11/16/...

(1b) Lucas-Kanade for motion tracking with affine motion.

$$I_x(x,y)u(x,y) + I_y(x,y) + I_t(x,y) = 0$$

Affine motion model:

Let the motion (u,v) at pixel (x,y) be

$$u(x,y) = a_1 x + b_1 y + c_1,$$

$$v(x,y) = a_2 x + b_2 y + c_2$$

$a_1, a_2, b_1, b_2, c_1, c_2$ are unknown parameters to be estimated

Define the parameter vector

$$p = \begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ a_2 \\ b_2 \\ c_2 \end{bmatrix}$$

→ Substitute affine motion into the optical flow equation.

For a pixel (x_i, y_i) in the window we have

→ Substitute $u(x_i, y_i)$ and $v(x_i, y_i)$:

$$I_x(x_i, y_i)(a_1 x_i + b_1 y_i + c_1) + I_y(x_i, y_i)(a_2 x_i + b_2 y_i + c_2) + I_t(x_i, y_i) = 0.$$

Expanding group terms:

11/16/2025

$(I_x x_i) a_1 + (I_x y_i) b_1 + (I_x) c_1 + (I_y x_i) a_2 + (I_y y_i) b_2 +$

$(I_y) c_2 = -I_t(x_i, y_i)$

writing it compactly:

$$[I_x x_i,\ I_x y_i,\ I_x,\ I_y x_i,\ I_y y_i,\ I_y] \begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ a_2 \\ b_2 \\ c_2 \end{bmatrix} - I_t(x_i, y_i).$$

$\Rightarrow$ Matrix form $Ap = b$

For all pixels $i = 1, \ldots, N$ in the window we stack these equations:

$Ap = b$

where each row of A is

$$A_i = [I_x(x_i, y_i),\ I_x(x_i, y_i) y_i,\ I_x(x_i, y_i),\ I_y(x_i, y_i),\ I_y(x_i, y_i) y_i,\ I_y(x_i, y_i)]$$

and

$$b_i = -I_t(x_i, y_i)$$

so

$$A = \begin{bmatrix} I_x x_1 & I_x y_1 & I_x & I_y x_1 & I_y y_1 & I_y \\ I_x x_2 & I_x y_2 & I_x & I_y x_2 & I_y y_2 & I_y \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ I_x x_N & I_x y_N & I_x & I_y x_N & I_y y_N & I_y \end{bmatrix},\quad b = \begin{bmatrix} I_t(x_1, y_1) \\ I_t(x_2, y_2) \\ \vdots \\ I_t(x_N, y_N) \end{bmatrix}$$

11/16/2025

Least - Squares solutions:

In general we have more pixels than parameters ($N >> 6$) so the system is overdetermined.
Lucas-Kanade computes the motion parameters "p" in the least-squares sense, by minimizing

$$\|Ap - b\|^2$$

The normal equations are

$$A^T A p = A^T b,$$

so the parameter estimate is

$$p = (A^T A)^{-1} A^T b$$

This gives the six affine motion parameters
$(a_1, b_2, c_1, a_2, b_1, c_2)^T$.