Divide and Conquer: Strassen's Algorithm, Fibonacci Numbers

Lecture 3

The divide-and-conquer design paradigm

1. *Divide* the problem (instance) into subproblems.

2. *Conquer* the subproblems by solving them recursively.

3. *Combine* subproblem solutions.

Example: merge sort

1. ***Divide:*** Trivial.
2. ***Conquer:*** Recursively sort 2 subarrays.
3. ***Combine:*** Linear-time merge.

$$T(n) = 2\,T(n/2) + O(n)$$

 # subproblems

subproblem size

work dividing and combining

Master theorem (reprise)

$$T(n) = a\,T(n/b) + f(n)$$

CASE 1: $f(n) = O(n^{\log_b a - \varepsilon})$
$\Rightarrow T(n) = \Theta(n^{\log_b a})$.

CASE 2: $f(n) = \Theta(n^{\log_b a}\lg^k n)$
$\Rightarrow T(n) = \Theta(n^{\log_b a}\lg^{k+1} n)$.

CASE 3: $f(n) = \Omega(n^{\log_b a + \varepsilon})$ and $af(n/b) \le cf(n)$
$\Rightarrow T(n) = \Theta(f(n))$.

Merge sort: $a = 2$, $b = 2 \Rightarrow n^{\log_b a} = n$
$\Rightarrow$ **CASE 2** $(k = 0) \Rightarrow T(n) = \Theta(n\lg n)$.

Binary search

Find an element in a sorted array:

1. ***Divide:*** Check middle element.
2. ***Conquer:*** Recursively search 1 subarray.
3. ***Combine:*** Trivial.

Example: Find 9

3	5	7	8	9	12	15

Binary search

Find an element in a sorted array:

1. Divide: Check middle element.

2. Conquer: Recursively search 1 subarray.

3. Combine: Trivial.

Example: Find 9

| 3 | 5 | 7 | 8 | 9 | 12 | 15 |

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Binary search

Find an element in a sorted array:

1. *Divide:* Check middle element.
2. *Conquer:* Recursively search 1 subarray.
3. *Combine:* Trivial.

Example: Find 9

<div align="center">

3   5   7   8   **9**   12   15

</div>

Binary search

Find an element in a sorted array:

1. Divide: Check middle element.

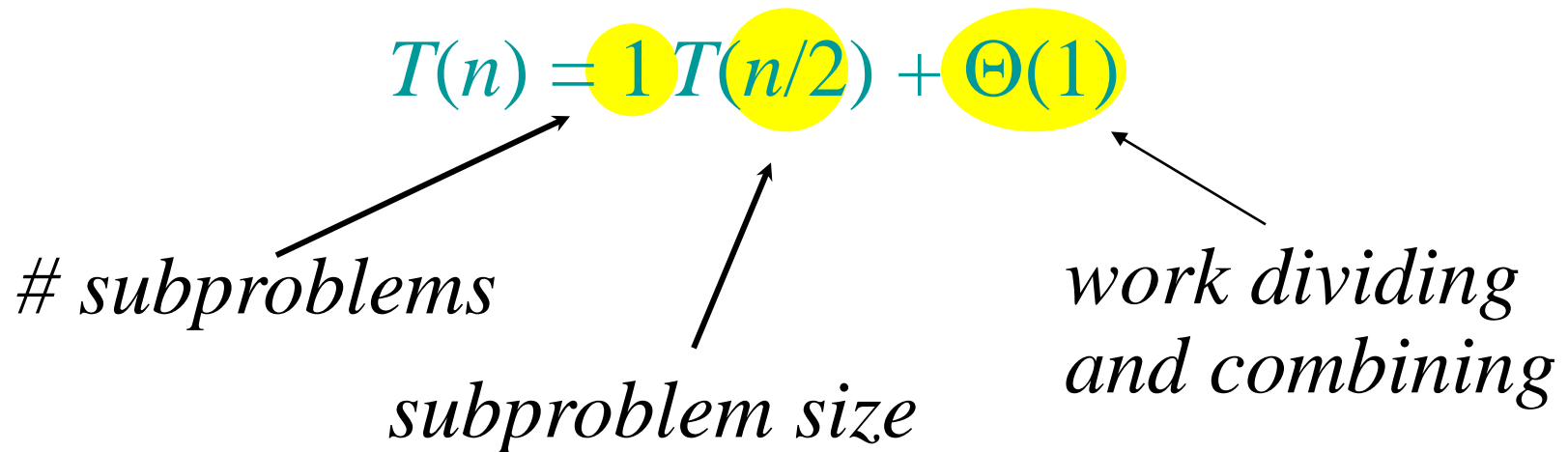*2. Conquer:* Recursively search 1 subarray.

3. Combine: Trivial.

Example: Find 9

3   5   7   8   **9**   12   15

Recurrence for binary search

$$T(n) = 1\,T(n/2) + \Theta(1)$$

 # subproblems

subproblem size

work dividing and combining

$$n^{\log_b a} = n^{\log_2 1} = n^0 = 1 \Rightarrow \text{CASE 2 } (k = 0)$$
$$\Rightarrow T(n) = \Theta(\lg n) \,.$$

Powering a number

Problem: Compute a^n, where $n \in \mathbf{N}$.

Naive algorithm: $\Theta(n)$.

Divide-and-conquer algorithm:

$$a^n = \begin{cases} a^{n/2} \cdot a^{n/2} & \text{if } n \text{ is even;} \\ a^{(n-1)/2} \cdot a^{(n-1)/2} \cdot a & \text{if } n \text{ is odd.} \end{cases}$$

$$T(n) = T(n/2) + \Theta(1) \implies T(n) = \Theta(\lg n) \,.$$

Fibonacci numbers

Recursive definition:

$$F_n = \begin{cases} 0 & \text{if } n = 0; \\ 1 & \text{if } n = 1; \\ F_{n-1} + F_{n-2} & \text{if } n \geq 2. \end{cases}$$

0 1 1 2 3 5 8 13 21 34 $\cdots$

Naive recursive algorithm: $\Omega(\phi^n)$ (exponential time), where $\phi = (1 + \sqrt{5})/2$ is the *golden ratio*.

Computing Fibonacci numbers

Naive recursive squaring:

$F_n = \phi^n/\sqrt{5}$ rounded to the nearest integer.

- Recursive squaring: $\Theta(\lg n)$ time.
- This method is unreliable, since floating-point arithmetic is prone to round-off errors.

Bottom-up:

- Compute F_0, F_1, F_2, …, F_n in order, forming each number by summing the two previous.
- Running time: $\Theta(n)$.

Recursive squaring

Theorem: $\begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n$.

Algorithm: Recursive squaring.
Time $= \Theta(\lg n)$.

Proof of theorem. (Induction on n.)

Base ($n = 1$): $\begin{bmatrix} F_2 & F_1 \\ F_1 & F_0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^1$.

Recursive squaring

Inductive step ($n \geq 2$):

$$\begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix} = \begin{bmatrix} F_n & F_{n-1} \\ F_{n-1} & F_{n-2} \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{n-1} \cdot \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n \quad \blacksquare$$

Matrix multiplication

Input: $\quad A = [a_{ij}],\ B = [b_{ij}].$
Output: $\ C = [c_{ij}] = A \cdot B.$ $\qquad\Bigg\}\ i,j = 1, 2, \ldots, n.$

$$\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}$$

$$c_{ij} = \sum_{k=1}^{n} a_{ik} \cdot b_{kj}$$

Standard algorithm

for $i \leftarrow 1$ **to** n

    do for $j \leftarrow 1$ **to** n

        do $c_{ij} \leftarrow 0$

            for $k \leftarrow 1$ **to** n

                do $c_{ij} \leftarrow c_{ij} + a_{ik} \cdot b_{kj}$

Running time $= \Theta(n^3)$

Divide-and-conquer algorithm

IDEA:
$n{\times}n$ matrix = $2{\times}2$ matrix of $(n/2){\times}(n/2)$ submatrices:

$$\begin{bmatrix} r & s \\ t & u \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

$$C \quad = \quad A \quad \cdot \quad B$$

$$\left.\begin{array}{l} r = ae + bg \\ s = af + bh \\ t = ce + dh \\ u = cf + dh \end{array}\right\}$$

8 mults of $(n/2){\times}(n/2)$ submatrices
4 adds of $(n/2){\times}(n/2)$ submatrices

Analysis of D&C algorithm

$$T(n) = 8\,T(n/2) + \Theta(n^2)$$

 # submatrices

submatrix size

work adding submatrices

$$n^{\log_b a} = n^{\log_2 8} = n^3 \implies \text{CASE 1} \implies T(n) = \Theta(n^3).$$

No better than the ordinary algorithm.

Strassen's idea

- Multiply 2×2 matrices with only 7 recursive mults.

$$P_1 = a \cdot (f - h)$$
$$P_2 = (a + b) \cdot h$$
$$P_3 = (c + d) \cdot e$$
$$P_4 = d \cdot (g - e)$$
$$P_5 = (a + d) \cdot (e + h)$$
$$P_6 = (b - d) \cdot (g + h)$$
$$P_7 = (a - c) \cdot (e + f)$$

$$r = P_5 + P_4 - P_2 + P_6$$
$$s = P_1 + P_2$$
$$t = P_3 + P_4$$
$$u = P_5 + P_1 - P_3 - P_7$$

7 mults, 18 adds/subs.
Note: No reliance on commutativity of mult!

Strassen's idea

- Multiply $2{\times}2$ matrices with only 7 recursive mults.

$$P_1 = a \cdot (f - h)$$
$$P_2 = (a + b) \cdot h$$
$$P_3 = (c + d) \cdot e$$
$$P_4 = d \cdot (g - e)$$
$$P_5 = (a + d) \cdot (e + h)$$
$$P_6 = (b - d) \cdot (g + h)$$
$$P_7 = (a - c) \cdot (e + f)$$

$$
\begin{aligned}
r &= P_5 + P_4 - P_2 + P_6 \\
&= (a + d)(e + h) \\
&\quad + d(g - e) - (a + b)h \\
&\quad + (b - d)(g + h) \\
&= ae + ah + de + dh \\
&\quad + dg - de - ah - bh \\
&\quad + bg + bh - dg - dh \\
&= ae + bg
\end{aligned}
$$

Strassen's algorithm

1. *Divide:* Partition A and B into $(n/2) \times (n/2)$ submatrices. Form terms to be multiplied using $+$ and $-$.

2. *Conquer:* Perform 7 multiplications of $(n/2) \times (n/2)$ submatrices recursively.

3. *Combine:* Form C using $+$ and $-$ on $(n/2) \times (n/2)$ submatrices.

$$T(n) = 7\,T(n/2) + \Theta(n^2)$$

Analysis of Strassen

$$T(n) = 7\,T(n/2) + \Theta(n^2)$$

$$n^{\log_b a} = n^{\log_2 7} \approx n^{2.81} \implies \text{CASE 1} \implies T(n) = \Theta(n^{\lg 7}).$$

The number 2.81 may not seem much smaller than 3, but because the difference is in the exponent, the impact on running time is significant. In fact, Strassen's algorithm beats the ordinary algorithm on today's machines for $n \geq 30$ or so.

Best to date (of theoretical interest only): $\Theta(n^{2.376\cdots})$.

Conclusion

- Divide and conquer is just one of several powerful techniques for algorithm design.

- Divide-and-conquer algorithms can be analyzed using recurrences and the master method (so practice this math).

- Can lead to more efficient algorithms