} = \frac{x+2y-3}{2x+y-3}$$

$\curvearrowleft f_0 \rightarrow 1^{\text{st}}$  form.

$h$	$k$	1	$h$	$k$
1	2	-3	1	2
2	1	-3	2	1

$$\frac{h}{-6+3} = \frac{k}{-6+3} = \frac{1}{1-4}$$

$$\frac{h}{-3} = \frac{k}{-3} = \frac{1}{-3}$$

$$h = 1, k = 1 \quad \text{or}$$

$$\boxed{x = x+1 \\ y = y+1} \quad \text{Ans.}$$

Q. Find the Substitution —

$$\frac{dy}{dx} = \frac{2x+3y+1}{4x+6y+1}$$

$\curvearrowleft = 0 \rightarrow 2^{\text{nd}}$  form

$$\frac{dy}{dx} = \frac{(2x+3y)+1}{2(2x+3y)+1}$$

$$\boxed{z = 2x+3y}$$

$$0. \frac{dy}{dx} = \frac{x+2y+3}{y+2x+5}$$

$$\frac{dy}{dx} = \frac{x+2y+3}{2x+y+5}$$

$\hookrightarrow \neq 0 \rightarrow \text{1st form.}$

$h$	$k$	1	$h$	$k$
1		2	3	1
2		1	5	2

$$\frac{h}{10-3} = \frac{k}{6-5} = \frac{1}{1-4}$$

$$\frac{h}{7} = \frac{k}{1} = \frac{1}{-3}$$

$$h = -\frac{7}{3}, k = -\frac{1}{3}$$

$x = x + \frac{7}{3}$	<del>A02</del>
$y = y - \frac{1}{3}$	

## # EXACT DIFFERENTIAL EQN:-

A diff. eqn of the form  $Mdx + Ndy = 0$  is said to

be exact if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Gen. Sol<sup>n</sup>  $\rightarrow \int^x Mdx + \int (\text{terms of } N \text{ not containing } x) dy = c$

Ex:-  $(x^2 - ay)dx + (y^2 - ax)dy = 0$

$$\frac{\partial M}{\partial y} = -a \quad , \quad \frac{\partial N}{\partial x} = -a$$

exact.

$$\int (x^2 - ay) dx + \int y^2 dy = c$$

$$\frac{x^3}{3} - axy + \frac{y^3}{3} = c$$

$$\Rightarrow x^3 + y^3 - 3axy = 3c \quad \underline{\text{Ans}}$$

MORE  
Efficient  
Shortcut

GENERALIZATION:-

Ex:-  $(ax+hy+g)dx + (hx+by+f)dy = 0$ .

$$\frac{\partial M}{\partial y} = h \quad , \quad \frac{\partial N}{\partial x} = h.$$

$$\left[ \frac{ax^2}{2} + hxy + gx + \frac{by^2}{2} + fy = c \right] \quad \underline{\text{Ans}}$$

a.  $e^x + \tan y dx + (1 + e^x) \sec^2 y dy = 0$

$$\frac{\partial M}{\partial y} = e^x \sec^2 y$$

$$\frac{\partial N}{\partial x} = e^x \sec^2 y$$

exact.

Sol<sup>n</sup>

$$e^x + \tan y + \cdot \tan y = c$$

$$\Rightarrow \boxed{(e^x + 1) \tan y = c}$$

$$Q. (x^2 + y^2)dx + 2xy dy = 0$$

$$\frac{\partial M}{\partial y} = 2y \quad , \quad \frac{\partial N}{\partial x} = 2y$$

Sol<sup>n</sup>

exact

$$\boxed{\frac{x^3}{3} + y^2 x = C}$$

$$Q. (1 + e^{\frac{x}{y}})dx + e^{\frac{x}{y}}(1 - \frac{x}{y})dy = 0$$

$$\begin{aligned} \text{exact } \left( \frac{\partial M}{\partial y} = e^{\frac{x}{y}} \cdot x \cdot \left(-\frac{1}{y^2}\right) \right. \\ \left. \frac{\partial N}{\partial x} = e^{\frac{x}{y}} \left(\frac{-1}{y}\right) + \left(1 - \frac{x}{y}\right) e^{\frac{x}{y}} \left(\frac{1}{y}\right) \right. \\ = e^{\frac{x}{y}} \left( \frac{-1}{y} + \frac{1}{y} - \frac{x}{y^2} \right) \end{aligned}$$

Sol<sup>n</sup>

$$\boxed{x + y e^{\frac{x}{y}} = C}$$

## # Non-Exact Differential Eq<sup>n</sup> :-

A diff eq<sup>n</sup>  $Mdx + Ndy = 0$  is said to be non-exact, if  $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

METHOD-1 (Method to form a suitable Integrating factor)

IMPORTANT INTEGRATING FACTORS :-

①  $x dy + y dx = d(xy)$

②  $\frac{x dy - y dx}{x^2} = d\left(\frac{y}{x}\right)$

③  $\frac{x dy - y dx}{xy} = d\left(\log \frac{y}{x}\right)$

④  $\frac{x dy - y dx}{x^2+y^2} = d\left(\tan^{-1} \frac{y}{x}\right)$

⑤  $\frac{y dx - x dy}{y^2} = d\left(\frac{x}{y}\right)$

⑥  $\frac{y dx - x dy}{xy} = d\left(\log \frac{x}{y}\right)$

⑦  $\frac{y dx - x dy}{x^2+y^2} = d\left(\tan^{-1} \frac{x}{y}\right)$

⑧  $\frac{ye^x dx - e^x dy}{y^2} = d\left(\frac{e^x}{y}\right)$

⑨  $\frac{2xy dx - x^2 dy}{y^2} = d\left(\frac{x^2}{y}\right)$

⑩  $\frac{x dx + y dy}{x^2+y^2} = d\left[\frac{1}{2} \log(x^2+y^2)\right]$

$$Q. \quad y(1+xy)dx + x(1-xy)dy = 0.$$

$$(y+xy^2)dx + (x-x^2y)dy = 0.$$

$$\frac{\partial M}{\partial y} = 1+2xy, \quad \frac{\partial N}{\partial x} = 1-2xy$$

Non exact

$\rightarrow$  we can make it perfect I.F but considering  
1st term carefully

$$ydx + xy^2dx + xdy - x^2ydy = 0$$

Perfect I.F

$$\frac{d(xy)}{(xy)^2} + \frac{xy[ydx - xdy]}{(xy)^2} = \frac{0}{(xy)^2}$$

$$\Rightarrow \int \frac{1}{(xy)^2} dxy + \int d(\log \frac{x}{y}) = 0$$

$$\Rightarrow \boxed{-\frac{1}{xy} + \log \frac{x}{y} = C} \quad \left\{ \because \frac{dx}{x^2} = -\frac{1}{x} \right\}$$

$$Q. ① \quad ydx - xdy + (1+x^2)dx + x^2 \sin y dy = 0.$$

by taking  $x^{-1}$  common it is also  
integrable

$$\frac{ydx - xdy}{x^2} + \frac{(1+x^2)dx}{x^2} + \frac{x^2 \sin y dy}{x^2}$$

$$\Rightarrow - \int \frac{(xdy - ydx)}{x^2} + \int \left( \frac{1}{x^2} + 1 \right) dx + \int \sin y dy = 0.$$

$$\Rightarrow - \int d\left(\frac{y}{x}\right) + \int \left( \frac{1}{x^2} + 1 \right) dx + \int \sin y dy = 0.$$

$$\Rightarrow \boxed{-\frac{y}{x} - \frac{1}{x} + x - \cos y = C.}$$

$$Q. \int \frac{\alpha xy^2 dx}{y} + \int \frac{ye^x dx - e^x dy}{y^2} = 0$$

$$\Rightarrow \int \alpha x dx + \int d\left(\frac{e^x}{y}\right) = 0$$

$$\Rightarrow \boxed{\alpha \frac{x^2}{2} + \frac{e^x}{y} = c} \quad \checkmark$$

$$Q. (y^2 e^x + 2xy) dx - x^2 dy = 0$$

$$\Rightarrow \frac{y^2 e^x dx}{y^2} + \frac{2xy dx - x^2 dy}{y^2} = 0$$

$$\Rightarrow \int e^x dx + \int d\left(\frac{x^2}{y}\right) = 0$$

$$\Rightarrow \boxed{-e^x + \frac{x^2}{y} = c} \quad \checkmark$$

$$Q. \frac{2x^2 y^2 dx}{y^2} + \frac{ye^x dx}{y^2} - \frac{e^x dy}{y^2} - \frac{y^3 dy}{y^2} = 0$$

$$\Rightarrow \int 2x^2 dx + \int d\left(\frac{e^x}{y}\right) - \int y dy = 0$$

$$\Rightarrow \boxed{\frac{2x^3}{3} + \frac{e^x}{y} - \frac{y^2}{2} = c} \quad \checkmark$$

## NON EXACT DIFFERENTIAL EQN - I

### METHOD - 2

If  $Mdx + Ndy = 0$  is non exact, but homogeneous and if

~~M~~  $Mx + Ny \neq 0$ , then  $\frac{1}{Mx+Ny}$  is an I.F.

After multiplying in diff. eqn  $\rightarrow$  non exact becomes exact.

Then for sol<sup>n</sup> will follow procedure of Exact diff. eqn.

$d=2$        $\frac{\text{Homogeneous}}{d=2}$

$$\rightarrow Q. ① \quad (x^2+y^2)dx - 2xydy = 0$$

$$\frac{\partial M}{\partial y} = +2y, \quad \frac{\partial N}{\partial x} = 2y$$

$$\therefore I.F. = \frac{1}{Mx+Ny} = \frac{1}{x(x^2+y^2) + y(-2xy)} = \frac{1}{x^3+xy^2-2xy^2} \\ = \frac{1}{x^3-xy^2} \\ = \frac{1}{x(x^2-y^2)}$$

Now multiply by I.F. —

$$\int \frac{x^2+y^2}{x(x^2-y^2)} dx - \frac{2xy}{x(x^2-y^2)} dy = 0.$$

$$\Rightarrow \int \left[ \frac{2x}{x^2-y^2} - \frac{1}{x} \right] dx = \log C$$

$$\Rightarrow \log \frac{x^2-y^2}{x} = \log C$$

$$\Rightarrow \boxed{x^2-y^2 = cx} \quad \checkmark$$

Calculate I.F -

④

$$① x^2ydx - (x^3 + y^3)dy = 0.$$

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(x^2y) = Mx + Ny \neq 0$$

$$\therefore x^3y - x^2y - y^4 \neq 0$$

$$\therefore \frac{1}{Mx + Ny} = -\frac{1}{y^4}$$

Note:-

In Exam option will be provided as  $\frac{1}{y^4}$

coz, (-) and const will not effect integration and ~~log~~  
in process of solving '-' or const can be taken out  
common and can be send to right where 0 is given.

$$② (x^2y - xy^2)dx - (x^3 - 2x^2y)dy$$

$$\begin{aligned} Mx + Ny &= x^3y - x^2y^2 - x^3y + 2x^2y^2 \\ &= x^2y^2 \end{aligned}$$

$$\text{I.F.} \doteq \frac{1}{Mx + Ny} = \frac{1}{x^2y^2}$$

$$③ xydx - (x^2 + 2y^2)dy = 0.$$

$$Mx + Ny = x^2y - x^2y - 2y^3$$

$$\therefore \text{I.F.} = \frac{1}{Mx + Ny} = \frac{1}{-2y^3} \underset{\text{no importance}}{\approx} \frac{1}{y^3} \quad \text{Ans}$$

## NON EXACT DIFFERENTIAL EQN :-

### METHOD - 3 :-

If  $Mdx + Ndy = 0$  is non exact, but is in the form of  
 ~~$y f(xy)dx + x g(xy)dy = 0$~~  and if  
 $Mx - Ny \neq 0$

then  $\frac{1}{Mx - Ny}$  is an I.F.

$$\Rightarrow y(1+xy)dx + x(1-xy) = 0$$

$$Mx - Ny = xy + x^2y^2$$

$$\therefore \frac{1}{Mx - Ny} = \frac{1}{xy(1+xy)}$$

$$Q. ① y(x^2y^2 + xy + 1)dx + x(x^2y^2 - xy + 1)dy = 0$$

for fast calculation of  $\frac{1}{Mx - Ny}$  - cancel ~~at~~ terms having same ~~de~~ order and some const. and increase power or multiply by  $\frac{1}{xy}$ .

$$\therefore \frac{1}{Mx - Ny} = \frac{1}{x^2y^2}$$

$$② y(x^3y^3 + x^2y^2 + xy + 1)dx + x(x^3y^3 - x^2y^2 - xy + 1)dy = 0$$

$$\frac{1}{Mx - Ny} = \frac{1}{x^3y^3 + x^2y^2}$$

$$③ y(x^2y^2 + 1)dx + x(y - 2x^2y^2)dy = 0$$

$$\frac{1}{Mx - Ny} = \frac{1}{x^3y^3}$$

$$④ y(\cos xy + \sin xy)dx + x[x \cancel{\sin xy} - \cos xy]dy = 0$$

$$\frac{1}{Mx - Ny}$$

## NON-EXACT DIFFERENTIAL EQN:-

### METHOD-4 :-

If  $Mdx + Ndy = 0$  is non exact and if  ~~$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$~~

Tip  
 $N$  is divisible  
में यह भी  
सिर्फ़ परिवर्तनी

$$\frac{1}{N} \left( \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right) = f(x) \quad \rightarrow \quad \text{IF } f(x) = \frac{1}{x} \rightarrow x$$

$$I.F = e^{\int f(x) dx}$$

Tip  
 $M$  is divisible  
में यह भी  
सिर्फ़ परिवर्तनी

$$\frac{1}{M} \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) = g(y)$$

$$I.F = e^{\int g(y) dy}$$

$$\begin{aligned} \frac{1}{x} &= \frac{1}{x} \\ \frac{2}{x} &= x^2 \\ -\frac{2}{x} &= \frac{1}{x^2} \\ \frac{3}{x} &= x^3 \\ -\frac{3}{x} &= \frac{1}{x^3} \end{aligned}$$

Q.  $(x^2 + y^2 + 1) dx - 2xy dy = 0$

$$\frac{\partial M}{\partial y} = 2y, \quad \frac{\partial N}{\partial x} = -2y.$$

1 term  $\therefore$  divisible by  $N$

$$\frac{1}{N} \left( \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right)$$

$$= -\frac{1}{2xy} (2y + 2y)$$

$$= -\frac{2}{2xy} = -\frac{1}{x} \rightarrow I.F = \frac{1}{x^2}$$

Q. Q.  $(y + xy^3) dx + 2(x^2y^2 + x + y^4) dy = 0$

$$\frac{\partial M}{\partial y} = 1 + 3xy^2$$

$$\frac{\partial N}{\partial x} = 2xy^2 + 1 + \cancel{y^4}$$

$$\cancel{+ 3xy^2} \cancel{+ 2xy^2} \cancel{+ y^4} \\ \cancel{+ 2xy^2} \cancel{+ y^4}$$

2 terms  
 $\therefore$  divisible by  $M$

$$\frac{1}{M} \left( \frac{\partial M}{\partial x} - \frac{\partial N}{\partial y} \right)$$

$$= \frac{1}{(y + xy^3)} [4xy^2 + 2 + 3xy^2] = \cancel{-} \frac{1}{y} \rightarrow I.F = \frac{1}{y}$$

$$(2) \quad (y^4 + 2y) dx + (xy^3 + 2y^4 - 4x) dy = 0$$

$$\frac{\partial M}{\partial y} = 4y^3 + 2 \quad \frac{\partial N}{\partial x} = y^3 - 4$$

2 terms  
 $\therefore$  divisible by M  
 as it M has also  
 2 terms

$$\begin{aligned} & \frac{1}{M} \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \\ &= \frac{1}{y^4 + 2y} \left( y^3 - 4 - 4y^3 - 2 \right) \\ &= \frac{1}{y^4 + 2y} \left( -3y^3 - 6 \right) \\ &= -\frac{3}{y} \rightarrow I.F = \frac{1}{y^3} \end{aligned}$$

$$(3) \quad \left( y + \frac{y^3}{2} + \frac{x^2}{2} \right) dx + \left( \frac{x + xy^2}{4} \right) dy = 0$$

$$\begin{aligned} \frac{\partial M}{\partial y} &= 1 + \frac{xy^2}{3}, \quad \frac{\partial N}{\partial x} = \frac{1}{4} + \frac{y^2}{4} \\ & \frac{1}{M} \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \\ &= \frac{4}{x + xy^2} \left( 1 + y^2 - \frac{1}{4} - \frac{y^2}{4} \right) \\ &= \frac{4}{x(1+y^2)} \left( \cancel{3} \left( \cancel{\frac{3}{4}} + \cancel{\frac{3y^2}{4}} \right) \right) \\ &= \frac{3}{x} \rightarrow I.F = \frac{x^3}{4} \end{aligned}$$

$$Q. (3x^2y^4 + 2xy)dx + (2x^3y^3 - x^2)dy = 0.$$

$$\frac{\partial M}{\partial y} = 12x^2y^3 + 2x$$

$$\frac{\partial N}{\partial x} = 6x^2y^3 - 2x$$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$  ↗ अवैराग्य + sign वाले एवं दूसरे चर  
devisible by M not N

$$\frac{1}{M} \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right)$$

$$= \frac{1}{3x^2y^4 + 2xy} (-6x^2y^3 + 4x)$$

$$= -\frac{2}{y} \rightarrow I.F \rightarrow \frac{1}{y^2}$$

## # Linear Eq<sup>n</sup>:

$\frac{dy}{dx} + Py = Q$ , Where P and Q are said to be a fn. of x, called

Linear eq<sup>n</sup>.

$$I.F = e^{\int P dx}$$

General sol<sup>n</sup>—

$$y e^{\int P dx} = \int (Q \cdot e^{\int P dx}) dx + C$$

Ex:-  
Q

$\frac{dx}{dy} + Px = Q$ , Where P and Q are said to be fn. of y

$$I.F = e^{\int P dy}$$

General sol<sup>n</sup>—

$$x e^{\int P dy} = \int (Q \cdot e^{\int P dy}) dy + C$$

$$\textcircled{1} \quad \frac{dy}{dx} + \frac{y}{x} = \frac{\log x}{x}$$

$$P = \frac{1}{x}$$

$$Q = \frac{\log x}{x}$$

$$I.F = e^{\int P dx} = e^{\int \frac{1}{x} dx} = x$$

sol<sup>n</sup>—

$$y \cdot x = \int \log x dx + C$$

$$\Rightarrow yx = x \log x - x + C \quad \underline{\text{Ans}}$$

$$\textcircled{2} \quad (2-x^2) \frac{dy}{dx} - 2xy = x^2$$

$$\frac{dy}{dx} = \frac{2xy}{2-x^2} = \frac{x^2}{2-x^2}$$

For I.F =  $e^{\int P dx} = e^{\int \frac{2x}{2-x^2} dx} = e^{\log(2-x^2)}$

$$= 2-x^2$$

G. Soln -

$$y(2-x^2) = \int \frac{x^2}{2-x^2} (2-x^2) dx + C$$

$$= \frac{x^3}{3} + C$$

$$\textcircled{3} \quad \frac{dy}{dx} + \frac{P}{Q} y = \left(\frac{-x^2}{e}\right), \quad y(0) = 1$$

$$\text{I.F} = e^{\int P dx} = e^{\int 2x dx} = e^{x^2} = e^{x^2}$$

G. Soln -

$$ye^{x^2} = \int e^{-x^2} \cdot e^{x^2} dx + C$$

$$\boxed{ye^{x^2} = x + C}$$

when  $y(0) = 1$

$$0 \cdot 1 \cdot e^0 = C \Rightarrow C = 1$$

$$\therefore \boxed{ye^{x^2} = x + 1}$$

$$④ \frac{dy}{dx} + \left(\frac{1}{x \log x}\right)^P y = \left(\frac{2 \log x}{x \log x}\right)^Q$$

$$I.F = e^{\int \frac{1}{x \log x} dx} = e^{\int \frac{1}{\log x} dx} = e^{\log(\log x)} = \log x$$

G. Soln

$$y \log x = \int \frac{2 \log x}{x \log x} \times \log x dx + C$$

$$y \log x = 2 \frac{(\log x)^2}{2} + C$$

~~y log x~~ = 2 f.

$$\boxed{y = \log x + \frac{C}{\log x}}$$

$$⑤ (x + 2y^3) \frac{dy}{dx} = y$$

$$\frac{dx}{dy} = \frac{x}{y} + \frac{2y^2}{y}$$

$$\frac{dx}{dy} - \left(\frac{P}{y}\right)x = \left(2y^2\right)^Q$$

$$I.F = e^{\int -\frac{1}{y} dy} = \frac{1}{y}$$

G. Soln

$$x \cdot \frac{1}{y} = \int 2y^2 \times \frac{1}{y} dy + C$$

$$\frac{x}{y} = y^2 + C$$

$$\Rightarrow \boxed{x = y^3 + Cy}$$

$$Q. 5 \quad (1+y^2)dx + (x - e^{\tan^{-1}y})dy = 0$$

$$(1+y^2)dx = (e^{\tan^{-1}y} - x)dy$$

$$\Rightarrow \frac{dx}{dy} = \frac{e^{\tan^{-1}y}}{1+y^2} - \frac{x}{1+y^2}$$

$$\Rightarrow \frac{dx}{dy} + \frac{x}{1+y^2} = \frac{e^{\tan^{-1}y}}{1+y^2}$$

$$\therefore I.F = e^{\int \frac{1}{1+y^2} dy} = e^{\frac{1}{2} \tan^{-1}y} = e^{\tan^{-1}y}$$

General Soln —

$$x \cancel{+} e^{\tan^{-1}y} = \int \frac{e^{\tan^{-1}y}}{1+y^2} \times e^{\tan^{-1}y} dy + C.$$

$$= \frac{1}{2} \int \frac{e^{2\tan^{-1}y}}{1+y^2} \times 2 dy + C$$

$$\boxed{x e^{\tan^{-1}y} = \frac{1}{2} e^{2\tan^{-1}y} + C.}$$

MOB: 9560127702  
NEW DELHI-110065  
WEST END LIAISON, SINGLALIA  
ADY, DEVA MARG, SINGLALIA  
ADY, PHOTOSTAT & BOOK PRINTERS  
www.notesguru.info

## # BERNOULLI'S EQN:-

A diff. eqn of the form  $\frac{dy}{dx} + Py = Q(y^n)$ , where

P and Q are fn. of x if said to be a bernoulli's eqn

$$\therefore y^{-n} \frac{dy}{dx} + P \cancel{y^{1-n}} = Q$$

$$\Rightarrow z = y^{1-n}$$

$$\frac{dz}{dx} = (1-n)y^{1-n-1} \frac{dy}{dx}$$

$$\frac{dy}{dx} \rightarrow z$$

$$y^{-n} \frac{dy}{dx} = \frac{1}{(1-n)} \frac{dz}{dx}$$

$$\frac{1}{(1-n)} \frac{dz}{dx} + Pz = Q.$$

$$\Rightarrow \frac{dz}{dx} + P(1-n)z = Q(1-n).$$

$$* I.F = e^{\int P(1-n)dx}$$

Gen. soln.

$$* y^{1-n} e^{\int P(1-n)dx} = \int (Q(1-n) e^{\int P(1-n)dx}) dx + C$$



$$\frac{dx}{dy} + Px = Q^n$$

$$I.F = e^{\int P(1-n)dy}$$

Gen. soln

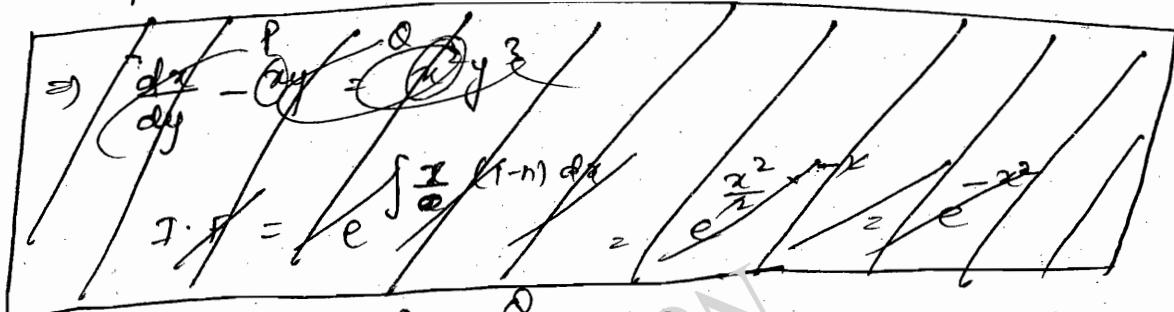
$$x^{1-n} \times I.F = \int (Q(1-n) \times I.F) dy + C$$

Find I.F.

$$\textcircled{1} \quad \frac{dy}{dx} + \frac{P}{x} = x^2 y^3$$

$$I.F. = e^{\int \frac{1}{x} (1-n) dx} = e^{(1-n) \log x} = x^{1-n} = x^{-2} = \frac{1}{x^2}$$

$$\textcircled{2} \quad \frac{dy}{dx} (xy + x^2 y^3) = 1$$



$$\Rightarrow \frac{dx}{dy} - \frac{P}{y} = \frac{Q}{x^2}$$

$$I.F. = e^{\int -\frac{1}{y} (1-n) dy} = e^{\int -y (-2) dy} = e^{\int y dy} = e^{\frac{y^2}{2}}$$

## # Differential Eq<sup>n</sup> with constant coefficient :-

A diff. eqn of the form -

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots + a_n y = Q.$$

Where  $a_0, a_1, a_2, \dots, a_n$  are constants and  $Q$  be the function of  $x$  is said to be differential equation with constant coefficient.

$$\text{Let } \frac{d}{dx} = D$$

$$(a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \dots + a_n) y = Q.$$

$$\boxed{f(D) y = Q}$$

This eq<sup>n</sup> may contain two types of sol<sup>n</sup>

- one is called complementary fn
- other is called particular integral.

Note:-

If  $Q=0$ , then complementary fn is called the general soln.  
 If  $Q \neq 0$ , then only particular integral must be existed.

# To Find Complimentary fn (C.F) :-

Replace 'D' by  $m$ ,  $f(m)=0$  is called auxiliary eqn.

$$a_0 m^n + a_1 m^{n-1} + a_2 m^{n-2} + \dots + a_n = 0.$$

Case - 1.

Roots are real and Unequal.

Let  $m = \alpha, \beta, \gamma, \dots$

$$\therefore y_c = C_1 e^{\alpha x} + C_2 e^{\beta x} + C_3 e^{\gamma x} + \dots$$

Case - 2

Roots are real and equal.

Let  $m = \alpha, \alpha, \beta$

$$y_c = (C_1 + C_2 x) e^{\alpha x} + C_3 e^{\beta x}$$

Let  $m = \alpha, \alpha, \alpha, \beta$ .

$$y_c = (C_1 + C_2 x + C_3 x^2) e^{\alpha x} + C_4 e^{\beta x}$$

Let  $m = \alpha, \alpha, \beta, \beta$

$$y_c = (C_1 + C_2 x) e^{\alpha x} + (C_3 + C_4 x^2) e^{\beta x}$$

Ex:-

$$\frac{d^2y}{dx^2} - 4 \frac{dy}{dx} + 4y = 0.$$

$$m^2 - 4m + 4 = 0.$$

$$(m-2)^2 = 0.$$

$$m = 2, 2$$

$$y = (C_1 + C_2 x) e^{2x}$$

### Case-3

Roots are complex —

Let  $m = \alpha \pm i\beta$

$$y_c = [c_1 \cos \beta x + c_2 \sin \beta x] e^{\alpha x}$$

Let  $m = \alpha \pm i\beta, \alpha \pm i\beta$ .

$$y_c = [(c_1 + c_2 x) \cos \beta x + (c_3 + c_4 x) \sin \beta x] e^{\alpha x}$$

Ex:-

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 4y = 0$$

$$m^2 - 2m + 4 = 0$$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{2 \pm 2\sqrt{-3}}{2}$$

$$= 1 \pm i\sqrt{3}$$

$$= 1 \pm \sqrt{3}i$$

$$\therefore y = [c_1 \cos \sqrt{3}x + c_2 \sin \sqrt{3}x] e^x$$

# To find the roots of Eq<sup>n</sup>:-

For 3<sup>rd</sup> degree  $y$ . Every 3<sup>rd</sup> degree eq<sup>n</sup> must contain at least 1 real root.

ii). If sum of all the coeff = 0, then  $m=1$  satisfies eq<sup>n</sup>

or  $(m-1)$  is a factor.

one root

iii). If sum of coeff. of even powers = sum of coeff. of odd powers  
then  $m=-1$  satisfies the eq<sup>n</sup>. or  $(m+1)$  will be factor.

one root

$$\textcircled{1} \quad (D^3 - 3D + 2)y = 0$$

$$m^3 - 3m + 2 = 0$$

~~sum = 0~~  $\rightarrow m=1$  is root ~~if~~

~~coeff.~~

$m=1$	<del>6</del>	0	-3	2
<del>0</del>	0	1	1	-2
<del>sum = 1</del>	1	1	-2	0
$m=1$	0	1	2	
	1	2	0	

$\rightarrow m^3$

$m=1$	0	1	2
<del>0</del>	1	2	
<del>sum = 1</del>	1	2	
	1	2	0

$\rightarrow m^2$

	1	2	0

$\rightarrow m$

$$m+2 = 0$$

$$m = -2$$

$\therefore m = 1, 1 \& -2$  are roots.

$$\textcircled{2} \quad [D^3 - 4D^2 + 5D - 2]y = 0.$$

$$m^3 - 4m^2 + 5m - 2 = 0$$

$$1 - 4 + 5 - 2 = 0.$$

$m=1$	1	-4	5	-2
<del>0</del>	0	1	-3	2
$m=1$	1	-3	2	
<del>0</del>	0	1	-2	
	1	-2	0	

$$m-2 = 0$$

$$m = 2$$

$$\therefore m = 1, 1, 2 \quad \underline{\Delta}$$

$$\textcircled{2} \quad [D^4 - 2D^3 + 2D - 1] y = 0.$$

$$m^4 - 2m^3 + 2m - 1 = 0$$

$$m^2 - 2m + 1 = 0.$$

$$\therefore m = 1 \quad \checkmark$$

$$\begin{array}{c|ccccc|c} m=1 & 1 & -2 & 0 & 2 & -1 \\ & 0 & 1 & -1 & -1 & 1 \\ \hline m=1 & 1 & -1 & -1 & 1 & 0 \\ & 0 & 1 & 0 & -1 & \\ \hline m=1 & 1 & 0 & -1 & 0 & \\ & 0 & 1 & 1 & & \\ \hline & 1 & 1 & 0 & & \end{array}$$

$$m + 1 = 0$$

$$m = -1.$$

$$\textcircled{3} \quad [D^3 + 6D^2 + 11D + 6] y = 0$$

$$\cancel{m^3 + 6m^2 + 11m + 6} = 0.$$

$$\cancel{12} = 12 \rightarrow m = -1$$

$$\begin{array}{c|ccc|c} m = -1 & 1 & 6 & 11 & 6 \\ & 0 & -1 & -5 & -6 \\ \hline & 1 & 5 & 6 & 0 \end{array}$$

$$m^2 + 5m + 6 = 0.$$

$$\cancel{m^2 + 3m} (m+3)(m+2) = -3, -2$$

$$m = -1, -2, -3$$

$$\textcircled{4} \quad (D^4 - D^3 - 9D^2 - 11D - 4) y = 0$$

$$m^4 - m^3 - 9m^2 - 11m - 4 = 0$$

$$\begin{array}{ccccccc} 1 & -1 & -9 & -11 & -4 \\ & \swarrow & \searrow & & & \\ & & -12 & & & \end{array} \Rightarrow m = -1$$

$$\begin{array}{c} m = -1 \\ \hline 1 & -1 & -9 & -11 & -4 \\ 0 & -1 & 2 & 7 & 4 \\ \hline 1 & -2 & -7 & -4 & \\ -6 & & -6 & & \\ \hline 0 & -1 & 3 & 4 & \\ 1 & -3 & -4 & & \\ -3 & & -3 & & \\ \hline 0 & -1 & 4 & & \\ 1 & -4 & & & \end{array}$$

$$m - 4 = 0$$

$$m = 4$$

$$\therefore m = -1, -1, -1, 4$$

$$\left. \begin{array}{l} y = (c_1 + c_2 x + c_3 x^2) e^{-x} \\ \quad \quad \quad + c_4 e^{4x} \end{array} \right\}$$

$$\textcircled{5} \quad (D^2 - 3D + 4) y = 0$$

$$m^2 - 3m + 4 = 0$$

$$\begin{aligned} (m-4)(m-1) &= 0 \\ m &= \frac{3 \pm \sqrt{9-16}}{2} \\ &= \frac{3 \pm \sqrt{-7}}{2} \end{aligned}$$

$$\left. \begin{array}{l} y = \left( c_1 \cos \frac{\sqrt{7}}{2} x \right) \\ \quad \quad \quad + \left( c_2 \sin \frac{\sqrt{7}}{2} x \right) e^{\frac{3x}{2}} \end{array} \right\}$$

$$\textcircled{6} \quad [D^2 - (a+b)D + ab] y$$

$$m^2 - (a+b)m + ab = 0$$

$$(m-a)(m-b) = 0$$

$$m = a, b$$

$$\therefore y = C_1 e^{ax} + C_2 e^{bx}$$

# To find the particular Integral:-

$$F(D)y = \theta$$

$$\text{then } y_p = PI = \frac{1}{F(D)} Q$$

Notes Guru



Notes Guru

Case-III: To find P.I. of the form  $\frac{1}{f(D)} \sin ax$ ,  $\frac{1}{f(D)} \cos ax$ , replace  $D^2$  by  $-a^2$  if  $f(-a^2) \neq 0$ .

$$\begin{aligned} \frac{1}{D^2 - 2D + 5} \sin 2x &= \frac{1}{-4 - 2D + 5} \sin 2x = \frac{1}{1 - 2D} \times \frac{1+2D}{1+2D} \sin 2x \\ &= \frac{1+2D}{1-4D^2} \sin 2x = \frac{(1+2D)}{1-4(-4)} \sin 2x \\ &= \frac{\sin 2x + 4 \cos 2x}{17} \end{aligned}$$

Case-IV: To find P.I. of the form  $\frac{1}{D^2 + a^2} \sin ax$ ,  $\frac{1}{D^2 + a^2} \cos ax$ ,

$$\frac{1}{D^2 + a^2} \sin ax = \frac{x}{2a} \left( \frac{\sin ax}{2D} \right) = \frac{x}{2} \left( -\frac{\cos ax}{a} \right)$$

$$\boxed{\begin{aligned} \frac{1}{D^2 + a^2} \sin ax &= -\frac{x}{2a} \cos ax \\ \frac{1}{D^2 + a^2} \cos ax &= \frac{x}{2a} \sin ax \end{aligned}}$$

$$\frac{1}{D^2 + 4} \sin 2x = -\frac{x}{4} \cos 2x$$

$$\frac{1}{D^2 + 9} \cos 3x = \frac{x}{6} \sin 3x$$

$$\frac{1}{D^2 + 16} \sin 4x = -\frac{x}{8} \cos 4x$$

$$\cancel{\frac{1}{D^2 + 1}} \cos 3x = \frac{1}{-9 + 6} \cos 3x = \frac{-1}{8} \cos 3x$$

$$\textcircled{1} \quad (\Delta^2 - \Delta - 2)y = \cos 2x$$

$$\textcircled{2} \quad (\Delta^2 - 4)y = \sin^2 x$$

$$\textcircled{3} \quad (\Delta^2 + 1)y = \sin 2x \cdot \cos x$$

$$\begin{aligned} \rightarrow \textcircled{1} \quad \frac{1}{\Delta^2 - \Delta - 2} \cos 2x &= \frac{1}{(\Delta - 4) - \Delta - 2} \cos 2x = \frac{1}{-\Delta + 4} \times \frac{-\Delta + 4}{-\Delta + 4} \cos 2x \\ &= \frac{-\Delta + 4}{36 - \Delta^2} \cos 2x \\ &= \left[ \frac{(-\Delta + 4)}{36 - (-4)} \cos 2x \right] \\ &= -\frac{6 \cos 2x - 2 \sin 2x}{40} \\ &= -\frac{x}{20} (3 \cos 2x + \sin 2x) \\ &= -\frac{1}{20} (3 \cos 2x + \sin 2x). \end{aligned}$$

$$\textcircled{2} \quad \frac{1}{\Delta^2 - 4} \sin^2 x = \frac{1}{\Delta^2 - 4} \left[ \frac{1 - \cos 2x}{2} \right] = \frac{1}{2} \left[ \frac{1}{\Delta^2 - 4} (1) - \frac{1}{\Delta^2 - 4} (\cos 2x) \right]$$

$$\boxed{\begin{aligned} &= \frac{1}{\Delta^2 - 4} e^{0x} \\ &= \frac{1}{\Delta^2 - 4} (5) = \frac{x}{2\Delta - 1} (5) \end{aligned}}$$

$$\begin{aligned} &= \frac{1}{2} \left[ \frac{1}{0-4} (1) - \frac{1}{-4-4} \cos 2x \right] \\ &= -\frac{1}{8} + \underline{\frac{1}{16} \cos 2x} \end{aligned}$$

$$\textcircled{3} \quad (D^2+1)y = \sin 2x \cdot \cos x$$

$$= \frac{1}{2(D^2+1)} 2 \sin 2x \cdot \cos x$$

$$= \frac{1}{2(D^2+1)} \left[ \frac{1}{2} \sin 3x + \frac{1}{D^2+1} \sin x \right]$$

$$= \frac{1}{2} \left[ \frac{1}{-9+1} \sin 3x - \frac{x}{2(1)} \cos x \right]$$

$$= -\frac{1}{16} \sin 3x - \frac{x}{4} \cos x.$$

SHORTCUT

$$\rightarrow \text{If } (D^2+a^2)y = \tan ax$$

$$y_p = -\frac{1}{a^2} \cos ax \cdot \log \left( \tan \left( \frac{\pi}{4} + \frac{ax}{2} \right) \right)$$

$$\rightarrow \text{If } (D^2+a^2)y = \sec ax$$

$$y_p = \frac{x}{a} \sin ax + \frac{1}{a^2} \cos ax \cdot \log(\cos ax)$$

$$\text{ex:- } (D^2+4)y = \tan 2x$$

$$y_p = -\frac{1}{4} \cos 2x \cdot \log \left( \tan \left( \frac{\pi}{4} + x \right) \right)$$

$$\text{ex:- } (D^2+9)y = \sec 3x$$

$$y_p = \frac{\pi}{3} \sin 3x + \frac{1}{9} \cos 3x \cdot \log(\cos 3x)$$

Case-II

To find the P.I. of the form  $\frac{1}{f(D)}(x^n)$

$$\textcircled{1} \quad (D^2 - 4)y = x^2$$

$$\frac{1}{D^2 - 4} x^2$$

$$= \frac{1}{-4 \left[ 1 - \frac{D^2}{4} \right]} x^2$$

$$= -\frac{1}{4} \left[ 1 - \frac{D^2}{4} \right]^{-1} x^2$$

$$= -\frac{1}{4} \left[ 1 + \frac{D^2}{4} + \underbrace{\left( \frac{D^2}{4} \right)^2}_{0} + \dots \right] x^2$$

$$= -\frac{1}{4} \left[ x^2 + \frac{1}{4^2} \right]$$

$$= -\frac{1}{4} \left[ x^2 + \frac{1}{16} \right].$$

$$\textcircled{2} \quad [D^3 + 8]y = x^4 + 2x + 1$$

$$= \frac{1}{D^3 + 8} (x^4 + 2x + 1)$$

$$= \frac{1}{8 \left[ 1 + \frac{D^3}{8} \right]} (x^4 + 2x + 1)$$

$$= \frac{1}{8} \left[ 1 + \frac{D^3}{8} \right]^{-1} (x^4 + 2x + 1)$$

$$= \frac{1}{8} \left[ 1 - \frac{D^3}{8} + \frac{D^6}{8^2} + \dots \right] (x^4 + 2x + 1)$$

$$= \frac{1}{8} [(x^4 + 2x + 1) - \frac{1}{8} (24x + 0 + 0)]$$

$$= \frac{1}{8} (x^4 - x + 1)$$

$$(1+D)^{-1} = 1 - D + D^2 - D^3 + \dots$$

$$(1-D)^{-1} = 1 + D + D^2 + D^3 + \dots$$

$$(1+D)^{-2} = 1 - 2D + 3D^2 - 4D^3 + \dots$$

$$(1-D)^{-2} = 1 + 2D + 3D^2 + 4D^3 + \dots$$

Remember

$$D^3(x^2) = 0$$

$$\boxed{D^n(x^n) = n!}$$

$$\Rightarrow D^{n+1}(x^n) = 0$$

Notes Guru

$$\begin{aligned}
 ③ \quad & (D^2 + 2D + 1)y = x^2 + x \\
 \Rightarrow & \frac{1}{(D+1)^2} (x^2 + x) \\
 = & [1+D]^{-2} (x^2 + x) \\
 = & [1 - 2D + 3D^2] (x^2 + x) \\
 = & (x^2 + x) - 2(2x + 1) + 3(2 + 0) \\
 = & x^2 + x - 4x - 2 + 6 \\
 = & x^2 - 3x + 4
 \end{aligned}$$

Case-VI: To find PI of the form  $\frac{1}{f(D)} e^{ax} \cdot v$ , where  
 $v$  is a fn. of  $x$ .

$$\boxed{\frac{1}{f(D)} e^{ax} \cdot v = e^{ax} \frac{1}{f(D+a)} \cdot v}$$

$$① \quad (D^2 - 2D + 1)y = x^2 e^x$$

$$② \quad (D^2 - 2D + 5)y = e^x \sin x$$

$$③ \quad (D^2 - 4D + 4)y = e^{2x} \cos 3x$$

$$④ \quad (D^3 + 3D^2 + 3D + 1)y = x^2 e^{-x}$$

$$\begin{aligned}
 \rightarrow ① \quad & \frac{1}{(D-1)^2} e^x x^2 = e^x \cdot \frac{1}{(D+1-x)^2} x^2 = e^x \left[ \int \frac{x^2}{D^2} \right] \\
 & = e^x \frac{x^4}{12}
 \end{aligned}$$

represents integration

$$\textcircled{2} \quad \frac{1}{D^2 - 2D + 5} (e^x \sin x)$$

$$= e^x \left( \frac{1}{(D+1)^2 - 2(D+1) + 5} \sin x \right)$$

$$= e^x \left[ \frac{1}{D^2 + 1 + 2D - 2 + 5} \sin x \right]$$

$$= e^x \left[ \frac{1}{D^2 + 4} \sin x \right]$$

$$= e^x \left[ \frac{1}{-1+4} \sin x \right]$$

$$= \frac{e^x \sin x}{3}$$

$$\textcircled{3} \quad \frac{1}{(D^2 - 4D + 4)} e^{2x} \cos 3x$$

$$= e^{2x} \left[ \frac{1}{(D-2)^2} \cos 3x \right]$$

$$= e^{2x} \left[ \frac{1}{D^2} \cos 3x \right]$$

$$= e^{2x} \frac{1}{-9} \cos 3x$$

$$= -\frac{e^{2x}}{9} \cos 3x =$$

$$\textcircled{f} \quad \frac{1}{(D^3 + 3D^2 + 3D + 1)} e^{-x} x^2$$

$$= \cancel{\frac{1}{(D+1)^3}} e^{-x} x^2$$

$$= e^{-x} \frac{1}{(D+1)^3} x^2$$

$$= e^{-x} \left[ \frac{1}{D^3} x^2 \right]$$

$$= e^{-x} \left( \frac{x^5}{60} \right) \cancel{+ x}$$

*w*

$$\boxed{\int \int \int x^2 = \frac{x^3}{3} - \frac{x^4}{12} + \frac{x^5}{60}}$$

Case-VII. To find the P.I. of the form  $\frac{1}{f(D)} x \cdot v$ , where  $v$  is a fn of  $x$  (Trigonometric fn)

$$\boxed{\frac{1}{f(D)} x \cdot v = x \cdot \frac{1}{f(D)} v - \frac{f'(D)}{[f(D)]^2} \cdot v}$$

$$\textcircled{1} \quad (D^2 + 4)y = x \cos x$$

$$\textcircled{2} \quad (D^2 - 2D + 1)y = x e^x \sin x$$

$$\textcircled{1} \quad \frac{1}{D^2 + 4} (x \cos x)$$

$$= x \cdot \frac{1}{(D^2 + 4)} \cos x - \frac{2D}{(D^2 + 4)^2} \cdot \cos x$$

$$= x \cdot \frac{1}{-1 + 4} \cos x - \frac{2(-\sin x)}{(-1 + 4)^2}$$

$$= \frac{x}{3} \cos x + \frac{2}{9} \sin x.$$

$$② \frac{1}{(D-1)^2} e^x \cdot x \cdot \sin x$$

$$= e^x \left[ \frac{1}{(D+1-1)^2} x \sin x \right]$$

$$= e^x \left[ \frac{1}{D^2} x \sin x \right]$$

$$= e^x \left[ x \frac{1}{-a^2} \sin x - \frac{2D}{(D-a^2)^2} \cdot \overset{\curvearrowleft}{\sin x} \right]$$

$$= e^x \left[ x \frac{1}{-1} \sin x - \frac{2 \cos x}{(-1)^2} \right]$$

$$= e^x (-x \sin x - 2 \cos x) -$$

## # Differential Eq<sup>n</sup> with variable co-efficients :-

or

Cauchy Euler form:-

→ A diff eq<sup>n</sup> is of the form —

$$a_0 x^n \frac{d^y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + a_2 x^{n-2} \frac{d^{n-2}y}{dx^{n-2}} + \dots + a_n y = Q$$

Where  $a_0, a_1, a_2, \dots, a_n$  are constants

and  $Q$  be the function of  $x$ .

is said to be Diff<sup>n</sup> Eq<sup>n</sup> with Variable Coefficients.

$$x = e^z \Rightarrow z = \log x$$

$$\frac{dz}{dx} = \frac{1}{x} \quad \& \quad \frac{d}{dx} = x$$

$$\frac{d}{dz} = \frac{d}{dx} \cdot \frac{dx}{dz} = x \frac{d}{dx} = xD = \frac{d}{dx} = \theta$$

Now—

$$\frac{d}{dz} \left( x \frac{d}{dx} \right) = \frac{d^2}{dz^2}$$

$$\frac{d}{dx} \left( x \frac{d}{dx} \right) \cdot \frac{dx}{dz} = \theta^2$$

$$x \left( x \frac{d^2}{dx^2} + \frac{d}{dx} \cdot 1 \right) = \theta^2$$

$$\therefore x^2 \frac{d^2}{dx^2} + x \frac{d}{dx} = \theta^2$$

$$x^2 \frac{d^2}{dx^2} + \theta = \theta^2$$

$$x^2 \frac{d^2}{dx^2} = \theta(\theta - 1)$$

$$x^2 \frac{d^2}{dx^2} = x^2 D^2 = \theta(\theta - 1)$$

~~Important Substitution to Remember~~ Let  $x \frac{d}{dx} = xD = \theta$

$$x^2 \frac{d^2}{dx^2} = x^2 D^2 = \theta(\theta-1) = \theta^2 - \theta$$

$$x^3 \frac{d^3}{dx^3} = x^3 D^3 = \theta(\theta-1)(\theta-2) = \theta^3 - 3\theta^2 + 2\theta$$

Variable Coeff.

$$\textcircled{1} \quad (x^2 D^2 - xD - 3)y = 0$$

$$= [\theta^2 - \theta - 3]y = 0$$

$$= [m^2 - 2m - 3] = 0$$

$$= (m-3)(m+1) = 0$$

$$m = 3, -1$$

$$Y = C_1 e^{3z} + C_2 e^{-z}$$

$$= C_1 (e^z)^3 + C_2 (e^z)^{-1}$$

$$\boxed{Y = C_1 z^3 + C_2 z^{-1}}$$

$$\textcircled{2} \quad [x^3 D^3 + 3x^2 D^2 - 2xD + 2]y = 0$$

$$[\theta^3 - 3\theta^2 + 2\theta + 3\theta^2 - 3\theta - 2\theta + 2]y = 0$$

$$[\theta^3 - 3\theta + 2]y = 0$$

$$m^3 - 3m + 2 = 0$$

~~$m^3 - 3m + 2 = 0$~~

$\therefore m=1$  satisfies.

$$\begin{array}{c|ccc|c} & 1 & 0 & -3 & 2 \\ m=1 & 0 & 1 & 1 & -2 \\ \hline & 1 & 1 & 1-2 & 0 \\ M=1 & 0 & & & \\ \hline & 1 & & & \end{array}$$

$$m+2=0 \Rightarrow m=-2,$$

$$m-3=0 \Rightarrow m=3,$$

$$m-1=0 \Rightarrow m=1,$$

$$m+1=0 \Rightarrow m=-1,$$

$$m-2=0 \Rightarrow m=2,$$

$$m-4=0 \Rightarrow m=4,$$

$$m-5=0 \Rightarrow m=5,$$

$$m-6=0 \Rightarrow m=6,$$

$$m-7=0 \Rightarrow m=7,$$

$$m-8=0 \Rightarrow m=8,$$

$$m-9=0 \Rightarrow m=9,$$

$$m-10=0 \Rightarrow m=10,$$

$$m-11=0 \Rightarrow m=11,$$

$$m-12=0 \Rightarrow m=12,$$

$$m-13=0 \Rightarrow m=13,$$

$$m-14=0 \Rightarrow m=14,$$

$$m-15=0 \Rightarrow m=15,$$

$$m-16=0 \Rightarrow m=16,$$

$$m-17=0 \Rightarrow m=17,$$

$$m-18=0 \Rightarrow m=18,$$

$$m-19=0 \Rightarrow m=19,$$

$$m-20=0 \Rightarrow m=20,$$

$$m-21=0 \Rightarrow m=21,$$

$$m-22=0 \Rightarrow m=22,$$

$$m-23=0 \Rightarrow m=23,$$

$$m-24=0 \Rightarrow m=24,$$

$$m-25=0 \Rightarrow m=25,$$

$$m-26=0 \Rightarrow m=26,$$

$$m-27=0 \Rightarrow m=27,$$

$$m-28=0 \Rightarrow m=28,$$

$$m-29=0 \Rightarrow m=29,$$

$$m-30=0 \Rightarrow m=30,$$

$$m-31=0 \Rightarrow m=31,$$

$$m-32=0 \Rightarrow m=32,$$

$$m-33=0 \Rightarrow m=33,$$

$$m-34=0 \Rightarrow m=34,$$

$$m-35=0 \Rightarrow m=35,$$

$$m-36=0 \Rightarrow m=36,$$

$$m-37=0 \Rightarrow m=37,$$

$$m-38=0 \Rightarrow m=38,$$

$$m-39=0 \Rightarrow m=39,$$

$$m-40=0 \Rightarrow m=40,$$

$$m-41=0 \Rightarrow m=41,$$

$$m-42=0 \Rightarrow m=42,$$

$$m-43=0 \Rightarrow m=43,$$

$$m-44=0 \Rightarrow m=44,$$

$$m-45=0 \Rightarrow m=45,$$

$$m-46=0 \Rightarrow m=46,$$

$$m-47=0 \Rightarrow m=47,$$

$$m-48=0 \Rightarrow m=48,$$

$$m-49=0 \Rightarrow m=49,$$

$$m-50=0 \Rightarrow m=50,$$

$$m-51=0 \Rightarrow m=51,$$

$$m-52=0 \Rightarrow m=52,$$

$$m-53=0 \Rightarrow m=53,$$

$$m-54=0 \Rightarrow m=54,$$

$$m-55=0 \Rightarrow m=55,$$

$$m-56=0 \Rightarrow m=56,$$

$$m-57=0 \Rightarrow m=57,$$

$$m-58=0 \Rightarrow m=58,$$

$$m-59=0 \Rightarrow m=59,$$

$$m-60=0 \Rightarrow m=60,$$

$$m-61=0 \Rightarrow m=61,$$

$$m-62=0 \Rightarrow m=62,$$

$$m-63=0 \Rightarrow m=63,$$

$$m-64=0 \Rightarrow m=64,$$

$$m-65=0 \Rightarrow m=65,$$

$$m-66=0 \Rightarrow m=66,$$

$$m-67=0 \Rightarrow m=67,$$

$$m-68=0 \Rightarrow m=68,$$

$$m-69=0 \Rightarrow m=69,$$

$$m-70=0 \Rightarrow m=70,$$

$$m-71=0 \Rightarrow m=71,$$

$$m-72=0 \Rightarrow m=72,$$

$$m-73=0 \Rightarrow m=73,$$

$$m-74=0 \Rightarrow m=74,$$

$$m-75=0 \Rightarrow m=75,$$

$$m-76=0 \Rightarrow m=76,$$

$$m-77=0 \Rightarrow m=77,$$

$$m-78=0 \Rightarrow m=78,$$

$$m-79=0 \Rightarrow m=79,$$

$$m-80=0 \Rightarrow m=80,$$

$$m-81=0 \Rightarrow m=81,$$

$$m-82=0 \Rightarrow m=82,$$

$$m-83=0 \Rightarrow m=83,$$

$$m-84=0 \Rightarrow m=84,$$

$$m-85=0 \Rightarrow m=85,$$

$$m-86=0 \Rightarrow m=86,$$

$$m-87=0 \Rightarrow m=87,$$

$$m-88=0 \Rightarrow m=88,$$

$$m-89=0 \Rightarrow m=89,$$

$$m-90=0 \Rightarrow m=90,$$

$$m-91=0 \Rightarrow m=91,$$

$$m-92=0 \Rightarrow m=92,$$

$$m-93=0 \Rightarrow m=93,$$

$$m-94=0 \Rightarrow m=94,$$

$$m-95=0 \Rightarrow m=95,$$

$$m-96=0 \Rightarrow m=96,$$

$$m-97=0 \Rightarrow m=97,$$

$$m-98=0 \Rightarrow m=98,$$

$$m-99=0 \Rightarrow m=99,$$

$$m-100=0 \Rightarrow m=100,$$

$$m-101=0 \Rightarrow m=101,$$

$$m-102=0 \Rightarrow m=102,$$

$$m-103=0 \Rightarrow m=103,$$

$$m-104=0 \Rightarrow m=104,$$

$$m-105=0 \Rightarrow m=105,$$

$$m-106=0 \Rightarrow m=106,$$

$$m-107=0 \Rightarrow m=107,$$

$$m-108=0 \Rightarrow m=108,$$

$$m-109=0 \Rightarrow m=109,$$

$$m-110=0 \Rightarrow m=110,$$

$$m-111=0 \Rightarrow m=111,$$

$$m-112=0 \Rightarrow m=112,$$

$$m-113=0 \Rightarrow m=113,$$

$$m-114=0 \Rightarrow m=114,$$

$$m-115=0 \Rightarrow m=115,$$

$$m-116=0 \Rightarrow m=116,$$

$$m-117=0 \Rightarrow m=117,$$

$$m-118=0 \Rightarrow m=118,$$

$$m-119=0 \Rightarrow m=119,$$

$$m-120=0 \Rightarrow m=120,$$

$$m-121=0 \Rightarrow m=121,$$

$$m-122=0 \Rightarrow m=122,$$

$$m-123=0 \Rightarrow m=123,$$

$$m-124=0 \Rightarrow m=124,$$

$$m-125=0 \Rightarrow m=125,$$

$$m-126=0 \Rightarrow m=126,$$

$$m-127=0 \Rightarrow m=127,$$

$$m-128=0 \Rightarrow m=128,$$

$$m-129=0 \Rightarrow m=129,$$

$$m-130=0 \Rightarrow m=130,$$

$$m-131=0 \Rightarrow m=131,$$

$$m-132=0 \Rightarrow m=132,$$

$$m-133=0 \Rightarrow m=133,$$

$$m-134=0 \Rightarrow m=134,$$

$$m-135=0 \Rightarrow m=135,$$

$$m-136=0 \Rightarrow m=136,$$

$$m-137=0 \Rightarrow m=137,$$

$$m-138=0 \Rightarrow m=138,$$

$$m-139=0 \Rightarrow m=139,$$

$$m-140=0 \Rightarrow m=140,$$

$$m-141=0 \Rightarrow m=141,$$

$$m-142=0 \Rightarrow m=142,$$

$$m-143=0 \Rightarrow m=143,$$

$$m-144=0 \Rightarrow m=144,$$

$$m-145=0 \Rightarrow m=145,$$

$$m-146=0 \Rightarrow m=146,$$

$$m-147=0 \Rightarrow m=147,$$

$$m-148=0 \Rightarrow m=148,$$

$$m-149=0 \Rightarrow m=149,$$

$$m-150=0 \Rightarrow m=150,$$

$$m-151=0 \Rightarrow m=151,$$

$$m-152=0 \Rightarrow m=152,$$

$$m-153=0 \Rightarrow m=153,$$

$$m-154=0 \Rightarrow m=154,$$

$$m-155=0 \Rightarrow m=155,$$

$$m-156=0 \Rightarrow m=156,$$

$$m-157=0 \Rightarrow m=157,$$

$$m-158=0 \Rightarrow m=158,$$

$$m-159=0 \Rightarrow m=159,$$

$$m-160=0 \Rightarrow m=160,$$

$$m-161=0 \Rightarrow m=161,$$

$$m-162=0 \Rightarrow m=162,$$

$$m-163=0 \Rightarrow m=163,$$

$$m-164=0 \Rightarrow m=164,$$

$$m-165=0 \Rightarrow m=165,$$

$$m-166=0 \Rightarrow m=166,$$

$$m-167=0 \Rightarrow m=167,$$

$$m-168=0 \Rightarrow m=168,$$

$$m-169=0 \Rightarrow m=169,$$

$$m-170=0 \Rightarrow m=170,$$

$$m-171=0 \Rightarrow m=171,$$

$$m-172=0 \Rightarrow m=172,$$

$$m-173=0 \Rightarrow m=173,$$

$$m-174=0 \Rightarrow m=174,$$

$$m-175=0 \Rightarrow m=175,$$

$$m-176=0 \Rightarrow m=176,$$

$$m-177=0 \Rightarrow m=177,$$

$$m-178=0 \Rightarrow m=178,$$

$$m-179=0 \Rightarrow m=179,$$

$$m-180=0 \Rightarrow m=180$$

$$\therefore y = (c_1 + c_2 z)e^z + c_3 e^{-2z}$$

$$\checkmark \boxed{y = (c_1 + c_2 \log x)x + c_3 x^{-2}}$$

$$③ [x^2 D^2 + xD - 4] y = 0, \quad y(0) = 0 \\ y(1) = 1$$

$$[D^2 + D - 4] y = 0$$

$$m^2 - 4 = 0$$

$$m = \pm \sqrt{4} = \pm 2 = 2, -2$$

$$\therefore y = c_1 e^{2z} + c_2 e^{-2z}$$

$$\Rightarrow y = c_1 x^2 + c_2 x^{-2}$$

$$\Rightarrow y = c_1 x^2 + \frac{c_2}{x^2}$$

$$\Rightarrow x^2 y = c_1 x^4 + c_2$$

$$\because x(0) = 0$$

$$\Rightarrow \underline{c_2 = c_1 x^4 + c_2}$$

$$0 \cdot 0 = c_1(0) + c_2$$

$$\Rightarrow \boxed{c_2 = 0}$$

$$y(1) = 1$$

$$1^2 \cdot 1 = c_1(1)^4 + c_2$$

$$1 = c_1 + 0$$

$$\boxed{c_1 = 1}$$

$$\therefore y = 1(x^2) + \frac{0}{x^2}$$

$$\checkmark \boxed{y = x^2}$$

$$Q. [x^4 D^3 + 2x^3 D^2 - x^2 D + x] y = 1$$

$$\Rightarrow (x^3 D^3 + 2x^2 D^2 - x D + 1) y = \frac{1}{x}$$

$$\Rightarrow (\theta^3 - 3\theta^2 + 3\theta + 2\theta^2 - 2\theta - \theta + 1) y = e^{-z}$$

$$\Rightarrow (\theta^3 - \theta^2 - \theta + 1) y = e^{-z}$$

$$m^3 - m^2 - m + 1 = 0$$

$$m^2(m-1) - 1(m-1) = 0$$

$$(m-1)(m^2-1) = 0$$

$$(m-1)(m-1)(m+1) = 0$$

$$m = 1, 1, -1$$

$$\therefore y_c = (c_1 + c_2 z) e^z + c_3 e^{-z}$$

$$\Rightarrow y_c = (c_1 + c_2 \log x) x + c_3 x^{-1}$$

$$\Rightarrow y_p = \frac{1}{\theta^3 - \theta^2 - \theta + 1} e^{-z}$$

$$= \frac{z}{3\theta^2 - 2\theta - 1} e^{-z}$$

$$= \frac{z}{3(-1)^2 - 2(-1) - 1} e^{-z} = \frac{z e^{-z}}{4}$$

$$= \frac{\log x}{4} \times \frac{1}{x} = \frac{1}{4x} \log x$$

$$\textcircled{5} \quad (x^2 D^2 - x D + 1)y = 0.$$

$$\theta^2 - \theta - \theta + 1 = 0$$

$$m^2 - 2m + 1 = 0$$

$$m = +1, +1$$

$$= (c_1 + c_2 z) e^{z^2}$$

$$= (c_1 + c_2 \log x) x$$

$$\textcircled{6} \quad x^2 \frac{d^3 y}{dx^3} + 2x \frac{d^2 y}{dx^2} - 2y \frac{dy}{dx} = 0$$

$$x^3 \frac{d^3 y}{dx^3} + 2x^2 \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} = 0.$$

$$\Rightarrow (x^3 D^3 + 2x^2 D^2 - 2x D) y = 0$$

$$\Rightarrow [D^3 - 3D^2 + 2D + 2D^2 - 2D - 2D] y = 0$$

$$\Rightarrow m^3 - 3m^2 - 2m = 0$$

$$\Rightarrow m(m^2 - m - 2) = 0$$

$$\Rightarrow m(m-2)(m+1) = 0$$

$$m = 0, 2, -1$$

$$\therefore y = c_1 e^{0z} + c_2 e^{2z} + c_3 e^{-z}$$

$$\boxed{y = c_1 + c_2 x^2 + c_3 x^{-1}}$$

$$\textcircled{7} \quad 4x^2y'' + 12xy' + 3y = 0$$

$$(4x^2D^2 + 12xD + 3)y = 0$$

$$\Rightarrow (4\theta^2 - 4\theta + 12\theta + 3)y = 0$$

$$\Rightarrow 4m^2 + 8m + 3 = 0$$

$$\boxed{\begin{array}{c} m = -\frac{8+16}{2} \\ m = -\frac{8-16}{2} \end{array}}$$

$$m = -\frac{1}{2}, -\frac{3}{2}$$

$$\therefore y = c_1 x^{-\frac{1}{2}} + c_2 x^{-\frac{3}{2}}$$

$$\textcircled{8} \quad x^2y'' + xy' - y = 0$$

$$[\theta^2 - \cancel{\theta} + \cancel{\theta} - 1] y = 0$$

$$m^2 - 1 = 0$$

$$m = \pm 1$$

$$y = c_1 e^x + c_2 e^{-x}$$

$$\boxed{y = c_1 x + c_2 x^{-1}}$$

✓

## # Orthogonal Trajectories :-

→ The family of curves which cuts every member of the given family orthogonally is called the orthogonal Trajectory of the given family.

$$\rightarrow f(x, y, c) = 0 \quad \text{--- } \textcircled{1}$$

Replace

$$\rightarrow F\left(x, y, \frac{dy}{dx}\right) = 0 \quad \text{--- } \textcircled{2}$$

replace  $\left(\frac{dy}{dx}\right)$  by  $\left(-\frac{dx}{dy}\right)$

$$\rightarrow g\left(x, y, -\frac{dx}{dy}\right) = 0 \quad \text{--- } \textcircled{3}$$

The sol<sup>n</sup> of eq<sup>n</sup> ③ is called the orthogonal trajectory of the given family.

Note:-

If eq<sup>n</sup> ② and eq<sup>n</sup> ③ are identical, then the curve is said to be self orthogonal.

Q. Find the Orthogonal Trajectory of family of lines passing through the origin—

i) family of lines

ii) family of Circles having the centre at the origin.

iii) family of Parabolas

iv) family of ellipses.

\* \* \* If fn given, then  
Steps —

find  $\frac{dy}{dx}$

replace  $\frac{dy}{dx}$  by  $-\frac{dx}{dy}$   
and integrate & solve.

that will represent  
orthogonal of that fn

$$\textcircled{1} \rightarrow y = mx \quad \text{--- } \textcircled{1}$$

$$\frac{dy}{dx} = m$$

$$y = x \frac{dy}{dx} \quad \text{--- } \textcircled{2}$$

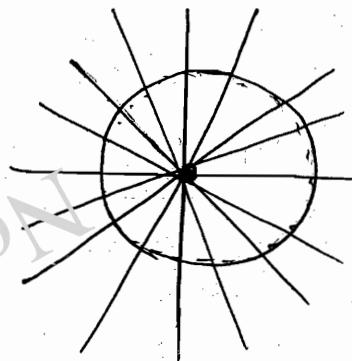
$$y = -x \frac{dx}{dy} \quad \text{--- } \textcircled{3}$$

$$\int y dy + x dx = 0$$

$$\frac{y^2}{2} + \frac{x^2}{2} = C$$

$$x^2 + y^2 = 2C$$

$$\boxed{x^2 + y^2 = r^2} \rightarrow \text{circle}$$



$$\textcircled{2} \quad y^2 = 4ax \quad \text{--- } \textcircled{1}$$

$$2y \frac{dy}{dx} = 4a$$

$$y^2 = 2y \frac{dy}{dx} x$$

~~$$2 \cancel{y} \frac{dy}{\cancel{y}} \Rightarrow \int \cancel{dx} \cancel{x}$$~~

$$2x \frac{dy}{dx} = y \quad \text{--- } \textcircled{2}$$

$$y = -2x \frac{dx}{dy} \quad \text{--- } \textcircled{3}$$

$$\int y dy = \int -2x dx$$

$$\frac{y^2}{2} = -2 \frac{x^2}{2} + C$$

$$\Rightarrow \boxed{\frac{x^2 + y^2}{2} = C} \rightarrow \text{Ellipse}$$

$$\rightarrow x^2 - y^2 = a^2 \quad \text{--- } \textcircled{1}$$

case

$$px - py \frac{dy}{dx} = 0$$

$$x + y \frac{dx}{dy} = 0$$

$$xdy + ydx = 0$$

$$\int d(xy) = \int 0$$

$$\boxed{xy = c}$$

Conclusion - Orthogonal Trajectories of family of -

circle  $\xrightarrow{\text{is}}$  st line.

st line  $\xrightarrow{\text{is}}$  circle

Parabola  $\xrightarrow{\text{is}}$  ellipse

Hyperbola  $\xrightarrow{\text{is}}$  Rectangular Hyperbola.

$$\textcircled{2} \quad y^2 = 4a(x+a) \quad \text{--- } \textcircled{1}$$

$$2yy_1 = 4a \cdot 1 \Rightarrow a = \frac{yy_1}{2}$$

$$y^2 = 2yy_1 \left( x + \frac{yy_1}{2} \right)$$

$$y = 2xy_1 + yy_1^2 \quad \text{--- } \textcircled{2}$$

replace  $y_1$  by  $-\frac{1}{y_1}$

$$y = -\frac{2x}{y_1} + \frac{y}{y_1^2}$$

$$yy_1^2 = -2xy_1 + y$$

$$yy_1^2 + 2xy_1 = y \quad \text{--- (3)}$$

$$\therefore \text{eq}^n (2) = \text{eq}^n (3)$$

$\therefore$  Curve is said to be self orthogonal.

#

$$\boxed{\frac{x^2}{a^2+\lambda^2} + \frac{y^2}{b^2+\lambda^2} = 1}$$

where  $\lambda$  is a parameter

Imp  
Result

is also self-Orthogonal.

#

$$x^2 + y^2 + 2gx + c = 0$$

Imp  
Concept

$$x^2 + y^2 + 2fy + c = 0$$

$$\text{O.T of } x^2 + y^2 + 2gx + c = 0 \text{ is } -$$

$$x^2 + y^2 - 2gy + 5 = 0$$

always take opposite coefficient  
sign

=

#

$$f(r, \theta, c) = 0 \quad \text{--- (1)}$$

$$F\left(r, \theta, \frac{dr}{d\theta}\right) = 0 \quad \text{--- (2)}$$

replace  $\frac{dr}{d\theta}$  by  $-r^2 \frac{d\theta}{dr}$

$$g\left(r, \theta, -r^2 \frac{d\theta}{dr}\right) = 0 \quad \text{--- (3)}$$

The sol<sup>n</sup> of eq<sup>n</sup> (3) is called the Orthogonal of the given family.

$$\textcircled{1} \quad r = a\theta$$

$$\log r = \log a + \log \theta$$

$$\frac{1}{r} \frac{dr}{d\theta} = 0 + \frac{1}{\theta}$$

$$\frac{1}{r} \times -r^2 \frac{d\theta}{dr} = \frac{1}{\theta}$$

$$\int \frac{dr}{r} = - \int \theta d\theta$$

$$\log r = -\frac{\theta^2}{2} + C$$

$$r = e^{-\frac{\theta^2}{2} + C}$$

$$r = e^{-\frac{\theta^2}{2}} \cdot e^C$$

$$\boxed{r = K e^{-\frac{\theta^2}{2}}}$$

$$\textcircled{2} \quad r = \frac{2a}{1-\cos\theta}$$

$$\log r = \log 2a - \log(1-\cos\theta)$$

$$\frac{1}{r} \frac{dr}{d\theta} = 0 - \frac{1}{1-\cos\theta} \sin\theta$$

$$\frac{1}{r} \times -r^2 \frac{d\theta}{dr} = -\frac{\sin\theta}{1-\cos\theta}$$

$$r \frac{d\theta}{dr} = \frac{\sin\theta}{1-\cos\theta} \cancel{\sin\frac{\theta}{2}}$$

$$\Rightarrow \frac{dr}{r} = \frac{1-\cos\theta}{\sin\theta} d\theta$$

$$\cancel{\sin\frac{\theta}{2} \cos\frac{\theta}{2}}$$

$$\Rightarrow \int \frac{dr}{r} = \int \frac{-2 \sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} d\theta$$

$$\Rightarrow \log r = -2 \log \cos\frac{\theta}{2} + \log C$$

$$\Rightarrow \log r = \log \frac{c}{\cos^2 \frac{\theta}{2}}$$

$$\Rightarrow r = \frac{c}{\cos^2 \frac{\theta}{2}}$$

$$\Rightarrow r = \frac{c}{\frac{1 + \cos \theta}{2}}$$

$$\Rightarrow r = \boxed{r = \frac{2c}{1 + \cos \theta}}$$

i.e.,  $\boxed{r = \frac{2a}{1 - \cos \theta}}$

Orthogonal

$$\leftrightarrow \boxed{r = \frac{2c}{1 + \cos \theta}}$$

③  $r^n = a^n \sin n\theta$

$$n \log r = n \log a + \log \sin n\theta.$$

$$\frac{n}{r} \frac{dr}{d\theta} = n \frac{\cos n\theta}{\sin n\theta}$$

$$\frac{n}{r} - r^2 \frac{d\theta}{dr} = n \frac{\cos n\theta}{\sin n\theta}$$

$$-nr \frac{d\theta}{dr} = n \frac{\cos n\theta}{\sin n\theta}$$

$$\Rightarrow \frac{dr}{r} = \frac{1}{n} \int -n \frac{\sin n\theta}{\cos n\theta} d\theta$$

$$\log r = \frac{1}{n} \log \cos n\theta + \log c$$

$$n \log r = \log \cos n\theta + n \log c$$

$$\log r^n = \log \cos n\theta + \log c^n$$

$$\boxed{r^n = c^n \cos n\theta}$$

i.e.,  $\boxed{r^n = a^n \sin n\theta}$

Orthogonal

$$\leftrightarrow \boxed{r^n = c^n \cos n\theta}$$

- Q. The diff. eq<sup>n</sup> of family of curves is given by  $\frac{dy}{dx} = \frac{x}{y}$   
 then the system of orthogonal Trajectories of the given family  
 of curves -

$$\rightarrow \frac{dy}{dx} = \frac{x}{y}$$

$$-\frac{dx}{dy} = \frac{y}{x}$$

$$-\frac{dx}{x} = \frac{dy}{y}$$

$$-\log x = \log y + \log c$$

$$\Rightarrow \boxed{xy = c}$$

This is the Orthogonal Trajectory of family  $\frac{dy}{dx} = \frac{x}{y}$  -

## # Differential Eq<sup>n</sup> of First Order but not first Degree:-

- A differential eq<sup>n</sup> of the form -

$$a_0 \left( \frac{dy}{dx} \right)^n + a_1 \left( \frac{dy}{dx} \right)^{n-1} + a_2 \left( \frac{dy}{dx} \right)^{n-2} + \dots + a_n = 0.$$

Where  $a_0, a_1, a_2, \dots, a_n$  are fun. of  $x$  and  $y$  or constants  
 is said to be differential eq<sup>n</sup> of 1<sup>st</sup> order but not first  
 degree.

$$\text{Let } \frac{dy}{dx} = P$$

$$\boxed{a_0 P^n + a_1 P^{n-1} + a_2 P^{n-2} + \dots + a_n = 0}$$

$$f(x, y, P) = 0$$

- This eq<sup>n</sup> containing 3 variables  $x, y$  and  $P$ , therefore it  
 may contain 3 types of solutions.

- It may be sol

solvable for  $y$ , may be solvable for  $x$ .

\* The eqn solvable for P -

→ If it is solvable for P, then the eqn must be factorized.

$$\rightarrow [P - f_1(x, y)] [P - f_2(x, y)] [P - f_3(x, y)] \dots [P - f_n(x, y)] = 0$$

then  $P - f_1(x, y) = 0$

replace by  $\frac{dy}{dx}$

$$\boxed{\frac{dy}{dx} = f_1(x, y)}$$

ex:- ①  $P^2 - 5P + 6 = 0$  where  $P = \frac{dy}{dx}$

$$(P-3)(P-2) = 0$$

$$P-2 = 0$$

$$\frac{dy}{dx} - 2 = 0$$

$$\int dy = \int 2 dx = \int 0$$

$$y - 2x - C_1 = 0 \quad \text{--- } ①$$

$$y - 3x - C_2 = 0 \quad \text{--- } ②$$

$$\Rightarrow \boxed{(y - 2x - C_1)(y - 3x - C_2) = 0}$$

②  $x^2 P^2 + xy P - 6y^2 = 0$

$$\Rightarrow x^2 P^2 + 3xyP - 2xyP - 6y^2 = 0$$

$$\Rightarrow xP(xP+3y) - 2y(xP+3y) = 0$$

$$\Rightarrow (xP+3y)(xP-2y) = 0$$

$$xP + 3y = 0$$

$$x \frac{dy}{dx} + 3y = 0$$

$$\frac{dy}{dx} = \frac{3}{x} y = 0$$

$$y = P \int P dx - \int e^{\int P dx} + C$$

$$y(x^3) - C_1 = 0 \quad \text{--- } ①$$

$$x \frac{dy}{dx} - 2y = 0.$$

$$\frac{dy}{dx} - \frac{2}{x} y = 0.$$

$$y\left(\frac{1}{x^2}\right) - C_2 = 0 \quad \text{--- } ②.$$

Sol<sup>n</sup>  
obtained

$$\boxed{(x^3y - C_1)(y - C_2x^2) = 0}$$

$$③ xyP^2 - (x^2 + y^2)P + xy = 0.$$

$$xyP^2 - x^2P - y^2P + xy = 0$$

$$xP(yP - x) - y(yP - x) = 0$$

$$(xP - y)(yP - x) = 0.$$

$$xP - y = 0.$$

$$x \frac{dy}{dx} - y = 0$$

$$\frac{dy}{dx} - \frac{1}{x}y = 0.$$

$$y\left(\frac{1}{x}\right) - C_2 = 0 \quad \text{--- } ①$$

$$yP - x = 0.$$

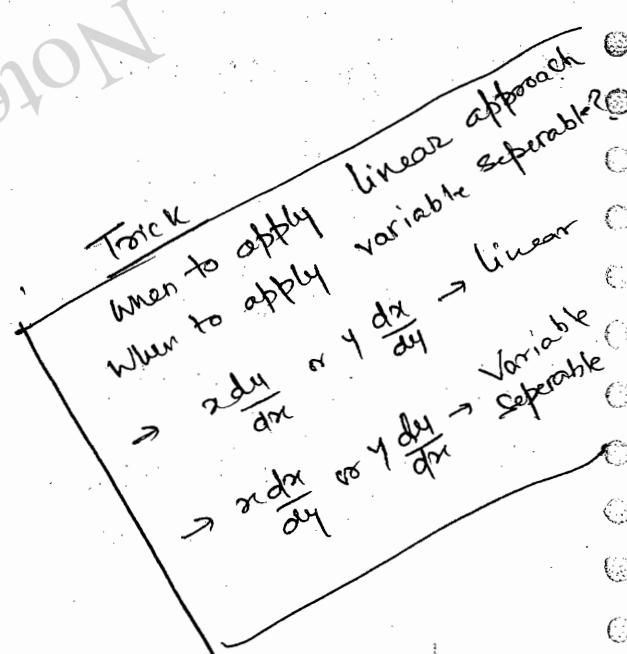
$$y \frac{dy}{dx} - x = 0$$

$$y dy - x dx = 0$$

$$\frac{y^2}{2} - \frac{x^2}{2} = C_2$$

$$\Rightarrow y^2 - x^2 - 2C_2 = 0 \quad \text{--- } ②$$

$$\therefore \boxed{(y - C_1)(y^2 - x^2 - 2C_2) = 0}$$



\* When factorization is not possible, then solvable for P is not possible  
two case arises —

When solvable for y ?

① The degree of y must be 1.

② Separate y from x & P

$$y = f(x, P)$$

③ diff. both side wrt 'x'

④ Replace  $\frac{dy}{dx}$  by P

⑤ neglect the terms or  
cancel the terms which  
does not contain  $\frac{dP}{dx}$  on  
both sides.

When  
solvable for x ?

① The degree of x must be 1.

② Separate x from y & P

$$x = f(y, P)$$

③ Diff. both side wrt 'y'.

④ Replace  $\frac{dx}{dy}$  by  $\frac{1}{P}$ .

⑤ Neglect the term or  
cancel the term which  
does not contain  $\frac{dP}{dy}$  on  
both sides.

Ex:-  $y + Px = P^2 x^4$  — ①

∴ degree of y = 1  $\rightarrow$  solvable for y ✓

∴ all coeff are 1, 1, 1  $\Rightarrow$  means factorization not possible  $\Rightarrow$  not solvable for P X

$$y = P^2 x^4 - Px$$

$$\frac{dy}{dx} = P^2(4x^3) + P^4 \left( 2P \frac{dP}{dx} \right) - P(1) - x \left( \frac{dP}{dx} \right)$$

$$P - 4P^2 x^3 + P = 2P \cdot (2Px^4 - x) \frac{dP}{dx}$$

$$2P(1 - 2Px^3) = -x(1 - 2Px^3) \frac{dP}{dx}$$

$$2P = \dots$$

$$\Rightarrow \int \frac{dp}{P} = - \int \frac{dx}{x}$$

$$\Rightarrow \log P = \log \frac{C}{x^2}$$

$$P = \frac{C}{x^2}$$

2 methods now to get sol<sup>n</sup> of diff eq<sup>n</sup> in Q.

either put  $P = \frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{C}{x^2}$$

and integrate and solve.

$$dy - \frac{C dx}{x^2} = 0$$

$$\boxed{y + \frac{C}{x} = C}$$

sol<sup>n</sup>

Put in eq<sup>n</sup> of Question.

$$\therefore y + \frac{C}{x} = x^4$$

$$\frac{C}{x^2} = (x^2)^2$$

$$\boxed{y + \frac{C}{x} = x^4}$$

sol<sup>n</sup> w

Aw

Q)  $y = 2px + y^2 p^3$

$$2px = y - y^2 p^3$$

$$x = \frac{y}{2p} - \frac{y^2 p^3}{2p}$$

$$\frac{dx}{dy}$$

$$2x = \frac{1}{p} \cdot y - y^2 p^2$$

$$2 \frac{dx}{dy} = \frac{1}{p} \cdot 1 + y \left( -\frac{1}{p^2} \frac{dp}{dy} \right) - 2y p^2 - y^2 \left( 2p \frac{dp}{dy} \right)$$

$$2 \cdot \frac{1}{p} = \frac{1}{p} - \frac{y}{p^2} \frac{dp}{dy} - 2y p^2 - 2p y^2 \frac{dp}{dy}$$

$$\frac{2}{p} - \frac{1}{p} + 2y p^2 = \left( -\frac{y}{p^2} - 2p y^2 \right) \frac{dp}{dy}$$

$$\frac{(1 + 2y p^3)}{p} = -y \left( \frac{1 + 2y p^3}{p^2} \right) \frac{dp}{dy}$$

$\frac{dy}{P}$

$$I = -\frac{y}{P} \frac{dP}{dy}$$

$$\int \frac{dP}{P} = \int -\frac{dy}{y}$$

$$\log P = \log \frac{C}{y}$$

$$\boxed{P = \frac{C}{y}} \quad \text{Put in Q.}$$

$$\therefore y = 2x - \frac{C}{y} + y^2 \frac{C^3}{y^3}$$

$$\boxed{y^2 = 2cx + c^3}$$

Shortcut

$$\boxed{y = Px + \phi(P)}$$

replace P by c

$$\boxed{y = cx + \phi(c)}$$

ex:- ①  $y = Px - P^2$

gen soln =  $\boxed{y = cx - c^2}$

②  $\log (Px - y) = P$

$$\therefore Px - y = e^P$$

$$y = Px - e^P$$

$$\boxed{y = cx - e^c}$$

## Numerical Methods

### I: Sol<sup>n</sup> of non-linear equation :-

Let  $f(x) = 0$  be any non linear equation to find the approximate root of  $f(x) = 0$  choose any two values of  $x = a, b$ , particularly adjacent values in such a way that their functional values must have opposite signs.

ex:-  $f(a) < 0, f(b) > 0$

Then, we can say that root lies betw<sup>n</sup> a and b.

This approximate root can be find by —

- ① method of bisection —
- ② Regular Falsi method
- ③ Newton - Raphson

#### \* Method of Bisection:-

$$x = a, b$$

$$f(a) < 0, f(b) > 0.$$

$$\frac{f(a) \quad 0 \quad f(b)}{ }$$

$$x_1 = \frac{a+b}{2}$$

$$f(x_1) < 0$$

$$f(a), f(x_1), f(b)$$

<      <      >

$$x_2 = \frac{x_1 + b}{2}$$

$$f(x_2) > 0$$

$x_1$

$$x_3 = \frac{x_1 + x_2}{2}$$

upto  $n$ th  $n+1$ th etc

non linear means ?

linear  $\rightarrow$  degree 1.

non-linear  $\rightarrow$  degree  $\neq 1$

The rate of convergence in bisection method is very - very slow.

→ Comparatively, the Bisection method, Regula - Falsi method has a little fast & rate of convergence.

### \* Regula - Falsi Method :-

$$x_1 = \frac{a f(b) - b f(a)}{f(b) - f(a)}$$

### \* Newton - Raphson method :-

Imp  
1-marks

→ has a very fast rate of Convergence.

→ and the convergence of Newton Raphson method is 2<sup>nd</sup> order convergence or Quadratic convergence.

$$x_1 = \frac{x_0 - f(x_0)}{f'(x_0)}$$

initial guess (99% chance given for  $x_0$ )  
if chance not given

then how to get  $x_0$

$$\begin{cases} \text{take } f(1) \rightarrow x_0 \\ f(2) \rightarrow x_0 \end{cases}$$

whatever is nearer to origin that value will be taken.

when  $f(1)$  and  $f(2)$  must have opposite signs.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\textcircled{1} \quad f(x) = x^3 + 2x - 5 = 0$$

$x_0 = 1$  find  $x_1$ :

$$\rightarrow f'(x) = 3x^2 + 2$$

$$x_1 = 1 - \frac{x^3 + 2x - 5}{3x^2 + 2}$$

$$= 1 - \left[ \frac{1+2-5}{5} \right]$$

$$= 1 + \frac{2}{5}$$

$$x_1 = 1.4 \underline{\text{Ans}}$$

$$\textcircled{2} \quad f(x) = x^3 + 3x - 7 = 0$$

$$x_0 = 1.$$

$$x_1 = ?.$$

$$\rightarrow x_1 = 1 - \frac{x^3 + 3 - 7}{3 \cdot 1 + 3}$$

$$= 1 - \frac{-8}{6}$$

$$= 1 + \frac{1}{2}$$

$$= 1.5 \underline{\text{Ans}}$$

$$\textcircled{3} \quad f(x) = x^3 - 2x - 5 = 0.$$

$$f(0) = -5 < 0$$

$$f(1) = 1 - 2 - 5 = -6 < 0$$

$$f(2) = 8 - 4 - 5 = -1 < 0 \rightarrow x_0 = 2 \text{ is chosen}$$

$$f(3) = 27 - 6 - 5 = 16 > 0.$$

$$x_0 = 2$$

$$x_1 = 2 - \frac{(-1)}{3(2^2) - 2} = 2 + \frac{1}{10} = 2.1$$

A

$$f(x) = e^x - 1 = 0$$

$$x_0 = -1$$

$$x_1 = ?$$

@ 0.71828  
@ 0.34567  
@ 0.12326  
@ 0.00000

$$\rightarrow f'(x) = e^x$$

$$x_1 = -1 - \frac{e^{-1} - 1}{e^{-1}}$$

$$= -1 - 1 + e$$

$$= e - 2$$

$$= 2.71 - 2$$

$$= \underline{\underline{0.71}}$$

⑤  $f(x) = x^2 - 2x - 1$

$$x_0 = 2$$

$$x_2 = \underline{\underline{\quad}}$$

@ 2.425    ⑥ 2.423    ⑦ 2.419    ⑧ 2.417

$$\rightarrow f'(x) = 2x - 2$$

$$x_1 = 2 - \frac{-1}{2} = 2 + 0.5 = 2.5$$

$$x_2 = 2.5 - \frac{6.25 - 5 - 1}{5 - 2}$$

$$= 2.5 - \frac{0.25}{3}$$

$$= 2.5 - 0.083$$

$$\Rightarrow \underline{\underline{2.417}}$$

$$= \underline{\underline{2.417}}$$

Q.  $f(x) = x^4 - 3x + 1 = 0$

$x_0 = 0$

$x_1 = \underline{\quad}$

- Ⓐ  $-\frac{1}{3}$  Ⓑ  $\frac{1}{3}$  Ⓒ 3 Ⓓ -3

$\rightarrow f'(x) = 4x^3 - 3$

$x_1 = 0 - \frac{1}{-3}$

$= \frac{1}{3} \underline{Am}$

Q: The approximate root of eq<sup>n</sup>.  $f(x) = (x-1)^2 + x - 3 = 0$  is to be solved by the Newton Raphson method. The method will fail in the very first iteration if the initial guess  $x_0 = \underline{\quad}$

- Ⓐ 0 Ⓑ 0.5 Ⓒ 1.0 Ⓓ 1.5

$\rightarrow f(x) = x^2 - 2x + 1 + x - 3$   
 $= x^2 - x - 2$

$f'(x) = 2x - 1$

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

$= x_0 - \frac{x_0^2 - x_0 - 2}{2x_0 - 1} \xrightarrow{0} \text{method will fail.}$

i.e. method will fail if  $2x_0 - 1 = 0$

$\Rightarrow x_0 = \underline{0.5}$

Q The Newton Raphson iteration  $x_{n+1} = \frac{1}{2} \left[ x_n + \frac{R}{x_n} \right]$  can be used to compute the

Imp Concept

compute the

a) sq. root of R

b) sq. of R

c) logarithm of R

d) Reciprocal of R.

$$\rightarrow x_{n+1} = x_n = \alpha.$$

$$\therefore \alpha = \frac{1}{2} \left[ \alpha + \frac{R}{\alpha} \right].$$

$$\Rightarrow 2\alpha - \alpha = \frac{R}{\alpha}$$

$$\Rightarrow \alpha^2 = R$$

$$\Rightarrow \alpha = \sqrt{R}$$

↳ used to compute sq. root of no.

Eg:-  $x_{n+1} = \frac{1}{2} \left[ x_n + \frac{117}{x_n} \right]$  converges to  $\sqrt{117}$ .

Q  $f(x) = x^3 - x^2 + 4x - 4 = 0$

$$x_0 = 2$$

$$x_1 = \underline{\hspace{2cm}}$$

or  $\frac{2}{3}, 4/3, 1, 4, \frac{3}{2}$ .

$$\rightarrow f'(x) = 3x^2 - 2x + 4$$

$$x_1 = 2 - \left[ \frac{\frac{8}{3}}{\frac{12}{3}} \right]$$

$$= \frac{4}{3}$$

=

Q. The recursion relation  $x = e^{-x}$  is to be solved by Newton-Raphson method, then which of the following is true?

a)  $x_{n+1} = e^{-x_n}$

b)  $x_{n+1} = x_n - e^{-x_n}$

c)  $x_{n+1} = \frac{(1+x_n)e^{-x_n}}{(1+e^{-x_n})}$

d)  $x_{n+1} = \frac{x_n^2 - e^{-x_n}(1+x_n) - 1}{x_n - e^{-x_n}}$

$\rightarrow f(x) = x - e^{-x}$

$f'(x) = 1 + e^{-x}$

$f'(x_n) = 1 + e^{-x_n} \rightarrow$  This will come into denominator.

$\therefore$  option c)

Q. Consider the series —

$$x_{n+1} = \frac{x_n}{2} + \frac{9}{8x_n}$$

$x_0 = 0.5$  is obtained from the Newton-Raphson method.

The series converges to —

$\rightarrow x_{n+1} = x_n = \alpha$  (Take it always when given Series Converges)

$$\Rightarrow \alpha = \frac{\alpha}{2} + \frac{9}{8\alpha}$$

$$\Rightarrow \alpha = \frac{4\alpha^2 + 9}{8\alpha}$$

$$\Rightarrow 4\alpha^2 = 9$$

$$\Rightarrow \alpha^2 = \frac{9}{4}$$

$$\Rightarrow \alpha = \sqrt{\frac{9}{4}} = \frac{3}{2} = 1.5 \text{ Ans}$$

a. The sol<sup>n</sup> of the variables  $x_1$  and  $x_2$  for the equations -

$$U = 10x_2 \sin x_1 - 0.8 = 0 \quad \text{--- (1)}$$

$$V = 10x_2^2 - 10x_2 \cos x_1 - 0.6 = 0 \quad \text{--- (2)}$$

is to be solved by the Newton Raphson method.

Assuming the initial value,  $x_1 = 0.0$

Jacobian Matrix =  $\begin{bmatrix} \dots & x_2 = 1.0 \end{bmatrix}$

→ Jacobian Matrix - 
$$\begin{bmatrix} \frac{\partial U}{\partial x_1} & \frac{\partial U}{\partial x_2} \\ \frac{\partial V}{\partial x_1} & \frac{\partial V}{\partial x_2} \end{bmatrix}$$

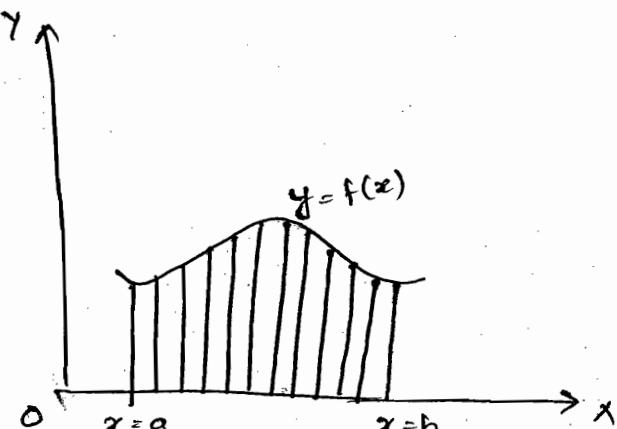
$$\frac{\partial U}{\partial x_1} = 10x_2 \cos x_1 = 10(1)(1) = 10$$

$$\frac{\partial U}{\partial x_2} = 10 \sin x_1 - 10(0) = 0.$$

$$\frac{\partial V}{\partial x_1} = 10x_2 \sin x_1 = 10(1)(0) = 0.$$

$$\frac{\partial V}{\partial x_2} = 20x_2 - 10 \cos x_1 = 20(1) - 10(1) = 10.$$

## # Numerical Integration:-



$$h = \frac{b-a}{n}$$

### ① Trapezoidal Rule:-

$$\int_a^b f(x) dx = \frac{h}{2} [(y_0 + y_n) + 2(y_1 + y_2 + y_3 + \dots + y_{n-1})]$$

### ② Simpson's $\frac{1}{3}$ rd Rule:-

$$\int_a^b f(x) dx = \frac{h}{3} [(y_0 + y_n) + 4(y_1 + y_3 + y_5 + \dots) + 2(y_2 + y_4 + y_6 + \dots)]$$

### ③ Simpson's $\frac{3}{8}$ th Rule:-

$$\int_a^b f(x) dx = \frac{3h}{8} [(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + y_7 + \dots) + 2(y_3 + y_6 + y_9 + \dots)]$$

Note:-

The accuracy of Trapezoidal rule or order of integration of Trapezoidal rule is  $O(h^2)$  → accuracy at 2<sup>nd</sup> place of decimal

The accuracy of Simpson's  $\frac{1}{4}$  rule =  $O(h^4)$  → accuracy upto 4<sup>th</sup> place of decimal

The accuracy of Simpson's  $\frac{3}{8}$ th rule is  $O(h^5)$  → accuracy upto 5<sup>th</sup> place of decimal

$$Q. \int_0^1 \frac{1}{1+x^2} dx$$

By Trapezoidal & Simpson's rule and dividing the interval  
into 4 equal parts and Derive the value of  $\pi$ .

$$\rightarrow \int_0^1 \frac{1}{1+x^2} dx$$

$$h = \frac{b-a}{n} = \frac{1-0}{4} = \frac{1}{4} = 0.25$$

$$y = \frac{1}{1+x^2}$$

$x$	0	0.25	0.50	0.75	1.
$y$	1.0000	0.9412	0.8000	0.6400	0.5000

$$\frac{1}{1+\frac{1}{16}}, \quad \frac{1}{1+\frac{4}{16}}, \quad \frac{1}{1+\frac{9}{16}}$$

$$\therefore \int_0^1 \frac{1}{1+x^2} dx = \frac{0.25}{2} [(1.0000 + 0.5000) + 2(0.9412 + 0.8000 + 0.6400)]$$

$$\left. \tan^{-1} x \right|_0^1 = 0.7828$$

$$\tan^{-1} 1 - \tan^{-1} 0 = 0.7828$$

$$\Rightarrow \frac{\pi}{4} = 0.7828$$

$$\Rightarrow \pi = 3.1312 \quad \text{we know } \pi = 3.1415$$

difference occurred at ~~4th~~ place  
grd

Simpson  $\frac{1}{3}$ rd Rule —

$$\int_0^1 \frac{1}{1+x^2} dx = \frac{0.25}{3} \left[ (1.0000 + 0.5000) + 4(0.9412 + 0.6400) + 2(0.8600) \right]$$

$$\tan^{-1} 1 = 0.7854$$

$$\Rightarrow \tan^{-1} - \cancel{\tan^0} = 0.7854$$

$$\Rightarrow \frac{\pi}{4} = 0.7854$$

$$\Rightarrow \pi = 0.7854 \times 4$$

$$\pi = 3.14 \cancel{16} \quad \text{we know } \pi = 3.14 \cancel{15}$$

diff occur at 4<sup>th</sup> place.

By Shortcut  
method.

Q.  $\int_0^6 \frac{1}{1+x^2} dx$  by Trapezoidal rule by dividing the interval into 6 equal parts.

a) 1.4132 b) 1.3756 c) 1.2326 d) 1.0986.

→ ~~tan~~

$$\int_0^6 \frac{1}{1+x^2} dx$$

$$= \tan^{-1} 6 - \cancel{\tan^0}$$

$\therefore = 1.40 \dots$

Compare with option → diff will be at 2<sup>nd</sup> place.

∴ option a) is correct.

Q.  $\int_1^3 \log_e x dx$ ,  $n=2$  = \_\_\_\_\_ By Simpson's  $\frac{1}{4}$  th rule.

- (A) 0.50
- (B) 0.80
- (C) 1.00
- (D) 1.29

→ Directly integrate and Compare the value at 4<sup>th</sup> place will differ.

Q.  $\int_0^{\pi/4} \frac{\sin x}{\cos^3 x} dx$ ,  $n=4$ , By Simpson's  $\frac{3}{8}$  rule.

- (A) 0.3883
- (B) 0.4344
- (C) 0.5026
- (D) 0.5589

$$\int_0^{\pi/4} \frac{\sin x}{\cos^3 x} dx = \int_0^{\pi/4} \frac{\sin x}{\cos x} \cdot \frac{1}{\cos^2 x} dx = \int_0^{\pi/4} \tan x \cdot \sec^2 x dx.$$

$$= \left( \frac{\tan^2 x}{2} \right)_0^{\pi/4} = 0.5000.$$

Q.  $\int_1^3 \frac{1}{x} dx$ ,  $n=2$ , By Simpson's  $\frac{1}{3}$  rd rule.

a) 1.000

b) 1.098

c) 1.111

d) 1.120

{ Comparison ⇒ Use only Numeric method  
Do not ever integrate or Shortcut method

$$y = \frac{1}{x}$$

x	1	2	3
y	1	0.50	0.333

$$\int_1^3 \frac{1}{x} dx = \frac{1}{3} [1.333 + 4(0.5)] \\ = \frac{3.333}{3} = 1.111,$$

Q. Torque exerted on a flywheel over a cycle is listed in the table —

Angle ( $^{\circ}$ )	$0^{\circ}$	$60^{\circ}$	$120^{\circ}$	$180^{\circ}$	$240^{\circ}$	$300^{\circ}$	$360^{\circ}$
Torque (Nm)	0	1066	323	0	-323	-355	0

Estimate the value of the fn by Simpson  $\frac{1}{3rd}$  rule.

- ② 542     ③ 993    ④ 1444    ⑤ 1986

$$\rightarrow \int_0^{2\pi} f(x) dx$$

$$= \frac{60^{\circ}}{3} \left[ (0+0) + 4(1066 + 0 - 355) + 2(323 - 323) \right]$$

$$= \frac{\pi}{3} \times 4(711)$$

$$= \frac{\pi}{9} \times 2844$$

$$= 992.7$$

$$\approx 993$$

## # Numerical Sol" of Ordinary Differential Eqn:-

$$\frac{dy}{dx} = f(x, y), \quad x = x_0, \quad y = y_0, \quad y(x_1) = \underline{\hspace{2cm}}$$

Methods -

- ① Taylor's series method
- ② Picard's method
- ③ Runge - Kutta method.

Taylor Series -

$$y = y_0 + (x-x_0)(y')_0 + \frac{(x-x_0)^2}{2!}(y'')_0 + \frac{(x-x_0)^3}{3!}(y''')_0 + \dots \infty.$$

Q.  $\frac{dy}{dx} = -xy, \quad y(0) = 1, \quad y(0.1) = \underline{\hspace{2cm}}$

$$\begin{aligned} x+y &= 1 \\ y &= 1-x \\ &\approx 1-0.1 \\ &= 0.99 \\ &\approx 0.995 \end{aligned}$$

→

$$y' = -xy$$

$$y'' = -x^2 y' - y \cdot 1$$

$$y''' = -x^3 y'' - 2x^2 y'$$

$$y^{(4)} = -x^4 y''' - 3x^3 y''$$

$$y = 1 + x(0) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(0) + \frac{x^4}{4!}(3) + \dots$$

$$= 1 - \frac{x^2}{2} + \frac{x^4}{8} \quad \xrightarrow{\text{Consider only upto 4th order rest can be neglected}}$$

$$y(0.1) = 1 - \frac{0.01}{2} + \frac{0.0001}{8} \quad \cancel{\text{neglected}}$$

$$= 1 - 0.005$$

$$= 0.995 \quad \underline{\text{Ans}}$$

Q) Picard's method:-

$$\frac{dy}{dx} = f(x, y), \quad x = x_0, \quad y = y_0.$$

$$\Rightarrow \int_{y_0}^y dy = \int_{x_0}^x f(x, y) dx.$$

$$\Rightarrow y - y_0 = \int_{x_0}^x f(x, y) dx$$

$$\Rightarrow y = y_0 + \int_{x_0}^x f(x, y) dx.$$

$$y_1 = y_0 + \int_{x_0}^x f(x, y_0) dx$$

$$y_2 = y_0 + \int_{x_0}^x f(x, y_1) dx$$

$$y_3 = y_0 + \int_{x_0}^x f(x, y_2) dx.$$

-----

$$y = ( ) + \dots$$

Q.  $\frac{dy}{dx} = -xy, \quad y(0) = 1, \quad y(0.1) = \underline{\hspace{2cm}}$

$$y_1 = 1 + \int_0^x -x y_0 dx$$

$$y_1 = \left( 1 - \frac{x^2}{2} \right)$$

$$y_2 = 1 + \int_0^x -x \left( 1 - \frac{x^2}{2} \right) dx$$

$$y_2 = 1 - \frac{x^2}{2} + \frac{x^4}{8}$$

$$\therefore y = 1 - \frac{x^2}{2} + \frac{x^4}{8} + \dots$$

Q.  $\frac{dy}{dx} = \frac{x^2}{1+y^2}$ ,  $y(0)=0$ ,  $y(0.25) = \underline{\hspace{2cm}}$

$$\rightarrow y_1 = 0 + \int \frac{x^2}{1+0^2} dx = \frac{x^3}{3}$$

$$y_2 = 0 + \int \frac{x^2}{1+\left(\frac{x^3}{3}\right)^2} dx \\ = \tan^{-1}\left(\frac{x^3}{3}\right)$$

$$y_3 = 0 + \int \frac{x^2}{1+\left(\tan^{-1}\frac{x^3}{3}\right)^2} dx \rightarrow \int \text{ becomes difficult.}$$

When derivatives are convenient  
Taylor Series

When Taylor Ser.  $\int$  is convenient  
Picard's method

2. 
$$\begin{array}{|c|} \hline y = \tan^{-1} \frac{x^3}{3} \\ \hline \end{array}$$

$$\boxed{y(0.25) = \tan^{-1} \left(\frac{0.25}{3}\right)^3}$$

Take first one as  
approx soln.

Q.  $\frac{dy}{dx} = -xy$ ,  $y(0)=1$ , &  $y(0.1)=\underline{\hspace{2cm}}$

$$\frac{dy}{dx} + xy = 0$$

$$I.F = e^{\int P dx} = e^{\int x dx} = e^{\frac{x^2}{2}}$$

$$y \cdot e^{\frac{x^2}{2}} = C$$

$$\therefore e^0 = C \Rightarrow C = 1$$

$$y e^{\frac{x^2}{2}} = 1$$

$$y = \frac{1}{e^{\frac{x^2}{2}}} = e^{-\frac{x^2}{2}} = e^{-\frac{(0.1)^2}{2}} = e^{-\frac{0.01}{2}} = e^{-0.005} \\ = 0.995$$

## # Runge-Kutta Method:-

$$y = y_0 + \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4)$$

where  $k_1 = h \cdot f(x_0, y_0)$   $\rightarrow$  where  $h = x_1 - x_0$ .

$$k_2 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$k_4 = h f(x_0 + h, y_0 + k_3)$$

Q. ①  $\frac{dy}{dx} = f(x, y)$   $y(0) = 1$ ,  $y(0.1) = ?$

$$h = x_1 - x_0 = 0.1 - 0 = 0.1$$

$$k_1 = h f(x_0, y_0) = 0.1(0+1) = 0.1$$

$$k_2 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.1 \left[0.05 + 1.05\right] = 0.11$$

$$k_3 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.1 \left[0.05 + 1.055\right] = 0.1105$$

$$k_4 = h f(x_0 + h, y_0 + k_3) = 0.1 [0.1 + 1.105] = 0.12105$$

Q Match the following —

E: Newton-Raphson  $\rightarrow$  Soln of non linear eqn

F: Runge-Kutta  $\rightarrow$  Soln of linear algebraic

G: Simpson rule  $\rightarrow$  Soln of O.D. eqn.

H: Gauss Elimination  $\rightarrow$  Soln of Linear Algebraic eqns  
Numeric Integration.

## Complex Variable

①  $z = x + iy$

$\bar{z} = x - iy$

$z \cdot \bar{z} = |z|^2 = x^2 + y^2$

②  $x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$  is called the modulus-Amplitude form  
or  
Polar form

$$r = |z| = \sqrt{x^2 + y^2}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

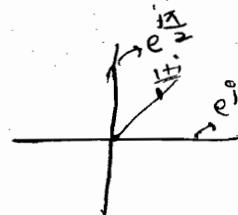
③  $\cos\theta + i\sin\theta = e^{i\theta}$

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

$$1 = \cos 0^\circ + i\sin 0^\circ = e^{i0}$$

$$i = \cos \frac{\pi}{2} + i\sin \frac{\pi}{2} = e^{i\pi/2}$$

$$1+i = \sqrt{2} \left[ \cos \frac{\pi}{4} + i\sin \frac{\pi}{4} \right] = \sqrt{2} e^{i\pi/4}$$



Q. (i)<sup>i</sup> = \_\_\_\_\_ where  $i = \sqrt{-1}$

(x)<sup>x</sup> = \_\_\_\_\_ where  $x = \sqrt{-1}$

$$(i)^i = (e^{i\pi/2})^i = e^{-\frac{\pi}{2}}$$

Q. ①  $(1+i)(2-5i) = _____$

②  $\left| \frac{3+4i}{1-2i} \right| = _____$

③  $\frac{i+3}{i+1} = _____$

④  $\frac{1+2i}{i-2} = _____$

⑤  $\frac{-5+i10}{3+i4} = _____$

$$\textcircled{1} \quad (1+i)(2-5i)$$

$$= 2+2i-5i-5i^2$$

$\downarrow +5$

$$= 7-3i$$

$$\textcircled{2} \quad \left| \frac{3+4i}{1-2i} \right| = \frac{\sqrt{9+16}}{\sqrt{1+4}} = \sqrt{\frac{25}{5}} = \sqrt{5}$$

$$\textcircled{3} \quad \frac{i+3}{i+1} \times \frac{i-1}{i-1}$$

$$= \frac{i^2 + 3i - i - 3}{-1 - 1}$$

$$= \frac{-4 + 2i}{-2}$$

$$= 2 - i$$

$$\textcircled{4} \quad \frac{1+2i}{i-2} = ?$$

$$= \frac{1+2i}{i-2} \times \frac{i+2}{i+2}$$

$$= \frac{i+2i+2+4i}{-1-4} = -i$$

$$\textcircled{5} \quad \frac{-5+i10}{3+i4}$$

$$= \frac{-5+i10}{3+i4} \times \frac{3-i4}{3-i4}$$

$$= \frac{-15+30i+20i-40i^2}{9+16}$$

$$= \frac{-15+50i+40}{25} = \frac{1+2i}{5}$$

Q. The value of  $(1+i)^8 = \underline{\hspace{2cm}}$

- Ⓐ 2 Ⓑ 4 Ⓒ 8 Ⓓ = 16.

$$\begin{aligned} &\rightarrow (1+i)^8 \\ &= [(1+i)^2]^4 \\ &= [1+2i]^4 \\ &= 16i^4 \\ &= 16 \underline{\text{Ans}} \end{aligned}$$

Q. The Polar form of  $2+2i$  is  $\underline{\hspace{2cm}}$

$$r = |z| = \sqrt{4+4} = 2\sqrt{2}.$$

$$\theta = \tan^{-1} 1 = \frac{\pi}{4}$$

$$\therefore 2+2i = 2\sqrt{2} e^{i\frac{\pi}{4}}$$

## # Analytic function:-

A single valued fn which is defined and differentiable at each point of domain  $D$  is said to be an analytic fn in that domain.

### Note:-

The necessary and sufficient cond' for a function

$f(z) = u(x, y) + i v(x, y)$  to be analytic is it should

satisfy the Cauchy-Riemann eqn.

*(Cond' to be analytic)*

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{cases} \quad \text{or} \quad \begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$$

a.  $f(z) = (x^2 - y^2) + i(2xy)$

$$u = x^2 - y^2, v = 2xy.$$

$$\begin{array}{ll} \frac{\partial u}{\partial x} = 2x & \frac{\partial v}{\partial y} = 2x \\ \frac{\partial u}{\partial y} = -2y & \frac{\partial v}{\partial x} = 2y \end{array}$$

Satisfying C.R. eqn

$\therefore$  Analytic in nature

## # Harmonic function:-

— Any function of  $x$  and  $y$  satisfies the Laplace's eq<sup>n</sup>  
is said to be an harmonic function.

↓ represented by

$\nabla^2 u = 0$

Ex:- ①  $u = x^2 - y^2$

$$\begin{array}{l|l} \frac{\partial u}{\partial x} = 2x & \frac{\partial u}{\partial y} = -2y \\ \frac{\partial^2 u}{\partial x^2} = 2 & \frac{\partial^2 u}{\partial y^2} = -2 \end{array}$$

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 - 2 = 0 \quad \therefore \text{Harmonic} \checkmark$$

Note:-

Real and Imj part of an Analytic eq<sup>n</sup> satisfies the Laplacian Equation.

i.e.,  $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

# if  $f(z) = u(x, y) + i v(x, y)$  (Ques. If  $u(x, y)$ ) Real Part given

then  $f(z) = 2u\left(\frac{z}{2}, \frac{z}{2i}\right) - u(0, 0) + ci$

If Real part Given then asked directly  $f(z)$  then how to get

Q.  $u = x^2 - y^2$

$$u\left(\frac{z}{2}, \frac{z}{2i}\right) = \left(\frac{z}{2}\right)^2 - \left(\frac{z}{2i}\right)^2$$

$$= \frac{z^2}{4} + \frac{z^2}{4}$$

$$= \frac{z^2}{2}$$

$$u(0, 0) = 0 - 0 = 0.$$

$$f(z) = 2 \times \frac{z^2}{2} - 0 + ci = z^2 + ci$$

$$u = x^3 + 3xy^2 + 3x + 1.$$

$$\begin{aligned} u\left(\frac{z}{2}, \frac{z}{2}\right) &= \frac{z^3}{8} - \frac{3z}{2} \left(\frac{z^2}{4i^2}\right) + \frac{3z}{2} + 1 \\ &= \frac{z^3}{8} + \frac{3z^3}{8} + \frac{3z}{2} + 1 \\ &= \cancel{\frac{z^3}{8}} + \frac{3z^3}{8} + \frac{3z}{2} + 1 \end{aligned}$$

$$u(0,0) = 0 - 0 + 0 + 1 = 1.$$

$$f(z) = 2\left(\frac{z^3}{8} + \frac{3z}{2} + 1\right) - 1 + Ci$$

$$f(z) = z^3 + 3z + 1 + Ci$$



$$\begin{aligned} Q. \quad f(z) &= z^2 \\ &= (x+iy)^2 \\ &= x^2 + i^2y^2 + 2ixy, \\ &= x^2 - y^2 + 2ixy. \end{aligned}$$

If real part given, how to get Imag part

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$$

$$dv = -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy$$

which is exact.

$$\text{Let } u = x^2 - y^2$$

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial u}{\partial y} = -2y$$

$$dv = 2ydx + 2xdy$$

↳ exact  
no need to  
check

Gen. Soln —  
 $\star = \int 2y dx + \int 0 dy$

$$V = 2xy \text{ AM}$$

$$Q. \quad u = x^3 - 3xy^2 + 3x + 1$$

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 3$$

$$\frac{\partial u}{\partial y} = -6xy$$

$$dv = \int 6xy dx + \int (\cancel{3x^2 - 3y^2 + 3}) dy \\ =$$

$$Q. \quad u = e^x \cos y$$

$$\frac{\partial u}{\partial x} = e^x \cos y$$

$$\frac{\partial u}{\partial y} = -e^x \sin y.$$

$$dv = -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy$$

$$\int dv = \int e^x \sin y dx + \int \cancel{e^x \cos y} dy$$

$$\boxed{V = e^x \sin y.}$$

\* If imaginary part given and asked real part then how to get?

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy.$$

or

$$\boxed{du = \frac{\partial v}{\partial y} dx - \frac{\partial v}{\partial x} dy}$$

which is exact - .

ex:- ①  $v = -xy$

$$\frac{\partial v}{\partial x} = -y, \quad \frac{\partial v}{\partial y} = -x.$$

$$\int du = \int -x dx + \int y dy$$

$$u = -\frac{x^2}{2} + \frac{y^2}{2} = \frac{1}{2}(y^2 - x^2).$$

② Given  $v = y^3 - 3x^2y$ , find  $u$ . Where  $f(z)$  is harmonic

$$\frac{\partial v}{\partial x} = -6xy \quad \frac{\partial v}{\partial y} = 3y^2 - 3x^2$$

$$\int du = \int (3y^2 - 3x^2) dx + \int 6xy dy$$

$$\boxed{u = 3xy^2 - x^3}$$

## # Singular Points → or Pole

Singular points are those at which the given function  $f(z)$  is not analytic.

These points are also called Singularities or poles.  
find Singular Points.

Q. ①  $f(z) = \frac{1-2z}{z(z-1)(z-2)}$

$$z = 0, 1, 2.$$

Q.  $f(z) = \frac{z}{z^2-z-2}$

$$f(z) = \frac{z}{(z-2)(z+1)}$$

$$\Rightarrow z = 2, -1.$$

Q.  $f(z) = \frac{z}{z^4-1}$

$$z = \pm 1, \pm i$$

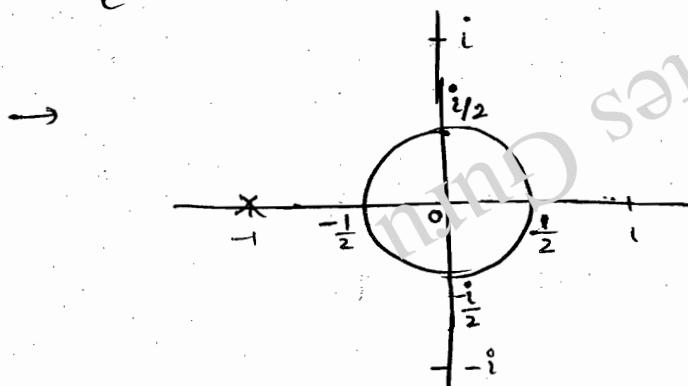
## # Cauchy's Theorem:-

If  $f(z)$  is an analytic fn and its derivative  $f'(z)$  is continuous at all points inside and on a simple and closed curve  $C$ , then -

$$\boxed{\oint_C f(z) dz = 0}$$

means  
if pole outside the limit, then  $\int = 0$

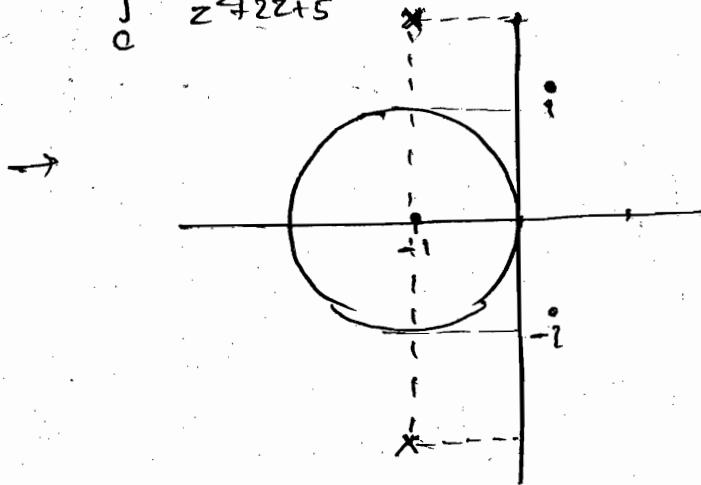
Q.  $\oint_C \frac{3z^2 + 7z + 1}{z+1} dz$ , where  $c$  is  $|z| = \frac{1}{2}$



$\therefore$  pole is outside curve.

$\therefore$  Ans is 0.

Q.  $\oint_C \frac{z+4}{z^2 + 2z + 5} dz$  where  $c$  is  $|z+1| = 1$

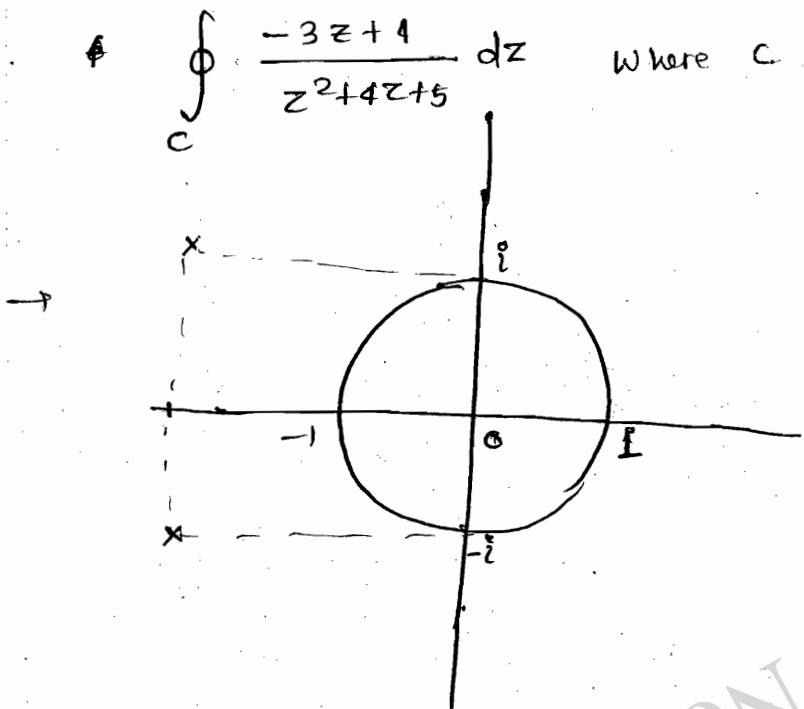


Pole  $\Rightarrow z^2 + 2z + 5 = 0$

$$\Rightarrow -2 \pm \sqrt{4-20} = \frac{-2 \pm 4i}{2} = -1 \pm 2i$$

Pole =  $-1 + 2i$

Q.  $\oint_C \frac{-3z+4}{z^2+4z+5} dz$  where C is  $|z|=1$



$$z^2 + 4z + 5 = 0$$

$$\frac{-4 \pm \sqrt{16 - 20}}{2}$$

$$= \frac{-4 \pm 2i}{2}$$

$$= -2 \pm i$$

Pole outside the curve  
∴ Answer is 0.

## # Cauchy's Integral formula

V.V Imp  
2 marks

If  $f(z)$  is analytic for inside and on a simple closed curve 'C' and if  $a$  is any point inside C then,

$$\oint_C \frac{f(z)}{z-a} dz = 2\pi i f(a)$$

$$\oint_C \frac{f(z)}{(z-a)^2} dz = 2\pi i f'(a)$$

$$\oint_C \frac{f(z)}{(z-a)^3} dz = \frac{2\pi i}{2!} f''(a)$$

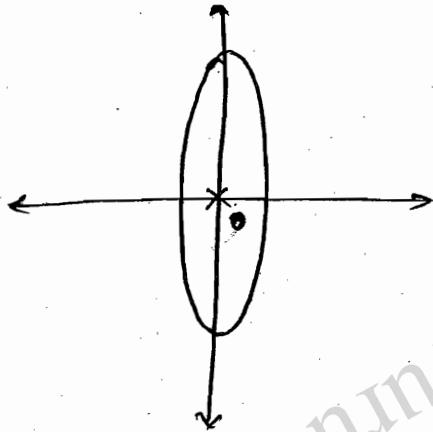
$\oint_C \frac{f(z)}{(z-a)^n} dz$	$= \frac{2\pi i}{(n-1)!} f^{(n-1)}(a)$
-----------------------------------	--

Q. ① The value of  $\oint_C \frac{\cos z}{z} dz$  where C is an ellipse

$$9x^2 + 4y^2 = 1$$

$$\rightarrow 9x^2 + 4y^2 = 1$$

$$\frac{x^2}{\left(\frac{1}{3}\right)^2} + \frac{y^2}{\left(\frac{1}{2}\right)^2} = 1$$



$\therefore$  Pole inside

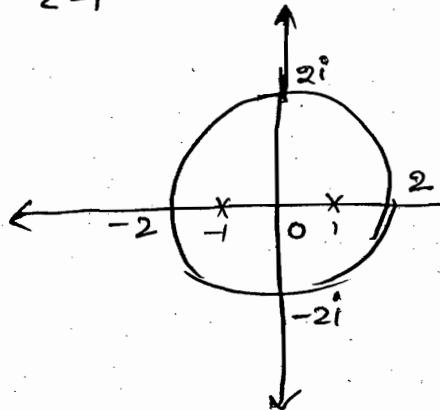
$\therefore$  Cauchy Integral formula applicable

$$\therefore \oint_C \frac{\cos z}{z} dz = 2\pi i \cos z \Big|_{z=0}$$

$$= 2\pi i \text{ Ans.}$$

Q.  $\oint_C \frac{1}{z^2-1} dz$  where C is the circle  $x^2+y^2=4$ .

$\rightarrow$



Both Poles inside

Partial fraction

Method-1

Partial fraction -

$$\oint_C \frac{1}{(z-1)(z+1)} dz = \frac{1}{2} \oint \left[ \frac{1}{z-1} - \frac{1}{z+1} \right] dz$$

$$= \frac{1}{2} \left( 2\pi i (1) - 2\pi i (-1) \right)$$

$$= 0.$$

Observation:-

Sometimes even also pole inside then  $\int = 0$

Method-2

Shortcut

$$\oint_C \frac{1}{z^2-1} dz = \oint_C \frac{1}{(z-1)(z+1)} dz$$

$$= \oint \frac{\left(\frac{1}{z+1}\right)}{(z-1)} + \frac{\left(\frac{1}{z-1}\right)}{(z+1)} dz$$

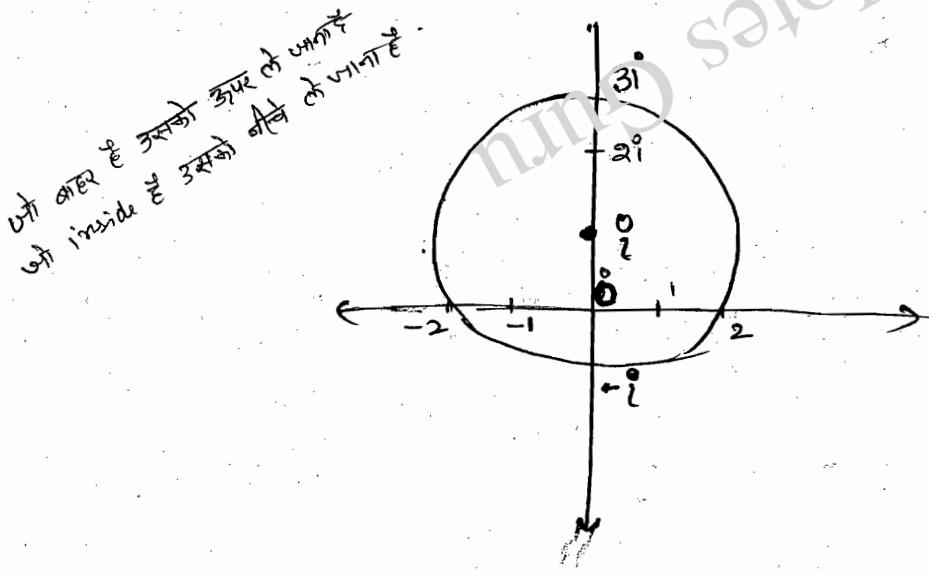
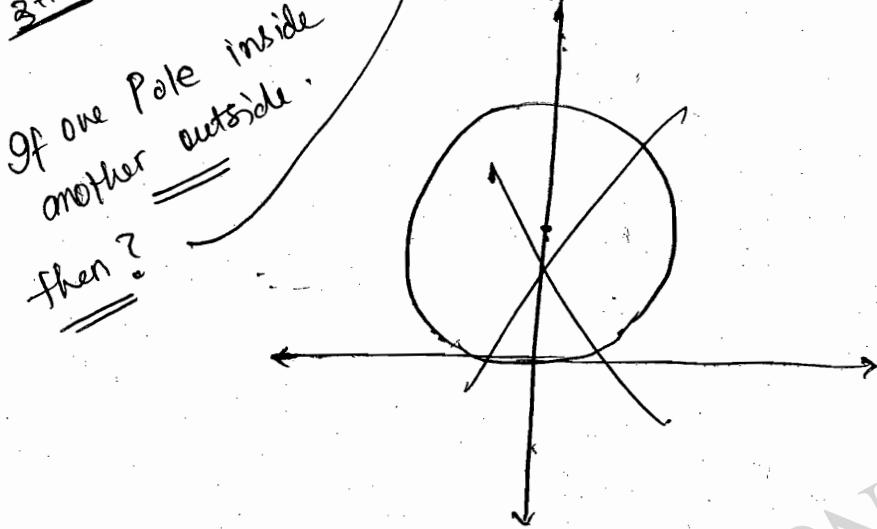
$$= 2\pi i \left(\frac{1}{z+1}\right)'_{z=1} + 2\pi i \left(\frac{1}{z-1}\right)_{z=-1}$$

$$= 2\pi i \left(\frac{1}{2}\right) + 2\pi i \left(-\frac{1}{2}\right)$$

$$= 0.$$

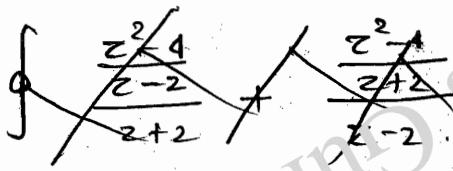
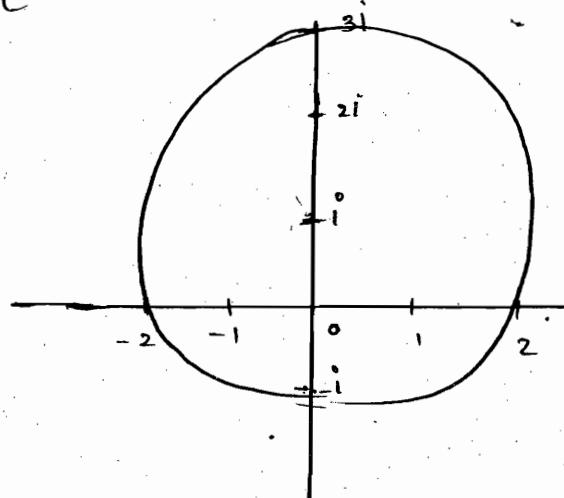
Q.  
Repeated  
3 times

$$\oint_C \frac{z-1}{(z+1)^2(z-2)} dz \quad \text{where } C \text{ is } |z-i| = 2$$



$$\begin{aligned}
 & \oint_C \frac{z-1}{(z+1)^2(z-2)} dz \\
 &= \oint_C \frac{\cancel{(z-1)}}{\cancel{(z-2)}} \frac{dz}{(z+1)^2} \quad f'(z) = \frac{-1}{(z-2)^2} \\
 &= 2\pi i \left[ \frac{-1}{(z-2)^2} \right]_{z=-1} \\
 &= \frac{-2\pi i}{(-1-2)^2} \\
 &= -\frac{2\pi i}{9}
 \end{aligned}$$

Q.  $\oint_C \frac{z^2 - 4}{z^2 + 4} dz$  where  $C$  is the circle  $|z - i| = 2$



=  ~~$2\pi i f(z)$~~

$$\oint_C \frac{z^2 - 4}{(z-2i)(z+2i)} dz$$

✓ x

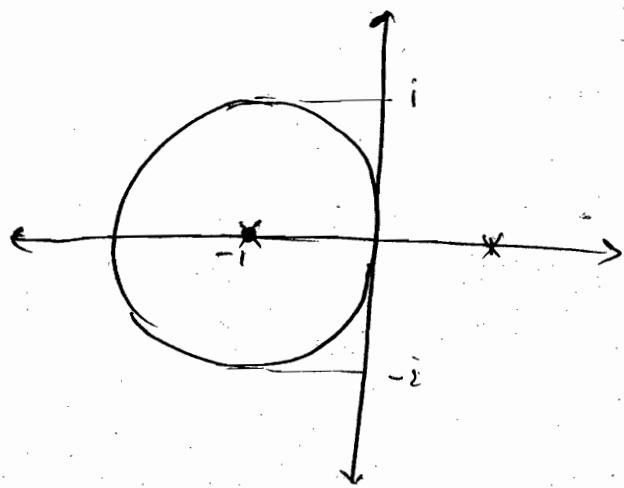
$$= \oint_C \frac{\frac{z^2 - 4}{z+2i}}{z-2i} dz$$

$$= 2\pi i \left( \frac{\frac{z^2 - 4}{z+2i}}{z-2i} \right)_{z=2i}$$

$$= 2\pi i \left( \frac{-4-4}{2i+2i} \right) = 2\pi i \left( \frac{-8}{4i} \right)$$

$$= -4\pi$$

Q.  $\oint_C \frac{z^2}{z^4+1} dz$  C is  $|z+1|=1$ .



$$\left\{ \begin{array}{l} \text{inside pole} \\ \text{outside pole} \end{array} \right. \quad \oint_C \frac{z^2}{(z^2-1)(z^2+1)} dz = \oint_C \frac{z^2}{(z-1)(z+1)(z-i)(z+i)} dz \quad \text{outside pole}$$

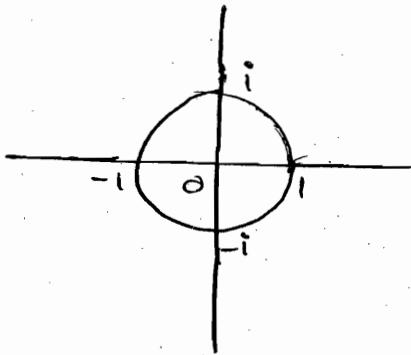
$$\oint_C \frac{\frac{z^2}{z^2}}{(z-1)(z^2+1)} dz$$

$$= 2\pi i \left( \frac{z^2}{(z-1)(z^2+1)} \right)_{z=-1}$$

$$= 2\pi i \left( \frac{1}{-2\pi/2} \right)$$

$$= -\frac{\pi i}{2}$$

Q.  $\oint \frac{5z^2 + 7z + 3}{(z-1)^3}$  where C is  $|z|=1$



$$\oint \frac{5z^2 + 7z + 3}{(z-1)^3}$$

$$f(z) = 5z^2 + 7z + 3.$$

$$f'(z) = 10z + 7.$$

$$f''(z) = 10.$$

$$\therefore \oint \frac{5z^2 + 7z + 3}{(z-1)^3} = \frac{10\pi i}{2!} (10)$$

$$= 10\pi i$$

~~\*\*\*~~ Residue at  $z=a$  where  $a$  is the pole of order  $n$ .

$$\lim_{z \rightarrow a} \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} \left[ (z-a)^n \cdot f(z) \right].$$

Q. find residue of  $f(z) = \frac{1-2z}{z(z-1)(z-2)}$

$$\text{at } z=0, \frac{1-0}{(0-1)(0-2)} = \frac{1}{2} =$$

$$\text{at } z=1, \frac{1-2}{1 \times 1} = \frac{1}{2} =$$

$$\text{at } z=2, \frac{1-4}{2 \times 1} = -\frac{3}{2} =$$

Q. ① find residue of f(z) =  $\frac{z^2}{(z-1)(z+1)(z-2)}$

$$\text{at } z=1 \quad \frac{1}{z-1} = -\frac{1}{2}$$

$$z=-1 \quad \frac{1}{-z-1} = \frac{1}{6}$$

$$z=2, \quad \frac{4}{1\times 3} = \frac{4}{3}$$

Q. ②  $f(z) = \frac{1}{z(z+2)^3}$

$$\frac{1}{z(z+2)^3} = \frac{A_0}{z} + \frac{A_1}{(z+2)} + \frac{A_2}{(z+2)^2} + \frac{A_3}{(z+2)^3}$$

f(z)  $\frac{(z+2)^3}{z} \rightarrow$   $\lim_{z \rightarrow -2} \frac{1}{2!} \frac{d^2}{dz^2} [(z+2)^3 \cdot \frac{1}{z(z+2)^3}]$

$$= \frac{1}{2} \frac{d^2}{dz^2} \left( \frac{1}{z} \right)$$

$$= \frac{1}{2} \times \frac{2}{z^3}$$

$$= \frac{1}{(-2)^3} = \frac{1}{8} =$$

Workbook

$$\textcircled{1} \quad A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \Delta = 1$$

$$\textcircled{2} \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \Delta = -1$$

$$\Delta = \pm 1$$

$$\textcircled{3} \quad xy = y$$

$$yx = x$$

$$x^2 + y^2 = ?$$

$$= xx + yy$$

$$= (yx)x + (xy)y$$

$$= (yx)(yx) + (xy)(xy)$$

$$= y(xy) + x(xy)$$

$$= yx + xy$$

$$= x + y. \quad \underline{\text{Ans}}$$

$$\textcircled{7} \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Total 8

$$\textcircled{8} \quad A = [a_{ij}]_{m \times n}$$

$$a_{ij} = i + j$$

$$A_{2 \times 2} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$$

→ 12 Check with option

⑨  $4^4$   
 $3 \times 4^4$

⑩ wrong Q.

⑪ 10

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

⑫ ⑬ ⑭  $5! = \underline{\underline{120}}$

⑮  $A_{ij} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$$2 \begin{bmatrix} 0 & -3 & -8 \\ 3 & 0 & -5 \\ 8 & 5 & 0 \end{bmatrix} \quad \because \Delta = 0$$

$\therefore$  inverse not possible.

⑯

Correct  $2x + 3y + z = \lambda z \rightarrow 3^{rd}$

⑰

⑱

(38)

$$\left[ \begin{array}{cc} \cancel{1} & 0 \\ 1 & 2 \end{array} \right]$$

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots + \frac{1}{n(n+1)}.$$

$$S_n = \sum T_n$$

$$= \sum \frac{1}{n(n+1)}$$

$$= \sum \frac{1}{n} - \frac{1}{n+1}$$

$$= 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \dots + \frac{1}{n} - \frac{1}{n+1}$$

$$= 1 - \frac{1}{n+1} \quad \underline{\underline{A}}$$

(39)

$$1 \cdot \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{4} \cdots \frac{1}{n}$$

$$= \frac{1}{n!} \quad \underline{\underline{A}}$$

(40)

$$A = \begin{bmatrix} 40 & -29 & -11 \\ -18 & 30 & -12 \\ 26 & 24 & -50 \end{bmatrix}$$

$$\lambda_1, \lambda_2 = ?$$

$$\lambda_1 + \lambda_2 + \lambda_3 = 20$$

$\Delta = 0$

$23^{\circ}0'$

$$\lambda_2 = 20 - \lambda_1 \quad \underline{\underline{A}}$$

(45)

$$3, 2, -1$$

$$B = A^2 - A$$

$$= 9 - 3 = 6$$

$$= 2^2 - 2 = 2$$

$$= (-1)^2 - (-1) = 2$$

$$|B| = 6 \times 2 \times 2 = 24$$

(46)

$$A = \begin{bmatrix} 1 & 0 & 0 \\ i & \frac{-1+i\sqrt{3}}{2} & 0 \\ 0 & 1+2i & \frac{-1-i\sqrt{3}}{2} \end{bmatrix}$$

Trace?   
 Trace is  
 ∑ principal diagonals

$$(A)^{102} = (1)^{102}, \left(\frac{-1+i\sqrt{3}}{2}\right)^{102}, \left(\frac{-1-i\sqrt{3}}{2}\right)^{102}$$

$$= 1, \left(e^{-\frac{i\pi}{3}}\right)^{102}, \left(e^{\frac{i\pi}{3}}\right)^{102}$$

$$= 1, e^{-34\pi i}, e^{34\pi i}$$

$$= 1 + 1 + 1 = 3$$

(48)

$$\lambda = -1, 1, 0$$

$$|A^{100} + 1|$$

$$2, 2, 1$$

$$|A^{100} + 1| = 2 \times 2 \times 1 = 4$$

(49)

$$\lambda_1 = 1, \lambda_2 = 4$$

$$x_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, x_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

(a)  $\begin{bmatrix} 4 & 8 \\ 5 & 9 \end{bmatrix}$

(b)  $\begin{bmatrix} 9 & 8 \\ 5 & -4 \end{bmatrix}$

(c)  $\begin{bmatrix} 9 & 2 \\ 1 & 2 \end{bmatrix}$

(d)  $\begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$

(51)

$$A = 1, -1, \lambda$$

$$A + 3I = 1+3, -1+3, \lambda+3.$$

$$A + 3I = 4, 2, \lambda+3.$$

$$A^2 + 3I = 1(4), -1(2), \lambda(\lambda+3).$$

$$= 4, -2, \lambda^2 + 3\lambda.$$

$$\lambda^2 + 3\lambda = 18$$

$$\lambda \neq 0$$

$$\Delta \neq 0$$

∴ Inverse exists.

Similarly  
inverse  
exists for A

Correction  $\rightarrow \frac{2x_2}{2x_2}$  not  $3x_2$

(62)

$$A^{-1} = I - 2A.$$

$$I = A - 2A^2$$

$$I = \lambda - 2\lambda^2$$

$$2\lambda^2 - \lambda + 1 = 0$$

$$\Delta = \lambda_1 \lambda_2$$

$$\therefore = \frac{1}{2} \quad \checkmark$$

(63)  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$a+d = ad - bc = 1$$

$$\text{Then } A^3 = ?$$

$$\rightarrow \lambda_1 + \lambda_2 = 1$$

$$\lambda_1 \lambda_2 = 1$$

$$\lambda_1 + \frac{1}{\lambda_1} = 1$$

$$\begin{aligned} \lambda_1^2 - 1 &= 1 - \lambda_1 \Rightarrow 0 \\ \lambda_1^2 + 1 &= \lambda_1 \\ \lambda_1^2 + 1 - \lambda_1 &= 0 \end{aligned}$$

$$\lambda_1^2 + 1 = \lambda_1$$

$$\boxed{A^2 = A - I}$$

$$A^3 = A^2 - A$$

$$= (A - I) A$$

$$= A^2 - A$$

$$= A - I - A$$

$$= -I$$

## Laplace Transform:

→ Let  $f(t)$  be any fn. of  $t$  defined & positive values of  $T$ , then the Laplace Transform of  $f(t)$  is denoted by  $\mathcal{L}[f(t)]$  and is defined as —

$$\mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t) dt = F(s)$$

$$\Rightarrow f(t) = \mathcal{L}^{-1}[F(s)]$$

is called the Inverse Laplace Transform of  $F(s)$ .

$$① \quad \mathcal{L}(t^n) = \int_0^\infty t^n e^{-st} dt$$

$$\text{Let } x = st \Rightarrow t = \frac{x}{s} \Rightarrow dt = \frac{dx}{s}$$

$$t \rightarrow \infty \Rightarrow x \rightarrow \infty$$

$$t \rightarrow 0 \Rightarrow x \rightarrow 0$$

$$= \int_0^\infty e^{-x} \left(\frac{x}{s}\right)^n \times \frac{dx}{s}$$

$$= \int_0^\infty e^{-x} \frac{x^n}{s^n} \cdot \frac{dx}{s}$$

$$= \frac{1}{s^{n+1}} \int_0^\infty e^{-x} x^n dx$$

$$= \frac{1}{s^{n+1}} \int_0^\infty e^{-x} x^{\overbrace{n+1-1}} dx$$

$$= \frac{\Gamma(n+1)}{s^{n+1}}$$

$$\therefore \mathcal{L}[t^n] = \frac{n!}{s^{n+1}}, \text{ if } n \in \mathbb{N}$$

$$\mathcal{L}[t^n] = \frac{n! \sqrt{n}}{s^{n+1}}, \text{ if } n > 0$$

$$\therefore L(x) = \frac{1}{s}$$

$$\mathcal{L}(t) = \frac{1}{s^2}$$

Improper Integral.

$$\Gamma_n = \int_0^\infty e^{-x} x^n dx$$

$$\Gamma_n = n\sqrt{n}, \text{ if } n > 0$$

$$\Gamma_{n+1} = n!, \text{ if } n \in \mathbb{N}$$

$$\Gamma_1 = 1$$

$$\Gamma_2 = \sqrt{\pi}$$

$$L(1) = \frac{1}{s}$$

$$L(t) = \frac{1}{s^2}$$

$$L(t^2) = \frac{2}{s^3}$$

$$L(e^{at}) = \frac{1}{s-a}$$

$$L(e^{-at}) = \frac{1}{s+a}$$

$$L(\sin at) = \frac{a}{s^2+a^2}$$

$$L(\cos at) = \frac{s}{s^2+a^2}$$

$$L(\sinh at) = \frac{a}{s^2-a^2}$$

$$L(\cosh at) = \frac{s}{s^2-a^2}$$

Q. find  $L(t^2+6t+8)$ ,  $L(\sqrt{t})$ ,  $L\left(\frac{1}{\sqrt{t}}\right)$ ,  $L(\sin^2 t)$ ,  $L(\sin^3 t)$ ,  $L(\sin 2t \cdot \cos t)$ .

$$\& f(t) = \begin{cases} 0, & 0 < t < 2 \\ 3, & t \geq 2 \end{cases}$$

find  $L[f(t)]$ .

$$\rightarrow ① L(t^2+6t+8) = \frac{8}{s^3} + \frac{6}{s^2} + \frac{8}{s}$$

$$② L(t^{\frac{1}{2}}) = \frac{\sqrt{\frac{1}{2}+1}}{s^{\frac{1}{2}+1}} = \frac{\frac{1}{2}\sqrt{\frac{1}{2}}}{s^{\frac{3}{2}}} = \frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}$$

$$③ L(t^{-\frac{1}{2}}) = \frac{\sqrt{\frac{1}{2}+1}}{s^{-\frac{1}{2}+1}} = \frac{\sqrt{\frac{1}{2}}}{s^{\frac{1}{2}}} = \frac{\sqrt{\pi}}{\sqrt{s}} = \sqrt{\frac{\pi}{s}}$$

$$④ L(\sin^2 t) = L\left(\frac{1-\cos 2t}{2}\right) = \frac{1}{2s} - \frac{1}{2} \cdot \frac{s}{(s^2+4)} = \frac{1}{2s} - \frac{s}{2(s^2+4)}$$

$$⑤ L(\sin^3 t) = L\left(\frac{3\sin t - \sin 3t}{4}\right) = \frac{3}{4} \left[ \frac{3}{s^2+1} - \frac{3}{s^2+9} \right] = \frac{3}{4} \left[ \frac{8s^2}{(s^2+1)(s^2+9)} \right] = \frac{6}{(s^2+1)(s^2+9)}$$

$$\textcircled{6} \quad L(\sin at \cdot \cos t) = \frac{1}{2} L[\sin 3t + \sin t] \\ = \frac{1}{2} \left[ \frac{3}{s^2+9} + \frac{1}{s^2+1} \right]$$

$$\textcircled{7} \quad f(t) = \begin{cases} 0, & 0 < t < 2 \\ 3, & t \geq 2 \end{cases}$$

$$L[f(t)] = \int_0^\infty f(t) e^{-st} dt \\ = \int_0^2 0 dt + \int_2^\infty 3 e^{-st} dt \\ = \left. -\frac{3}{s} e^{-st} \right|_2^\infty \\ = -\frac{3}{s} [0 - e^{-2s}] \\ = \frac{3e^{-2s}}{s}$$

# Shifting Theorem:-

$$\text{If } L[f(t)] = \int_0^\infty e^{-st} f(t) dt = F(s)$$

$$\text{then } L[e^{at} f(t)] = F(s-a)$$

$$\text{L.H.S.} = L[e^{at} f(t)] = \int_0^\infty f(t) e^{-t(s-a)} dt \\ = F(s-a)$$

formulae:-

$$L(t^n) = \frac{n!}{s^{n+1}}$$

$$L(e^{at} t^n) = \frac{n!}{(s-a)^{n+1}}$$

$$L[e^{-at} t^n] = \frac{n!}{(s+a)^{n+1}}$$

$$L[e^{at} \sin bt] = \frac{b}{(s-a)^2 + b^2}$$

$$L[e^{at} \cos bt] = \frac{(s-a)}{(s-a)^2 + b^2}$$

$$L[e^{at} \sinh bt] = \frac{b}{(s+a)^2 + b^2}$$

$$L[e^{at} \cosh bt] = \frac{s+a}{(s+a)^2 + b^2}$$

$$L[e^{at} \sinh bt] = \frac{b}{(s-a)^2 - b^2}$$

$$L[e^{at} \cosh bt] = \frac{b}{(s+a)^2 - b^2}$$

$$L(t \sin at)$$

$$= I.P \text{ of } L(t e^{ait}) \quad L(t) = \frac{1}{s^2}$$

$$= I.P \text{ of } \left[ \frac{1}{(s-ai)^2} \times \frac{(sta^2)^2}{(s+ai)^2} \right]$$

$$= I.P \text{ of } \left[ \frac{s^2-a^2+2ias}{(s^2+a^2)^2} \right]$$

$$\left. \begin{aligned} L(t \sin at) &= \frac{2as}{(s^2+a^2)^2} \\ L(t \cos at) &= \frac{s^2-a^2}{(s^2+a^2)^2} \end{aligned} \right\}$$

$$L(t \sin t) = \frac{2s}{(s^2+1)^2}$$

$$L(t \cos t) = \frac{s^2-1}{(s^2+1)^2}$$

~~marks Q.~~

## # Inverse Laplace Transform:-

$$L(t^n) = \frac{n!}{s^{n+1}}$$

$$L\left(\frac{t^n}{n!}\right) = \frac{1}{s^{n+1}}$$

$$L^{-1}\left(\frac{1}{s^{n+1}}\right) = \frac{t^n}{n!}$$

$$\therefore L^{-1}\left[\frac{1}{s}\right] = 1$$

$$L^{-1}\left[\frac{1}{s^2}\right] = t$$

$$L^{-1}\left[\frac{1}{s^3}\right] = \frac{t^2}{2}$$

$$L^{-1}\left(\frac{1}{s-a}\right) = e^{at}$$

$$L^{-1}\left(\frac{1}{s+a}\right) = e^{-at}$$

$$L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{1}{a} \sin at$$

$$L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at$$

$$L^{-1}\left(\frac{1}{s^2-a^2}\right) = \frac{t}{a} \sinh at$$

$$L^{-1}\left(\frac{s}{s^2-a^2}\right) = \cosh at$$

Q. Find  $L^{-1}\left[\frac{1}{2s-5}\right]$ ,  $L^{-1}\left[\frac{1}{s(s+1)}\right]$ ,  $L^{-1}\left[\frac{1}{(s+1)^2}\right]$

$$L^{-1}\left[\frac{s}{(s+a)^2}\right], L^{-1}\left[\frac{s+23}{s^2+4s+13}\right], L^{-1}\left[\frac{s-2}{s^2+2s+2}\right]$$

$$\rightarrow \textcircled{1} L^{-1}\left[\frac{1}{2s-5}\right] = \frac{1}{2} L^{-1}\left[\frac{1}{s-\frac{5}{2}}\right] = \frac{1}{2} e^{\frac{5}{2}t}$$

$$\textcircled{2} L^{-1}\left[\frac{1}{s(s+1)}\right] = L^{-1}\left[\frac{1}{s} - \frac{1}{s+1}\right] = 1 - e^{-t}$$

$$\textcircled{3} L^{-1}\left[\frac{1}{(s+1)^2}\right] = t e^{-t}$$

$$\textcircled{4} L^{-1}\left(\frac{s}{(s+a)^2}\right) = L^{-1}\left[\frac{s+a-a}{(s+a)^2}\right] = e^{-at} - a t e^{-at} \\ = e^{-at}(1-at)$$

$$\textcircled{5} L^{-1}\left[\frac{s+23}{s^2+4s+13}\right]$$

$$= L^{-1}\left[\frac{s+23}{s^2+2\times 2s+4+13-4}\right]$$

$$= L^{-1}\left[\frac{s-2+25}{(s-2)^2+3^2}\right]$$

$$= e^{2t} \cos 3t + \frac{25}{3} e^{2t} \sin 3t$$

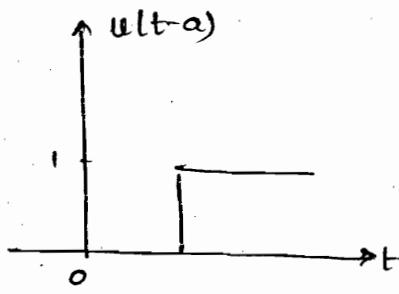
$$\textcircled{6} L^{-1}\left[\frac{s-2}{s^2+2s+1+1+2-1}\right]$$

$$= L^{-1}\left[\frac{s+1-3}{(s+1)^2+1}\right]$$

$$= e^{-t} \cos t - 3 e^{-t} \sin t$$

$$= e^{-t} [\cos t - 3 \sin t]$$

## # Unit Step function:-



The unit step function

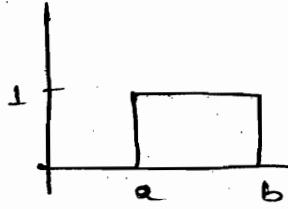
$u(t-a)$  is defined by —

$$u(t-a) = \begin{cases} 0, & 0 < t < a \\ 1, & t \geq a \end{cases}$$

$$\begin{aligned} \text{Then } L[u(t-a)] &= \int_0^{\infty} e^{-st} u(t-a) dt \\ &= \int_0^a 0 dt + \int_a^{\infty} 1 e^{-st} dt \\ &= \frac{1}{s} [e^{-as} - e^{-as}] \end{aligned}$$

$$\boxed{L[u(t-a)] = \frac{e^{-as}}{s}}$$

①



$$= u(t-a) - u(t-b)$$

$$= \frac{e^{-as}}{s} - \frac{e^{-bs}}{s} = \frac{1}{s} [e^{-as} - e^{-bs}]$$

## # Periodic function:-

→ A fn.  $f(t)$  is said to be periodic with the period ' $T$ ', if  
 $f(t+T) = f(t)$ , where  $T$  is the least positive number.

ex:-  $f(t) = \sin t$   
 $f(t+2\pi) = \sin(t+2\pi)$ ,  
=  $\sin t$ .  
 $f(t+2\pi) = f(t)$

## # Laplace Transform of Periodic function:-

Let  $f(t)$  be a periodic fn with period  $T$ , then the Laplace Transform of the periodic fn is defined by —

$$L[f(t)] = \frac{1}{1-e^{-sT}} \int_0^{sT} e^{-st} \cdot f(t) dt$$

ex:-  $f(t) = \begin{cases} 3, & 0 < t < 2 \\ 0, & 2 < t < 4 \end{cases}$   
and  $f(t+4) = f(t)$ , then find  $L[f(t)]$ .

$$\begin{aligned} \rightarrow L[f(t)] &= \frac{1}{1-e^{-sT}} \left[ \int_0^2 e^{-st} \times 3 dt + \int_2^4 0 dt \right] \\ &= \frac{1}{1-e^{-4s}} \left[ -\frac{3}{s} [e^{-2t} - 1] \right] \\ &= \frac{3(1/e^{-8s})}{s(1-e^{-4s})} \cancel{(1/e^{-2s})} (1+e^{-2s}) \\ &= \frac{3}{s(1+e^{-2s})} \end{aligned}$$

$$Q. f(t) = \begin{cases} 1, & 0 < t < b \\ -1, & b < t < 2b \end{cases}, \quad f(t+2b) = f(t), \text{ find } L[f(t)]$$

Top Result

$$\rightarrow L[f(t)] = \frac{1}{1-e^{-2bs}} \left[ \int_0^b e^{-st} dt + \int_b^{2b} e^{-st} dt \right]$$

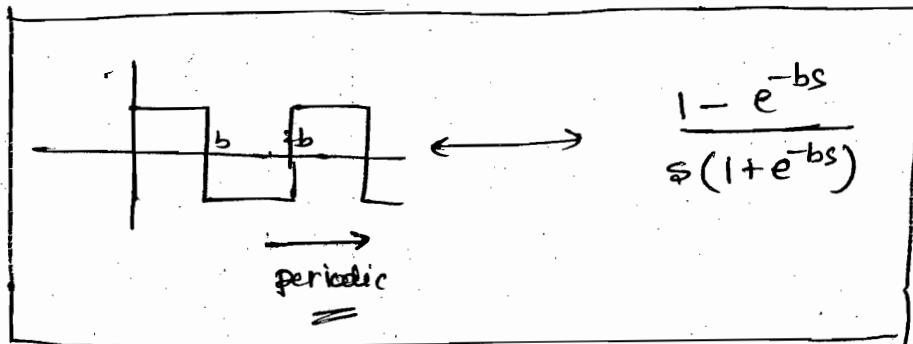
$$= \frac{1}{1-e^{-2bs}} \left[ \frac{1-e^{-bs}}{s} + \frac{e^{-2bs}-e^{-bs}}{s} \right]$$

$$= \frac{1}{(1-e^{-2bs})} \cdot \frac{1-e^{-bs} + e^{-2bs}-e^{-bs}}{s}$$

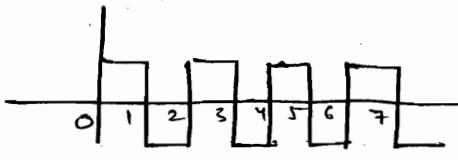
$$= \frac{1}{(1-e^{-2bs})} \left[ \frac{1-2e^{-bs}+e^{-2bs}}{s} \right]$$

$$= \frac{(1-e^{-bs})(1+e^{-bs})}{s(1+e^{-bs})}$$

Remember i.e.  
Top Result



ex:-



$$L[f(t)] = ?$$

Here  $b = 1$

$$L[f(t)] = \frac{1 - e^{-s}}{s(1 + e^{-s})}$$

Periodic

Q. ~~Find L[f(t)]~~

$$L[f(t)] = F(s) = \frac{5s^2 + 23s + 6}{s(s^2 + 2s + 2)}$$

then  $Lt f(t) = \lim_{t \rightarrow \infty} f(t)$

- Ⓐ 3 Ⓑ 5 Ⓒ  $\frac{17}{2}$  Ⓓ  $\infty$

Q.  $f(t) = L^{-1} \left[ \frac{3s+1}{s^3 + 4s^2 + (K-3)s} \right]$

$$\lim_{t \rightarrow \infty} f(t) = 1$$

then  $K = \underline{\hspace{2cm}}$

$$\rightarrow \text{A/C to F.V.T} - \frac{1}{K-3} = 1 \Rightarrow K = \underline{\hspace{2cm}}$$

Q. The Laplace Transform —

$$L[f(t)] = \frac{1}{s^2(s+1)}$$

$$f(t) = ?$$

$$\rightarrow \frac{1}{s^2(s+1)} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{s+1}$$

$$\frac{1}{s^2} = 1 + \frac{1}{s} + \frac{1}{s+1}$$

$$a_1 = -1$$

$$= t^2 - 1 + e^{-t}$$

## Probability

→ If an experiment is conducted under essentially given conditions upto n times, let m cases are favourable to an event E then the Probability of E is denoted by  $P(E)$  and is defined by -

$$\boxed{P(E) = \frac{m}{n}}$$

$$P(\bar{E}) = \frac{n-m}{n} = 1 - \frac{m}{n} = 1 - P(E).$$

$$\therefore \boxed{P(E) + P(\bar{E}) = 1}$$

Sum of Success & failure is always 1.

## Axioms of Probability :-

① Axiom of Positivity :-  $0 \leq P(E) \leq 1$ .

② Certainty :-  $P(S) = 1$ .

③ Union :-  $P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$

Note :- If  $E_1$  and  $E_2$  are mutually Exclusive or Disjoint events  
i.e.,  $E_1 \cap E_2 = \emptyset$

$$\text{then } P(E_1 \cup E_2) = P(E_1) + P(E_2)$$

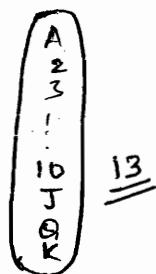
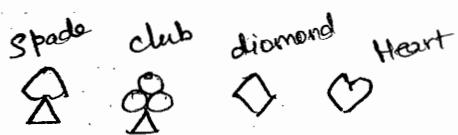
## Sample Space :-

The set of all possible outcomes in an experiment is said to be the Sample Space.

2 coins tossed  
 $S = \{HH, HT, TH, TT\}$ .

1 dice thrown  
 $S = \{1, 2, 3, 4, 5, 6\}$ .

2 dice thrown  
 $S = \{(1,1), (1,2), (1,3), \dots, (6,6)\} \rightarrow$  Total 36



## # Conditional Probability:-

Let  $S$  be the sample space, and Event  $E_1, E_2 \subset S$  then the conditional Probability of  $E_1$  after the occurrence of  $E_2$  is denoted by -

$$P\left(\frac{E_1}{E_2}\right) = \frac{P(E_1 \cap E_2)}{P(E_2)}, \quad (P(E_2) \neq 0)$$

$$P\left(\frac{E_2}{E_1}\right) = \frac{P(E_1 \cap E_2)}{P(E_1)}, \quad (P(E_1) \neq 0).$$

## # Random Variable:-

Mean :-  $\sum x_i P(x=x_i)$  if exists, it is called the mean of R.V.

$$\mu = \sum x_i P(x=x_i).$$

Variance of R.V  $\sum (x_i - \mu)^2 P(x=x_i)$  if exists, it is called the variance of R.V

$$\sigma^2 = \sum (x_i - \mu)^2 P(x=x_i)$$

$$= \sum (x_i^2 + \mu^2 - 2x_i\mu) P(x=x_i)$$

$$= \sum x_i^2 P(x=x_i) + \sum \mu^2 \cdot P(x=x_i) - \sum 2x_i \mu P(x=x_i)$$

$$= \sum x_i^2 P(x=x_i) + \mu^2 \sum P(x=x_i) - 2\mu \sum x_i P(x=x_i)$$

$$= \sum x_i^2 [P(x=x_i)] + \mu^2 - 2\mu^2$$

$$\boxed{\sigma^2 = \sum x_i^2 [P(x=x_i)] - \mu^2}$$

### ③ Standard Deviation:-

The standard deviation is the sq. root of variance and is denoted by  $\sigma$ .

ABOVE WERE FOR GENERAL DISTRIBUTION:-

### # Binomial Distribution:-

$n$  = no. of trials

$p$  = prob. of success.

$q$  = prob. of failures.

$$p+q=1$$

$$P(X=k) = {}^n C_k q^{n-k} p^k$$

$$\text{Mean} = n \cdot p$$

$$\text{Variance} = npq$$

$$S.D = \sqrt{npq}$$

$$\text{If } p=q$$

$$P(X=k) = {}^n C_k p^k$$

### # Poisson Distribution:-

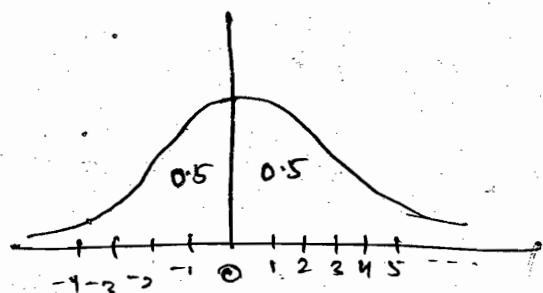
$$P(X=k) = \frac{\lambda^k}{k!} e^{-\lambda}, \text{ where } \lambda \text{ is a parameter}$$

$$\text{Mean} = \lambda$$

$$\text{Variance} = \lambda$$

$$S.D = \sqrt{\lambda}$$

### # Normal Distribution:-



$$Z = \frac{X-\mu}{\sigma}$$

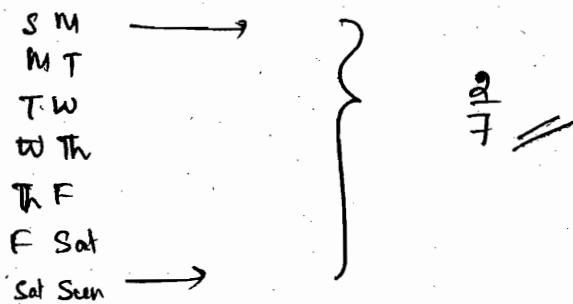
$\mu$  = mean.

$\sigma$  = S.D

Q. What is the chance that a leap year will have 53 Sundays.

$$\rightarrow 52 \times 7 = 364$$

for rest 2 days -



$$\text{for Non leap year} - P(53 \text{ Sunday}) = \frac{1}{7} \approx$$

Q. from a deck of playing Cards - 2 cards are drawn at Random  
What will be the probab. that both Cards will be Kings?  
if -

① the first Card is not replaced

② Replacement is allowed:

$$\rightarrow ① \frac{4C_1}{52C_1} \times \frac{3}{51}$$

$$= \frac{4}{52} \times \frac{3}{51} = \frac{1}{221}.$$

$$② \frac{4}{52} \times \frac{4}{52} \approx = \frac{1}{169}$$

Q. A Bag Contain 3 red & white and 7 black balls, what is the Prob that 2 balls drawn are white and Black.

$\rightarrow$

8 3 7  
(W B)

$$\frac{6C_1 \times 7C_1}{16C_2} = \frac{6 \times 7}{\frac{16 \times 15}{2}} = \frac{7}{20}.$$

Q. A problem in electronics is given to 3 students A, B, C whose chances of solving it are  $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$  respectively. What is the prob. that the prob. will be solved.

$$\rightarrow P(A) = \frac{1}{2} \\ P(B) = \frac{1}{3} \\ P(C) = \frac{1}{4}$$

individual  
 / then  
 $P(\bar{A}), P(\bar{B}), P(\bar{C})$

$$1 - P(\bar{A} \cap \bar{B} \cap \bar{C}) \\ = 1 - P(\bar{A}) P(\bar{B}) P(\bar{C}) \\ = 1 - \frac{1}{2} \times \frac{1}{3} \times \frac{1}{4} \\ = \frac{3}{4} \\ = 75\% \text{ chances.}$$

Q. A husband & wife appear in interview for 2 vacancies for the same post. The prob. of husband selection is  $\frac{2}{7}$  & that of wife selection is  $\frac{1}{5}$ .

- What is the Prob. that —
- ① both are selected.
  - ② Only one of them is selected.
  - ③ None of them is selected.

$$\rightarrow P(A) = \frac{2}{7} \\ P(B) = \frac{1}{5}$$

$$\textcircled{1} \quad P(A \cap B) = P(A) \cdot P(B) = \frac{2}{7} \times \frac{1}{5} = \frac{2}{35}.$$

$$\textcircled{2} \quad \cancel{P(A \cup B)}$$

$$P(A \cap \bar{B}) + P(\bar{A} \cap B)$$

$$= \frac{2}{7} \times \frac{4}{5} + \frac{5}{7} \times \frac{1}{5}$$

$$= \frac{18}{35} = \frac{2}{7}.$$

$$\textcircled{3} \quad P(\bar{A} \cap \bar{B}) = P(\bar{A}) \cdot P(\bar{B}) \\ = \frac{5}{7} \times \frac{4}{5} = \frac{2}{7}$$

Q. An integer is selected at random from the first 200 digits. What is the prob. that the integer chosen is divisible by either 6 or 8?

$$\rightarrow 1, 2, 3, \dots, 200$$

$$\div 6 \rightarrow 6, 12, 18, \dots, 198$$

$$\div 8 \rightarrow 8, 16, 24, \dots, 200$$

$$\div 6 \& \div 8 \rightarrow 24, 48, \dots, 192$$

$$\text{no. of integers } \div 6 = \frac{198}{6} = 33 = n(E_1)$$

$$\text{no. of integer } \div 8 = \frac{200}{8} = 25 = n(E_2)$$

$$\text{no. of integers } \div 6 \& \div 8 = \frac{192}{8} = 24 = n(E_1 \cap E_2)$$

$$\begin{aligned} n(E_1 \cup E_2) &= n(E_1) + n(E_2) - n(E_1 \cap E_2) \\ &= 33 + 25 - 24 \\ &= 50. \end{aligned}$$

$$\therefore P(E_1 \cup E_2) = \frac{50}{200} = \frac{1}{4}$$

Q. A hits a target 4 times in 5 shots  $\rightarrow$  B  $\rightarrow$  3 times in 4 shots  
C  $\rightarrow$  2 in 3 shots.

The fire a volley. What is Prob. that 9 shots at least hit the

target?  $\uparrow$  means  
A, B, C then  
again ABC  
continuously

$$\rightarrow A \rightarrow 4 \text{ in } 5$$

$$B \rightarrow 3 \text{ in } 4$$

$$C \rightarrow 2 \text{ in } 3$$

for at least 2 shots —

$$P(A \cap B \cap \bar{C}) + P(A \cap \bar{B} \cap C) + P(\bar{A} \cap B \cap C) + P(A \cap B \cap C)$$

$$= \frac{1}{5} \times \frac{3}{4} \times \frac{1}{3} + \frac{4}{5} \times \frac{1}{4} \times \frac{2}{3} + \frac{1}{5} \times \frac{3}{4} \times \frac{2}{3} + \frac{4}{5} \times \frac{3}{4} \times \frac{2}{3}$$

$$= \frac{12}{60} + \frac{8}{60} + \frac{5}{60} = \frac{5}{6}$$

For at least 1 shot  $\rightarrow 1 - P(\bar{A} \cap \bar{B} \cap \bar{C})$

Q On a college 25% of student fail in Math, 15% failed in Chemistry, 10% failed in both Maths and Chemistry if a student is selected at random —

① If he failed in Chem. then what is the probability that he fails in Math.

② Vice-versa

③ What is the prob. that he failed neither Mathematics,  
→ nor Chemistry?

$$\rightarrow P(M) = 0.25$$

$$P(C) = 0.15$$

$$P(M \cap C) = 0.10$$

$$\textcircled{1} \quad P\left(\frac{M}{C}\right) = \frac{P(M \cap C)}{P(C)} = \frac{0.10}{0.15} = \frac{2}{3}$$

$$\textcircled{2} \quad P\left(\frac{C}{M}\right) = \frac{P(C \cap M)}{P(M)} = \frac{0.10}{0.25} = \frac{2}{5}$$

$$\begin{aligned} \textcircled{3} \quad P(\overline{M \cup C}) &= 1 - P(M \cup C) \\ &= 1 - [P(M) + P(C) - P(M \cap C)] \\ &= 1 - [\cancel{0.25} + 0.15 - 0.10] \\ &= 1 - 0.3 \\ &= 0.7 \end{aligned}$$

Q. A single dice is thrown twice, what is the probability that their sum is neither 8 nor more than 9.

$$\rightarrow n(s) = 36$$

$$n(E_1) = \{(2,6), (3,5), (4,4), (5,3), (6,2)\}$$

$$\therefore P(E_1) = \frac{5}{36}.$$

$$n(E_2) = \{(3,6), (6,5), (5,4), (4,3)\}$$

$$P(E_2) = \frac{4}{36}.$$

$\therefore$  These are mutually exclusive  $\Rightarrow$  8 और 9 की समानता 9 और 8 की समानता

$$\therefore P(\overline{E_1 \cup E_2}) = 1 - P(E_1 \cup E_2)$$

$$= 1 - [P(E_1) + P(E_2)]$$

$$= 1 - \left[ \frac{5}{36} + \frac{4}{36} \right]$$

$$= \frac{\frac{27}{36}}{4} = \frac{3}{4}$$

Q. A pair of fair dice is thrown, if the two numbers appearing are different, find the Prob that -

- ① The sum is 6
- ② an ace appears
- ③ The sum is 4 or less

$$\rightarrow n(s) = 36 - 6 = 30.$$

Some no  
appearing.

for Sum is 6 -

$$n(E_1) = \{(1,5), (2,4), (5,1), (4,2)\} \Rightarrow P(E_1) = \frac{4^2}{30} = \frac{2}{15}$$

for Ace Appears -  $P(E) = 0$  if option available

But Here Ace means 1

$$n(E_2) = \{(1,2), (1,3), (1,4), (1,5), (1,6), \dots, (2,1)\} = 10$$

$$\therefore P(E_2) :$$

④ the sum is 4 or less

$$n(E) = \{(1,2), (1,3), (2,1), (3,1)\} = 4$$

Binomial  
Type - 2

$$P(E) = \frac{4^2}{20} = \frac{2}{15}$$

Q. Team A has probability  $\frac{2}{3}$  of winning whenever it plays.

Suppose A plays 4 games. find the prob. that A wins more than half of its games.

$$\rightarrow \text{Team A} \rightarrow \frac{2}{3}$$

$$n=4$$

$$p = \frac{2}{3}$$

$$q = \frac{1}{3}$$

$$P(X > 2) = P(X=3) + P(X=4)$$

$$= {}^4C_3 \left(\frac{2}{3}\right)^3 \left(\frac{1}{3}\right)^1 + {}^4C_4 \left(\frac{2}{3}\right)^4 \left(\frac{1}{3}\right)^0$$

$$= {}^4C_3 \left(\frac{2}{3}\right)^3 \left(\frac{1}{3}\right)^1 + {}^4C_4 \left(\frac{2}{3}\right)^4 \left(\frac{1}{3}\right)^0$$

$$= \frac{4 \times 8}{81} + \frac{1 \times 16}{81} = \frac{48}{81} = \frac{16}{27}$$

How to know Binomial?

n choose k if all  
k

Given not imp  
asked imp, a/c to that  
p is selected

Q. In a bank 60% of all customers applied for a loan are rejected. If 4 new loans applications received. find the prob that 3 of them are accepted.

$$\rightarrow n=4  
p=0.4  
q=0.6$$

$$P(X=3) = {}^4C_3 (0.6)^1 (0.4)^3  
= 4 (0.6)(0.4)^3  
= 0.1536.$$

- B. A player tosses 2 fair coins. He wins \$2 if 2 heads occurs and wins \$1 if 1 head occurs. On the other hand he loses \$3, if no head occurs. Find the expected value 'E' of the game.

$$\begin{aligned} \rightarrow & \$2 \rightarrow 2H \\ & \$1 \rightarrow 1H \\ & -\$3 \rightarrow 0H \end{aligned}$$

$$S = \{(H,H), (H,T), (T,H), (T,T)\}$$

$x_i$	\$2	\$1	$-\$3$
$P_i$	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{1}{4}$

$$E(x) = \sum x_i P(x=x_i)$$

$$= 2\left(\frac{1}{4}\right) + 1 \times \left(\frac{2}{4}\right) - 3\left(\frac{1}{4}\right)$$

$$= \frac{1}{2} + \frac{1}{2} - \frac{3}{4}$$

$$= \frac{1}{4} > 0 \rightarrow \text{favourable to player}$$

- Q. 4 fair coins are tossed simultaneously. find the prob. that at least one head and one tail turn up.

$$\rightarrow n = 4$$

$$P = 2/2$$

$$P(X=1) + P(X=2) + P(X=3)$$

$$= \left( {}^4C_1 + {}^4C_2 + {}^4C_3 \right) \left( \frac{1}{2} \right)$$

$$= \frac{4+6+4}{16} = \frac{14}{16} = \frac{7}{8}$$

$$= 1 - [P(X=4) + P(X=0)]$$

$$= 1 - \left[ \frac{1+1}{16} \right]$$

$$= 1 - \frac{1}{8} = \frac{7}{8}$$

Q. Five fair coins are tossed simultaneously. Find the prob. that at least one head turn up.

$$\begin{aligned} \rightarrow & 1 - P(X=0) \\ & = 1 - {}^5C_0 \left(\frac{1}{2}\right)^5 \\ & = 1 - \frac{1}{32} = \frac{31}{32} \end{aligned}$$

Q. 4 fair coins are tossed simultaneously. Find the Prob. of event the no. of times heads show up is more than the no. of times tails show up.

$$\begin{aligned} \rightarrow & P(X=3) + P(X=4) \\ & = {}^4C_3 + {}^4C_4 + \left(\frac{1}{2}\right)^4 \\ & = \frac{4+1}{16} = \frac{5}{16} \end{aligned}$$

Q. There are 5 duplicate and 10 original items in an automobile shop. 3 items are bought up by customer at random. find the Prob. that none of the items is duplicate.

$\rightarrow$  Means all original.

$$= \frac{{}^{10}C_3}{{}^{15}C_3} = \frac{10 \times 9 \times 8}{15 \times 14 \times 13} = \frac{24}{91} //$$

Q. Out of 800 families with 4 children each how many families would be expected to have —

- ① 2 boys and 2 girls.
- ② At least one boy
- ③ At most 2 girls.

$\rightarrow$  for 1 family —

$$n=4$$

$$P=q=\frac{1}{2}$$

$$\therefore P(X=2) = {}^4C_2 \left(\frac{1}{2}\right)^4 = 6 \times \frac{1}{16} \times 800 = 300 //$$

Now  
for 800 family

Poisson distribution  
Gaussian distribution  
 $n$  is large (very large)  
But discrimination  
individual ref &  
Not applicable

$$\begin{aligned}
 \textcircled{2} \quad P(X \geq 1) &= 1 - P(X=0) \\
 &= 1 - {}^4C_0 \left(\frac{1}{2}\right)^4 \\
 &\approx 1 - \frac{1}{16} \\
 &= \frac{15}{16} \times 800 \\
 &\approx 750
 \end{aligned}$$

$$\begin{aligned}
 \textcircled{3} \quad P(X \leq 2) &= P(X=0) + P(X=1) + P(X=2) \\
 &= \left( {}^4C_0 + {}^4C_1 + {}^4C_2 \right) \left(\frac{1}{2}\right)^4 \\
 &= \frac{1+4+6}{16} \\
 &= \frac{11}{16} \times 800 \\
 &= \underline{\underline{550}} \ldots
 \end{aligned}$$

- Q. A book of 300 pages containing 30 pointing mistakes. Assuming that these errors are randomly distributed throughout the book and  $X$  is the no. of errors per page has a poisson's distribution. What is the prob. that 20 pages selected at random will be free of errors.

$$\rightarrow \text{error in page } P = \frac{30}{300} = \frac{1}{10}$$

$$\text{Given } n = 20$$

$$\begin{aligned}
 \lambda &= np \\
 &= 20 \times \frac{1}{10} \\
 &= 2
 \end{aligned}$$

$$\boxed{\lambda = 2}$$

$$P(X=0) = \frac{\lambda^0}{0!} e^{-\lambda} = e^{-2} = 0.135$$

Tip  
 When in Poisson distribution  
 $\lambda$  is not mentioned,  
 we will find it by Binomial  
 help.  
 i.e.  $\lambda = np$   
 $= 20 \times \frac{1}{10} = 2$   
 $\therefore \lambda^2 = 2$

Q. 1000 students had written an examination with the mean of test = 35  $\rightarrow$  and  $S.D = 5$ . Assuming the distribution to be normal. find how many student's marks —

- (a) lie between 25 & 40.
- (b) How many got more than 40.
- (c) How many get below 20.

$$\text{Given } P(0 \leq Z \leq 1) = 0.3415$$

$$P(0 \leq Z \leq 2) = 0.4772$$

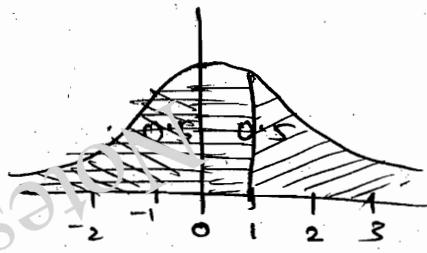
$$P(0 \leq Z \leq 3) = 0.449$$

$$\rightarrow \mu = 35$$

$$\sigma = 5$$

$$Z = \frac{x - \mu}{\sigma}$$

$$P(x_1 \leq x \leq x_2)$$



$$\text{for } x_1 = 25 \rightarrow Z_1 = \frac{25 - 35}{5} = -2$$

$$\text{for } x_2 = 40 \rightarrow Z_2 = \frac{40 - 35}{5} = 1$$

$$\therefore P(25 \leq x \leq 40) = P(-2 \leq Z \leq 1)$$

$$= P(-2 \leq Z \leq 0) + P(0 \leq Z \leq 1)$$

$$= P(0 \leq Z \leq 2) + P(0 \leq Z \leq 1)$$

$$= 0.4772 + 0.3415$$

$$= 0.8187$$

$$= 0.8187 \times 1000 \xrightarrow{\text{Total Students}}$$

$$= 818.7$$

$$P(x \geq 40) = P(Z \geq 1)$$

$$= 0.5 - P(0 \leq Z \leq 1)$$

$$= 0.5 - 0.3415$$

$$= 0.1585$$

$$= 0.1585 \times 1000$$

$$= 158.5$$

Q.  $P(X \leq 20)$

$$Z = \frac{20 - 35}{5} = \frac{-15}{5} = -3.$$

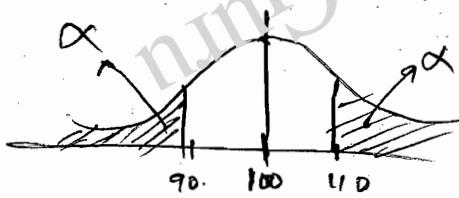
$$\begin{aligned} P(Z \leq 3) &= P(X < 20) = 0.5 - P(0 \leq Z \leq 3) \\ &= 0.5 - 0.499 \\ &\approx 0.001 \times 1000 \\ &= 1 \text{ student.} \end{aligned}$$

Q. for a Random Variable —

( $-\infty < x < \infty$ ) following a normal distribution with the ~~mean~~

Mean is 100. If the prob. of ~~is~~  $P(X \geq 110) = \alpha$

then  $P(90 \leq x \leq 110) = \underline{\quad}$

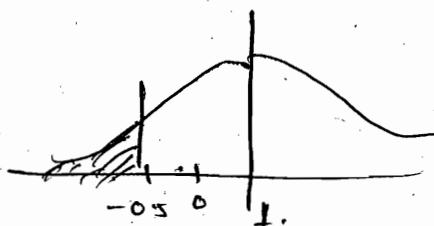


$$P(X \geq 110) = \alpha$$

$$P(90 \leq x \leq 110) = 1 - 2\alpha.$$

Q. Let  $X$  be the normal variant with Mean = 1 &  $\sigma^2 = 4$   
then the Prob. of  $P(X < 0)$  —

- (a) 0.5
- (b) >0 and < 0.5
- (c) >0.5 and < 1
- (d) 1.



$$\mu = 1$$

$$\sigma^2 = 4$$

$$\sigma = 2$$

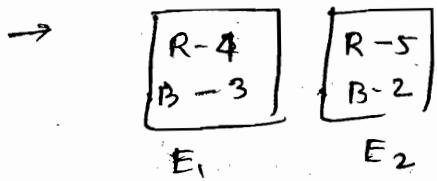
$$Z = \frac{x - \mu}{\sigma} = \frac{-1}{2} = -0.5$$

$$P(X < 0)$$

$$= P(Z < -0.5)$$

$$= 0.5 - P(0 \leq Z)$$

Q. A box contain 3 blue and 4 red ball. Another identical Box contain 2 blue ball and 5 red ball. One ball is selected at Random from one of the two boxes and it is red. The Prob. that it come from the first box —



$$P(E_1) = P(E_2) = \frac{1}{2}$$

$$P\left(\frac{A}{E_1}\right) = \frac{4}{7}, \quad P\left(\frac{A}{E_2}\right) = \frac{5}{7}$$

$$\begin{aligned} P\left(\frac{E_1}{A}\right) &= \frac{P(E_1) \cdot P(A/E_1)}{P(E_1) \cdot P(A/E_1) + P(E_2) \cdot P(A/E_2)} \\ &= \frac{\frac{1}{2} \left( \frac{4}{7} \right)}{\frac{1}{2} \left( \frac{4}{7} \right) + \frac{1}{2} \left( \frac{5}{7} \right)} \\ &= \frac{\frac{4}{7}}{\frac{9}{7}} = \frac{4}{9}. \end{aligned}$$

Q. Box P has 2 Red & 3 blue balls. Box Q has 3 Red & 1 Blue ball. The Prob. of selecting boxes P and Q are  $\frac{1}{3}$  and  $\frac{2}{3}$  respectively. One ball is selected from one of two boxes and it is red, the Prob that it came from box P.

$$\rightarrow P(P) = \frac{1}{3}, \quad P(Q) = \frac{2}{3}, \quad P\left(\frac{A}{P}\right) = \frac{2}{5}, \quad P\left(\frac{A}{Q}\right) = \frac{3}{4}.$$

$$P\left(\frac{P}{A}\right) = \frac{P(P) \cdot P\left(\frac{A}{P}\right)}{P(P) \cdot P\left(\frac{A}{P}\right) + P(Q) \cdot P\left(\frac{A}{Q}\right)} = \frac{\frac{1}{3} \cdot \frac{2}{5}}{\frac{1}{3} \cdot \frac{2}{5} + \frac{2}{3} \cdot \frac{3}{4}} = \frac{1}{9} =$$

Q. The Probability density fn of the form  $P(x) = k e^{-\alpha|x|}$ ,  $x \in (-\infty, \infty)$ , then the value of k —

$$\begin{aligned} \rightarrow \int_{-\infty}^{\infty} P(x) dx = 1 &\Rightarrow \int_{-\infty}^{\infty} k e^{-\alpha|x|} dx = 1 \Rightarrow 2 \int_0^{\infty} k e^{-\alpha x} dx = 1 \\ &\Rightarrow \frac{2k}{\alpha} [e^{-\infty} - e^0] = 1 \\ &\Rightarrow \frac{-2k}{\alpha} (0-1) = 1 \\ &\Rightarrow k = \frac{\alpha}{2} = 0.5\alpha. \end{aligned}$$

Q. Two Bag Contain 10 coins each and the colors for each bag are numbered from 1 to 10. One coin is drawn at random from each bag. What is the prob that one of the coin has value (1, 2, 3, 4) while the other has value (7, 8, 9 or 10) ...

## Differential Eq<sup>n</sup>:

- Formation of Diff<sup>n</sup> eq<sup>n</sup> → diff upto no of constants.
- Solution of Diff eq<sup>n</sup>
  - Variable - Separable
    - normal
    - reducible

Notes Guru

**FOR MORE MATERIAL EE,  
ME, CIVIL,CSE, ECE, EEE, GS  
VISIT  
NOTESGURU.INFO**

**ALL IN ONE PLATFORM FOR ENGINEERING  
STUDENTS AND GATE ASPIRANTS**

**SEE ALSO :**

**COLLEGE NOTES**

---

**UNIVERSITY NOTES**

---

**COMPETITION NOTES**

---

**STUDY MATERIAL**

---

**NON TECH NOTES**

---

**Thanks for choosing  
notesguru**



*Do not copy this material. If you find any website or blog publishing this material with 'notesguru.info' logo or watermark kindly report us to notesguru01@gmail.com*