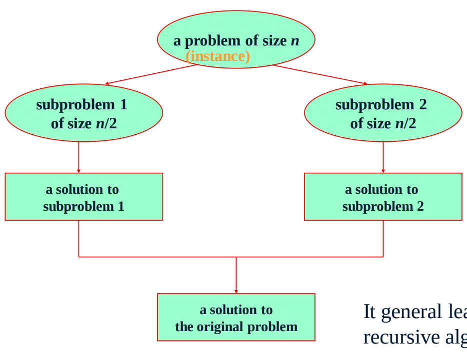
Divide and Conquer
Strassen’s Matrix Multiplication,
Closest pair

Divide-and-Conquer

The most-well known algorithm design strategy:

1. Divide instance of problem into two or more smaller instances
2. Solve smaller instances recursively
3. Obtain solution to original (larger) instance by combining these solutions

Divide-and-Conquer Technique (cont.)



It general leads to a recursive algorithm

Multiplication of Large Integers

Consider the problem of multiplying two (large) n-digit integers represented by arrays of their digits such as:

A = 12345678901357986429 B = 87654321284820912836

The grade-school algorithm:

$$
\begin{array}{r}
a_1\ a_2 \ldots\ a_n \\
\underline{b_1\ b_2 \ldots\ b_n} \\
(d_{10})\ d_{11}d_{12} \ldots\ d_{1n} \\
(d_{20})\ d_{21}d_{22} \ldots\ d_{2n} \quad \\
\underline{\ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots} \quad\quad \\
(d_{n0})\ d_{n1}d_{n2} \ldots\ d_{nn} \quad\quad\quad\quad
\end{array}
$$

Efficiency: $\Theta(n^2)$ single-digit multiplications

First Divide-and-Conquer Algorithm

A small example: $A * B$ where $A = 2135$ and $B = 4014$

$A = (21\cdot 10^2 + 35)$, $B = (40 \cdot 10^2 + 14)$

So, $A * B = (21 \cdot 10^2 + 35) * (40 \cdot 10^2 + 14)$

$= 21 * 40 \cdot 10^4 + (21 * 14 + 35 * 40) \cdot 10^2 + 35 * 14$

In general, if $A = A_1A_2$ and $B = B_1B_2$ (where **A and B are *n*-digit, A_1, A_2, B_1, B_2 are *n*/2-digit numbers**),

$\mathbf{A * B = A_1 * B_1 \cdot 10^n + (A_1 * B_2 + A_2 * B_1) \cdot 10^{n/2} + A_2 * B_2}$

Recurrence for the number of one-digit multiplications $M(n)$:

$\mathbf{M(n) = 4M(n/2), \quad M(1) = 1}$

Solution: $M(n) = n^2$

Second Divide-and-Conquer Algorithm

$A * B = A_1 * B_1 \cdot 10^n + (A_1 * B_2 + A_2 * B_1) \cdot 10^{n/2} + A_2 * B_2$

The idea is to decrease the **number of multiplications from 4 to 3**:

$(A_1 + A_2) * (B_1 + B_2) = A_1 * B_1 + \mathbf{(A_1 * B_2 + A_2 * B_1)} + A_2 * B_2,$

I.e., $\mathbf{(A_1 * B_2 + A_2 * B_1)} = (A_1 + A_2) * (B_1 + B_2) - A_1 * B_1 - A_2 * B_2,$ which requires only 3 multiplications at the expense of (4-1) extra add/sub.

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Multiplication of n-digit numbers requires three multiplications of n/2-digit numbers

Recurrence for the number of multiplications $M(n)$:

$\mathbf{M(n) = 3M(n/2), \quad M(1) = 1}$

Solution: $\mathbf{M(n) = 3^{\log_2 n} = n^{\log_2 3} \approx n^{1.585}}$

What if we count both multiplications and additions?

Example of Large-Integer Multiplication

2135 ∗ 4014

= (21*10^2 + 35) * (40*10^2 + 14)

= (21*40)*10^4 + **c1***10^2 + 35*14

where **c1 = (21+35)*(40+14) - 21*40 - 35*14**, and

21*40 = (2*10 + 1) * (4*10 + 0)

= (2*4)*10^2 + **c2***10 + 1*0

where **c2 = (2+1)*(4+0) - 2*4 - 1*0,** etc.

This process requires 9 digit multiplications as opposed to 16.

Conventional Matrix Multiplication

- Brute-force algorithm

$$\begin{pmatrix} c_{00} & c_{01} \\ c_{10} & c_{11} \end{pmatrix} = \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} * \begin{pmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{pmatrix}$$

$$= \begin{pmatrix} a_{00} * b_{00} + a_{01} * b_{10} & a_{00} * b_{01} + a_{01} * b_{11} \\ a_{10} * b_{00} + a_{11} * b_{10} & a_{10} * b_{01} + a_{11} * b_{11} \end{pmatrix}$$

8 multiplications

4 additions

Efficiency class in general: Θ (n^3)

Strassen's Matrix Multiplication

- Strassen's algorithm for two 2x2 matrices (1969):

$$\begin{pmatrix} c_{00} & c_{01} \\ c_{10} & c_{11} \end{pmatrix} = \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} * \begin{pmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{pmatrix}$$

$$= \begin{pmatrix} m_1 + m_4 - m_5 + m_7 & m_3 + m_5 \\ m_2 + m_4 & m_1 + m_3 - m_2 + m_6 \end{pmatrix}$$

- $m_1 = (a_{00} + a_{11}) * (b_{00} + b_{11})$
- $m_2 = (a_{10} + a_{11}) * b_{00}$
- $m_3 = a_{00} * (b_{01} - b_{11})$
- $m_4 = a_{11} * (b_{10} - b_{00})$
- $m_5 = (a_{00} + a_{01}) * b_{11}$
- $m_6 = (a_{10} - a_{00}) * (b_{00} + b_{01})$
- $m_7 = (a_{01} - a_{11}) * (b_{10} + b_{11})$

7 multiplications

18 additions

Strassen's Matrix Multiplication

Strassen observed [1969] that the product of two matrices can be computed in general as follows:

$$\begin{pmatrix} C_{00} & C_{01} \\ C_{10} & C_{11} \end{pmatrix} = \begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix} * \begin{pmatrix} B_{00} & B_{01} \\ B_{10} & B_{11} \end{pmatrix}$$

$$= \begin{pmatrix} M_1 + M_4 - M_5 + M_7 & M_3 + M_5 \\ M_2 + M_4 & M_1 + M_3 - M_2 + M_6 \end{pmatrix}$$

Formulas for Strassen's Algorithm

$M_1 = (A_{00} + A_{11}) * (B_{00} + B_{11})$

$M_2 = (A_{10} + A_{11}) * B_{00}$

$M_3 = A_{00} * (B_{01} - B_{11})$

$M_4 = A_{11} * (B_{10} - B_{00})$

$M_5 = (A_{00} + A_{01}) * B_{11}$

$M_6 = (A_{10} - A_{00}) * (B_{00} + B_{01})$

$M_7 = (A_{01} - A_{11}) * (B_{10} + B_{11})$

Analysis of Strassen's Algorithm

If n is not a power of 2, matrices can be padded with zeros.

Number of multiplications:

$$\mathbf{M}(n) = 7\mathbf{M}(n/2), \quad \mathbf{M}(1) = 1$$

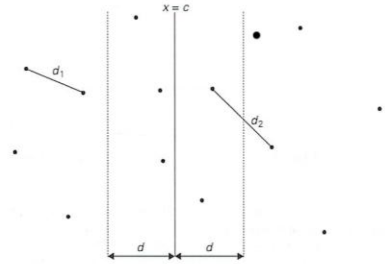
Solution: $\mathbf{M}(n) = 7^{\log_2 n} = n^{\log_2 7} \approx n^{2.807}$ **vs.** n^3 of brute-force alg.

Algorithms with better asymptotic efficiency are known but they are even more complex and not used in practice.

Closest-Pair Problem by Divide-and-Conquer

Step 0 Sort the points by x (list one) and then by y (list two).

Step 1 Divide the points given into two subsets S_1 and S_2 by a vertical line $x = c$ so that half the points lie to the left or on the line and half the points lie to the right or on the line.



Closest Pair by Divide-and-Conquer (cont.)

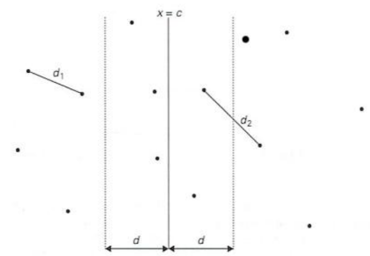
Step 2 Find recursively the closest pairs for the left and right subsets.

Step 3 Set $d = \min\{d_1, d_2\}$

We can limit our attention to the points in the symmetric vertical strip of width $2d$ as possible closest pair. Let C_1 and C_2 be the subsets of points in the left subset S_1 and of the right subset S_2, respectively, that lie in this vertical strip. The points in C_1 and C_2 are stored in increasing order of their y coordinates, taken from the second list.

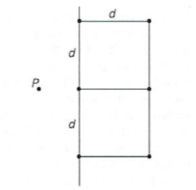
Step 4 For every point $P(x,y)$ in C_1, we inspect points in C_2 that may be closer to P than d. There can be no more than 6 such points (because $d \le d_2$)

- Unfortunately, d is not necessarily the smallest distance between all pairs of points in S1 and S2 because a closer pair of points can lie on the opposite sides separating the line.
- When we combine the two sets, we must examine such points. (Illustrate this on the diagram)



Closest Pair by Divide-and-Conquer: Worst Case

The worst case scenario is depicted below:



Efficiency of the Closest-Pair Algorithm

Running time of the algorithm (without sorting) is:

$$T(n) = 2T(n/2) + M(n), \text{ where } M(n) \in \Theta(n)$$

By the Master Theorem (with $a = 2$, $b = 2$, $d = 1$)

$$T(n) \in \Theta(n \log n)$$

So the total time is $\Theta(n \log n)$.