Thm 6.30

Let $\{a_k\}$ and $\{b_k\}$ be real sequence.

For all pairs of integers $n \geq m \geq 1$ set $A(n,m) \coloneqq \sum_{k=m}^{n} a_k$

Then $\sum_{k=m}^{n} a_k b_k = A(n,m)b_n - \sum_{k=m}^{n-1} A(k,m)(b_{k+1} - b_k)$

Thm 7.21.ii

$S(x) = \sum_{k=0}^{\infty} a_k\,(x - x_0)^k$ *be a power series centered at* x_0

(ii) S(x) converges uniformly on any closed interval $[a,b] \subset (x_0 - R, x_0 + R)$

Thm 7.26

If $f(x) := \sum_{k=0}^{\infty} a_k (x - x_0)^k$ is a power series with positive radius of convergence R, then f is continuous on $(x_0 - R, x_0 + R)$

Thm 7.27 [ABEL's Theorem]

Suppose that [a,b] is nondegenerate. If $f(x) := \sum_{k=0}^{\infty} a_k (x - x_0)^k$ converges on [a,b], then f(x) is continuous and converges uniformly on [a,b].

Pf.

By thm 7.21.ii and thm 7.26, we may suppose that f has positive, finite radius of convergence R, and by symmetry , that $a = x_0$ and $b = x_0 + R$.
Thus suppose that f(x) converges at $x = x_0 + R$ and fix $x1 \in (x_0, x_0 + R]$.

Set $b_k = a_k R^k$ and $c_k = (x_1 - x_0)^k / R^k$ for $k \in N$

By hypothesis, $f(x) := \sum_{k=0}^{\infty} a_k (x - x_0)^k$ converges

But $f(b) = \sum_{k=0}^{\infty} a_k (x_0 + R - x_0)^k = \sum_{k=0}^{\infty} a_k R^k \quad \therefore \sum_{k=1}^{\infty} b_k$ converges

Hence, $\begin{cases} \forall \varepsilon > 0 \\ \exists N_0 > 1 \end{cases} st. k > m \geq N_0 \ implies \left|\sum_{j=m}^{k} b_j\right| < \varepsilon$

Since $0 < x_1 - x_0 \leq R$, the sequence $\{c_k\}$ is decreasing.

Applying Abel's Formula and telescoping (by page 174, thm 6.30)

(相當於現在用b_k, c_k代入人家的a_k, b_k)

(先定義這裡 $A(n,m) = \sum_{k=m}^{n} b_k$)

$$\left|\sum_{k=m}^{n} a_k (x_1 - x_0)^k\right| = \left|\sum_{k=m}^{n} b_k c_k\right| = \left|A(n,m)c_n - \sum_{m}^{n-1} A(k,m)(c_{k+1} - c_k)\right| = \left|c_n \sum_{k=m}^{n} b_k + \sum_{k=m}^{n-1} [(c_k - c_{k+1}) \sum_{j=m}^{k} b_j]\right|$$

絕對值內兩項，第一項單純，仔細拆解第二項

$$第一項 = c_n \times \sum_{k=m}^{n} b_k < c_n \times \varepsilon$$

$$第二項 = (c_m - c_{m+1}) \sum_{j=m}^{m} b_j + (c_{m+1} - c_{m+2}) \sum_{j=m}^{m+1} b_j + (c_{m+1} - c_{m+2}) \sum_{j=m}^{m+2} b_j + \cdots\cdots\cdots + (c_{n-1} - c_n) \sum_{j=m}^{n-1} b_j$$

$$< \left(\sum_{j=m}^{n} b_j\right) \times [(c_m - c_{m+1}) + (c_{m+1} - c_{m+2}) + \cdots\cdots\cdots (c_{n-1} - c_n)]$$

$$= \varepsilon \times (c_m - c_n)$$

$$\therefore |第一項 + 第二項| < \varepsilon \times c_n + \varepsilon(c_m - c_n) = \varepsilon \times c_m$$

Since $c_m \leq c_1 \leq \frac{R}{R} = 1$ $ie. c_m \leq 1$, it follows that

$$\left|\sum_{k=m}^{n} a_k (x_1 - x_0)^k\right| < \varepsilon \; for \; \forall x_1 \in (x_0, x_0 + R]$$

Since this inequality also holds for $x_0 = x_1$

We conclude that $\sum_{k=m}^{n} a_k (x_1 - x_0)^k$ converge uniformly on $[x_0, x_0 + R]$. <proof is over>

我的疑問是：有網底的部分真的會成立嗎？

我的癥結是：確定$\left|\sum_{j=m}^{k} b_j\right| < \varepsilon$以後，就能確定$\left|\sum_{j=m}^{m} b_j\right| < \varepsilon$, $\left|\sum_{j=m}^{m+1} b_j\right| < \varepsilon$, $\left|\sum_{j=m}^{m+2} b_j\right| < \varepsilon$, $\left|\sum_{j=m}^{m+3} b_j\right| < \varepsilon$, $\left|\sum_{j=m}^{m+4} b_j\right| < \varepsilon \cdots\cdots\cdots$, $\left|\sum_{j=m}^{k-1} b_j\right| < \varepsilon$ 嗎?

因為我們可以很自由的操作$\{a_k\}$

比如 $R = \frac{1}{2}$, $a_5 = 100, a_6 = (-1) = a_7 = a_8 = \cdots$ (*一直到無限項*)

$$b_5 = 100 * \frac{1}{32} = 3.125, b_6 = \frac{-1}{64}, b_7 = \frac{-1}{128}, b_8 = \frac{-1}{256}, b_9 = \frac{-1}{512}, \dots,$$

此時對 b 數列加總應該會越多項越小吧?

所以此時$\left|\sum_{j=5}^{k} b_j\right| \doteqdot 3.125 + \frac{\frac{-1}{64}}{1-\frac{1}{2}} = 3.09375$ (for large k)

所以假設我們取$\varepsilon = 3.09375$

如果只加總第一項，就變成$\left|\sum_{j=5}^{5} b_j\right| = 3.125 < 3.09375$（矛盾）

所以紅色的式子應該改成

$$\begin{cases} \forall \varepsilon > 0 \\ \exists N_0 > 1 \end{cases} st.k > m \geq N_0 \;\; implies \begin{cases} \left|\sum_{j=m}^{k} b_j\right| < \varepsilon \\ \left|\sum_{j=m}^{m} b_j\right| < \varepsilon \\ \left|\sum_{j=m}^{m+1} b_j\right| < \varepsilon \\ ,\left|\sum_{j=m}^{m+2} b_j\right| < \varepsilon \\ \left|\sum_{j=m}^{m+3} b_j\right| < \varepsilon \\ \vdots \\ \left|\sum_{j=m}^{k-1} b_j\right| < \varepsilon \end{cases}$$