ACM/ICPC CheatSheet

Puzzles

Contents

1 STL Useful Tips

1.1 Common libraries

```
/*** Functions ***/
#include<algorithm>
#include<functional> // for hash
#include<climits> // all useful constants
#include<cmath>
#include<cstdio>
#include<cstdlib> // random
#include<ctime>
#include<iostream>
#include<sstream>
/*** Data Structure ***/
#include<deque> // double ended queue
#include<list>
#include<queue> // including priority_queue
#include<stack>
#include<string>
#include<vector>
```

1.2 Useful constant

```
INT_MIN
INT_MAX
LONG_MIN
LONG_MAX
LLONG_MIN
LLONG_MAX
(~0u) // infinity (for long and long long)
      // use (~0u)>>2 for int.
```

1.3 Space waster

```
// consider to redefine data types to void data range problem
#define int long long // make everyone long long
#define double long double // make everyone long double

// function definitions

#undef int // main must return int
int main(void)
#define int long long // redefine int

// rest of program
```

1.4 Initialize array with predefined value

```
// for 1d array, use STL fill_n or fill to initialize array
fill(a, a+size_of_a, value)
fill_n(a, size_of_a, value)
// for 2d array, if want to fill in 0 or -1
memset(a, 0, sizeof(a));
  // otherwise, use a loop of fill or fill_n through every a[i]
  fill(a[i], a[i]+size_of_ai, value) // from 0 to number of row.
```

1.5 Modifying sequence operations

```
void copy(first, last, result);
void swap(a,b);
void swap(first1, last1, first2); // swap range
void replace(first, last, old_value, new_value); // replace in range
void replace_if(first, last, pred, new_value); // replace in conditions
  // pred can be represented in function
  // e.x. bool IsOdd (int i) { return ((i%2)==1); }
void reverse(first, last); // reverse a range of elements
void reverse_copy(first, last, result); // copy a reverse of range of elements
void random_shuffle(first, last); // using built-in random generator to shuffle array
```

1.6 Merge

```
// merge sorted ranges
void merge(first1, last1, first2, last2, result, comp);
// union of two sorted ranges
void set_union(first1, last1, first2, last2, result, comp);
// intersection of two sorted ranges
void set_interaction(first1, last1, first2, last2, result, comp);
// difference of two sorted ranges
void set_difference((first1, last1, first2, last2, result, comp);
```

1.7 String

```
// Searching
unsigned int find(const string &s2, unsigned int pos1 = 0);
unsigned int rfind(const string &s2, unsigned int pos1 = end);
unsigned int find_first_of(const string &s2, unsigned int pos1 = 0);
unsigned int find_last_of(const string &s2, unsigned int pos1 = end);
unsigned int find_first_not_of(const string &s2, unsigned int pos1 = 0);
unsigned int find_last_not_of(const string &s2, unsigned int pos1 = end);
// Insert, Erase, Replace
string& insert(unsigned int pos1, const string &s2);
string& insert(unsigned int pos1, unsigned int repetitions, char c);
string& erase(unsigned int pos = 0, unsigned int len = npos);
string& replace(unsigned int pos1, unsigned int len1, const string &s2);
string& replace(unsigned int pos1, unsigned int len1, unsigned int repetitions, char c);
// String streams
stringstream s1;
int i = 22;
s1 << "Hello world! " << i;
cout << s1.str() << endl;
```

1.8 Heap

```
template <class RandomAccessIterator>
  void push_heap (RandomAccessIterator first, RandomAccessIterator last);
template <class RandomAccessIterator, class Compare>
  void push_heap (RandomAccessIterator first, RandomAccessIterator last,
          Compare comp);
```

```
template <class RandomAccessIterator>
  void pop_heap (RandomAccessIterator first, RandomAccessIterator last);
template <class RandomAccessIterator, class Compare>
  void pop_heap (RandomAccessIterator first, RandomAccessIterator last,
         Compare comp);

template <class RandomAccessIterator>
  void make_heap (RandomAccessIterator first, RandomAccessIterator last);
template <class RandomAccessIterator, class Compare>
  void make_heap (RandomAccessIterator first, RandomAccessIterator last,
         Compare comp );

template <class RandomAccessIterator>
  void sort_heap (RandomAccessIterator first, RandomAccessIterator last);
template <class RandomAccessIterator, class Compare>
  void sort_heap (RandomAccessIterator first, RandomAccessIterator last,
         Compare comp);
template <class RandomAccessIterator>
  RandomAccessIterator is_heap_until (RandomAccessIterator first,
                   RandomAccessIterator last);
template <class RandomAccessIterator, class Compare>
  RandomAccessIterator is_heap_until (RandomAccessIterator first,
                   RandomAccessIterator last
                   Compare comp);
```

1.9 Sort

```
void sort(iterator first, iterator last);
void sort(iterator first, iterator last, LessThanFunction comp);
void stable_sort(iterator first, iterator last);
void stable_sort(iterator first, iterator last, LessThanFunction comp);
void partial_sort(iterator first, iterator middle, iterator last);
void partial_sort(iterator first, iterator middle, iterator last, LessThanFunction comp);
bool is_sorted(iterator first, iterator last);
bool is_sorted(iterator first, iterator last, LessThanOrEqualFunction comp);
// example for sort, if have array x, start_index, end_index;
sort(x+start_index, x+end_index);
```

1.10 Permutations

```
bool next_permutation(iterator first, iterator last);
bool next_permutation(iterator first, iterator last, LessThanOrEqualFunction comp);
bool prev_permutation(iterator first, iterator last);
bool prev_permutation(iterator first, iterator last, LessThanOrEqualFunction comp);
```

1.11 Searching

```
// will return address of iterator, call result as *iterator;
iterator find(iterator first, iterator last, const T &value);
iterator find_if(iterator first, iterator last, const T &value, TestFunction test);
bool binary_search(iterator first, iterator last, const T &value);
bool binary_search(iterator first, iterator last, const T &value, LessThanOrEqualFunction comp);
```

1.12 Random algorithm

```
srand(time(NULL));
// generate random numbers between [a,b)
rand() % (b - a) + a;
// generate random numbers between [0,b)
rand() % b;
// generate random permutations
random_permutation(anArray, anArray + 10);
random_permutation(aVector, aVector + 10);
```

2 Number Theory

2.1 Max or min

```
int max(int a, int b) { return a>b ? a:b; }
int min(int a, int b) { return a<b ? a:b; }
```

2.2 Greatest common divisor — GCD

```
int gcd(int a, int b)
{
  if (b==0) return a;
  else return gcd(b, a%b);
}
```

2.3 Least common multiple — LCM

```
int lcm(int a, int b)
{
  return a*b/gcd(a,b);
}
```

2.4 If prime number

```
bool prime(int n)
{
  if (n<2) return false;
  if (n<=3) return true;
  if (!(n%2) || !(n%3)) return false;
  for (int i=5;i*i<=n;i+=6)
    if (!(n%i) || !(n%(i+2))) return false;

  return true;
}
```

2.5 Prime factorization

```
// smallest prime factor of a number.
function factor(int n)
{
    int a;
    if (n%2==0)
        return 2;
    for (a=3;a<=sqrt(n);a++++)
    {
```

```
    if (n%a==0)
    return a;
  }
  return n;
}

// complete factorization
int r;
while (n>1)
{
    r = factor(n);
    printf("%d ", r);
    n /= r;
}
```

2.6 Leap year

```
bool isLeap(int n)
{
  if (n%100==0)
    if (n%400==0) return true;
    else return false;

  if (n%4==0) return true;
  else return false;
}
```

2.7 Binary exponential

```
int binpow (int a, int n)
{
  int res = 1;
  while (n)
    if (n & 1)
    {
      res *= a;
      --n;
    }
    else
    {
      a *= a;
      n >>= 1;
    }
  return res;
}
```

2.8 a^b mod p

```
long powmod(long base, long exp, long modulus) {
  base %= modulus;
  long result = 1;
  while (exp > 0) {
    if (exp & 1) result = (result * base) % modulus;
    base = (base * base) % modulus;
    exp >>= 1;
  }
```

```
  return result;
}
```

2.9 Factorial mod

```
//n! mod p
int factmod (int n, int p) {
  long long res = 1;
  while (n > 1) {
    res = (res * powmod (p-1, n/p, p)) % p;
    for (int i=2; i<=n%p; ++i)
      res=(res*i) %p;
    n /= p;
  }
  return int (res % p);
}
```

2.10 Generate combinations

```
// n>=m, choose M numbers from 1 to N.
void combination(int n, int m)
{
  if (n<m) return ;
  int a[50]={0};
  int k=0;

  for (int i=1;i<=m;i++) a[i]=i;
  while (true)
  {
    for (int i=1;i<=m;i++)
      cout << a[i] << " ";
    cout << endl;

    k=m;
    while ((k>0) && (n-a[k]==m-k)) k--;
    if (k==0) break;
    a[k]++;
    for (int i=k+1;i<=m;i++)
      a[i]=a[i-1]+1;
  }
}
```

2.11 10-ary to m-ary

```
char a[16]={'0','1','2','3','4','5','6','7','8','9',
    'A','B','C','D','E','F'};

string tenToM(int n, int m)
{
  int temp=n;
  string result="";
  while (temp!=0)
  {
    result=a[temp%m]+result;
    temp/=m;
  }
```

```
  return result;
}
```

2.12 m-ary to 10-ary

```
string num="0123456789ABCDE";

int mToTen(string n, int m)
{
  int multi=1;
  int result=0;

  for (int i=n.size()-1;i>=0;i--)
  {
    result+=num.find(n[i])*multi;
    multi*=m;
  }

  return result;
}
```

2.13 Binomial coefficient

```
#define MAXN 100 // largest n or m
long binomial_coefficient(n,m) // compute n choose m
int n,m;
{
    int i,j;
    long bc[MAXN][MAXN];
    for (i=0; i<=n; i++) bc[i][0] = 1;
    for (j=0; j<=n; j++) bc[j][j] = 1;
    for (i=1; i<=n; i++)
        for (j=1; j<i; j++)
            bc[i][j] = bc[i-1][j-1] + bc[i-1][j];
    return bc[n][m];
}
```

3 Catalan numbers

$$C_n = \sum_{k=0}^{n-1} C_k C_{n-1-k} = \frac{1}{n+1}\binom{n}{k} \tag{1}$$

The first terms of this sequence are 2, 5, 14, 42, 132, 429, 1430 when $C_0 = 1$. This is the number of ways to build a balanced formula from n sets of left and right parentheses. It is also the number of triangulations of a convex polygon, the number of rooted binary tress on $n+1$ leaves and the number of paths across a lattice which do not rise above the main diagonal.

3.1 Eulerian numbers

$$\left\langle {n \atop k} \right\rangle = k \left\langle {n-1 \atop k} \right\rangle + (n-k+1) \left\langle {n-1 \atop k-1} \right\rangle \tag{2}$$

```
// This is the number of permutations of length n with exactly k ascending sequences or runs.
// Basis: k=0 has value 1
#define MAXN 100 // largest n or k
```

```
long eularian(n,k)
int n,m;
{
    int i,j;
    long e[MAXN][MAXN];
    for (i=0; i<=n; i++) e[i][0] = 1;
    for (j=0; j<=n; j++) e[0][j] = 0;
    for (i=1; i<=n; i++)
        for (j=1; j<i; j++)
           e[i][j] = k*e[i-1][j] + (i-j+1)*e[i-1][j-1];
    return e[n][k];
}
```

3.2 Karatsuba algorithm in Java

```
// fast algorithm to find multiplication of two big numbers.
import java.math.BigInteger;
import java.util.Random;

class Karatsuba {
  private final static BigInteger ZERO = new BigInteger("0");
  public static BigInteger karatsuba(BigInteger x, BigInteger y)
  {
    int N = Math.max(x.bitLength(), y.bitLength());
    if (N <= 2000) return x.multiply(y);
    N=(N/2)+(N %2);
    BigInteger b = x.shiftRight(N);
    BigInteger a = x.subtract(b.shiftLeft(N));
    BigInteger d = y.shiftRight(N);
    BigInteger c = y.subtract(d.shiftLeft(N));
    BigInteger ac = karatsuba(a, c);
    BigInteger bd = karatsuba(b, d);
    BigInteger abcd = karatsuba(a.add(b), c.add(d));
    return ac.add(abcd.subtract(ac).subtract(bd).shiftLeft(N)).add(bd.shiftLeft(2*N));
  }

  public static void main(String[] args)
  {
    long start, stop, elapsed;
    Random random = new Random();
    int N = Integer.parseInt(args[0]);
    BigInteger a = new BigInteger(N, random);
    BigInteger b = new BigInteger(N, random);
    start = System.currentTimeMillis();
    BigInteger c = karatsuba(a, b);
    stop = System.currentTimeMillis();
    System.out.println(stop - start);
    start = System.currentTimeMillis();
    BigInteger d = a.multiply(b);
    stop = System.currentTimeMillis();
    System.out.println(stop - start);
    System.out.println((c.equals(d)));
  }
}
```

3.3 Euler's totient function

```
// the positive integers less than or equal to n that are relatively prime to n.
int phi (int n)
{
  int result = n;
  for (int i=2; i*i<=n; ++i)
    if(n %i==0)
    {
      while(n %i==0)
        n /= i;
      result -= result / i;
    }
  if (n > 1)
    result -= result / n;
  return result;
}
```

4 Searching Algorithms

4.1 Find rank k in array

```
int find(int l, int r, int k)
{
  int i=0,j=0,x=0,t=0;

  if (l==r) return a[l];
  x=a[(l+r)/2];
  t=a[x]; a[x]=a[r]; a[r]=t;
  i=l-1;

  for (int j=l; j<=r-1;j++)
    if (a[j]<=a[r])
    {
      i++;
      t=a[i]; a[i]=a[j]; a[j]=t;
    }
  i++;
  t=a[i]; a[i]=a[r]; a[r]=t;
  if (i==k) return a[i];
  if (i<k) return find(i+1, r,k);

  return find(l, i-1, k);
}
```

4.2 KMP Algorithm

```
#include <iostream>
#include <string>
#include <vector>

using namespace std;

typedef vector<int> VI;

void buildTable(string& w, VI& t)
{
  t = VI(w.length());
  int i = 2, j = 0;
```

```
  t[0] = -1; t[1] = 0;

  while(i < w.length())
  {
    if(w[i-1] == w[j]) { t[i] = j+1; i++; j++; }
    else if(j > 0) j = t[j];
    else { t[i] = 0; i++; }
  }
}

int KMP(string& s, string& w)
{
  int m = 0, i = 0;
  VI t;

  buildTable(w, t);
  while(m+i < s.length())
  {
    if(w[i] == s[m+i])
    {
      i++;
      if(i == w.length()) return m;
    }
    else
    {
      m += i-t[i];
      if(i > 0) i = t[i];
    }
  }
  return s.length();
}

int main(void)
{
  string a = (string) "The example above illustrates the general technique for assembling "+
    "the table with a minimum of fuss. The principle is that of the overall search: "+
    "most of the work was already done in getting to the current position, so very "+
    "little needs to be done in leaving it. The only minor complication is that the "+
    "logic which is correct late in the string erroneously gives non-proper "+
    "substrings at the beginning. This necessitates some initialization code.";

  string b = "table";

  int p = KMP(a, b);
  cout << p << ": " << a.substr(p, b.length()) << " " << b << endl;

  return 0;
}
```

5 Dynamic Programming

5.1 0/1 Knapsack problems

```
#include<iostream>

using namespace std;

int f[1000]={0};
```

```
int n=0, m=0;

int main(void)
{
  cin >> n >> m;

  for (int i=1;i<=n;i++)
  {
    int price=0, value=0;
    cin >> price >> value;

    for (int j=m;j>=price;j--)
      if (f[j-price]+value>f[j])
        f[j]=f[j-price]+value;
  }

  cout << f[m] << endl;

  return 0;
}
```

5.2 Complete Knapsack problems

```
#include<iostream>

using namespace std;

int f[1000]={0};
int n=0, m=0;

int main(void)
{
  cin >> n >> m;

  for (int i=1;i<=n;i++)
  {
    int price=0, value=0;
    cin >> price >> value;

    for (int j=price; j<=m; j++)
      if (f[j-price]+value>f[j])
        f[j]=f[j-price]+value;
  }

  cout << f[m] << endl;

  return 0;
}
```

5.3 Longest common subsequence (LCS)

```
int dp[1001][1001];

int lcs(const string &s, const string &t)
{
  int m = s.size(), n = t.size();
  if (m == 0 || n == 0) return 0;
```

```
  for (int i=0; i<=m; ++i)
    dp[i][0] = 0;
  for (int j=1; j<=n; ++j)
    dp[0][j] = 0;
  for (int i=0; i<m; ++i)
    for (int j=0; j<n; ++j)
      if (s[i] == t[j])
            dp[i+1][j+1] = dp[i][j]+1;
          else
            dp[i+1][j+1] = max(dp[i+1][j], dp[i][j+1]);
  return dp[m][n];
}
```

5.4 Longest increasing common sequence (LICS)

```
#include<iostream>

using namespace std;

int a[100]={0};
int b[100]={0};
int f[100]={0};
int n=0, m=0;

int main(void)
{
  cin >> n;
  for (int i=1;i<=n;i++) cin >> a[i];
  cin >> m;
  for (int i=1;i<=m;i++) cin >> b[i];

  for (int i=1;i<=n;i++)
  {
    int k=0;
    for (int j=1;j<=m;j++)
    {
      if (a[i]>b[j] && f[j]>k) k=f[j];
      else if (a[i]==b[j] && k+1>f[j]) f[j]=k+1;
    }
  }

  int ans=0;
  for (int i=1;i<=m;i++)
    if (f[i]>ans) ans=f[i];

  cout << ans << endl;

  return 0;
}
```

5.5 Longest Increasing Subsequence (LIS)

```
#include<iostream>

using namespace std;

int n=0;
```

```
int a[100]={0}, f[100]={0}, x[100]={0};

int main(void)
{
  cin >> n;
  for (int i=1;i<=n;i++)
  {
    cin >> a[i];
    x[i]=INT_MAX;
  }

  f[0]=0;

  int ans=0;
  for(int i=1;i<=n;i++)
  {
    int l=0, r=i;

    while (l+1<r)
    {
      int m=(l+r)/2;
      if (x[m]<a[i]) l=m; else r=m;
      // change to x[m]<=a[i] for non-decreasing case
    }

    f[i]=l+1;
    x[l+1]=a[i];
    if (f[i]>ans) ans=f[i];
  }

  cout << ans << endl;

  return 0;
}
```

5.6 Maximum submatrix

```
// URAL 1146 Maximum Sum
#include<iostream>

using namespace std;

int a[150][150]={0};
int c[200]={0};

int maxarray(int n)
{
   int b=0, sum=-100000000;
   for (int i=1;i<=n;i++)
   {
      if (b>0) b+=c[i];
      else b=c[i];
      if (b>sum) sum=b;
   }

   return sum;
}
```

```
int maxmatrix(int n)
{
   int sum=-100000000, max=0;

   for (int i=1;i<=n;i++)
   {
      for (int j=1;j<=n;j++)
         c[j]=0;

      for (int j=i;j<=n;j++)
      {
         for (int k=1;k<=n;k++)
            c[k]+=a[j][k];
         max=maxarray(n);
         if (max>sum) sum=max;
      }
   }

   return sum;
}

int main(void)
{
   int n=0;
   cin >> n;
   for (int i=1;i<=n;i++)
      for (int j=1;j<=n;j++)
         cin >> a[i][j];

   cout << maxmatrix(n);
   return 0;
}
```

5.7 Partitions of integers

```
#define MAXN 100 // largest n or m
long int_coefficient(n,k) // compute f(n,k)
int n,m;
{
    int i,j;
    long f[[MAXN][MAXN];
    f [1][1] = 1;
    for (i=0;i<=n;i++) f[i][0] = 0;
    for (i=1; i<=n; i++)
        for (j=1; j<i; j++)
           if (i-j <= 0)
               f[i][j] = f[i][k-1];
           else
               f[i][j] = f[i-j][k]+f[i][k-1];
    return f[n][k];
}
```

5.8 Partitions of sets

Number of ways to partition $n+1$ items into k sets.

$$\left\{ n \atop k \right\} = k \left\{ n-1 \atop k \right\} + \left\{ n-1 \atop k-1 \right\} \tag{3}$$

where

$$\left\{ {n \atop 1} \right\} = \left\{ {n \atop n} \right\} = 1 \tag{4}$$

6 Trees

6.1 Tree traversal

```
int L[100]={0};
int R[100]={0};

void DLR(int m)
{
  cout << m << " ";
  if (L[m]!=0) DLR(L[m]);
  if (R[m]!=0) DLR(R[m]);
}

void LDR(int m)
{
  if (L[m]!=0) LDR(L[m]);
  cout << m << " ";
  if (R[m]!=0) LDR(R[m]);
}

void LRD(int m)
{
  if (L[m]!=0) LRD(L[m]);
  if (R[m]!=0) LRD(R[m]);
  cout << m << " ";
}

int main(void)
{
  cin >> n;
  for (int i=1;i<=n;i++)
    cin >> L[i] >> R[i];

  DLR(1); cout << endl;
  LDR(1); cout << endl;
  LRD(1); cout << endl;

  return 0;
}
```

6.2 Depth and width of tree

```
#include <iostream>
#include <queue>
#include <stack>

using namespace std;

int l[100]={0};
int r[100]={0};
stack<int> mystack;
int n=0;
int w=0;
```

```cpp
int d=0;

int depth(int n)
{
  if (l[n]==0 && r[n]==0)
    return 1;

  int depthl=depth(l[n]);
  int depthr=depth(r[n]);
  int dep=depthl>depthr ? depthl:depthr;
  return dep+1;
}

void width(int n)
{
  if (n<=d)
  {
    int t=0,x;
    stack<int> tmpstack;
    while (!mystack.empty())
    {
      x=mystack.top();
      mystack.pop();
      if (x!=0)
      {
        t++;
        tmpstack.push(l[x]);
        tmpstack.push(r[x]);
      }
    }

    w=w>t?w:t;
    mystack=tmpstack;
    width(n+1);
  }
}

int main(void)
{
  cin >> n;

  for (int i=1;i<=n;i++)
    cin >> l[i] >> r[i];

  d=depth(1);
  mystack.push(1);
  width(1);

  cout << w << " " << d << endl;

  return 0;
}
```

7 Graph Theory

7.1 Graph representation

```
// The most common way to define graph is to use adjacency matrix
// example:
//      (1) (2) (3) (4) (5)
// (1)  2   0   5   0   0
// (2)  4   2   0   0   1
// (3)  3   0   0   1   4
// (4)  6   9   0   0   0
// (5)  1   1   1   1   5
// it's always a square matrix.
// suppose a graph has n nodes, if given exactly adjacency matrix
for (int i=1;i<=n;i++)
  for (int j=1;i<=n;j++)
  {
    cin << a[i][j] << endl;
  }
// Usually will go like this representation in data
// start_node end_node weight
// suppose m lines
for (int i=1;i<=m;i++)
{
  int x=0, y=0, t=0;
  cin >> x >> y >> t;
  a[x][y]=t;
  // if undirected graph
  a[y][x]=t;
}
// another variant: on the ith line, has data as
// end_node weight
// when you read data, you can assign matrix as
a[i][x]=t;
// if undirected graph
a[x][i]=t;

// Initialization of graph !!!IMPORTANT
// Depends on usage, normally initialize as 0 for all elements in matrix.
// so that 0 means no connection, non-0 means connection
// (for problem without weight, use weight as 1)
// If weights are important in this context (especially searching for path)
// Initialize graph as infinity for all elements in matrix.

// Another way to store graph is Adjacency list
// No space advantage if using array (unknown maximum number for in-degree).
// Big space advantage if using dynamic data structure (like list, vector).
// each row represent a node and its connectivity.
// we don't need it so much due to it's search efficiency.
// let's define a node as
struct Node{
  int id; // node id
  int w;  // weight
};
// suppose n nodes and m lines of inputs as
// start_node end_node weight
// assume using <vector> in this example
// g is a vector, and each element of g is also a vector of Node
for (int i=1;i<=m;i++)
{
  int x=0, y=0, t=0;
  cin >> x >> y >> t;
  Node temp; temp.id=y; temp.w=t;
```

```
  g[x].push_back(temp);
  // if undirected
  temp.id=x;
  g[y].push_back(temp);
}
// Note that you don't need this node structure if graph has only connectivity information.

/**** Special Structure ****/

// Special structure here is usually not a typical graph, like city-blocks, triangles
// They are represented in 2-d array and shows weights on nodes instead of edges.
// Note that in this case travel through edge has no cost, but visit node has cost.

// Triangles: Read data like this
// 1
// 1 2
// 4 2 7
// 7 3 1 5
// 6 2 9 4 6
for (int i=1;i<=n;i++)
  for (int j=i;j<=n;j++)
    cin >> a[i][j];

// Simple city-blocks: it's just like first form of adjacency matrix, but this time
// represents weights on nodes, may not be square matrix.
// 1 2 4 5 6
// 2 4 5 1 3
// 4 5 2 3 6
for (int i=1;i<=n;i++)
  for (int j=1;<=m;j++)
    cin >> a[i][j];

// More complex data structures: typical city-block structure may has some constraints on
// questions, but it has no boundaries. However, some questions requires to form a maze.
// In these cases, data structures can be very flexible, it totally depends on how the question
// presents the data. A usual way is to record it's adjacent blocks information:
struct Block{
  bool l[4]; // if has 8 neighbors then use bool l[8];
             // label them as your favor, e.x.
             //   1    1 2 3
             // 4 x 2  8 x 4
             //   3    7 6 5
             // true if there is path, false if there is boundary
  // other informations (optional)
  int weight;
  int component_id;
  // etc.
};

// Note that usually we use array from index 1 instead of 0 because sometimes
// you need index 0 as your boundary, and start from index 1 will give you
// advantage on locating nodes or positions
```

7.2 Flood fill algorithm

```
//component(i) denotes the
//component that node i is in
void flood_fill(new_component)
```

```
  do
    num_visited = 0
    for all nodes i
      if component(i) = -2
      num_visited = num_visited + 1
      component(i) = new_component

    for all neighbors j of node i
      if component(j) = nil
        component(j) = -2
  until num_visited = 0

void find_components()
  num_components = 0
  for all nodes i
    component(node i) = nil
  for all nodes i
    if component(node i) is nil
      num_components = num_components + 1
      component(i) = -2
      flood_fill(component num_components)
```

7.3 SPFA — shortest path

```
int q[3001]={0}; // queue for node
int d[1001]={0}; // record shortest path from start to ith node
bool f[1001]={0};
int a[1001][1001]={0}; // adjacency list
int w[1001][1001]={0}; // adjacency matrix

int main(void)
{
  int n=0, m=0;
  cin >> n >> m;

  for (int i=1;i<=m;i++)
  {
    int x=0, y=0, z=0;
    cin >> x >> y >> z; // node x to node y has weight z
    a[x][0]++;
    a[x][a[x][0]]=y;
    w[x][y]=z;
    /*
    // for undirected graph
    a[x][0]++;
    a[y][a[y][0]]=x;
    w[y][x]=z;
    */
  }

  int s=0, e=0;
  cin >> s >> e; // s: start, e: end
  SPFA(s);
  cout << d[e] << endl;

  return 0;
}
```

```
void SPFA(int v0)
{
  int t,h,u,v;
  for (int i=0;i<1001;i++) d[i]=INT_MAX;
  for (int i=0;i<1001;i++) f[i]=false;
  d[v0]=0;
  h=0; t=1; q[1]=v0; f[v0]=true;

  while (h!=t)
  {
    h++;
    if (h>3000) h=1;
    u=q[h];
    for (int j=1; j<=a[u][0];j++)
    {
      v=a[u][j];
      if (d[u]+w[u][v]<d[v]) // change to > if calculating longest path
      {
        d[v]=d[u]+w[u][v];
        if (!f[v])
        {
          t++;
          if (t>3000) t=1;
          q[t]=v;
          f[v]=true;
        }
      }
    }
    f[u]=false;
  }
}
```

7.4 Floyd-Warshall algorithm – shortest path of all pairs

```
// map[i][j]=infinity at start
void floyd()
{
  for (int k=1; k<=n; k++)
    for (int i=1; i<=n; i++)
      for (int j=1; j<=n; j++)
        if (i!=j && j!=k && i!=k)
          if (map[i][k]+map[k][j]<map[i][j])
            map[i][j]=map[i][k]+map[k][j];
}
```

7.5 Prim — minimum spanning tree

```
int d[1001]={0};
bool v[1001]={0};
int a[1001][1001]={0};

int main(void)
{
  int n=0;
  cin >> n;
  for (int i=1;i<=n;i++)
  {
```

```
    int x=0, y=0, z=0;
    cin >> x >> y >> z;
    a[x][y]=z;
  }
  for (int i=1;i<=n;i++)
    for (int j=1;j<=n;j++)
      if (a[i][j]==0) a[i][j]=INT_MAX;

  cout << prim(1,n) << endl;
}
int prim(int u, int n)
{
  int mst=0,k;

  for (int i=0;i<d.length;i++) d[i]=INT_MAX;
  for (int i=0;i<v.length;i++) v[i]=false;

  d[u]=0;
  int i=u;

  while (i!=0)
  {
    v[i]=true;k=0;
    mst+=d[i];
    for (int j=1;j<=n;j++)
      if (!v[j])
      {
        if (a[i][j]<d[j]) d[j]=a[i][j];
        if (d[j]<d[k]) k=j;
      }
    i=k;
  }
  return mst;
}
```

7.6 Eulerian circuit

```
// USACO Fence
#include<iostream>

using namespace std;

int f[100]={0}, ans[100]={0};
bool g[100][100]={0}, v[100]={0};
int n=0, m=0, c=0;

void dfs(int k)
{
  for (int i=1;i<=n;i++)
    if (g[k][i])
    {
      g[k][i]=false;
      g[i][k]=false;
      dfs(i);
    }
  m++;
  ans[m]=k;
}
```

```
int main(void)
{
  cin >> n >> m;

  for (int i=1;i<=m;i++)
  {
    int x=0, y=0;
    g[x][y]=true;
    g[y][x]=true;
    f[x]++;
    f[y]++;
  }

  m=0;
  int k1=0;
  for (int i=1;i<=n;i++)
  {
    if (f[i]%2==1) k1++;
    if (k1>2)
    {
      cout << "error" << endl;
      return 0;
    }
    if (f[i]%2 && c==0) c=i;
  }

  if (c==0) c=1;
  dfs(x);

  for (int i=m;i>=1;i--) cout << ans[i] << endl;
  return 0;
}
```

7.7 Topological sort

```
// Find any solution of topological sort.
#include<iostream>

using namespace std;

int f[100]={0}, ans[100]={0};
bool g[100][100]={0}, v[100]={0};
int n=0, m=0;

void dfs(int k)
{
  int i=0;
  v[k]=true;
  for (int i=1;i<=n;i++)
    if (g[k][i] && !v[i]) dfs(i);

  m++;
  ans[m]=k;
}

int main(void)
{
```

```
  cin >> n >> m;

  for (int i=1;i<=m;i++)
  {
    int x=0, y=0;
    cin >> x >> y;
    g[y][x]=true;
  }

  m=0;
  for (int i=1;i<=n;i++)
    if (!v[i]) dfs(i);

  for (int i=1;i<=n;i++) cout << ans[i] << endl;
  return 0;
}
```

```
// Find the order of topological sort is dictionary minimum
#include<iostream>

using namespace std;

int f[100]={0}, ans[100]={0};
bool g[100][100]={0}, v[100]={0};
int n=0, m=0;

int main(void)
{
  cin >> n >> m;

  for (int i=1;i<=m;i++)
  {
    int x=0, y=0;
    cin >> x >> y;
    g[x][y]=true;
    f[y]++;
  }
  for (int i=1;i<=n;i++)
  {
    for (int j=1;j<=n;j++)
    {
      if (f[j]==0 && !v[j]) break;

      if (f[j]!=0)
      {
        cout << "error" << endl;
        return 0;
      }

      ans[i]=j;
      v[j]=true;
      for (int k=1;k<=n;k++)
        if (g[j][k]) f[k]--;
    }
  }
  for (int i=1;i<=n;i++) cout << ans[i] << endl;
  return 0;
}
```

Theoretical Computer Science Cheat Sheet

Definitions	
$f(n) = O(g(n))$	iff $\exists$ positive c, n_0 such that $0 \le f(n) \le cg(n)$ $\forall n \ge n_0$.
$f(n) = \Omega(g(n))$	iff $\exists$ positive c, n_0 such that $f(n) \ge cg(n) \ge 0$ $\forall n \ge n_0$.
$f(n) = \Theta(g(n))$	iff $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$.
$f(n) = o(g(n))$	iff $\lim_{n\to\infty} f(n)/g(n) = 0$.
$\lim\limits_{n\to\infty} a_n = a$	iff $\forall \epsilon > 0$, $\exists n_0$ such that $\lvert a_n - a\rvert < \epsilon$, $\forall n \ge n_0$.
$\sup S$	least $b \in \mathbb{R}$ such that $b \ge s$, $\forall s \in S$.
$\inf S$	greatest $b \in \mathbb{R}$ such that $b \le s$, $\forall s \in S$.
$\liminf\limits_{n\to\infty} a_n$	$\lim\limits_{n\to\infty} \inf\{a_i \mid i \ge n, i \in \mathbb{N}\}$.
$\limsup\limits_{n\to\infty} a_n$	$\lim\limits_{n\to\infty} \sup\{a_i \mid i \ge n, i \in \mathbb{N}\}$.
$\binom{n}{k}$	Combinations: Size k subsets of a size n set.
$\left[{n \atop k}\right]$	Stirling numbers (1st kind): Arrangements of an n element set into k cycles.
$\left\{{n \atop k}\right\}$	Stirling numbers (2nd kind): Partitions of an n element set into k non-empty sets.
$\left\langle{n \atop k}\right\rangle$	1st order Eulerian numbers: Permutations $\pi_1\pi_2\ldots\pi_n$ on $\{1, 2, \ldots, n\}$ with k ascents.
$\left\langle\!\left\langle{n \atop k}\right\rangle\!\right\rangle$	2nd order Eulerian numbers.
C_n	Catalan Numbers: Binary trees with $n+1$ vertices.

Series

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}, \qquad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}.$$

In general:

$$\sum_{i=1}^{n} i^m = \frac{1}{m+1}\left[(n+1)^{m+1} - 1 - \sum_{i=1}^{n}\left((i+1)^{m+1} - i^{m+1} - (m+1)i^m\right)\right]$$

$$\sum_{i=1}^{n-1} i^m = \frac{1}{m+1}\sum_{k=0}^{m}\binom{m+1}{k}B_k n^{m+1-k}.$$

Geometric series:

$$\sum_{i=0}^{n} c^i = \frac{c^{n+1}-1}{c-1}, \quad c \ne 1, \qquad \sum_{i=0}^{\infty} c^i = \frac{1}{1-c}, \qquad \sum_{i=1}^{\infty} c^i = \frac{c}{1-c}, \quad |c| < 1,$$

$$\sum_{i=0}^{n} ic^i = \frac{nc^{n+2} - (n+1)c^{n+1} + c}{(c-1)^2}, \quad c \ne 1, \qquad \sum_{i=0}^{\infty} ic^i = \frac{c}{(1-c)^2}, \quad |c| < 1.$$

Harmonic series:

$$H_n = \sum_{i=1}^{n}\frac{1}{i}, \qquad \sum_{i=1}^{n} iH_i = \frac{n(n+1)}{2}H_n - \frac{n(n-1)}{4}.$$

$$\sum_{i=1}^{n} H_i = (n+1)H_n - n, \qquad \sum_{i=1}^{n}\binom{i}{m}H_i = \binom{n+1}{m+1}\left(H_{n+1} - \frac{1}{m+1}\right).$$

1. $\displaystyle\binom{n}{k} = \frac{n!}{(n-k)!k!}$, **2.** $\displaystyle\sum_{k=0}^{n}\binom{n}{k} = 2^n$, **3.** $\displaystyle\binom{n}{k} = \binom{n}{n-k}$,

4. $\displaystyle\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$, **5.** $\displaystyle\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$,

6. $\displaystyle\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}$, **7.** $\displaystyle\sum_{k=0}^{n}\binom{r+k}{k} = \binom{r+n+1}{n}$,

8. $\displaystyle\sum_{k=0}^{n}\binom{k}{m} = \binom{n+1}{m+1}$, **9.** $\displaystyle\sum_{k=0}^{n}\binom{r}{k}\binom{s}{n-k} = \binom{r+s}{n}$,

10. $\displaystyle\binom{n}{k} = (-1)^k\binom{k-n-1}{k}$, **11.** $\displaystyle\left\{{n \atop 1}\right\} = \left\{{n \atop n}\right\} = 1$,

12. $\displaystyle\left\{{n \atop 2}\right\} = 2^{n-1} - 1$, **13.** $\displaystyle\left\{{n \atop k}\right\} = k\left\{{n-1 \atop k}\right\} + \left\{{n-1 \atop k-1}\right\}$,

14. $\displaystyle\left[{n \atop 1}\right] = (n-1)!$, **15.** $\displaystyle\left[{n \atop 2}\right] = (n-1)!H_{n-1}$, **16.** $\displaystyle\left[{n \atop n}\right] = 1$, **17.** $\displaystyle\left[{n \atop k}\right] \ge \left\{{n \atop k}\right\}$,

18. $\displaystyle\left[{n \atop k}\right] = (n-1)\left[{n-1 \atop k}\right] + \left[{n-1 \atop k-1}\right]$, **19.** $\displaystyle\left\{{n \atop n-1}\right\} = \left[{n \atop n-1}\right] = \binom{n}{2}$, **20.** $\displaystyle\sum_{k=0}^{n}\left[{n \atop k}\right] = n!$, **21.** $\displaystyle C_n = \frac{1}{n+1}\binom{2n}{n}$,

22. $\displaystyle\left\langle{n \atop 0}\right\rangle = \left\langle{n \atop n-1}\right\rangle = 1$, **23.** $\displaystyle\left\langle{n \atop k}\right\rangle = \left\langle{n \atop n-1-k}\right\rangle$, **24.** $\displaystyle\left\langle{n \atop k}\right\rangle = (k+1)\left\langle{n-1 \atop k}\right\rangle + (n-k)\left\langle{n-1 \atop k-1}\right\rangle$,

25. $\displaystyle\left\langle{0 \atop k}\right\rangle = \begin{cases}1 & \text{if } k = 0,\\ 0 & \text{otherwise}\end{cases}$ **26.** $\displaystyle\left\langle{n \atop 1}\right\rangle = 2^n - n - 1$, **27.** $\displaystyle\left\langle{n \atop 2}\right\rangle = 3^n - (n+1)2^n + \binom{n+1}{2}$,

28. $\displaystyle x^n = \sum_{k=0}^{n}\left\langle{n \atop k}\right\rangle\binom{x+k}{n}$, **29.** $\displaystyle\left\langle{n \atop m}\right\rangle = \sum_{k=0}^{m}\binom{n+1}{k}(m+1-k)^n(-1)^k$, **30.** $\displaystyle m!\left\{{n \atop m}\right\} = \sum_{k=0}^{n}\left\langle{n \atop k}\right\rangle\binom{k}{n-m}$,

31. $\displaystyle\left\langle{n \atop m}\right\rangle = \sum_{k=0}^{n}\left\{{n \atop k}\right\}\binom{n-k}{m}(-1)^{n-k-m}k!$, **32.** $\displaystyle\left\langle\!\left\langle{n \atop 0}\right\rangle\!\right\rangle = 1$, **33.** $\displaystyle\left\langle\!\left\langle{n \atop n}\right\rangle\!\right\rangle = 0 \quad \text{for } n \ne 0$,

34. $\displaystyle\left\langle\!\left\langle{n \atop k}\right\rangle\!\right\rangle = (k+1)\left\langle\!\left\langle{n-1 \atop k}\right\rangle\!\right\rangle + (2n-1-k)\left\langle\!\left\langle{n-1 \atop k-1}\right\rangle\!\right\rangle$, **35.** $\displaystyle\sum_{k=0}^{n}\left\langle\!\left\langle{n \atop k}\right\rangle\!\right\rangle = \frac{(2n)^{\underline{n}}}{2^n}$,

36. $\displaystyle\left\{{x \atop x-n}\right\} = \sum_{k=0}^{n}\left\langle\!\left\langle{n \atop k}\right\rangle\!\right\rangle\binom{x+n-1-k}{2n}$, **37.** $\displaystyle\left\{{n+1 \atop m+1}\right\} = \sum_{k}\binom{n}{k}\left\{{k \atop m}\right\} = \sum_{k=0}^{n}\left\{{k \atop m}\right\}(m+1)^{n-k}$,

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Identities Cont.

38. $\genfrac{[}{]}{0pt}{}{n+1}{m+1} = \sum_k \genfrac{[}{]}{0pt}{}{n}{k}\binom{k}{m} = \sum_{k=0}^{n} \genfrac{[}{]}{0pt}{}{k}{m} n^{\underline{n-k}} = n!\sum_{k=0}^{n}\frac{1}{k!}\genfrac{[}{]}{0pt}{}{k}{m},$

39. $\genfrac{[}{]}{0pt}{}{x}{x-n} = \sum_{k=0}^{n} \left\langle\!\!\left\langle {n \atop k} \right\rangle\!\!\right\rangle \binom{x+k}{2n},$

40. $\genfrac{\{}{\}}{0pt}{}{n}{m} = \sum_k \binom{n}{k}\genfrac{\{}{\}}{0pt}{}{k+1}{m+1}(-1)^{n-k},$

41. $\genfrac{[}{]}{0pt}{}{n}{m} = \sum_k \genfrac{[}{]}{0pt}{}{n+1}{k+1}\binom{k}{m}(-1)^{m-k},$

42. $\genfrac{\{}{\}}{0pt}{}{m+n+1}{m} = \sum_{k=0}^{m} k\genfrac{\{}{\}}{0pt}{}{n+k}{k},$

43. $\genfrac{[}{]}{0pt}{}{m+n+1}{m} = \sum_{k=0}^{m} k(n+k)\genfrac{[}{]}{0pt}{}{n+k}{k},$

44. $\binom{n}{m} = \sum_k \genfrac{\{}{\}}{0pt}{}{n+1}{k+1}\genfrac{[}{]}{0pt}{}{k}{m}(-1)^{m-k},$

45. $(n-m)!\binom{n}{m} = \sum_k \genfrac{[}{]}{0pt}{}{n+1}{k+1}\genfrac{\{}{\}}{0pt}{}{k}{m}(-1)^{m-k}, \quad \text{for } n \ge m,$

46. $\genfrac{\{}{\}}{0pt}{}{n}{n-m} = \sum_k \binom{m-n}{m+k}\binom{m+n}{n+k}\genfrac{[}{]}{0pt}{}{m+k}{k},$

47. $\genfrac{[}{]}{0pt}{}{n}{n-m} = \sum_k \binom{m-n}{m+k}\binom{m+n}{n+k}\genfrac{\{}{\}}{0pt}{}{m+k}{k},$

48. $\genfrac{\{}{\}}{0pt}{}{n}{\ell+m}\binom{\ell+m}{\ell} = \sum_k \genfrac{\{}{\}}{0pt}{}{k}{\ell}\genfrac{\{}{\}}{0pt}{}{n-k}{m}\binom{n}{k},$

49. $\genfrac{[}{]}{0pt}{}{n}{\ell+m}\binom{\ell+m}{\ell} = \sum_k \genfrac{[}{]}{0pt}{}{k}{\ell}\genfrac{[}{]}{0pt}{}{n-k}{m}\binom{n}{k}.$

Trees

Every tree with n vertices has $n-1$ edges.

Kraft inequality: If the depths of the leaves of a binary tree are $d_1, \dots, d_n$:

$$\sum_{i=1}^{n} 2^{-d_i} \le 1,$$

and equality holds only if every internal node has 2 sons.

Recurrences

Master method:

$$T(n) = aT(n/b) + f(n), \quad a \ge 1, b > 1$$

If $\exists\epsilon > 0$ such that $f(n) = O(n^{\log_b a - \epsilon})$ then

$$T(n) = \Theta(n^{\log_b a}).$$

If $f(n) = \Theta(n^{\log_b a})$ then

$$T(n) = \Theta(n^{\log_b a}\log_2 n).$$

If $\exists\epsilon > 0$ such that $f(n) = \Omega(n^{\log_b a + \epsilon})$, and $\exists c < 1$ such that $af(n/b) \le cf(n)$ for large n, then

$$T(n) = \Theta(f(n)).$$

Substitution (example): Consider the following recurrence

$$T_{i+1} = 2^{2^i} \cdot T_i^2, \quad T_1 = 2.$$

Note that T_i is always a power of two. Let $t_i = \log_2 T_i$. Then we have

$$t_{i+1} = 2^i + 2t_i, \quad t_1 = 1.$$

Let $u_i = t_i/2^i$. Dividing both sides of the previous equation by 2^{i+1} we get

$$\frac{t_{i+1}}{2^{i+1}} = \frac{2^i}{2^{i+1}} + \frac{t_i}{2^i}.$$

Substituting we find

$$u_{i+1} = \tfrac{1}{2} + u_i, \qquad u_1 = \tfrac{1}{2},$$

which is simply $u_i = i/2$. So we find that T_i has the closed form $T_i = 2^{i2^{i-1}}$. Summing factors (example): Consider the following recurrence

$$T(n) = 3T(n/2) + n, \quad T(1) = 1.$$

Rewrite so that all terms involving T are on the left side

$$T(n) - 3T(n/2) = n.$$

Now expand the recurrence, and choose a factor which makes the left side "telescope"

$$1\big(T(n) - 3T(n/2) = n\big)$$
$$3\big(T(n/2) - 3T(n/4) = n/2\big)$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$3^{\log_2 n - 1}\big(T(2) - 3T(1) = 2\big)$$

Let $m = \log_2 n$. Summing the left side we get $T(n) - 3^m T(1) = T(n) - 3^m = T(n) - n^k$ where $k = \log_2 3 \approx 1.58496$. Summing the right side we get

$$\sum_{i=0}^{m-1} \frac{n}{2^i} 3^i = n\sum_{i=0}^{m-1}\left(\tfrac{3}{2}\right)^i.$$

Let $c = \frac{3}{2}$. Then we have

$$\begin{aligned} n\sum_{i=0}^{m-1} c^i &= n\left(\frac{c^m - 1}{c - 1}\right) \\ &= 2n(c^{\log_2 n} - 1) \\ &= 2n(c^{(k-1)\log_c n} - 1) \\ &= 2n^k - 2n, \end{aligned}$$

and so $T(n) = 3n^k - 2n$. Full history recurrences can often be changed to limited history ones (example): Consider

$$T_i = 1 + \sum_{j=0}^{i-1} T_j, \quad T_0 = 1.$$

Note that

$$T_{i+1} = 1 + \sum_{j=0}^{i} T_j.$$

Subtracting we find

$$\begin{aligned} T_{i+1} - T_i &= 1 + \sum_{j=0}^{i} T_j - 1 - \sum_{j=0}^{i-1} T_j \\ &= T_i. \end{aligned}$$

And so $T_{i+1} = 2T_i = 2^{i+1}$.

Generating functions:

1. Multiply both sides of the equation by x^i.
2. Sum both sides over all i for which the equation is valid.
3. Choose a generating function $G(x)$. Usually $G(x) = \sum_{i=0}^{\infty} x^i g_i$.
3. Rewrite the equation in terms of the generating function $G(x)$.
4. Solve for $G(x)$.
5. The coefficient of x^i in $G(x)$ is g_i.

Example:

$$g_{i+1} = 2g_i + 1, \quad g_0 = 0.$$

Multiply and sum:

$$\sum_{i\ge0} g_{i+1}x^i = \sum_{i\ge0} 2g_i x^i + \sum_{i\ge0} x^i.$$

We choose $G(x) = \sum_{i\ge0} x^i g_i$. Rewrite in terms of $G(x)$:

$$\frac{G(x) - g_0}{x} = 2G(x) + \sum_{i\ge0} x^i.$$

Simplify:

$$\frac{G(x)}{x} = 2G(x) + \frac{1}{1-x}.$$

Solve for $G(x)$:

$$G(x) = \frac{x}{(1-x)(1-2x)}.$$

Expand this using partial fractions:

$$\begin{aligned} G(x) &= x\left(\frac{2}{1-2x} - \frac{1}{1-x}\right) \\ &= x\left(2\sum_{i\ge0} 2^i x^i - \sum_{i\ge0} x^i\right) \\ &= \sum_{i\ge0}(2^{i+1} - 1)x^{i+1}. \end{aligned}$$

So $g_i = 2^i - 1$.

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$\pi \approx 3.14159,\quad e \approx 2.71828,\quad \gamma \approx 0.57721,\quad \phi = \frac{1+\sqrt{5}}{2} \approx 1.61803,\quad \hat{\phi} = \frac{1-\sqrt{5}}{2} \approx -.61803$

i	2^i	p_i
1	2	2
2	4	3
3	8	5
4	16	7
5	32	11
6	64	13
7	128	17
8	256	19
9	512	23
10	1,024	29
11	2,048	31
12	4,096	37
13	8,192	41
14	16,384	43
15	32,768	47
16	65,536	53
17	131,072	59
18	262,144	61
19	524,288	67
20	1,048,576	71
21	2,097,152	73
22	4,194,304	79
23	8,388,608	83
24	16,777,216	89
25	33,554,432	97
26	67,108,864	101
27	134,217,728	103
28	268,435,456	107
29	536,870,912	109
30	1,073,741,824	113
31	2,147,483,648	127
32	4,294,967,296	131

Pascal's Triangle

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1

General

Bernoulli Numbers ($B_i = 0$, odd $i \neq 1$):

$$B_0 = 1,\ B_1 = -\tfrac{1}{2},\ B_2 = \tfrac{1}{6},\ B_4 = -\tfrac{1}{30},$$
$$B_6 = \tfrac{1}{42},\ B_8 = -\tfrac{1}{30},\ B_{10} = \tfrac{5}{66}.$$

Change of base, quadratic formula:

$$\log_b x = \frac{\log_a x}{\log_a b}, \qquad \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Euler's number e:

$$e = 1 + \tfrac{1}{2} + \tfrac{1}{6} + \tfrac{1}{24} + \tfrac{1}{120} + \cdots$$
$$\lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = e^x.$$
$$\left(1 + \tfrac{1}{n}\right)^n < e < \left(1 + \tfrac{1}{n}\right)^{n+1}.$$
$$\left(1 + \tfrac{1}{n}\right)^n = e - \frac{e}{2n} + \frac{11e}{24n^2} - O\left(\frac{1}{n^3}\right).$$

Harmonic numbers:

$$1,\ \tfrac{3}{2},\ \tfrac{11}{6},\ \tfrac{25}{12},\ \tfrac{137}{60},\ \tfrac{49}{20},\ \tfrac{363}{140},\ \tfrac{761}{280},\ \tfrac{7129}{2520}, \ldots$$

$$\ln n < H_n < \ln n + 1,$$
$$H_n = \ln n + \gamma + O\left(\frac{1}{n}\right).$$

Factorial, Stirling's approximation:

$$1, 2, 6, 24, 120, 720, 5040, 40320, 362880, \ldots$$

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1 + \Theta\left(\frac{1}{n}\right)\right).$$

Ackermann's function and inverse:

$$a(i,j) = \begin{cases} 2^j & i = 1 \\ a(i-1, 2) & j = 1 \\ a(i-1, a(i, j-1)) & i, j \geq 2 \end{cases}$$
$$\alpha(i) = \min\{j \mid a(j,j) \geq i\}.$$

Binomial distribution:

$$\Pr[X = k] = \binom{n}{k} p^k q^{n-k}, \qquad q = 1 - p,$$
$$\mathrm{E}[X] = \sum_{k=1}^{n} k \binom{n}{k} p^k q^{n-k} = np.$$

Poisson distribution:

$$\Pr[X = k] = \frac{e^{-\lambda}\lambda^k}{k!}, \quad \mathrm{E}[X] = \lambda.$$

Normal (Gaussian) distribution:

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}, \quad \mathrm{E}[X] = \mu.$$

The "coupon collector": We are given a random coupon each day, and there are n different types of coupons. The distribution of coupons is uniform. The expected number of days to pass before we to collect all n types is

$$nH_n.$$

Probability

Continuous distributions: If

$$\Pr[a < X < b] = \int_a^b p(x)\,dx,$$

then p is the probability density function of X. If

$$\Pr[X < a] = P(a),$$

then P is the distribution function of X. If P and p both exist then

$$P(a) = \int_{-\infty}^{a} p(x)\,dx.$$

Expectation: If X is discrete

$$\mathrm{E}[g(X)] = \sum_x g(x) \Pr[X = x].$$

If X continuous then

$$\mathrm{E}[g(X)] = \int_{-\infty}^{\infty} g(x)p(x)\,dx = \int_{-\infty}^{\infty} g(x)\,dP(x).$$

Variance, standard deviation:

$$\mathrm{VAR}[X] = \mathrm{E}[X^2] - \mathrm{E}[X]^2,$$
$$\sigma = \sqrt{\mathrm{VAR}[X]}.$$

For events A and B:

$$\Pr[A \vee B] = \Pr[A] + \Pr[B] - \Pr[A \wedge B]$$
$$\Pr[A \wedge B] = \Pr[A] \cdot \Pr[B],$$

iff A and B are independent.

$$\Pr[A|B] = \frac{\Pr[A \wedge B]}{\Pr[B]}$$

For random variables X and Y:

$$\mathrm{E}[X \cdot Y] = \mathrm{E}[X] \cdot \mathrm{E}[Y],$$

if X and Y are independent.

$$\mathrm{E}[X + Y] = \mathrm{E}[X] + \mathrm{E}[Y],$$
$$\mathrm{E}[cX] = c\,\mathrm{E}[X].$$

Bayes' theorem:

$$\Pr[A_i|B] = \frac{\Pr[B|A_i]\Pr[A_i]}{\sum_{j=1}^{n} \Pr[A_j]\Pr[B|A_j]}.$$

Inclusion-exclusion:

$$\Pr\left[\bigvee_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \Pr[X_i] + \sum_{k=2}^{n} (-1)^{k+1} \sum_{i_i < \cdots < i_k} \Pr\left[\bigwedge_{j=1}^{k} X_{i_j}\right].$$

Moment inequalities:

$$\Pr\left[|X| \geq \lambda\,\mathrm{E}[X]\right] \leq \frac{1}{\lambda},$$
$$\Pr\left[\left|X - \mathrm{E}[X]\right| \geq \lambda \cdot \sigma\right] \leq \frac{1}{\lambda^2}.$$

Geometric distribution:

$$\Pr[X = k] = pq^{k-1}, \qquad q = 1 - p,$$
$$\mathrm{E}[X] = \sum_{k=1}^{\infty} kpq^{k-1} = \frac{1}{p}.$$

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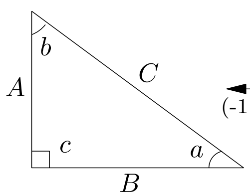
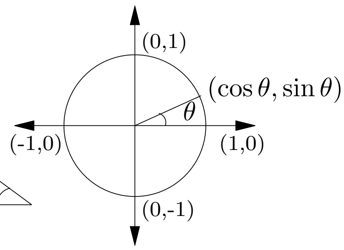
Trigonometry





Pythagorean theorem:
$$C^2 = A^2 + B^2.$$

Definitions:
$$\sin a = A/C, \quad \cos a = B/C,$$
$$\csc a = C/A, \quad \sec a = C/B,$$
$$\tan a = \frac{\sin a}{\cos a} = \frac{A}{B}, \quad \cot a = \frac{\cos a}{\sin a} = \frac{B}{A}.$$

Area, radius of inscribed circle:
$$\tfrac{1}{2}AB, \quad \frac{AB}{A+B+C}.$$

Identities:
$$\sin x = \frac{1}{\csc x}, \qquad \cos x = \frac{1}{\sec x},$$
$$\tan x = \frac{1}{\cot x}, \qquad \sin^2 x + \cos^2 x = 1,$$
$$1 + \tan^2 x = \sec^2 x, \qquad 1 + \cot^2 x = \csc^2 x,$$
$$\sin x = \cos\left(\tfrac{\pi}{2} - x\right), \qquad \sin x = \sin(\pi - x),$$
$$\cos x = -\cos(\pi - x), \qquad \tan x = \cot\left(\tfrac{\pi}{2} - x\right),$$
$$\cot x = -\cot(\pi - x), \qquad \csc x = \cot \tfrac{x}{2} - \cot x,$$
$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y,$$
$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y,$$
$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y},$$
$$\cot(x \pm y) = \frac{\cot x \cot y \mp 1}{\cot x \pm \cot y},$$
$$\sin 2x = 2 \sin x \cos x, \qquad \sin 2x = \frac{2\tan x}{1 + \tan^2 x},$$
$$\cos 2x = \cos^2 x - \sin^2 x, \qquad \cos 2x = 2\cos^2 x - 1,$$
$$\cos 2x = 1 - 2\sin^2 x, \qquad \cos 2x = \frac{1 - \tan^2 x}{1 + \tan^2 x},$$
$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}, \qquad \cot 2x = \frac{\cot^2 x - 1}{2\cot x},$$
$$\sin(x+y)\sin(x-y) = \sin^2 x - \sin^2 y,$$
$$\cos(x+y)\cos(x-y) = \cos^2 x - \sin^2 y.$$

Euler's equation:
$$e^{ix} = \cos x + i \sin x, \qquad e^{i\pi} = -1.$$

Matrices

Multiplication:
$$C = A \cdot B, \quad c_{i,j} = \sum_{k=1}^{n} a_{i,k} b_{k,j}.$$

Determinants: $\det A \neq 0$ iff A is non-singular.
$$\det A \cdot B = \det A \cdot \det B,$$
$$\det A = \sum_{\pi} \prod_{i=1}^{n} \operatorname{sign}(\pi) a_{i,\pi(i)}.$$

2×2 and 3×3 determinant:
$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc,$$
$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = g \begin{vmatrix} b & c \\ e & f \end{vmatrix} - h \begin{vmatrix} a & c \\ d & f \end{vmatrix} + i \begin{vmatrix} a & b \\ d & e \end{vmatrix}$$
$$= \begin{matrix} aei + bfg + cdh \\ -ceg - fha - ibd. \end{matrix}$$

Permanents:
$$\operatorname{perm} A = \sum_{\pi} \prod_{i=1}^{n} a_{i,\pi(i)}.$$

Hyperbolic Functions

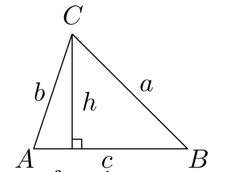
Definitions:
$$\sinh x = \frac{e^x - e^{-x}}{2}, \qquad \cosh x = \frac{e^x + e^{-x}}{2},$$
$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \qquad \operatorname{csch} x = \frac{1}{\sinh x},$$
$$\operatorname{sech} x = \frac{1}{\cosh x}, \qquad \coth x = \frac{1}{\tanh x}.$$

Identities:
$$\cosh^2 x - \sinh^2 x = 1, \qquad \tanh^2 x + \operatorname{sech}^2 x = 1,$$
$$\coth^2 x - \operatorname{csch}^2 x = 1, \qquad \sinh(-x) = -\sinh x,$$
$$\cosh(-x) = \cosh x, \qquad \tanh(-x) = -\tanh x,$$
$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y,$$
$$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y,$$
$$\sinh 2x = 2 \sinh x \cosh x,$$
$$\cosh 2x = \cosh^2 x + \sinh^2 x,$$
$$\cosh x + \sinh x = e^x, \qquad \cosh x - \sinh x = e^{-x},$$
$$(\cosh x + \sinh x)^n = \cosh nx + \sinh nx, \quad n \in \mathbb{Z},$$
$$2\sinh^2 \tfrac{x}{2} = \cosh x - 1, \qquad 2\cosh^2 \tfrac{x}{2} = \cosh x + 1.$$

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
0	0	1	0
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$	1	0	∞

… in mathematics you don't understand things, you just get used to them.
– J. von Neumann

More Trig.



Law of cosines:
$$c^2 = a^2 + b^2 - 2ab\cos C.$$

Area:
$$A = \tfrac{1}{2}hc,$$
$$= \tfrac{1}{2}ab\sin C,$$
$$= \frac{c^2 \sin A \sin B}{2\sin C}.$$

Heron's formula:
$$A = \sqrt{s \cdot s_a \cdot s_b \cdot s_c},$$
$$s = \tfrac{1}{2}(a + b + c),$$
$$s_a = s - a,$$
$$s_b = s - b,$$
$$s_c = s - c.$$

More identities:
$$\sin \tfrac{x}{2} = \sqrt{\frac{1 - \cos x}{2}},$$
$$\cos \tfrac{x}{2} = \sqrt{\frac{1 + \cos x}{2}},$$
$$\tan \tfrac{x}{2} = \sqrt{\frac{1 - \cos x}{1 + \cos x}},$$
$$= \frac{1 - \cos x}{\sin x},$$
$$= \frac{\sin x}{1 + \cos x},$$
$$\cot \tfrac{x}{2} = \sqrt{\frac{1 + \cos x}{1 - \cos x}},$$
$$= \frac{1 + \cos x}{\sin x},$$
$$= \frac{\sin x}{1 - \cos x},$$
$$\sin x = \frac{e^{ix} - e^{-ix}}{2i},$$
$$\cos x = \frac{e^{ix} + e^{-ix}}{2},$$
$$\tan x = -i\frac{e^{ix} - e^{-ix}}{e^{ix} + e^{-ix}},$$
$$= -i\frac{e^{2ix} - 1}{e^{2ix} + 1},$$
$$\sin x = \frac{\sinh ix}{i},$$
$$\cos x = \cosh ix,$$
$$\tan x = \frac{\tanh ix}{i}.$$

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Theoretical Computer Science Cheat Sheet

Number Theory

The Chinese remainder theorem: There exists a number C such that:

$$\begin{aligned} C &\equiv r_1 \bmod m_1 \\ \vdots \;\; &\vdots \;\; \vdots \\ C &\equiv r_n \bmod m_n \end{aligned}$$

if m_i and m_j are relatively prime for $i \neq j$.

Euler's function: $\phi(x)$ is the number of positive integers less than x relatively prime to x. If $\prod_{i=1}^n p_i^{e_i}$ is the prime factorization of x then

$$\phi(x) = \prod_{i=1}^{n} p_i^{e_i-1}(p_i - 1).$$

Euler's theorem: If a and b are relatively prime then

$$1 \equiv a^{\phi(b)} \bmod b.$$

Fermat's theorem:

$$1 \equiv a^{p-1} \bmod p.$$

The Euclidean algorithm: if $a > b$ are integers then

$$\gcd(a, b) = \gcd(a \bmod b, b).$$

If $\prod_{i=1}^n p_i^{e_i}$ is the prime factorization of x then

$$S(x) = \sum_{d|x} d = \prod_{i=1}^{n} \frac{p_i^{e_i+1} - 1}{p_i - 1}.$$

Perfect Numbers: x is an even perfect number iff $x = 2^{n-1}(2^n - 1)$ and $2^n - 1$ is prime.

Wilson's theorem: n is a prime iff

$$(n-1)! \equiv -1 \bmod n.$$

Möbius inversion:

$$\mu(i) = \begin{cases} 1 & \text{if } i = 1. \\ 0 & \text{if } i \text{ is not square-free.} \\ (-1)^r & \text{if } i \text{ is the product of} \\ & r \text{ distinct primes.} \end{cases}$$

If

$$G(a) = \sum_{d|a} F(d),$$

then

$$F(a) = \sum_{d|a} \mu(d) G\left(\frac{a}{d}\right).$$

Prime numbers:

$$p_n = n \ln n + n \ln\ln n - n + n \frac{\ln\ln n}{\ln n} + O\left(\frac{n}{\ln n}\right),$$

$$\pi(n) = \frac{n}{\ln n} + \frac{n}{(\ln n)^2} + \frac{2!n}{(\ln n)^3} + O\left(\frac{n}{(\ln n)^4}\right).$$

Graph Theory

Definitions:

Loop	An edge connecting a vertex to itself.
Directed	Each edge has a direction.
Simple	Graph with no loops or multi-edges.
Walk	A sequence $v_0 e_1 v_1 \ldots e_\ell v_\ell$.
Trail	A walk with distinct edges.
Path	A trail with distinct vertices.
Connected	A graph where there exists a path between any two vertices.
Component	A maximal connected subgraph.
Tree	A connected acyclic graph.
Free tree	A tree with no root.
DAG	Directed acyclic graph.
Eulerian	Graph with a trail visiting each edge exactly once.
Hamiltonian	Graph with a cycle visiting each vertex exactly once.
Cut	A set of edges whose removal increases the number of components.
Cut-set	A minimal cut.
Cut edge	A size 1 cut.
k-Connected	A graph connected with the removal of any $k-1$ vertices.
k-Tough	$\forall S \subseteq V, S \neq \emptyset$ we have $k \cdot c(G - S) \leq \lvert S \rvert$.
k-Regular	A graph where all vertices have degree k.
k-Factor	A k-regular spanning subgraph.
Matching	A set of edges, no two of which are adjacent.
Clique	A set of vertices, all of which are adjacent.
Ind. set	A set of vertices, none of which are adjacent.
Vertex cover	A set of vertices which cover all edges.
Planar graph	A graph which can be embeded in the plane.
Plane graph	An embedding of a planar graph.

$$\sum_{v \in V} \deg(v) = 2m.$$

If G is planar then $n - m + f = 2$, so

$$f \leq 2n - 4, \quad m \leq 3n - 6.$$

Any planar graph has a vertex with degree ≤ 5.

Notation:

$E(G)$	Edge set
$V(G)$	Vertex set
$c(G)$	Number of components
$G[S]$	Induced subgraph
$\deg(v)$	Degree of v
$\Delta(G)$	Maximum degree
$\delta(G)$	Minimum degree
$\chi(G)$	Chromatic number
$\chi_E(G)$	Edge chromatic number
G^c	Complement graph
K_n	Complete graph
K_{n_1,n_2}	Complete bipartite graph
$\mathrm{r}(k, \ell)$	Ramsey number

Geometry

Projective coordinates: triples (x, y, z), not all x, y and z zero.

$$(x, y, z) = (cx, cy, cz) \quad \forall c \neq 0.$$

Cartesian	Projective
(x, y)	$(x, y, 1)$
$y = mx + b$	$(m, -1, b)$
$x = c$	$(1, 0, -c)$

Distance formula, L_p and L_∞ metric:

$$\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2},$$

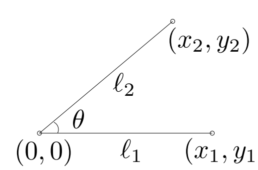
$$\left[|x_1 - x_0|^p + |y_1 - y_0|^p\right]^{1/p},$$

$$\lim_{p \to \infty} \left[|x_1 - x_0|^p + |y_1 - y_0|^p\right]^{1/p}.$$

Area of triangle (x_0, y_0), (x_1, y_1) and (x_2, y_2):

$$\frac{1}{2} \operatorname{abs} \begin{vmatrix} x_1 - x_0 & y_1 - y_0 \\ x_2 - x_0 & y_2 - y_0 \end{vmatrix}.$$

Angle formed by three points:



$$\cos\theta = \frac{(x_1, y_1) \cdot (x_2, y_2)}{\ell_1 \ell_2}.$$

Line through two points (x_0, y_0) and (x_1, y_1):

$$\begin{vmatrix} x & y & 1 \\ x_0 & y_0 & 1 \\ x_1 & y_1 & 1 \end{vmatrix} = 0.$$

Area of circle, volume of sphere:

$$A = \pi r^2, \qquad V = \tfrac{4}{3}\pi r^3.$$

If I have seen farther than others, it is because I have stood on the shoulders of giants.
– Issac Newton

Theoretical Computer Science Cheat Sheet

π

Wallis' identity:
$$\pi = 2 \cdot \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdots}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdots}$$

Brouncker's continued fraction expansion:
$$\frac{\pi}{4} = 1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \cfrac{7^2}{2 + \cdots}}}}$$

Gregrory's series:
$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$$

Newton's series:
$$\frac{\pi}{6} = \frac{1}{2} + \frac{1}{2 \cdot 3 \cdot 2^3} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 5 \cdot 2^5} + \cdots$$

Sharp's series:
$$\frac{\pi}{6} = \frac{1}{\sqrt{3}}\left(1 - \frac{1}{3^1 \cdot 3} + \frac{1}{3^2 \cdot 5} - \frac{1}{3^3 \cdot 7} + \cdots\right)$$

Euler's series:
$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots$$
$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \cdots$$
$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \cdots$$

Partial Fractions

Let $N(x)$ and $D(x)$ be polynomial functions of x. We can break down $N(x)/D(x)$ using partial fraction expansion. First, if the degree of N is greater than or equal to the degree of D, divide N by D, obtaining
$$\frac{N(x)}{D(x)} = Q(x) + \frac{N'(x)}{D(x)},$$
where the degree of N' is less than that of D. Second, factor $D(x)$. Use the following rules: For a non-repeated factor:
$$\frac{N(x)}{(x-a)D(x)} = \frac{A}{x-a} + \frac{N'(x)}{D(x)},$$
where
$$A = \left[\frac{N(x)}{D(x)}\right]_{x=a}.$$
For a repeated factor:
$$\frac{N(x)}{(x-a)^m D(x)} = \sum_{k=0}^{m-1} \frac{A_k}{(x-a)^{m-k}} + \frac{N'(x)}{D(x)},$$
where
$$A_k = \frac{1}{k!}\left[\frac{d^k}{dx^k}\left(\frac{N(x)}{D(x)}\right)\right]_{x=a}.$$

The reasonable man adapts himself to the world; the unreasonable persists in trying to adapt the world to himself. Therefore all progress depends on the unreasonable.
– George Bernard Shaw

Calculus

Derivatives:

1. $\displaystyle \frac{d(cu)}{dx} = c\frac{du}{dx}$, **2.** $\displaystyle \frac{d(u+v)}{dx} = \frac{du}{dx} + \frac{dv}{dx}$, **3.** $\displaystyle \frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$,

4. $\displaystyle \frac{d(u^n)}{dx} = nu^{n-1}\frac{du}{dx}$, **5.** $\displaystyle \frac{d(u/v)}{dx} = \frac{v\left(\frac{du}{dx}\right) - u\left(\frac{dv}{dx}\right)}{v^2}$, **6.** $\displaystyle \frac{d(e^{cu})}{dx} = ce^{cu}\frac{du}{dx}$,

7. $\displaystyle \frac{d(c^u)}{dx} = (\ln c)c^u\frac{du}{dx}$, **8.** $\displaystyle \frac{d(\ln u)}{dx} = \frac{1}{u}\frac{du}{dx}$,

9. $\displaystyle \frac{d(\sin u)}{dx} = \cos u\frac{du}{dx}$, **10.** $\displaystyle \frac{d(\cos u)}{dx} = -\sin u\frac{du}{dx}$,

11. $\displaystyle \frac{d(\tan u)}{dx} = \sec^2 u\frac{du}{dx}$, **12.** $\displaystyle \frac{d(\cot u)}{dx} = \csc^2 u\frac{du}{dx}$,

13. $\displaystyle \frac{d(\sec u)}{dx} = \tan u \,\sec u\frac{du}{dx}$, **14.** $\displaystyle \frac{d(\csc u)}{dx} = -\cot u \,\csc u\frac{du}{dx}$,

15. $\displaystyle \frac{d(\arcsin u)}{dx} = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$, **16.** $\displaystyle \frac{d(\arccos u)}{dx} = \frac{-1}{\sqrt{1-u^2}}\frac{du}{dx}$,

17. $\displaystyle \frac{d(\arctan u)}{dx} = \frac{1}{1+u^2}\frac{du}{dx}$, **18.** $\displaystyle \frac{d(\text{arccot}\, u)}{dx} = \frac{-1}{1+u^2}\frac{du}{dx}$,

19. $\displaystyle \frac{d(\text{arcsec}\, u)}{dx} = \frac{1}{u\sqrt{1-u^2}}\frac{du}{dx}$, **20.** $\displaystyle \frac{d(\text{arccsc}\, u)}{dx} = \frac{-1}{u\sqrt{1-u^2}}\frac{du}{dx}$,

21. $\displaystyle \frac{d(\sinh u)}{dx} = \cosh u\frac{du}{dx}$, **22.** $\displaystyle \frac{d(\cosh u)}{dx} = \sinh u\frac{du}{dx}$,

23. $\displaystyle \frac{d(\tanh u)}{dx} = \text{sech}^2 u\frac{du}{dx}$, **24.** $\displaystyle \frac{d(\coth u)}{dx} = -\text{csch}^2 u\frac{du}{dx}$,

25. $\displaystyle \frac{d(\text{sech}\, u)}{dx} = -\text{sech}\, u \,\tanh u\frac{du}{dx}$, **26.** $\displaystyle \frac{d(\text{csch}\, u)}{dx} = -\text{csch}\, u \,\coth u\frac{du}{dx}$,

27. $\displaystyle \frac{d(\text{arcsinh}\, u)}{dx} = \frac{1}{\sqrt{1+u^2}}\frac{du}{dx}$, **28.** $\displaystyle \frac{d(\text{arccosh}\, u)}{dx} = \frac{1}{\sqrt{u^2-1}}\frac{du}{dx}$,

29. $\displaystyle \frac{d(\text{arctanh}\, u)}{dx} = \frac{1}{1-u^2}\frac{du}{dx}$, **30.** $\displaystyle \frac{d(\text{arccoth}\, u)}{dx} = \frac{1}{u^2-1}\frac{du}{dx}$,

31. $\displaystyle \frac{d(\text{arcsech}\, u)}{dx} = \frac{-1}{u\sqrt{1-u^2}}\frac{du}{dx}$, **32.** $\displaystyle \frac{d(\text{arccsch}\, u)}{dx} = \frac{-1}{|u|\sqrt{1+u^2}}\frac{du}{dx}$.

Integrals:

1. $\displaystyle \int cu\,dx = c\int u\,dx$, **2.** $\displaystyle \int (u+v)\,dx = \int u\,dx + \int v\,dx$,

3. $\displaystyle \int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1$, **4.** $\displaystyle \int \frac{1}{x}dx = \ln x$, **5.** $\displaystyle \int e^x\,dx = e^x$,

6. $\displaystyle \int \frac{dx}{1+x^2} = \arctan x$, **7.** $\displaystyle \int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$,

8. $\displaystyle \int \sin x\,dx = -\cos x$, **9.** $\displaystyle \int \cos x\,dx = \sin x$,

10. $\displaystyle \int \tan x\,dx = -\ln|\cos x|$, **11.** $\displaystyle \int \cot x\,dx = \ln|\cos x|$,

12. $\displaystyle \int \sec x\,dx = \ln|\sec x + \tan x|$, **13.** $\displaystyle \int \csc x\,dx = \ln|\csc x + \cot x|$,

14. $\displaystyle \int \arcsin \tfrac{x}{a}dx = \arcsin \tfrac{x}{a} + \sqrt{a^2 - x^2}, \quad a > 0$,

Calculus Cont.

15. $\int \arccos\frac{x}{a}dx = \arccos\frac{x}{a} - \sqrt{a^2-x^2}, \quad a>0,$

16. $\int \arctan\frac{x}{a}dx = x\arctan\frac{x}{a} - \frac{a}{2}\ln(a^2+x^2), \quad a>0,$

17. $\int \sin^2(ax)dx = \frac{1}{2a}\big(ax - \sin(ax)\cos(ax)\big),$

18. $\int \cos^2(ax)dx = \frac{1}{2a}\big(ax + \sin(ax)\cos(ax)\big),$

19. $\int \sec^2 x\,dx = \tan x,$

20. $\int \csc^2 x\,dx = -\cot x,$

21. $\int \sin^n x\,dx = -\frac{\sin^{n-1}x\cos x}{n} + \frac{n-1}{n}\int \sin^{n-2}x\,dx,$

22. $\int \cos^n x\,dx = \frac{\cos^{n-1}x\sin x}{n} + \frac{n-1}{n}\int \cos^{n-2}x\,dx,$

23. $\int \tan^n x\,dx = \frac{\tan^{n-1}x}{n-1} - \int \tan^{n-2}x\,dx, \quad n\neq 1,$

24. $\int \cot^n x\,dx = -\frac{\cot^{n-1}x}{n-1} - \int \cot^{n-2}x\,dx, \quad n\neq 1,$

25. $\int \sec^n x\,dx = \frac{\tan x\sec^{n-1}x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2}x\,dx, \quad n\neq 1,$

26. $\int \csc^n x\,dx = -\frac{\cot x\csc^{n-1}x}{n-1} + \frac{n-2}{n-1}\int \csc^{n-2}x\,dx, \quad n\neq 1,$

27. $\int \sinh x\,dx = \cosh x,$

28. $\int \cosh x\,dx = \sinh x,$

29. $\int \tanh x\,dx = \ln|\cosh x|,$

30. $\int \coth x\,dx = \ln|\sinh x|,$

31. $\int \operatorname{sech} x\,dx = \arctan\sinh x,$

32. $\int \operatorname{csch} x\,dx = \ln\left|\tanh\frac{x}{2}\right|,$

33. $\int \sinh^2 x\,dx = \frac{1}{4}\sinh(2x) - \frac{1}{2}x,$

34. $\int \cosh^2 x\,dx = \frac{1}{4}\sinh(2x) + \frac{1}{2}x,$

35. $\int \operatorname{sech}^2 x\,dx = \tanh x,$

36. $\int \operatorname{arcsinh}\frac{x}{a}dx = x\operatorname{arcsinh}\frac{x}{a} - \sqrt{x^2+a^2}, \quad a>0,$

37. $\int \operatorname{arctanh}\frac{x}{a}dx = x\operatorname{arctanh}\frac{x}{a} + \frac{a}{2}\ln|a^2-x^2|,$

38. $$\int \operatorname{arccosh}\frac{x}{a}dx = \begin{cases} x\operatorname{arccosh}\dfrac{x}{a} - \sqrt{x^2+a^2}, & \text{if } \operatorname{arccosh}\frac{x}{a}>0 \text{ and } a>0,\\ x\operatorname{arccosh}\dfrac{x}{a} + \sqrt{x^2+a^2}, & \text{if } \operatorname{arccosh}\frac{x}{a}<0 \text{ and } a>0,\end{cases}$$

39. $\int \frac{dx}{\sqrt{a^2+x^2}} = \ln\left(x+\sqrt{a^2+x^2}\right), \quad a>0,$

40. $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\arctan\frac{x}{a}, \quad a>0,$

41. $\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\arcsin\frac{x}{a}, \quad a>0,$

42. $\int (a^2-x^2)^{3/2}dx = \frac{x}{8}(5a^2-2x^2)\sqrt{a^2-x^2} + \frac{3a^4}{8}\arcsin\frac{x}{a}, \quad a>0,$

43. $\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a}, \quad a>0,$

44. $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right|,$

45. $\int \frac{dx}{(a^2-x^2)^{3/2}} = \frac{x}{a^2\sqrt{a^2-x^2}},$

46. $\int \sqrt{a^2\pm x^2}\,dx = \frac{x}{2}\sqrt{a^2\pm x^2} \pm \frac{a^2}{2}\ln\left|x+\sqrt{a^2\pm x^2}\right|,$

47. $\int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left|x+\sqrt{x^2-a^2}\right|, \quad a>0,$

48. $\int \frac{dx}{ax^2+bx} = \frac{1}{a}\ln\left|\frac{x}{a+bx}\right|,$

49. $\int x\sqrt{a+bx}\,dx = \frac{2(3bx-2a)(a+bx)^{3/2}}{15b^2},$

50. $\int \frac{\sqrt{a+bx}}{x}dx = 2\sqrt{a+bx} + a\int \frac{1}{x\sqrt{a+bx}}dx,$

51. $\int \frac{x}{\sqrt{a+bx}}dx = \frac{1}{\sqrt{2}}\ln\left|\frac{\sqrt{a+bx}-\sqrt{a}}{\sqrt{a+bx}+\sqrt{a}}\right|, \quad a>0,$

52. $\int \frac{\sqrt{a^2-x^2}}{x}dx = \sqrt{a^2-x^2} - a\ln\left|\frac{a+\sqrt{a^2-x^2}}{x}\right|,$

53. $\int x\sqrt{a^2-x^2}\,dx = -\frac{1}{3}(a^2-x^2)^{3/2},$

54. $\int x^2\sqrt{a^2-x^2}\,dx = \frac{x}{8}(2x^2-a^2)\sqrt{a^2-x^2} + \frac{a^4}{8}\arcsin\frac{x}{a}, \quad a>0,$

55. $\int \frac{dx}{\sqrt{a^2-x^2}} = -\frac{1}{a}\ln\left|\frac{a+\sqrt{a^2-x^2}}{x}\right|,$

56. $\int \frac{x\,dx}{\sqrt{a^2-x^2}} = -\sqrt{a^2-x^2},$

57. $\int \frac{x^2\,dx}{\sqrt{a^2-x^2}} = -\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\arcsin\frac{x}{a}, \quad a>0,$

58. $\int \frac{\sqrt{a^2+x^2}}{x}dx = \sqrt{a^2+x^2} - a\ln\left|\frac{a+\sqrt{a^2+x^2}}{x}\right|,$

59. $\int \frac{\sqrt{x^2-a^2}}{x}dx = \sqrt{x^2-a^2} - a\arccos\frac{a}{|x|}, \quad a>0,$

60. $\int x\sqrt{x^2\pm a^2}\,dx = \frac{1}{3}(x^2\pm a^2)^{3/2},$

61. $\int \frac{dx}{x\sqrt{x^2+a^2}} = \frac{1}{a}\ln\left|\frac{x}{a+\sqrt{a^2+x^2}}\right|,$

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Calculus Cont.

62. $\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\arccos\frac{a}{|x|}, \quad a>0,$ **63.** $\int \frac{dx}{x^2\sqrt{x^2\pm a^2}} = \mp\frac{\sqrt{x^2\pm a^2}}{a^2x},$

64. $\int \frac{x\,dx}{\sqrt{x^2\pm a^2}} = \sqrt{x^2\pm a^2},$ **65.** $\int \frac{\sqrt{x^2\pm a^2}}{x^4}\,dx = \mp\frac{(x^2+a^2)^{3/2}}{3a^2x^3},$

66. $$\int \frac{dx}{ax^2+bx+c} = \begin{cases} \frac{1}{\sqrt{b^2-4ac}}\ln\left|\frac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}}\right|, & \text{if } b^2>4ac, \\ \frac{2}{\sqrt{4ac-b^2}}\arctan\frac{2ax+b}{\sqrt{4ac-b^2}}, & \text{if } b^2<4ac, \end{cases}$$

67. $$\int \frac{dx}{\sqrt{ax^2+bx+c}} = \begin{cases} \frac{1}{\sqrt{a}}\ln\left|2ax+b+2\sqrt{a}\sqrt{ax^2+bx+c}\right|, & \text{if } a>0, \\ \frac{1}{\sqrt{-a}}\arcsin\frac{-2ax-b}{\sqrt{b^2-4ac}}, & \text{if } a<0, \end{cases}$$

68. $$\int \sqrt{ax^2+bx+c}\,dx = \frac{2ax+b}{4a}\sqrt{ax^2+bx+c} + \frac{4ax-b^2}{8a}\int\frac{dx}{\sqrt{ax^2+bx+c}},$$

69. $$\int \frac{x\,dx}{\sqrt{ax^2+bx+c}} = \frac{\sqrt{ax^2+bx+c}}{a} - \frac{b}{2a}\int\frac{dx}{\sqrt{ax^2+bx+c}},$$

70. $$\int \frac{dx}{x\sqrt{ax^2+bx+c}} = \begin{cases} \frac{-1}{\sqrt{c}}\ln\left|\frac{2\sqrt{c}\sqrt{ax^2+bx+c}+bx+2c}{x}\right|, & \text{if } c>0, \\ \frac{1}{\sqrt{-c}}\arcsin\frac{bx+2c}{|x|\sqrt{b^2-4ac}}, & \text{if } c<0, \end{cases}$$

71. $$\int x^3\sqrt{x^2+a^2}\,dx = (\tfrac{1}{3}x^2-\tfrac{2}{15}a^2)(x^2+a^2)^{3/2},$$

72. $$\int x^n\sin(ax)\,dx = -\tfrac{1}{a}x^n\cos(ax) + \tfrac{n}{a}\int x^{n-1}\cos(ax)\,dx,$$

73. $$\int x^n\cos(ax)\,dx = \tfrac{1}{a}x^n\sin(ax) - \tfrac{n}{a}\int x^{n-1}\sin(ax)\,dx,$$

74. $$\int x^n e^{ax}\,dx = \frac{x^n e^{ax}}{a} - \tfrac{n}{a}\int x^{n-1}e^{ax}\,dx,$$

75. $$\int x^n\ln(ax)\,dx = x^{n+1}\left(\frac{\ln(ax)}{n+1} - \frac{1}{(n+1)^2}\right),$$

76. $$\int x^n(\ln ax)^m\,dx = \frac{x^{n+1}}{n+1}(\ln ax)^m - \frac{m}{n+1}\int x^n(\ln ax)^{m-1}\,dx.$$

$x^1 =$	$x^{\underline{1}}$	$=$	$x^{\overline{1}}$
$x^2 =$	$x^{\underline{2}}+x^{\underline{1}}$	$=$	$x^{\overline{2}}-x^{\overline{1}}$
$x^3 =$	$x^{\underline{3}}+3x^{\underline{2}}+x^{\underline{1}}$	$=$	$x^{\overline{3}}-3x^{\overline{2}}+x^{\overline{1}}$
$x^4 =$	$x^{\underline{4}}+6x^{\underline{3}}+7x^{\underline{2}}+x^{\underline{1}}$	$=$	$x^{\overline{4}}-6x^{\overline{3}}+7x^{\overline{2}}-x^{\overline{1}}$
$x^5 =$	$x^{\underline{5}}+15x^{\underline{4}}+25x^{\underline{3}}+10x^{\underline{2}}+x^{\underline{1}}$	$=$	$x^{\underline{5}}-15x^{\overline{4}}+25x^{\overline{3}}-10x^{\overline{2}}+x^{\overline{1}}$

$x^{\overline{1}} =$	x^1	$x^{\underline{1}} =$	x^1
$x^{\overline{2}} =$	x^2+x^1	$x^{\underline{2}} =$	x^2-x^1
$x^{\overline{3}} =$	$x^3+3x^2+2x^1$	$x^{\underline{3}} =$	$x^3-3x^2+2x^1$
$x^{\overline{4}} =$	$x^4+6x^3+11x^2+6x^1$	$x^{\underline{4}} =$	$x^4-6x^3+11x^2-6x^1$
$x^{\overline{5}} =$	$x^5+10x^4+35x^3+50x^2+24x^1$	$x^{\underline{5}} =$	$x^5-10x^4+35x^3-50x^2+24x^1$

Finite Calculus

Difference, shift operators:

$$\Delta f(x) = f(x+1) - f(x),$$
$$\mathrm{E}\, f(x) = f(x+1).$$

Fundamental Theorem:

$$f(x) = \Delta F(x) \Leftrightarrow \sum f(x)\delta x = F(x) + C.$$
$$\sum_a^b f(x)\delta x = \sum_{i=a}^{b-1} f(i).$$

Differences:

$\Delta(cu) = c\Delta u, \qquad \Delta(u+v) = \Delta u + \Delta v,$

$\Delta(uv) = u\Delta v + \mathrm{E}\,v\Delta u,$

$\Delta(x^{\underline{n}}) = nx^{\underline{n-1}},$

$\Delta(H_x) = x^{\underline{-1}}, \qquad \Delta(2^x) = 2^x,$

$\Delta(c^x) = (c-1)c^x, \qquad \Delta\binom{x}{m} = \binom{x}{m-1}.$

Sums:

$\sum cu\,\delta x = c\sum u\,\delta x,$

$\sum(u+v)\,\delta x = \sum u\,\delta x + \sum v\,\delta x,$

$\sum u\Delta v\,\delta x = uv - \sum \mathrm{E}\,v\Delta u\,\delta x,$

$\sum x^{\underline{n}}\,\delta x = \frac{x^{\underline{n+1}}}{m+1}, \qquad \sum x^{\underline{-1}}\,\delta x = H_x,$

$\sum c^x\,\delta x = \frac{c^x}{c-1}, \qquad \sum\binom{x}{m}\,\delta x = \binom{x}{m+1}.$

Falling Factorial Powers:

$x^{\underline{n}} = x(x-1)\cdots(x-n+1), \quad n>0,$

$x^{\underline{0}} = 1,$

$x^{\underline{n}} = \frac{1}{(x+1)\cdots(x+|n|)}, \quad n<0,$

$x^{\underline{n+m}} = x^{\underline{m}}(x-m)^{\underline{n}}.$

Rising Factorial Powers:

$x^{\overline{n}} = x(x+1)\cdots(x+n-1), \quad n>0,$

$x^{\overline{0}} = 1,$

$x^{\overline{n}} = \frac{1}{(x-1)\cdots(x-|n|)}, \quad n<0,$

$x^{\overline{n+m}} = x^{\overline{m}}(x+m)^{\overline{n}}.$

Conversion:

$x^{\underline{n}} = (-1)^n(-x)^{\overline{n}} = (x-n+1)^{\overline{n}}$

$\quad = 1/(x+1)^{\overline{-n}},$

$x^{\overline{n}} = (-1)^n(-x)^{\underline{n}} = (x+n-1)^{\underline{n}}$

$\quad = 1/(x-1)^{\underline{-n}},$

$$x^n = \sum_{k=1}^n \left\{ n \atop k \right\} x^{\underline{k}} = \sum_{k=1}^n \left\{ n \atop k \right\}(-1)^{n-k}x^{\overline{k}},$$

$$x^{\underline{n}} = \sum_{k=1}^n \left[n \atop k \right](-1)^{n-k}x^k,$$

$$x^{\overline{n}} = \sum_{k=1}^n \left[n \atop k \right]x^k.$$

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Series

Taylor's series:

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2}f''(a) + \cdots = \sum_{i=0}^{\infty} \frac{(x-a)^i}{i!} f^{(i)}(a).$$

Expansions:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots = \sum_{i=0}^{\infty} x^i,$$

$$\frac{1}{1-cx} = 1 + cx + c^2x^2 + c^3x^3 + \cdots = \sum_{i=0}^{\infty} c^i x^i,$$

$$\frac{1}{1-x^n} = 1 + x^n + x^{2n} + x^{3n} + \cdots = \sum_{i=0}^{\infty} x^{ni},$$

$$\frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + 4x^4 + \cdots = \sum_{i=0}^{\infty} i x^i,$$

$$\sum_{k=0}^{n} \left\{ n \atop k \right\} \frac{k! z^k}{(1-z)^{k+1}} = x + 2^n x^2 + 3^n x^3 + 4^n x^4 + \cdots = \sum_{i=0}^{\infty} i^n x^i,$$

$$e^x = 1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + \cdots = \sum_{i=0}^{\infty} \frac{x^i}{i!},$$

$$\ln(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 - \cdots = \sum_{i=1}^{\infty} (-1)^{i+1} \frac{x^i}{i},$$

$$\ln \frac{1}{1-x} = x + \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 + \tfrac{1}{4}x^4 + \cdots = \sum_{i=1}^{\infty} \frac{x^i}{i},$$

$$\sin x = x - \tfrac{1}{3!}x^3 + \tfrac{1}{5!}x^5 - \tfrac{1}{7!}x^7 + \cdots = \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i+1}}{(2i+1)!},$$

$$\cos x = 1 - \tfrac{1}{2!}x^2 + \tfrac{1}{4!}x^4 - \tfrac{1}{6!}x^6 + \cdots = \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i}}{(2i)!},$$

$$\tan^{-1} x = x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \tfrac{1}{7}x^7 + \cdots = \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i+1}}{(2i+1)},$$

$$(1+x)^n = 1 + nx + \tfrac{n(n-1)}{2}x^2 + \cdots = \sum_{i=0}^{\infty} \binom{n}{i} x^i,$$

$$\frac{1}{(1-x)^{n+1}} = 1 + (n+1)x + \tbinom{n+2}{2}x^2 + \cdots = \sum_{i=0}^{\infty} \binom{i+n}{i} x^i,$$

$$\frac{x}{e^x - 1} = 1 - \tfrac{1}{2}x + \tfrac{1}{12}x^2 - \tfrac{1}{720}x^4 + \cdots = \sum_{i=0}^{\infty} \frac{B_i x^i}{i!},$$

$$\frac{1}{2x}(1 - \sqrt{1-4x}) = 1 + x + 2x^2 + 5x^3 + \cdots = \sum_{i=0}^{\infty} \frac{1}{i+1} \binom{2i}{i} x^i,$$

$$\frac{1}{\sqrt{1-4x}} = 1 + 2x + 6x^2 + 20x^3 + \cdots = \sum_{i=0}^{\infty} \binom{2i}{i} x^i,$$

$$\frac{1}{\sqrt{1-4x}} \left(\frac{1 - \sqrt{1-4x}}{2x} \right)^n = 1 + (2+n)x + \tbinom{4+n}{2}x^2 + \cdots = \sum_{i=0}^{\infty} \binom{2i+n}{i} x^i,$$

$$\frac{1}{1-x} \ln \frac{1}{1-x} = x + \tfrac{3}{2}x^2 + \tfrac{11}{6}x^3 + \tfrac{25}{12}x^4 + \cdots = \sum_{i=1}^{\infty} H_i x^i,$$

$$\frac{1}{2} \left(\ln \frac{1}{1-x} \right)^2 = \tfrac{1}{2}x^2 + \tfrac{3}{4}x^3 + \tfrac{11}{24}x^4 + \cdots = \sum_{i=2}^{\infty} \frac{H_{i-1} x^i}{i},$$

$$\frac{x}{1-x-x^2} = x + x^2 + 2x^3 + 3x^4 + \cdots = \sum_{i=0}^{\infty} F_i x^i,$$

$$\frac{F_n x}{1 - (F_{n-1} + F_{n+1})x - (-1)^n x^2} = F_n x + F_{2n} x^2 + F_{3n} x^3 + \cdots = \sum_{i=0}^{\infty} F_{ni} x^i.$$

Ordinary power series:

$$A(x) = \sum_{i=0}^{\infty} a_i x^i.$$

Exponential power series:

$$A(x) = \sum_{i=0}^{\infty} a_i \frac{x^i}{i!}.$$

Dirichlet power series:

$$A(x) = \sum_{i=1}^{\infty} \frac{a_i}{i^x}.$$

Binomial theorem:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$

Difference of like powers:

$$x^n - y^n = (x-y) \sum_{k=0}^{n-1} x^{n-1-k} y^k.$$

For ordinary power series:

$$\alpha A(x) + \beta B(x) = \sum_{i=0}^{\infty} (\alpha a_i + \beta b_i) x^i,$$

$$x^k A(x) = \sum_{i=k}^{\infty} a_{i-k} x^i,$$

$$\frac{A(x) - \sum_{i=0}^{k-1} a_i x^i}{x^k} = \sum_{i=0}^{\infty} a_{i+k} x^i,$$

$$A(cx) = \sum_{i=0}^{\infty} c^i a_i x^i,$$

$$A'(x) = \sum_{i=0}^{\infty} (i+1) a_{i+1} x^i,$$

$$xA'(x) = \sum_{i=1}^{\infty} i a_i x^i,$$

$$\int A(x)\,dx = \sum_{i=1}^{\infty} \frac{a_{i-1}}{i} x^i,$$

$$\frac{A(x) + A(-x)}{2} = \sum_{i=0}^{\infty} a_{2i} x^{2i},$$

$$\frac{A(x) - A(-x)}{2} = \sum_{i=0}^{\infty} a_{2i+1} x^{2i+1}.$$

Summation: If $b_i = \sum_{j=0}^{i} a_i$ then

$$B(x) = \frac{1}{1-x} A(x).$$

Convolution:

$$A(x)B(x) = \sum_{i=0}^{\infty} \left(\sum_{j=0}^{i} a_j b_{i-j} \right) x^i.$$

God made the natural numbers;
all the rest is the work of man.
– Leopold Kronecker

Theoretical Computer Science Cheat Sheet

Series

Expansions:

$$\frac{1}{(1-x)^{n+1}}\ln\frac{1}{1-x} = \sum_{i=0}^{\infty}(H_{n+i}-H_n)\binom{n+i}{i}x^i,$$

$$\left(\frac{1}{x}\right)^{\overline{-n}} = \sum_{i=0}^{\infty}\left\{ {i \atop n} \right\}x^i,$$

$$x^{\overline{n}} = \sum_{i=0}^{\infty}\left[{n \atop i} \right]x^i,$$

$$(e^x-1)^n = \sum_{i=0}^{\infty}\left\{ {i \atop n} \right\}\frac{n!x^i}{i!},$$

$$\left(\ln\frac{1}{1-x}\right)^n = \sum_{i=0}^{\infty}\left[{i \atop n} \right]\frac{n!x^i}{i!},$$

$$x\cot x = \sum_{i=0}^{\infty}\frac{(-4)^iB_{2i}x^{2i}}{(2i)!},$$

$$\tan x = \sum_{i=1}^{\infty}(-1)^{i-1}\frac{2^{2i}(2^{2i}-1)B_{2i}x^{2i-1}}{(2i)!},$$

$$\zeta(x) = \sum_{i=1}^{\infty}\frac{1}{i^x},$$

$$\frac{1}{\zeta(x)} = \sum_{i=1}^{\infty}\frac{\mu(i)}{i^x},$$

$$\frac{\zeta(x-1)}{\zeta(x)} = \sum_{i=1}^{\infty}\frac{\phi(i)}{i^x},$$

$$\zeta(x) = \prod_p\frac{1}{1-p^{-x}},$$

$$\zeta^2(x) = \sum_{i=1}^{\infty}\frac{d(i)}{x^i} \quad \text{where } d(n)=\textstyle\sum_{d|n}1,$$

$$\zeta(x)\zeta(x-1) = \sum_{i=1}^{\infty}\frac{S(i)}{x^i} \quad \text{where } S(n)=\textstyle\sum_{d|n}d,$$

$$\zeta(2n) = \frac{2^{2n-1}|B_{2n}|}{(2n)!}\pi^{2n}, \quad n\in\mathbb{N},$$

$$\frac{x}{\sin x} = \sum_{i=0}^{\infty}(-1)^{i-1}\frac{(4^i-2)B_{2i}x^{2i}}{(2i)!},$$

$$\left(\frac{1-\sqrt{1-4x}}{2x}\right)^n = \sum_{i=0}^{\infty}\frac{n(2i+n-1)!}{i!(n+i)!}x^i,$$

$$e^x\sin x = \sum_{i=1}^{\infty}\frac{2^{i/2}\sin\frac{i\pi}{4}}{i!}x^i,$$

$$\sqrt{\frac{1-\sqrt{1-x}}{x}} = \sum_{i=0}^{\infty}\frac{(4i)!}{16^i\sqrt{2}(2i)!(2i+1)!}x^i,$$

$$\left(\frac{\arcsin x}{x}\right)^2 = \sum_{i=0}^{\infty}\frac{4^ii!^2}{(i+1)(2i+1)!}x^{2i}.$$

Escher's Knot

Stieltjes Integration

If G is continuous in the interval $[a,b]$ and F is nondecreasing then

$$\int_a^b G(x)\,dF(x)$$

exists. If $a\le b\le c$ then

$$\int_a^c G(x)\,dF(x) = \int_a^b G(x)\,dF(x) + \int_b^c G(x)\,dF(x).$$

If the integrals involved exist

$$\int_a^b \big(G(x)+H(x)\big)\,dF(x) = \int_a^b G(x)\,dF(x) + \int_a^b H(x)\,dF(x),$$

$$\int_a^b G(x)\,d\big(F(x)+H(x)\big) = \int_a^b G(x)\,dF(x) + \int_a^b G(x)\,dH(x),$$

$$\int_a^b c\cdot G(x)\,dF(x) = \int_a^b G(x)\,d\big(c\cdot F(x)\big) = c\int_a^b G(x)\,dF(x),$$

$$\int_a^b G(x)\,dF(x) = G(b)F(b)-G(a)F(a)-\int_a^b F(x)\,dG(x).$$

If the integrals involved exist, and F possesses a derivative F' at every point in $[a,b]$ then

$$\int_a^b G(x)\,dF(x) = \int_a^b G(x)F'(x)\,dx.$$

Cramer's Rule

If we have equations:

$$\begin{aligned} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n &= b_1\\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n &= b_2\\ \vdots \qquad\quad \vdots \qquad\qquad\quad \vdots &\\ a_{n,1}x_1 + a_{n,2}x_2 + \cdots + a_{n,n}x_n &= b_n \end{aligned}$$

Let $A=(a_{i,j})$ and B be the column matrix (b_i). Then there is a unique solution iff $\det A\ne 0$. Let A_i be A with column i replaced by B. Then

$$x_i = \frac{\det A_i}{\det A}.$$

00	47	18	76	29	93	85	34	61	52
86	11	57	28	70	39	94	45	02	63
95	80	22	67	38	71	49	56	13	04
59	96	81	33	07	48	72	60	24	15
73	69	90	82	44	17	58	01	35	26
68	74	09	91	83	55	27	12	46	30
37	08	75	19	92	84	66	23	50	41
14	25	36	40	51	62	03	77	88	99
21	32	43	54	65	06	10	89	97	78
42	53	64	05	16	20	31	98	79	87

Fibonacci Numbers

$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$

Definitions:

$$F_i = F_{i-1}+F_{i-2}, \quad F_0=F_1=1,$$

$$F_{-i} = (-1)^{i-1}F_i,$$

$$F_i = \tfrac{1}{\sqrt{5}}\left(\phi^i-\hat{\phi}^i\right),$$

Cassini's identity: for $i>0$:

$$F_{i+1}F_{i-1}-F_i^2 = (-1)^i.$$

Additive rule:

$$F_{n+k} = F_kF_{n+1}+F_{k-1}F_n,$$

$$F_{2n} = F_nF_{n+1}+F_{n-1}F_n.$$

Calculation by matrices:

$$\begin{pmatrix} F_{n-2} & F_{n-1}\\ F_{n-1} & F_n \end{pmatrix} = \begin{pmatrix} 0 & 1\\ 1 & 1 \end{pmatrix}^n.$$

The Fibonacci number system: Every integer n has a unique representation

$$n = F_{k_1}+F_{k_2}+\cdots+F_{k_m},$$

where $k_i \ge k_{i+1}+2$ for all i, $1\le i<m$ and $k_m\ge 2$.

Improvement makes strait roads, but the crooked roads without Improvement, are roads of Genius.
– William Blake (The Marriage of Heaven and Hell)