Functions and Approximation

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1 Functions - Derivatives and Approximation and Integrals

A real valued function $f(x)$ of one variable x can be thought of as an algorithm (or method - as in Java) that accepts x as input and outputs the value $f(x)$. Functions of one variable can often be visualised as a graph - using x and y axes as references. In many cases these graphs are continuous curves in the $x - y$ plane. In some sense when we view the graph of a function we are looking at the method in action. As is the case with objects and their methods, sometimes we know the details of the method - we know what's going on in that "black box" that takes x as input and that produces $f(x)$ as output. In other situations (if we are re-using software (as encouraged to do so in OO programming languages) then we may only be able to see the graph of the input/output without knowing the details of what is going on inside the black box). One reasonable question is : can we reverse engineer a method given only the information about input/output?
In some cases we can mimic the behaviour of a method - up to a point without being sure that "our" method is identical to the one that produces the observed input/output values.
In some cases we may have reasons to assume something about the form of the relationship between the input and output, in which case we may feel more confident about our method being the same as the observed method regarding input/output. Observations of values and understanding of correlation/causation may let us be confident that observed input/output values are connected via a function of a specified form.If we have agreed on the form of a function then there may be a way of specifying the particular values of parameters that appear in the function definition.
Once we have a functional description of input/output behaviour (with all parameters determined) then we may reasonably ask questions about maximum values, minimum values and average values of the function. Such questions often depend on knowledge of the derivative of the function and/or integral of the function for their answers.
Recall the definition of the derivative of a function $f(x)$ -

$$\frac{df}{dx} = \lim \frac{f(x+h) - f(x)}{h}, h \to 0$$

The derivative is a measure of the rate of change of the output relative to (with respect to) the input.

Example: Suppose we have the following funtion describing input and output:

$$f(x) = 3x^2 - 12x + 7$$

where $0 \leq x \leq 5$. Find the maximum and minimum values of the output.

Notice that the range of input is restricted to between 0 and 5.

First we might compute the derivative and determine when it is zero. Why ?

$$\frac{df}{dx} = 6x - 12$$

and

$$6x - 12 = 0 \Longrightarrow x = 2$$

which is within the range of input. Hence the minimum occurs when $x = 2$ which means the minimum value of the output is $3.2^2 - 12.2 + 7 = -5$.

As for the maximum value - this must occur at an extreme of the input - why?

When $x = 0$ we have $f(0) = 7$ and when $x = 5$ we have $f(5) = 22$.

2 Another example

Example: Let us consider the function $f(p) = p(p-1)$ on the interval of values $0 \leq p \leq 1$. This may be written as $f(p) = p - p^2$.

This is a quadratic function in p and has negative p^2 coefficient. So the graph is an up-turned bowl having a global maximum where $\frac{df}{dp} = 1 - 2p = 0$. Hence the max value occurs at $2p = 1$ or $p = .5$ and the max value is $f_{max} = .5(.5) = .25$. At the end points the value of the function is 0.

3 Rules for differentiation

Product - for use when a function $h(x)$ is a product of simpler functions $f(x)$ and $g(x)$. The rule gives us a formula for determining the derivative of $h(x)$ assuming we know how to differentiate the simpler functions. It reads as

$$\frac{dh}{dx} = f(x)\frac{dg}{dx} + g(x)\frac{df}{dx}$$

Composition rule - for use when a function $h(x) = f(g(x))$ is a composition of two simpler functions - the value of $h(x)$ is computed in two stages. Many functions are of this kind - for example

$$h(x) = (7x^2 + 3x + 5)^{100}$$

Stage 1: $g(x) = 7x^2 + 3x + 5$

Stage 2: $f() = ()^{100}$

The output of stage 1 is used as input for stage 2.

$$= (100(g(x))^{99})(14x + 3)$$

$$= (100(7x^2 + 3x + 5)^{99})(14x + 3)$$

Generally: If $h(x) = f(g(x))$ then

$$\frac{dh}{dx} = \frac{df}{dg}\frac{dg}{dx}$$

Quotient Rule - for use when a function $h(x)$ is a quotient of two simpler funtions

$$h(x) = \frac{fx}{g(x)}$$

then

$$\frac{dh}{dx} = \frac{g(x)\frac{df}{dx} - f(x)\frac{dg}{dx}}{g(x)^2}$$

Recall that the integral of a function $f(x)$ over an interval $[a, b]$ can be interpreted a couple of ways. In the first case we interpret the value of the integral as the "signed" area bounded between the graph of the function and the x axis over the interval. The second interpretation is that the integral when divided by the length of the interval $(b - a)$ is the average value of the function over that range of input values. That is

$$\bar{f} = \frac{1}{b-a}\int_a^b f(x)dx$$

If we can, we evaluate the integral of $f(x)$ by finding a function $F(x)$ whose derivative is $f(x)$. Then

$$F(b) - F(a) = \int_a^b f(x)dx$$

If we cant find such an $F(x)$ then we determine the value of the integral of $f(x)$ using some other method - looking up value in a book of tables or approximating the value somehow.

Calculating the integral of a function is in general a more difficult task than finding the derivative of a function.There are no product, quotient or composition rules for determining the integral of a function which is a product, quotient or composition of simpler functions.

Example: Calculate the average value of the function $f(x) = 1/x$ over the interval [1,3].

The average of f is

$$\bar{f} = \frac{1}{3-1}\int_1^3 f(x)dx$$

$$\bar{f} = \frac{1}{2}ln(x)|_1^3 = \frac{ln(3)}{2} = .5493$$