Mathematical Essentials

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1 Mathematics as a language and aid to thought

Consider the following problem: Two numbers add up to 110. If the larger number is 100 more than the smaller number what are the two numbers?
What about: Paul is 3 years older than Mary. What are Paul's and Mary's ages?
What about: Paul is 5 times Mary's age. The square root of Paul's age is Mary's age. What are their ages?
And: The sum of Paul's and Mary's current accounts is the same as half of Paul's current account on its own. If Paul has 50 E in his account how much in Mary's?
The more quantities (variables) there are the harder it becomes to keep track of these things when relationships are described verbally. The point of algebra is to express relationships between things as expressions or through using equations. These expressions can then be manipulated using mathematical laws in order to extract information about the quantities.

2 Sets - Notation and Terminology

A set is a collection of well-defined objects. Sets of objects are usually expressed using upper-case characters such as : A , B , C
The objects that comprise a set A are usually called the elements of the set A. If a is an element of the set A this is written as: $a \in A$.
If an object x is not an element of a set A then write $x \notin A$.
There is a special set called the empty set, denoted by $\emptyset$, that has no elements in it. It is **like** an empty container.
Like numbers sets come more alive when they are combined in different ways. The rules for combining sets are a little bit like the rules (general) for combining numbers. A little!
We write $A \subseteq B$ provided every element of A is an element of B. We say that A is a subset of B. The symbol $\subseteq$ is a bit like the familiar $\leq$ which is used for numbers: $a \leq b$ means the value a is less than or equal b.
Sets A and B are equal provided $A \subseteq B$ and $B \subseteq A$.
When talking about sets there is normally a context in which the discussion is taking place - this serves to set the scene for the discussion. From a set point of view this is

manifested using a "universal set" - the overarching set of things (objects) that encompass the discussion. The universal set may be written a $\mathcal{U}$ - note again, this changes from discussion to discussion. Today we may be talking about "numbers of people in queues" - tomorrow we may be discussing the "bank balances" of the people in the queues. $\mathcal{U}$ for the first situation might be $\mathbb{N}$ - the set of natural numbers $0, 1, 2, 3, 4, ...$
The second situation might use $\mathcal{U} = \mathbb{R}$ the set of real numbers.
The programming versions of these mathematical sets of numbers, $\mathbb{N}$ (or $\mathbb{Z}$)and $\mathbb{R}$ would be int and float or double types. Clearly, the computer version of these sets are quite different from the mathematical (conceptual) versions. For example, one method often used for visualising the set of real numbers $\mathbb{R}$ is to associate each element in $\mathbb{R}$ with a point on a straight line. In this visualisation larger numbers (or more precisely, points representing larger numbers)lie to the right (on the line) of those points representing smaller numbers.When we talk about the real number line it is this visualisation that is meant.

3 Combining Sets

We can add two sets A and B by forming the union $A \cup B$. The set $A \cup B$ consists of all elements that come either from A or from B. In forming unions we do not repeat elements - that is, if $x \in A$ and $x \in B$ then when we form $A \cup B$ we say $x \in (A \cup B)$ once - not twice.

We can multiply sets by forming the intersection of the two sets: $A \cap B$ consisting of all things that are both in A and B.

Note the terminology - addition - multiplication being used here as synonyms for union and intersection ! In this way of thinking about the operations $\cup$ and $\cap$ it turns out to be reasonable to think of $\mathcal{U}$ and $\emptyset$ as being the equivalents of 1 and 0 respectively (when we are thinking of numbers).
Given a set A and a universal set in which A sits then we can define the complement of A ; denoted $\overline{A}$. This is **like** the negative of the set A. It consists of all elements in $\mathcal{U}$ that are not in A.
Consider the following : $\overline{A \cup B} = \overline{A} \cap \overline{B}$. This equation shows how the metaphor of addition, multiplication and negation goes only so far. This is the case with other metaphors in other situations too.

4 Special Sets

The set of natural numbers $\mathbb{N} = \{0, 1, 2, 3....\}$

The set of integers $\mathbb{Z} = \{0, \pm1, \pm2, \pm3, ...\}$

The set of **ratio** nal numbers $= \{\frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0\}$

The set of real numbers $\mathbb{R}$: all numbers with decimal expansions

The set of complex numbers $\mathbb{C} = \{a + b\mathbf{i} | a, b \in \mathbb{R}$ and $\mathbf{i}^{2=-1}\}$

5 Visualisation of the Real Numbers

We often use a straight line (horizontal) as a representation of the set of real numbers where each (idealised - geometric) point on the line corresponds to a unique real number and every real number can be associated with a point on the line. Furthermore we assume that the points running continuously from left to right correspond to values in ascending order. Hence points to the left of points to the right represent smaller numerical values. We can put two lines at right angles to form the coordinate system for points on the plane. A coordinate system allows us to describe points in a numerical way. This gives us $\mathbb{R}^2 = \{(a, b)|a, b \in \mathbb{R}\}$.
In like manner we can form higher dimensional spaces simply by taking more and more real numbers in the array. Hence $\mathbb{R}^3$ gives us three dimensional space.

$$\mathbb{R}^N = \{(r_1, r_2, r_3, ...r_N)\}$$

- this is just an array with N elements! In maths it is referred to as N-dimensional Euclidean space.