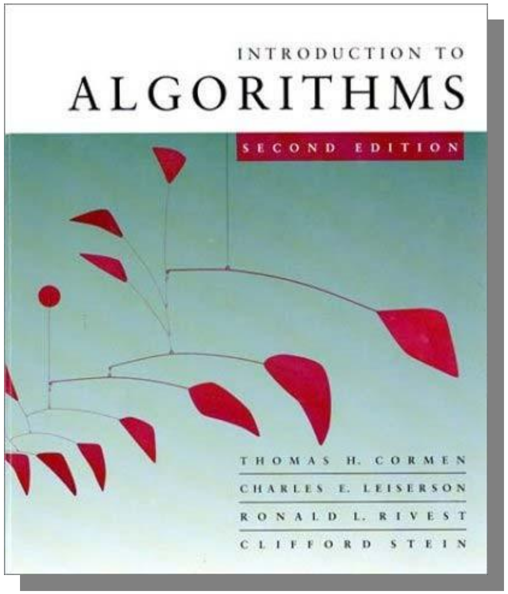
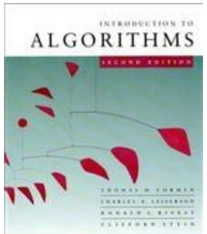
Introduction to Algorithms
6.046J/18.401J



LECTURE 1
Analysis of Algorithms
- Insertion sort
- Merge sort

Prof. Charles E. Leiserson

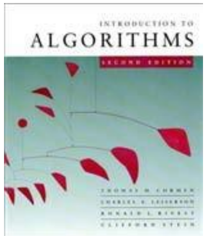


Analysis of algorithms

The theoretical study of computer-program performance and resource usage.

What's more important than performance?

- modularity
- correctness
- maintainability
- functionality
- robustness
- user-friendliness
- programmer time
- simplicity
- extensibility
- reliability



The problem of sorting

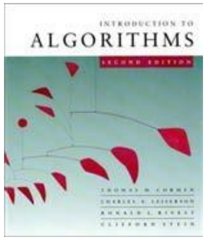
Input: sequence $\langle a_1, a_2, \ldots, a_n \rangle$ of numbers.

Output: permutation $\langle a'_1, a'_2, \ldots, a'_n \rangle$ such that $a'_1 \le a'_2 \le \cdots \le a'_n$.

Example:

Input: 8 2 4 9 3 6

Output: 2 3 4 6 8 9



Insertion sort

"pseudocode"

INSERTION-SORT (A, n) $\quad \triangleleft \; A[1 \,.\,.\, n]$

$\quad$ **for** $j \leftarrow 2$ **to** n

$\qquad$ **do** $key \leftarrow A[\,j]$

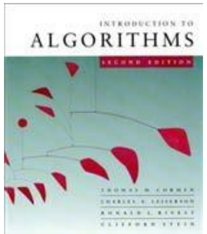
$\qquad\quad i \leftarrow j - 1$

$\qquad\quad$ **while** $i > 0$ and $A[i] > key$

$\qquad\qquad$ **do** $A[i+1] \leftarrow A[i]$

$\qquad\qquad\quad i \leftarrow i - 1$

$\qquad\quad A[i+1] = key$



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for $j \leftarrow 2$ **to** n

  do $key \leftarrow A[j]$

    $i \leftarrow j - 1$

    while $i > 0$ and $A[i] > key$

      do $A[i+1] \leftarrow A[i]$

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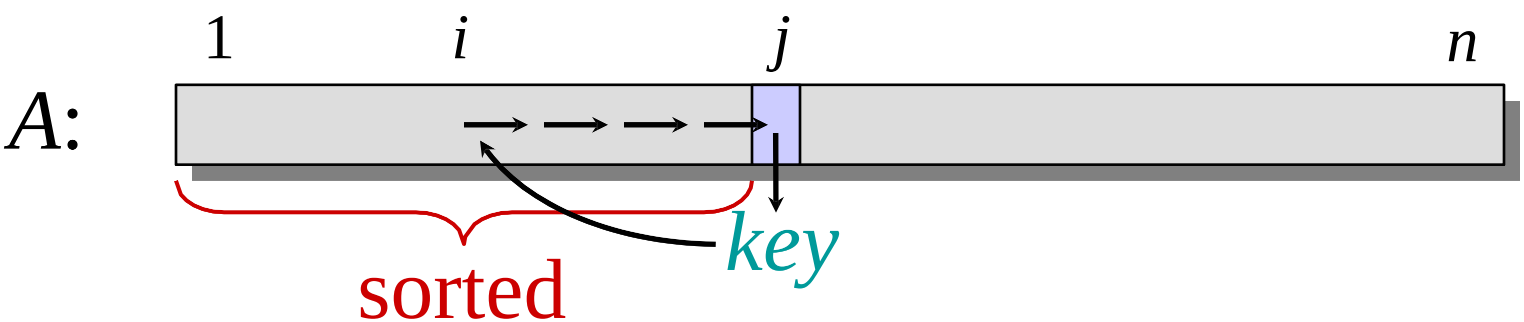
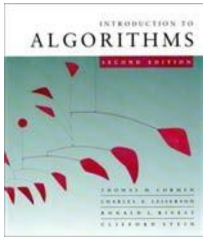
    $A[i+1] = key$





Example of insertion sort

8 2 4 9 3 6

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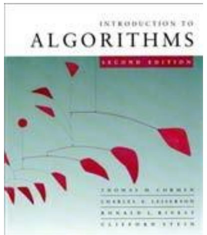

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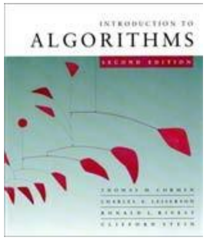

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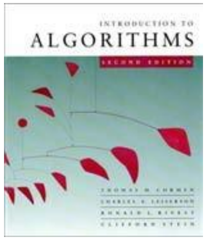

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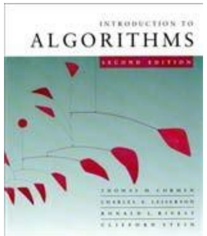

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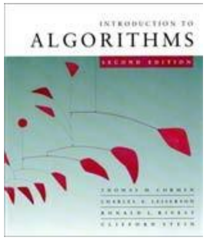

Example of insertion sort

8	2	4	9	3	6
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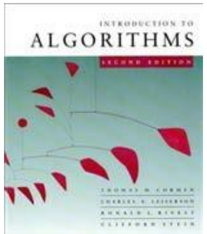
2	4	8	9	3	6
2	4	8	9	3	6



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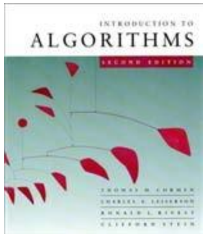

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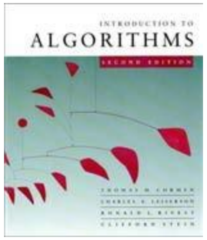

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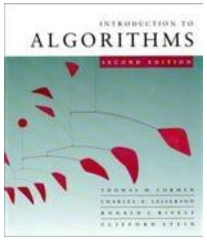
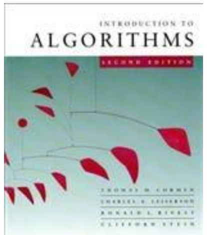


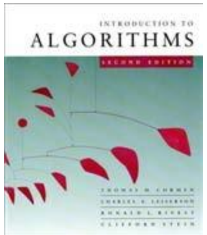
Example of insertion sort

8	2	4	9	3	6	
2	8	4	9	3	6	
2	4	8	9	3	6	
2	4	8	9	3	6	
2	3	4	8	9	6	
2	3	4	6	8	9	*done*



Running time

- The running time depends on the input: an already sorted sequence is easier to sort.
- Parameterize the running time by the size of the input, since short sequences are easier to sort than long ones.
- Generally, we seek upper bounds on the running time, because everybody likes a guarantee.



Kinds of analyses

Worst-case: (usually)

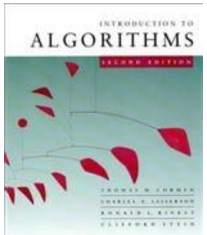
- $T(n)$ = maximum time of algorithm on any input of size n.

Average-case: (sometimes)

- $T(n)$ = expected time of algorithm over all inputs of size n.
- Need assumption of statistical distribution of inputs.

Best-case: (bogus)

- Cheat with a slow algorithm that works fast on *some* input.



Machine-independent time

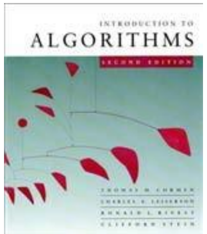
What is insertion sort's worst-case time?

- It depends on the speed of our computer:
  - relative speed (on the same machine),
  - absolute speed (on different machines).

BIG IDEA:

- Ignore machine-dependent constants.
- Look at ***growth*** of $T(n)$ as $n \to \infty$.

"Asymptotic Analysis"



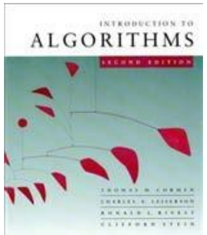
Θ-notation

Math:

$\Theta(g(n)) = \{ f(n)$: there exist positive constants c_1, c_2, and n_0 such that $0 \le c_1 g(n) \le f(n) \le c_2 g(n)$ for all $n \ge n_0 \}$

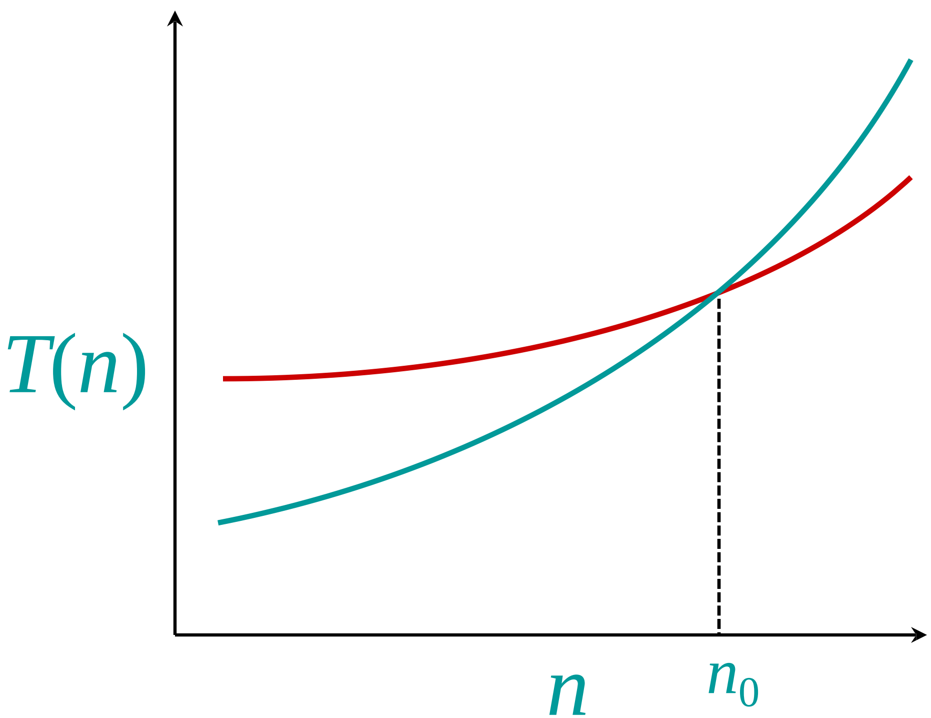
Engineering:

- Drop low-order terms; ignore leading constants.
- Example: $3n^3 + 90n^2 - 5n + 6046 = \Theta(n^3)$

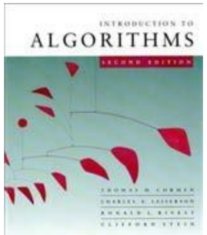


Asymptotic performance

When n gets large enough, a $\Theta(n^2)$ algorithm *always* beats a $\Theta(n^3)$ algorithm.



- We shouldn't ignore asymptotically slower algorithms, however.
- Real-world design situations often call for a careful balancing of engineering objectives.
- Asymptotic analysis is a useful tool to help to structure our thinking.



Insertion sort analysis

Worst case: Input reverse sorted.

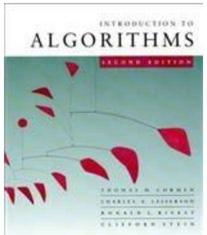
$$T(n)=\sum_{j=2}^{n}\Theta(j)=\Theta\left(n^2\right) \qquad \text{[arithmetic series]}$$

Average case: All permutations equally likely.

$$T(n)=\sum_{j=2}^{n}\Theta(j/2)=\Theta\left(n^2\right)$$

Is insertion sort a fast sorting algorithm?

- Moderately so, for small n.
- Not at all, for large n.

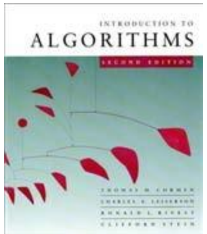


Merge sort

MERGE-SORT $A[1 \,..\, n]$

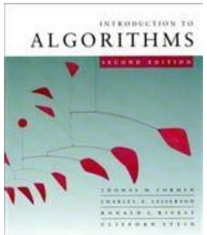
1. If $n = 1$, done.
2. Recursively sort $A[\,1 \,..\, \lceil n/2 \rceil\,]$ and $A[\,\lceil n/2 \rceil + 1 \,..\, n\,]$.
3. "***Merge***" the 2 sorted lists.

Key subroutine: **MERGE**



Merging two sorted arrays

20	12
13	11
7	9
2	1



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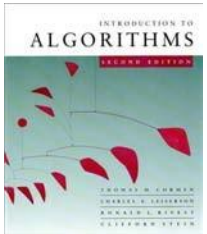
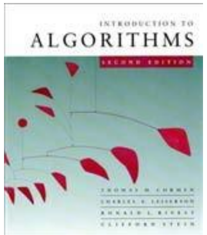


Merging two sorted arrays

20	12		20	12
13	11		13	11
7	9		7	9
2	1		2	

1



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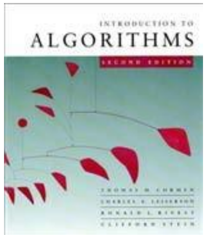
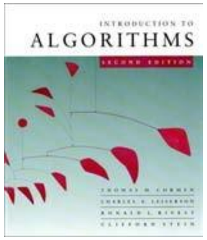
13	11		13	11
7	9		7	9
2	1		2	

1 2



Merging two sorted arrays

20	12	20	12	20	12
13	11	13	11	13	11
7	9	7	9	7	9
2	1	2			
1		2			



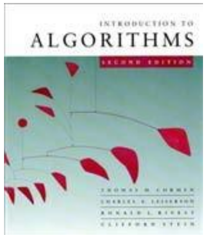
Merging two sorted arrays

20	12	20	12	20	12
13	11	13	11	13	11
7	9	7	9	7	9
2	1	2			
1		2		7	

Merging two sorted arrays

20	12	20	12	20	12	20	12
13	11	13	11	13	11	13	11
7	9	7	9	7	9		9
2	1	2					

1 2 7



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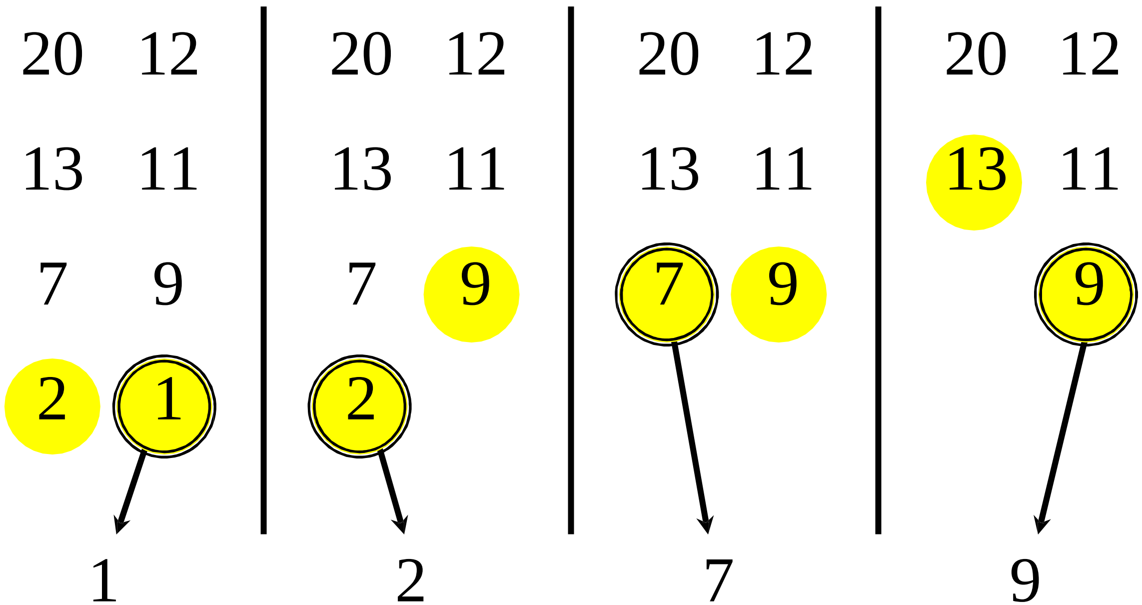

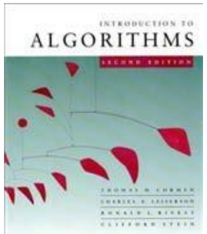

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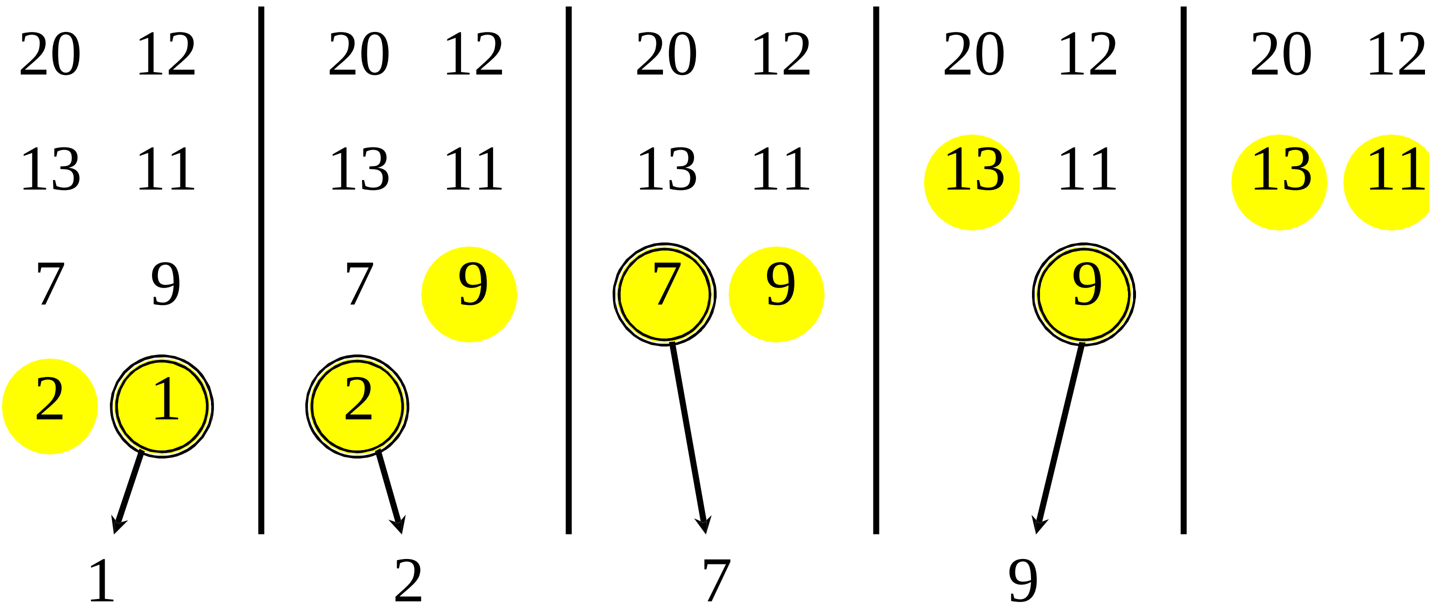

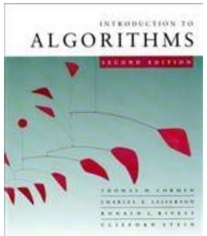

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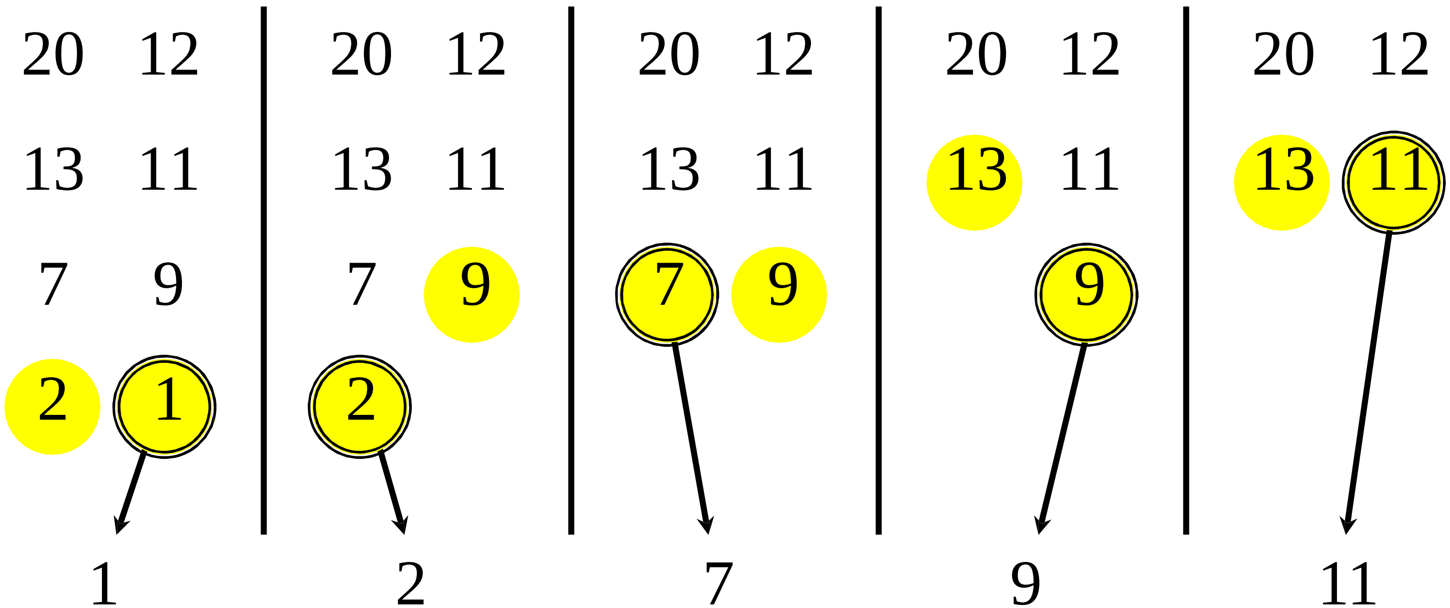
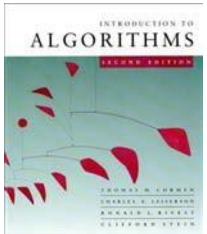
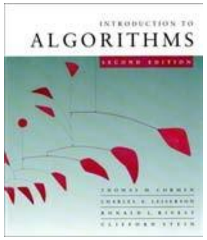




Merging two sorted arrays

20	12	20	12	20	12	20	12	20	12	20	12
13	11	13	11	13	11	13	11	13	11	13	
7	9	7	9	7	9		9				
2	1	2									
1		2		7		9		11			



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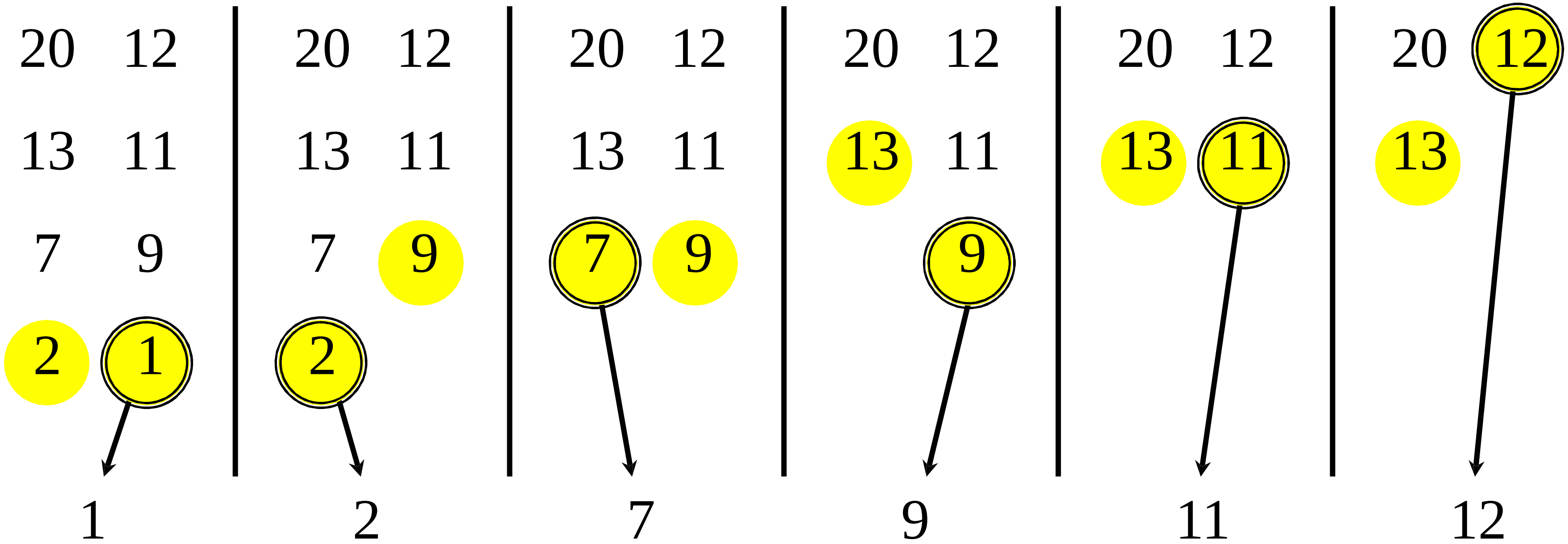

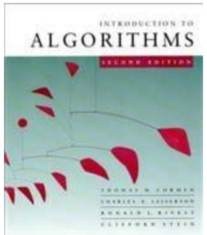

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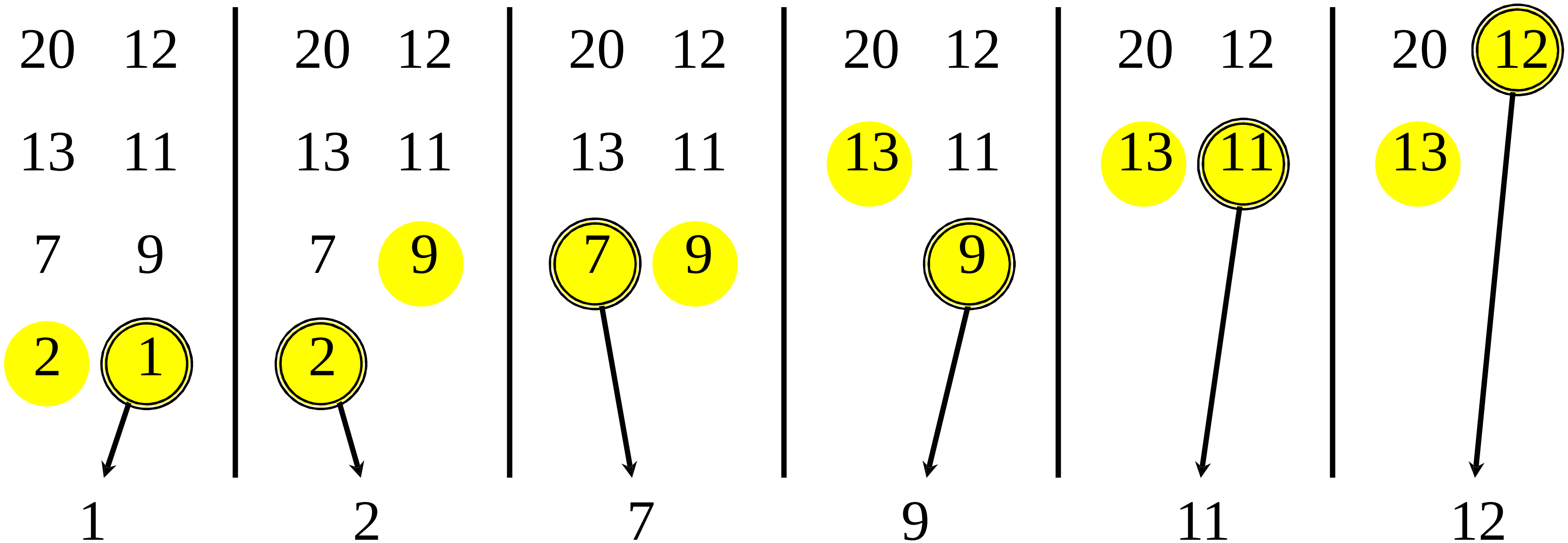
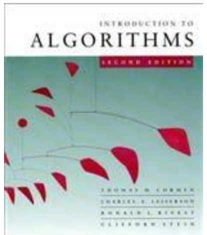


Time = $\Theta(n)$ to merge a total of n elements (linear time).

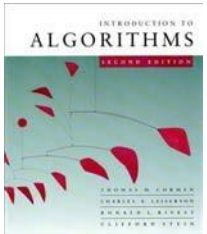


Analyzing merge sort

$T(n)$	**MERGE-SORT** $A[1 \,..\, n]$
$\Theta(1)$	1. If $n = 1$, done.
$2T(n/2)$	2. Recursively sort $A[\,1 \,..\, \lceil n/2 \rceil\,]$ and $A[\,\lceil n/2 \rceil + 1 \,..\, n\,]$.
$\Theta(n)$	**3.** ***"Merge"*** the 2 sorted lists

Abuse

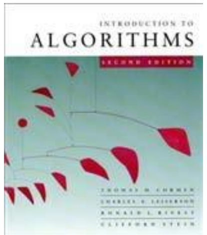
Sloppiness: Should be $T(\lceil n/2 \rceil) + T(\lfloor n/2 \rfloor)$, but it turns out not to matter asymptotically.



Recurrence for merge sort

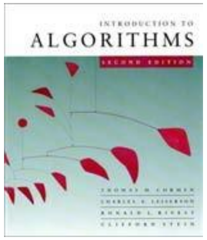
$$T(n) = \begin{cases} \Theta(1) \text{ if } n = 1; \\ 2T(n/2) + \Theta(n) \text{ if } n > 1. \end{cases}$$

- We shall usually omit stating the base case when $T(n) = \Theta(1)$ for sufficiently small n, but only when it has no effect on the asymptotic solution to the recurrence.
- CLRS and Lecture 2 provide several ways to find a good upper bound on $T(n)$.



Recursion tree

Solve $T(n) = 2T(n/2) + cn$, where $c > 0$ is constant.



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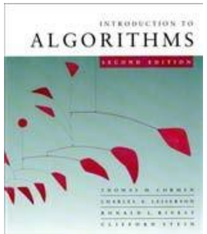
$$T(n)$$



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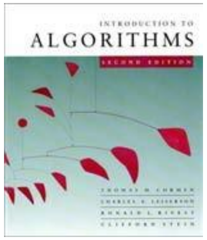
cn

$T(n/2)$ $T(n/2)$



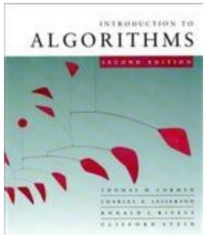
Recursion tree

Solve $T(n) = 2T(n/2) + cn$, where $c > 0$ is constant.

$$cn$$

$$cn/2 \qquad\qquad cn/2$$

$$T(n/4) \qquad T(n/4) \qquad T(n/4) \qquad T(n/4)$$



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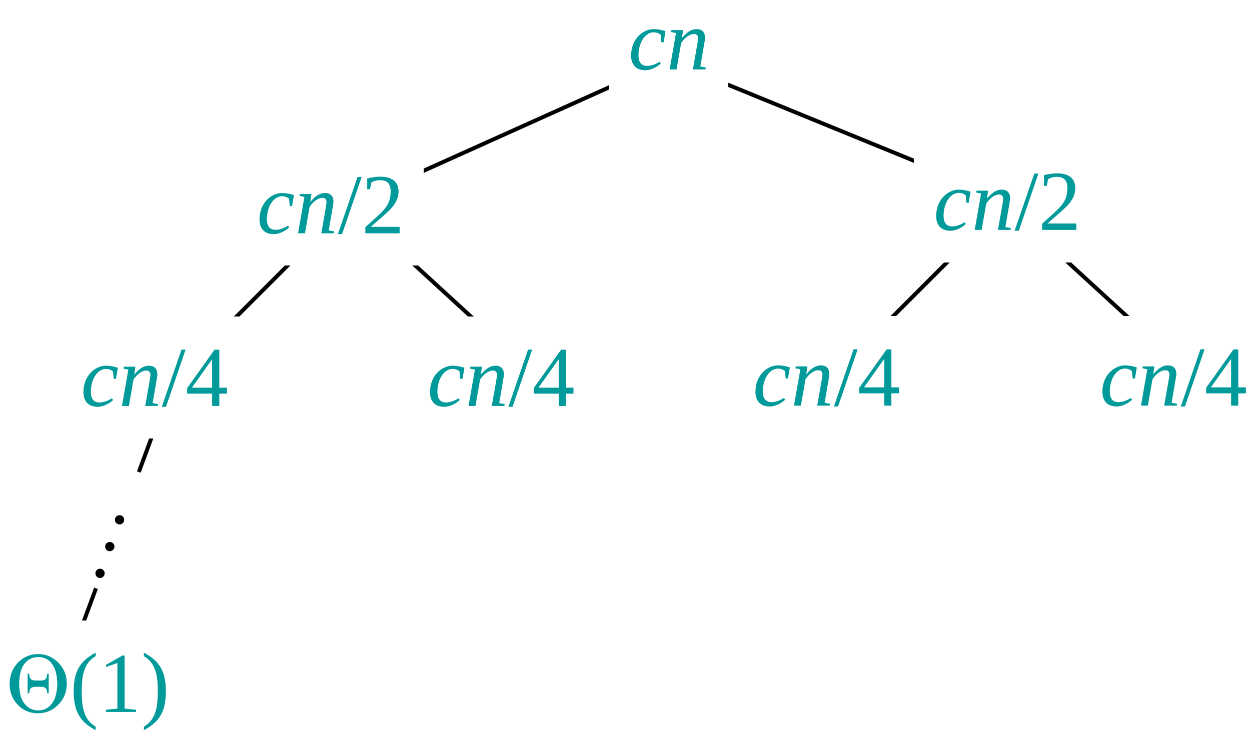

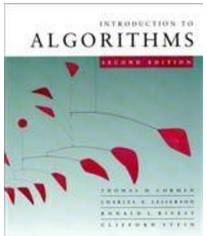

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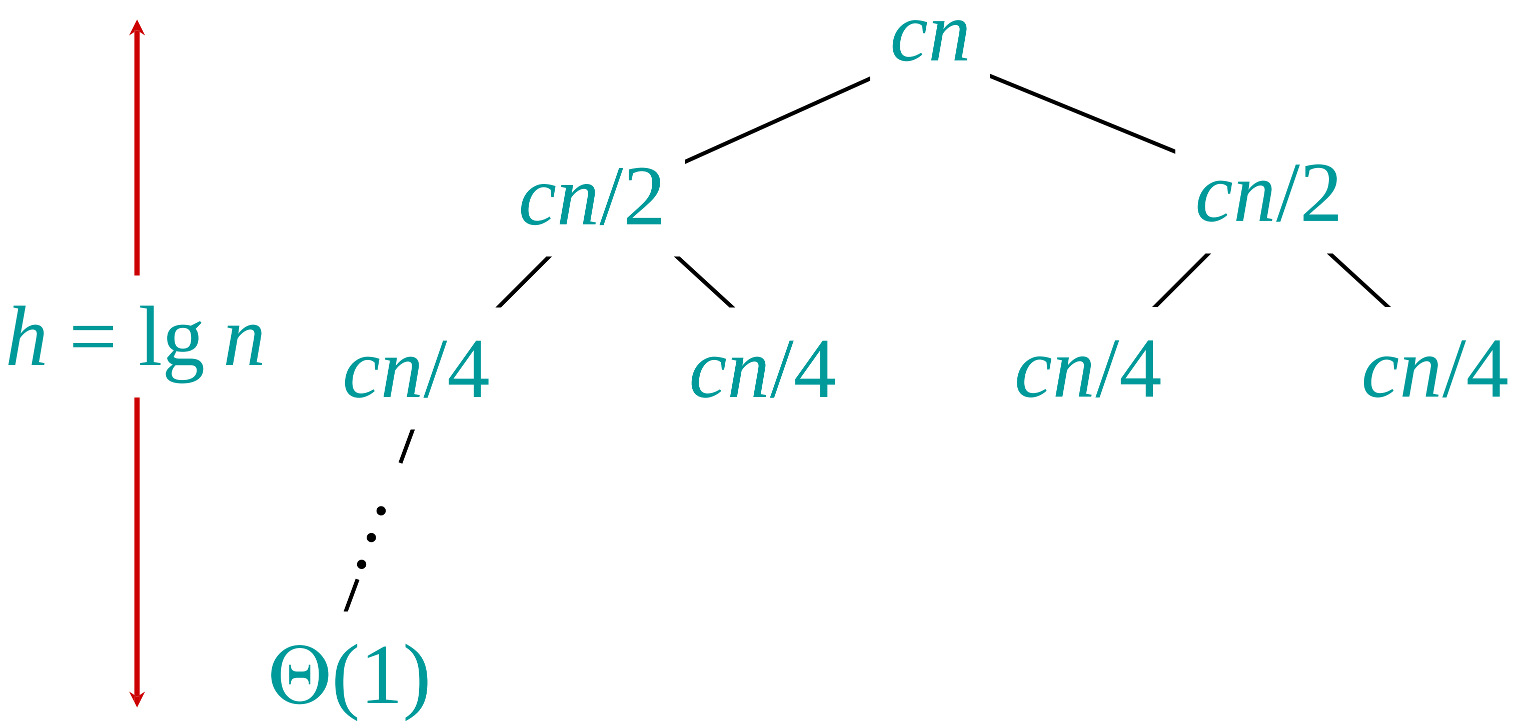

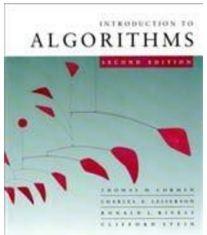

Recursion tree

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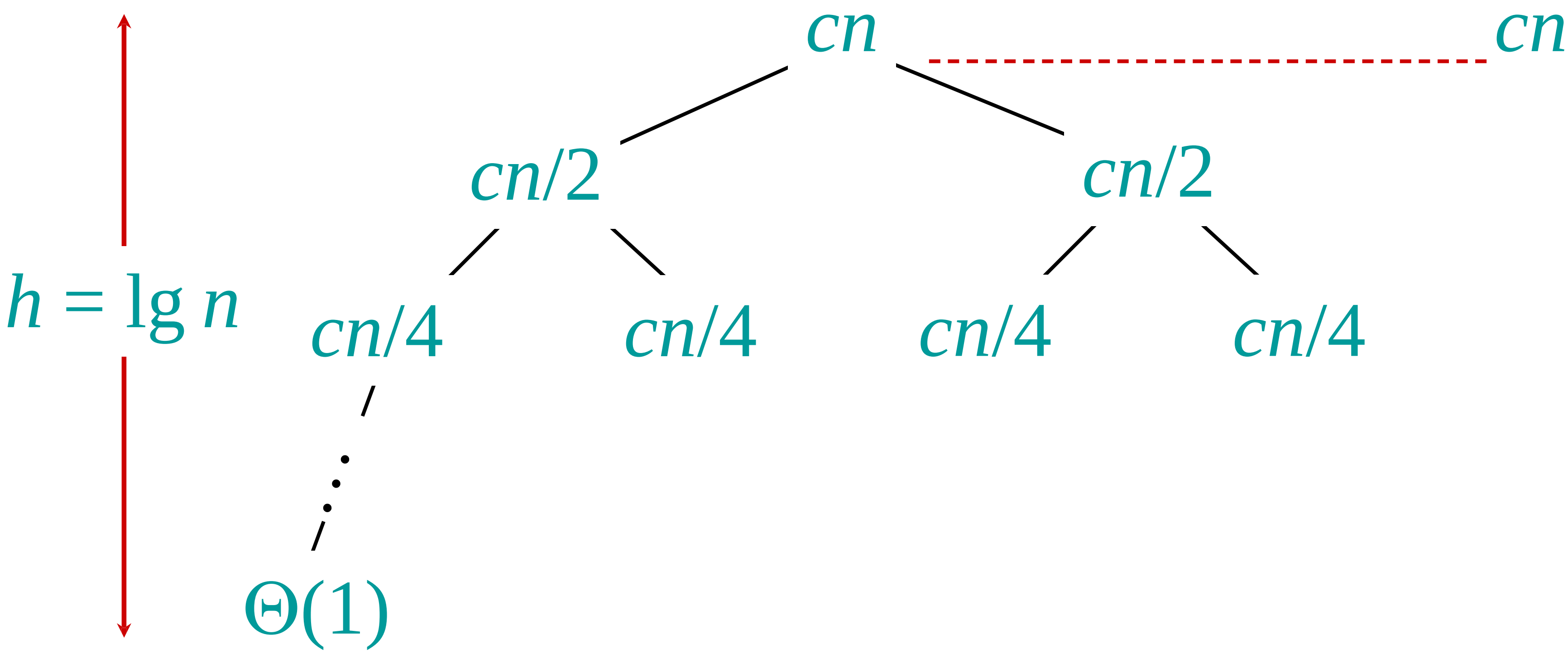

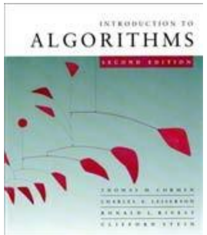

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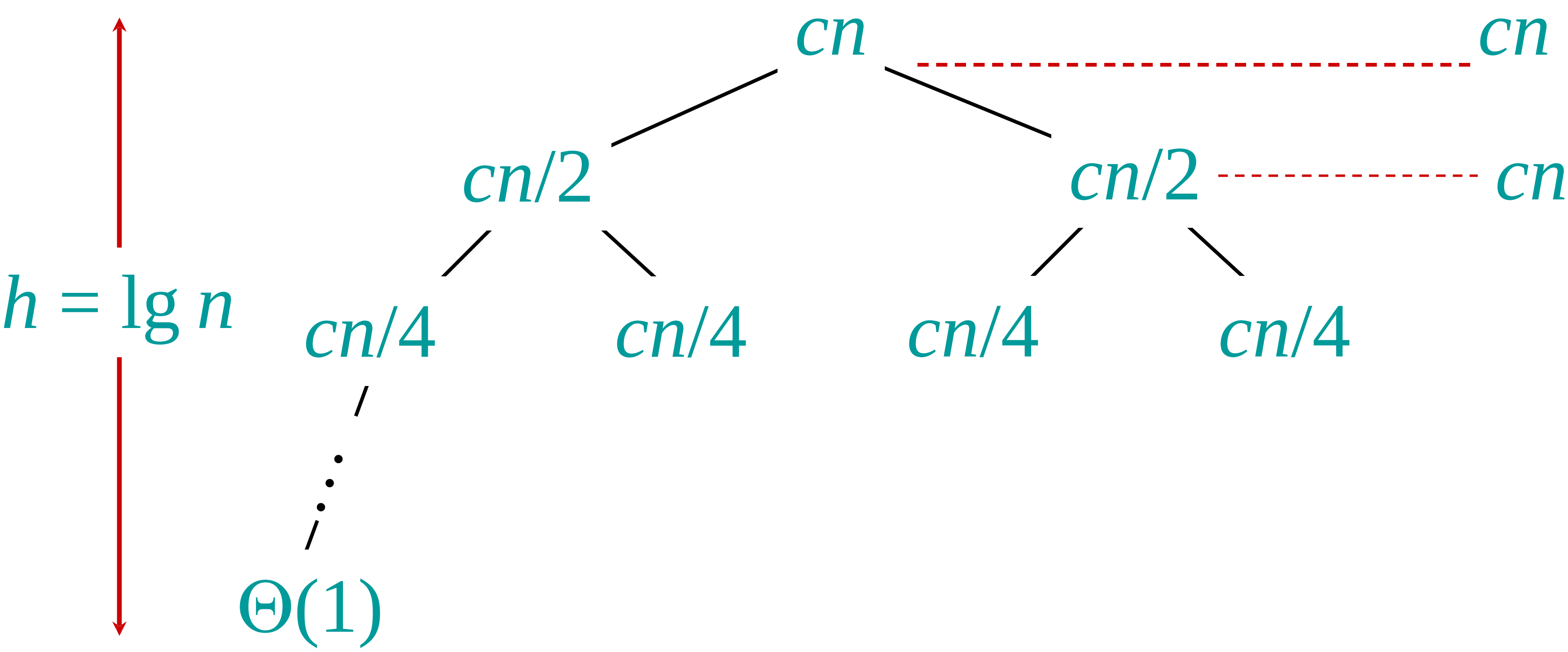

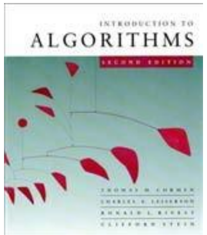

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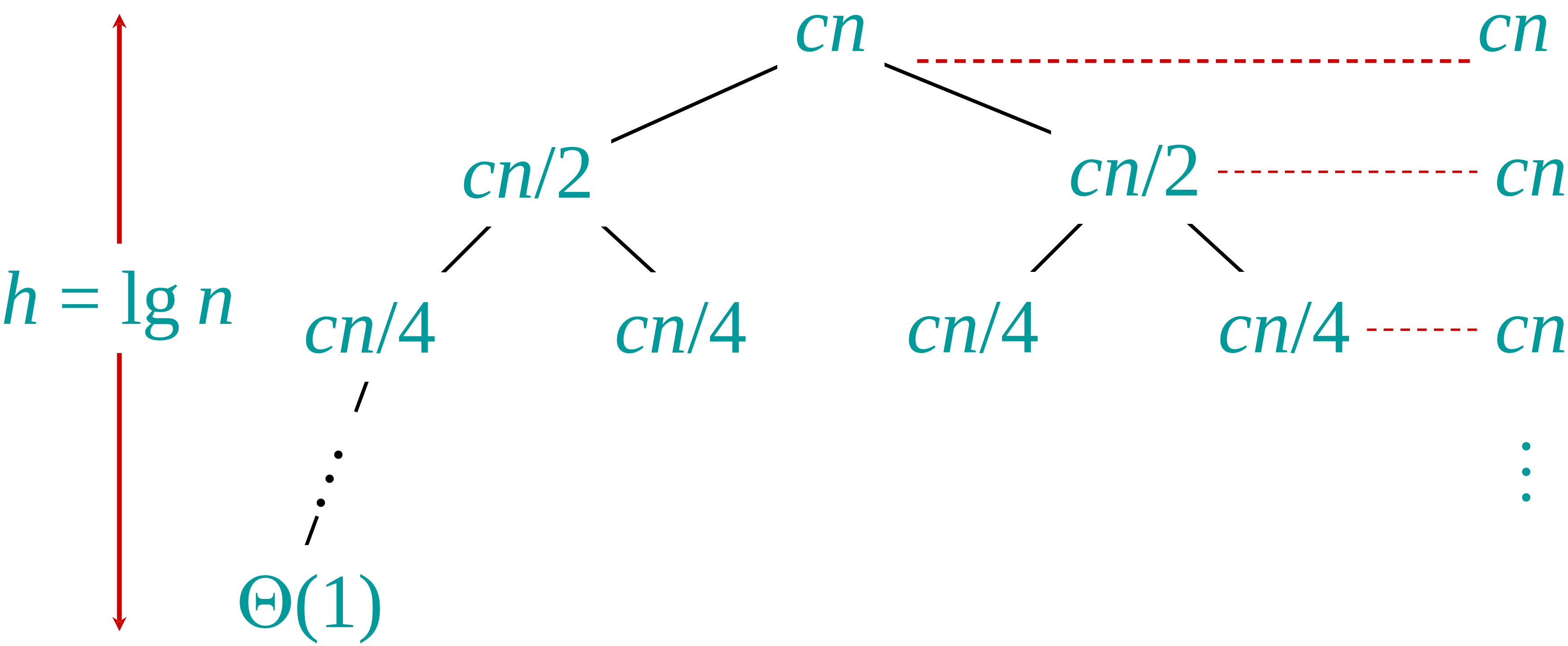

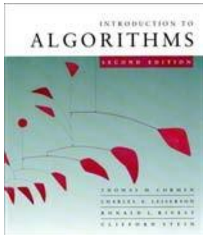

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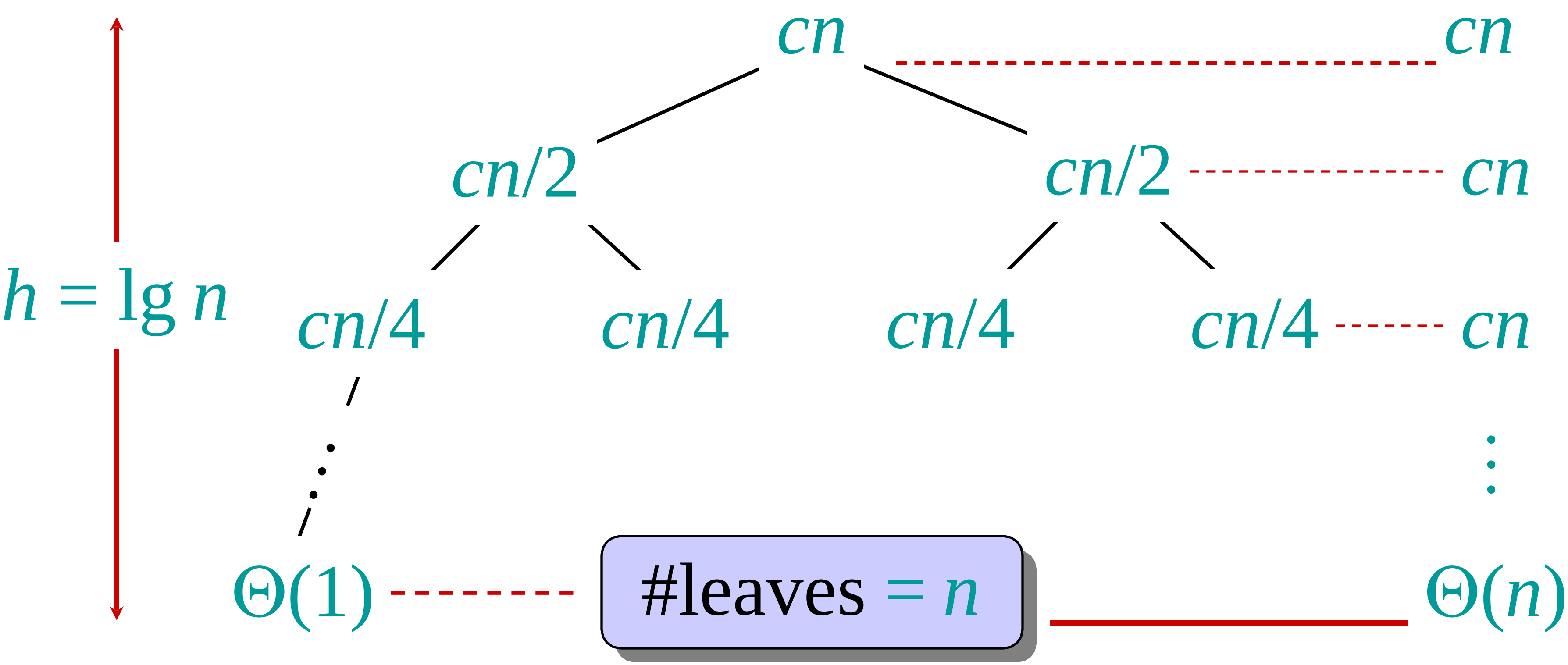

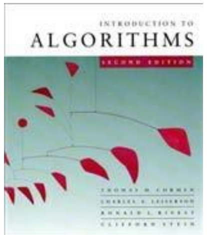

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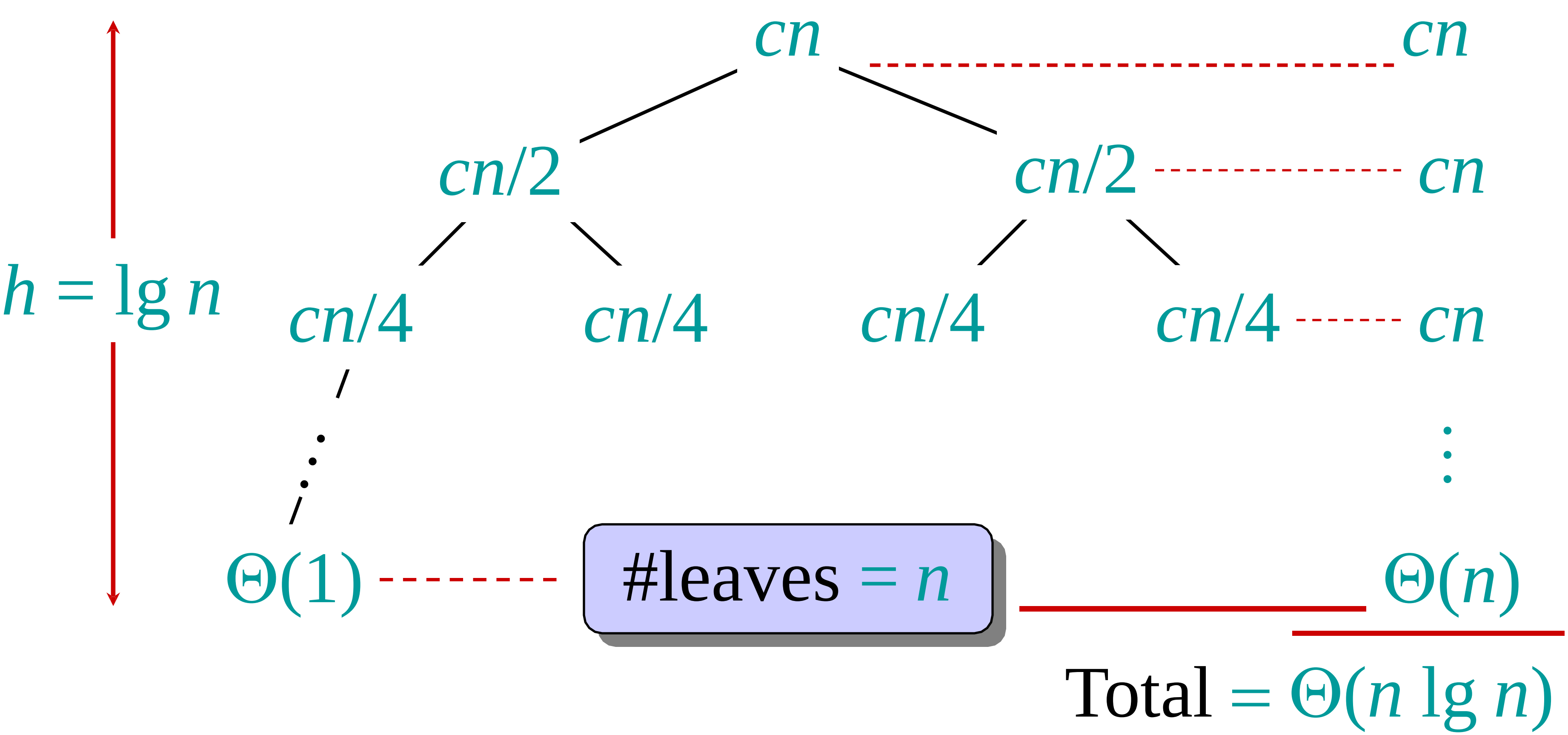

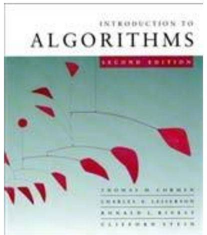

Conclusions

- $\Theta(n \lg n)$ grows more slowly than $\Theta(n^2)$.
- Therefore, merge sort asymptotically beats insertion sort in the worst case.
- In practice, merge sort beats insertion sort for $n > 30$ or so.
- Go test it out for yourself!