Mathematical Essentials 2

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1 Functions and Graphs - Standard Functions

A function f from $\mathbb{R}$ to $\mathbb{R}$, is denoted $f : \mathbb{R} \longrightarrow \mathbb{R}$. A function accepts input from some set, called the domain of the function (it is the acceptable set of inputs to the function) and produces output (based on some algorithm). The set of all possible outputs from a function is called the range of the function. We write $f(x)$ to denote the value outputted by the function for an input value x - and read it as "f of x". A function defines a relationship between the input and output values. This relationship between input and output values can sometimes be visualised as a graph of points in the Cartesion plane, each point corresponding to an input-output pair of values.
Example: $f(x) = 3$ is a constant function that outputs the value 3 no matter what the input value. It is called a constant function. The graph of this is a horizontal line, parallel to the x-axis at a height 3 above the x-axis.
Example: $f(x) = ax + b$. This function is called a linear function since its graph is a straight line with a slope a and an intercept on the y-axis at b. Any line in the Cartesian plane can be described by a function of this form. A point (x, y) will be on the line if the values of x and y satisfy the equation $y = ax + b$ and vice versa. Thus, we have an algebraic way of deciding if a point lies on the geometric shape (line in this case)without having to draw perfect pictures.
Note; the slope of a line is determined according to slope $= \frac{rise}{run} = \frac{y_2 - y_1}{x_2 - x_1}$ where (x_1, y_1) and (x_2, y_2) are any two distinct points on the line and assuming $x_1 \neq x_2$ (that is to say vertical lines are not ascribed a slope value - when $x_1 = x_2$ the line is vertical - the "run" is 0).
The function $f(x) = ax^2 + bx + c$ is a quadratic function and its graph is a parabola - a bowl-shaped profile if $a > 0$ and upturned profile if $a < 0$. Decide what happens if $a = 0$! The parabola is symmetric about the (vertical) line passing through the low point of the bowl ($a > 0$) or the high point on the upturned bowl shape if $a < 0$. Drawing the graph of a particular quadratic function becomes a problem of drawing a bowl shape positioned at the correct point where the low point occurs and gauging correctly the "sweep" of the wings of the parabola. The low point is called the minimum. Location of this point is done using the so-called derivative of the function and determining for what value(s) of x the derivative expression has a value 0.

Example: Sketch the graph of $f(x) = x^2 + 2x + 5$. We draw the graph by collating the following information: The function f is a quadratic function - hence the graph is a parabola. The multiple of the x^2 term is 1 (also called the coefficient of x^2) which is greater than 0 and hence it is bowl shaped. The minimum occurs when $2x+2=0$ since this is the derivative of the function $f(x) = x^2 + 2x + 5$. The equation

$$2x + 2 = 0$$

has solution $x = -1$. When the x value is -1 then f has value (y value)

$$y = f(-1) = (-1)^2 + 2(-1) + 5 = 4$$

This is the minimum y value and the low point is $(-1, 4)$. The spread of the wings of the parabola is indicated by one other point. Choose $x = 0$, say, then $f(0) = 0^2 + 2.0 + 5 = 5$. In short - low point $(-1, 4)$ - one other guide point $(0, 5)$. Now sketch the graph keeping all these remarks in mind.

2 General Graphing Procedure

Given the problem of graphing a function $f(x)$ we attempt the following:

- Determine the derivative of $f(x)$ - usually using some helpful rules for dealing with complex expressions. Denote the derivative as $\frac{df}{dx}$.
- try to find where $\frac{df}{dx} = 0$. the resulting x-values naturally divide the x -axis into portions where the graph of the function behaves in ways that are predictable throughout the range from one such x value and the next - that is, between two such x values the function either increases throughout or decreases throughout "it doesn't get to change its mind".
- On the basis of having found all x-values for which $\frac{df}{dx} = 0$ now calculate the corresponding y-values. In fact this step and the last one go hand-in-hand.
- Plot these "extreme points" and keep in mind the behaviour between them as you draw the graph. If there is a value x for which the derivative is undefined you should treat this with special care - it might correspond to an "extreme point" or it may correspond to a point that is actually missing from the graph of $f(x)$ - that is, a value for which the function itself is undefined!

For example:

- Sketch $f(x) = \frac{1}{x^2} = x^{-2}$.
- Then $\frac{df}{dx} = -2.x^{-3}$.
- Set $\frac{df}{dx} = \frac{-2}{x^3} = 0$

- This last equation has no solution AND $\frac{df}{dx}$ is undefined when $x = 0$, as is $f(0)$!
- The function either increases or decreases on the region $(-\infty, 0)$ and on the region $(0, +\infty)$. The point where $x = 0$ is not on the graph as $f(0)$ is undefined.
- Sketch using this information and by inferring behaviour as $x \longrightarrow \infty$.

3 The Derivative

The slope $\frac{df}{dx}$ of the graph of a function $f(x)$ has been used to help us draw the graph. The algebraic quantity $\frac{df}{dx}$ is called the derivative of f with respect to the variable x. The symbols $\frac{df}{dx}$ is read as "dee f dee x" There are other short-hand ways of expressing this quantity, such as f', $\dot{f}$. Note with these expressions the name of the input variable is not present in the notation. One needs to be clear about the name of the input (independent) variable as there may be many variables floating about. The derivative tells us the rate at which the dependent variable changes relative to change in the input variable.
If $p(t)$ is a function that describes the position of a point moving along a straight line path in terms of time then the derivative of $p(t)$ (or more concisely, the derivative of p) is a function that describes the velocity of the point at any time "t".
Furthermore if we consider the derivative of the derivative of p we have a function that describes the acceleration of the point. This so-called "second derivative" is written $\frac{d^2p}{dt^2}$ and this is read as "dee 2 p dee t squared".
The notations $\frac{d^2p}{dt^2}$ for the second derivative and $\frac{df}{dx}$ for the first derivative remind us that the derivative is computed as a ratio of terms:

$$\frac{df}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

For $h \neq 0$ the term on the right is simply the slope of a line that is "close" to the tangent line to the curve at point $(x, f(x))$.

4 Exercises

Find the minimum value of the function $f(x) = x^2 + 4x - 3$. Hence sketch the graph of the function.
Find the maximum value of the function $f(x) = -x^2 + 3x + 2$. Hence sketch the graph of $f(x)$.
Sketch the graph of the function $f(x) = \frac{x^3}{3} - x + 5$. Remember to use the fact the function is a "cubic" function of "x" and hence you know its basic shape.
(Challenge Problem)Suppose you are given a certain length L of fencing that is to be used to cordon off a rectangular area. How can you make sure you cordon off the largest possible area? That is, determine the dimensions of the rectangle that contains the largest area - for the given amount of fencing.