Counting money: Greedy and Dynamic Programming

Example: Counting money

- Suppose you want to count out a certain amount of money, using the fewest possible bills and coins
- A greedy algorithm would do this would be: At each step, take the largest possible bill or coin that does not overshoot
  - Example: To make $6.39, you can choose:
    - a $5 bill
    - a $1 bill, to make $6
    - a 25¢ coin, to make $6.25
    - A 10¢ coin, to make $6.35
    - four 1¢ coins, to make $6.39
- For US money, the greedy algorithm always gives the optimum solution

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A failure of the greedy algorithm

- In some (fictional) monetary system, “krons” come in 1 kron, 7 kron, and 10 kron coins
- Using a greedy algorithm to count out 15 krons, you would get
  - A 10 kron piece
  - Five 1 kron pieces, for a total of 15 krons
  - This requires six coins
- A better solution would be to use two 7 kron pieces and one 1 kron piece
  - This only requires three coins
- The greedy algorithm results in a solution, but not in an optimal solution

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Counting coins

- To find the minimum number of US coins to make any amount, the greedy method always works
  - At each step, just choose the largest coin that does not overshoot the desired amount: 31¢=25
- The greedy method would not work if we did not have 5¢ coins
  - For 31 cents, the greedy method gives seven coins (25+1+1+1+1+1+1), but we can do it with four (10+10+10+1)
- The greedy method also would not work if we had a 21¢ coin
  - For 63 cents, the greedy method gives six coins (25+25+10+1+1+1), but we can do it with three (21+21+21)
- How can we find the minimum number of coins for any given coin set?

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Coin set for examples

- For the following examples, we will assume coins in the following denominations:
  1¢ 5¢ 10¢ 21¢ 25¢
- We'll use 63¢ as our goal

A simple solution

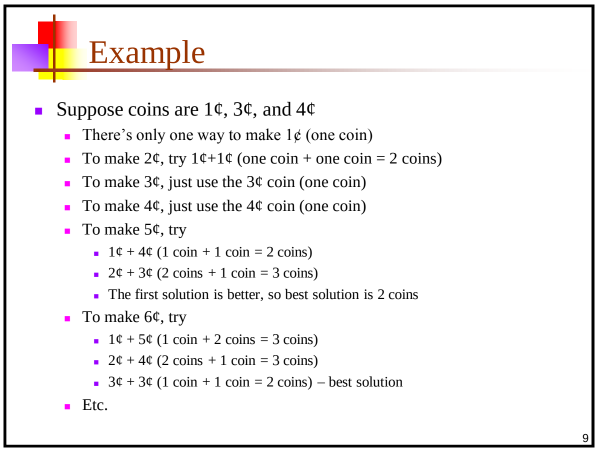
- We always need a 1¢ coin, otherwise no solution exists for making one cent
- To make K cents:
  - If there is a K-cent coin, then that one coin is the minimum
  - Otherwise, for each value i < K,
    - Find the minimum number of coins needed to make i cents
    - Find the minimum number of coins needed to make K - i cents
  - Choose the i that minimizes this sum
- This algorithm can be viewed as divide-and-conquer, or as brute force
  - This solution is recursive
  - It requires exponential work
  - It is *infeasible* to solve for 63¢

Another solution

- We can reduce the problem recursively by choosing the first coin, and solving for the amount that is left
- For 63¢:
  - One 1¢ coin plus the best solution for 62¢
  - One 5¢ coin plus the best solution for 58¢
  - One 10¢ coin plus the best solution for 53¢
  - One 21¢ coin plus the best solution for 42¢
  - One 25¢ coin plus the best solution for 38¢
- Choose the best solution from among the 5 given above
- Instead of solving 62 recursive problems, we solve 5
- Still a very expensive algorithm

A dynamic programming solution

- Idea: Solve first for one cent, then two cents, then three cents, etc., up to the desired amount
  - *Save each answer in an array !*
- For each new amount N, compute all the possible pairs of previous answers which sum to N
  - For example, to find the solution for 13¢,
    - First, solve for all of 1¢, 2¢, 3¢, ..., 12¢
    - Next, choose the best solution among:
      - Solution for 1¢ + solution for 12¢
      - Solution for 2¢ + solution for 11¢
      - Solution for 3¢ + solution for 10¢
      - Solution for 4¢ + solution for 9¢
      - Solution for 5¢ + solution for 8¢
      - Solution for 6¢ + solution for 7¢



Example

- Suppose coins are 1¢, 3¢, and 4¢
  - There’s only one way to make 1¢ (one coin)
  - To make 2¢, try 1¢+1¢ (one coin + one coin = 2 coins)
  - To make 3¢, just use the 3¢ coin (one coin)
  - To make 4¢, just use the 4¢ coin (one coin)
  - To make 5¢, try
    - 1¢ + 4¢ (1 coin + 1 coin = 2 coins)
    - 2¢ + 3¢ (2 coins + 1 coin = 3 coins)
    - The first solution is better, so best solution is 2 coins
  - To make 6¢, try
    - 1¢ + 5¢ (1 coin + 2 coins = 3 coins)
    - 2¢ + 4¢ (2 coins + 1 coin = 3 coins)
    - 3¢ + 3¢ (1 coin + 1 coin = 2 coins) – best solution
  - Etc.

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